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Volodymyr MAZORCHUK & Vanessa MIEMIETZ

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SERRE FUNCTORS FOR LIE ALGEBRAS AND SUPERALGEBRAS

by Volodymyr MAZORCHUK & Vanessa MIEMIETZ

Abstract. — We propose a new realization, using Harish-Chandra bimodules, of the Serre functor for the BGG category $\mathcal{O}$ associated to a semi-simple complex finite dimensional Lie algebra. We further show that our realization carries over to classical Lie superalgebras in many cases. Along the way we prove that category $\mathcal{O}$ and its parabolic generalizations for classical Lie superalgebras are categories with full projective functors. As an application we prove that in many cases the endomorphism algebra of the basic projective-injective module in (parabolic) category $\mathcal{O}$ for classical Lie superalgebras is symmetric. As a special case we obtain that in these cases the algebras describing blocks of the category of finite dimensional modules are symmetric. We also compute the latter algebras for the superalgebra $\mathfrak{q}(2)$.

Résumé. — Nous proposons une nouvelle réalisation du foncteur de Serre pour la catégorie $\mathcal{O}$ de BGG associée à une algèbre de Lie semi-simple complexe de dimension finie, en utilisant les bimodules d'Harish-Chandra. De plus, nous démontrons que dans beaucoup de cas notre réalisation s'applique aux super algèbres de Lie classiques. Pour cela, nous prouvons que la catégorie $\mathcal{O}$ et ses généralisations paraboliques pour les super-algèbres de Lie classiques sont des catégories avec foncteurs pleins projectifs. Comme application, nous montrons que, dans beaucoup de cas, l'algèbre d'endomorphismes du module projectif-injectif basique de la catégorie $\mathcal{O}$ (parabolique) pour les super-algèbres de Lie est symétrique. En particulier, dans ce cas, les algèbres décrivant les blocs de la catégorie de modules de dimension finie sont symétriques. Nous calculons ces dernières algèbres pour la super algèbre de Lie $\mathfrak{q}(2)$.

1. Introduction and description of the results

The category of finite dimensional modules over a semi-simple complex finite dimensional Lie algebra is semi-simple and hence completely understood. For Lie superalgebras the situation is rather more complicated. Although many blocks of the category of finite dimensional modules over a

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Lie superalgebra are still semi-simple, a general block might have infinitely many simple modules or infinite global dimension. The recent preprint [8] gives a combinatorial description of associative algebras whose module categories are equivalent to blocks of the category of finite dimensional $\mathfrak{gl}(m,n)$-modules. This description implies many nice properties of these algebras, in particular, it turns out that these algebras are symmetric and Koszul (this is proved in earlier articles of the series). To solve a similar problem for other Lie superalgebras is an open and seemingly difficult task.

Understanding the category of finite dimensional modules is the first step towards understanding the representation theory of a Lie superalgebra. The next natural step is to understand the analogue of the BGG category $\mathcal{O}$ (and its numerous generalizations). Various questions related to the general theory of highest weight modules over finite dimensional Lie superalgebras have been studied in [6, 5, 15, 9], see also references therein. However, as in the case of finite dimensional modules, most of the problems in this theory are still to be solved.

Among the classical Lie superalgebras the queer Lie superalgebra $\mathfrak{q}(n)$ stands out in many ways. While it looks deceptively easy, consisting only of an even and an odd copy of $\mathfrak{gl}_n$, its representation theory is surprisingly intricate. One of the roots of this problem is that the classical Cartan subalgebra of $\mathfrak{q}(n)$ is non-commutative. Two major achievements in the representation theory of $\mathfrak{q}(n)$ were a description of characters of simple finite dimensional $\mathfrak{q}(n)$-modules given in [30, 29], and a relation of character formulae for $\mathfrak{q}(n)$ to Kazhdan-Lusztig combinatorics in [5]. The latter article contains, in particular, conjectural combinatorics for the whole category $\mathcal{O}$. For "easy" blocks this conjecture was proved in [11].

Motivated by the ultimate goal of proving the conjectures from [5] and extending the results of [8] to $\mathfrak{q}(n)$, in the present article we take some first steps in this direction. We consider a classical Lie superalgebra $\mathfrak{g}$ from the list $\mathfrak{gl}(m,n)$, $\mathfrak{sl}(m,n)$, $\mathfrak{osp}(m,n)$, $\mathfrak{psl}(n,n)$, $\mathfrak{q}(n)$, $\mathfrak{pq}(n)$, $\mathfrak{sq}(n)$, $\mathfrak{psq}(n)$. We show that a block of the category of finite dimensional $\mathfrak{g}$-modules, related in a certain way to a simple strongly typical module which is not stable under parity change, is described by a symmetric algebra. For the first four superalgebras this covers all blocks and for the last four this covers roughly half of the blocks.

To this end we study Serre functors for the category $\mathcal{O}$ and its parabolic generalizations. The Lie algebraic counterpart of this theory was developed in [25] and is heavily based on the combinatorial description of blocks of $\mathcal{O}$ for Lie algebras, which does not yet exist for Lie superalgebras. Therefore we are forced to reprove the main results from [25] in a completely different way. The paper [25] establishes the important property that the Serre functor naturally commutes with projective functors (in the sense of [20]). This silently uses the fact that the category $\mathcal{O}$ for Lie algebras is a category with full projective functors in the sense of [20].

We show that, under some natural assumptions, the category $\mathcal{O}$ for Lie superalgebras as well as its parabolic generalizations are categories with full projective functors. This setup involves a choice of the so-called "dominant object" Δ, which in our case turns out to be a (parabolic) Verma module, whose highest weight satisfies certain conditions. We observe that a convenient way to define functors naturally commuting with projective functors is to use Harish-Chandra bimodules. For two $\mathfrak{g}$-modules M and N the space $\mathcal{L}(M,N)$ of all linear maps from M to N which are locally finite with respect to the adjoint action of the even part has a natural $\mathfrak{g}$-bimodule structure. Extending results of [2] and [27] we show that certain direct summands $\mathcal{O}^{\mathfrak{p}}_{\underline{\lambda}}$ of the (parabolic) category $\mathcal{O}^{\mathfrak{p}}$ for Lie superalgebras are equivalent to categories of Harish-Chandra bimodules. On the latter we have the usual restricted duality $\_^{\circledast}$. As the main result of the paper we prove the following.

Theorem 1.1. — *For category $\mathcal{O}^{\mathfrak{p}}_{\underline{\lambda}}$, the left derived of the functor*

$$\mathcal{L}(\_,\Delta)^{\circledast}\otimes_{U(\mathfrak{g})}\Delta$$

is a Serre functor on the corresponding category of perfect complexes.

Even in the case of Lie algebras this description of the Serre functor is new. The proof relies heavily on methods developed by Gorelik in [13], which we adapt to our more general situation. As an application we obtain that the endomorphism algebra of the basic projective-injective module in $\mathcal{O}^{\mathfrak{p}}_{\underline{\lambda}}$ is symmetric. For Lie superalgebras of type I this reproves and strengthens [3, Theorem 3.9.1] which says that the endomorphism algebra of a basic projective-injective finite dimensional module is weakly symmetric. We use the latter result to compute the associative algebras describing blocks of the category of finite dimensional $\mathfrak{q}(2)$-modules.

The article is organized as follows: In Section 2 we collect necessary preliminaries on modules over $\Bbbk$-linear categories, Serre functors and symmetric algebras. In Section 3 we recall the theory of categories with full projective functors following [20]. In Section 4 and Section 5 we prove the main results for Lie algebras and superalgebras, respectively. Finally, Section 6 is devoted to the above mentioned example of $\mathfrak{q}(2)$.

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2. Serre functors for $\Bbbk$-linear categories

2.1. Conventions

For an abstract category $\mathscr{C}$ and two objects $x, y \in \mathscr{C}$ we denote by $\mathscr{C}(x, y)$ the set of morphisms from x to y in $\mathscr{C}$. At the same time, for categories of modules and the corresponding derived categories, we will use the usual Hom notation. In particular, working with $\mathscr{C}$-modules (that is functors from $\mathscr{C}$ to some fixed category), $\mathrm{Hom}_{\mathscr{C}}$ will mean morphisms in the category of $\mathscr{C}$-modules.

For an abelian category $\mathcal{A}$ we denote by $\mathcal{D}^-(\mathcal{A})$ and $\mathcal{D}^b(\mathcal{A})$ the derived categories of complexes (of objects in $\mathcal{A}$) bounded from the right and from both sides, respectively. If $\mathcal{A}$ is the category of $\mathscr{C}$-modules for some category $\mathscr{C}$, the corresponding derived categories $\mathcal{D}^-(\mathcal{A})$ and $\mathcal{D}^b(\mathcal{A})$ will be denoted simply by $\mathcal{D}^-(\mathscr{C})$ and $\mathcal{D}^b(\mathscr{C})$, respectively (and we will never consider categories of modules over abelian categories). For a right exact functor F on $\mathcal{A}$ we denote by $\mathscr{L}\mathrm{F}$ the corresponding left derived functor.

For the rest of the paper we fix an algebraically closed field $\Bbbk$ and denote by $\_^*$ the usual duality $\mathrm{Hom}_{\Bbbk}(\_, \Bbbk)$. All categories in the paper are assumed to be $\Bbbk$-linear (i.e. enriched over $\Bbbk$-mod) and all functors are supposed to be $\Bbbk$-linear and additive. If not stated otherwise, a functor always means a *covariant* functor.

2.2. Serre functors

Let $\mathscr{C}$ be a $\Bbbk$-linear additive category with finite dimensional morphism spaces. A *Serre functor* on $\mathscr{C}$ is an additive auto-equivalence F of $\mathscr{C}$ together with isomorphisms

$$\Psi_{x,y} : \mathscr{C}(x, \mathrm{F}y) \cong \mathscr{C}(y, x)^*, \tag{2.1}$$

natural in x and y (see [4]). If a Serre functor exists, it is unique (up to isomorphism) and commutes with all auto-equivalences of $\mathscr{C}$.

For example, let A be a finite dimensional associative $\Bbbk$-algebra of finite global dimension. Then the category $\mathcal{D}^b(A)$ always has a Serre functor, which is given by the left derived of the *Nakayama functor*

$$A^* \otimes_A {}_- : A\text{-mod} \to A\text{-mod}$$

(see [16]). Note that we have an isomorphism of functors $A^* \otimes_A {}_- \cong \mathrm{Hom}_A({}_-, A)^*$ (as both are right exact and agree on the projective generator A). The last example is typical in the sense that to have a Serre functor one usually has to extend the original abelian category A-mod to the triangulated category $\mathcal{D}^b(A)$ (or some other triangulated category, for example, the category of perfect complexes, see Subsection 2.4).

2.3. Infinite dimensional setup

In what follows we will often work with abelian categories having infinitely many isoclasses of simple objects. Therefore we will need an "infinite dimensional" generalization of the last example. Consider a small $\Bbbk$-linear category $\mathcal{C}$ which satisfies the following assumptions:

(I) $\mathcal{C}$ is *basic* in the sense that different objects from $\mathcal{C}$ are not isomorphic;
(II) for any $x, y \in \mathcal{C}$ the $\Bbbk$-vector space $\mathcal{C}(x, y)$ is finite dimensional;
(III) for any $x \in \mathcal{C}$ there exist only finitely many $y \in \mathcal{C}$ such that $\mathcal{C}(x, y) \neq 0$;
(IV) for any $x \in \mathcal{C}$ there exist only finitely many $y \in \mathcal{C}$ such that $\mathcal{C}(y, x) \neq 0$;
(V) for any $x \in \mathcal{C}$ the endomorphism algebra $\mathcal{C}(x, x)$ is local and basic.

We will call such categories *strongly locally finite*, or simply *slf-categories*.

A *left $\mathcal{C}$-module* is a covariant functor $M : \mathcal{C} \to \Bbbk\text{-Mod}$. A $\mathcal{C}$-module M is said to be *finite dimensional* provided that $\sum_{x \in \mathcal{C}} \dim M(x) < \infty$. We denote by $\mathcal{C}$-mod the category of all finite dimensional $\mathcal{C}$-modules. The corresponding notion and category mod-$\mathcal{C}$ of *right* modules are defined similarly using contravariant functors. The functor ${}_-^*$ induces a duality between $\mathcal{C}$-mod and mod-$\mathcal{C}$.

Because of our assumptions (I)-(V), indecomposable projective $\mathcal{C}$-modules are of the form $\mathcal{C}(x, {}_-)$, $x \in \mathcal{C}$, and belong to $\mathcal{C}$-mod. Hence $\mathcal{C}$-mod is an abelian length category (i.e. every object has finite length) with enough projectives and injectives (because of $\_^*$). Let $\overline{\mathcal{C}}$ denote the full subcategory of $\mathcal{C}$-mod consisting of the indecomposable projectives $\mathcal{C}(x, \_)$, $x \in \mathcal{C}$. Then $\mathcal{C}$-mod is equivalent to the category of finite-dimensional $\overline{\mathcal{C}}^{\mathrm{op}}$-modules, moreover, the category $\overline{\mathcal{C}}^{\mathrm{op}}$ is isomorphic to $\mathcal{C}$.

Remark 2.1. — For an slf-category $\mathcal{C}$ denote by $A_{\mathcal{C}}$ the path algebra of $\overline{\mathcal{C}}^{\mathrm{op}}$. Then the category $\mathcal{C}$-mod is equivalent to the category of finite-dimensional $A_{\mathcal{C}}$-modules. The algebra $A_{\mathcal{C}}$ can be axiomatically described by the following properties:

- $A_{\mathcal{C}}$ is equipped with a system $\{e_{\mathtt{i}} : \mathtt{i} \in \mathscr{C}\}$ of primitive orthogonal idempotents (the identity morphisms for objects of $\mathscr{C}$);
- each $A_{\mathcal{C}} e_{\mathtt{i}}$ and $e_{\mathtt{i}} A_{\mathcal{C}}$ is finite dimensional (this combines (II)–(IV));
- $A_{\mathcal{C}} = \displaystyle\bigoplus_{\mathtt{i},\mathtt{j} \in \mathscr{C}} e_{\mathtt{i}} A_{\mathcal{C}} e_{\mathtt{j}}$.

A $\mathcal{C}$*-bimodule* is a bifunctor $B = B(\_, \_)$ from $\mathcal{C}$ to $\Bbbk$-Mod, contravariant in the first (left) argument and covariant in the second (right) argument. A typical example of a bimodule is the regular bimodule $\mathcal{C} = \mathcal{C}(\_, \_)$.

The Nakayama functor $\mathrm{N} := \mathcal{C}^* \otimes_{\mathcal{C}} \_ \cong \mathrm{Hom}_{\mathcal{C}}(\_, \mathcal{C})^*$ is an endofunctor of $\mathcal{C}$-mod (for more details on tensor products see [24, 2.2]). Consider the endofunctor $\mathscr{L}\mathrm{N}$ of $\mathcal{D}^-(\mathcal{C})$. Let $\mathscr{P}(\mathcal{C})$ denote the full subcategory of $\mathcal{D}^-(\mathcal{C})$ consisting of objects isomorphic to finite complexes of projective objects (the so-called *perfect complexes*). Our first easy observation is the following (compare with [25, 4.3]):

Proposition 2.2. — *Assume that all injective $\mathcal{C}$-modules are of finite projective dimension. Then $\mathscr{L}\mathrm{N}$ is a Serre functor on $\mathscr{P}(\mathcal{C})$.*

Proof. — To start with we claim that $\mathscr{L}\mathrm{N}$ preserves $\mathscr{P}(\mathcal{C})$. Indeed, the functor $\mathscr{L}\mathrm{N}$ is a triangle functor and $\mathscr{P}(\mathcal{C})$ is generated, as a triangulated category, by indecomposable projective $\mathcal{C}$-modules. The functor N maps an indecomposable projective $\mathcal{C}$-module to an indecomposable injective $\mathcal{C}$-module, and the latter has finite projective dimension and hence is an object of $\mathscr{P}(\mathcal{C})$.

That $\mathscr{L}\mathrm{N}$ has the property given by (2.1) is checked by the following standard computation: For any $N \in \mathscr{P}(\mathcal{C})$ and projective $P \in \mathcal{C}$-mod we have natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{P}(\mathcal{C})}(N, \mathscr{L}\mathrm{N}\, P)^* &= \mathrm{Hom}_{\mathscr{P}(\mathcal{C})}(N, \mathrm{Hom}_{\Bbbk}(\mathrm{Hom}_{\mathcal{C}}(P, \mathcal{C}), \Bbbk))^* \\
\text{(by adjunction)} &= \mathrm{Hom}_{\Bbbk}(\mathrm{Hom}_{\mathcal{C}}(P, \mathcal{C}) \otimes_{\mathcal{C}} N, \Bbbk)^* \\
\text{(as } (_^*)^* = \mathrm{Id}) &= \mathrm{Hom}_{\mathcal{C}}(P, \mathcal{C}) \otimes_{\mathcal{C}} N \\
\text{(by projectivity of } P) &= \mathrm{Hom}_{\mathcal{C}}(P, N).
\end{aligned}$$

Using the triangle property for $\mathscr{L}\mathrm{N}$ this extends to the whole of $\mathscr{P}(\mathcal{C})$ (in the second variable) and proves (2.1). □

2.4. Serre functors and symmetric algebras

Let $\mathcal{C}$ be as in the previous subsection. The category $\mathcal{C}$ is called *symmetric* provided that the $\mathcal{C}$-$\mathcal{C}$-bimodules $\mathcal{C}$ and $\mathcal{C}^*$ are isomorphic. Our second observation is the following infinite-dimensional generalization of [25, Lemma 3.1]:

Proposition 2.3. — *Assume that all injective $\mathcal{C}$-modules are of finite projective dimension. Then $\mathcal{C}$ is symmetric if and only if the Serre functor on $\mathscr{P}(\mathcal{C})$ is isomorphic to the identity.*

Proof. — By Proposition 2.2, under our assumptions the functor $\mathscr{L}\mathrm{N}$ is the Serre functor on $\mathscr{P}(\mathcal{C})$. If $\mathcal{C}$ is symmetric, then $\mathrm{N} = \mathcal{C}^* \otimes_{\mathcal{C}} {}_- \cong \mathcal{C} \otimes_{\mathcal{C}} {}_-$, the latter being isomorphic to the identity functor on $\mathcal{C}$-mod. Hence $\mathscr{L}\mathrm{N}$ is isomorphic to the identity functor.

Conversely, if $\mathscr{L}\mathrm{N}$ is isomorphic to the identity functor, then N is isomorphic to the identity functor, when restricted to the additive subcategory of all projective modules in $\mathcal{C}$-mod. The latter is certainly true for the functor $\mathcal{C} \otimes_{\mathcal{C}} {}_-$. As both N and $\mathcal{C} \otimes_{\mathcal{C}} {}_-$ are right exact and agree on projective modules, they must be isomorphic, which implies that the bimodules $\mathcal{C}^*$ and $\mathcal{C}$ are isomorphic. □

3. Categories with full projective functors

3.1. Definitions

Let $\mathscr{A}$ be an abelian category, $M \in \mathscr{A}$ and $\mathcal{F} := \{\mathrm{F}_i : i \in I\}$ a full subcategory of the category of right exact endofunctors of $\mathscr{A}$. We assume that $\mathcal{F}$ is closed (up to isomorphism) under direct sums and composition. Following [20] we say that $(\mathscr{A}, M, \mathcal{F})$ is a *category with full projective functors*, or, simply, an *fpf-category*, provided that

(i) $\mathrm{Id}_{\mathscr{A}} \in \mathcal{F}$;
(ii) for every $i \in I$ the object $\mathrm{F}_i\, M$ is projective in $\mathscr{A}$;
(iii) every $N \in \mathscr{A}$ is a quotient of $\mathrm{F}_i\, M$ for some $i \in I$;
(iv) for all $i, j \in I$ the evaluation map $\mathrm{ev}_M : \mathcal{F}(\mathrm{F}_i, \mathrm{F}_j) \to \mathscr{A}(\mathrm{F}_i\, M, \mathrm{F}_j\, M)$ is surjective.

Functors in $\mathcal{F}$ are called *projective functors* and M is called the *dominant object.*

3.2. Basic examples

Let $\mathfrak{g}$ be a semi-simple finite-dimensional complex Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Consider the BGG category $\mathcal{O}$ associated with this triangular decomposition (for details on $\mathcal{O}$ we refer the reader to [17]). Let Θ be the set of weights of all finite dimensional $\mathfrak{g}$-modules. For $\lambda \in \mathfrak{h}^*$ consider the coset $\overline{\lambda} = \lambda + \Theta$ and denote by $\mathcal{O}_{\overline{\lambda}}$ the full subcategory of $\mathcal{O}$ containing all modules M such that $\operatorname{supp} M \subset \overline{\lambda}$. Fix any dominant regular $\mu \in \overline{\lambda}$ and consider the corresponding Verma module $\Delta(\mu)$ (with highest weight μ).

For every finite-dimensional $\mathfrak{g}$-module E the functor $E \otimes_{\mathbb{C}} {}_-$ preserves $\mathcal{O}_{\overline{\lambda}}$. Let $\mathcal{F}$ denote the family of all such functors. Then $(\mathcal{O}_{\overline{\lambda}}, \Delta(\mu), \mathcal{F})$ is an fpf-category, see [20, Proposition 16].

If $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h} \oplus \mathfrak{n}_+$, we have the full subcategory $\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$ of $\mathcal{O}_{\overline{\lambda}}$ consisting of all modules on which the action of $\mathfrak{p}$ is locally finite. Every functor in $\mathcal{F}$ preserves $\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$. Let $\Delta(\mu)^{\mathfrak{p}}$ denote the maximal quotient of $\Delta(\mu)$ which lies in $\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$. Then $(\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}, \Delta(\mu)^{\mathfrak{p}}, \mathcal{F})$ is an fpf-category, see [20, Proposition 22]. For further examples of categories with full projective functors see [20].

3.3. Functors naturally commuting with projective functors

Assume that $(\mathscr{A}, M, \mathcal{F})$ is an fpf-category as in Subsection 3.1. An endofunctor G of $\mathscr{A}$ is said to *naturally commute* with projective functors if for every $i \in I$ there is an isomorphism $\eta_i : \mathrm{F}_i \circ \mathrm{G} \to \mathrm{G} \circ \mathrm{F}_i$ such that for any $i, j \in I$ and any $\alpha \in \mathcal{F}(\mathrm{F}_i, \mathrm{F}_j)$ the diagram

$$\begin{tikzcd}
\mathrm{F}_i \circ \mathrm{G} \arrow[r, "\alpha_{\mathrm{G}}"] \arrow[d, "\eta_i"'] & \mathrm{F}_j \circ \mathrm{G} \arrow[d, "\eta_j"] \\
\mathrm{G} \circ \mathrm{F}_i \arrow[r, "\mathrm{G}(\alpha)"] & \mathrm{G} \circ \mathrm{F}_j
\end{tikzcd}$$

commutes. A functor naturally commuting with projective functors is determined uniquely (up to isomorphism) by its image on the dominant object M, see [20]. For examples of functors naturally commuting with projective functors (in the situations described in Subsection 3.2) we again refer the reader to [20]. A natural question to ask is under what assumptions the Nakayama functor naturally commutes with projective functors.

4. Serre functors for Lie algebras

4.1. Harish-Chandra bimodules and category $\mathcal{O}$

From now on $\Bbbk = \mathbb{C}$ and we abbreviate $\otimes_{\mathbb{C}}$ by $\otimes$. Consider the setup of Subsection 3.2 and denote by $\mathcal{H}$ the category of *Harish-Chandra bimodules* for $\mathfrak{g}$. This category consists of all finitely generated $\mathfrak{g}$-bimodules on which the adjoint action of $\mathfrak{g}$ is locally finite (see [2] or [18, Kapitel 6]).

Let $U = U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. For a primitive ideal I of U let $\mathcal{H}_I^1$ denote the full subcategory of $\mathcal{H}$ which consists of all bimodules annihilated by I from the right. For $\lambda \in \mathfrak{h}^*$ let I_λ denote the annihilator of $\Delta(\lambda)$. If λ is dominant and I is a primitive ideal containing I_λ, we denote by $\Delta^I(\lambda)$ the quotient $\Delta(\lambda)/I\Delta(\lambda)$. We denote by $\mathcal{O}_I$ the full subcategory of $\mathcal{O}_{\overline{\lambda}}$ consisting of all modules which are isomorphic to quotients of modules of the form $E \otimes \Delta^I(\lambda)$, where E is a finite-dimensional $\mathfrak{g}$-module (the categories $\mathcal{O}_0^{\hat{\mathbf{R}}}$ from [26, 4.3] are direct summands of $\mathcal{O}_I$). Let $\_^\star$ be the usual duality on $\mathcal{O}$. The duality $\_^\star$ restricts to $\mathcal{O}_I$.

For two $\mathfrak{g}$-modules M and N, let $\mathcal{L}(M,N)$ denote the $\mathfrak{g}$-bimodule of all linear maps from M to N which are locally finite with respect to the adjoint action of $\mathfrak{g}$ (see [18, Kapitel 6]). For $\lambda \in \mathfrak{h}^*$ dominant and regular we have an equivalence of categories as follows (see [2, Theorem 5.9] for the category $\mathcal{O}$ and [27, Theorem 3.1] for the general statement):

$$\mathcal{O}_I \underset{\_\otimes_U \Delta^I(\lambda)}{\overset{\mathcal{L}(\Delta^I(\lambda),\_)}{\rightleftarrows}} \mathcal{H}_I^1 \tag{4.1}$$

Fix one representative in every isomorphism class of indecomposable projective objects in $\mathcal{O}_I$ and denote by $\mathcal{C}_I$ the full subcategory of $\mathcal{O}_I$ generated by these fixed objects. Then $\mathcal{C}_I$ is an slf-category and $\mathcal{O}_I$ is equivalent to $\mathcal{C}_I$-mod (note that $\mathcal{C}_I \cong \mathcal{C}_I^{\mathrm{op}}$ because of $\star$). The canonical equivalence between $\mathcal{O}_I$ and $\mathcal{C}_I^{\mathrm{op}}$-mod sends $M \in \mathcal{O}_I$ to the $\mathcal{C}_I^{\mathrm{op}}$-module $\mathrm{Hom}_{\mathfrak{g}}(\_, M)$ (which is a functor from $\mathcal{C}_I^{\mathrm{op}}$ to $\mathbb{C}$-mod) and is defined on morphisms in the natural way.

Let σ denote the Chevalley anti-involution on $\mathfrak{g}$. For a $\mathfrak{g}$-bimodule X we denote by X^s the bimodule defined as follows: $X^s = X$ as a vector space and for $a, b \in \mathfrak{g}$, $x \in X$, we have $a \cdot_s x \cdot_s b := \sigma(b)x\sigma(a)$. We also denote by X^* the dual bimodule of X. If X is a Harish-Chandra bimodule, we denote by $X^{\circledast}$ the subbimodule of X^* consisting of all elements on which the adjoint action of $\mathfrak{g}$ is locally finite.

4.2. Serre functor for category $\mathcal{O}$

The Serre functor for the regular block of the category $\mathcal{O}$ was described in [1] and [25]. The latter also contains a description of the Serre functor for the regular block of $\mathcal{O}^{\mathfrak{p}}$.

Theorem 4.1. — *Let λ be dominant and regular and I be a primitive ideal containing I_λ. Set $\Delta := \Delta^I(\lambda)$. Then for any $P, N \in \mathcal{O}_I$ with P projective there is an isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}\big(N, \mathcal{L}(P,\Delta)^{\circledast} \otimes_U \Delta\big) \cong \mathrm{Hom}_{\mathfrak{g}}(P,N)^*,$$

natural in both P and N.

Proof. — In the proof we will need the following statement:

Proposition 4.2. — *For any $P, N \in \mathcal{O}_I$ with P projective there is an isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}\text{-}\mathfrak{g}}(\mathcal{L}(\Delta,P),\mathcal{L}(\Delta,N)) \cong \mathcal{L}(P,\Delta) \otimes_{U\text{-}U} \mathcal{L}(\Delta,N) \tag{4.2}$$

natural in both P and N (here $\otimes_{U\text{-}U}$ denotes the tensor product over $U \otimes U^{\mathrm{op}}$).

Proof. — We start with the following observation:

Lemma 4.3. — *In the case $P = \Delta$ both sides of* (4.2) *are finite dimensional vector spaces of the same dimension.*

Proof. — If $P = \Delta$, then $\mathcal{L}(\Delta,\Delta) \cong U/I$, see [18, 6.9], and the left hand side of (4.2) (which is clearly finite dimensional) is isomorphic to $\mathcal{L}(\Delta,N)^{\mathfrak{g}}$, the set of $\mathfrak{g}$-invariants of $\mathcal{L}(\Delta,N)$, see [2, Lemma 2.2]. Further, we have the inclusion

$$\begin{aligned}
U/I \otimes_{U\text{-}U} \mathcal{L}(\Delta,N) &\subset \mathrm{Hom}_{\mathbb{C}}\big(U/I \otimes_{U\text{-}U} \mathcal{L}(\Delta,N),\mathbb{C}\big)^* \\
\text{(by adjunction)} &\cong \mathrm{Hom}_{\mathfrak{g}\text{-}\mathfrak{g}}\big(U/I,\mathcal{L}(\Delta,N)^*\big)^* \\
&\cong \mathrm{Hom}_{\mathfrak{g}\text{-}\mathfrak{g}}\big(U/I,\mathcal{L}(\Delta,N)^{\circledast}\big)^* \\
\text{(by the above)} &\cong \big((\mathcal{L}(\Delta,N)^{\circledast})^{\mathfrak{g}}\big)^* \\
&\cong \mathcal{L}(\Delta,N)^{\mathfrak{g}},
\end{aligned}$$

where the third step is justified by the facts that the image of any homomorphism from U/I to $\mathcal{L}(\Delta,N)^*$ belongs to $\mathcal{L}(\Delta,N)^{\circledast}$, and the fifth step is justified by the fact that $(\mathcal{L}(\Delta,N)^{\circledast})^{\mathfrak{g}} \cong (\mathcal{L}(\Delta,N)^{\mathfrak{g}})^*$ as the canonical non-degenerate pairing $\mathcal{L}(\Delta,N) \times \mathcal{L}(\Delta,N)^{\circledast} \to \mathbb{C}$ restricts to a non-degenerate pairing $\mathcal{L}(\Delta, N)^{\mathfrak{g}} \times (\mathcal{L}(\Delta, N)^{\circledast})^{\mathfrak{g}} \to \mathbb{C}$. Since the vector space $\mathcal{L}(\Delta, N)^{\mathfrak{g}}$ is finite dimensional, the original inclusion

$$U/I \otimes_{U\text{-}U} \mathcal{L}(\Delta, N) \subset \mathrm{Hom}_{\mathbb{C}}\big(U/I \otimes_{U\text{-}U} \mathcal{L}(\Delta, N), \mathbb{C}\big)^*$$

is, in fact, an isomorphism. The claim follows. □

Write $\mathcal{L}(\Delta, N) \cong \mathcal{L}(\Delta, N)^{\mathfrak{g}} \oplus X$, where X is stable with respect to the adjoint action of $\mathfrak{g}$. In the case $P = \Delta$ we have $\mathcal{L}(\Delta, \Delta) = U/I$ and $U/I \otimes_{U-U} \mathcal{L}(\Delta, N)$ is the 0-degree Hochschild homology of $\mathcal{L}(\Delta, N)$, which equals

$$U/I \otimes_{U-U} \mathcal{L}(\Delta, N) \cong \mathcal{L}(\Delta, N)/\mathcal{K}\mathcal{L}(\Delta, N), \tag{4.3}$$

where

$$\mathcal{K}\mathcal{L}(\Delta, N) := \{uv - vu : u \in U/I, v \in \mathcal{L}(\Delta, N)\}.$$

As an adjoint module, X is a direct sum of finite-dimensional simple modules, none of which is isomorphic to the trivial module. Therefore $X \subset \mathcal{K}\mathcal{L}(\Delta, N)$. From Lemma 4.3 it follows that $X = \mathcal{K}\mathcal{L}(\Delta, N)$ comparing the dimensions. Composing (4.3) with the projection of $\mathcal{L}(\Delta, N)$ onto $\mathcal{L}(\Delta, N)^{\mathfrak{g}}$ along X we obtain the following natural isomorphism:

$$U/I \otimes_{U-U} \mathcal{L}(\Delta, N) \cong \mathcal{L}(\Delta, N)/\mathcal{K}\mathcal{L}(\Delta, N) \cong \mathcal{L}(\Delta, N)^{\mathfrak{g}}. \tag{4.4}$$

Consider now the case where $P = E \otimes \Delta$ for some finite-dimensional $\mathfrak{g}$-module E. By [18, 6.2] we have canonical isomorphisms

$$\mathcal{L}(E \otimes \Delta, \Delta) \cong \mathcal{L}(\Delta, \Delta) \otimes E^*, \qquad \mathcal{L}(\Delta, E \otimes \Delta) \cong E \otimes \mathcal{L}(\Delta, \Delta). \tag{4.5}$$

Denote by $\check{E}$ the right $\mathfrak{g}$-module defined as follows: $\check{E} = E$ as a vector space and $v \check{\cdot} g := -gv$ for $v \in E$ and $g \in \mathfrak{g}$ (and similarly for going from right to left modules). Then the usual adjunction (see [2, 2.1(d)]) gives, by restriction, the isomorphism

$$\mathrm{Hom}_{\mathfrak{g}\text{-}\mathfrak{g}}(E \otimes U/I, X) \cong \mathrm{Hom}_{\mathfrak{g}\text{-}\mathfrak{g}}(U/I, \check{E}^* \otimes X). \tag{4.6}$$

A straightforward computation also shows that

$$(U/I \otimes E^*) \otimes_{U\text{-}U} X \cong U/I \otimes_{U\text{-}U} (\check{E}^* \otimes X) \tag{4.7}$$

via $u \otimes v \otimes x \mapsto u \otimes v \otimes x$.

Combining (4.4), (4.7) and (4.6) we obtain a required isomorphism in the case $P = E \otimes \Delta$. Now the claim follows from additivity of all functors and naturality of all constructions. □

The theorem is now proved via the following chain of natural isomorphisms:

$$\begin{array}{rcl}
\mathrm{Hom}_{\mathfrak{g}}(P,N) & \stackrel{(4.1)}{\cong} & \mathrm{Hom}_{\mathfrak{g}\text{-}\mathfrak{g}}(\mathcal{L}(\Delta,P),\mathcal{L}(\Delta,N)) \\
\text{(by Proposition 4.2)} & \cong & \mathcal{L}(P,\Delta)\otimes_{U\text{-}U}\mathcal{L}(\Delta,N) \\
\text{(taking the double dual)} & \cong & \mathrm{Hom}_{\mathbb{C}}\big(\mathcal{L}(P,\Delta)\otimes_{U\text{-}U}\mathcal{L}(\Delta,N),\mathbb{C}\big)^* \\
\text{(by adjunction)} & \cong & \mathrm{Hom}_{\mathfrak{g}\text{-}\mathfrak{g}}\big(\mathcal{L}(\Delta,N),\mathcal{L}(P,\Delta)^*\big)^* \\
 & \cong & \mathrm{Hom}_{\mathfrak{g}\text{-}\mathfrak{g}}\big(\mathcal{L}(\Delta,N),\mathcal{L}(P,\Delta)^{\circledast}\big)^* \\
\text{(by (4.1))} & \cong & \mathrm{Hom}_{\mathfrak{g}}\big(N,\mathcal{L}(P,\Delta)^{\circledast}\otimes_U\Delta\big)^*
\end{array}$$

Here taking the double dual is justified by the fact that the vector space in question is finite dimensional (see Lemma 4.3), and the penultimate step is justified by the fact that the image of every morphism from $\mathcal{L}(\Delta,N)$ to $\mathcal{L}(P,\Delta)^*$ belongs to $\mathcal{L}(P,\Delta)^{\circledast}$. □

Corollary 4.4. — *Assume that λ, I and Δ are as above.*

(a) *The functor $\mathcal{L}({}_{-},\Delta)^{\circledast}\otimes_U\Delta$ is isomorphic to the Nakayama functor on $\mathcal{O}_I$.*

(b) *If we additionally assume that all injective modules in $\mathcal{O}_I$ have finite projective dimension, then the left derived of the functor $\mathcal{L}({}_{-},\Delta)^{\circledast}\otimes_U\Delta$ is a Serre functor on $\mathscr{P}(\mathcal{C}_I)$.*

We would like to point out that the hypothesis of Corollary 4.4(b) is satisfied if $\mathcal{O}_I$ is a direct summand of the usual or parabolic category $\mathcal{O}$. Note that $\mathcal{O}_I$ is a direct summand of $\mathcal{O}$ if I is a minimal primitive ideal. Furthermore, $\mathcal{O}_I$ is a direct summand of some parabolic $\mathcal{O}$ if I is the annihilator of some dominant parabolic Verma module.

4.3. Alternative descriptions and applications

Corollary 4.5. — *Under the assumptions of Corollary 4.4*(a) *the Nakayama functor on $\mathcal{O}_I$ naturally commutes with projective functors.*

Proof. — By Corollary 4.4(a), the Nakayama functor on $\mathcal{O}_I$ is isomorphic to the functor $\mathcal{L}({}_{-},\Delta)^{\circledast}\otimes_U\Delta$. Applying ${}_{-}^{\circledast}$ to the left isomorphism in (4.5) we get that $\mathcal{L}({}_{-},\Delta)^{\circledast}\otimes_U\Delta$ commutes with projective functors and naturality follow from the definition of projective functors. □

Under the assumptions of Corollary 4.4(a) consider the endofunctor C of $\mathcal{O}_I$ of *partial coapproximation* with respect to projective-injective modules, see [21, 2.5]. This functor is the unique (up to isomorphism) right exact functor which sends a projective module P to the submodule of P

generated by images of all possible morphisms from projective-injective modules to P and acts on morphisms via restriction.

The module $\Delta = \Delta^I(\lambda)$ has, by [23, Corollary 3], simple socle which we denote by $K = K^I(\lambda)$ (it is the only simple subquotient of Δ of maximal Gelfand-Kirillov (GK) dimension). We have a canonical embedding $\varphi : U/\mathrm{Ann}_U K \hookrightarrow \mathcal{L}(K, K)$. The question of surjectivity of φ is known as *Kostant's problem* for K (see [19] and references therein). It is known (see [19, 4.1]) that Kostant's problem has a positive answer in the case when $\mathcal{O}_I$ is the usual or parabolic category $\mathcal{O}$ (and in many other cases as well).

Corollary 4.6. — *Assume that λ, I and Δ are as above and that φ is surjective. Then the Nakayama functor on $\mathcal{O}_I$ is isomorphic to* C^2.

Proof. — The fact that functor C (and hence also C^2) naturally commutes with projective functors repeats verbatim the proof of [25, Claim 2, Page 153]. The Nakayama functor naturally commutes with projective functors by Corollary 4.5. Therefore to complete the proof we only need to show that C^2 maps Δ to the corresponding injective module.

First we claim that $\mathrm{C}\Delta \cong K$, where K is the simple socle of Δ (as in Subsection 4.2). Indeed, Δ is projective and hence $\mathrm{C}\Delta$ coincides with the trace of projective-injective modules in Δ. By [23, 3.2], this trace coincides with K.

By [23, Corollary 12], the surjectivity of φ is equivalent to the existence of a two step resolution $\Delta \hookrightarrow X_0 \to X_1$ with X_0 and X_1 projective-injective. Applying $\star$ we obtain

$$X_1^\star \to X_0^\star \twoheadrightarrow \Delta^\star \tag{4.8}$$

where both $X_0^\star$ and $X_1^\star$ are again projective-injective. As the kernel of the natural projection $\Delta^\star \twoheadrightarrow K^\star \cong K$ is killed by C, it follows that $\mathrm{C}K \cong \mathrm{C}\Delta^\star$ and from (4.8) we obtain $\mathrm{C}\Delta^\star \cong \Delta^\star$, which is exactly what we needed. □

Remark 4.7. — If $\mathcal{O}_I$ is a direct summand of the usual or parabolic category $\mathcal{O}$ one can give yet another description for the Nakayama and Serre functors in terms of Arkhipov's twisting functors, see [25] for details.

Fix one representative in every isomorphism class of indecomposable projective-injective objects in $\mathcal{O}_I$ and denote by $\mathcal{P}_I$ the full subcategory of $\mathcal{O}_I$ generated by these fixed objects. The category $\mathcal{P}_I$ is an slf-category.

Corollary 4.8. — *Under the assumptions of Corollary 4.6 the category $\mathcal{P}_I$ is symmetric.*

Proof. — By our construction of $\mathcal{P}_I$, every projective $\mathcal{P}_I$-module is also injective. Thus the Nakayama functor preserves $\mathscr{P}(\mathcal{P}_I)$ and hence gives rise to a Serre functor on this category. From Corollary 4.6 we obtain that the Serre functor on $\mathscr{P}(\mathcal{P}_I)$ is the left derived of C^2. The functor C obviously induces the identity functor when restricted to projective-injective modules. Now the claim follows from Proposition 2.3. □

In the case when I is the annihilator of some parabolic Verma module, Corollary 4.8 implies (and reproves) [25, Theorem 4.6]. If $\mathfrak{g}$ is of type A, then for an arbitrary I the claim of Corollary 4.8 can be deduced combining [25, Theorem 4.6] and [26, Theorem 18]. In all other cases Corollary 4.8 seems to be new.

5. Serre functors for Lie superalgebras

5.1. Generalities on Lie superalgebras

In this section we denote by $\mathfrak{g}$ one of the following Lie superalgebras: $\mathfrak{gl}(m,n)$, $\mathfrak{sl}(m,n)$, $\mathfrak{osp}(m,n)$, $\mathfrak{psl}(n,n)$, $\mathfrak{q}(n)$, $\mathfrak{pq}(n)$, $\mathfrak{sq}(n)$, $\mathfrak{psq}(n)$. For all these superalgebras the Lie algebra $\mathfrak{g}_0$ is reductive. The first four superalgebras admit an even $\mathfrak{g}$-invariant bilinear form which is non-degenerate on $[\mathfrak{g},\mathfrak{g}]$ (these superalgebras are called *basic*). The last four superalgebras are called *queer* Lie superalgebras, or Lie superalgebras of type $\mathfrak{q}$. In the following we denote by $\mathfrak{g}$-sMod the category of all $\mathfrak{g}$-supermodules, where morphisms are homogeneous of degree zero (and by a module over a superalgebra we always mean a supermodule). The category $\mathfrak{g}$-sMod is abelian. Denote by Π the parity change autoequivalence of $\mathfrak{g}$-sMod and by $U = U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.

Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ of $\mathfrak{g}$, where $\mathfrak{h}$ is a Cartan subalgebra, and let $\mathcal{O}$ denote the full subcategory of $\mathfrak{g}$-sMod consisting of finitely generated modules M such that the action of $\mathfrak{h}_0$ on M is diagonalizable and the action of $U(\mathfrak{n}^+)$ on M is locally finite. Note that $\mathfrak{h} = \mathfrak{h}_0$ if and only if $\mathfrak{g}$ is basic.

If $\mathfrak{p} \subset \mathfrak{g}$ is a parabolic subsuperalgebra containing $\mathfrak{h} \oplus \mathfrak{n}^+$, we denote by $\mathcal{O}^{\mathfrak{p}}$ the full subcategory of $\mathcal{O}$ consisting of all modules M on which the action of $U(\mathfrak{p})$ is locally finite. In particular, the category $\mathcal{O}^{\mathfrak{g}}$ coincides with the category of finite dimensional $\mathfrak{h}_0$-diagonalizable $\mathfrak{g}$-modules. Abusing language, in the following by finite dimensional $\mathfrak{g}$-modules we will mean finite dimensional $\mathfrak{h}_0$-diagonalizable $\mathfrak{g}$-modules (the two categories coincide if $\mathfrak{g}_0$ is a semi-simple Lie algebra). Let $\_^{\star}$ denote the usual duality on $\mathcal{O}$ (which is simple preserving for basic $\mathfrak{g}$ and simple preserving up to Π if $\mathfrak{g}$ is of type $\mathfrak{q}$). This duality restricts to $\mathcal{O}^{\mathfrak{p}}$.

We have the usual induction and restriction functors Ind and Res between $\mathfrak{g}$-sMod and $\mathfrak{g}_0$-sMod. They are adjoint in the usual way $(\mathrm{Ind}, \mathrm{Res})$. We also have the adjointness $(\mathrm{Res}, \Pi^{\dim \mathfrak{g}_1} \circ \mathrm{Ind})$, see [12, 3.2.4] and [10, Proposition 2.2]. Note that $\dim \mathfrak{g}_1$ is even dimensional if $\mathfrak{g}$ is basic and also for $\mathfrak{q}(n)$ and $\mathfrak{pq}(n)$ when n is even, and for $\mathfrak{sq}(n)$ and $\mathfrak{psq}(n)$ when n is odd.

Since $U(\mathfrak{g})$ is a finite extension of $U(\mathfrak{g}_0)$, we define the category $\mathcal{H}$ of *Harish-Chandra* $\mathfrak{g}$*-bimodules* as the full subcategory of the category of $\mathfrak{g}$-bimodules which become Harish-Chandra $\mathfrak{g}_0$-bimodules after restriction. For any graded ideal I of $U(\mathfrak{g})$ we denote by $\mathcal{H}_I^1$ the full subcategory of $\mathcal{H}$ which consists of all bimodules annihilated by I from the right. If M and N are $\mathfrak{g}$-modules, we denote by $\mathcal{L}(M, N)$ the $\mathfrak{g}_0$-bimodule $\mathcal{L}(\mathrm{Res}\, M, \mathrm{Res}\, N)$ which is also a $\mathfrak{g}$-bimodule as $U(\mathfrak{g})$ is a finite extension of $U(\mathfrak{g}_0)$.

5.2. Structure of the category $\mathcal{O}$

Consider the category $\mathfrak{h}$-dmod of finite dimensional $\mathfrak{h}$-modules on which the action of $\mathfrak{h}_0$ is diagonalizable. In case $\mathfrak{h} = \mathfrak{h}_0$ (i.e. $\mathfrak{g}$ is basic) this category is semi-simple and its simple objects are naturally parameterized by pairs (λ, ε), where $\lambda \in \mathfrak{h}_0^*$ (which describes the action of $\mathfrak{h}_0$) and $\varepsilon \in \{0, 1\}$ (which determines the parity of the module). If $\mathfrak{g}$ is of type $\mathfrak{q}$, then the category $\mathfrak{h}$-dmod is not semi-simple, it has enough projectives (see [7, Section 3]) and its simple objects are naturally parameterized by pairs (λ, ε), where $\lambda \in \mathfrak{h}_0^*$ (which describes the action of $\mathfrak{h}_0$) and ε is either in $\{+\}$ or in $\{+, -\}$ depending on λ as prescribed by the theory of Clifford algebras (see [15, Appendix]). If we have two different simple modules for some λ, they differ by a parity change.

For a simple $V \in \mathfrak{h}$-dmod set $\mathfrak{n}^+ V = 0$ and define the corresponding *Verma* module as

$$\Delta(V) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}^+)} V.$$

Every object in $\mathcal{O}$ has finite length (already as a $\mathfrak{g}_0$-module as $U(\mathfrak{g})$ is a finite extension of $U(\mathfrak{g}_0)$). To distinguish Lie algebras from superalgebras, we denote by $\tilde{\mathcal{O}}$ the category $\mathcal{O}$ for $\mathfrak{g}_0$ (defined with respect to the triangular decomposition of $\mathfrak{g}_0$ obtained from the above triangular decomposition of $\mathfrak{g}$ by restriction).

Category $\mathcal{O}$ has enough projectives (which may be obtained as direct summands of modules induced from projectives in $\tilde{\mathcal{O}}$), every projective has a filtration by Verma modules (see [7]). As induction is exact and adjoint (from both sides) to exact functors, it sends injectives to injectives and projectives to projectives. In particular, it follows that all injective modules in $\mathcal{O}$ have finite projective dimension (since this is true for $\tilde{\mathcal{O}}$).

For any parabolic $\mathfrak{p}$ (as in Subsection 5.1), the usual projection from $\mathcal{O}$ onto $\mathcal{O}^{\mathfrak{p}}$ obviously commutes with both induction and restriction. Hence, using [31] and [32], we similarly get that injective modules in $\mathcal{O}^{\mathfrak{p}}$ have finite projective dimension. We denote by $\Delta^{\mathfrak{p}}(V)$ the image of $\Delta(V)$ in $\mathcal{O}^{\mathfrak{p}}$.

5.3. Equivalence to Harish-Chandra bimodules

Let T denote the generator of the anticenter of $U(\mathfrak{g})$, see [12]. An element $\lambda \in \mathfrak{h}_0^*$ is called *strongly typical* if T does not annihilate the Verma module $\Delta(V)$, where V is a simple $\mathfrak{h}$-module on which $\mathfrak{h}_0$ acts via λ. Note that λ strongly typical implies that V is projective in $\mathfrak{h}$-dmod.

Let Θ denote the set of weights of all simple finite dimensional $\mathfrak{g}$-modules. Fix a strongly typical $\lambda \in \mathfrak{h}_0^*$ which is regular and dominant with respect to the dot-action of W and set $\overline{\lambda} = \lambda + \Theta$. Call such λ *generic* provided that the $\mathfrak{g}_0$-module $\operatorname{Res}\Delta(V)$ is a direct sum of Verma modules and non-isomorphic direct summands of $\operatorname{Res}\Delta(V)$ correspond to different central characters. Denote by $\mathcal{O}_{\overline{\lambda}}$ the full subcategory of $\mathcal{O}$ consisting of all M such that the $\mathfrak{g}_0$-support of M belongs to $\overline{\lambda}$. Define $\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$ correspondingly. For V as above the indecomposable direct summand of $\mathcal{O}$ containing $\Delta(V)$ is equivalent to a direct summand of $\tilde{\mathcal{O}}$, see [14, 11].

As in the Lie algebra situation, we have *projective endofunctors* on $\mathcal{O}$ given as direct summands of the functors of the form $E \otimes {}_-$, where E is a finite dimensional $\mathfrak{g}$-module. The functor $E\otimes {}_-$ is both left and right adjoint to $\check{E}^* \otimes {}_-$, in particular, it is exact and sends projectives to projectives. It is easy to see that every indecomposable projective in $\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$ is a direct summand of $E\otimes\Delta(V)$ for some finite dimensional E, in particular, $\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$ coincides with the category $\operatorname{coker}\langle E \otimes \Delta(V)\rangle$ which consists of all modules M having a two step resolution

$$X_1 \to X_0 \to M \to 0,$$

where X_0, X_1 belong to the additive closure of modules of the form $E \otimes \Delta(V)$ for finite dimensional E. Denote by $\mathcal{F}$ the full subcategory of the category of right exact endofunctors of $\mathcal{O}$ whose objects are projective functors. Now we can formulate our first main result for superalgebras.

Theorem 5.1. — *Let $\lambda \in \mathfrak{h}_0^*$ be as above, V a simple $\mathfrak{h}_0$-module of weight λ and $I = \operatorname{Ann}_U \Delta^{\mathfrak{p}}(V)$.*

(a) *The triple $(\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}, \Delta^{\mathfrak{p}}(V), \mathcal{F})$ is an fpf-category.*

(b) *Assume that* $\Pi V \ncong V$ *and that* λ *is generic. Then the natural injective map* $U/I \hookrightarrow \mathcal{L}(\Delta^{\mathfrak{p}}(V), \Delta^{\mathfrak{p}}(V))$ *is surjective.*

(c) *Assume that* $\Pi V \ncong V$ *and that* λ *is generic. Then we have the following mutually inverse equivalences of categories:*

$$\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}} \underset{-\otimes_U \Delta^{\mathfrak{p}}(V)}{\overset{\mathcal{L}(\Delta^{\mathfrak{p}}(V),-)}{\rightleftarrows}} \mathcal{H}^1_I \tag{5.1}$$

Note that for our choice of λ the module $\Delta^{\mathfrak{p}}(V)$ is obviously projective in $\mathcal{O}^{\mathfrak{p}}$. The condition $\Pi V \ncong V$ is equivalent to the condition that $\dim \mathfrak{g}_1$ is even.

5.4. Proof of Theorem 5.1(a)

We only need to check condition (iv). Assume first that $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}^+$. For any finite dimensional $\mathfrak{g}$-module E we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{O} \arrow[r, "E\otimes -"] \arrow[d, "\mathrm{Res}"'] & \mathcal{O} \arrow[d, "\mathrm{Res}"] \\ \tilde{\mathcal{O}} \arrow[r, "\mathrm{Res}E\otimes -"] & \tilde{\mathcal{O}} \end{tikzcd} \tag{5.2}$$

Denote by $\mathcal{O}_V$ the block of $\mathcal{O}$ containing $\Delta(V)$. For any $\mu \in \mathfrak{h}_0^*$ denote by $\tilde{\mathcal{O}}_\mu$ the block of $\tilde{\mathcal{O}}$ containing $\Delta(\mu)$. By [14] and [11], to our λ there corresponds a dominant regular $\lambda' \in \mathfrak{h}_0^*$ such that the appropriate direct summand of (5.2) has the form

$$\begin{tikzcd} \mathcal{O}_V \arrow[r, "\mathrm{F}"] \arrow[d, "\mathrm{G}"'] & \mathcal{O}_V \arrow[d, "\mathrm{G}"] \\ \tilde{\mathcal{O}}_{\lambda'} \arrow[r, "\mathrm{F}'"] & \tilde{\mathcal{O}}_{\lambda'}, \end{tikzcd} \tag{5.3}$$

where G is an equivalence. In particular, for any projective functor $\mathrm{F} : \mathcal{O}_V \to \mathcal{O}_V$ there is a projective functor $\mathrm{F}' : \tilde{\mathcal{O}}_{\lambda'} \to \tilde{\mathcal{O}}_{\lambda'}$ such that (5.3) commutes. Since condition (iv) is satisfied for $\tilde{\mathcal{O}}_{\lambda'}$ and G is an equivalence, it follows that (iv) is satisfied for $\mathcal{O}_V$ as well. This shows that $\mathcal{O}_V$ is an fpf-category.

The corresponding statement for the category $\mathcal{O}_{\overline{\lambda}}$ reduces to $\mathcal{O}_V$ using the adjointness of $E \otimes -$ and $\check{E}^* \otimes -$ as follows. We have

$$\mathrm{Hom}_{\mathfrak{g}}(E_1 \otimes \Delta(V), E_2 \otimes \Delta(V)) \cong \mathrm{Hom}_{\mathfrak{g}}(\Delta(V), \check{E}_1^* \otimes E_2 \otimes \Delta(V))$$

by adjointness and the homomorphism space on the right hand side can be computed inside $\mathcal{O}_V$. Similarly for the morphisms of the corresponding projective functors. Now the claim follows from the observation that the evaluation map from (iv) commutes with adjunction.

The statement for the category $\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$ follows from that for the category $\mathcal{O}_{\overline{\lambda}}$ since every homomorphism between projective modules in $\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$ comes from a homomorphism between projective modules in $\mathcal{O}_{\overline{\lambda}}$.

5.5. Proof of Theorem 5.1(b) and (c)

Let us first assume that $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}^+$. If $\mathfrak{g}$ is one of the superalgebras $\mathfrak{gl}(m,n)$, $\mathfrak{sl}(m,n)$, $\mathfrak{osp}(m,n)$ or $\mathfrak{psl}(n,n)$, the claim of Theorem 5.1(b) is proved in [13, Proposition 5.1(ii)]. The idea of the following proof for type $\mathfrak{q}$-superalgebras was suggested by Maria Gorelik and follows closely [13, Section 8]. We are going to define some analogue of Gorelik's notion of a *perfect mate*, show that it exists for type $\mathfrak{q}$-superalgebras and use it to prove our statement.

Denote by χ_λ the $U(\mathfrak{g})$-central character of $\Delta(V)$ and by χ^0_λ the $U(\mathfrak{g}_0)$-central character of $\Delta(\lambda)$.

Lemma 5.2. — *Let L be a simple $U(\mathfrak{g})$-module with central character χ_λ. Then* $\mathrm{Res}\, L$ *has a non-zero component with $U(\mathfrak{g}_0)$-central character χ^0_λ.*

Proof. — Set $J := \mathrm{Ann}_U L$. Then J is a primitive ideal of U and hence, by [28], coincides with the annihilator of some simple highest weight module$N$. Since L has central character χ_λ, we can choose $N \in \mathcal{O}_V$. From (5.3) (with $\lambda' = \lambda$ by [11]) it follows that $\mathrm{Res}\, N$ has a non-zero component with central character χ^0_λ. Let $\chi_1 = \chi^0_\lambda$, $\chi_2, \dots, \chi_k$ be all central characters occurring in $\mathrm{Res}\, N$, and $\mathtt{m}_1, \dots, \mathtt{m}_k$ the corresponding maximal ideals in $Z(\mathfrak{g}_0)$. Let $l_1, \dots, l_k$ be minimal possible such that $\prod_{i=1}^k \mathtt{m}_i^{l_i}$ annihilates N. Then $\prod_{i=2}^k \mathtt{m}_i^{l_i}$ does not annihilate N and thus it does not annihilate L either. At the same time, the nonzero space $\prod_{i=2}^k \mathtt{m}_i^{l_i} L$ is annihilated by $\mathtt{m}_1^{l_1}$. The claim follows. □

Denote $\mathtt{m} = \mathtt{m}_1$, where $\mathtt{m}_1$ is as in the proof of Lemma 5.2. Consider the set

$$X := \{v \in \mathcal{L}(\Delta(\lambda), \Delta(\lambda)) : v\mathtt{m}^l = \mathtt{m}^l v = 0, l \gg 0\}$$

and define $B := UXU$, which is a U-subbimodule of $\mathcal{L}(\Delta(\lambda), \Delta(\lambda))$.

Proposition 5.3. — *We have $B = \mathcal{L}(\Delta(\lambda), \Delta(\lambda))$.*

Proof. — Let B' denote the cokernel of the embedding

$$B \hookrightarrow \mathcal{L}(\Delta(\lambda), \Delta(\lambda)).$$

Then B' satisfies

$$\{v \in B' : v\mathtt{m}^l = \mathtt{m}^l v = 0, l \gg 0\} = 0$$

by construction of B. The fact that $B' = 0$ is now proved mutatis mutandis [13, 8.4.2]. □

Let M denote the $\mathfrak{g}_0$-submodule of $\Delta(V)$, annihilated by $\mathtt{m}$. As λ is generic by assumption, the module M is isomorphic to a direct sum of copies of $\Delta(\lambda)$. The direct summands of M are indexed by a basis of V and hence the number of direct summands equals $\dim V$. Due to Proposition 5.3, to complete the proof of Theorem 5.1(b) it is enough to show that U surjects onto the $\mathfrak{g}_0$-bimodule

$$\mathcal{L}(M, M) \cong \mathrm{End}_{\mathbb{C}}(V) \otimes \mathcal{L}(\Delta(\lambda), \Delta(\lambda)).$$

We know that $U(\mathfrak{g}_0)$ surjects onto $\mathcal{L}(\Delta(\lambda), \Delta(\lambda))$ (see [18, 6.9(10)]). By [15, A.3.2], under our assumption that $\Pi V \ncong V$ the algebra $U(\mathfrak{h})$ surjects onto the matrix algebra $\mathrm{End}_{\mathbb{C}}(V)$. The claim follows.

For an arbitrary $\mathfrak{p}$ the claim of Theorem 5.1(b) follows from the case $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}^+$ similarly to [18, 6.9].

The claim of Theorem 5.1(c) follows from Theorem 5.1(b) mutatis mutandis [27, Theorem 3.1].

5.6. Some conjectures following Theorem 5.1

In this subsection $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}_+$, $\lambda \in \mathfrak{h}_0^*$, V is a simple $\mathfrak{h}_0$-module of weight λ and $I = \mathrm{Ann}_U \Delta(V)$.

Conjecture 5.4. — Assume λ is strongly typical and regular. Then, the adjoint $\mathfrak{g}$-module $(U/I)^{\mathrm{ad}}$ is a direct sum of injective finite-dimensional modules.

By [18, 6.8(3)], for every finite dimensional $\mathfrak{g}$-module E there is a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(E, \mathcal{L}(\Delta(\lambda), \Delta(\lambda))^{\mathrm{ad}}) \cong \mathrm{Hom}_{\mathfrak{g}}(\Delta(\lambda), \check{E}^* \otimes \Delta(\lambda)).$$

Since the functor $\mathrm{Hom}_{\mathfrak{g}}(\Delta(\lambda), (\_)^* \otimes \Delta(\lambda))$ is exact on the category of finite dimensional $\mathfrak{g}$-modules (as tensoring over $\mathbb{C}$ is exact and $\Delta(\lambda)$ is projective), it follows that the adjoint $\mathfrak{g}$-module $\mathcal{L}(\Delta(\lambda), \Delta(\lambda))^{\mathrm{ad}}$ is a direct sum of injective finite dimensional modules. In particular, Theorem 5.1(b) implies Conjecture 5.4 in the case $V \ncong \Pi V$.

Conjecture 5.5. — Assume λ is strongly typical and regular. Then the bimodule U/I is a direct summand of $\mathcal{L}(\Delta(\lambda),\Delta(\lambda))$.

Conjecture 5.5 obviously implies Conjecture 5.4. Denote by φ the automorphism of $U(\mathfrak{g})$, which multiplies elements from $\mathfrak{g}_0$ with 1 and elements from $\mathfrak{g}_1$ with -1. For a U-bimodule B denote by B^{φ} the bimodule, obtained by twisting the right action of U with φ.

Conjecture 5.6. — Assume λ is strongly typical and regular, and $V\cong \Pi V$. Then we have the following bimodule decomposition: $\mathcal{L}(\Delta(\lambda),\Delta(\lambda))\cong U/I\oplus(U/I)^{\varphi}$.

From [15, A.3.2] it follows that in the case $V\cong\Pi V$ the algebra $U(\mathfrak{h})$ does not surject onto the matrix algebra $\mathrm{End}_{\mathbb{C}}(V)$, in fact, the image of $U(\mathfrak{h})$ in $\mathrm{End}_{\mathbb{C}}(V)$ has dimension $\frac{1}{2}\dim\mathrm{End}_{\mathbb{C}}(V)$. It follows that in this case the image of U in $\mathcal{L}(\Delta(\lambda),\Delta(\lambda))$ is only "one half" of $\mathcal{L}(\Delta(\lambda),\Delta(\lambda))$.

5.7. Annihilators and Kostant's problem

In this subsection we work under the assumptions of Theorem 5.1(b). Due to the equivalences from [14] and [11], the module $\Delta^{\mathfrak{p}}(V)$ has simple socle, say K, which is the unique simple subquotient of $\Delta^{\mathfrak{p}}(V)$ of maximal GK-dimension. Thanks to (5.3), for every projective functor θ we have

$$\dim\mathrm{Hom}_{\mathfrak{g}}(\theta\Delta^{\mathfrak{p}}(V),\Delta^{\mathfrak{p}}(V))=\dim\mathrm{Hom}_{\mathfrak{g}}(\theta K,K). \tag{5.4}$$

Proposition 5.7. —

(a) *The natural inclusion*

$$U/\mathrm{Ann}_U(K)\hookrightarrow\mathcal{L}(K,K)$$

is surjective.

(b) $\mathrm{Ann}_U(K)=\mathrm{Ann}_U(\Delta^{\mathfrak{p}}(V))$.

(c) *We have* $\mathrm{Ann}_U(N)=\mathrm{Ann}_U(\Delta^{\mathfrak{p}}(V))$ *for any nonzero submodule* N *of* $\Delta^{\mathfrak{p}}(V)$.

Proof. — From (5.4) we have $\mathcal{L}(K,K)\cong\mathcal{L}(\Delta^{\mathfrak{p}}(V),\Delta^{\mathfrak{p}}(V))$ using the same argument as in [19, Lemma 11]. Now (a) follows from the fact that U surjects onto $\mathcal{L}(\Delta^{\mathfrak{p}}(V),\Delta^{\mathfrak{p}}(V))$, established in Theorem 5.1(b). Since

$$\mathrm{Ann}_U(\Delta^{\mathfrak{p}}(V))\subset\mathrm{Ann}_U(K),$$

claim (b) follows from (a). As K is the simple socle of $\Delta^{\mathfrak{p}}(V)$, claim (c) follows from (b). □

COROLLARY 5.8. — *All Verma modules in $\mathcal{O}_V$ have the same annihilator and for each such module N we have $U/\mathrm{Ann}_U(N) \cong \mathcal{L}(N,N)$.*

Proof. — The first claim follows from Proposition 5.7(c) since all Verma modules in $\mathcal{O}_V$ are submodules of $\Delta(V)$.

To prove the second claim we need some notation. Let W denote the Weyl group. Then Verma modules in $\mathcal{O}_V$ are naturally indexed by elements of W. For $w \in W$ let $\Delta(w)$ denote the corresponding Verma module (with the convention $\Delta(e) = \Delta(V)$). Since all $\Delta(w)$ have the same annihilator by the first claim, to prove the second claim it is enough to show that for any simple reflection $s \in W$ we have

$$\mathcal{L}(\Delta(w), \Delta(w)) = \mathcal{L}(\Delta(sw), \Delta(sw)). \tag{5.5}$$

Without loss of generality we may assume that $\Delta(sw) \hookrightarrow \Delta(w) \twoheadrightarrow C$, where C is just the cokernel. Then C is s-finite while the simple top and the simple socle of $\Delta(sw)$ are s-infinite. Using this, (5.5) is proved similarly to [19, Lemma 11]. The claim follows. □

5.8. Serre functor for category $\mathcal{O}_{\overline{\lambda}}$

Fix one representative in every isomorphism class of indecomposable projective objects in $\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$ and denote by $\mathcal{C}^{\mathfrak{p}}_{\overline{\lambda}}$ the full subcategory of $\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$ which these fixed objects generate. The category $\mathcal{C}^{\mathfrak{p}}_{\overline{\lambda}}$ is an slf-category and $\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$ is equivalent to $\mathcal{C}^{\mathfrak{p}}_{\overline{\lambda}}$-mod (note that $\mathcal{C}^{\mathfrak{p}}_{\overline{\lambda}} \cong (\mathcal{C}^{\mathfrak{p}}_{\overline{\lambda}})^{\mathrm{op}}$ because of $\star$). Now we are ready to formulate our main result.

THEOREM 5.9. — *Let $\lambda \in \mathfrak{h}_0^*$ be strongly typical, regular, dominant and generic. Let further V be a simple $\mathfrak{h}_0$-module of weight λ and assume that $\Pi V \not\cong V$. Set $\Delta := \Delta^{\mathfrak{p}}(V)$. Then we have:*

(a) *For any $P, N \in \mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$ with P projective there is an isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}\big(N, \mathcal{L}(P,\Delta)^{\circledast} \otimes_U \Delta\big) \cong \mathrm{Hom}_{\mathfrak{g}}(P,N)^*,$$

natural in both P and N.

(b) *The left derived of the functor $\mathcal{L}({}_{-},\Delta)^{\circledast} \otimes_U \Delta$ is a Serre functor on $\mathscr{P}(\mathcal{C}^{\mathfrak{p}}_{\overline{\lambda}})$.*

Proof. — Using Theorem 5.1(c) we will prove claim (a) along the lines of the proof of Theorem 4.1. In fact, we need only to prove an analogue of Proposition 4.2, the rest of the proof is identical to that of Theorem 4.1.

Our first observation is that it is enough to prove Proposition 4.2 under the assumption that N is projective. The case of general N reduces to the case of projective N by taking the first two steps of the projective resolution (because both sides of (4.2) are right exact in N).

If N is projective, then $N \cong \theta\Delta$ for some projective functor θ. Similarly to Subsection 5.6 one shows that the adjoint $\mathfrak{g}$-module $\mathcal{L}(\Delta, \theta\Delta)^{\mathrm{ad}}$ is a direct sum of injective finite dimensional modules. Let $L_{\mathfrak{g}}$ and $L_{\mathfrak{g}_0}$ denote the trivial $\mathfrak{g}$- and $\mathfrak{g}_0$-modules, respectively. Note that $\mathrm{Res}\, L_{\mathfrak{g}} \cong L_{\mathfrak{g}_0}$. Let $I(0)$ denote the injective hull of $L_{\mathfrak{g}}$ in the category of finite dimensional modules.

Lemma 5.10. — *The top of $I(0)$ is isomorphic to the trivial module.*

Proof. — The assumption $\Pi V \not\cong V$ is equivalent to the assumption that Ind is isomorphic to coinduction. By adjunction we have

$$\mathrm{Hom}_{\mathfrak{g}}(L_{\mathfrak{g}}, \mathrm{Ind}\, L_{\mathfrak{g}_0}) \cong \mathrm{Hom}_{\mathfrak{g}_0}(\mathrm{Res}\, L_{\mathfrak{g}}, L_{\mathfrak{g}_0}) \neq 0,$$

which implies that $I(0)$ is a direct summand of $\mathrm{Ind}\, L_{\mathfrak{g}_0}$. Similarly,

$$\mathrm{Hom}_{\mathfrak{g}}(\mathrm{Ind}\, L_{\mathfrak{g}_0}, L_{\mathfrak{g}}) \cong \mathrm{Hom}_{\mathfrak{g}_0}(L_{\mathfrak{g}_0}, \mathrm{Res}\, L_{\mathfrak{g}}) \neq 0$$

and hence $L_{\mathfrak{g}}$ appears in the top of $\mathrm{Ind}\, L_{\mathfrak{g}_0}$.

At the same time, we claim that $L_{\mathfrak{g}}$ does not appear in the top of any other indecomposable injective module I. Indeed, if L is the simple socle of I, then $L \neq L_{\mathfrak{g}}$ and hence L has a nontrivial simple $\mathfrak{g}_0$-submodule, say $N_{\mathfrak{g}_0}$. Then, by adjunction,

$$\mathrm{Hom}_{\mathfrak{g}}(L, \mathrm{Ind}\, N_{\mathfrak{g}_0}) \cong \mathrm{Hom}_{\mathfrak{g}_0}(\mathrm{Res}\, L, N_{\mathfrak{g}_0}) \neq 0$$

and hence I appears as a direct summand in $\mathrm{Ind}\, N_{\mathfrak{g}_0}$. At the same time, again by adjunction,

$$\mathrm{Hom}_{\mathfrak{g}}(\mathrm{Ind}\, N_{\mathfrak{g}_0}, L_{\mathfrak{g}}) \cong \mathrm{Hom}_{\mathfrak{g}_0}(N_{\mathfrak{g}_0}, \mathrm{Res}\, L_{\mathfrak{g}}) = 0.$$

The claim follows. □

In the case $P = \Delta$ we have $\mathcal{L}(\Delta, \Delta) = U/I$ by Theorem 5.1(b). It follows that in this case

$$\mathrm{Hom}_{\mathfrak{g}\text{-}\mathfrak{g}}(\mathcal{L}(\Delta, P), \mathcal{L}(\Delta, N)) \cong \mathcal{L}(\Delta, N)_0^{\mathfrak{g}}. \tag{5.6}$$

Similarly to the proof of Lemma 4.3, we obtain that

$$\mathcal{L}(P, \Delta) \otimes_{U\text{-}U} \mathcal{L}(\Delta, N) \cong ((\mathcal{L}(\Delta, N)^{\circledast})_0^{\mathfrak{g}})^*. \tag{5.7}$$

As $\mathcal{L}(\Delta, N)^{\mathrm{ad}}$ is a direct sum of indecomposable injective modules and $I(0)$ has isomorphic top and socle, the multiplicities of $L_{\mathfrak{g}}$ in the top and the socle of $\mathcal{L}(\Delta, N)^{\mathrm{ad}}$ coincide. This implies that the vector spaces in (5.6) and (5.7) have the same dimension, say k, giving us an analogue of Lemma 4.3.

Let Y be the isotypic component of $I(0)$ in $\mathcal{L}(\Delta, N)^{\mathrm{ad}}$. It is easy to see that $\mathcal{KL}(\Delta, N)$ contains both the complement Y' of Y and the radical $\mathrm{Rad}\, Y$ of Y. The dimension argument implies that $\mathcal{KL}(\Delta, N)$ coincides with $Y' \oplus \mathrm{Rad}\, Y$. This means that $\mathcal{L}(\Delta, N)/\mathcal{KL}(\Delta, N)$ is the top of Y. Clearly, $\mathcal{L}(\Delta, N)_0^{\mathfrak{g}}$ is the socle of Y. Let $u \in U(\mathfrak{g})$ be any element which annihilates the radical of $I(0)$ but not $I(0)$. Applying u provides a unique (up to a nonzero scalar) isomorphism from the top of Y to the socle of Y. This gives us an analogue of the isomorphism (4.4). The rest of the proof of Proposition 4.2 carries over analogously. Claim (a) follows.

Claim (b) follows directly from (a). This completes the proof. □

5.9. Applications

In this subsection we work under the assumptions of Theorem 5.9.

Corollary 5.11. — *The Nakayama functor on $\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$ naturally commutes with projective functors.*

Proof. — Mutatis mutandis Corollary 4.5. □

Consider the endofunctor C of $\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$ of partial coapproximation with respect to projective-injective modules.

Corollary 5.12. — *The Nakayama functor on $\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$ is isomorphic to* C^2.

Proof. — Mutatis mutandis Corollary 4.6. □

Fix one representative in every isomorphism class of indecomposable projective-injective objects in $\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$ and denote by $\mathcal{P}^{\mathfrak{p}}_{\overline{\lambda}}$ the full subcategory of $\mathcal{O}^{\mathfrak{p}}_{\overline{\lambda}}$ generated by these fixed objects. The category $\mathcal{P}^{\mathfrak{p}}_{\overline{\lambda}}$ is an slf-category.

Corollary 5.13. — *The category $\mathcal{P}^{\mathfrak{p}}_{\overline{\lambda}}$ is symmetric.*

Proof. — Mutatis mutandis Corollary 4.8. □

As a special case of Corollary 5.13 we obtain that the algebra $\mathcal{P}^{\mathfrak{g}}_{\overline{\lambda}}$ describing finite dimensional modules in $\mathcal{O}_{\overline{\lambda}}$ is symmetric. For the superalgebra $\mathfrak{gl}(m, n)$ this is proved in [8]. In the case $\Pi V \cong V$ this is no longer true in general (it is easy to see that the projective cover and the injective hull of the trivial $\mathfrak{q}(1)$-module are not isomorphic). Instead we propose the following:

Conjecture 5.14. — Under the assumption $\Pi V \cong V$ the functor Π is the Serre functor on $\mathscr{P}(\mathcal{P}^{\mathfrak{g}}_{\overline{\lambda}})$.

6. Finite dimensional $\mathfrak{q}(2)$-modules

6.1. The result

In this section we use our results to describe all blocks of the category of finite dimensional $\mathfrak{q}(2)$-modules.

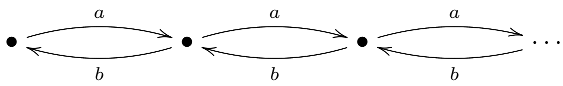
Theorem 6.1. — *Every block of the category of finite dimensional $\mathfrak{q}(2)$-modules is equivalent to the category of finite dimensional modules over one of the following algebras given by quiver and relations:*

(a)

(b)

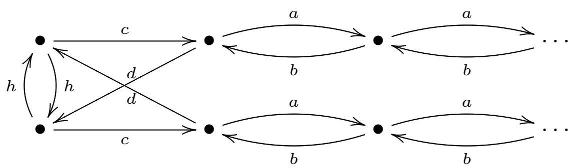
$$a^2 = 0.$$

(c)



$$a^2 = b^2 = 0, ab = ba.$$

(d)



$$a^2 = b^2 = 0, ab = ba;$$
$$chd = ba, dch = hdc;$$
$$h^2 = ac = db = cd = 0.$$

It is worth emphasizing that the algebra in Theorem 6.1(d) is not quadratic (not even homogeneous) and hence not Koszul either (in contrast to algebras from [8]). Note that all algebras in Theorem 6.1 are special biserial, in particular, they are tame.

6.2. Notation and the typical case

We fix the standard matrix unit basis $\{e_{11}, e_{12}, e_{21}, e_{22}, \overline{e}_{11}, \overline{e}_{12}, \overline{e}_{21}, \overline{e}_{22}\}$, where the overlined elements are odd.

Write $\mathfrak{q}(2)$-weights in the form $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ with respect to the dual basis of $\mathfrak{h}_0$. Simple finite dimensional $\mathfrak{q}(2)$-modules are determined uniquely up to parity change by their highest weight. If λ is the highest weight of a simple finite dimensional $\mathfrak{q}(2)$-module L, then either $\lambda = 0$ or

$\lambda_1 - \lambda_2 \in \mathbb{N}$. The module L satisfies $L \cong \Pi L$ if and only if exactly one of λ_1 and λ_2 equals zero. Denote by $\mathcal{W}$ the set of highest weights of all simple finite dimensional $\mathfrak{q}(2)$-modules. We will loosely denote these modules by $L(\lambda)$ and $\Pi L(\lambda)$ (but $L(0)$ is the trivial module).

The weight $\lambda \in \mathcal{W}$ is atypical if and only if it is of the form $(k, -k)$ for some $k \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$. The facts that strongly typical blocks are semi-simple (and hence described by Theorem 6.1(a)) and that other typical blocks are described by Theorem 6.1(b) follow from [10, 6.2].

6.3. The non-integral atypical block

For $k \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$ set $\lambda^k = (k, -k)$ and let $N(\lambda^k)$ denote the simple $\mathfrak{g}_0$-module with highest weight λ^k. It has dimension $2k + 1$. We denote by $P(\lambda^k)$ the projective cover of $L(\lambda^k)$ (in the category of finite dimensional modules).

Recall from [30, 22] that simple atypical $\mathfrak{q}(2)$-modules look as follows: we have the trivial module $L(0)$, its parity changed $\Pi L(0)$, and for every $k \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$ the even part of both $L(\lambda^k)$ and $\Pi L(\lambda^k)$ is isomorphic to $N(\lambda^k)$. In particular, using adjunction we obtain that the module $\operatorname{Ind} N(\lambda^0)$ is indecomposable, while $\operatorname{Ind} N(\lambda^k) \cong P(\lambda^k) \oplus \Pi P(\lambda^k)$ for all $k \neq 0$.

For any $\mathfrak{g}_0$-module M we have $\operatorname{Ind} M \cong \bigwedge \mathfrak{g}_1 \otimes M$ as $\mathfrak{g}_0$-modules. The $\mathfrak{g}_0$-module $\bigwedge \mathfrak{g}_1$ is a direct sum of four copies of the trivial module and four copies of $N(\lambda^1)$. Moreover, the odd part of $\bigwedge \mathfrak{g}_1$ is isomorphic to the even part.

Assume now that $k \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$. In this case the even part of $P(\lambda^k)$ is isomorphic, as a $\mathfrak{g}_0$-module, to the module

$$(N(0) \oplus N(\lambda^1)) \otimes N(\lambda^k) \cong \begin{cases} N(\lambda^k) \oplus N(\lambda^{k+1}) \oplus N(\lambda^k), & k = \frac{1}{2}; \\ N(\lambda^k) \oplus N(\lambda^{k-1}) \oplus N(\lambda^{k+1}) \oplus N(\lambda^k), & \text{otherwise.} \end{cases}$$

This implies that $P(\lambda^k)$ has length three if $k = \frac{1}{2}$ and length four otherwise. By Corollary 5.13, $P(\lambda^k)$ has isomorphic top and socle, which are thus isomorphic to $L(\lambda^k)$. The other composition factor of $P(\lambda^{\frac{1}{2}})$ has highest weight $\lambda^{\frac{3}{2}}$. For $k \neq \frac{1}{2}$, the other two composition factors of $P(\lambda^k)$ have highest weight λ^{k+1} and λ^{k-1}. Interchanging $L(\lambda^k)$ and $\Pi L(\lambda^k)$ for some k, if necessary, we may assume that all composition factors of $P(\lambda^k)$ have the form $L(\lambda^k)$ or $L(\lambda^{k\pm 1})$. This means that we have the following Loewy filtrations of projective modules:

$$
\begin{array}{ccc}
P(\lambda^{\frac{1}{2}}) & \qquad\qquad & P(\lambda^{k}) \\
\begin{tikzcd}
L(\lambda^{\frac{1}{2}}) \arrow[d, dash] \\
L(\lambda^{\frac{3}{2}}) \arrow[d, dash] \\
L(\lambda^{\frac{1}{2}})
\end{tikzcd}
& &
\begin{tikzcd}
 & L(\lambda^{k}) \arrow[dl, dash] \arrow[dr, dash] & \\
L(\lambda^{k-1}) \arrow[dr, dash] & & L(\lambda^{k+1}) \arrow[dl, dash] \\
 & L(\lambda^{k}) &
\end{tikzcd}
\end{array}
$$

It is now easy to see that the full Serre subcategory generated by $L(\lambda^k)$, $k \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$, forms a block which is equivalent to the category of finite dimensional modules over the algebra given in Theorem 6.1(c).

6.4. The principal block

Similarly to the previous subsection one shows that for $k \in \{2, 3, 4, \dots\}$ the projective module $P(\lambda^k)$ has length four, isomorphic simple top and socle, and two other composition factors which may be chosen to be isomorphic to $L(\lambda^{k-1})$ and $L(\lambda^{k+1})$. So we need only to determine the structure of $P(0)$ and $P(\lambda^1)$.

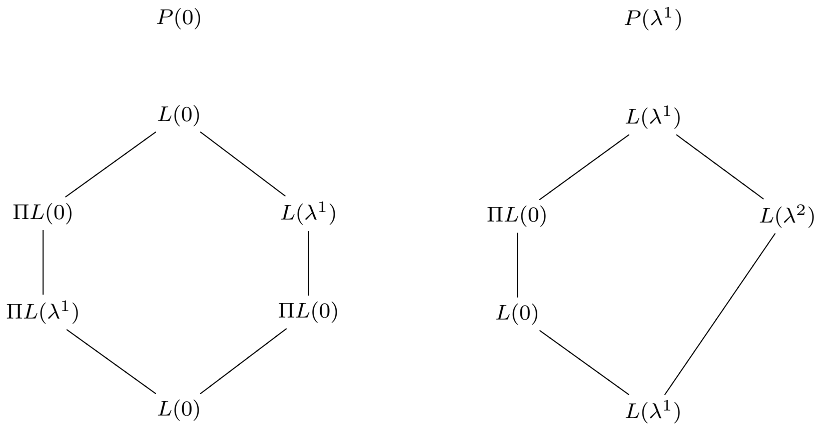
A similar argument for $P(\lambda^1)$ shows that it has length five with top and socle isomorphic to $L(\lambda^1)$ and the other three composition factors isomorphic to $L(0)$, $\Pi L(0)$ and $L(\lambda^2)$.

By a character argument, the module $P(0)$ has length six with top and socle isomorphic to $L(0)$, two other composition factors isomorphic to $\Pi L(0)$, and the two remaining composition factors having highest weight λ^1. Since $P(\lambda^1)$ contains both $L(0)$ and $\Pi L(0)$, the module $\Pi P(\lambda^1)$ contains both $L(0)$ and $\Pi L(0)$ as well. This implies that both $L(\lambda^1)$ and $\Pi L(\lambda^1)$ must appear in the injective hull of $L(0)$. Since we work with a symmetric algebra, it follows that $P(0)$ contains both $L(\lambda^1)$ and $\Pi L(\lambda^1)$, that is we now know the composition factors of both $P(0)$ and $P(\lambda^1)$.

Lemma 6.2. — *We have* $\mathrm{Ext}^1_{\mathfrak{g}}(L(0), \Pi L(0)) \cong \mathbb{C}$.

Proof. — The elements $e_{12}, e_{21}, \overline{e}_{12}$ and $\overline{e}_{21}$ obviously annihilate any extension of $L(0)$ by $\Pi L(0)$. An extension of $L(0)$ by $\Pi L(0)$ is thus given by specifying two scalars α and β which represent the action of $\overline{e}_{11}$ and $\overline{e}_{22}$, respectively. These scalars must satisfy $\alpha - \beta = 0$ as $[\overline{e}_{12}, e_{21}] = \overline{e}_{11} - \overline{e}_{22}$. On the other hand, it is straightforward to verify that any α and β satisfying $\alpha - \beta = 0$ give rise to a nontrivial extension of $L(0)$ by $\Pi L(0)$. This proves the lemma. □

From Lemma 6.2 it follows that $P(0)$ has Loewy length at least four for otherwise the top of $\operatorname{Rad} P(0)$ would contain two copies of $\Pi L(0)$. Therefore the top of $\operatorname{Rad} P(0)$ has at most two composition factors, one of which must be isomorphic to $\Pi L(0)$. The parity change argument implies that the top of $\operatorname{Rad} P(0)$ must contain more than $\Pi L(0)$. Without loss of generality we may assume that it contains $L(\lambda^1)$. Since we already know that $P(\lambda^1)$ does not contain any $\Pi L(\lambda^1)$ and that $L(0)$ does not extend $L(0)$, it follows that $P(0)$ has the composition structure as shown on the left part of the following picture:



We see that the image of the unique up to scalar nonzero map from $P(\lambda^1)$ to $P(0)$ contains an extension of $\Pi L(0)$ by $L(0)$. This implies that the composition structure of $P(\lambda^1)$ is as shown on the right hand side of the above picture. Now it is easy to see that the full Serre subcategory generated by $L(\lambda^k)$ and $\Pi L(\lambda^k)$, $k \in \{0, 1, 2, \ldots\}$ forms a block which is equivalent the category of finite dimensional modules over the algebra given in Theorem 6.1(d). Since this is the block containing the trivial module, it is called the *principal block*.

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Volodymyr MAZORCHUK
Uppsala University
Department of Mathematics
Box 480
751 06, Uppsala (Sweden)
mazor@math.uu.se

Vanessa MIEMIETZ
University of East Anglia
School of Mathematics
Norwich
NR4 7TJ (United Kingdom)
v.miemietz@uea.ac.uk