Pierre DERBEZ

Local rigidity of aspherical three-manifolds

<http://aif.cedram.org/item?id=AIF_2012____62_1_393_0>


L’accès aux articles de la revue « Annales de l’institut Fourier » (http://aif.cedram.org/), implique l’accord avec les conditions générales d’utilisation (http://aif.cedram.org/legal/). Toute reproduction en tout ou partie cet article sous quelque forme que ce soit pour tout usage autre que l’utilisation à fin strictement personnelle du copiste est constitutive d’une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.
LOCAL RIGIDITY OF ASPHERICAL THREE-MANIFOLDS

by Pierre DERBEZ (*)

Abstract. — In this paper we construct, for each aspherical oriented 3-manifold $M$, a 2-dimensional class in the $l_1$-homology of $M$ whose norm combined with the Gromov simplicial volume of $M$ gives a characterization of those nonzero degree maps from $M$ to $N$ which are homotopic to a covering map. As an application we characterize those degree one maps which are homotopic to a homeomorphism in term of isometries between the bounded cohomology groups of $M$ and $N$.

Résumé. — Dans ce papier nous construisons, pour chaque variété de dimension trois close orientable et asphérique $M$, une classe d’homologie $l_1$ de dimension deux dans $M$ dont la norme permet avec le volume simplicial de $M$ de caractériser les applications de degré non-nul de $M$ dans $N$ qui sont homotopes à un revêtement. Comme conséquence, nous donnons un critère d’homéomorphisme pour les applications de degré un en terme d’isométries entre les groupes de cohomologie bornée de $M$ et $N$.

1. Introduction

Throughout this paper all manifolds are orientable. Given a topological space $X$ we denote by $(C_*(X), \partial)$ its real singular chain complex endowed with the $l_1$-norm defined by $\|\sigma\|_1 = \sum_i |a_i|$ if $\sigma = \sum_i a_i \sigma_i$, where $\sigma_i$ are singular simplices.

Any finite covering map $f: M \to N$ between closed orientable 3-manifolds induces an isometry $f^\#: H_3(M; \mathbb{R}) \to H_3(N; \mathbb{R})$ with respect to the $l_1$ (semi) norm induced by the $l_1$-norms on the real singular chains of $M$ and $N$.

For hyperbolic manifolds this condition is sufficient to characterize covering maps by Gromov and Thurston’s works. However, since the Gromov...
simplicial volume of a 3-manifold $M$, which is the $l_1$-norm $|[M]|_1$ of a generator $[M]$ of $H_3(M;\mathbb{Z}) \subset H_3(M;\mathbb{R})$, does not detect the "non-hyperbolic part" of 3-manifolds one can construct, using pinching maps, many pairwise non-homeomorphic 3-manifolds with the same Gromov simplicial volume related by a degree one map.

When $M$ is a surface bundle over the circle with a fiber of negative Euler characteristic, M. Boileau and S. Wang gave in [3, Theorem 2.1, Corollary 2.3] a characterization of nonzero degree maps $f : M \to N$ into an irreducible 3-manifold which are homotopic to a covering map in terms of isometry with respect to the Thurston’s norm in the second homology group of the manifolds. The purpose of this paper is to extend [3, Theorem 2.1] to aspherical 3-manifolds.

According to the Geometrization Theorem of Perelman, any closed aspherical 3-manifold $M$ admits a JSJ-splitting along a family of characteristic tori $\mathcal{T}_M$ such that each component of $M \setminus \mathcal{T}_M$ either admits a Seifert fibration or has a complete finite volume hyperbolic interior.

We say that $M$ is orientable* if $M$ is orientable and if each Seifert component of $M \setminus \mathcal{T}_M$ admits a fibration over an orientable surface. This condition is satisfied for example when $M$ contains no embedded Klein bottle or when $M$ is obtained from a holomorphic function $f : (\mathbb{C}^3,0) \to (\mathbb{C},0)$ with an isolated singularity at 0 by taking the boundary of the singularity of $f$ at 0 defined by $f^{-1}(0) \cap S(\varepsilon)$, where $S(\varepsilon)$ is a Milnor sphere centered at 0 in $\mathbb{C}^3$ with radius $\varepsilon$ (see [18]). Notice that this orientation* condition is also satisfied when $M$ is a surface bundle with a fiber of negative Euler characteristic ([3]).

In [3, Theorem 2.1], a key point, is that when $M$ is a surface bundle, there there exists a class $\alpha_M \in H_2(M) \setminus \{0\}$, namely the class of the fiber, "passing non-trivially through the whole manifold". Of course, such a fiber class, does not exist in the homology of a general 3-manifold because if we try to define local classes in $M$ there are often homological obstructions which do not allow to glue them together in order to define a global class. However these obstructions disappear considering the $l_1$-completion $H_2^{l_1}(M)$ of $H_2(M)$ and a fiber class $\alpha_M$ can be defined in $H_2^{l_1}(M)$ as follows. Let $M$ be a closed orientable aspherical 3-manifold:

When $M$ is a geometric 3-manifold, set $\alpha_M = 0$ excepted when $M$ is a $\widetilde{\text{SL}_2(\mathbb{R})}$-manifold. In this case, $M$ admits a finite covering $p : \widetilde{M} \to M$ which is a (true) circle bundle $\xi : \widetilde{M} \to \widetilde{F}$ over a smooth surface. Then we set

$$\alpha_M = p_\sharp \circ \xi_\sharp^{-1} \left( \left[ \overline{F} \right]_1 \right)$$
where $[\tilde{F}]_1$ denotes the $l_1$-class of the $l_1$-cycle $\tilde{F}$. This makes sense since by [9, Mapping Theorem] $\xi$ induces an isometric isomorphism $\xi_2: H^{l_1}_2(\tilde{M}) \to H^{l_2}_2(\tilde{F})$.

When $M$ is not a geometric 3-manifold, each Seifert component of $M \setminus \mathcal{T}_M$ admits either a Euclidean geometry or a $\mathbb{H}^2 \times \mathbb{R}$-geometry. For each $\mathbb{H}^2 \times \mathbb{R}$-component $P_i$, $i = 1, \ldots, l$, of $M \setminus \mathcal{T}_M$ we choose a horizontal properly embedded incompressible surface $F_i$ in $P_i$ and we set

$$\alpha_M = \sum_{i=1}^{l} \frac{1}{k_i} \alpha_M(F_i)$$

where $k_i$ denotes the intersection number between $F_i$ and the generic fiber of $P_i$ and where $\alpha_M(F_i)$ denote the $l_1$-class of $F_i$ in $M$ which makes sense since the relative cycle $F_i$ of $P_i$ can be "filled" in a natural way giving a $l_1$-cycle in $M$ (see paragraph 2). If $M \setminus \mathcal{T}_M$ contains no $\mathbb{H}^2 \times \mathbb{R}$-components we just set $\alpha_M = 0$.

**Remark 1.1.** — Obviously, it follows from our construction that our fiber class does not need to be unique, as well as the fiber class of a surface bundle when the rank of the homology is distinct from 1, by a result of [17]. On the other hand, it follows from our proof of Theorem 1.2 that our results hold for any choices of a fiber class.

The main result of this paper states as follows

**Theorem 1.2.** — Let $f: M \to N$ be a nonzero degree map from a closed orientable* aspherical 3-manifold into a closed orientable irreducible 3-manifold such that $\|f_x([M])\|_1 = \|[M]\|_1$ and $\|f_x(\alpha_M)\|_1 = \|\alpha_M\|_1$ for some fiber class $\alpha_M$. Then $f$ is homotopic to a deg($f$)-fold covering map.

To make the hypothesis $\|f_x(\alpha_M)\|_1 = \|\alpha_M\|_1$ more concrete one can compare it with a condition given in [6] where we introduce an invariant denoted by $\text{vol}(M)$ and defined as the sum of the absolute value of the orbifolds Euler characteristic of the Seifert pieces of $M$. This volume is used to state rigidity results, see [6, Theorems 1.3 and 1.6]. Using sections 2 and 3 of this paper and results in [6] one can easily check that $\|\alpha_M\|_1 = \text{vol}(M)$ and if $\|f_x([M])\|_1 = \|[M]\|_1$, meaning that $\|M\| = |\text{deg}(f)||N|$, then $\|\alpha_M\|_1 = \text{vol}(M) \geq \|f_2\alpha_M\|_1 \geq \|\alpha_N\|_1 = \text{vol}(N)$.

As far as we know, there are no general results to characterize local isometries for aspherical 3-manifolds excepted when the sectional curvature is negative. In the situation we deal with, the best metric we can hope, in many cases, is a metric with non-positive curvature by [14] and
our manifolds contain many totally geodesic surfaces where the curvature vanishes. From the point of view of maps \( f : M \to N \) there are more flexibility when the sectional curvature of \( M \) vanishes and so it is more difficult to control the behavior of \( f \). On the other hand, we hope that our results offer an application of the theory of bounded cohomology and \( l_1 \)-homology.

Notice that if \( M \) and \( N \) are both orientable* then the isometry condition is also necessary (see Lemma 2.2 and Proposition 2.4). If \( N \) is not orientable* the condition is not necessary. Indeed, consider for \( N \) the trivial orientable \( S^1 \)-bundle over the genus \(-3\) surface \( \mathbb{RP}(2) \# \mathbb{RP}(2) \# \mathbb{RP}(2) \) and for \( M \) the trivial bundle \( \Sigma_2 \times S^1 \) which is a 2-fold covering \( p : M \to N \), where \( \Sigma_2 \) is the genus 2-surface. Let \( \alpha_M \) denote the class of \( \Sigma_2 \) in \( H^3_2(M; \mathbb{R}) \). Then it follows from the arguments of section 2 that \( \|\alpha_M\|_1 > 0 \) and \( p_\sharp(\alpha_M) = 0 \).

By the Hahn-Banach Theorem, for each fiber class \( \alpha_M \) with \( \|\alpha_M\|_1 > 0 \), there exists a class \( \beta_M \) in the second bounded cohomology group of \( M \), denoted by \( H^2_2(M; \mathbb{R}) \) and endowed with the semi-norm \( \|\cdot\|_\infty \), such that \( \langle \beta_M, \alpha_M \rangle = 1 \) and \( \|\beta_M\|_\infty = \frac{1}{\|\alpha_M\|_1} \). When \( \|\alpha_M\|_1 = 0 \), just set \( \beta_M = 0 \). Thus we deduce the following

**Corollary 1.3.** — Let \( f : M \to N \) be a nonzero degree map from a closed orientable* aspherical 3-manifold into a closed orientable irreducible 3-manifold such that \( \|f_\sharp([M])\|_1 = \|[M]\|_1 \). If there exists a class \( \beta \in H^2_6(N; \mathbb{R}) \) such that \( f_\sharp(\beta) = \beta_M \) and \( \|\beta_M\|_\infty = \|\beta\|_\infty \) then \( f \) is homotopic to a covering map.

We give the following corollary answering positively to a question of Professor M. Boileau.

**Corollary 1.4.** — A degree one map \( f : M \to N \) from a closed orientable* aspherical 3-manifold into a closed orientable irreducible 3-manifold is homotopic to a homeomorphism iff

(i) \( f_\sharp : H_3(M; \mathbb{R}) \to H_3(N; \mathbb{R}) \) is an isometry with respect to the \( l_1 \)-norms and

(ii) \( f \) induces an isometric isomorphism \( f_\sharp^2 : H^2_6(N; \mathbb{R}) \to H^2_6(M; \mathbb{R}) \), resp. an isometry \( f_\sharp^2 : H^1_2(M; \mathbb{R}) \to H^1_2(N; \mathbb{R}) \).

**Theorem 1.5.** — A nonzero degree map \( f : M \to N \) from a closed orientable aspherical 3-manifold into a closed orientable irreducible 3-manifold is homotopic to a covering map iff it induces a homomorphism \( f_* : \pi_1 M \to \pi_1 N \) with amenable kernel.
We end this section by mentioning a related result for self maps which is a direct consequence of [24] and [13, Theorem 0.7] using a standard covering space argument suggested by Professor W. Lück:

**THEOREM 1.6.** — Any nonzero degree map \( f: M \to M \) from a closed orientable aspherical 3-manifold to itself is homotopic to a \( \deg(f) \)-fold covering.

**Organization of the paper.** This paper is organized as follows. In Section 2 we collect some technical results which will be used in the proof of the theorem. More precisely we compute the \( l_1 \)-norm of certain classes in \( H^{l_1}_2(M) \) which come from classical integral homology classes and we study some isometric properties of finite coverings with respect to the \( l_1 \)-norms. Section 3 is devoted to the proof of Theorems 1.2 and 1.5.

**2. Norm of surfaces in aspherical 3-manifolds**

To fix the notations we recall the basic definitions of \( l_1 \)-homology and bounded cohomology according to the main papers of [16] and [9]. For a topological space \( X \), denote by \( C^{l_1}_n(X) \) the \( l_1 \)-completion of the real singular chains \( C_n(X) \). Then

\[
C^{l_1}_n(X) = \left\{ c = \sum_{i=1}^{\infty} a_i \sigma_i \text{ s.t. } \|c\|_1 = \sum_{i=1}^{\infty} |a_i| < \infty \right\}
\]

where \( a_i \in \mathbb{R} \) and \( \sigma_i: \Delta_n \to X \) is a singular \( n \)-simplex. We will denote by \( S_n(X) \) the set of singular \( n \)-simplices. The topological dual of \( C^{l_1}_n(X) \) is given by the set

\[
C^n_b(X) = \left\{ w \in C^n(X) \text{ s.t. } \|w\|_\infty = \sup_{\sigma \in S_n(X)} |\langle w, \sigma \rangle| < \infty \right\}
\]

Note that the \( \partial \) and \( \delta \) operators are bounded so that \( (C^{l_1}_n(X), \partial) \) and \( (C^n_b(X), \delta) \) are chain, resp. cochain, complexes. We denote by \( H^{l_1}_n(X) \), resp. by \( H^n_b(X) \), the homology, resp. cohomology, of this chain, resp. cochain, complex. The vector spaces \( H^{l_1}_n(X) \) and \( H^n_b(X) \) are endowed with the quotient semi-norm that we still denote by \( \|\cdot\|_1 \) and \( \|\cdot\|_\infty \) respectively. In the same way it is a standard fact that one can define the \( l_1 \)-homology and bounded cohomology of a pair of topological spaces \((X,A)\). Denote by \( i: A \to X \) the natural inclusion and by \( j: C^{l_1}_n(X) \to C^{l_1}_n(X,A) \) the projection. Then we get the classic long exact sequence

\[
\ldots \to H^{l_1}_{n-1}(A) \xrightarrow{i_*} H^{l_1}_{n}(A) \xrightarrow{j_*} H^{l_1}_{n}(X,A) \xrightarrow{\partial} H^{l_1}_{n-1}(A) \to \ldots
\]
If moreover each component of \( A \) has an amenable fundamental group then by [16, Corollary 2.5] we know that \( H^1_n(A) = \{0\} \) for any \( n \geq 1 \) and thus \( j \) admits an inverse \( j^{-1} : H^1_n(X, A) \to H^1_n(X) \) for \( n \geq 2 \) defined by \( j^{-1}([z]) = [z + u] \) where \( z \) is a relative cycle in \( (X, A) \) and \( u \) is any \( l_1 \)-chain in \( A \) such that \( \partial u = -\partial z \). It follows from the definition that any continuous map of pairs \( f : (X, A) \to (Y, B) \) induces a bounded homomorphism \( f_\# : H_n(X, A) \to H_n(Y, B) \) such that \( \|f_\#\| \leq 1 \). On the other hand, when \( M \) is compact orientable \( n \)-manifold with (possibly empty) boundary we will denote by \([M]\) its fundamental class in \( H_n(M, \partial M) \), by \([M]_1\) the image of \([M]\) under the homomorphism \( H_n(M, \partial M) \to H^1_n(M, \partial M) \) induced by the completion and by \( \|M\| \) its Gromov simplicial volume. For technical reasons we need the following

**Lemma 2.1.** — Let \( p : \tilde{X} \to X \) be a regular covering map with finite Galois group \( \Gamma \). For any \( \Gamma \)-invariant class \( \alpha \in H^1_n(\tilde{X}) \) then \( \|p_\#(\alpha)\|_1 = \|\alpha\|_1 \).

**Proof.** — We use the averaging retraction \( A : C^n_b(\tilde{X}) \to C^n_b(X) \) defined in [9] by

\[
\langle A(\gamma), \sigma \rangle = \frac{\sum_{g \in \Gamma} g^\# \gamma \sigma}{\text{Card}(\Gamma)}
\]

where \( \tilde{\sigma} : \Delta^n \to \tilde{X} \) denotes a lifting of \( \sigma : \Delta^n \to X \). This definition does not depend one the choice of the lifting \( \tilde{\sigma} \) since the covering is regular. By construction, \( A \) commutes with the differentials so that it induces a homomorphism \( \tilde{A} : H^n_b(\tilde{X}) \to H^n_b(X) \) such that \( \|\tilde{A}\| \leq 1 \). Let \( \alpha \in H^1_n(\tilde{X}) \) such that \( g^\#(\alpha) = \alpha \) for any \( g \in \Gamma \). If \( \|\alpha\|_1 \neq 0 \) then by the Hahn-Banach Theorem, there exists \( \beta \in H^n_b(\tilde{X}) \) such that \( \langle \beta, \alpha \rangle = 1 \) and \( \|\beta\|_\infty = \frac{1}{\|\alpha\|_1} \). Since \( \alpha \) is \( \Gamma \)-invariant then by the definition of the averaging we have \( \langle \tilde{A}(\beta), p_\#(\alpha) \rangle = 1 \) and thus using the Hölder inequality and the fact that \( \|\tilde{A}\| \leq 1 \) we deduce \( \|p_\#(\alpha)\|_1 \geq \|\alpha\|_1 \). This proves the lemma.

\[\Box\]

**2.1. \( \widetilde{SL}_2(\mathbb{R}) \)-manifolds**

Let \( M \) be an orientable* 3-manifold admitting a \( \widetilde{SL}_2(\mathbb{R}) \)-geometry. If moreover \( M \) is a (true) circle bundle, with projection \( \xi \) and base \( F \) then by [9, Mapping Theorem] \( \xi \) induces an isometric isomorphism \( \xi_\#: H^1_2(M) \to H^1_2(F) \). Denote by \( \alpha_M(F) \) the class \( \xi_\#^{-1}([F]_1) \).
Lemma 2.2. — Let \( M \) be an orientable* \( SL_2(\mathbb{R}) \)-manifold. 

(i) If \( M \) is a (true) circle bundle with base \( F \) then 
\[
\| \alpha_M(F) \|_1 = \| F \|
\]

(ii) Otherwise, for any finite covering \( p: \tilde{M} \to M \) such that \( \tilde{M} \) is a (true) circle bundle over a surface \( F \) and projection \( \xi: \tilde{M} \to F \) then 
\[
\| p_* \alpha_{\tilde{M}}(F) \| = \| F \|.
\]

(iii) Moreover when \( \tilde{M} \) is a circle bundle, the vector space generated by 
\( p_* \alpha_{\tilde{M}}(F) \) does not depend on the choice of the finite covering \( p: \tilde{M} \to M \).

Proof. — We first check point (i). The inequality 
\[
\| \alpha_M(F) \|_1 \leq \| F \|
\]
follows from the definition. To check the converse inequality we use exactly the same construction as in [2]. Fix a complete hyperbolic metric on \( F \). Since \( F \) is orientable we can define a bounded \( 2 \)-cocyle \( \Omega_F \) in the following way: for each \( 2 \)-simplex \( \sigma: \Delta^2 \to F \), where \( \Delta^2 \) denotes the standard \( 2 \)-simplex, we set 
\[
\langle \Omega_F, \sigma \rangle = A(st(\sigma)),
\]
where \( st(\sigma) \) denotes the geodesic simplex obtained from \( \sigma \) after straightening and \( A \) denotes the algebraic area with respect to the fixed hyperbolic metric. In particular we get, if \( z \) denotes a \( 2 \)-cycle representing the fundamental class of \( F \), 
\[
\langle \Omega_F, z \rangle = \text{Area}(F)
\]
and since \( \| \Omega_F \|_1 \leq \| f \| \leq \pi \) then by the Hölder inequality we get 
\[
\| \alpha_M(F) \|_1 \geq \| F \|.
\]
This proves point (i). We now prove point (ii). We consider two cases depending on whether the covering is regular or not.

Case 1. Assume that \( p \) is regular. Denote by \( \Gamma \) the Galois group of the covering. Note that since \( M \) is a Seifert bundle with orientable base \( 2 \)-orbifold then any \( g \in \Gamma \) induces an orientation preserving homeomorphism \( \overline{g}: F \to F \) such that \( \xi \circ g = \overline{g} \circ \xi \) and thus \( \alpha_{\tilde{M}}(F) \) is \( \Gamma \)-invariant and point (ii) of the lemma follows from Lemma 2.1 and point (i). This completes the proof of point (ii) in Case 1.

Case 2. If \( p \) is not regular then consider a finite covering \( q: \tilde{M} \to \tilde{M} \) such that \( p \circ q \) is regular. Since \( q_* \left( \langle \alpha_{\tilde{M}}(\tilde{F}) \rangle \right) = \langle \alpha_{\tilde{M}}(F) \rangle \), where \( \langle v \rangle \) denotes the vector space generated by the vector \( v \) and where \( \tilde{F} \) is the base of the bundle \( \tilde{M} \), then point (ii) in Case 2 follows from Case 1.

To check point (iii) it suffices to consider a common covering \( \tilde{M} \to \tilde{M}_1 \) and \( \tilde{M}_2 \) (which corresponds for example to \( (p_1)_*(\pi_1 \tilde{M}_1) \cap (p_2)_*(\pi_1 \tilde{M}_2) \)). This completes the proof of the lemma. \( \Box \)
2.2. Aspherical 3-manifolds

Let $M$ be a closed orientable* aspherical 3-manifold. We fix an orientation on $M$. In the following we will assume that $\mathbb{H}^2$ and $\mathbb{R}$ are oriented with the usual convention. Let $P$ denote a component of $M \setminus T_M$ whose interior admits a $\mathbb{H}^2 \times \mathbb{R}$-geometry. Since $M$ is orientable* then $P$ admits a Seifert fibration over an orientable basis and we denote by $h_P$ the fiber of $P$. We orient the fiber $h_P$ in such a way that the universal covering $p: \mathbb{H}^2 \times \mathbb{R} \to P$ induces an orientation preserving map $\mathbb{R} \to h_P$. Let $\mathcal{F}$ be an oriented surface and let $f: (\mathcal{F}, \partial \mathcal{F}) \to (P, \partial P)$ be a proper map. For any $x \in \mathbb{R}$ we denote by $\alpha_M(x, f)$ the class defined by $k_j f - 1 f_1$ following the composition of homomorphisms:

$$H_2^1(\mathcal{F}, \partial \mathcal{F}) \xrightarrow{f_*} H_2^1(P, \partial P) \xrightarrow{j_*} H_2^1(P) \xrightarrow{k} H_2^1(M)$$

where $k: P \to M$ denotes the inclusion.

**Lemma 2.3.** — We have $\|\alpha_M(x, f)\|_1 \leq |x|\|\mathcal{F}\|$ for any $x \in \mathbb{R}$.

**Proof.** — The proof follows from [9, Equivalence Theorem] combined with [16, Theorem 2.3].

Consider now a proper map $f: (\mathcal{F}, \partial \mathcal{F}) \to (P, \partial P)$ transverse to the fibers of $P$. We choose always the orientation of each component of $\mathcal{F}$ so that $f$ is orientation preserving which means that the orientation of $f(\mathcal{F})$ followed by the orientation of $h_P$ matches the orientation induced by $M$. The main purpose of this section is to check the following

**Proposition 2.4.** — Let $M$ be a closed aspherical orientable* 3-manifold and denote by $P_1, \ldots, P_l$ a collection of pairwise distinct Seifert components of $M \setminus T_M$ whose interior admits a $\mathbb{H}^2 \times \mathbb{R}$-geometry. For each $i = 1, \ldots, l$ assume that we are given an orientation preserving proper embedding $f_i: (\mathcal{F}_i, \partial \mathcal{F}_i) \to (P_i, \partial P_i)$. Then

(i) **Isometry:** for any $i = 1, \ldots, l$ we have the equality

$$\|\alpha_M(\mathcal{F}_i, f_i)\|_1 = \|\mathcal{F}_i\|$$

(ii) **Additivity under JSJ-splitting:**

$$\|\alpha_M(x_1 \mathcal{F}_1, f_1) + \ldots + \alpha_M(x_l \mathcal{F}_l, f_l)\|_1$$

$$= \|\alpha_M(x_1 \mathcal{F}_1, f_1)\|_1 + \ldots + \|\alpha_M(x_l \mathcal{F}_l, f_l)\|_1$$

where $x_1, \ldots, x_l$ are positive real numbers.

(iii) **Let $f: M \to N$ be a covering map with $N$ orientable*. If $\alpha = \alpha_M(x_1 \mathcal{F}_1, f_1) + \ldots + \alpha_M(x_l \mathcal{F}_l, f_l)$ then $\|f_*(\alpha)\|_1 = \|\alpha\|_1$.**
To prove this proposition we need two intermediate results. Hypothesis are the same as in Proposition 2.4.

**Lemma 2.5.** — Suppose that \{P_i\}_{i \in I} is a family of circle bundles components of \( M \setminus T_M \) admitting a \( \mathbb{H}^2 \times \mathbb{R} \)-geometry. For any \( i \in I \) there exists a bounded 2-cocyle \( \Omega_{P_i} \) in \( M \) satisfying the following properties:

(i) \( k_i^* (\Omega_{P_i}) \) is a relative 2-cocycle in \( (P_i, \partial P_i) \) where \( k_i: P_i \hookrightarrow M \) denotes the natural inclusion and \( k_i^* (\Omega_{P_j}) = 0 \) if \( i \neq j \),

(ii) \( \langle [\Omega_{P_i}], \alpha_M (F_i, f_i) \rangle \rangle = \text{Area}(F_i) \) where \( \text{Area}(F_i) \) denotes the area of \( \text{int}(F_i) \) endowed with a complete hyperbolic metric.

(iii) \( \| \sum_{i \in I} \Omega_{P_i} \|_\infty = \pi \), where \([\Omega_{P_i}]\) denotes the class of \( \Omega_{P_i} \) in \( H^2_b(M; \mathbb{R}) \).

**Remark 2.6.** — The above result is stated for Seifert pieces which are circle bundles only for convenience. This lemma remains true if we consider a family of Seifert pieces admitting a geometry \( \mathbb{H}^2 \times \mathbb{R} \) with an orientable base 2-orbifold. Notice that the bounded class \( \Omega_{P_i} \) cannot be defined for Seifert pieces with non-orientable basis.

To prove this lemma we need the reduction of singular chains with respect to the JSJ-splitting of aspherical 3-manifolds. This chain map is stated for example in [8]. Since this construction is crucial for our purpose we recall it and fix notation.

Let M be a closed aspherical orientable 3-manifold. Denote by \( P_1, ..., P_l \) the components of \( M \setminus T_M \). As in [8], we consider a chain map \( \rho: C_n(M) \to C_n(M) \) defined as follows:

**0-simplices.** If \( n = 0 \) then \( \rho \) is the identity.

**1-simplices.** If \( n = 1 \) let \( \tau: [v_0, v_1] \to M \) be a 1-simplex. Since \( T_M \) is incompressible, the map \( \tau \) is homotopic, rel. \( \{v_0, v_1\} \), to a reduced 1-simplex i.e. a map \( \tau_1: [v_0, v_1] \to M \) such that either \( \tau_1([v_0, v_1]) \subset T_M \) or \( \tau_1([v_0, v_1]) \) is transverse to \( T_M \) and for each component \( J \) of \( \tau_1^{-1}(P_i) \) then \( \tau_1|J \) is not homotopic rel. \( \partial J \) into \( \partial P_i \). Then we set \( \rho(\tau) = \tau_1 \) and we extend \( \rho \) by linearity.

**2-simplices.** If \( n = 2 \) let \( \sigma: \Delta^2 = [v_0, v_1, v_2] \to M \) be a 2-simplex. Then \( \sigma \) is homotopic rel. \( V(\Delta^2) = \{v_0, v_1, v_2\} \) to a reduced 2-simplex \( \sigma_1 \) such that either \( \sigma_1(\Delta^2) \subset T_M \) or \( \sigma_1|\text{int}(\Delta^2) \) is transverse to \( T_M \), the 1-simplex \( \sigma_1|e \) is reduced for each edge \( e \) of \( \Delta^2 \) and \( \sigma_1^{-1}(T_M) \) contains no loop components. Thus if \( J \) is a component of \( \sigma_1^{-1}(T_M) \) such that \( J \cap \text{int}(\Delta^2) \neq \emptyset \) then \( J \) is a proper arc in \( \Delta^2 \) connecting two distinct edges (see figure 2.1). Then we set \( \rho(\sigma) = \sigma_1 \) and we extend \( \rho \) by linearity.
Remark 2.7. — Suppose that $\sigma: \Delta^2 \to M$ is a reduced 2-simplex. If $\sigma(e)$ is not contained in $\mathcal{T}_M$ for any edge $e$ of $\Delta^2$ then there exists a unique component, denoted by $\text{Core}(\sigma)$, of $\Delta^2 \setminus \sigma^{-1}(\mathcal{T}_M)$ which meets the three edges of $\Delta^2$ (see [8]).

3-simplices. If $n = 3$ let $\sigma: \Delta^3 = [v_0, v_1, v_2, v_3] \to M$ be a 3-simplex. Then $\sigma$ is homotopic rel. $V(\Delta^3) = \{v_0, v_1, v_2, v_3\}$ to a reduced 3-simplex $\sigma_1$ such that either $\sigma_1(\Delta^3) \subset \mathcal{T}_M$ or $\sigma_1|\text{int}(\Delta^3)$ is transverse to $\mathcal{T}_M$, the 2-simplex $\sigma_1|\Delta^2_i$ is reduced for each face $\Delta^2_i$ of $\Delta^3$ and if $D$ is a component of $\sigma_1^{-1}(\mathcal{T}_M)$ such that $D \cap \text{int}(\Delta^3) \neq \emptyset$ then $D$ is either a normal triangle or a normal rectangle (see figure 2.2). Then we set $\rho(\sigma) = \sigma_1$ and we extend $\rho$ by linearity. Notice that the reduction is stable under the $\partial$-operator.

Proof of Lemma 2.5. — We use here the technique developed in [1].

Step 1: Crushing $M$ into $P_i$. Denote by $p_i: \widetilde{M}_i \to M$ the covering map corresponding to the subgroup $(k_i)_*(\pi_1 P_i)$ of $\pi_1 M$, fix a lifting $\tilde{k}_i: P_i \to \widetilde{M}_i$ of $k_i: P_i \to M$ and denote by $\tilde{P}_i$ the image of $\tilde{k}_i$. There exists a retraction $r_i: \widetilde{M}_i \to P_i$ crushing each component of $\widetilde{M}_i \setminus \tilde{P}_i$ to the corresponding component of $\partial\tilde{P}_i$ such that $r_i|\tilde{P}_i = \tilde{k}_i^{-1}$. Denote by $F_i$ the base surface.
of the circle bundle $P_i$ and by $\xi_i: P_i \to F_i$ the projection. Fix a complete hyperbolic metric on $\text{int}(F_i)$, crush each component of $\partial F_i$ to a point, denote by $\widehat{F}_i$ the new surface and by $q_i: F_i \to \widehat{F}_i$ the natural crushing map. This construction is equivalent to adding a limit parabolic point to each component $C$ of $\partial F_i$. This parabolic point corresponds to the fixed point of the parabolic isometry generating $\pi_1 C$.

**Step 2: Straightening simplices on surfaces with boundary.** Let $\sigma: \Delta^2 = [v_0, v_1, v_2] \to \widehat{F}_i$ be a (singular) 2-simplex. Consider an edge $\tau = \sigma|[v_i, v_j]: [v_i, v_j] \to \widehat{F}_i$ and denote by $\widetilde{\tau}: [v_i, v_j] \to \overline{H}^2$ a lifting of $\tau$ in the hyperbolic space union its boundary. Then $\widetilde{\tau}$ is homotopic by a homotopy fixing the end points to the unique geodesic arc (which may be constant) connecting the end points of $\widetilde{\tau}$. Denote by $\text{st}(\widetilde{\tau})$ the new straight 1-simplex and by $\text{st}(\tau)$ the projection of $\text{st}(\widetilde{\tau})$ into $\widehat{F}_i$. We straighten each edge of $\sigma$ and finally we homotop $\sigma$ to a straight 2-simplex $\text{st}(\sigma)$. As in the proof of Lemma 2.2 we define a bounded 2-cocyle $\widehat{\omega}_i$ on $\widehat{F}_i$ by setting $\langle \widehat{\omega}_i, \sigma \rangle = \mathcal{A}(\text{st}(\sigma))$, the algebraic area of $\text{st}(\sigma)$. Thus $q_i^\mathcal{A}(\widehat{\omega}_i)$ defines a relative 2-cocyle on $(F_i, \partial F_i)$ such that $\langle q_i^\mathcal{A}(\widehat{\omega}_i), z_i \rangle = \text{Area}(F_i)$, where $z_i$ is a relative 2-cycle representing the fundamental class of $F_i$. 
Step 3: Lifting the singular chains. Let \( \mu = \sum_l a_l \mu_l \) be a \( n \)-chain for \( n = 2,3 \) where \( a_l \in \mathbb{R} \) and \( \mu_l: \Delta^n \to M \) is a singular \( n \)-simplex. We choose a decomposition of each component of \( \Delta^n \setminus \rho(\mu_l)^{-1}(T_M) \) into \( n \)-simplices \( \nabla^j_l, j = 1, \ldots, n_l \) (recall that \( \rho \) denotes the reduction operator). Next we replace \( \mu \) by the \( n \)-chain \( \sigma = \sum_{l,j} a_l \rho(\mu_l) | \nabla^j_l \). Denote by \( \tilde{\sigma} \) the preimage of \( \sigma \) in \( \tilde{M}_i \). Then \( \tilde{\sigma} \) is a locally finite \( n \)-chain in \( \tilde{M}_i \). Since \( \tilde{P}_i \) is compact then we define a finite \( n \)-chain \( \tilde{\sigma}_i \) in \( \tilde{M}_i \) by taking only the simplices of \( \tilde{\sigma} \) which meet \( \tilde{P}_i \).

Step 4: Definition of a bounded cocyle satisfying the conclusion of the lemma. Keeping the same notation as in Step 3 we define a 2-cochain \( \Omega_{P_i} \) in \( M \) by setting

\[
\langle \Omega_{P_i}, \mu \rangle = \left\langle g^*_i \tilde{w}_i, \tilde{\sigma}_i \right\rangle
\]

where \( \mu \) is a singular 2-simplex and where \( g_i = q_i \circ \xi_i \circ r_i \). By construction \( \| \Omega_{P_i} \| \leq \pi \). Indeed let \( \sigma: \Delta^2 \to M \) be a singular 2-simplex. By construction of \( \Omega_{P_i} \) we may assume that \( \sigma \) is reduced. First note that it follows from the construction that for each triangle \( \Delta \) of \( \Delta^2 \setminus \sigma^{-1}(T_M) \) (given in the decomposition of Step 3) whose an edge is a component of \( \sigma^{-1}(T_M) \) then \( \langle \Omega_{P_i}, \sigma | \Delta \rangle = 0 \) (the simplices of \( \sigma | \Delta \) are sent into a point or a geodesic arc after straightening in \( \hat{P}_i \)). On the other hand there exist at most one triangle \( \Delta_\sigma \) of \( \Delta^2 \setminus \sigma^{-1}(T_M) \) whose no edge is a component of \( \sigma^{-1}(T_M) \). This triangle necessarily lives in \( \text{Core}(\sigma) \). Since there exists at most one component of \( \sigma | \Delta_\sigma \) which meets \( \tilde{P}_i \) then the inequality \( \| \Omega_{P_i} \| \leq \pi \) follows.

We check the cocycle condition \( \langle \delta \Omega_{P_i}, \sigma \rangle = 0 \) for each 3-simplex \( \sigma: \Delta^3 \to M \). Since \( \langle \delta \Omega_{P_i}, \sigma \rangle = \langle \Omega_{P_i}, \partial \sigma \rangle \) then we may assume that \( \sigma \) is reduced. Consider the 3-chain \( \sum_j \sigma | \nabla_j \), where \( \nabla_j \) is the decomposition (given in Step 3) of \( \Delta^3 \setminus \sigma^{-1}(T_M) \) into 3-simplices. The 2-faces of \( \nabla_j \) are made of interior triangles which consist of the triangles whose interiors are in the interior of \( \Delta^3 \) and of triangles which define the 2-simplices of a decomposition of \( \partial \Delta^3 \setminus \sigma^{-1}(T_M) \cap \partial \Delta^3 \). Since each interior triangle is the face of two adjacent tetrahedra then one can replace \( \sigma \) by \( \sum_j \sigma | \nabla_j \). Denote still \( \sum_j \sigma | \nabla_j \) by \( \sigma \). The 2-chain of \( \tilde{M}_i \) defined by

\[
\partial \tilde{\sigma}_i - \left( \tilde{\partial} \sigma \right)_i
\]

consists of an alternate sum of 2-simplices of \( \partial \tilde{\sigma} \) which does not meet \( \tilde{P}_i \). Since the retraction \( r_i \) crush each component of \( \tilde{M}_i \setminus \tilde{P}_i \) to \( \partial \tilde{P}_i \) then by construction

\[
\left\langle g^*_i \tilde{w}_i, \partial \tilde{\sigma}_i - \left( \tilde{\partial} \sigma \right)_i \right\rangle = 0
\]

(\( ** \))

\begin{flushright}
\textsc{Annales de l’Institut Fourier}
\end{flushright}
On the other hand by the definition
\[ \langle \delta \Omega_{P_i}, \sigma \rangle = \left\langle g_i^* \bar{\omega}_i, \left( \bar{\partial} \sigma \right)_i \right\rangle \]
Thus using (*) and (**) we get, since \( g_i^* \bar{\omega}_i \) is a cocycle by construction,
\[ \langle \delta \Omega_{P_i}, \sigma \rangle = \left\langle g_i^* \bar{\omega}_i, \partial \sigma_i \right\rangle = 0 \]
On the other hand it is easily checked from the construction that \( k^*_i \Omega_{P_i} \) is a relative cocycle of \((P_i, \partial P_i)\) and \( k^*_i(\Omega_{P_i}) = 0 \) for any \( i \neq j \).
We check point (ii). First note that \( \alpha_M(F_i, f_i) = [(k_i)_z((f_i)_z(\mu_i) + u)] \) where \( \mu_i \) is a relative 2-cycle representing the fundamental class of \( F_i \) and \( u \) is a \( l_1 \)-chain in \( \partial P_i \) such that \( \partial u = -\partial f_i(\mu_i) \). Thus the construction yields
\[ \langle [\Omega_{P_i}], \alpha_M(F_i, f_i) \rangle = \left\langle g_i^* \bar{\omega}_i, (\tilde{k}_i)_z((f_i)_z(\mu_i)) \right\rangle = \left\langle q_i^* \bar{\omega}_i, (\xi_i \circ f_i)_z(\mu_i) \right\rangle \]
But since \( \xi_i \circ f_i \) is a finite covering, with positive degree denoted by \( d_i \) then \( (\xi_i \circ f_i)_z([F_i]) = d_i[f_i] \) and thus we get (see Step 2)
\[ \langle [\Omega_{P_i}], \alpha_M(F_i, f_i) \rangle = d_i \left\langle q_i^* \bar{\omega}_i, z_i \right\rangle = d_i \text{Area}(F_i) = \text{Area}(F_i) \]
To complete the proof of the lemma it remains to compute the norm of the classes defined by \( \Omega_{P_i} \). Denote by \( \Omega \) the sum \( \sum_i \Omega_i \). We first check that \( \|\sum_i \Omega_{P_i}\|_{\infty} \leq \pi \). Let \( \sigma: \Delta^2 \to M \) be a singular 2-simplex. If there exists an edge \( e \) of \( \Delta^2 \) such that \( \rho \sigma(e) \subset \mathcal{T}_M \) then \( \langle \sum_{i \in I} \Omega_{P_i}, \sigma \rangle = 0 \).
If for any edge \( e \) of \( \Delta^2 \) we have \( \rho \sigma(e) \notin \mathcal{T}_M \) then there exists a unique component \( \text{Core}(\sigma) \) of \( (\rho \sigma)^{-1}(M \setminus \mathcal{T}_M) \) which meets the three edges of \( \Delta^2 \).
Denote by \( P_\nu \) the component of \( M \setminus \mathcal{T}_M \) such that \( \rho \sigma(\text{Core}(\sigma)) \subset \text{int}(P_\nu) \).
If \( \nu \in I \) then we have \( \left\| \sum_{i \in I} \Omega_{P_i}, \sigma \right\| = \|\Omega_{P_\nu}, \sigma \| \leq \pi \) and if \( \nu \notin I \) then \( \left\| \sum_{i \in I} \Omega_{P_i}, \sigma \right\| = 0 \). This proves that \( \|\sum_{i \in I} \Omega_{P_i}\|_{\infty} \leq \pi \). Using lemma 2.3 and points (i) and (ii) of the Lemma, we get the following equalities
\[ \left\| [\Omega], \alpha_M(F_i, f_i) \right\| = \text{Area}(F_i) \leq \|\Omega\|_{\infty} \|\alpha_M(F_i, f_i)\|_1 \leq \|\Omega\|_{\infty} \|F_i\|_1 \]
this completes the proof of Lemma 2.5 since \( \text{Area}(F_i) = \pi \|F_i\|_1 \). \( \square \)

**Lemma 2.8.** — Let \( M \) be a closed aspherical orientable* 3-manifold and let \( p: \tilde{M} \to M \) denote a finite regular covering whose each Seifert piece is a circle bundle with \( \mathbb{H}^2 \times \mathbb{R} \)-geometry. Assume that we are given orientation preserving proper embeddings \( f_i: (\tilde{F}_i, \partial \tilde{F}_i) \to (\tilde{P}_i, \partial \tilde{P}_i) \) where \( \{\tilde{P}_i\}_{i \in I} \) is a collection of Seifert pieces of \( \tilde{M} \). Then we have the equality
\[ \left\| p^*_i \left( \sum_i \alpha_M \left( x_i \bar{F}_i, f_i \right) \right) \right\|_1 = \left\| \sum_i \alpha_M \left( x_i \bar{F}_i, f_i \right) \right\|_1 \]
where the \( x_i \) are positive real numbers.
Proof. — Denote by $\Gamma$ the automorphism group of $p: \tilde{M} \to M$. Let $\tilde{\alpha}$ be the element $\sum \alpha_{\tilde{M}} (x_i, \tilde{F}_i, f_i)$ and denote by $\text{Av}(\tilde{\alpha})$ the class obtained by averaging $\tilde{\alpha}$ defined by $\text{Av}(\tilde{\alpha}) = \sum_{g \in \Gamma} g \tilde{\alpha}$. For a Seifert piece $\tilde{P}$ of $\tilde{M}$ denote by $\Omega_{\tilde{P}}$ the bounded 2-cocycle of $\tilde{M}$ constructed in Lemma 2.5 and denote by $\Omega$ the sum $\sum_{\tilde{P}} \Omega_{\tilde{P}}$. Notice that each $g \in \Gamma$ acts on $\tilde{M}$ as an orientation preserving homeomorphism which preserves the JSJ-splitting. In particular for each Seifert piece $\tilde{P}$ of $\tilde{M}$ then there exists a unique Seifert piece $\tilde{P}'$ such that $g(\tilde{P}) = \tilde{P}'$ and $g|\tilde{P}: (\tilde{P}, \partial \tilde{P}) \to (\tilde{P}', \partial \tilde{P}')$ is a homeomorphism. Moreover since each Seifert piece of $M$ has an orientable basis then $g|\tilde{P}: (\tilde{P}, \partial \tilde{P}) \to (\tilde{P}', \partial \tilde{P}')$ induces an orientation preserving homeomorphism between the bases of $\tilde{P}$ and $\tilde{P}'$. Then we get

$$
\langle [\Omega], \text{Av}(\tilde{\alpha}) \rangle = \text{Card}(\Gamma) \sum_{i \in I} x_i \text{Area}(\tilde{F}_i) \leq \pi \|\text{Av}(\tilde{\alpha})\|_1
$$

which proves that

$$
\|\text{Av}(\tilde{\alpha})\|_1 \geq \text{Card}(\Gamma) \sum_{i \in I} x_i \|\tilde{F}_i\|
$$

Since $\text{Av}(\tilde{\alpha})$ is $\Gamma$-invariant then by Lemma 2.1 $\|p_2(\text{Av}(\tilde{\alpha}))\|_1 = \|\text{Av}(\tilde{\alpha})\|_1$. Moreover using the definitions and Lemma 2.3

$$
\|p_2(\text{Av}(\tilde{\alpha}))\|_1 \leq \sum_{g \in \Gamma} \|p_2 g \tilde{\alpha}\|_1 \leq \sum_{g \in \Gamma} \|g \tilde{\alpha}\|_1 \leq \text{Card}(\Gamma) \sum_{i \in I} x_i \|\tilde{F}_i\|
$$

We deduce that $\sum_{g \in \Gamma} \|p_2 g \tilde{\alpha}\|_1 = \sum_{g \in \Gamma} \|g \tilde{\alpha}\|_1$. On the other hand, since we know that $\|p_2 g \tilde{\alpha}\|_1 \leq \|g \tilde{\alpha}\|_1$ for any $g \in \Gamma$ then we get in particular $\|p_2(\tilde{\alpha})\|_1 = \|\tilde{\alpha}\|_1$. □

Proof of Proposition 2.4. — To complete the proof of Proposition 2.4 it remains to check the following points

(i) $\|\alpha_M(F_i, f_i)\|_1 \geq \|F_i\|$ for $i = 1, \ldots, l$,
(ii) $\|\sum \alpha_M (x_i, F_i, f_i)\| \geq \sum \|\alpha_M (x_i, F_i, f_i)\|$, and
(iii) the covering property.

We first check points (i) and (ii). To this purpose we consider two cases.

Case 1. Assume that each $P_i$, $i = 1, \ldots, l$ is homeomorphic to a circle bundle. By Lemma 2.5 we know that there exists a bounded 2-cocycle $\Omega_{P_i}$ such that $\|\Omega_{P_i}\|_\infty = \pi$ and $|[\Omega_{P_i}], \alpha_M (F_i, f_i)| = \text{Area}(F_i)$. Then point (i) follows from Hölder inequality.

We check point (ii). Again, applying Lemma 2.5 we know that for each $i \in \{1, \ldots, l\}$ there exists a bounded 2-cocycle $\Omega_i$ in $M$ such that

$$
\langle [\Omega_i], \alpha_M (x_j, F_j, f_j) \rangle = \delta_{ij} x_j \text{Area}(F_j)
$$
for any \(i, j\) in \(\{1, \ldots, l\}\). Thus we get

\[
\left\langle \sum_i [\Omega_i], \sum_j \alpha_M(x_j, f_j) \right\rangle = \sum_i x_i \text{Area}(F_i) \leq \pi \left\| \sum_j \alpha_M(x_j, f_j) \right\|_1
\]

Hence

\[
\left\| \sum_i \alpha_M(x_i, F_i, f_i) \right\|_1 \geq \sum_i x_i \|F_i\| \geq \sum_i \|\alpha_M(x_i, F_i, f_i)\|_1
\]

This proves point (ii) in Case 1.

Case 2. We consider now the general case. Let \(\widetilde{p}: \widetilde{M} \to M\) be a finite regular covering of \(M\) whose each Seifert piece (in particular each component \(\widetilde{P}_i\) over \(P_i\) for \(i = 1, \ldots, l\)) is a circle bundle (such a covering exists by [15, Proposition 4.4]). For each \(i = 1, \ldots, l\) consider a covering \(\tilde{f}_i: (\widetilde{F}_i, \partial \widetilde{F}_i) \to (P_i, \partial P_i)\) (obtained by considering the group \((\tilde{f}_i)^{-1}(p_*(\pi_1 P_i))\)). By construction \(\tilde{f}_i\) is an orientation preserving embedding. Denote by \(d_i > 0\) the degree of the covering \(p_i: \widetilde{F}_i \to F_i\). By Case 1 we know that \(\|\alpha_{\widetilde{M}}(\widetilde{F}_i, \tilde{f}_i)\|_1 = \|\tilde{F}_i\|\) for \(i = 1, \ldots, l\). On the other hand by Lemma 2.8 we know that \(\|p_*(\alpha_{\widetilde{M}}(\widetilde{F}_i, \tilde{f}_i))\|_1 = \|\alpha_{\widetilde{M}}(\widetilde{F}_i, \tilde{f}_i)\|_1\). Since any continuous map induces a chain map then

\[
p_*(\alpha_{\widetilde{M}}(\widetilde{F}_i, \tilde{f}_i)) = d_i \alpha_M(F_i, f_i)
\]

which implies that

\[
\|d_i \alpha_M(F_i, f_i)\|_1 = \|\tilde{F}_i\|
\]

and thus \(\|\alpha_M(F_i, f_i)\|_1 = \|F_i\|\) for \(i = 1, \ldots, l\). To check point (ii) we know from Case 1 that

\[
\left\| \sum M (x_i, F_i, f_i) \right\| = \sum \left\| \alpha_{\widetilde{M}} \left( \frac{x_i}{d_i} \tilde{F}_i, \tilde{f}_i \right) \right\|
\]

Then using Lemma 2.8 in the right and left hand side, we get

\[
\left\| \sum \alpha_M (x_i, F_i, f_i) \right\| = \sum \|\alpha_M (x_i, F_i, f_i)\|.
\]

We check point (iii). Let \(f: M \to N\) denote a finite covering map and let \(\alpha = \sum \alpha_M (x_i, F_i, f_i)\). Using the construction of Case 2 with the same notations then \(\alpha = p_*(\tilde{\alpha})\) where \(\tilde{\alpha} = \sum \alpha_{\widetilde{M}} \left( \frac{x_i}{d_i} \tilde{F}_i, \tilde{f}_i \right)\). Possibly passing to some finite covering there are no loss of generality assuming \(f \circ p\) is regular. Hence we get using Lemma 2.8

\[
\|f_2(\alpha)\| = \|f_2 p_2(\tilde{\alpha})\| = \|\tilde{\alpha}\| \geq \|\alpha\|
\]

This completes the proof of Proposition 2.4. \(\square\)
3. Characterizations of covering maps

Given a closed irreducible orientable 3-manifold $M$ we denote by $\mathcal{H}(M)$ (resp. $\mathcal{S}(M)$) the disjoint union of the hyperbolic (resp. Seifert) components of $M \setminus \mathcal{T}_M$ (see [11], [12] and [21]). In order to prove Theorem 1.2 we first check the following technical result.

**Proposition 3.1.** — Let $M$ be a closed aspherical orientable 3-manifold. Any $\pi_1$-surjective nonzero degree map $f: M \to N$ into a closed irreducible orientable 3-manifold satisfying the following conditions

(i) Each Seifert component of $M \setminus \mathcal{T}_M$, resp. of $N \setminus \mathcal{T}_N$, is homeomorphic to a product, resp. to a $S^1$-bundle over an orientable surface, each Seifert component of $M \setminus \mathcal{T}_M$ has at least two boundary components (if $\mathcal{T}_M \neq \emptyset$) and each component of $\mathcal{T}_M$ is shared by two distinct components of $M \setminus \mathcal{T}_M$,
(ii) $\|f_*[M]\|_1 = \|[M]\|_1$, where $[M] \in H_3(M; \mathbb{R})$ is the fundamental class
(iii) $\|f_*\alpha_\mathcal{M}(\mathcal{F},g)\|_1 = \|\alpha_\mathcal{M}(\mathcal{F},g)\|_1$ for each orientation preserving proper embedding $g: \mathcal{F} \to \mathcal{P}$ when $\mathcal{P}$ runs over the Seifert pieces of $M$

is homotopic to a homeomorphism.

**3.1. Proof of Proposition 3.1**

Throughout this section we always assume that the map $f: M \to N$ and the manifolds $M, N$ satisfy the hypothesis of Proposition 3.1. Notice that we may assume in addition that $M$ is not a virtual torus bundle by [23]. Thus since each Seifert piece $P$ of $M$ is homeomorphic to a product we may assume that $P$ is a $H^2 \times \mathbb{R}$-manifold. Hence this implies, using hypothesis (ii) and (iii), that either $\|N\| \neq 0$ or $H^2_1(N; \mathbb{R})/\ker ||\|_1 \neq \{0\}$. Hence either $N$ is non-geometric or admits a geometry $H^3, H^2 \times \mathbb{R}$ or $\widetilde{SL}(2, \mathbb{R})$. The proof of Proposition 3.1 will come from the following sequence of claims.

**Claim 3.2.** — The map $f|T: T \to N$ is $\pi_1$-injective for any characteristic torus $T$ in $M$. Moreover, $f_*(\pi_1 P)$ is a non-abelian group for each Seifert piece $P$ of $M$.

**Proof.** — Let $T$ be a characteristic torus of $M$. From the Rigidity Theorem of Soma [20] and from hypothesis (ii) it is sufficient to consider the case when $T$ is shared by two distinct Seifert components $P$ and $P'$ of $M$. Denote by $h$ and $h'$ the $S^1$-fiber of $P$ and of $P'$ respectively. If $f|T: T \to N$ is not $\pi_1$-injective then we may assume, since $h$ and $h'$ generate a rank 2
subgroup of $\pi_1 T$ (by minimality of the JSJ-decomposition), that $P$ (for example) contains a simple closed curve $c$ distinct from the fiber $h$ such that $[c] \in \ker(f|T)_*$. Moreover since $\partial P$ is not connected then there exists an orientation preserving proper embedding $j : (F, \partial F) \to (P, \partial P)$ where $F$ is a connected surface such that $c$ is a boundary component of $j(F)$.

Indeed, denote by $T_1 = T$ the component of $\partial P$ which contains $c$ and by $T_2, \ldots, T_r$ the other components of $\partial P$ with $r \geq 2$. For each $i = 1, \ldots, r$ fix a basis $(s_i, h)$, where $s_i$ is a section of $T_i$ with respect to the $S^1$-fibration of $P$ such that $s_1 + \ldots + s_r$ is null-homologous in $P$ and where $h$ denotes the fiber of $P$. Denote by $(\alpha, \beta)$ the coprime integers with $\alpha \neq 0$ such that $c = \alpha[s_1] + \beta[h]$. Then

$$[c] + \alpha[s_2] + \ldots + \alpha[s_r] - \beta[h] = 0 \text{ in } H_1(P; \mathbb{Z})$$

Thus there exists a nontrivial class $\eta$ in $H_2(P, \partial P; \mathbb{Z})$ such that

$$\partial \eta = ((\alpha, \beta), (\alpha, 0), \ldots, (\alpha, 0), (\alpha, -\beta))$$

in $H_1(\partial P) = H_1(T_1) \oplus H_1(T_2) \oplus \ldots \oplus H_1(T_{r-1}) \oplus H_1(T_r)$. Since $P$ is aspherical, it follows from [22] that each class in $H_2(P, \partial P; \mathbb{Z})$ can be represented by a properly embedded incompressible surface. This can be argued as follows. By the Poincaré Duality, $H_2(P, \partial P; \mathbb{Z}) \simeq H^1(P; \mathbb{Z})$, there exists a homomorphism $\rho : \pi_1 P \to \mathbb{Z} = \pi_1 S^1$ corresponding to $\eta$. Since the spaces are aspherical the homomorphism is induced by a map $f_\eta : P \to S^1$. Taking the pre-image of a regular value $\theta \in S^1$ and using the construction given in [10, Chapter 6] we may arrange $f_\eta$ by a homotopy so that the components of $f_\eta^{-1}(\theta)$ are properly embedded incompressible surfaces. Denote by $F$ such a surface. Then $F$ is a horizontal surface and $c$ is parallel to some components of $\partial F$.

Denote by $T \times [-1, 1]$ a regular neighborhood of $T$ such that $T = T \times \{0\}$ and parametrize $T = S^1 \times S^1$ such that $c = S^1 \times \{\ast\}$. As in [19], consider the relation $\sim$ on $M$ defined by $z \sim z'$ iff $z = z'$ or $z = (x, y, t) \in T \times I$, $z' = (x', y', t') \in T \times I$ and $y = y'$, $t = t'$. Denote by $X = M/ \sim$ the quotient space and by $\pi : M \to X$ the quotient map. Then the map $f$ factors through $X$. Denote by $g : X \to N$ the map such that $f = g \circ \pi$. Denote by $\widehat{P}$ the image of $P$ under $\pi$. Topologically $\widehat{P}$ is obtained from $P$ after Dehn filling along $T$, identifying the meridian of a solid torus $V$ to $c$ so that the Seifert fibration of $P$ extends to a Seifert fibration of $\widehat{P}$ and the image $\widehat{F}$ of $F$ is a surface in $\widehat{P}$ obtained from $F$ after gluing a 2-disk along each component of $\partial F$ parallel to $c$. Consider the following commutative
where $\hat{j}$ is induced by $j$ and where $k: P \to M$ is the inclusion and $\hat{k}: \hat{P} \to X$ denotes the induced inclusion. Note that it follows from our construction, using standard homological arguments, that

$$\pi_\sharp(\alpha_M(F,j)) = \alpha_X(\hat{F},\hat{j}) \in H^1_{\mathbb{Z}}(X; \mathbb{R}) \quad (*)$$

where $\alpha_X(\hat{F},\hat{j})$ is defined by $\hat{k}_*\alpha_{\hat{P}}(\hat{F},\hat{j})$. We deduce, using hypothesis (iii) of Proposition 3.1, the following equalities:

$$\|\alpha_M(F,j)\|_1 \geq \|\pi_*\alpha_M(F,j)\|_1 \geq \|f_*\alpha_M(F,j)\|_1 = \|\alpha_M(F,j)\|_1$$

Thus using Lemma 2.3, equality (*) and Proposition 2.4(i) we get:

$$\|\hat{F}\| \geq \|\alpha_X(\hat{F},\hat{j})\|_1 = \|F\|$$

A contradiction. This proves the $\pi_1$-injectivity of the map $f|T$. It remains to check that $f_*(\pi_1 P)$ is a non-abelian group for each Seifert piece $P$. Assume that $f_*(\pi_1 P)$ is an abelian subgroup of $\pi_1 N$. Then the map $f|P: P \to N$ factors through a space $X$ with abelian fundamental group. Since $H^1_{\mathbb{Z}}(X)$ is trivial then we get a contradiction with hypothesis (iii) of Proposition 3.1 using point (i) of Proposition 2.4.

□

Claim 3.3. — There is a map $g$ homotopic to $f$ such that for each Seifert piece $\Sigma$ of $N$ then each component of $g^{-1}(\Sigma)$ is a Seifert piece of $M$.

Proof. — By hypothesis (ii) one can apply [20, Rigidity Theorem]. Thus one may assume that $f$ induces a deg($f$)-covering map from $\mathcal{H}(M)$ to $\mathcal{H}(N)$. Next, by Claim 3.2 one can apply [11, Mapping Theorem] which implies that one can arrange $f$ by a homotopy so that for each canonical torus $U$ of $N$ then $f^{-1}(U)$ is a disjoint union of canonical tori of $M$. Hence for each Seifert piece $\Sigma$ of $N$ the space $f^{-1}(\Sigma)$ is a canonical graph submanifold of $M$ (i.e. a submanifold which is the union of some Seifert pieces of $M$). If a component $G$ of $f^{-1}(\Sigma)$ is not a Seifert manifold then there exists a canonical torus $T$ in the interior of $G$ which is shared by two distinct Seifert pieces $\Sigma_1$ and $\Sigma_2$ of $G$. Since by Claim 3.2 the group $f_*(\pi_1 \Sigma_i)$ is non-abelian, for $i = 1, 2$, then using [11, Addendum to Theorem VI.I.6] we know that $f|\Sigma_i: \Sigma_i \to \Sigma$ is homotopic to a fiber preserving map.
Since $f|T$ is $\pi_1$-injective we get a contradiction by the minimality of the JSJ-decomposition. This proves the claim.

Since $f$ is $\pi_1$-surjective then to complete the proof of Proposition 3.1 it remains to check the following

**Claim 3.4.** — There is a map $g$ homotopic to $f$, rel. to $\mathcal{H}(M)$, such that for each Seifert piece $\Sigma$ of $N$ and for each component $G$ of $g^{-1}(\Sigma)$ then $g|G: G \to \Sigma$ is a covering map.

**Proof.** — First of all we know that for each component $G$ of $f^{-1}(\Sigma)$ then $f|G: G \to \Sigma$ is fiber preserving and non-degenerate in the sense of [11]. On the other hand, notice that $\Sigma$ is necessarily homeomorphic to a product. Indeed if $\partial \Sigma \neq \emptyset$ this is obvious and if $\partial \Sigma = \emptyset$ this comes from the following argument: first note that in this case $\Sigma = N$ and $G = M$, thus if $\Sigma$ is not homeomorphic to a product then the bundle has a non-zero Euler number and using the Seifert volume in [5, Theorem 3 and Lemma 3] and in [4, Theorem 4] we get a contradiction (since $G$ has a zero Euler number and $\deg(f) \neq 0$). Thus after choosing appropriate sections we identify $G$ with $K \times S^1$, resp. $\Sigma$ with $F \times S^1$, where $K$, resp. $F$, is a hyperbolic surface.

Let $\mathcal{F}$ denote a component of $(f|G)^{-1}(F)$. Arrange $f$ so that $\mathcal{F}$ is incompressible in $G$. Since $f$ is non-degenerate and fiber preserving then the inclusion $i: \mathcal{F} \to G$ is necessarily an orientation preserving proper embedding and $f|\mathcal{F}: \mathcal{F} \to F$ descends to a map $\pi: K \to F$. Therefore we get

$$f_\sharp(\alpha_M(\mathcal{F},i)) = \deg(f|\mathcal{F}: \mathcal{F} \to F)\alpha_N(F,j)$$

where $j: F \to \Sigma$ is the inclusion. This implies that

$$\|\mathcal{F}\| = |\deg(f|\mathcal{F}: \mathcal{F} \to F)| \times \|F\|$$

Thus we get the equality

$$\|K\| = \deg(\pi) \times \|F\|$$

Hence $\pi$ is homotopic to a covering map which implies that $f|G$ is also homotopic to a covering map. This proves the claim and completes the proof of Proposition 3.1. \hfill $\square$

### 3.2. Proof of Theorem 1.2

We first check that the condition is necessary.

When $\|\alpha_M\|_1 = 0$ there is nothing to prove. So let’s assume $\|\alpha_M\|_1 > 0$
Then either $M$ is a $\widetilde{SL}_2(\mathbb{R})$-manifold and Lemma 2.2 applies or $M$ is not a $\widetilde{SL}_2(\mathbb{R})$-manifold and Proposition 2.4 applies.

We verify now that the condition is sufficient. First of all note that according to [23] we may assume that $M$ is not a virtual torus bundle. In the sequel it will be convenient to consider the following commutative diagram

\[
\begin{array}{c}
M_2 \\ f_2 \downarrow \quad q \downarrow \\
N_2 \\
M_1 \\ f_1 \downarrow \\
N_1 \\
M \downarrow \\
f \downarrow \\
N 
\end{array}
\]

obtained as follows. The map $s: M_1 \to M$ is a finite covering such that each Seifert piece of $M_1$ is a circle bundle over an orientable surface with at least two boundary components if $T_{M_1} \neq \emptyset$, and each canonical torus of $M_1$ is shared by two distinct components of $M_1 \setminus T_{M_1}$ (for the existence of such a covering see [7, Lemmas 3.2 and 3.5]), the map $r: N_1 \to N$ is a finite covering corresponding to the subgroup $f_*s_*(\pi_1 M_1)$ in $\pi_1 N$, which is of finite index since $\deg(f) \neq 0$, the map $f_1: M_1 \to N_1$ is a lifting of $f \circ s$ which exists by our construction, the map $p: N_2 \to N_1$ is a finite covering such that each Seifert piece of $N_2$ is a $S^1$-bundle over an orientable surface and $f_2: M_2 \to N_2$ is the finite covering of $f_1$ corresponding to $p$, and $q: M_2 \to M_1$ is the covering corresponding to the subgroup $(f_1)_*^{-1}(p_* \pi_1 N_2)$. Notice that it follows from the construction that $f_1$ and $f_2$ are $\pi_1$-surjective.

**Claim 3.5.** — *The map $f_2$ is homotopic to a homeomorphism.*

**Proof.** — Assume that $M$ is a $\widetilde{SL}(2, \mathbb{R})$-manifold. Since $f$ has nonzero degree then $f$ is homotopic to a non degenerate fiber preserving map and $N$ is also a $\widetilde{SL}(2, \mathbb{R})$-manifold. Thus $f_2$ is a $\pi_1$-surjective nonzero degree map between circle bundle with nonzero Euler numbers. Denote by $F_2$, resp. $G_2$, the base of $M_2$, resp. $N_2$. It follows from the hypothesis of the theorem combined with Lemma 2.2 that $\|(f_2)_*(\alpha_{M_2})\|_1 = \|\alpha_{M_2}\|_1$. Thus $f_2$ induces a map $g: F_2 \to G_2$ such that $\xi \circ f_2 = g \circ \pi$ where $\pi: M_2 \to F_2$ and $\xi: N_2 \to G_2$ denote the bundle projections. Since by definition $\alpha_{M_2} = \pi_*^{-1}([F_2]_{t_1})$ then condition $\|(f_2)_*(\alpha_{M_2})\|_1 = \|\alpha_{M_2}\|_1$ implies

$$\|[F_2]_{t_1}\|_1 = \|g_*([F_2]_{t_1})\|_1 = \deg(g)\|[G_2]_{t_1}\|_1$$

and thus $\|F_2\| = \deg(g)\|[G_2]\|$. This proves that $g$ and hence $f_2$ is homotopic to a homeomorphism (recall that $f_2$ is $\pi_1$-surjective).
Assume now that $M$ is not a $\tilde{S}\text{L}(2,\mathbb{R})$-manifold. Then using point (ii) of Proposition 2.4 (additivity property) and the isometry hypothesis we have

$$\|f_2^*\alpha_M(\mathcal{F}_i, f_i)\|_1 = \|\alpha_M(\mathcal{F}_i, f_i)\|_1$$

for any $i = 1, \ldots, l$.

Indeed, by hypothesis we know that $\|f_2^*\alpha_M\|_1 = \|\alpha_M\|_1$ then by point (ii) of Proposition 2.4 (additivity property) and using the definition of $\alpha_M$ we have

$$\|f_2^*\alpha_M\|_1 = \|\alpha_M\|_1 = \sum_i \|\alpha_M\left(\frac{1}{k_i} \mathcal{F}_i, f_i\right)\|_1$$

Since, by paragraph 2, any continuous map induces a contraction with respect to the $l_1$-norm we get

$$\|f_2^*\alpha_M\|_1 = \|\alpha_M\|_1 \geq \sum_i \|f_2^*\alpha_M\left(\frac{1}{k_i} \mathcal{F}_i, f_i\right)\|_1 \geq \left\|\sum_i f_2^*\alpha_M\left(\frac{1}{k_i} \mathcal{F}_i, f_i\right)\right\|_1 = \|f_2^*\alpha_M\|_1$$

Hence we get

$$\sum_i \left(\|\alpha_M\left(\frac{1}{k_i} \mathcal{F}_i, f_i\right)\|_1 - \|f_2^*\alpha_M\left(\frac{1}{k_i} \mathcal{F}_i, f_i\right)\|_1\right) = 0$$

Again, since $f_2^*$ is a contraction, then each term of the sum is non-negative and thus $\|f_2^*\alpha_M(\mathcal{F}_i, f_i)\|_1 = \|\alpha_M(\mathcal{F}_i, f_i)\|_1$ for any $i = 1, \ldots, l$.

Note that if $g_i : \mathcal{G}_i \to P_i$ is any orientation preserving proper map of a surface $\mathcal{G}_i$ then

$$\|f_2^*\alpha_M(\mathcal{G}_i, g_i)\|_1 = \|\alpha_M(\mathcal{G}_i, g_i)\|_1$$

This comes from the following observation: by [25, Lemma 6] there are rational numbers $r_i, s_i$ and a vertical surface $W_i$ in $P_i$ (i.e. an incompressible properly embedded surface in $P_i$ which is fibered by the $S^1$-fibers of $P_i$) such that

$$(g_i)_2[\mathcal{G}_i] = r_i(f_i)_2[\mathcal{F}_i] + s_i[W_i] \in H_2(P_i, \partial P_i)$$

and since $W_i$ has zero simplicial volume the equality follows. In order to apply Proposition 3.1 to the map $f_2$ it remains to check hypothesis (iii).

Let $g : F_2 \to P_2$ be an orientation preserving embedding of a surface into a Seifert piece $P_2$ of $M_2$. Denote by $P$ the Seifert piece of $M$ such that $P_2$ is over $P$. Then by the above equality, applied to $s \circ q \circ g : F_2 \to P$, we have

$$\|f_2^*\alpha_M(F_2, s \circ q \circ g)\|_1 = \|\alpha_M(F_2, s \circ q \circ g)\|_1$$

On the other hand, using point (iii) of Proposition 2.4 we know that

$$\|\alpha_M(F_2, s \circ q \circ g)\|_1 = \|\alpha_{M_2}(F_2, f_2)\|_1$$
By the commutativity of the diagram we have
\[ f_2(\alpha_M(F_2, s \circ q \circ g)) = r_2 p_2^*(f_2)(\alpha_{M_2}(F_2, f_2)) \]
Therefore, this yields
\[ \|\alpha_{M_2}(F_2, f_2)\|_1 = \|r_2 p_2^*(f_2)(\alpha_{M_2}(F_2, f_2))\|_1 \leq \|(f_2)_2^*(\alpha_{M_2}(F_2, f_2))\|_1 \]
Accordingly we deduce that \( f_2 \) satisfies hypothesis of Proposition 3.1 which implies that \( f_2 \) is homotopic to a homeomorphism. □

Since \( M \) is an aspherical 3-manifolds then it has a torsion free fundamental group ([10]). Since \( p, q, r, s \) are finite covering maps then they induce injective homomorphisms at the \( \pi_1 \)-level and since \( f_2 \) induces an isomorphism \( f_\ast \) (\( \pi_1 M \)). Then \( f \) lifts to a map \( \tilde{f} : M \to \tilde{N} \) inducing an isomorphism at the \( \pi_1 \)-level. We deduce from this point using [13, Theorem 0.7] that \( \tilde{f} \) is a homeomorphism. This implies that \( f \) is a covering map and completes the proof of Theorem 1.2.

### 3.3. Proof of Theorem 1.5

By the Mapping Theorem of [9] the map \( f \) induces an isometry \( f_2 : H_3^1(M) \to H_3^1(N) \). On the other hand, using the same construction as in the proof of Lemma 2.5 in dimension three (instead of dimension 2) one deduces that the natural map \( H_3(M) \to H_3^1(M) \) is an isometry. Indeed, if \( \|M\| = 0 \) there is nothing to prove and if \( \|M\| > 0 \) this means that \( M \) contains some hyperbolic pieces \( H_1, \ldots, H_l \) in its geometric decomposition. Thus by the straightening technique used in the proof of Lemma 2.5 one can in the same way construct an element \( \Omega \in H_2^3(M) \) such that \( \langle \Omega, [M] \rangle = \text{vol}(H_1) + \ldots + \text{vol}(H_l) \) with \( \|\Omega\|_\infty \leq V_3 \), where \( V_3 \) denotes the supremum of the volume of geodesic 3-simplices in the hyperbolic 3-space. Hence the \( l_1 \)-norm of \( [M] \) is \( \|M\| \) in \( H_3^1(M) \), proving that \( H_3(M) \to H_3^1(M) \) is an isometry. This implies that \( f_2 : H_3(M; \mathbb{R}) \to H_3(N; \mathbb{R}) \) is an isometry.

Using the same covering argument as above one can assume, without loss of generality, that \( f \) is \( \pi_1 \)-surjective. If \( M \) is orientable* then Corollary 1.5 follows from Theorem 1.2 by the Mapping Theorem of [9]. If \( M \) is not orientable* then there exists a 2-fold finite covering \( p : M_2 \to M \) such that \( M_2 \) is orientable*. Note that the composition \( g = f \circ p_2 \) is not \( \pi_1 \)-surjective. Indeed if \( g \) is \( \pi_1 \)-surjective then \( f \circ p_2 \) is homotopic to a homeomorphism because since \( f_\ast \) has an amenable kernel then so is \( \ker(g_\ast) \) and thus \( g \)
induces an isometric isomorphism $g_\# : H^1_2(M_2) \to H^1_2(N)$. Since moreover $\|M_2\| = 2\|N\|$ one can apply Theorem 1.2. A contradiction. Since $g$ is not $\pi_1$-surjective then there exists a 2-fold covering $f' : M_2 \to N_2$ of the map $f$. Again, since $f_*$ has an amenable kernel then so is $\ker(f'_*)$. Moreover $f'$ is $\pi_1$-surjective and thus it induces an isometric isomorphism $f'_\# : H^1_2(M_2) \to H^1_2(N_2)$ and $\|M_2\| = \deg(f')\|N_2\|$. Hence by Theorem 1.2 the $f'$ is homotopic to a homeomorphism. Hence $f$ is homotopic to a homeomorphism. This completes the proof of the corollary.

**BIBLIOGRAPHY**


Manuscrit reçu le 31 mars 2010, accepté le 8 février 2011.

Pierre DERBEZ
LATP, UMR 6632,
Centre de Mathématiques et d’Informatique,
Technopole de Chateau-Gombert,
39, rue Frédéric Joliot-Curie -
13453 Marseille Cedex 13
derbez@cmi.univ-mrs.fr