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Ikuo SATAKE & Atsushi TAKAHASHI

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GROMOV–WITTEN INVARIANTS FOR MIRROR ORBIFOLDS OF SIMPLE ELLIPTIC SINGULARITIES

by Ikuo SATAKE & Atsushi TAKAHASHI

Abstract. — We consider a mirror symmetry of simple elliptic singularities. In particular, we construct isomorphisms of Frobenius manifolds among the one from the Gromov–Witten theory of a weighted projective line, the one from the theory of primitive forms for a universal unfolding of a simple elliptic singularity and the one from the invariant theory for an elliptic Weyl group. As a consequence, we give a geometric interpretation of the Fourier coefficients of an eta product considered by K. Saito.

Résumé. — Nous considérons une symétrie miroir des singularités elliptiques simples. En particulier, nous construisons des isomorphismes de variétés de Frobenius entre celui de la théorie de Gromov–Witten d'une droite projective à poids, celui de la théorie des formes primitives pour un déploiement universel d'une singularité elliptique simple et celui de la théorie des invariants pour un groupe de Weyl elliptique. Comme conséquence, nous donnons une interprétation géométrique des coefficients de Fourier d'un produit eta considéré par K. Saito.

Introduction

Mirror symmetry can be understood as a duality between algebraic geometry and symplectic geometry. It is an interesting problem to understand based on the philosophy of mirror symmetry some mysterious correspondences among isolated singularities, root systems and discrete groups such as Schwartz's triangle groups.

Let $f(x,y,z)$ be a holomorphic function which has an isolated singularity only at the origin $0 \in \mathbb{C}^3$. A distinguished basis of vanishing cycles in the Milnor fiber of f can be categorified to an A_∞-category $\mathrm{Fuk}^{\to}(f)$ called

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the directed Fukaya category whose derived category $D^b\mathrm{Fuk}^{\to}(f)$ is, as a triangulated category, an invariant of the holomorphic function f.

If $f(x,y,z)$ is a weighted homogeneous polynomial then one can consider another interesting triangulated category, the category of a maximally-graded singularity $D^{L_f}_{Sg}(R_f)$:

$$D^{L_f}_{Sg}(R_f) := D^b(\mathrm{gr}^{L_f}\text{-}R_f)/D^b(\mathrm{proj}^{L_f}\text{-}R_f), \tag{0.1}$$

where $R_f := \mathbb{C}[x,y,z]/(f)$ and L_f is the maximal grading (see section one of [6] for the definition) of f. This category $D^{L_f}_{Sg}(R_f)$ is considered as an analogue of the bounded derived category of coherent sheaves on a smooth proper algebraic variety.

In this setting, homological mirror symmetry conjectures can be stated as follows:

Conjecture ([6][23]). —

(i) *Let $f(x,y,z)$ be an invertible polynomial (see section one of [6] for the definition). There should exist a quiver with relations (Q,I) and triangulated equivalences*

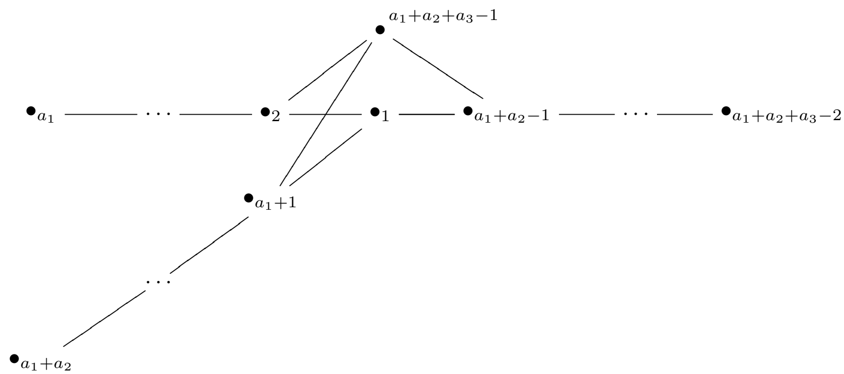
$$D^{L_f}_{Sg}(R_f) \simeq D^b(\mathrm{mod}\text{-}\mathbb{C}Q/I) \simeq D^b\mathrm{Fuk}^{\to}(f^t), \tag{0.2}$$

where f^t denotes the Berglund–Hübsch transpose of f.

(ii) *There should exist triangulated equivalences*

$$D^b\mathrm{coh}(\mathbb{P}^1_{a_1,a_2,a_3}) \simeq D^b(\mathrm{mod}\text{-}\mathbb{C}Q_{a_1,a_2,a_3}/I') \simeq D^b\mathrm{Fuk}^{\to}(T_{a_1,a_2,a_3}), \tag{0.3}$$

where $\mathbb{P}^1_{a_1,a_2,a_3}$ is the orbifold $\mathbb{P}^1$ with 3 isotropic points of orders a_1, a_2, a_3, Q_{a_1,a_2,a_3} is a quiver given by the following graph



with the orientation from vertices with smaller indices to those with larger indices and I' is the ideal generated by two generic paths from the 1-st vertex to the $a_1+a_2+a_3-1$-th vertex, and $T_{a_1,a_2,a_3} := x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - cx_1x_2x_3$, $c \in \mathbb{C}^*$.

It is natural to expect the following from (ii) of the above homological mirror symmetry conjectures since their "complexified Kähler moduli spaces" should be isomorphic and there should exist Frobenius structures (K. Saito's flat structures) on them:

Conjecture. — *There should exist isomorphisms of Frobenius manifolds* (*see for example* [3, 19] *for the definition*) *among*

(i) $M_{\mathbb{P}^1_{a_1,a_2,a_3}}$, *the one constructed from the Gromov–Witten theory of* $\mathbb{P}^1_{a_1,a_2,a_3}$,

(ii) $M_{(Q_{a_1,a_2,a_3},I')}$, *the one constructed from the invariant theory of the reflection group associated to the quiver with relations* (Q_{a_1,a_2,a_3},I'),

(iii) $M_{T_{a_1,a_2,a_3},\infty}$, *the one constructed from the universal unfolding of* T_{a_1,a_2,a_3} *by the choice of primitive form "at* $c=\infty$*".*

Remark 0.1. — It is also a part of conjecture that there exist Frobenius manifolds $M_{(Q_{a_1,a_2,a_3},I')}$ for $1/a_1+1/a_2+1/a_3<1$.

Rossi shows in [14] that Conjecture holds under the condition $1/a_1 + 1/a_2+1/a_3>1$. The next case to consider is when the triple (a_1,a_2,a_3) satisfies the condition $1/a_1+1/a_2+1/a_3=1$, in other words, the case when the polynomial f defines a simple elliptic singularity (see [6] for this relation between (a_1,a_2,a_3) and f). In particular, in this paper we shall give a proof of the above Conjecture for $(a_1,a_2,a_3)=(3,3,3)$ with the explicit presentation of the potential which gives us interesting quasi-modular forms based on the uniqueness of the solution of the WDVV equation. The following is our main result in this paper:

Theorem. — *We have isomorphisms of Frobenius manifolds*

$$M_{\mathbb{P}^1_{3,3,3}} \simeq M_{E_6^{(1,1)}} \simeq M_{T_{3,3,3},\infty},$$

where $M_{E_6^{(1,1)}}$ *denotes the Frobenius manifold constructed from the invariant theory of the elliptic Weyl group of type* $E_6^{(1,1)}$.

Moreover, the genus zero Gromov–Witten potential $F_0^{\mathbb{P}^1_{3,3,3}}$ *and the genus one Gromov–Witten potential* $F_1^{\mathbb{P}^1_{3,3,3}}$, *which is also considered as the G-function* (*see* [4] *for the definition*) *on* $M_{E_6^{(1,1)}}$ *and as the one on* $M_{T_{3,3,3},\infty}$, *are expressed by quasi-modular forms.*

An important consequence of this theorem is that we can give a geometric interpretation of the Fourier coefficients of an eta product considered by K. Saito [18]:

Theorem. — *Denote by $\eta(\tau)$ the Dedekind's eta function*

$$\eta(\tau) := e^{\frac{2\pi\sqrt{-1}\tau}{24}} \prod_{n\geqslant 1}\left(1-e^{2\pi\sqrt{-1}n\tau}\right), \quad \tau\in\mathbb{H} := \{\tau\in\mathbb{C} \mid \mathrm{Im}\tau > 0\}.$$

The eta product $\eta(3\tau)^3/\eta(\tau)$ is a generating function of Gromov–Witten invariants of $\mathbb{P}^1_{3,3,3}$. More precisely, the Fourier coefficient c_k defined by

$$\frac{\eta(3\tau)^3}{\eta(\tau)} = e^{\frac{2\pi\sqrt{-1}\tau}{3}} \sum_{k\geqslant 0} c_k e^{2\pi\sqrt{-1}k\tau} \tag{0.4}$$

is the Gromov–Witten invariant

$$\int_{[\overline{\mathcal{M}}_{0,0,3k[\mathbb{P}^1_{3,3,3}]}(\mathbb{P}^1_{3,3,3})]^{vir}} ev_1^*\gamma_1\wedge ev_2^*\gamma_2\wedge ev_3^*\gamma_3,$$

where γ_i is an element of $H^{2/3}_{orb}(\mathbb{P}^1_{3,3,3},\mathbb{Q})$ corresponding to the i-th isotropic point on $\mathbb{P}^1_{3,3,3}$.

We can also apply the same method to prove the Conjecture for the two other cases when $(a_1,a_2,a_3) = (2,4,4),(2,3,6)$. However, we omit them here since the number of monomials in those potentials are large (more than 50 for $(2,4,4)$ and more than 200 for $(2,3,6)$) we can not give the explicit presentation of the potential in this paper and we could understand not all but a few of interesting quasi-modular forms appearing in those potentials.

We can also consider a similar problem for which we do not have a hypersurface singularity:

Theorem. — *We have an isomorphism of Frobenius manifolds*

$$M_{\mathbb{P}^1_{2,2,2,2}} \simeq M_{D_4^{(1,1)}},$$

where $\mathbb{P}^1_{2,2,2,2}$ denotes an orbifold $\mathbb{P}^1$ with four isotropic points of orders 2 and $M_{D_4^{(1,1)}}$ denotes the Frobenius manifold constructed from the invariant theory of the elliptic Weyl group of type $D_4^{(1,1)}$.

Moreover, the genus zero Gromov–Witten potential $F_0^{\mathbb{P}^1_{2,2,2,2}}$ and the genus one Gromov–Witten potential $F_1^{\mathbb{P}^1_{2,2,2,2}}$, which is also considered as the G-function on $M_{D_4^{(1,1)}}$, are expressed by quasi-modular forms.

Note that in order to obtain the mirror isomorphism we have to develop the theory of primitive forms for a pair consisting in a singularity and its symmetry group. Once we have such a theory, we may apply it for the pair $(T_{2,4,4},\mathbb{Z}/2\mathbb{Z})$, for example.

If the triple (a_1, a_2, a_3) satisfies the condition $1/a_1 + 1/a_2 + 1/a_3 = 1$, then we have the triangulated equivalence $D_{Sg}^{L_f}(R_f) \simeq D^b\mathrm{coh}(\mathbb{P}^1_{a_1,a_2,a_3})$ of Buchweitz–Orlov type (see [24]). Also note that a mathematical formulation of the topological A-model for Landau–Ginzburg orbifold theory is considered in [7], which is called Fan–Jarvis–Ruan–Witten (FJRW) theory. Therefore, it is also natural to consider the following:

CONJECTURE. — *Let T_{a_1,a_2,a_3} be a polynomial which defines a simple elliptic singularity $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. There should exist an isomorphism of Frobenius manifolds between*

(i) $M_{(T_{a_1,a_2,a_3},\mathbb{Z}/d\mathbb{Z}),FJRW}$*, the one constructed from the FJRW theory for the pair $(T_{a_1,a_2,a_3}, \mathbb{Z}/d\mathbb{Z})$ where $d = 6, 7, 8$ for $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ respectively,*

(ii) $M_{T_{a_1,a_2,a_3},0}$*, the one constructed from the universal unfolding of T_{a_1,a_2,a_3} by the choice of primitive form "at $c = 0$".*

The authors are notified that Krawitz–Shen [8] gives a proof of this Conjecture based on the calculations of $M_{T_{a_1,a_2,a_3},0}$ by Noumi–Yamada [11] and Milanov–Ruan [9] prove a generalization of this, namely, the one for all genus potentials and their quasi-modularity.

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1. Gromov–Witten theory for orbifolds

Gromov–Witten theory is generalized for orbifolds (smooth proper Deligne–Mumford stacks). It is first studied by Chen–Ruan [2] in symplectic geometry and later by Abramovich-Graber–Vistoli [1] in algebraic geometry. In order to generalize Gromov–Witten theory for manifolds to the one for orbifolds, one also needs to count the number of "stable maps from orbifold curves". For this purpose, in [2] the notion of orbifold stable maps is introduced and in [1] the notion of twisted stable maps is introduced. These two constructions are quite different, however, as the usual Gromov–Witten theory for manifolds, they are expected to give the same Gromov–Witten invariants since they have common philosophy. In this paper, we will introduce Gromov–Witten invariants following [1] for simplicity.

Let $\mathcal{X}$ be an orbifold (or a smooth proper Deligne–Mumford stack over $\mathbb{C}$). Then, for $g \in \mathbb{Z}_{\geqslant 0}$, $n \in \mathbb{Z}_{\geqslant 0}$ and $\beta \in H_2(\mathcal{X}, \mathbb{Z})$, the moduli space (stack) $\overline{\mathcal{M}}_{g,n,\beta}(\mathcal{X})$ of orbifold (twisted) stable maps of genus g with n-marked points of degree β is defined. There exists a virtual fundamental class $[\overline{\mathcal{M}}_{g,n,\beta}(\mathcal{X})]^{vir}$ and Gromov–Witten invariants of genus g with n-marked points of degree β are defined as usual except for that we have to use the orbifold cohomology group $H^*_{orb}(\mathcal{X}, \mathbb{Q})$:

$$\langle \gamma_1, \ldots, \gamma_n \rangle^{\mathcal{X}}_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n,\beta}(\mathcal{X})]^{vir}} ev_1^*\gamma_1 \wedge \ldots ev_n^*\gamma_n, \qquad \gamma_1, \ldots, \gamma_n \in H^*_{orb}(\mathcal{X}, \mathbb{Q}),$$

where $ev_i^* : H^*_{orb}(\mathcal{X}, \mathbb{Q}) \longrightarrow H^*(\overline{\mathcal{M}}_{g,n,\beta}(\mathcal{X}), \mathbb{Q})$ denotes the induced homomorphism by the evaluation map. We also consider the generating function

$$F_g^{\mathcal{X}} := \sum_{n,\beta} \frac{1}{n!} \langle \mathbf{t}, \ldots, \mathbf{t} \rangle^{\mathcal{X}}_{g,n,\beta} q^{\beta}, \quad \mathbf{t} = \sum_i t_i \gamma_i$$

and call it the genus g potential where $\{\gamma_i\}$ denotes a $\mathbb{Q}$-basis of $H^*_{orb}(\mathcal{X}, \mathbb{Q})$. The main result in [1] and [2] tell us that we can treat the Gromov–Witten theory defined for orbifolds as if $\mathcal{X}$ were usual manifold. In particular, we have the point axiom, the divisor axiom for a class in $H^2(\mathcal{X}, \mathbb{Q})$ and the associativity of the quantum product, namely, the WDVV equation (see, for example, [2] for details of these axioms.), which gives a (formal) Frobenius manifold. These axioms enable us to calculate genus zero Gromov–Witten potential $F_0^{\mathcal{X}}$ easily.

In this paper, we shall only consider the case when $\mathcal{X}$ is $\mathbb{P}^1_{2,2,2,2}$ or $\mathbb{P}^1_{3,3,3}$, the orbifold $\mathbb{P}^1$ with 4 isotropic points of order 2 or the orbifold $\mathbb{P}^1$ with 3 isotropic points of order 3. Note that both are given by the global quotient of an elliptic curve $\mathbf{E}$, more precisely, we have $\mathbb{P}^1_{2,2,2,2} = [\mathbf{E}/(\mathbb{Z}/2\mathbb{Z})]$ and $\mathbb{P}^1_{3,3,3} = [\mathbf{E}/(\mathbb{Z}/3\mathbb{Z})]$. For these examples, by the uniqueness result on genus zero and one potentials, we shall see that the two definitions of Gromov–Witten invariants by [1] and [2] coincides.

2. Explicit calculations for $\mathbb{P}^1_{2,2,2,2}$

The orbifold cohomology group of $\mathbb{P}^1_{2,2,2,2}$ is, as a vector space, just the singular cohomology group of the inertia orbifold

$$\mathcal{I}\mathbb{P}^1_{2,2,2,2} = \mathbb{P}^1_{2,2,2,2} \bigsqcup B(\mathbb{Z}/2\mathbb{Z}) \bigsqcup B(\mathbb{Z}/2\mathbb{Z}) \bigsqcup B(\mathbb{Z}/2\mathbb{Z}) \bigsqcup B(\mathbb{Z}/2\mathbb{Z}),$$

and the orbifold Poincaré pairing is given by twisting the usual Poincaré pairing:

$$\int_{\mathbb{P}^1_{2,2,2,2}} \alpha \cup_{orb} \beta := \int_{\mathcal{I}\mathbb{P}^1_{2,2,2,2}} \alpha \cup I\beta,$$

where I is the involution defined in [1, 2]. Therefore, we can choose a basis $\gamma_0, \dots, \gamma_5$ of the orbifold cohomology group $H^*_{orb}(\mathbb{P}^1_{2,2,2,2}, \mathbb{Q})$ such that

$$H^0_{orb}(\mathbb{P}^1_{2,2,2,2}, \mathbb{Q}) \simeq \mathbb{Q}\gamma_0, \quad H^1_{orb}(\mathbb{P}^1_{2,2,2,2}, \mathbb{Q}) \simeq \bigoplus_{i=1}^{4} \mathbb{Q}\gamma_i,$$
$$H^2_{orb}(\mathbb{P}^1_{2,2,2,2}, \mathbb{Q}) \simeq \mathbb{Q}\gamma_5,$$

and

$$\int_{\mathbb{P}^1_{2,2,2,2}} \gamma_0 \cup \gamma_5 = 1, \quad \int_{\mathbb{P}^1_{2,2,2,2}} \gamma_i \cup \gamma_j = \frac{1}{2}\delta_{i,j}, \ i,j = 1, \dots 4.$$

Denote by $t_0, \dots, t_5$ the dual coordinates of the $\mathbb{Q}$-basis $\gamma_0, \dots, \gamma_5$. In discussion below, by applying the divisor axiom, we consider $\log q$ as a flat coordinate instead of t_5.

2.1. Genus zero potential

Theorem 2.1. — *The genus zero Gromov–Witten potential $F_0^{\mathbb{P}^1_{2,2,2,2}}$ of $\mathbb{P}^1_{2,2,2,2}$ is given as follows:*

$$\begin{aligned} F_0^{\mathbb{P}^1_{2,2,2,2}} =& \frac{1}{2} t_0^2 \log q + \frac{1}{4} t_0 (t_1^2 + t_2^2 + t_3^2 + t_4^2) \\ &+ (t_1 t_2 t_3 t_4) \cdot f_0(q) + \frac{1}{4}(t_1^4 + t_2^4 + t_3^4 + t_4^4) \cdot f_1(q) \\ &+ \frac{1}{6}(t_1^2 t_2^2 + t_1^2 t_3^2 + t_1^2 t_4^2 + t_2^2 t_3^2 + t_2^2 t_4^2 + t_3^2 t_4^2) \cdot f_2(q), \end{aligned}$$

where

$$f_0(q) := \frac{1}{2}\left(f(q) - f(-q)\right), \tag{2.1}$$

$$f_1(q) := f(q^4), \tag{2.2}$$

$$f_2(q) := f(q) - f_0(q) - f_1(q), \tag{2.3}$$

$$f(q) := -\frac{1}{24} + \sum_{n=1}^{\infty} n \frac{q^n}{1-q^n} = -q\frac{d}{dq}\log(\eta(q)), \tag{2.4}$$

$$\eta(q) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n). \tag{2.5}$$

Proof. — We can deduce Theorem 2.1 from the following uniqueness property of the potential:

LEMMA 2.2. — *There exists a unique 6-dimensional formal Frobenius structure with flat coordinates $t_0, t_1, t_2, t_3, t_4, t$ satisfying the following conditions:*

(i) *The Euler vector field E is given by $E = t_0 \frac{\partial}{\partial t_0} + \sum_{k=1}^4 \frac{1}{2} t_k \frac{\partial}{\partial t_k}$.*

(ii) *The Frobenius potential F_0 is given by*

$$\begin{aligned} F_0 =& \frac{1}{2} t_0^2 t + \frac{1}{4} t_0 (t_1^2 + t_2^2 + t_3^2 + t_4^2) \\ &+ (t_1 t_2 t_3 t_4) \cdot f_0(e^t) + \frac{1}{4}(t_1^4 + t_2^4 + t_3^4 + t_4^4) \cdot f_1(e^t) \\ &+ \frac{1}{6}(t_1^2 t_2^2 + t_1^2 t_3^2 + t_1^2 t_4^2 + t_2^2 t_3^2 + t_2^2 t_4^2 + t_3^2 t_4^2) \cdot f_2(e^t), \end{aligned}$$

where $f_0(q), f_1(q), f_2(q)$ have the following formal power series expansions:

$$f_0(q) = \sum_{n=1}^{\infty} a_n q^n \text{ with } a_1 = 1, \quad f_1(q) = \sum_{n=0}^{\infty} b_n q^n, \quad f_2(q) = \sum_{n=0}^{\infty} c_n q^n.$$

Proof. — We can show that the WDVV equation is equivalent to the following differential equations:

$$q \frac{d}{dq} f_0(q) = \frac{8}{3} f_0(q) f_2(q) - 24 f_0(q) f_1(q), \tag{2.6}$$

$$q \frac{d}{dq} f_1(q) = -\frac{2}{3} f_0(q)^2 - \frac{16}{3} f_1(q) f_2(q) + \frac{8}{9} f_2(q)^2, \tag{2.7}$$

$$q \frac{d}{dq} f_2(q) = 6 f_0(q)^2 - \frac{8}{3} f_2(q)^2. \tag{2.8}$$

Hence, we have the following recursion relations for a_n, b_n, c_n:

$$n a_n = \frac{8}{3} \sum_{k=1}^{n} a_k c_{n-k} - 24 \sum_{k=1}^{n} a_k b_{n-k}, \tag{2.9}$$

$$n b_n = -\frac{2}{3} \sum_{k=1}^{n-1} a_k a_{n-k} - \frac{16}{3} \sum_{k=0}^{n} b_k c_{n-k} + \frac{8}{9} \sum_{k=0}^{n} c_k c_{n-k}, \tag{2.10}$$

$$n c_n = 6 \sum_{k=1}^{n-1} a_k a_{n-k} - \frac{8}{3} \sum_{k=0}^{n} c_k c_{n-k}. \tag{2.11}$$

In particular, by setting $n = 0, 1$, we get $c_0 = 0$ and $b_0 = -1/24$. Therefore, the above recursion relations have the unique solution. □

Next, we construct the analytic solution to the WDVV equation as follows.

Lemma 2.3. — *Put*

$$f_0(q) := \frac{1}{2}\left(f(q) - f(-q)\right), \tag{2.12}$$

$$f_1(q) := f(q^4), \tag{2.13}$$

$$f_2(q) := f(q) - f_0(q) - f_1(q), \tag{2.14}$$

$$f(q) := -\frac{1}{24} + \sum_{n=1}^{\infty} n\frac{q^n}{1-q^n} = -q\frac{d}{dq}\log(\eta(q)). \tag{2.15}$$

Then the functions $f_0(q)$, $f_1(q)$, $f_2(q)$ *satisfies the following differential equations*:

$$q\frac{d}{dq}f_0(q) = \frac{8}{3}f_0(q)f_2(q) - 24f_0(q)f_1(q), \tag{2.16}$$

$$q\frac{d}{dq}f_1(q) = -\frac{2}{3}f_0(q)^2 - \frac{16}{3}f_1(q)f_2(q) + \frac{8}{9}f_2(q)^2, \tag{2.17}$$

$$q\frac{d}{dq}f_2(q) = 6f_0(q)^2 - \frac{8}{3}f_2(q)^2. \tag{2.18}$$

Proof. — Put

$$\vartheta_2(q) := \sum_{m\in\mathbb{Z}} q^{(m+\frac{1}{2})^2},$$

$$\vartheta_3(q) := \sum_{m\in\mathbb{Z}} q^{m^2},$$

$$\vartheta_4(q) := \sum_{m\in\mathbb{Z}} (-1)^m q^{m^2},$$

$$X_i(q) := q\frac{d}{dq}\log\vartheta_i \ (i=2,3,4).$$

Then the following differential relations

$$\frac{1}{2}q\frac{d}{dq}(X_2(q)+X_3(q)) = 2X_2(q)X_3(q), \tag{2.19}$$

$$\frac{1}{2}q\frac{d}{dq}(X_3(q)+X_4(q)) = 2X_3(q)X_4(q), \tag{2.20}$$

$$\frac{1}{2}q\frac{d}{dq}(X_4(q)+X_2(q)) = 2X_4(q)X_2(q) \tag{2.21}$$

are classically known as Halphen's equations (see [12]). For the proof of Lemma 2.3, we should only prove that

$$X_2(q) = -6f_1(q) + \frac{2}{3}f_2(q), \tag{2.22}$$

$$X_3(q) = 2f_0(q) - \frac{4}{3}f_2(q), \tag{2.23}$$

$$X_4(q) = -2f_0(q) - \frac{4}{3}f_2(q). \tag{2.24}$$

We have

$$X_2(q) = q\frac{d}{dq}\log[2\eta(q^2)^{-1}\eta(q^4)^2], \tag{2.25}$$

$$X_3(q) = q\frac{d}{dq}\log[\eta(q)^{-2}\eta(q^2)^5\eta(q^4)^{-2}], \tag{2.26}$$

$$X_4(q) = q\frac{d}{dq}\log[\eta(q)^2\eta(q^2)^{-1}] \tag{2.27}$$

by Jacobi's triple product formula (see [10]).

For $f_0(q)$, $f_1(q)$, $f_2(q)$, we prepare the following Sub-Lemma.

Sub-Lemma 2.4. — *For $f(q)$, we have*

$$\frac{1}{2}(f(q) + f(-q)) = 3f(q^2) - 2f(q^4). \tag{2.28}$$

Proof. — We define $\sigma(n)\,(n \geqslant 1)$ by

$$f(q) = -\frac{1}{24} + \sum_{n=1}^{\infty}\sigma(n)q^n.$$

If $n = 2^k m$ with m odd, then $\sigma(n) = (1+2+\cdots+2^k)\sigma(m) = (2^{k+1}-1)\sigma(m)$. Thus we have $\sigma(2n) = 3\sigma(n)$ if n is odd. Also if n is general, we have $\sigma(4n) = 3\sigma(2n) - 2\sigma(n)$. Then we have (2.28). □

By (2.28), we have

$$f_0(q) = -q\frac{d}{dq}\log[\eta(q)\eta(q^2)^{-\frac{3}{2}}\eta(q^4)^{\frac{1}{2}}], \tag{2.29}$$

$$f_1(q) = -q\frac{d}{dq}\log[\eta(q^4)^{\frac{1}{4}}], \tag{2.30}$$

$$f_2(q) = -q\frac{d}{dq}\log[\eta(q^2)^{\frac{3}{2}}\eta(q^4)^{-\frac{3}{4}}]. \tag{2.31}$$

From (2.25)–(2.27) and (2.29)–(2.31), we have (2.22)–(2.24). □

It is easy to show that, by our choice of basis $\gamma_0, \ldots, \gamma_5$ of $H^*_{orb}(\mathbb{P}^1_{2,2,2,2}, \mathbb{C})$ and their dual coordinates $t_0, \ldots, t_5 = \log q$ in the beginning of this section, the Gromov–Witten potential is of the form in Lemma 2.2 except for the condition $a_1 = 1$. The condition $a_1 = 1$ follows from the fact that the Gromov–Witten invariant a_1 counts the number of morphisms from $\mathbb{P}^1_{2,2,2,2}$ to $\mathbb{P}^1_{2,2,2,2}$ of degree one, which is exactly the identity map. Hence, we have $a_1 = 1$. Now, the statement in Theorem 2.1 follows from the uniqueness of the potential. □

By Theorem 2.1, the Gromov–Witten potential $F_0^{\mathbb{P}^1_{2,2,2,2}}$ converges on the domain $|q| < 1$. Thus it gives a Frobenius manifold $M_{\mathbb{P}^1_{2,2,2,2}} \simeq \{z \in \mathbb{C} \,|\, \mathrm{Re} z < 0\} \times \mathbb{C}^5$ with flat coordinates $(\log q, t_0, t_1, t_2, t_3, t_4)$.

For the elliptic root system of type $D_4^{(1,1)}$ ([17]), the domain $\mathbb{E}_{D_4^{(1,1)}}$ and the elliptic Weyl group $W_{D_4^{(1,1)}}$ are defined and the quotient space $M_{D_4^{(1,1)}} := \mathbb{E}_{D_4^{(1,1)}} // W_{D_4^{(1,1)}} \simeq \{z \in \mathbb{C} \,|\, \mathrm{Re} z < 0\} \times \mathbb{C}^5$ has a structure of the Frobenius manifold ([17], [21]). Its potential is explicitly calculated in [20] as follows:

Lemma 2.5. — ([20]) *By choosing the flat coordinates $t, e_0, e_1, e_3, e_4, e_2$ of $M_{D_4^{(1,1)}}$, the potential $F_0^{D_4^{(1,1)}}$ is expressed as*

$$
\begin{aligned}
F_0^{D_4^{(1,1)}} =& \frac{1}{2} t(e_2)^2 \\
&+ \frac{1}{4} e_2[e_0^2 + e_1^2 + e_3^2 + e_4^2] \\
&+ (e_0 e_1 e_3 e_4) \cdot h_0(t) \\
&+ \frac{1}{4}(e_0^4 + e_1^4 + e_3^4 + e_4^4) \cdot h_1(t) \\
&+ \frac{1}{6}(e_0^2 e_1^2 + e_0^2 e_3^2 + e_0^2 e_4^2 + e_1^2 e_3^2 + e_1^2 e_4^2 + e_3^2 e_4^2) \cdot h_2(t),
\end{aligned}
$$

where

$$
\begin{aligned}
h_0(t) &= \frac{1}{8} \Theta_{\omega_1,1}(e^t), \\
h_1(t) &= -\frac{1}{2}\left[\frac{1}{2}\frac{\frac{d}{dt}[\eta(e^{2t})]}{\eta(e^{2t})} + \frac{1}{24}\Theta_{0,1}(e^t)\right], \\
h_2(t) &= -\frac{3}{2}\left[\frac{1}{2}\frac{\frac{d}{dt}[\eta(e^{2t})]}{\eta(e^{2t})} - \frac{1}{24}\Theta_{0,1}(e^t)\right], \\
\Theta_{0,1}(q) &= \sum_{\gamma \in M} q^{(\gamma,\gamma)} = 1 + \cdots, \\
\Theta_{\omega_1,1}(q) &= \sum_{\gamma \in M + \omega_1} q^{(\gamma,\gamma)} = 8q + \cdots,
\end{aligned}
$$

where M is the coroot lattice of D_4 and ω_1 is the first fundamental weight in the notation of Bourbaki.

Remark 2.6. — We remark that the correspondence of the above coordinates with the ones in [20] is

$$t=\pi\sqrt{-1}\tau, e_0=c_0, e_1=c_1, e_3=c_3, e_4=c_4, e_2=\frac{-1}{2(2\pi\sqrt{-1})^2}c_2$$

and we take the intersection form of the Frobenius manifold as $\frac{-1}{(2\pi\sqrt{-1})^2}I^*$ instead of I^*.

Since the potential $F_0^{D_4^{(1,1)}}$ satisfies the assumptions of the Lemma 2.2, we have

Theorem 2.7. — *The Frobenius manifold $M_{\mathbb{P}^1_{2,2,2,2}}$ and the Frobenius manifold $M_{D_4^{(1,1)}}$ are isomorphic as Frobenius manifolds.*

2.2. Genus one potential

We shall also give the genus one Gromov–Witten potential.

Theorem 2.8. — *The genus one Gromov–Witten potential $F_1^{\mathbb{P}^1_{2,2,2,2}}$ of $\mathbb{P}^1_{2,2,2,2}$ is given as*

$$F_1^{\mathbb{P}^1_{2,2,2,2}}=-\frac{1}{2}\log(\eta(q^2)). \tag{2.32}$$

Proof. — The first derivative of the genus one Gromov–Witten potential $q\frac{d}{dq}F_1^{\mathbb{P}^1_{2,2,2,2}}$ is an element of $\mathbb{Q}[[q]]$ since the Euler vector field is given by $E=t_0\frac{\partial}{\partial t_0}+\sum_{k=1}^4\frac{1}{2}t_k\frac{\partial}{\partial t_k}$, $E\left(q\frac{d}{dq}F_1^{\mathbb{P}^1_{2,2,2,2}}\right)=0$ and we have the divisor axiom. Therefore, we only have to consider the (orbifold) stable maps with one marked point from smooth elliptic curves to $\mathbb{P}^1_{2,2,2,2}=[\mathbf{E}/(\mathbb{Z}/2\mathbb{Z})]$, which factor through the elliptic curve $\mathbf{E}$ by definition. In particular, the number of coverings of degree n from an elliptic curve to $\mathbf{E}$ is given by $\sigma(n):=\sum_{k|n}k$. Hence, we have

$$q\frac{d}{dq}F_1^{\mathbb{P}^1_{2,2,2,2}}=f(q^2)=-\frac{1}{24}+\sum_{n=1}^{\infty}\sigma(n)q^{2n}.$$

One may also obtain the statement by Dubrovin–Zhang's Virasoro constraint [5]. Indeed, Proposition 4 in [5] gives us the equation

$$q\frac{d}{dq}F_1^{\mathbb{P}^1_{2,2,2,2}} = f_1(q) + \frac{1}{3}f_2(q).$$

By Sub-Lemma 2.4, we have $f_1(q) + \frac{1}{3}f_2(q) = f(q^2)$. $\square$

The proof of Theorem 2.8 also shows that the genus one potential is uniquely reconstructed from the genus zero potential. In particular, this implies the G-function of $M_{D_4^{(1,1)}}$ coincides with $F_1^{\mathbb{P}^1_{2,2,2,2}}$.

3. Explicit calculations for $\mathbb{P}^1_{3,3,3}$

The orbifold cohomology group of $\mathbb{P}^1_{3,3,3}$ is, as a vector space, just the singular cohomology group of the inertia orbifold

$$\mathcal{I}\mathbb{P}^1_{3,3,3} = \mathbb{P}^1_{3,3,3} \bigsqcup B(\mathbb{Z}/3\mathbb{Z}) \bigsqcup B(\mathbb{Z}/3\mathbb{Z}) \bigsqcup B(\mathbb{Z}/3\mathbb{Z}),$$

and the orbifold Poincaré pairing is given by twisting the usual Poincaré pairing:

$$\int_{\mathbb{P}^1_{3,3,3}} \alpha \cup_{orb} \beta := \int_{\mathcal{I}\mathbb{P}^1_{3,3,3}} \alpha \cup I\beta,$$

where I is the involution defined in [1, 2]. Therefore, we can choose a $\mathbb{Q}$-basis $\gamma_0, \dots, \gamma_7$ of the orbifold cohomology group $H^*_{orb}(\mathbb{P}^1_{3,3,3}, \mathbb{Q})$ such that

$$H^0_{orb}(\mathbb{P}^1_{3,3,3}, \mathbb{Q}) \simeq \mathbb{Q}\gamma_0, \quad H^{\frac{2}{3}}_{orb}(\mathbb{P}^1_{3,3,3}, \mathbb{Q}) \simeq \bigoplus_{i=1}^{3} \mathbb{Q}\gamma_i,$$

$$H^{\frac{4}{3}}_{orb}(\mathbb{P}^1_{3,3,3}, \mathbb{Q}) \simeq \bigoplus_{i=4}^{6} \mathbb{Q}\gamma_i, \quad H^2_{orb}(\mathbb{P}^1_{3,3,3}, \mathbb{Q}) \simeq \mathbb{Q}\gamma_7,$$

and

$$\int_{\mathbb{P}^1_{3,3,3}} \gamma_0 \cup \gamma_7 = 1, \quad \int_{\mathbb{P}^1_{3,3,3}} \gamma_i \cup \gamma_j = \frac{1}{3}\delta_{i+j-7,0}, \ i,j = 1,\dots,6.$$

Denote by $t_0, \dots, t_7$ the dual coordinates of the $\mathbb{Q}$-basis $\gamma_0, \dots, \gamma_7$. In the discussion below, by applying the divisor axiom, we consider $\log q$ as a flat coordinate instead of t_7.

3.1. Genus zero potential

Theorem 3.1. — *The genus zero Gromov–Witten potential* $F_0^{\mathbb{P}^1_{3,3,3}}$ *of* $\mathbb{P}^1_{3,3,3}$ *is given as follows:*

$$\begin{aligned}
F_0^{\mathbb{P}^1_{3,3,3}} =& \frac{1}{2}t_0^2\log q + \frac{1}{3}t_0(t_1t_6+t_2t_5+t_3t_4) + (t_1t_2t_3)\cdot f_0(q) \\
&+ \frac{1}{6}(t_1^3+t_2^3+t_3^3)\cdot f_1(q) + (t_1t_2t_5t_6+t_1t_3t_4t_6+t_2t_3t_4t_5)\cdot f_2(q) \\
&+ \frac{1}{2}(t_1^2t_4t_5+t_2^2t_4t_6+t_3^2t_5t_6)\cdot f_3(q) \\
&+ \frac{1}{2}(t_1t_2t_4^2+t_1t_3t_5^2+t_2t_3t_6^2)\cdot f_4(q) + \frac{1}{4}(t_1^2t_6^2+t_2^2t_5^2+t_3^2t_4^2)\cdot f_5(q) \\
&+ \frac{1}{6}\left[t_1t_6(t_4^3+t_5^3)+t_2t_5(t_4^3+t_6^3)+t_3t_4(t_5^3+t_6^3)\right]\cdot f_6(q) \\
&+ \frac{1}{2}(t_1t_4t_5t_6^2+t_2t_4t_5^2t_6+t_3t_4^2t_5t_6)\cdot f_7(q) \\
&+ \frac{1}{4}(t_1t_4^2t_5^2+t_2t_4^2t_6^2+t_3t_5^2t_6^2)\cdot f_8(q) + \frac{1}{24}(t_1t_6^4+t_2t_5^4+t_3t_4^4)\cdot f_9(q) \\
&+ \frac{1}{36}(t_4^3t_5^3+t_4^3t_6^3+t_5^3t_6^3)\cdot f_{10}(q) + \frac{1}{24}(t_4t_5t_6^4+t_4t_5^4t_6+t_4t_5t_6^4)\cdot f_{11}(q) \\
&+ \frac{1}{8}(t_4^2t_5^2t_6^2)\cdot f_{12}(q) + \frac{1}{720}(t_4^6+t_5^6+t_6^6)\cdot f_{13}(q),
\end{aligned}$$

where $f_i(q), i=0,\dots,13$ *are given by*

$$f_0(q) = \frac{1}{3}\left(\frac{q\frac{d}{dq}a(q)}{1-a(q)^3}\right)^{\frac{1}{2}} = \frac{\eta(q^9)^3}{\eta(q^3)}, f_1(q) = a(q)f_0(q),$$

$$f_2(q) = -\frac{1}{9}\frac{q\frac{d}{dq}f_0}{f_0} + a(q)^2f_0(q)^2,$$

$$f_3(q) = f_0(q)^2, f_4(q) = a(q)f_0(q)^2,$$

$$f_5(q) = -\frac{2}{9}\frac{q\frac{d}{dq}f_0}{f_0} + a(q)^2f_0(q)^2,$$

$$f_6(q) = f_0(q)^3, f_7(q) = a(q)f_0(q)^3, f_8(q) = a(q)^2f_0(q)^3,$$

$$f_9(q) = a(q)^3f_0(q)^3, f_{10}(q) = 3a(q)f_0(q)^4, f_{11}(q) = 3a(q)^2f_0(q)^4,$$

$$f_{12}(q) = (2+a(q)^3)f_0(q)^4, f_{13}(q) = 3a(q)(2-a(q)^3)f_0(q)^4,$$

and

$$a(q) = 1 + \frac{1}{3}\left(\frac{\eta(q)}{\eta(q^9)}\right)^3 = \frac{1}{3}q^{-1}(1 + 5q^3 - 7q^6 + 3q^9 + \dots).$$

Proof. — We can deduce the Theorem from the following uniqueness property of the potential:

LEMMA 3.2. — *Let $F_0(t_0, \cdots, t_6, t, f_0, \cdots, f_{13})$ be a polynomial defined by*

$$\begin{aligned}
&F_0(t_0, \cdots, t_6, t, f_0, \cdots, f_{13}) \\
:=&\frac{1}{2}t_0^2 t + \frac{1}{3}t_0(t_1t_6 + t_2t_5 + t_3t_4) + (t_1t_2t_3)\cdot f_0 \\
&+ \frac{1}{6}(t_1^3 + t_2^3 + t_3^3)\cdot f_1 + (t_1t_2t_5t_6 + t_1t_3t_4t_6 + t_2t_3t_4t_5)\cdot f_2 \\
&+ \frac{1}{2}(t_1^2t_4t_5 + t_2^2t_4t_6 + t_3^2t_5t_6)\cdot f_3 \\
&+ \frac{1}{2}(t_1t_2t_4^2 + t_1t_3t_5^2 + t_2t_3t_6^2)\cdot f_4 + \frac{1}{4}(t_1^2t_6^2 + t_2^2t_5^2 + t_3^2t_4^2)\cdot f_5 \\
&+ \frac{1}{6}\left[t_1t_6(t_4^3 + t_5^3) + t_2t_5(t_4^3 + t_6^3) + t_3t_4(t_5^3 + t_6^3)\right]\cdot f_6 \\
&+ \frac{1}{2}(t_1t_4t_5t_6^2 + t_2t_4t_5^2t_6 + t_3t_4^2t_5t_6)\cdot f_7 \\
&+ \frac{1}{4}(t_1t_4^2t_5^2 + t_2t_4^2t_6^2 + t_3t_5^2t_6^2)\cdot f_8 + \frac{1}{24}(t_1t_6^4 + t_2t_5^4 + t_3t_4^4)\cdot f_9 \\
&+ \frac{1}{36}(t_4^3t_5^3 + t_4^3t_6^3 + t_5^3t_6^3)\cdot f_{10} + \frac{1}{24}(t_4t_5t_6^4 + t_4t_5^4t_6 + t_4t_5t_6^4)\cdot f_{11} \\
&+ \frac{1}{8}(t_4^2t_5^2t_6^2)\cdot f_{12} + \frac{1}{720}(t_4^6 + t_5^6 + t_6^6)\cdot f_{13}.
\end{aligned}$$

(i) *For the holomorphic functions $f_0(t), \cdots, f_{13}(t)$, the holomorphic function $F_0(t_0, \cdots, t_6, t, f_0(t), \cdots, f_{13}(t))$ is a potential of an 8-dimensional Frobenius structure with flat coordinates $t_0, t_1, t_2, t_3, t_4, t_5, t_6, t$ such that the Euler vector field E is given by*

$$E = t_0\frac{\partial}{\partial t_0} + \sum_{k=1}^{3}\frac{2}{3}t_k\frac{\partial}{\partial t_k} + \sum_{k=4}^{6}\frac{1}{3}t_k\frac{\partial}{\partial t_k}$$

if and only if there exists $A \in \mathbb{C}^*$ such that

$$f_0(t) = A\left(\frac{a(t)'}{1-a(t)^3}\right)^{1/2}, \tag{3.1}$$

$$\begin{aligned}
&f_1(t) = a(t)f_0(t),\ \ f_2(t) = -\frac{1}{2\cdot 3^2}\left(\frac{a(t)''}{a(t)'}+\frac{a(t)^2a(t)'}{1-a(t)^3}\right),\\
&f_3(t) = \frac{1}{3^2}\frac{a(t)'}{1-a(t)^3},\ \ f_4(t) = \frac{1}{3^2}\frac{a(t)a(t)'}{1-a(t)^3},\\
&f_5(t) = -\frac{1}{3^2}\left(\frac{a(t)''}{a(t)'}+\frac{2a(t)^2a(t)'}{1-a(t)^3}\right),\\
&f_6(t) = \frac{1}{3^4}A^{-4}f_0(t)^3, f_7(t) = a(t)f_6(t), f_8(t) = a(t)^2f_6(t),\\
&f_9(t) = a(t)^3f_6(t), f_{10}(t) = \frac{1}{3^5}A^{-6}a(t)f_0(t)^4, f_{11}(t) = a(t)f_{10}(t),\\
&f_{12}(t) = \frac{1}{3^6}A^{-6}(2+a(t)^3)f_0(t)^4, f_{13}(t) = (2-a(t)^3)f_{10}(t),
\end{aligned} \tag{3.2}$$

and

$$\frac{a(t)'''}{a(t)'} - \frac{3}{2}\left(\frac{a(t)''}{a(t)'}\right)^2 = -\frac{1}{2}\frac{8+a(t)^3}{(1-a(t)^3)^2}a(t)\cdot(a(t)')^2, \tag{3.3}$$

where $a(t) = f_1(t)/f_0(t)$ and $' = \frac{d}{dt}$.

(ii) *There exist unique formal power series:*

$$\widetilde{f_0}(q) = \sum_{n=1}^{\infty} a_0(n)q^n, \quad \widetilde{f_i}(q) = \sum_{n=0}^{\infty} a_i(n)q^n, \quad i = 1,\dots,13, \tag{3.4}$$

with $a_0(1) = 1$ and $a_1(0) = \frac{1}{3}$ such that $F_0(t_0,\cdots,t_6,t,\widetilde{f_0}(e^t),\cdots,\widetilde{f_{13}}(e^t))$ is the potential of an 8-dimensional Frobenius structure with flat coordinates $t_0, t_1,\dots,t_6,t$, and Euler vector field $E = t_0\frac{\partial}{\partial t_0} + \sum_{k=1}^{3}\frac{2}{3}t_k\frac{\partial}{\partial t_k} + \sum_{k=4}^{6}\frac{1}{3}t_k\frac{\partial}{\partial t_k}$.

Proof. — The assertion (i) is a direct consequence of WDVV equations and discussed already in [25]. For the proof of (ii), we need the following Sub-Lemma.

Sub-Lemma 3.3. — *There exists a unique formal Laurent series*

$$f(q) = \sum_{n=-1}^{\infty} a_n q^n$$

satisfying the following conditions:

(i) *The first coefficient* $a_{-1} = \frac{1}{3}$.

(ii) $f(q)$ *satisfies the following differential equation:*

$$\frac{f(q)'''}{f(q)'} - \frac{3}{2}\left(\frac{f(q)''}{f(q)'}\right)^2 = -\frac{1}{2}\frac{8+f(q)^3}{(1-f(q)^3)^2} f(q)\cdot (f(q)')^2, \tag{3.5}$$

where $' = q\frac{d}{dq}$.

Proof. — Put

$$S(q) := (1-f(q)^3)^2[f(q)'\cdot f(q)''' - \frac{3}{2}(f(q)'')^2] + \frac{1}{2}(8+f(q)^3)\cdot f(q)\cdot (f(q)')^4.$$

Condition (ii) is equivalent to all the coefficients of the q-expansion of $S(q)$ being zero. For the cases of $n \leqslant 0$, the coefficients of q^{-8+n} of $S(q)$ equal to zero. For the cases of $n \geqslant 1$, the coefficients of q^{-8+n} of $S(q)$ are of the form

$$-n^3 a_{-1}^7 a_{n-1} + \text{ a polynomial in } a_{-1},\cdots,a_{n-2}.$$

Since we have $a_{-1} = 1/3$, the coefficients $a_0, a_1, \cdots$ are uniquely determined inductively. □

We first construct $\widetilde{f}_0(q),\cdots,\widetilde{f}_{13}(q)$. Take a formal Laurent series $\widetilde{f}(q)$ as the one which is constructed in Sub-Lemma 3.3. We take $A \in \mathbb{C}^*$ such that the formal power series: $A(\frac{q\frac{d}{dq}\widetilde{f}(q)}{1-\widetilde{f}(q)^3})^{1/2}$ has an expansion $q+\cdots$. Then A^2 must be $1/9$. We define the following formal power series:

$$\widetilde{f}_0(q) := A(\frac{q\frac{d}{dq}\widetilde{f}(q)}{1-\widetilde{f}(q)^3})^{1/2},\ \ \widetilde{f}_1(q) := \widetilde{f}(q)\widetilde{f}_0(q),$$
$$\widetilde{f}_2(q) := -\frac{1}{2\cdot 3^2}\left(\frac{(q\frac{d}{dq})^2\widetilde{f}(q)}{q\frac{d}{dq}\widetilde{f}(q)} + \frac{\widetilde{f}(q)^2 q\frac{d}{dq}\widetilde{f}(q)}{1-\widetilde{f}(q)^3}\right),\cdots$$

in a parallel manner as in (3.2). By (i) of this Lemma, we see that $\widetilde{f}_i(q) (i = 0,\cdots,13)$ satisfy the conditions of (ii).

We show the uniqueness of $\widetilde{f}_i(q)$ $(i=0,\cdots,13)$. We assume that $\widehat{f}_i(q)$ $(i = 0,\cdots,13)$ also satisfy the conditions of (ii). Put $\widehat{f}(q) := \widehat{f}_1(q)/\widehat{f}_0(q)$. By (i) of this Lemma, we see that

(i) $\widehat{f}(e^t)$ must satisfy the differential equation (3.3).

(ii) $\exists \widehat{A} \in \mathbb{C}^*$ such that

$$\widehat{f}_0(e^t) = \widehat{A}\left(\frac{\frac{d}{dt}\widehat{f}(e^t)}{1-\widehat{f}(e^t)^2}\right)^{1/2}.$$

From (i), $\widehat{f}(q)$ satisfies (3.5). Since $\widehat{f}(q)$ has the expansion $\frac{1}{3}q^{-1}+\cdots$, $\widehat{f}(q)$ must be $\widetilde{f}(q)$ by Sub-Lemma 3.3. From (ii) and a comparison of the leading term of q-expansions of $\widetilde{f}_0(q)$ and $\widehat{f}_0(q)$, we have $\widetilde{f}_0(q) = \widehat{f}_0(q)$ and $\widehat{A}^2 = A^2$. Since $\widehat{f}_i(e^t)$ $(i = 1,\cdots,13)$ must satisfy (3.2), we have $\widetilde{f}_i(q) = \widehat{f}_i(q)$ $(i = 1,\cdots,13)$. Thus we obtain Lemma 3.2. □

Next, we construct the analytic solution to the WDVV equation as follows.

Lemma 3.4. — *Put*

$$h(q) = 1 + \frac{1}{3}\left(\frac{\eta(q)}{\eta(q^9)}\right)^3 = \frac{1}{3}q^{-1}+\cdots. \tag{3.6}$$

Then $h(q)$ has the following properties:

(i) *$h(q)$ satisfies the following differential equation.*

$$\frac{h(q)'''}{h(q)'} - \frac{3}{2}\left(\frac{h(q)''}{h(q)'}\right)^2 = -\frac{1}{2}\frac{8+h(q)^3}{(1-h(q)^3)^2}h(q)\cdot(h(q)')^2,$$

where $' = q\frac{d}{dq}$.

(ii) *$h(q)$ satisfies the following equation*:

$$-\frac{1}{64}\frac{h(q)^3(8+h(q)^3)^3}{(1-h(q)^3)^3} = J(q) \tag{3.7}$$

where $J(q)$ is the Laurent series characterized by the conditions that

(a) $J(q) = \frac{1}{1728}(q^{-3}+744+\cdots)$,

(b) $J(\exp(\frac{2\pi\sqrt{-1}\tau}{3}))$ *is the elliptic modular function on the upper half plane* $\mathbb{H} = \{\tau\in\mathbb{C} \,|\, \mathrm{Im}\tau > 0\}$.

(iii) *$h(q)$ has the following expressions*:

$$h(q) = \omega + \frac{1}{3}\left(\frac{\eta(q\omega^{-2})}{\eta(q^9)}\right)^3\cdot\exp(\frac{2\pi\sqrt{-1}}{12}) = \omega^2 + \frac{1}{3}\left(\frac{\eta(q\omega^{-1})}{\eta(q^9)}\right)^3\cdot\exp(\frac{2\pi\sqrt{-1}}{24}), \tag{3.8}$$

where $\omega = \exp(\frac{2\pi\sqrt{-1}}{3})$.

Proof. — The uniformization of the Hesse pencil:

$$x_0^3 + x_1^3 + x_2^3 - 3ax_0x_1x_2 = 0$$

is classically studied and we refer to [13]. In [13], the parameter a is described as a holomorphic function $a(\tau)$ on the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} \,|\, \mathrm{Im}\tau > 0\}$ as

$$a(\tau) = 1 + 9\left(\frac{\eta(\exp(2\pi\sqrt{-1}3\tau))}{\eta(\exp(\frac{2\pi\sqrt{-1}\tau}{3}))}\right)^3.$$

By the modular property of $\eta(\exp(2\pi\sqrt{-1}\tau))$, we have

$$a(-\frac{1}{\tau}) = h(\exp(\frac{2\pi\sqrt{-1}\tau}{3})). \tag{3.9}$$

Then we can deduce Lemme 3.4 from the corresponding results for $a(\tau)$, which are classically known and written in [13]. $\square$

Finally, we give two important formulas for the function $h(q)$ in Lemma 3.4:

Lemma 3.5. — *We have the following equations*:

$$(1)\ \frac{1}{3^3}\frac{(q\frac{d}{dq}h(q))^6}{(h(q)^3-1)^3} = \eta(q^3)^{24}. \tag{3.10}$$

$$(2)\ \frac{q\frac{d}{dq}h(q)}{1-h(q)^3} = 3^2\left(\frac{\eta(q^9)^3}{\eta(q^3)}\right)^2. \tag{3.11}$$

Proof. — We have

$$\frac{1}{2^6\cdot 3^9}\frac{(q\frac{d}{dq}J(q))^6}{J(q)^4(J(q)-1)^3} = \eta(q^3)^{24}, \tag{3.12}$$

because the leading terms of the q-expansions coincide and if we put $q = \exp(\frac{2\pi\sqrt{-1}\tau}{3})$, then both sides are cusp forms of weight 12 with respect to the $SL(2,\mathbb{Z})$ action and therefore they are uniquely determined by the leading terms of the q-expansions.

By (3.7) and (3.12), we have (3.10).

We could easily check that

$$\exp(\frac{2\pi\sqrt{-1}}{24})\eta(q)\eta(q\omega^{-1})\eta(q\omega^{-2})\eta(q^9) = (\eta(q^3))^4. \tag{3.13}$$

By (3.6), (3.8), (3.13), we have

$$h(q)^3 - 1 = \frac{1}{3^3}\left(\frac{\eta(q^3)}{\eta(q^9)}\right)^{12}. \tag{3.14}$$

By (3.10), (3.14) and the comparison of the leading terms of q-expansions, we have (3.11). $\square$

It is easy to show that, by our choice of basis $\gamma_0, \dots, \gamma_7$ of the orbifold cohomology group $H^*_{orb}(\mathbb{P}^1_{3,3,3}, \mathbb{Q})$ and their dual coordinates $t_0, \dots, t_7$ in the beginning of this section, the Gromov–Witten potential is of the form in Lemma 3.2 except for the condition $a_0(1) = 1$. Indeed, we can choose elements $\gamma_1, \gamma_6 \in H^*_{orb}(\mathbb{P}^1_{3,3,3}, \mathbb{C})$ contained in the basis which will correspond to coordinates t_1, t_6 such that $\gamma_1 \circ \gamma_1 = \gamma_6$ and $\int_{\mathbb{P}^1_{3,3,3}} \gamma_1 \cup \gamma_6 = \frac{1}{3}$ where $\circ$ denotes the orbifold cohomology ring structure on $H^*_{orb}(\mathbb{P}^1_{3,3,3}, \mathbb{C})$. This gives us $a_1(0) = \frac{1}{3}$. The condition $a_0(1) = 1$ follows from the fact that the Gromov–Witten invariant $a_0(1)$ counts the number of morphisms from $\mathbb{P}^1_{3,3,3}$ to $\mathbb{P}^1_{3,3,3}$ of degree one, which is exactly the identity map. Hence, we have $a_0(1) = 1$. Now, the statement in Theorem 3.1 follows from the uniqueness of the potential. $\square$

Now, we consider the Frobenius structure on the base space of the universal unfolding of simple elliptic singularity of type $\widetilde{E}_6 : W_{\widetilde{E}_6}(x_1, x_2, x_3) := x_1^3 + x_2^3 + x_3^3 - 3ax_1x_2x_3$. It is easily obtained once we fix a primitive form (see [19] for example). It is proven by K. Saito in [15] that there exists a primitive form for $W_{\widetilde{E}_6}(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3 - 3ax_1x_2x_3$ and it is given by choosing a cycle in the corresponding elliptic curve $\{W_{\widetilde{E}_6}(x_1, x_2, x_3) = 0\} \subset \mathbb{P}^2$.

Denote by $M_{\widetilde{E}_6,\infty}$ the Frobenius manifold with the choice of the primitive form associated to the cycle in the elliptic curve which vanishes when the parameter a goes to infinity. In view of (i) of Lemma 3.2, we only have to calculate the holomorphic function $a(t)$ in order to describe the potential for $M_{\widetilde{E}_6,\infty}$. However, it is also easy to see from the result in [15] that we can choose the uniformization parameter $\tau/3$ as the flat coordinate t for our choice of primitive form and hence we have $a(\tau) = h(\exp(\frac{2\pi\sqrt{-1}\tau}{3}))$ as in the equation (3.9). By rescaling other flat coordinates suitably, it is possible to set $A = 1/3$ (in the notation of (i) of Lemma 3.2). Therefore, we can apply the uniqueness of the potential, (ii) of Lemma 3.2, and hence we obtain an isomorphism $M_{\mathbb{P}^1_{3,3,3}} \simeq M_{\widetilde{E}_6,\infty}$ as Frobenius manifolds.

On the other hand, for the elliptic root system of type $E_6^{(1,1)}$ ([17]), the domain $\mathbb{E}_{E_6^{(1,1)}}$ and the elliptic Weyl group $W_{E_6^{(1,1)}}$ are defined and the quotient space $M_{E_6^{(1,1)}} := \mathbb{E}_{E_6^{(1,1)}} // W_{E_6^{(1,1)}} \simeq \{z \in \mathbb{C} \,|\, \mathrm{Re} z < 0\} \times \mathbb{C}^7$ has a Frobenius manifold structure isomorphic to $M_{\widetilde{E}_6,\infty}$ ([16], [17], [21]). To summarize, we obtain the following

THEOREM 3.6. — *We have isomorphisms of Frobenius manifolds*

$$M_{\mathbb{P}^1_{3,3,3}} \simeq M_{\widetilde{E}_6,\infty} \simeq M_{E_6^{(1,1)}}.$$

3.2. Genus one potential

We shall also give the genus one Gromov–Witten potential.

THEOREM 3.7. — *The genus one Gromov–Witten potential* $F_1^{\mathbb{P}^1_{3,3,3}}$ *of* $\mathbb{P}^1_{3,3,3}$ *is given as*

$$F_1^{\mathbb{P}^1_{3,3,3}} = -\frac{1}{3}\log(\eta(q^3)). \tag{3.15}$$

Proof. — The proof is similar to the one for $F_1^{\mathbb{P}^1_{2,2,2,2}}$. It is easy to see that the genus one Gromov–Witten potential $F_1^{\mathbb{P}^1_{3,3,3}}$ is an element of $\mathbb{Q}[[q]]$ since the Euler vector field is given by $E = t_0\frac{\partial}{\partial t_0} + \sum_{k=1}^{3}\frac{2}{3}t_k\frac{\partial}{\partial t_k} + \sum_{k=4}^{6}\frac{1}{3}t_k\frac{\partial}{\partial t_k}$. Therefore, we only have to consider the (orbifold) stable maps with one marked point from smooth elliptic curves to $\mathbb{P}^1_{3,3,3} = [\mathbf{E}/(\mathbb{Z}/3\mathbb{Z})]$, which factor through $\mathbf{E}$ by definition. Hence, we have that

$$q\frac{d}{dq}F_1^{\mathbb{P}^1_{2,2,2,2}} = -\frac{1}{24} + \sum_{n=1}^{\infty}\sigma(n)q^{3n}.$$

One may also obtain the statement by Dubrovin–Zhang's Virasoro constraint [5]. Indeed, Proposition 4 in [5] gives us the equation

$$\begin{aligned} q\frac{d}{dq}F_1^{\mathbb{P}^1_{3,3,3}} &= \frac{3}{4}f_2(q) + \frac{3}{8}f_5(q) \\ &= -\frac{1}{12}q\frac{d}{dq}\log\left(q\frac{d}{dq}h(q)\right) - \frac{1}{8}\frac{q\frac{d}{dq}h(q)\cdot h(q)^2}{1-h(q)^3}. \end{aligned}$$

By the equation (3.10) in Lemma 3.5, we have

$$-\frac{1}{12}q\frac{d}{dq}\log(q\frac{d}{dq}h(q)) - \frac{1}{8}\frac{q\frac{d}{dq}h(q)\cdot h(q)^2}{1-h(q)^3} = -\frac{1}{3}q\frac{d}{dq}\log(\eta(q^3)). \qquad \square$$

Strachan [22] calculates the G-function for the Frobenius structure on the universal unfolding of simple elliptic singularities of type $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ with the choice of the primitive form "at $a = 0$". If we use the primitive form "at $a = \infty$" instead, then G-functions for $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$ can be obtained as $-\frac{1}{3}\log(\eta(q^3))$, $-\frac{1}{4}\log(\eta(q^4))$ and $-\frac{1}{4}\log(\eta(q^4))$ respectively. This is consistent with our calculation of Gromov–Witten invariants and mirror symmetry.

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Ikuo SATAKE
Faculty of Education,
Kagawa University,
1-1 Saiwai-cho Takamatsu Kagawa,
760-8522, Japan
satakeikuo@gmail.com

Atsushi TAKAHASHI
Department of Mathematics,
Graduate School of Science,
Osaka University,
Toyonaka Osaka, 560-0043, Japan
takahashi@math.sci.osaka-u.ac.jp