ANNALES

DE

L'INSTITUT FOURIER

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Tome 61, nº 7 (2011), p. 2745-2750.

<http://aif.cedram.org/item?id=AIF_2011__61_7_2745_0>

Ann. Inst. Fourier, Grenoble
61, 7 (2011) 2745-2750

A REMARK ON A CONJECTURE OF HAIN AND LOOIJENGA

by Carel FABER

Abstract. — We show that the natural generalization of a conjecture of Hain and Looijenga to the case of pointed curves holds for all g and n if and only if the tautological rings of the moduli spaces of curves with rational tails and of stable curves are Gorenstein.

Résumé. — Nous montrons que la généralisation naturelle d'une conjecture de Hain et Looijenga au cas des courbes épointées tient pour tout g et n si et seulement si les anneaux tautologiques des espaces des modules des courbes à queues rationnelles et des courbes stables sont des anneaux de Gorenstein.

Let $M_{g,n}$ (resp. $\overline{M}_{g,n}$) be the moduli space of smooth (resp. stable) n-pointed curves of genus g and let $M^{ct}_{g,n}$ be the moduli space of pointed curves of compact type, the complement of the boundary divisor Δ_{irr} of irreducible singular curves and their degenerations. Let $M^{rt}_{g,n}$ be the moduli space of pointed curves with rational tails; for $g \geqslant 2$, it is the inverse image of M_g under the natural morphism $\overline{M}_{g,n} \to \overline{M}_g$, while $M^{rt}_{1,n} = M^{ct}_{1,n}$ and $M^{rt}_{0,n} = \overline{M}_{0,n}$ by definition. Here, (g,n) is a pair of nonnegative integers such that $2g-2+n > 0$. There is a natural partial ordering of these pairs: $(h,m) \leqslant (g,n)$ if and only if $h \leqslant g$ and $2h-2+m \leqslant 2g-2+n$, or, in other words, if and only if there exists a stable n-pointed curve of genus g whose dual graph contains a vertex of genus h with valency m.

We recall the definition of the tautological algebras $R^\bullet(\overline{M}_{g,n})$ from [4]: the system $\{R^\bullet(\overline{M}_{g,n})\}_{(g,n)}$ is defined as the set of smallest $\mathbb{Q}$-subalgebras of the rational Chow rings $A^\bullet(\overline{M}_{g,n})$ that is closed under push-forward via all maps forgetting markings and all standard gluing maps. The well-known ψ-, κ-, and λ-classes are tautological. The system is also closed under pull-back via the forgetting and gluing maps. The successive quotients $R^\bullet(M^{ct}_{g,n})$, $R^\bullet(M^{rt}_{g,n})$, and $R^\bullet(M_{g,n})$ are defined as the restrictions

Keywords: Moduli spaces of curves, tautological ring, Gorenstein ring.
Math. classification: 14H10, 13H10.

to the respective open subsets. (Observe that it is in general not known whether the corresponding tautological localization sequences are exact in the middle.)

The following results are known:

(a) $R^\bullet(M^{rt}_{g,n})$ vanishes in degrees $> g-2+n-\delta_{0g}$ and is 1-dimensional in degree $g-2+n-\delta_{0g}$.
(b) $R^\bullet(M^{ct}_{g,n})$ vanishes in degrees $> 2g-3+n$ and is 1-dimensional in degree $2g-3+n$.
(c) $R^\bullet(\overline{M}_{g,n})$ (vanishes in degrees $> 3g-3+n$ and) is 1-dimensional in degree $3g-3+n$.

Statement (a) was proved by Looijenga [10] and Faber [1], [3]. Statements (b) and (c) were proved by Graber and Vakil [6], [7] and Faber and Pandharipande [4].

Recall the following three conjectures:

(A) $R^\bullet(M^{rt}_{g,n})$ is Gorenstein with socle in degree $g-2+n-\delta_{0g}$.
(B) $R^\bullet(M^{ct}_{g,n})$ is Gorenstein with socle in degree $2g-3+n$.
(C) $R^\bullet(\overline{M}_{g,n})$ is Gorenstein with socle in degree $3g-3+n$.

(For a graded $\mathbb{Q}$-algebra $R^\bullet$, to be Gorenstein with socle in degree m means that it vanishes in degrees $> m$, that R^m is isomorphic to $\mathbb{Q}$, and that the pairings $R^i \times R^{m-i} \to R^m$ are perfect.)

In the case $g = 0$, the three conjectures coincide and have been proved by Keel [9]. Conjecture (A) in the case $n = 0$ is due to the author [1] and is true for $g \leqslant 23$. Hain and Looijenga [8] raised (C) as a question and (A), (B), and (C) were formulated in [11] (see also [3], [2]).

Hain and Looijenga also introduce a compactly supported version of the tautological algebra: they define $R^\bullet_c(M_{g,n})$ as the set of elements in $R^\bullet(\overline{M}_{g,n})$ that restrict trivially to the Deligne-Mumford boundary (i.e., the pull-back via any standard map from a product of moduli spaces $\overline{M}_{g_i,n_i}$ onto the closure of a boundary stratum vanishes). It is a graded ideal in $R^\bullet(\overline{M}_{g,n})$ and a module over $R^\bullet(M_{g,n})$. They then formulate the following conjecture in the case $n = 0$:

Conjecture 1 (Hain and Looijenga [8]). — *The intersection pairings*

$$R^k(M_g) \times R_c^{3g-3-k}(M_g) \to R_c^{3g-3}(M_g) \cong \mathbb{Q}, \qquad k = 0, 1, 2, \ldots$$

are perfect (Poincaré duality) and $R^\bullet_c(M_g)$ is a free $R^\bullet(M_g)$-module of rank one.

Observe that $\lambda_1 \in R^1_c(M_{1,1})$ and $\lambda_g\lambda_{g-1} \in R^{2g-1}_c(M_g)$ for $g > 1$. (The author's proof of the nonvanishing of $R^{g-2+n}(M^{rt}_{g,n})$ for $g > 0$ uses this fact.) So this class is supposed to be a generator of the $R^\bullet(M_g)$-module $R^\bullet_c(M_g)$ (the unique generator of degree $2g-1$ up to a scalar).

However, the pull-backs of these classes to $\overline{M}_{g,n}$ don't lie in $R^\bullet_c(M_{g,n})$ for $n \geqslant 2$, since they don't vanish on the boundary strata corresponding to curves with rational tails. Let us therefore define $R^\bullet_c(M^{rt}_{g,n})$ as the set of elements in $R^\bullet(\overline{M}_{g,n})$ that restrict trivially to $\overline{M}_{g,n} \setminus M^{rt}_{g,n}$. Consider the following conjectures:

(D) The intersection pairings

$$R^k(M^{rt}_{g,n}) \times R^{3g-3+n-k}_c(M^{rt}_{g,n}) \to R^{3g-3+n}_c(M^{rt}_{g,n}) \cong \mathbb{Q}$$

are perfect for $k \geqslant 0$.

(E) In addition to (D), $R^\bullet_c(M^{rt}_{g,n})$ is a free $R^\bullet(M^{rt}_{g,n})$-module of rank one.

Conjecture (E) appears to be the natural generalization of Conjecture 1 to the case $n > 0$. For reasons that will become clear in a moment, we also include the weaker statement (D). Observe that (E) implies that $\lambda_g\lambda_{g-1}$ is a generator of $R^\bullet_c(M^{rt}_{g,n})$ for $g > 0$ (the unique one of degree $2g-1$ up to a scalar), by (a) above.

Theorem 1. — *Conjectures (A) and (C) are true for all (g,n) if and only if Conjecture (E) is true for all (g,n). More precisely,*

$$A_{(g,n)} \text{ and } C_{(g,n)} \Rightarrow E_{(g,n)} \Rightarrow A_{(g,n)} \text{ and } D_{(g,n)}$$

and

$$\{D_{(g',n')}\}_{(g',n')\leqslant(g,n)} \Rightarrow \{C_{(g',n')}\}_{(g',n')\leqslant(g,n)} .$$

Proof. Suppose first that (C) is not true for all (g,n) and let a minimal counterexample be given by $0 \neq \alpha \in R^\bullet(\overline{M}_{g,n})$, i.e., $R^\bullet(\overline{M}_{g',n'})$ is Gorenstein for all $(g',n') < (g,n)$ and $\deg(\alpha\beta) = 0$ for all $\beta \in R^\bullet(\overline{M}_{g,n})$. (We write deg for the degree homomorphism on $R_0(\overline{M}_{g,n})$ and its extension by zero to all of $R^\bullet(\overline{M}_{g,n})$). It follows that $g > 0$.

Let π denote the standard map $\overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ onto the boundary divisor Δ_{irr}. Let $\gamma \in R^\bullet(\overline{M}_{g-1,n+2})$ be arbitrary. Then

$$\deg((\pi^*\alpha)\gamma) = \deg(\pi_*((\pi^*\alpha)\gamma)) = \deg(\alpha\pi_*\gamma) = 0,$$

since $\pi_*\gamma$ is tautological. Since $R^\bullet(\overline{M}_{g-1,n+2})$ is Gorenstein, it follows that $\pi^*\alpha = 0$.

Next, let π denote one of the standard maps $\overline{M}_{g_1,n_1} \times \overline{M}_{g_2,n_2} \to \overline{M}_{g,n}$ onto a boundary component parametrizing reducible singular curves ($g_1 + g_2 = g$ and $n_1 + n_2 = n + 2$). We have the push-forward map

$$\pi_* : R^\bullet(\overline{M}_{g_1,n_1}) \otimes_{\mathbb{Q}} R^\bullet(\overline{M}_{g_2,n_2}) \to R^\bullet(\overline{M}_{g,n})$$

and the pull-back map in the other direction (cf. [5]). The tensor product is Gorenstein, with perfect pairing given by

$$\deg((\beta_1 \otimes \beta_2)(\gamma_1 \otimes \gamma_2)) = \deg(\beta_1\gamma_1)\deg(\beta_2\gamma_2).$$

Let γ_1 resp. γ_2 be arbitrary elements of $R^\bullet(\overline{M}_{g_1,n_1})$ resp. $R^\bullet(\overline{M}_{g_2,n_2})$. Then

$$\deg((\pi^*\alpha)(\gamma_1 \otimes \gamma_2)) = \deg(\pi_*((\pi^*\alpha)(\gamma_1 \otimes \gamma_2))) = \deg(\alpha\pi_*(\gamma_1 \otimes \gamma_2)) = 0,$$

since $\pi_*(\gamma_1 \otimes \gamma_2)$ is tautological. Again, it follows that $\pi^*\alpha = 0$.

Therefore, $0 \neq \alpha \in R_c^\bullet(M_{g,n})$ and a fortiori $0 \neq \alpha \in R_c^\bullet(M_{g,n}^{rt})$. But it pairs to zero with all β and this contradicts $D_{(g,n)}$. The implication in the second display follows as an immediate consequence.

The next step is to prove the implication $E_{(g,n)} \Rightarrow A_{(g,n)}$. As mentioned above, if $g > 0$ and $E_{(g,n)}$ holds, then $\lambda_g\lambda_{g-1}$ generates $R_c^\bullet(M_{g,n}^{rt})$ freely. Suppose that $A_{(g,n)}$ fails: let $0 \neq \alpha \in R^\bullet(M_{g,n}^{rt})$ be such that it pairs to zero with all $\beta \in R^\bullet(M_{g,n}^{rt})$, i.e., $\deg(\alpha\beta\lambda_g\lambda_{g-1}) = 0$ for all β (note that $g > 0$). From $D_{(g,n)}$, it follows that $\alpha\lambda_g\lambda_{g-1} = 0$, but this contradicts $E_{(g,n)}$. This proves the second implication in the first display.

To prove the first implication, we first show that $A_{(g,n)}$ and $C_{(g,n)}$ imply $D_{(g,n)}$. Assume that $D_{(g,n)}$ fails; the perfect pairing may fail on either side. Suppose first that $0 \neq \alpha \in R_c^\bullet(M_{g,n}^{rt})$ pairs to zero with all of $R^\bullet(M_{g,n}^{rt})$. We know that $\pi^*\alpha = 0$, for every standard map π associated to a stratum in $\overline{M}_{g,n} \setminus M_{g,n}^{rt}$. This means that the product of α and a Chow class pushed forward via such a map is zero (hence the pairing is well-defined). Since α pairs to zero with all of $R^\bullet(M_{g,n}^{rt})$, it gives a counterexample to $C_{(g,n)}$. If instead $0 \neq \alpha \in R^\bullet(M_{g,n}^{rt})$ pairs to zero with all of $R_c^\bullet(M_{g,n}^{rt})$, then it pairs to zero with all classes of the form $\beta\lambda_g\lambda_{g-1}$, for $\beta \in R^\bullet(M_{g,n}^{rt})$ (note that $g > 0$). In this case, α gives a counterexample to $A_{(g,n)}$.

We conclude by showing that $A_{(g,n)}$ and $C_{(g,n)}$ imply $E_{(g,n)}$. We already have $D_{(g,n)}$. If $E_{(g,n)}$ doesn't hold, then $g > 0$ and certainly $\lambda_g\lambda_{g-1}$ fails to be a basis for $R_c^\bullet(M_{g,n}^{rt})$, i.e., multiplication by $\lambda_g\lambda_{g-1}$ fails to be surjective or injective. From $A_{(g,n)}$ and $D_{(g,n)}$, it follows that the surjectivity and injectivity of this map are equivalent (recall from [5], Cor. 1, that $R^\bullet(\overline{M}_{g,n})$ is finite-dimensional). But if $0 \neq \alpha \in R^\bullet(M_{g,n}^{rt})$ and $\alpha\lambda_g\lambda_{g-1} = 0$, then $A_{(g,n)}$ fails. □

There is an analogous result in the compact type case. Begin by defining $R_c^\bullet(M_{g,n}^{ct})$ as the set of elements in $R^\bullet(\overline{M}_{g,n})$ that pull back to zero via the standard map $\overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ onto Δ_{irr}. Conjectures (D) and (E) have obvious analogues (D^{ct}) and (E^{ct}). We have that

$$B_{(g,n)} \text{ and } C_{(g,n)} \Rightarrow E^{ct}_{(g,n)} \Rightarrow B_{(g,n)} \text{ and } D^{ct}_{(g,n)}$$

and

$$\{D^{ct}_{(g',n')}\}_{(g',n')\leqslant(g,n)} \Rightarrow \{C_{(g',n')}\}_{(g',n')\leqslant(g,n)}\,.$$

The proof proceeds entirely analogously; the class λ_g now plays the role of $\lambda_g\lambda_{g-1}$ (it is no longer necessary to treat the case $g=0$ separately).

Acknowledgements. The author thanks Eduard Looijenga, Rahul Pandharipande, and Michael Shapiro for useful discussions. The author is supported by the Göran Gustafsson Foundation for Research in Natural Sciences and Medicine and grant 622-2003-1123 from the Swedish Research Council.

Note added in the second version (November 2010). Tavakol [13] has proved that the tautological ring of $M^{ct}_{1,n} = M^{rt}_{1,n}$ is Gorenstein with socle in degree $n-1$ (Conjectures (A) and (B) for $g=1$). From Theorem 1, the tautological rings $R^\bullet(\overline{M}_{1,n})$ are Gorenstein if and only if $E_{(1,n)}$ holds for all $n \geqslant 1$, in other words, if and only if $R^\bullet_c(M^{rt}_{1,n})$ is generated by λ_1 as an $R^\bullet(M^{rt}_{1,n})$-module.

Note added in the third version (April 2012). Tavakol [14] has now also proved Conjecture (A) for $g=2$: the tautological ring of $M^{rt}_{2,n}$ is Gorenstein with socle in degree n.

Note added in the fourth version (June 2012). Petersen [12] has proved that the tautological ring of $\overline{M}_{1,n}$ is Gorenstein with socle in degree n (Conjecture (C) for $g=1$).

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Manuscrit reçu le 6 décembre 2010,
accepté le 25 avril 2012.

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