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Jet schemes of complex plane branches and equisingularity


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JET SCHEMES OF COMPLEX PLANE BRANCHES
AND EQUISINGULARITY

by Hussein MOURTADA (*)

Abstract. — For \( m \in \mathbb{N} \), we determine the irreducible components of the \( m \)-th Jet Scheme of a complex branch \( C \) and we give formulas for their number \( N(m) \) and for their codimensions, in terms of \( m \) and the generators of the semigroup of \( C \). This structure of the Jet Schemes determines and is determined by the topological type of \( C \).

1. Introduction

Let \( K \) be an algebraically closed field. The space of arcs \( X_\infty \) of an algebraic \( K \)-variety \( X \) is a non-Noetherian scheme in general. It has been introduced by Nash in [10]. Nash has initiated its study by looking at its image by the truncation maps \( X_\infty \to X_m \) in the jet schemes of \( X \). The \( m \)-th jet scheme \( X_m \) of \( X \) is a \( K \)-scheme of finite type which parametrizes morphisms \( \text{Spec} \ K[t]/(t)^{m+1} \to X \). From now on, we assume \( \text{char} \ K = 0 \).

In [10], Nash has derived from the existence of a resolution of singularities of \( X \), that the number of irreducible components of the Zariski closure of the set of the \( m \)-truncations of arcs on \( X \) that send 0 into the singular locus of \( X \) is constant for \( m \) large enough. Besides a theorem of Kolchin asserts that if \( X \) is irreducible, then \( X_\infty \) is also irreducible. More recently,

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the jet schemes have attracted attention from various viewpoints. In [9], Mustata has characterized the locally complete intersection varieties having irreducible $X_m$ for $m \geq 0$. In [2], a formula comparing the codimensions of $Y_m$ in $X_m$ with the log canonical threshold of a pair $(X, Y)$ is given. In this work, we consider a curve $C$ in the complex plane $\mathbb{C}^2$ with a singularity at 0 at which it is analytically irreducible (i.e. the formal neighborhood $(C, 0)$ of $C$ at 0 is a branch). We determine the irreducible components of the space $C_m^0 := \pi_m^{-1}(0)$ where $\pi_m : C_m \rightarrow C$ is the canonical projection, and we show that their number is not bounded as $m$ grows. More precisely, let $x$ be a transversal parameter in the local ring $\mathcal{O}_{\mathbb{C}^2, 0}$, i.e. the line $x = 0$ is transversal to $C$ at 0 and following [2], for $e \in \mathbb{N}$, let

$$\text{Cont}_e(x)_m (\text{resp. Cont}_{>e}(x)_m) := \{ \gamma \in C_m \mid \text{ord}_x \circ \gamma = e(\text{resp.} > e) \},$$

where $\text{Cont}$ stands for contact locus. Let $\Gamma(C) = < \bar{\beta}_0, \cdots, \bar{\beta}_g >$ be the semigroup of the branch $(C, 0)$ and let $e_i = \gcd(\bar{\beta}_0, \cdots, \bar{\beta}_i)$, $0 \leq i \leq g$. Recall that $\Gamma(C)$ and the topological type of $C$ near 0 are equivalent data and characterize the equisingularity class of $(C, 0)$ as defined by Zariski in [13]. We show in theorem 4.9 that the irreducible components of $C_m^0$ are

$$C_{m\kappa_1} = \text{Cont}^{\kappa \bar{\beta}_0}(x)_m,$$

for $1 \leq \kappa$ and $\kappa \bar{\beta}_0 + e_1 \leq m$,

$$C_{m\kappa_2}^{j} = \text{Cont}^{\kappa \bar{\beta}_j}(x)_m$$

for $2 \leq j \leq g, 1 \leq \kappa, \kappa \neq 0 \mod \frac{e_j - 1}{e_j}$ and $\kappa \bar{\beta}_0 \bar{\beta}_1 + e_1 \leq m < \kappa \bar{\beta}_j$,

$$B_m = \text{Cont}^{> \bar{\beta}_0 \bar{\beta}_1 \bar{\beta}_1 \cdots}(x)_m,$$

if $q \bar{\beta}_0 + \bar{\beta}_1 + e_1 \leq m < (q + 1)n_1 \bar{\beta}_1 + e_1$.

These irreducible components give rise to infinite and finite inverse systems represented by a tree. We recover $< \bar{\beta}_0, \cdots, \bar{\beta}_g >$ from the tree and the multiplicity $\bar{\beta}_0$ in corollary 4.13, and we give formulas for the number of irreducible components of $C_m^0$ and their codimensions in terms of $m$ and $(\bar{\beta}_0, \cdots, \bar{\beta}_g)$ in proposition 4.7 and corollary 4.10. We recover the fact coming from [2] and [6] that

$$\min_m \frac{\text{codim}(C_m^0, C_m^2)}{m + 1} = \frac{1}{\bar{\beta}_0} + \frac{1}{\bar{\beta}_1}.$$

The structure of the paper is as follows: The basics about Jet schemes and the results that we will need are presented in section 2. In section 3
we present the definitions and the results we will need about branches. The last section is devoted to the proof of the main result and corollaries.

2. Jet schemes

Let $\mathbb{K}$ be an algebraically closed field of arbitrary characteristic. Let $X$ be a $\mathbb{K}$–scheme of finite type over $k$ and let $m \in \mathbb{N}$. The functor $F_m : \mathbb{K} \text{– Schemes} \rightarrow \text{Sets}$ which to an affine scheme defined by a $\mathbb{K}$–algebra $A$ associates

$$F_m(\text{Spec}(A)) = \text{Hom}_K(\text{Spec}A[t]/(t^{m+1}), X)$$

is representable by a $\mathbb{K}$–scheme $X_m$ [12]. $X_m$ is the $m$–th jet scheme of $X$, and $F_m$ is isomorphic to its functor of points. In particular the closed points of $X_m$ are in bijection with the $\mathbb{K}[t]/(t^{m+1})$ points of $X$.

For $m, p \in \mathbb{N}, m > p$, the truncation homomorphism $A[t]/(t^{m+1}) \rightarrow A[t]/(t^{p+1})$ induces a canonical projection $\pi_{m,p} : X_m \rightarrow X_p$. These morphisms clearly verify $\pi_{m,p} \circ \pi_{q,m} = \pi_{q,p}$ for $p < m < q$.

Note that $X_0 = X$. We denote the canonical projection $\pi_{m,0} : X_m \rightarrow X_0$ by $\pi_m$.

**Example 2.1.** — Let $X = \text{Spec} \frac{\mathbb{K}[x_0, \ldots, x_n]}{(f_1, \ldots, f_r)}$ be an affine $\mathbb{K}$–scheme. For a $\mathbb{K}$-algebra $A$, to give a $A$-point of $X_m$ is equivalent to give a $\mathbb{K}$–algebra homomorphism

$$\varphi : \frac{\mathbb{K}[x_0, \ldots, x_n]}{(f_1, \ldots, f_r)} \rightarrow A[t]/(t^{m+1}).$$

The map $\varphi$ is completely determined by the image of $x_i, i = 0, \cdots, n$

$$x_i \mapsto \varphi(x_i) = x_i^{(0)} + x_i^{(1)}t + \cdots + x_i^{(m)}t^m$$

such that $f_l(\varphi(x_0), \ldots, \varphi(x_n)) \in (t^{m+1}), l = 1, \cdots, r$.

If we write

$$f_l(\varphi(x_0), \cdots, \varphi(x_n)) = \sum_{j=0}^{m} F_l^{(j)}(x^{(0)}, \cdots, x^{(j)}) t^j \mod (t^{m+1})$$

where $x^{(j)} = (x_0^{(j)}, \cdots, x_n^{(j)})$, then

$$X_m = \text{Spec} \frac{\mathbb{K}[x_0^{(0)}, \ldots, x_n^{(m)}]}{(F_l^{(j)})_{j=0,\ldots,m}}$$
Example 2.2. — From the above example, we see that the m-th jet scheme of the affine space $A^n_K$ is isomorphic to $A^{(m+1)n}_K$ and that the projection $\pi_{m,m-1} : (A^n_K)_m \to (A^n_K)_{m-1}$ is the map that forgets the last $n$ coordinates.

Let $\text{char}(K) = 0$, $S = K[x_0, \ldots, x_n]$ and $S_m = K[x^{(0)}, \ldots, x^{(m)}]$. Let $D$ be the $K$-derivation on $S_m$ defined by $D(x^{(j)}) = x^{(j+1)}$ if $0 \leq j < m$, and $D(x^{(m)}) = 0$. For $f \in S$ let $f^{(1)} := D(f)$ and we recursively define $f^{(m)} = D(f^{(m-1)})$.

**Proposition 2.3.** — Let $X = \text{Spec}(S/(f_1, \ldots, f_r)) = \text{Spec}(R)$ and $R_m = \Gamma(X_m)$. Then

$$R_m = \text{Spec}(K[x^{(0)}, \ldots, x^{(m)}]/(f^{(j)})_{j=0, \ldots, m, i=1, \ldots, r}).$$

**Proof.** — For a $K$-algebra $A$, to give an $A$-point of $X_m$ is equivalent to give an homomorphism

$$\phi : K[x_0, \ldots, x_n] \to A[t]/(t^{m+1})$$

which can be given by

$$x_i \to x^{(0)}_i/0! + x^{(1)}_i/1! t + \cdots + x^{(m)}_i/m! t^m.$$  

Then for a polynomial $f \in S$, we have

$$\phi(f) = \sum_{j=0}^{m} \frac{f^{(j)}(x^{(0)}, \ldots, x^{(j)})}{j!} t^j.$$ 

To see this, it is sufficient to remark that it is true for $f = x_i$, and that both sides of the equality are additive and multiplicative in $f$, and the proposition follows. \hfill $\square$

**Remark 2.4.** — Note that the proposition shows the linearity of the equations $F^{(j)}_i(x^{(0)}, \ldots, x^{(j)})$ defining $X_m$ with respect to the new variables i.e $x^{(j)}$. We can deduce from this that if $X$ is a nonsingular $K$-variety of dimension $n$, then the projections $\pi_{m,m-1} : X_m \to X_{m-1}$ are locally trivial fibrations with fiber $A^n_K$. In particular, $X_m$ is a non-singular variety of dimension $(m + 1)n$.

### 3. Semigroup of complex branches

The main references for this section are [14],[8],[1],[11],[5],[4],[7]. Let $f \in \mathbb{C}[[x,y]]$ be an irreducible power series, which is $y$-regular (i.e $f(0, y) = $
$y^{n}u(y)$ where $u$ is invertible in $\mathbb{C}[[y]]$ and such that $\text{mult}_0 f = \beta_0$ and let $C$ be the analytically irreducible plane curve(branch for short) defined by $f$ in $\text{Spec} \ \mathbb{C}[[x, y]]$. By the Newton-Puiseux theorem, the roots of $f$ are

$$y = \sum_{i=0}^{\infty} a_i w^i x^{\beta_0}$$

where $w$ runs over the $\beta_0$-th-roots of unity in $\mathbb{C}$. This is equivalent to the existence of a parametrization of $C$ of the form

$$x(t) = t^{\beta_0}$$

$$y(t) = \sum_{i \geq \beta_0} a_i t^i.$$ We recursively define

$$\beta_i = \min\{i, a_i \neq 0, \gcd(\beta_0, \cdots, \beta_{i-1}) \text{ is not a divisor of } i\}.$$ Let $e_0 = \beta_0$ and $e_i = \gcd(e_{i-1}, \beta_i), i \geq 1$. Since the sequence of positive integers

$$e_0 > e_1 > \cdots > e_i > \cdots$$

is strictly decreasing, there exists $g \in \mathbb{N}$, such that $e_g = 1$. The sequence $(\beta_1, \cdots, \beta_g)$ is the sequence of Puiseux exponents of $C$. We set

$$n_i := \frac{e_{i-1}}{e_i}, m_i := \frac{\beta_i}{e_i}, i = 1, \cdots, g$$

and by convention, we set $\beta_{g+1} = +\infty$ and $n_{g+1} = 1$. On the other hand, for $h \in \mathbb{C}[[x, y]]$, we define the intersection number

$$(f, h)_0 = (C, C_h)_0 := \dim_{\mathbb{C}} \frac{\mathbb{C}[[x, y]]}{(f, h)} = \text{ord}_t h(x(t), y(t))$$

where $C_h$ is the Cartier divisor defined by $h$ and $\{x(t), y(t)\}$ is as above. The mapping $v_f : \frac{\mathbb{C}[[x, y]]}{(f)} \to \mathbb{N}, h \mapsto (f, h)_0$ defines a divisorial valuation. We define the semigroup of $C$ to be the semigroup of $v_f$, i.e. $\Gamma(C) = \Gamma(v_f)$ = \{(f, h)_0 \in \mathbb{N}, h \neq 0 \mod(f)\}. The following propositions and theorem from [14] characterize the structure of $\Gamma(C)$.

**Proposition 3.1.** — There exists a unique sequence of $g + 1$ positive integers $(\beta_0, \cdots, \beta_g)$ such that:

i) $\beta_0 = \beta_0$,

ii) $\beta_i = \min\{\Gamma(C) \setminus \langle \beta_0, \cdots, \beta_{i-1} \rangle, 1 \leq i \leq g$,

iii) $\Gamma(C) = \langle \beta_0, \cdots, \beta_g \rangle$,

where for $i = 1, \cdots, g + 1, \langle \beta_0, \cdots, \beta_{i-1} \rangle$ is the semigroup generated by $\beta_0, \cdots, \beta_{i-1}$. By convention, we set $\beta_{g+1} = +\infty$.  

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Proposition 3.2. — The sequence \((\widehat{\beta}_0, \ldots, \widehat{\beta}_g)\) verifies:

i) \(e_i = \gcd(\beta_0, \ldots, \beta_i), 0 \leq i \leq g,\)

ii) \(\widehat{\beta}_0 = \beta_0, \widehat{\beta}_1 = \beta_1\) and \(\widehat{\beta}_i = n_{i-1}\beta_{i-1} + \beta_i - \beta_{i-1} \in \mathbb{C}.\) In particular \(n_i \beta_i < \widehat{\beta}_{i+1},\)

for \(i = 2, \ldots, g.\)

Theorem 3.3. — The sequence \((\widehat{\beta}_0, \ldots, \widehat{\beta}_g)\) and the sequence \((\beta_0, \ldots, \beta_g)\) are equivalent data. They determine and are determined by the topological type of \(C.\)

Then from the appendix of [14], [1] or [11], we can choose a system of approximate roots (or a minimal generating sequence) \(\{x_0, \ldots, x_{g+1}\}\) of the divisorial valuation \(v_f.\) We set \(x = x_0, y = x_1;\) for \(i = 2, \ldots, g + 1, x_i \in \mathbb{C}[[x, y]]\) is irreducible; for \(1 \leq i \leq g,\) the analytically irreducible curve \(C_i = \{x_i = 0\}\) has \(i - 1\) Puiseux exponents and \(C_{g+1} = C.\) This sequence also verifies

i) \(v_f(x_i) = \hat{\beta}_0, 0 \leq i \leq g,\)

ii) \(\Gamma(C_i) = \{\hat{\beta}_0 / e_{i-1}, \ldots, \hat{\beta}_{i-1} / e_{i-1}\} > 1 \leq i \leq g + 1.\)

iii) \(2 \leq i \leq g,\) there exists a unique system of nonnegative integers \(b_{ij}, 0 \leq j < i\) such that for \(1 \leq j < i, b_{ij} < n_j\) and \(n_i \hat{\beta}_i = \Sigma_{0 \leq j < i} b_{ij} \hat{\beta}_j.\)

Furthermore, for \(1 \leq i \leq g,\) one can choose \(x_i\) such that they satisfy identities of the form

\[
x_{i+1} = x_i^{n_i} - c_i x_0^{b_{i0}} \cdots x_{i-1}^{b_{i(i-1)}} - \sum_{\gamma = (\gamma_0, \ldots, \gamma_i) \neq 0} c_i, \gamma x_0^{\gamma_0} \cdots x_i^{\gamma_i}, (*)
\]

with, \(0 \leq \gamma_j < n_j, 1 \leq j \leq i,\) and \(\Sigma_j \gamma_j \hat{\beta}_j > n_i \hat{\beta}_i\) and with \(c_i, \gamma, c_i \in \mathbb{C}\) and \(c_i \neq 0.\) These last equations (*) let us realize \(C\) as a complete intersection in \(\mathbb{C}^{g+1} = \text{Spec } \mathbb{C}[[x_0, \ldots, x_g]]\) defined by the equations

\[
f_i = x_{i+1} - (x_i^{n_i} - c_i x_0^{b_{i0}} \cdots x_{i-1}^{b_{i(i-1)}} - \sum_{\gamma = (\gamma_0, \ldots, \gamma_i) \neq 0} c_i, \gamma x_0^{\gamma_0} \cdots x_i^{\gamma_i})
\]

for \(1 \leq i \leq g,\) with \(x_{g+1} = 0\) by convention.

Let \(h \in \mathbb{C}[[x, y]]\) be a \(y\)-regular irreducible power series with multiplicity \(p = \text{ord}_y h(0, y).\) Let \(f(x, y^1)\) and \(z(x^1)\) be respectively roots of \(f\) and \(h\) as in (1). We call contact order of \(f\) and \(h\) in their Puiseux series the following rational number

\[
o_f(h) := \max\{\text{ord}_x (y(wx_0^{1/\beta_0}) - z(\lambda x^1)); w^{\beta_0} = 1, \lambda^p = 1\} = \\
\max\{\text{ord}_x (y(wx_0^{1/\beta_0}) - z(x^1)); w^{\beta_0} = 1\} = \\
\max\{\text{ord}_x (y(x_0^{1/\beta_0}) - z(\lambda x^1)); \lambda^p = 1\} = o_h(f).
\]

The following formula is from [8], see also [5].
Proposition 3.4. — Assume that \( f \) and \( h \) are as above; let \( (\beta_1, \cdots, \beta_g) \) the sequence of Puiseux exponents of \( f \) and let \( i \leq g + 1 \) be the smallest strictly positive integer such that \( o_f(h) \leq \frac{\beta_i}{\beta_0} \). Then
\[
\frac{(f, h)_0}{p} = \sum_{k=1}^{i-1} \frac{e_{k-1} - e_k}{\beta_0} \beta_k + e_{i-1} o_f(h) = (\beta_{i-1} e_{i-2} + (\beta_0 o_f(h) - \beta_{i-1}) e_{i-1}) \frac{1}{\beta_0}.
\]

Corollary 3.5. — [1][5] Let \( i > 0 \) be an integer. Then \( o_f(h) \leq \frac{\bar{\beta}_i}{\bar{\beta}_0} \) iff \( \frac{(f, h)_0}{p} \leq e_{i-1} \frac{\bar{\beta}_i}{\bar{\beta}_0} \). Moreover \( o_f(h) = \frac{\bar{\beta}_i}{\bar{\beta}_0} \) iff \( \frac{(f, h)_0}{p} = e_{i-1} \frac{\bar{\beta}_i}{\bar{\beta}_0} \). In particular \( o_f(x_i) = \frac{\bar{\beta}_i}{\bar{\beta}_0}, 1 \leq i \leq g \). We say that \( C_i x_i = 0 \) has maximal contact with \( C \).

4. Jet schemes of complex branches

We keep the notations of sections 2 and 3. We consider a curve \( C \subset \mathbb{C}^2 \) with a branch of multiplicity \( \beta_0 > 1 \) at 0, defined by \( f \). Note that in suitable coordinates we can write
\[
f(x_0, x_1) = (x_1^{n_1} - cx_0^{m_1}) e_1 + \sum_{a \beta_0 + b \beta_1 > \beta_0 \beta_1} c_{ab} x_0^a x_1^b; c \in \mathbb{C}^* \text{ and } c_{ab} \in \mathbb{C}. \tag{\odot}
\]
We look for the irreducible components of \( C_m := (\pi_m^{-1}(0)) \) for every \( m \in \mathbb{N} \), where \( \pi_m : C_m \to C \) is the canonical projection. Let \( J_m \) be the radical of the ideal defining \( (\pi_m^{-1}(0)) \) in \( \mathbb{C}^2 \).
In the sequel, we will denote the integral part of a rational number \( r \) by \([r]\).

Proposition 4.1. — For \( 0 < m < n_1 \bar{\beta}_1 \), we have that
\[
(C_m^0)_{\text{red}} = (\pi_m^{-1}(0))_{\text{red}} = \text{Spec} \left[ \mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, x_1^{(0)}, x_1^{(1)}, \cdots, x_1^{(m)}] \right. \left/ \langle x_0^{(0)}, \cdots, x_0^{(\lfloor \frac{m}{\bar{\beta}_1} \rfloor)}, x_1^{(0)}, \cdots, x_1^{(\lfloor \frac{m}{\bar{\beta}_0} \rfloor)} \rangle \right. ,
\]
and
\[
(C_{n_1 \bar{\beta}_1}^0)_{\text{red}} = (\pi_{n_1 \bar{\beta}_1}^{-1}(0))_{\text{red}} = \text{Spec} \left[ \mathbb{C}[x_0^{(0)}, \cdots, x_0^{(n_1 \bar{\beta}_1)}, x_1^{(0)}, x_1^{(1)}, \cdots, x_1^{(n_1 \bar{\beta}_1)}] \right. \left/ \langle x_0^{(0)}, \cdots, x_0^{(n_1 - 1)}, x_1^{(0)}, \cdots, x_1^{(n_1 - 1)}, x_1^{(n_1)} - cx_0^{(n_1 m_1)} \rangle \right. .
\]

Proof. — We write \( f = \Sigma_{(a, b)} c_{ab} f_{ab} \) where \( (a, b) \in \mathbb{N}^2, f_{ab} = x_0^a x_1^b, c_{ab} \in \mathbb{C} \) and \( a \beta_0 + b \bar{\beta}_1 \geq \beta_0 \beta_1 \) (the segment \([0, \beta_0)(\beta_1, 0])\) is the Newton Polygon of \( f \). Let \( \text{supp}(f) = \{(a, b) \in \mathbb{N}^2; c_{ab} \neq 0\} \).
For $0 < m < n_1 \bar{\beta}_1$, the proof is by induction on $m$. For $m = 1$, we have that

$$F^{(1)} = \sum_{(a,b) \in \text{supp}(f)} c_{ab} F^{(1)}_{ab}$$

where $(F^{(0)}, \ldots, F^{(i)})$ (resp. $(F^{(0)}_{ab}, \ldots, F^{(i)}_{ab})$) is the ideal defining the $i$-th jet scheme $C_i$ of $C$ (resp. $C^a_{i}$, the $i$-th jet scheme of $C^a_{i}$) in $\mathbb{C}^2_i$.

Then we have

$$F^{(1)}_{ab} = \sum_{i_k = 1} x_0^{(i_1)} \cdots x_0^{(i_a)} x_1^{(i_a+1)} \cdots x_1^{(i_a+b)},$$

where $\bar{\beta}_1(a + b) \geq a \beta_0 + b \bar{\beta}_1 > \beta_0 \bar{\beta}_1$ so $a + b \geq \beta_0 > 1$. Then for every $(a, b) \in \text{supp}(f)$ and every $(i_1, \ldots, i_a, \ldots, i_a+b) \in \mathbb{N}^{a+b}$ such that $\sum_{k=1}^{a+b} i_k = 1$ there exists $1 \leq k \leq a + b$ such that $i_k \neq 0$, this means that $F^{(1)}_{ab} \in (x_0^{(0)}, x_1^{(0)})$ and since we are looking over the origin, we have that $(x_0^{(0)}, x_1^{(0)}) \subseteq J_1$ therefore $(\pi^{-1}(0))_{\text{red}} = \text{Spec} \frac{\mathbb{C}[x_0^{(0)}, x_1^{(0)}, x_1^{(m-1)}]}{(x_0^{(0)}, \cdots, x_0^{(\frac{m-1}{\beta_1})}, x_1^{(0)}, \cdots, x_1^{(\frac{m-1}{\beta_0})})}$ (In fact this is nothing but the Zariski tangent space of $C$ at 0).

Suppose that the lemma holds until $m - 1$ i.e.

$$(\pi_{m-1}^{-1}(0))_{\text{red}} = \text{Spec} \frac{\mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m-1)}, x_1^{(0)}, \cdots, x_1^{(m-1)}]}{(x_0^{(0)}, \cdots, x_0^{(\frac{m-1}{\beta_1})}, x_1^{(0)}, \cdots, x_1^{(\frac{m-1}{\beta_0})})}.$$

**First case:** If $[\frac{m-1}{\beta_1}] = [\frac{m}{\beta_1}]$ and $[\frac{m-1}{\beta_0}] = [\frac{m}{\beta_0}]$. We have

$$F^{(m)} = \sum_{(a,b) \in \text{supp}(f)} c_{ab} \sum_{i_k = m} x_0^{(i_1)} \cdots x_0^{(i_a)} x_1^{(i_a+1)} \cdots x_1^{(i_a+b)}$$

Let $(a, b) \in \text{supp}(f)$; if for every $k = 1, \ldots, a$, we had $i_k \geq [\frac{m}{\beta_1}] + 1$, and for every $k = a + 1, \ldots, a + b$, we had $i_k \geq [\frac{m}{\beta_0}] + 1$, then

$$m \geq a([\frac{m}{\beta_1}] + 1) + b([\frac{m}{\beta_0}] + 1) \geq \frac{m}{\beta_1} a + \frac{m}{\beta_0} b = m \frac{a \beta_0 + b \beta_1}{\beta_0 \beta_1} \geq m.$$  

The contradiction means that there exists $1 \leq k \leq a$ such that $i_k \leq [\frac{m}{\beta_1}]$ or there exists $a + 1 \leq k \leq a + b$ such that $i_k \leq [\frac{m}{\beta_0}]$. So $F^{(m)}$ lies in the ideal generated by $J_{m-1}^0$ in $\mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, x_1^{(0)}, \cdots, x_1^{(m)}]$ and $J_m = J_{m-1}^0 \mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, x_1^{(0)}, \cdots, x_1^{(m)}]$.

**Second case:** If $[\frac{m-1}{\beta_1}] = [\frac{m}{\beta_1}]$ and $[\frac{m-1}{\beta_0}] + 1 = [\frac{m}{\beta_0}]$ (i.e. $\beta_0$ divides $m$). We have that

$$F^{(m)} = F^{(m)}_{0, \beta_0} + \sum_{(a,b) \in \text{supp}(f); (a,b) \neq (0, \beta_0)} F^{(m)}_{ab}.$$  

(\*)
where
\[
F_{0\beta_0}^{(m)} = \sum_{i_k = m} x_1^{(i_1)} \cdots x_1^{(i_{\beta_0})} = x_1^{(\frac{m}{\beta_0})} + \sum_{i_k = m; (i_1, \cdots, i_{\beta_0}) \neq (\frac{m}{\beta_0}, \cdots, \frac{m}{\beta_0})} x_1^{(i_1)} \cdots x_1^{(i_{\beta_0})},
\]
but \(\sum i_k = m\) and \((i_1, \cdots, i_{\beta_0}) \neq (\frac{m}{\beta_0}, \cdots, \frac{m}{\beta_0})\) implies that there exists \(1 \leq k \leq \beta_0\) such that \(i_k < \frac{m}{\beta_0}\), so
\[
\sum_{i_k = m; (i_1, \cdots, i_{\beta_0}) \neq (\frac{m}{\beta_0}, \cdots, \frac{m}{\beta_0})} x_1^{(i_1)} \cdots x_1^{(i_{\beta_0})} \in J^{0}_{m-1} \mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, x_1^{(0)}, \cdots, x_1^{(m)}].
\]
For the same reason as above, we have that
\[
\sum_{(a,b) \in \text{supp}(f); (a,b) \neq (0,\beta_0)} F^{(m)} \in J^{0}_{m-1} \mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, x_1^{(0)}, \cdots, x_1^{(m)}].
\]
From (**) we deduce that \(x_1^{(\frac{m}{\beta_0})} \in J^{0}_{m}\) and
\[
F^{(m)} \in (x_0^{(0)}, \cdots, x_0^{(\lfloor \frac{m}{\beta_1} \rfloor)}, x_1^{(0)}, \cdots, x_1^{(\frac{m}{\beta_0})}).
\]
Then \(J^{0}_{m} = (x_0^{(0)}, \cdots, x_0^{(\lfloor \frac{m}{\beta_1} \rfloor)}, x_1^{(0)}, \cdots, x_1^{(\frac{m}{\beta_0})})\).
The third case i.e. if \(\lfloor \frac{m-1}{\beta_1} \rfloor + 1 = \lfloor \frac{m}{\beta_1} \rfloor\) and \(\lfloor \frac{m-1}{\beta_0} \rfloor = \lfloor \frac{m}{\beta_0} \rfloor\) is discussed as the second one. Note that these are the only three possible cases since \(m < n_1 \beta_1 = \text{lcm}(\beta_0, \beta_1)\)(here \(\text{lcm}\) stands for the least common multiple).

For \(m = n_1 \beta_1\), we have that \(F^{(m)}\) is the coefficient of \(tm\) in the expansion of
\[
f(x_0^{(0)} + x_0^{(1)} t + \cdots + x_0^{(m)} t^m, x_1^{(0)} + x_1^{(1)} t + \cdots + x_1^{(m)} t^m).
\]
But since we are interested in the radical of the ideal defining the \(m\)-th jet scheme, and we have found that \(x_0^{(0)}, \cdots, x_0^{(n_1-1)}, x_1^{(0)}, \cdots, x_1^{(m_1-1)} \in J^{0}_{m-1} \subseteq J^{0}_{m}\), we can annihilate \(x_0^{(0)}, \cdots, x_0^{(n_1-1)}, x_1^{(0)}, \cdots, x_1^{(m_1-1)}\) in the above expansion. Using (\(\circ\)), we see that the coefficient of \(tm\) is \((x_1^{(m_1)})^{n_1} - cx_0^{(n_1)m_1})e_1.

\[\square\]

In the sequel if \(A\) is a ring, \(I \subseteq A\) an ideal and \(f \in A\), we denote by \(V(I)\) the subvariety of \(\text{Spec} A\) defined by \(I\) and by \(D(f)\) the open set in \(\text{Spec} A\), \(D(f) := \text{Spec} A_f\).

The proof of the following corollary is analogous to that of proposition 4.1.
Corollary 4.2. — Let \( m \in \mathbb{N} \); let \( k \geq 1 \) be such that \( m = \ell \bar{\beta}_1 + i \); \( 1 \leq i \leq \ell \bar{\beta}_1 \). Then if \( i < \ell \bar{\beta}_1 \), we have that
\[
\text{Cont}^{\ell \bar{\beta}_1} (x_0)_m = (\pi_{m, \ell \bar{\beta}_1}^{-1} (V(x_0^{(0)}, \ldots, x_0^{(\ell \bar{\beta}_1)})))_{\text{red}} = \text{Spec} \left( \mathbb{C}[x_0^{(0)}, \ldots, x_0^{(m)}, x_1^{(0)}, \ldots, x_1^{(m)}] \right)
\]
and if \( i = \ell \bar{\beta}_1 \)
\[
\left( \pi_{m, \ell \bar{\beta}_1}^{-1} (V(x_0^{(0)}, \ldots, x_0^{(\ell \bar{\beta}_1)})) \right)_{\text{red}} = \text{Spec} \left( \mathbb{C}[x_0^{(0)}, \ldots, x_0^{((k+1)m_1-1)}, x_1^{(0)}, \ldots, x_1^{((k+1)m_1-1)}] \right)
\]

We now consider the case of a plane branch with one Puiseux exponent.

Lemma 4.3. — Let \( C \) be a plane branch with one Puiseux exponent. Let \( m, k \in \mathbb{N} \), such that \( k \neq 0 \) and \( m \geq \ell \bar{\beta}_1 + 1 \), and let \( \pi_{m, \ell \bar{\beta}_1} : C_m \to C_{\ell \bar{\beta}_1} \) be the canonical projection. Then
\[
C_{m}^{k} := \pi_{m, \ell \bar{\beta}_1}^{-1} (V(x_0^{(0)}, \ldots, x_0^{(k-1)}) \cap D(x_0^{(k)}))_{\text{red}}
\]
is irreducible of codimension \( k(m_1 + n_1) + 1 + (m - \ell \bar{\beta}_1) \) in \( \mathbb{C}^2_m \).

Proof. — First note that since \( e_1 = 1 \), we have \( m_1 = \frac{\beta_1}{e_1} = \beta_1 \). Let \( I_m^k \) be the ideal defining \( C_{m}^{k} \) in \( \mathbb{C}^2 \cap D(x_0^{(k)}) \). Since \( m \geq \ell \bar{\beta}_1 \), by corollary 4.2, \( x_0^{(0)}, \ldots, x_0^{(k-1)} \) \( \in I_m^k \). So \( I_m^k \) is the radical of the ideal \( I_m^{*ok} := (x_0^{(0)}, \ldots, x_0^{(m_1-1)}, x_1^{(0)}, \ldots, x_1^{(m_1-1)}, F(0), \ldots, F(m)) \). Now it follows from \( \diamond \) and proposition 2.3 that
\[
F^{(l)} \in (x_0^{(0)}, \ldots, x_0^{(k-1)}, x_1^{(0)}, \ldots, x_1^{(k-1)}) \text{ for } 0 \leq l < \ell m_1,
\]
\[
F^{(\ell m_1)} \equiv x_1^{(m_1)n_1} \
- cx_0^{(m_1)n_1} \mod (x_0^{(0)}, \ldots, x_0^{(k-1)}, x_1^{(0)}, \ldots, x_1^{(k-1)}),
\]
\[
F^{(\ell m_1 + l)} \equiv \ell m_1 x_1^{(m_1)n_1-1} x_1^{(k-1)} x_0^{(m_1)l} + m_1 cx_0^{(m_1)n_1-1} x_0^{(k-1)} \mod (x_0^{(0)}, \ldots, x_0^{(k-1)}, x_1^{(0)}, \ldots, x_1^{(k-1)}),
\]
for \( 1 \leq l \leq m - \ell m_1 \). This implies that
\[
I_m^{*ok} = (x_0^{(0)}, \ldots, x_0^{(k-1)}, x_1^{(0)}, \ldots, x_1^{(k-1)}, F^{(m_1)}, \ldots, F^{(m)}).
\]
Moreover the subscheme of $\mathbb{C}^2_m \cap D(x_0^{(kn_1)})$ defined by $I_{m}^{*0k}$ is isomorphic to the product of $\mathbb{C}^* (\mathbb{C}^* \text{ is isomorphic to the regular locus of } x_1^{(km_1)} - cx_0^{(kn_1)m_1})$ by an affine space and its codimension is $k(m_1 + n_1) + 1 + (m - kn_1m_1)$; so it is reduced and irreducible, and it is nothing but $C_m^k$, or equivalently $I_{m}^{0k} = I_{m}^{*0k}$.

**Corollary 4.4.** — Let $C$ be a plane branch with one Puiseux exponent. Let $m \in \mathbb{N}, m \neq 0$. Let $q \in \mathbb{N}$ be such that $m = q n_1 \beta_1 + i; 0 < i \leq n_1 \beta_1$. Then $C_m^0 = \pi_{m-1}^{-1}(0)$ has $q + 1$ irreducible components which are:

$$C_{mkI} = \overline{C_m^k}, 1 \leq k \leq q,$$

and $B_m = \text{Cont}^{> q_1}(x) = \pi_{m,qn_1 \beta_1}^{-1}(V(x_0^{(0)}, \ldots, x_0^{(q_1)})).$

We have that

$$\text{codim}(C_{mkI}, \mathbb{C}^2_m) = k(m_1 + n_1) + 1 + (m - kn_1m_1)$$

and

$$\text{codim}(B_m, \mathbb{C}^2_m) = q(m_1 + n_1) + \left\lceil \frac{i}{\beta_0} \right\rceil + \left\lceil \frac{i}{\beta_1} \right\rceil + 2 = \left\lceil \frac{m}{\beta_0} \right\rceil + \left\lceil \frac{m}{\beta_1} \right\rceil + 2 \text{ if } i < n_1 \beta_1$$

$$\text{codim}(B_m, \mathbb{C}^2_m) = (q + 1)(m_1 + n_1) + 1 \text{ if } i = n_1 \beta_1.$$

**Proof.** — The codimensions and the irreducibility of $B_m$ and $C_{mkI}$ follow from corollary 4.2 and lemma 4.3. This shows that if $1 \leq k < k' \leq q$, we have $\text{codim}(C_{mkI}, \mathbb{C}^2_m) < \text{codim}(C_{mk'I}, \mathbb{C}^2_m)$, then $C_{mk'I} \not\subseteq C_{mkI}$. On the other hand, since $C_{mk'I} \subseteq V(x_0^{(kn_1)})$ and $C_{mkI} \not\subseteq V(x_0^{(kn_1)})$, we have that $C_{mkI} \not\subseteq C_{mk'I}$. This also shows that $\dim B_m \geq \dim C_{mkI}$ for $1 \leq k \leq q$, therefore $B_m \not\subseteq C_{mkI}, 1 \leq k \leq q$. But $C_{mkI} \not\subseteq B_m$ because $B_m \subseteq V(x_0^{(q_1)})$ and $C_{mkI} \not\subseteq V(x_0^{(q_1)})$ for $1 \leq k \leq q$. We thus have that $C_{mkI} \not\subseteq B_m$ and $B_m \not\subseteq C_{mkI}$. We conclude the corollary from the fact that by construction $C_m^0 = \bigcup_{k=1}^q C_{mkI} \cup B_m$.

To understand the general case, i.e. to find the irreducible components of $C_m^0$, where $C$ has a branch with $g$ Puiseux exponents at 0, since for $kn_1 \beta_1 < m \leq (k+1)n_1 \beta_1, m, k \in \mathbb{N}$ we know by corollary 4.2 the structure of the $m$-jets that project to $V(x_0^{(0)}, \ldots, x_0^{(kn_1)}) \cap C_{m,kn_1 \beta_1}^0,$ we have to understand for $m > kn_1 \beta_1$ the $m$-jets that projects to $V(x_0^{(0)}, \ldots, x_0^{(kn_1-1)}) \cap D(x_0^{(kn_1)}),$ i.e. $C_m^k := \pi_{m,kn_1 \beta_1}^{-1}(V(x_0^{(0)}, \ldots, x_0^{(kn_1-1)}) \cap D(x_0^{(kn_1)}))_{\text{red}}.$

Let $m, k \in \mathbb{N}$ be such that $m \geq kn_1 \beta_1$. Let $j = \max\{l, n_2 \cdots n_{l-1} \text{ divides } k\}$ (we set $j = 2$ if the greatest common divisor $(k, n_2) = 1$ or if $g = 1$). Set $\kappa$ such that $k = \kappa n_2 \cdots n_{j-1},$ then we have $kn_1 = \kappa \frac{\beta_0}{n_j \cdots n_g}$. 

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Indeed, we have that \( \pi_i \leq g + 1 \); for \( i = 2, \ldots, g \), and \( kn_1 \beta_1 < m < \kappa e_i - 1 \beta_i \), we have that
\[
C_m^k = \pi_{m, [\frac{m}{m-n_i}]}(C_{m, [\frac{m}{m-n_i}]}),
\]
where \( \pi_{m, [\frac{m}{m-n_i}]} : C_m^2 \rightarrow C_{\frac{m}{m-n_i}}^2 \) is the canonical map. For \( j < g + 1 \) and \( m \geq \kappa \beta_j \), we have that
\[
C_m^k = \emptyset
\]

**Proof.** — Let \( \phi \in C_m^k \). Let \( \tilde{\phi} : \text{Spec } \mathbb{C}[[t]] \rightarrow (\mathbb{C}^2, 0) \) be such that \( \phi = \tilde{\phi} \mod t^{m+1} \). Let \( \tilde{f} \in \mathbb{C}[[x, y]] \) be a function that defines the branch \( \tilde{C} \) image of \( \tilde{\phi} \). We may assume that the map \( \text{Spec } \mathbb{C}[[t]] \rightarrow \tilde{C} \) induced by \( \tilde{\phi} \) is the normalization of \( \tilde{C} \). Since \( \text{ord}_i x_0 \circ \tilde{\phi} = kn_1, \text{ord}_i x_1 \circ \tilde{\phi} = km_1 \) the multiplicity \( m(\tilde{f}) \) of \( \tilde{C} \) at the origin is \( \text{ord}_x \tilde{f}(0, x_1) = kn_1 = \kappa \beta_0 \).

**Claim:** If \( (f, \tilde{f})_0 < \kappa e_i - 1 \beta_i \), then \( (f, \tilde{f})_0 = n_i \cdots n_g (x_i, \tilde{f})_0 \).

Indeed, we have that \( \frac{(f, \tilde{f})_0}{\text{ord}_y f(0, y)} < e_i - 1 \beta_i \beta_0 \), therefore by corollary 3.5 we have that
\[
o_f(\tilde{f}) < \frac{\beta_i}{\beta_0} = o_f(x_i).
\]
We will prove that \( o_f(\tilde{f}) = o_{x_i}(\tilde{f}) \). (It was pointed by the referee that this follows from [1]. For the convenience of the reader we give a detailed proof below.)

Let \( y(x^{\frac{1}{\beta_0}}), z(x^{\frac{1}{\beta_1}}) \) and \( u(x^{\frac{1}{\beta_0}}) \) be respectively Puiseux-roots of \( f, x_i \) and \( \tilde{f} \). There exist \( w, \lambda \in \mathbb{C} \) such that \( w^{\frac{1}{m-n_i}} = 1, \lambda^m(\tilde{f}) = 1 \) and
\[
o_f(\tilde{f}) = \text{ord}_x (u(\lambda x^{\frac{1}{m-n_i}}) - y(x^{\frac{1}{\beta_0}}))
\]
and
\[
o_f(x_i) = \text{ord}_x (y(x^{\frac{1}{\beta_0}}) - z(wx^{\frac{1}{m-n_i}})).
\]
Since \( o_f(\tilde{f}) < o_f(x_i) \), we have that
\[
o_f(\tilde{f}) = \text{ord}_x (u(\lambda x^{\frac{1}{m-n_i}}) - y(x^{\frac{1}{\beta_0}}) + y(x^{\frac{1}{\beta_0}}) - z(wx^{\frac{1}{m-n_i}}))
\]
\[
= \text{ord}_x (u(\lambda x^{\frac{1}{m-n_i}}) - z(wx^{\frac{1}{m-n_i}})) \leq o_f(x_i, \tilde{f}).
\]

On the other hand, there exist \( \lambda \) and \( \delta \in \mathbb{C} \), such that \( \lambda^m(\tilde{f}) = 1, \delta^\beta_0 = 1 \) and such that
\[
o_{x_i}(\tilde{f}) = \text{ord}_x (u(\lambda x^{\frac{1}{m-n_i}}) - z(x^{\frac{1}{m-n_i}}))
\]
and
\[
o_f(x_i) = \text{ord}_x (y(\delta x^{\frac{1}{\beta_0}}) - z(x^{\frac{1}{m-n_i}})).
\]
We have then that
\[ o_{x_i}(\tilde{f}) = \text{ord}_x(u(\lambda x^{-\frac{1}{m(f)}}) - y(\delta x^{\frac{1}{\rho_0}}) + y(\delta x^{\frac{1}{\rho_0}}) - z(wx^{\frac{1}{n_1-n_i-1}})) \].

Now
\[ \text{ord}_x(u(\lambda x^{-\frac{1}{m(f)}}) - y(\delta x^{\frac{1}{\rho_0}})) \leq o_f(\tilde{f}) \]
\[ < o_f(x_i) = \text{ord}_x(y(\delta x^{\frac{1}{\rho_0}}) - z(wx^{\frac{1}{n_1-n_i-1}})). \]

So
\[ o_{x_i}(\tilde{f}) = \text{ord}_x(u(\lambda x^{-\frac{1}{m(f)}}) - y(\delta x^{\frac{1}{\rho_0}})) \leq o_f(\tilde{f}). \]

We conclude that \( o_f(\tilde{f}) = o_{x_i}(\tilde{f}) \), and since the sequence of Puiseux exponents of \( C_i \) is \((\frac{\beta_0}{n_1\cdots n_g}, \cdots, \frac{\beta_i-1}{n_i\cdots n_g})\), applying proposition 3.4 to \( C \) and \( C_i \), we find that \((f, \tilde{f})_0 = n_i \cdots n_g(x_i, \tilde{f})_0\) and claim follows.

On the other hand by the corollary 3.5 applied to \( f \) and \( \tilde{f}, (f, \tilde{f})_0 \geq \kappa e_{i-1} \frac{\beta_i}{e_{j-1}} \) if and only if \( o_f(\tilde{f}) \geq \frac{\beta_i}{\beta_0} = o_{x_i}(f) = o_f(x_i) \) so \( o_f(\tilde{f}) \geq \frac{\beta_i}{\beta_0} \) if and only if \( o_{x_i}(\tilde{f}) \geq \frac{\beta_i}{\beta_0} \), therefore \((x_i, \tilde{f})_0 \geq \kappa \frac{\beta_i}{e_{j-1}} \). This proves the first assertion.

The second assertion is a direct consequence of lemma 5.1 in [5]. \( \square \)

To further analyse the \( C_m^k \)'s, we realize, as in section 3, \( C \) as a complete intersection in \( \mathbb{C}^{g+1} = \text{Spec} \mathbb{C}[x_0, \cdots, x_g] \) defined by the ideal \((f_1, \cdots, f_g)\) where
\[ f_i = x_{i+1} - (x_i^{n_i} - c_i x_0^{b_{i0}} \cdots x_{i-1}^{b_{i(i-1)}} - \sum_{\gamma=(\gamma_0, \cdots, \gamma_i)} c_i, \gamma x_0^{\gamma_0} \cdots x_i^{\gamma_i}) \]
for \( 1 \leq i \leq g \) and \( x_{g+1} = 0 \). This will let us see the \( C_m^k \)'s as fibrations over some reduced scheme that we understand well.

We keep the notations above and let \( I_{m}^0 \) be the radical of the ideal defining \( C_m^0 \) in \( \mathbb{C}^{g+1} \) and let \( I_{m}^{0k} \) be the ideal defining
\[ C_m^k = (V(I_{m}^0, x_0^{(k)}, \cdots, x_0^{(kn_1-1)}) \cap D(x_0^{(kn_1)}))_{\text{red}} \text{ in } D(x_0^{(kn_1)}). \]

**Lemma 4.6.** — Let \( k \neq 0, j \), and \( \kappa \) as above. For \( 1 \leq i < j \leq g \) (resp. \( 1 \leq i < j = 1 = g \)) and for \( \kappa n_i \cdots n_{j-1} \beta_i \leq m < \kappa n_i+1 \cdots n_{j-1} \beta_i+1 \), we have
\[
I_{m}^{0k} = (x_0^{(\frac{n_0}{n_j-n_g}-1)}, x_0^{(\frac{n_1}{n_j-n_g}-1)}), \]
\[ x_0^{(\frac{n_i}{n_j-n_g}-1)}, \cdots, x_0^{(\frac{n_j}{n_j-n_g}-1)}, F_{l}^{(\frac{m}{n_j-n_g})}, \cdots, F_{l}^{(m)}, 1 \leq l \leq i, \]
\[ x_0^{(0)}, \cdots, x_0^{(\frac{m}{n_j-n_g})}, x_{i+1}^{(\frac{m}{n_j-n_g})}, \]
\[ F_{l}^{(0)}, \cdots, F_{l}^{(m)}, i + 1 \leq l \leq g - 1. \]
Moreover for \(1 \leq l \leq i\),
\[
F_l^{(\{n_j \beta_j\}_{j \neq g})} \equiv - (x_l \{(n_j \beta_j\}_{j \neq g})^{b_l} - c_l x_0^{\{n_j \beta_j\}_{j \neq g}} \ldots x_{l-1}\}
\]
mod \((x_l^{(0)}, \ldots, x_{l-1}^{(0)})_{0 \leq l \leq i}, x_{l+1}^{(0)}, \ldots, x_{m+1}^{(0)}\),
for \(1 \leq l < i\) and \(\kappa \frac{n_j \beta_j}{n_j \cdots n_g} < \kappa \frac{n_{j+1} \beta_j}{n_j \cdots n_g}\) (resp. \(l = i\) and \(\kappa \frac{n_j \beta_j}{n_j \cdots n_g} < \kappa \frac{n_{j+1} \beta_j}{n_j \cdots n_g}\)),
\[
F_l^{(n)} \equiv -(n_l x_l^{(0)}, \ldots, x_{l-1}^{(0)})_{0 \leq l \leq i}, x_{l+1}^{(0)}, \ldots, x_{m+1}^{(0)}\)
\]
mod \((x_l^{(0)}, \ldots, x_{m+1}^{(0)})_{0 \leq l \leq i}, x_{l+1}^{(0)}, \ldots, x_{m+1}^{(0)}\),
for \(1 \leq l < i\) and \(\kappa \frac{n_j \beta_j}{n_j \cdots n_g} < \kappa \frac{n_{j+1} \beta_j}{n_j \cdots n_g}\) (resp. \(l = i\) and \(\kappa \frac{n_j \beta_j}{n_j \cdots n_g} < \kappa \frac{n_{j+1} \beta_j}{n_j \cdots n_g}\)),
or \(i + 1 \leq l \leq g - 1\) and \(0 \leq n \leq m\),
\[
F_l^{(n)} = x_{l+1}^{(n)} + H_l(x_0^{(0)}, \ldots, x_0^{(n)}, \ldots, x_l^{(0)}, \ldots, x_l^{(n)}).
\]
For \(i = j - 1 = g\) and \(m \geq \kappa n_g \beta_g\),
\[
F_l^{(0)} = (x_0^{(0)}, \ldots, x_0^{(\kappa \beta_0 - 1)}),
\]
\[
x_l^{(0)}, \ldots, x_l^{(\kappa \beta_l - 1)}, F_l^{(\kappa n_g \beta_g)}, \ldots, F_l^{(m)}), 1 \leq l \leq g,
\]
where for \(1 \leq l < g\) and \(\kappa n_g \beta_g \leq n \leq m\), the above formula for \(F_l^{(n)}\) remains valid,
\[
F_l^{(\kappa n_g \beta_g)} \equiv - (x_g^{(\kappa \beta_g)} g \ldots c_g x_0^{(\kappa \beta_0)} g_0 \ldots x_{g-1}^{(\kappa \beta_{g-1})} g_{g-1})
\]
mod \((x_l^{(0)}, \ldots, x_l^{(\kappa \beta_l - 1)})_{0 \leq l \leq g}\)
and for \(\kappa n_g \beta_g < n \leq m\),
\[
F_l^{(n)} \equiv -(n_g x_g^{(\kappa \beta_g)} \ldots c_g x_0^{(\kappa \beta_0)} g \ldots x_{g-1}^{(\kappa \beta_{g-1})} g_{g-1})
\]
\[
\sum_{0 \leq h \leq g-1} b_h x_0^{(\kappa \beta_0)} g \ldots x_h^{(\kappa \beta_h)} g_h \ldots x_{g-1}^{(\kappa \beta_{g-1})} g_{g-1}
\]
mod \((x_l^{(0)}, \ldots, x_l^{(\kappa \beta_l - 1)})_{0 \leq l \leq g}\)
Proof. — First assume that $\kappa n_i \cdots n_{j-1} \beta_i \leq m < \kappa n_i+1 \cdots n_{j-1} \beta_{i+1}$ for $1 \leq i < j \leq g$ (resp. $1 \leq i < j-1 = g$). By proposition 4.5, we have that $C_m^k = \tilde{\pi}_{m,1\cdots n_g}^{-1}(C_{i+1,1\cdots n_g}^k)$ where $\pi_{m,1\cdots n_g}^{-1} : \mathbb{C}_m^{2} \to \mathbb{C}_m^{2}$ is the canonical map. Now $\mathbb{C}^2 = \text{Spec} \mathbb{C}[x_0, x_1]$ (resp. $C_1^+ = V(x_{j+1})$) is realized as the complete intersection in $\mathbb{C}^{g+1} = \text{Spec} \mathbb{C}[x_0, \cdots, x_g]$ defined by the ideal $(f_1, \cdots, f_{g-1})$ (resp. $(f_1, \cdots, f_{g-1}, x_{j+1})$). So since $m \geq \kappa n_i \beta_i$, $I_m^{ok}$ is the radical of the ideal $I_m^{ok} =$

$$(x_0^{(0)}, \cdots, x_0^{(k_1-1)}, x_1^{(0)}, \cdots, x_1^{(k_1-1)}, F_1^{(0)}, \cdots, F_1^{(m)},$

$$\cdots, F_{g-1}^{(0)}, \cdots, F_{g-1}^{(m)}, x_{i+1}^{(0)}, \cdots, x_{i+1}^{(\lceil \frac{m}{n_{i+1} \cdots n_g} \rceil)}).$$

We first observe that $F_1^{(n)} \equiv x_2^{(n)} \mod (x_0^{(0)}, \cdots, x_0^{(k_1-1)}, x_1^{(0)}, \cdots, x_1^{(k_1-1)})$ for $0 \leq n < \kappa n_1 \beta_1$. Now since $\frac{m}{n_{2 \cdots n_g}} \geq \frac{m}{n_{2 \cdots n_g}} \geq \kappa n_1 m_1$, we have

$$F_1^{(k_1 m_1)} \equiv -x_1^{(k_1 m_1)} - c_1 x_0^{(k_1 m_1)} \mod (x_0^{(0)}, \cdots, x_0^{(k_1-1)}, x_1^{(0)}, \cdots, x_1^{(k_1-1)}, x_0^{(0)}, \cdots, x_2^{(\lceil \frac{m}{n_{i+1} \cdots n_g} \rceil)})$$

and

$$F_1^{(n)} \equiv -n_1 x_1^{(k_1 m_1 - 1)} x_1^{(k_1 m_1 - n - k_1 m_1)} - c_1 x_0^{(k_1 m_1 - 1)} x_0^{(k_1 m_1 - n - k_1 m_1)} + H(x_0^{(0)}, \cdots, x_0^{(k_1 m_1 - 1)}, x_1^{(0)}, \cdots, x_1^{(k_1 m_1 - 1)}$$

$$\mod (x_0^{(0)}, \cdots, x_0^{(k_1 m_1 - 1)}, x_1^{(0)}, \cdots, x_1^{(k_1 m_1 - 1)}, x_0^{(0)}, \cdots, x_2^{(\lceil \frac{m}{n_{i+1} \cdots n_g} \rceil)})$$

for $k n_1 \beta_1 < n \leq \frac{m}{n_{2 \cdots n_g}}$. Finally, for $l = 1$ and $\frac{m}{n_{2 \cdots n_g}} < n \leq m$, or $2 \leq l \leq g - 1$ and $0 \leq n \leq m$, we have

$$F_l^{(n)} = x_{l+1}^{(n)} + H_l(x_0^{(0)}, \cdots, x_0^{(n)}), \cdots, x_l^{(0)}, \cdots, x_l^{(n)}).$$

As a consequence for $i = 1$, the subscheme of $\mathbb{C}^{g+1} \cap D(x_0^{(k_1)})$ defined by $I_m^{ok}$ is isomorphic to the product of $\mathbb{C}^*$ by an affine space, so it is reduced and irreducible and $I_m^{ok} = I_m^{ok}$ is a prime ideal in $\mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, \cdots, x_g^{(0)}, \cdots, x_g^{(m)}]_{x_0^{(k_1)}}$, generated by a regular sequence, i.e the proposition holds for $i = 1$.

Assume that it holds for $i < j - 1 < g$ (resp. $i < j - 1 = g - 1$). For $\kappa n_{i+1} \cdots n_{j-2} \beta_{i+1} \leq m < \kappa n_{i+2} \cdots n_j \beta_{i+2}$, the ideal in $\mathbb{C}[x_0^{(0)}, \cdots, x_0^{(m)}, \cdots, x_g^{(0)}, \cdots, x_g^{(m)}]_{x_0^{(k_1)}}$ generated by $I_m^{ok}$ is contained in $I_m^{ok}$. By the inductive hypothesis, $x_l^{(0)}, \cdots, x_l^{(n_{j-2} \cdots n_g - 1)} \in$
$I_{\kappa n_{i+1} \cdots n_{j-1} \bar{\beta}_{i+1}}$ for \( l = 1, \ldots, i + 1 \). So $I_{\kappa}$ is the radical of

\[
I_{\kappa} \cap (x_{0}^{(0)}, \ldots, x_{0}^{(\frac{\kappa \bar{\beta}_{0}}{n_{j-n_{g}}}-1)}, x_{1}^{(0)}, \ldots, x_{l}^{(\frac{\kappa \bar{\beta}_{l}}{n_{j-n_{g}}}-1)}, F_{l}^{(0)}, \ldots, F_{l}^{(m)}, 1 \leq l \leq i + 1,
\]

\[
x_{i+2}^{(0)}, \ldots, x_{i+2}^{(\frac{m}{n_{i+2 \cdots n_{g}}})}, F_{i+2}^{(0)}, \ldots, F_{l}^{(m)}, i + 2 \leq l \leq g - 1.
\]

Now for \( 0 \leq n < \frac{\kappa n_{j} \bar{\beta}_{l}}{n_{j \cdots n_{g}}} \), we have

\[
F_{l}^{(n)} \equiv x_{l+1}^{(n)} \mod (x_{0}^{(0)}, \ldots, x_{l}^{(\frac{\kappa \bar{\beta}_{0}}{n_{j-n_{g}}}-1)}, x_{l}^{(0)}, \ldots, x_{l}^{(\frac{\kappa \bar{\beta}_{l}}{n_{j-n_{g}}}-1)}, 1 \leq l \leq i + 1).
\]

Here since \( \bar{\beta}_{i+1} > n_{i} \bar{\beta}_{l} \), for \( 1 \leq l \leq i \) and \( \frac{m}{n_{i+2 \cdots n_{g}}} \geq \frac{[\frac{m}{n_{i+2 \cdots n_{g}}}]}{\frac{\kappa n_{i+1} \bar{\beta}_{i+1}}{n_{j \cdots n_{g}}}} \), we can delete \( F_{l}^{(n)} \), \( 1 \leq l \leq i + 1, 0 \leq n < \frac{\kappa n_{j} \bar{\beta}_{l}}{n_{j \cdots n_{g}}} \) from the above generators of \( I_{\kappa}^{*} \). The identities relative to the \( F_{l}^{(n)} \) for \( 1 \leq l \leq i + 1, \frac{\kappa n_{j} \bar{\beta}_{l}}{n_{j \cdots n_{g}}} \leq n < \frac{\kappa n_{j} \bar{\beta}_{l}}{n_{j \cdots n_{g}}} \) or \( i + 2 \leq l \leq g - 1 \) and \( 0 \leq n \leq m \) follow immediately from \((\phi)\). Hence the subscheme of \( \mathbb{C}^{n+1} \cap D(x_{0}^{(k_{n_{1}})}) \) defined by \( I_{\kappa}^{*} \) is isomorphic to the product of \( \mathbb{C}^{n} \) by an affine space, so it is reduced and irreducible and \( I_{\kappa} = I_{\kappa}^{*} \) is a prime ideal in \( \mathbb{C}[x_{0}^{(0)}, \ldots, x_{0}^{(m)}, \ldots, x_{g}^{(0)}, \ldots, x_{g}^{(m)}]_{x_{0}^{(k_{n_{1}})}} \), generated by a regular sequence, i.e the proposition holds for \( i + 1 \).

The case \( i = j - 1 = g \) and \( m \geq \kappa n_{g} \bar{\beta}_{l} \) follows by similar arguments. \( \square \)

As an immediate consequence we get

**Proposition 4.7.** — Let \( C \) be a plane branch with \( g \) Puiseux exponents. Let \( k \neq 0, j \) and \( \kappa \) as above. For \( m \geq \kappa n_{1} \bar{\beta}_{1} \), let \( \pi_{m,k_{n_{1}} \bar{\beta}_{1}} : C_{m} \rightarrow C_{\kappa n_{1} \bar{\beta}_{1}} \) be the canonical projection and let \( C_{m}^{k} := \pi_{m,k_{n_{1}} \bar{\beta}_{1}}^{-1}(D(x_{0}(k_{n_{1}})) \cap V(x_{0}(0), \ldots, x_{0}(k_{n_{1}})))_{\text{red}} \). Then for \( 1 \leq i < j \leq g \) (resp.\( 1 \leq i < j - 1 = g \)) and \( \kappa n_{i} \cdots n_{j-1} \bar{\beta}_{i} \leq m < \kappa n_{i+1} \cdots n_{j-1} \bar{\beta}_{i+1} \), \( C_{m}^{k} \) is irreducible of codimension

\[
\frac{\kappa}{n_{j} \cdots n_{g}}(\bar{\beta}_{0} + \bar{\beta}_{1} + \sum_{l=1}^{i-1}(\bar{\beta}_{l+1} - n_{i} \bar{\beta}_{l})) + ([\frac{m}{n_{i+1} \cdots n_{g}}] - \frac{\kappa n_{i} \bar{\beta}_{i}}{n_{j} \cdots n_{g}}) + 1
\]

in \( \mathbb{C}_{m}^{n} \). (We suppose that the sum in the formula is equal to 0 when \( i = 1 \).) For \( j \leq g \) and \( m \geq \kappa \bar{\beta}_{j} \) (resp.\( j = g + 1 \) and \( m \geq \kappa n_{g} \bar{\beta}_{g} \)),

\[ C_{m}^{k} = \emptyset \]
(resp. \( C_m^k \) is of codimension
\[
\kappa(\bar{\beta}_0 + \bar{\beta}_1 + \sum_{l=1}^{g-1} (\bar{\beta}_{l+1} - n_l \bar{\beta}_l)) + m - \kappa n g \bar{\beta}_g + 1
\]
in \( C_m^2 \).

The referee kindly pointed out that for \( m \in \mathbb{N} \) such that \( \kappa n_i \cdots n_{j-1} \bar{\beta}_i \leq m < \kappa n_{i+1} \cdots n_{j-1} \bar{\beta}_{i+1} \), the codimension of \( C_m^k \) can also be written as:
\[
\frac{\kappa}{e_{j-1}} (\bar{\beta}_0 + \bar{\beta}_{i+1} - \bar{\beta}_{i+1}) + \frac{m}{e_i} + 1.
\]

For \( k' \geq k \) and \( m \geq k' n_1 \bar{\beta}_1 \), we now compare \( \text{codim}(C_m^k, C_m^2) \) and \( \text{codim}(C_m^{k'}, C_m^2) \).

**Corollary 4.8.** — For \( k' \geq k \geq 1 \) and \( m \geq k' n_1 \bar{\beta}_1 \), if \( C_m^k \) and \( C_m^{k'} \) are nonempty, we have
\[
\text{codim}(C_m^{k'}, C_m^2) \leq \text{codim}(C_m^k, C_m^2).
\]

**Proof.** — Let \( \gamma^k : [kn_1 \bar{\beta}_1, \infty[ \to [k(n_1 + m_1), \infty[ \) be the piecewise linear function given by
\[
\gamma^k(m) = \frac{k}{e_1} (\bar{\beta}_0 + \bar{\beta}_1 + \sum_{l=1}^{i-1} (\bar{\beta}_{l+1} - n_l \bar{\beta}_l)) + \left( \frac{m}{e_i} - \frac{kn_i \bar{\beta}_i}{e_1} \right) + 1
\]
for \( 1 \leq i \leq g \) and \( \frac{k \bar{\beta}_1}{n_2 \cdots n_{i-1}} \leq m < \frac{k \bar{\beta}_{i+1}}{n_2 \cdots n_i} \). (Recall that by convention \( \bar{\beta}_{g+1} = \infty \)).

In view of proposition 4.7, we have that \( \text{codim}(C_m^k, C_m^2) = [\gamma^k(m)] \) for \( k \equiv 0 \mod n_2 \cdots n_{j-1} \) and \( k \not\equiv 0 \mod n_2 \cdots n_j \) with \( 2 \leq j \leq g \) and any integer \( m \in [kn_1 \bar{\beta}_1, \frac{k \bar{\beta}_{i+1}}{n_2 \cdots n_{j-1}}] \) or for \( k \equiv 0 \mod n_2 \cdots n_g \) and any integer \( m \geq kn_1 \bar{\beta}_1 \). Similarly we define \( \gamma^{k'} : [k'n_1 \bar{\beta}_1, \infty[ \to [k'(n_1 + m_1), \infty[ \) by changing \( k \) to \( k' \).

Let \( \Gamma^k (\text{resp.} \Gamma^{k'}) \) be the graph of \( \gamma^k (\text{resp.} \gamma^{k'}) \) in \( \mathbb{R}^2 \). Now let \( \tau : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( \tau(a, b) = (a, b - 1) \) and let \( \lambda^{k'/k} : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( \lambda^{k'/k}(a, b) = \frac{k'}{k} (a, b) \). We note that \( \tau(\Gamma^{k'}) = \lambda^{k'/k}(\tau(\Gamma^k)) \); we also note that the endpoints of \( \tau(\Gamma^k) \) and \( \tau(\Gamma^{k'}) \) lie on the line through 0 with slope \( \frac{\beta_0 + \beta_1}{e_1 n_1 \bar{\beta}_1} = \frac{1}{e_1} \frac{n_1 + m_1}{n_1 m_1} < \frac{1}{e_1} \). Since \( \frac{k'}{k} \geq 1 \), the image of \( \tau(\Gamma^k) \) by \( \lambda^{k'/k} \) lies in the interior subset of \( \mathbb{R}^2_{\geq 0} \) whith boundary the union of \( \tau(\Gamma^k) \), of the segment joining its endpoint \([kn_1 \bar{\beta}_1, \frac{k}{k} (\bar{\beta}_0 + \bar{\beta}_1)]\) to \([kn_1 \bar{\beta}_1, 0)\) and of \([kn_1 \bar{\beta}_1, \infty[ \times 0 \). This implies that \( \gamma^{k'}(m) \leq \gamma^k(m) \) for \( m \geq k' n_1 \bar{\beta}_1 \), hence \( [\gamma^{k'}(m)] \leq [\gamma^k(m)] \) and the claim. \( \square \)
Theorem 4.9. — Let $C$ be a plane branch with $g \geq 2$ Puiseux exponents. Let $m \in \mathbb{N}$. For $1 \leq m < n_1 \beta_1 + e_1, C_m^0 = \text{Cont}^{>0}(x_0)_m$ is irreducible. For $qn_1 \beta_1 + e_1 \leq m < (q + 1)n_1 \beta_1 + e_1$, with $q \geq 1$ in $\mathbb{N}$, the irreducible components of $C_m^0$ are:

$$C_{m\kappa} = \overline{\text{Cont}^{\kappa\beta_0}(x_0)_m}$$

for $1 \leq \kappa$ and $\kappa\beta_0 \beta_1 + e_1 \leq m$,

$$C^{j}_{m\kappa v} = \overline{\text{Cont}^{\kappa\beta_0}(x_0)_m}$$

for $j = 2, \ldots, g$, $1 \leq \kappa$ and $\kappa \not\equiv 0 \mod n_j$ and such that $\kappa n_1 \cdots n_{j-1} \beta_1 + e_1 \leq m < \kappa \beta_j$,

$$B_m = \text{Cont}^{>n_1 q}(x_0)_m.$$ 

Proof. — We first observe that for any integer $k \neq 0$ and any $m \geq kn_1 \beta_1$,

$$(C^0_m)^{\text{red}} = \bigcup_{1 \leq h \leq \kappa} C^h_m \cup \text{Cont}^{>kn_1}(x_0)_m$$

where $C^h_m := \text{Cont}^{hn_1}(x_0)_m$. Indeed, for $k = 1$, we have that $(C^0_m)^{\text{red}} \subseteq V(x_0^{(0)}, \ldots, x_0^{(n_1 - 1)})$ by proposition 4.1. Arguing by induction on $k$, we may assume that the claim holds for $m \geq (k - 1)n_1 \beta_1$. Now by corollary 4.2, we know that for $m \geq kn_1 \beta_1$, $\text{Cont}^{(k-1)n_1}(x_0)_m \subseteq V(x_0^{(0)}, \ldots, x_0^{(kn_1 - 1)})$, hence the claim for $m \geq kn_1 \beta_1$.

We thus get that for $qn_1 \beta_1 + e_1 \leq m < (q + 1)n_1 \beta_1 + e_1$,

$$(C^0_m)^{\text{red}} = \bigcup_{1 \leq k \leq q} C^k_m \cup \text{Cont}^{>qn_1}(x_0)_m.$$ 

By proposition 4.7, for $1 \leq k \leq q, C^k_m$ is either irreducible or empty. We first note that if $C^k_m \neq 0$, then $C^{k'}_m \not\subseteq \text{Cont}^{>qn_1}(x_0)_m$ Similarly, if $1 \leq k < k'$ and if $C^k_m$ and $C^{k'}_m$ are nonempty, then $C^k_m \not\subseteq C^{k'}_m$. On the other hand by corollary 4.8, we have that $\text{codim}(C^{k'}_m, C^2_m) \leq \text{codim}(C^k_m, C^2_m)$. So $\overline{C^k_m} \not\subseteq \overline{C^2_m}$. Finally we will show that $\text{Cont}^{>qn_1}(x_0)_m \not\subseteq \overline{C^k_m}$ if $C^k_m \neq \emptyset$ for $1 \leq k \leq q$. To do so, it is enough to check that $\text{codim}(C^k_m, C^2_m) \geq \text{codim}(\text{Cont}^{>qn_1}(x_0)_m, C^2_m)$. For $m \in[qn_1 \beta_1 + e_1, (q + 1)n_1 \beta_1]$, we have

$$\delta'(m) := \text{codim}(\text{Cont}^{>qn_1}(x_0)_m, C^2_m)$$

$$= 2 + q(n_1 + m_1) + \left[\frac{m - qn_1 \beta_1}{\beta_0}\right] + \left[\frac{m - qn_1 \beta_1}{\beta_1}\right]$$
by corollary 4.2. Let \( \lambda^q : [qn_1 \beta_1 + e_1] \rightarrow [q(n_1 + m_1), \infty] \) be the function given by \( \lambda^q(m) = q(n_1 + m_1) + \frac{q_1 \beta_1}{e_1} + 1 \). For simplicity, set \( i = m - qn_1 \beta_1 \).

For any integer \( i \) such that \( e_1 \leq n_1 \beta_1 = n_1 m_1 e_1 \), we have \( 1 + \left[ \frac{i}{n_1 e_1} \right] + \left[ \frac{i}{m_1 e_1} \right] \leq \left[ \frac{i}{e_1} \right] \). Indeed this is true for \( i = e_1 \) and it follows by induction on \( i \) from the fact that for any pair of integers \((b, a)\), we have \([\frac{b+1}{a}] = [\frac{b}{a}]\) if and only if \( b + 1 \equiv 0 \) mod \( a \) and \([\frac{b+1}{a}] = [\frac{b}{a}] + 1\) otherwise, since \( i < n_1 m_1 e_1 \). So \( \delta^q(m) \leq [\lambda^q(m)] \).

But in the proof of corollary 4.8, we have checked that if \( C^k_m \neq \emptyset \), then \( \text{codim}(C^k_m, C^2_m) = [\gamma^k(m)] \). We have also checked that for \( q \geq k \) and \( m \geq qn_1 \beta_1 \), \( \gamma^k(m) \geq \gamma^q(m) \).

Finally in view of the definitions of \( \gamma^q \) and \( \lambda^q \), we have \( \gamma^q(m) \geq \lambda^q(m) \), so \( [\gamma^q(m)] \geq [\lambda^q(m)] \geq \delta^q(m) \).

For \( m = (q + 1)n_1 \beta_1 \), we have \( \delta^q(m) = (q + 1)(n_1 + m_1) + 1 \) by corollary 4.2. For \( m \in [(q + 1)n_1 \beta_1, (q + 1)n_1 \beta_1 + e_1] \), we have

\[
\text{Cont}^{q(n_1)}(x_0)_m = C^{q+1}_m \cup \text{Cont}^{(q+1)n_1}(x_0)_m
\]

and

\[
\text{Cont}^{(q+1)n_1}(x_0)_m = V(x_0^{(0)}, \ldots, x_0^{(q+1)n_1}, x_1^{(0)}, \ldots, x_1^{((q+1)m_1)})
\]

again by corollary 4.2. If in addition we have \( m < (q + 1)\beta_2 \), then by proposition 4.5 \( C^{q+1}_m = V(x_0^{(0)}, \ldots, x_0^{((q+1)n_1-1)}, x_1^{((q+1)n_1-1)} - c_1 x_0^{((q+1)n_1)} \cap D(x_0^{(q+1)n_1}) \), thus we have \( \text{Cont}^{q(n_1)}(x_0)_m = C^{q+1}_m \) and \( \delta^q(m) = (q + 1)(n_1 + m_1) + 1 \). We have \((q + 1)n_1 \beta_1 + e_1 \leq (q + 1)\beta_2\) if \( q + 1 \geq n_2 \), because \( \beta_2 - n_1 \beta_1 \equiv 0 \) mod \((e_2)\). If not, we may have \((q + 1)\beta_2 < (q + 1)n_1 \beta_1 + e_1\), so for \((q + 1)\beta_2 \leq m < (q + 1)n_1 \beta_1 + e_1\), we have \( C^{q+1}_m = \emptyset \), \( \text{Cont}^{q(n_1)}(x_0)_m = \text{Cont}^{(q+1)n_1}(x_0)_m \) and \( \delta^q(m) = (q + 1)(n_1 + m_1) + 2 \).

In both cases, for \( m \in [(q + 1)n_1 \beta_1, (q + 1)n_1 \beta_1 + e_1] \), we have \( \delta^q(m) \leq (q + 1)(n_1 + m_1 + 1) \). Since \([\lambda^q(m)] = q(n_1 + m_1) + n_1 m_1 + 1\), we conclude that \([\lambda^q(m)] = \delta^q(m)\), so for \( 1 \leq k \leq q \), if \( C^k_m \neq \emptyset \), we have \([\gamma^k(m)] = \delta^q(m)\). This proves that the irreducible components of \( C^0_m \) are the \( C^k_m \) for \( 1 \leq k \leq q \) and \( C^k_m \neq \emptyset \), and \( \text{Cont}^{q(n_1)}(x_0)_m \), hence the claim in view of the characterization of the nonempty \( C^k_m \)'s given in proposition 4.5. $$\square$$

**Corollary 4.10.** — Under the assumption of theorem 4.9, let \( q_0 + 1 = \min\{\alpha \in \mathbb{N}; \alpha(\beta_2 - n_1 \beta_1) \geq e_1\} \). Then \( 0 \leq q_0 < n_2 \). For \( 1 \leq m < (q_0 + 1)n_1 \beta_1 + e_1 \), \( C^0_m \) is irreducible and we have \( \text{codim}(C^0_m, C^2_m) = 2 + [\frac{m}{\beta_0}] + [\frac{m}{\beta_1}] \) for \( 0 \leq q < q_0 \) and \( qn_1 \beta_1 + e_1 \leq m < (q + 1)n_1 \beta_1 \) or \( 0 \leq q < q_0 \) and \((q + 1)\beta_2 \leq m < (q + 1)n_1 \beta_1 + e_1\).
$$1 + \left[\frac{m}{\beta_0}\right] + \left[\frac{m}{\beta_1}\right] \text{ for } 0 \leq q < q_0 \text{ and } (q+1)n_1\beta_1 \leq m < (q+1)\beta_2$$

or \( (q_0+1)n_1\beta_1 \leq m < (q_0+1)n_1\beta_1 + e_1 \).

For \( q \geq q_0 + 1 \) in \( \mathbb{N} \) and \( qn_1\beta_1 + e_1 \leq m < (q+1)n_1\beta_1 + e_1 \), the number of irreducible components of \( C^0_m \) is:

$$N(m) = q + 1 - \sum_{j=2}^{g} \left( \left\lfloor \frac{m}{\beta_j} \right\rfloor - \left\lfloor \frac{m}{n_j\beta_j} \right\rfloor \right)$$

and \( \text{codim}(C^0_m, C^2_m) = 2 + \left[\frac{m}{\beta_0}\right] + \left[\frac{m}{\beta_1}\right] \text{ for } qn_1\beta_1 + e_1 \leq m < (q+1)n_1\beta_1 \).

$$1 + \left[\frac{m}{\beta_0}\right] + \left[\frac{m}{\beta_1}\right] \text{ for } (q+1)n_1\beta_1 \leq m < (q+1)n_1\beta_1 + e_1.$$
Finally, assume that $qn_1\beta_1 + e_1 \leq m < (q+1)n_1\beta_1 + e_1$ with $q \geq 1$ and $q \leq q_0$. Since $q_0 < n_2$, for $1 \leq k \leq q$ we have $k \equiv 0 \mod(n_2)$ and $m \geq qn_1\beta_1 + e_1 > q\beta_2$, hence for $1 \leq k \leq q$, $C_m^k = \emptyset$ and $C_m^0 = \text{Cont}^{qn_1}(x_0)m$ is irreducible. (The case $q = q_0$ was already known). So for $n_1\beta_1 \leq m < (q_0+1)n_1\beta_1 + e_1$, $C_m^0$ is irreducible. (Recall that for $1 \leq m < q_0n_1\beta_1 + e_1$, the irreducibility of $C_m^0$ is already known). It only remains to check the codimensions of $C_m^0$ for $1 \leq m \leq q_0n_1\beta_1 + e_1$. Here again we have seen in the proof of Theorem 4.9 that $\text{codim}(C_m^0, C_m^2) = \text{codim}(\text{Cont}^{qn_1}(x_0)m, C_m^2) =: \delta^q(m)$ for any $q \geq 1$ and $qn_1\beta_1 + e_1 \leq m < (q+1)n_1\beta_1$ and that

$$
\delta^q(m) = 2 + \left[ \frac{m}{\beta_0} \right] + \left[ \frac{m}{\beta_1} \right] \text{ for any } q \geq 1 \text{ and } qn_1\beta_1 + e_1 \leq m < (q+1)n_1\beta_1
$$

$$(q+1)(n_1 + m_1) + 1 =
1 + \left[ \frac{m}{\beta_0} \right] + \left[ \frac{m}{\beta_1} \right] \text{ for } q < q_0 \text{ and } (q+1)n_1\beta_1 \leq m < (q+1)\beta_2
$$

$$(q+1)(n_1 + m_1) + 2 =
2 + \left[ \frac{m}{\beta_0} \right] + \left[ \frac{m}{\beta_1} \right] \text{ for } q < q_0 \text{ and } (q+1)\beta_2 \leq m < (q+1)n_1\beta_1 + e_1.
$$

This completes the proof.

In [6], Igusa has shown that the log-canonical threshold of the pair $((\mathbb{C}^2, 0), (C, 0))$ is $\frac{1}{\beta_0} + \frac{1}{\beta_1}$. Here $((\mathbb{C}^2, 0), (C, 0))$ is the formal neighborhood of $\mathbb{C}^2$ (resp. $C$) at 0. Corollary 4.10 allows to recover Corollary B of [2] in this special case.

**Corollary 4.11.** — If the plane curve $C$ has a branch at 0, with multiplicity $\beta_0$, and first Puiseux exponent $\beta_1$, then

$$
\min_m \frac{\text{codim}(C_m^0, C_m^2)}{m + 1} = \frac{1}{\beta_0} + \frac{1}{\beta_1}.
$$

**Proof.** — For any $m, p \neq 0$ in $\mathbb{N}$, we have $m - p \left[ \frac{m}{p} \right] \leq p - 1$ and $m - p \left[ \frac{m}{p} \right] = p - 1$ if and only if $m + 1 \equiv 0 \mod(p)$; so for any $m \in \mathbb{N}$, $2 + \left[ \frac{m}{\beta_0} \right] + \left[ \frac{m}{\beta_1} \right] \geq (m + 1)(\frac{1}{\beta_0} + \frac{1}{\beta_1})$ and we have equality if and only if $m + 1 \equiv 0 \mod(\beta_0)$ and mod $\beta_1$ or equivalently $m + 1 \equiv 0 \mod(n_1\beta_1)$ since $n_1\beta_1$ is the least common multiple of $\beta_0$ and $\beta_1$. If not we have $1 + \left[ \frac{m}{\beta_0} \right] + \left[ \frac{m}{\beta_1} \right] \geq (m + 1)(\frac{1}{\beta_0} + \frac{1}{\beta_1})$. Now if $(q+1)n_1\beta_1 \leq m < (q+1)n_1\beta_1 + e_1$ with $q \in \mathbb{N}$, we have $(q+1)n_1\beta_1 < m + 1 \leq (q+1)n_1\beta_1 + e_1 < (q+2)n_1\beta_1$, so $m + 1 \neq 0 \mod(n_1\beta_1)$. If $(q+1)n_1\beta_1 \leq m < (q+1)\beta_2$ with $q \in \mathbb{N}$ and $q < q_0$, then $(q+1)n_1\beta_1 < m + 1 \leq (q+1)n_1\beta_1 + e_1 < (q+2)n_1\beta_1$, so $m + 1 \neq 0 \mod
(n_1 \bar{\beta}_1). So in both cases, we have 1 + \left[ \frac{m}{\beta_0} \right] + \left[ \frac{m}{\beta_1} \right] \geq (m + 1)(\frac{1}{\beta_0} + \frac{1}{\beta_1}). The claim follows from corollary 4.10. \hfill \Box

It also follows immediately from corollary 4.10.

**Corollary 4.12.** — Let \( q_0 \in \mathbb{N} \) as in corollary 4.10. There exists \( n_1 \bar{\beta}_1 \) linear functions, \( L_0, \cdots, L_{n_1 \bar{\beta}_1 - 1} \) such that \( \dim(C^0_m) = L_i(m) \) for any \( m \equiv i \mod (n_1 \bar{\beta}_1) \) such that \( m \geq q_0 n_1 \bar{\beta}_1 + e_1 \).

The canonical projections \( \pi_{m+1,m} : C^0_{m+1} \to C^0_m \), \( m \geq 1 \), induce infinite inverse systems

\[ \cdots B_{m+1} \to B_m \cdots \to B_1 \]

\[ \cdots C_{m+1,\kappa I} \to C_{m\kappa I} \cdots \to C_{(\kappa \bar{\beta}_0 \bar{\beta}_1 + e_1) \kappa I} \to B_{\kappa \bar{\beta}_0 \bar{\beta}_1 + e_1 - 1} \]

and finite inverse systems

\[ C_j^{(\kappa \bar{\beta}_1 - 1) \kappa v} \to C_{m\kappa I} \cdots \to C_j^{(\kappa n - n_j - 1 \bar{\beta}_1 + e_1) \kappa v} \to B_{\kappa n - n_j - 1 \bar{\beta}_1 + e_1 - 1} \]

for \( 2 \leq j \leq g \), and \( \kappa \neq 0 \mod (n_j) \).

We get a tree \( T_{C,0} \) by representing each irreducible component of \( C^0_m \), \( m \geq 1 \), by a vertex \( v_{i,m}, 1 \leq i \leq N(m) \), and by joining the vertices \( v_{i,m+1} \) and \( v_{i,0,m} \) if \( \pi_{m+1,m} \) induces one of the above maps between the corresponding irreducible components.

This tree only depends on the semigroup \( \Gamma \).

Conversely, we recover \( \bar{\beta}_0, \cdots, \bar{\beta}_g \) from this tree and \( \max\{m, \text{codim}(B_m, C^2_m) = 2\} = \bar{\beta}_0 - 1 \). Indeed the number of edges joining two vertices from which an infinite branch of the tree starts is \( \beta_0 \bar{\beta}_1 \). We thus recover \( \bar{\beta}_1 \) and \( e_1 \). We recover \( \bar{\beta}_2 - n_1 \bar{\beta}_1, \cdots, \bar{\beta}_j - n_1 \cdots n_{j-1} \bar{\beta}_1, \cdots, \bar{\beta}_g - n_1 \cdots n_{g-1} \bar{\beta}_1 \), hence \( \bar{\beta}_2, \cdots, \bar{\beta}_g \) from the number of edges in the finite branches.

**Corollary 4.13.** — Let \( C \) be a plane branch with \( g \geq 1 \) Puiseux exponents. The tree \( T_{C,0} \) described above and \( \max\{m, \text{dim } C^0_m = 2m\} \) determines the sequence \( \bar{\beta}_0, \cdots, \bar{\beta}_g \) or equivalently the equisingularity class of \( C \) and conversely.

We represent below the tree for the branch defined by

\[ f(x,y) = (y^2 - x^3)^2 - 4x^6y - x^9 = 0, \]

whose semigroup is \( < \bar{\beta}_0 = 4, \bar{\beta}_1 = 6, \bar{\beta}_2 = 15 > \), and for which we have \( e_1 = 2, e_2 = 1 \) and \( n_1 = n_2 = 2 \).
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