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THE CHOW RING OF THE STACK OF CYCLIC COVERS OF THE PROJECTIVE LINE

by Damiano FULGHESU & Filippo VIVIANI (*)

Abstract. — In this paper we compute the integral Chow ring of the stack of smooth uniform cyclic covers of the projective line and we give explicit generators.

Résumé. — Dans ce travail nous calculons l'anneau d'intersection avec des coefficients entiers du champ des revêtements cycliques lisses et uniformes de la droite projective. Nous explicitons aussi tous les générateurs.

1. Introduction

The study of moduli spaces of curves has been greatly enhanced by the introduction of algebraic stacks in the work of Deligne and Mumford [4]. The moduli stack $\mathcal{M}_g$ of curves of genus g and its compactification $\overline{\mathcal{M}}_g$ via stable curves are of Deligne-Mumford type. More precisely, they admits a stratification into locally closed substacks that are quotients of a scheme by an algebraic group acting with finite stabilizers. By using this local structure, Mumford laid down the basis of enumerative computations in [11]. Unfortunately, very little is known in general about the integral Chow ring $A^*(\mathcal{M}_g)$. The intersection rings of $\mathcal{M}_{1,1}$ and $\mathcal{M}_2$ are computed in [6] and [15]. Even with rational coefficients the ring $A^*(\mathcal{M}_g)$ is known only up to $g = 5$ (see [7] and [9]).

In this paper, we give a complete answer for the stack $\mathcal{H}_{\rm sm}(1, r, d)$ of smooth uniform μ_r-cyclic covers of the projective line, whose branch divisor has degree $N = rd$. In particular, when $r = 2$, $d \geqslant 3$, and the characteristic of the base field k is different from 2, we get the closed substack $\mathcal{H}_g$ of

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$\mathcal{M}_g$, whose geometric points are the hyperelliptic curves. The main result we use in order to compute the Chow ring $A^*(\mathcal{H}_{\mathrm{sm}}(1,r,d))$, is the explicit structure of global quotient stack given by Arsie and Vistoli in [2]

$$\mathcal{H}_{\mathrm{sm}}(1,r,d) = \left[(\mathrm{Sym}^N(V^*) \smallsetminus \Delta)/G\right],$$

where V is the standard representation of GL_2 (so that $\mathrm{Sym}^N(V^*)$ is the vector space of homogeneous binary forms of degree N), Δ is the discriminant (*i.e.* the closed subset of binary forms with at least one multiple root), $G = GL_2$ when d is odd, and $G = \mathbb{G}_m \times PGL_2$ when d is even. We are therefore able to apply equivariant intersection theory developed by Edidin-Graham [6] and Totaro [13], getting that

$$A^*\left(\mathcal{H}_{\mathrm{sm}}(1,r,d)\right) = A^*_G(\mathrm{Sym}^N(V^*) \smallsetminus \Delta).$$

The first step of our computation is to pass to the projectivization (see Section 3). We reduce to the computation of $A^*_G\left(\mathbb{P}\left(\mathrm{Sym}^N(V^*) \smallsetminus \Delta\right)\right)$ after showing that

$$A^*_G(\mathrm{Sym}^N(V^*) \smallsetminus \Delta) = A^*_G\left(\mathbb{P}\left(\mathrm{Sym}^N(V^*) \smallsetminus \Delta\right)\right)/(t - c_1(\mathcal{D})),$$

where $\mathcal{D}$ is a one dimensional representation of G and t is the hyperplane class. Afterward we consider the exact sequence

$$\begin{aligned} A^G_*(\mathbb{P}(\Delta)) \xrightarrow{i_*} A^*_G(\mathbb{P}(\mathrm{Sym}^N(V^*))) \cong A^*_G[t]/p_N(t) \\ \longrightarrow A^*_G\left(\mathbb{P}\left(\mathrm{Sym}^N(V^*) \smallsetminus \Delta\right)\right) \end{aligned}$$

to get (see 3.4)

$$A^*(\mathcal{H}_{\mathrm{sm}}(1,r,d)) = \frac{A^*_G[t]}{\langle t - c_1(\mathcal{D}), p_N(c_1(\mathcal{D})), \mathrm{Im}(i_*)\rangle}. \tag{1.1}$$

where $p_N(t)$ is a monic polynomial in the hyperplane class t of degree $N+1$ with coefficients in A^*_G. Then we have two cases.

(1) When d is odd, we follow [5] to show that $(\mathrm{Im}(i_*))$ is generated by two elements and that the class $p_N(c_1(\mathcal{D}))$ belongs to the ideal $(\mathrm{Im}(i_*), t - c_1(\mathcal{D}))$. If $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > N$, we get (Theorem 4.2)

$$A^*(\mathcal{H}_{\mathrm{sm}}(1,r,d)) = \frac{\mathbb{Z}[c_1, c_2]}{\langle r(N-1)c_1, \frac{(N-r)(N-r-2)}{4}c_1^2 - N(N-2)c_2\rangle}.$$

where c_1 and c_2 are (the pull-back of) the Chern classes of the standard representation V of GL_2.

(2) When d is even, the group G is $\mathbb{G}_m \times PGL_2$ and A^*_G is $\mathbb{Z}[c_1, c_2, c_3]/ < 2c_3 >$. Since $\mathbb{G}_m$ acts trivially on $\mathbb{P}(\mathrm{Sym}^N(V^*))$, we can consider only the action of PGL_2. In Proposition 5.3, we show that $(\mathrm{Im}(i_*))$ has two generators (which we compute in Proposition 5.2). It is enough to prove the statement first with coefficients in $\mathbb{Z}[1/2]$ (5.2), then in the localization $\mathbb{Z}_{(2)}$ (5.2). The last step (see Section 6) is to check if the class $p_N(c_1(\mathcal{D}))$ belongs to the ideal $I = (\mathrm{Im}(i_*), t - c_1(\mathcal{D}))$. The answer depends on r. More precisely $p_N(c_1(\mathcal{D})) \in I$ if and only if r is even. We compute the ring $A^*(\mathcal{H}_{\mathrm{sm}}(1, r, d))$ in Theorem 5.7. If $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > N$, we have

$$A^*(\mathcal{H}_{\mathrm{sm}}(1, r, d)) = \frac{\mathbb{Z}[c_1, c_2, c_3]}{\left\langle p_N(-rc_1), 2c_3, 2r(N-1)c_1, 2r^2c_1^2 - \frac{N(N-2)}{2}c_2 \right\rangle}$$

from where we can remove $p_N(-rc_1)$ if r is even.

Remark 1.1. — In particular, we get that the Picard group of $\mathcal{H}_{\mathrm{sm}}(1, r, d)$ is cyclic of order $r(N-1)$ if d is odd, and of order $2r(N-1)$ if d is even. Therefore, we recover the result of [2, Thm. 5.1].

In Section 7 we give an explicit description for the generators of the Chow ring $A^*(\mathcal{H}_{\mathrm{sm}}(1, r, d))$ as Chern classes of natural vector bundles (see Theorem 7.1 and 7.2). Moreover (Proposition 7.4), we express the tautological λ classes of Mumford ([11]) in terms of the explicit generators. In the hyperelliptic case we recover the result of [8].

Recently, Bolognesi and Vistoli ([3]) have described the stack $\mathcal{T}_g$ of trigonal curves of genus g as a quotient stack and have used this explicit presentation to compute the integral Picard group of $\mathcal{T}_g$. It is likely that some of techniques of this note could be adapted to compute the integral Chow ring of $\mathcal{T}_g$.

2. Notations

Throughout this paper we fix two positive integers r and d, and we let $N = rd$. We work over a field k in which r is invertible.

Let us review the description of the stack $\mathcal{H}_{\mathrm{sm}}(1, r, d)$ of smooth uniform μ_r-cyclic covers of the projective line with branch divisor of degree $N = rd$, following Arsie and Vistoli (see [2]). We need first to recall the definition of a smooth uniform cyclic cover.

Definition 2.1 ([2], Def. 2.1, 2.4).

(1) *A uniform μ_r-cyclic cover of a scheme Y consists of a morphism of schemes $f\colon X \to Y$ together with an action of the group scheme μ_r on X, such that for each point $q \in Y$, there is an affine neighborhood $V = \operatorname{Spec} R$ of $q \in Y$, together with an element $h \in R$ that is not a zero divisor, and an isomorphism of V-schemes $f^{-1}(V) \cong \operatorname{Spec} R[x]/(x^r - h)$ which is μ_r-equivariant, when the right hand side is given the obvious actions. The branch divisor Δ_f of f is the Cartier divisor on Y whose restriction to* $\operatorname{Spec}(R)$ *has local equation $\{h = 0\}$.*

(2) *A uniform μ_r-cyclic cover $f\colon X \to Y$ is said to be smooth if Y and Δ_f are smooth or, equivalently, if Y and X are smooth.*

The above definition admits a relative version.

Definition 2.2 ([2], Def. 2.3, 2.4).

(1) *Let $Y \to S$ be a morphism of schemes. A relative uniform μ_r-cyclic cover of $Y \to S$ is a uniform μ_r-cyclic cover $f\colon X \to Y$ such that the branch divisor Δ_f is flat over S.*

(2) *A relative uniform μ_r-cyclic cover $f\colon X \to Y$ of $Y \to S$ is said to be smooth if Y and Δ_f are smooth over S or, equivalently, if Y and X are smooth over S.*

Finally, we need to recall the definition of a Brauer-Severi scheme.

Definition 2.3. — *Let S be a scheme. A Brauer-Severi scheme of relative dimension n over S is a smooth morphism $P \to S$ whose geometric fibers are isomorphic to the projective space of dimension n.*

We can now define $\mathcal{H}_{\mathrm{sm}}(1, r, d)$ as a category fibered in groupoids over the category of k-schemes.

Definition 2.4. — *We denote by $\mathcal{H}_{\mathrm{sm}}(1, r, d)$ the category fibered in groupoids over the category of k-schemes, defined as follows.*

An object of $\mathcal{H}_{\mathrm{sm}}(1, r, d)(S)$ over S is a smooth relative uniform μ_r-cyclic cover $X \xrightarrow{f} P \to S$ over a Brauer-Severi scheme $P \to S$ of relative dimension one such that the branch divisor Δ_f has relative degree $N = rd$ over S.

A morphism from $(X \xrightarrow{f} P \to S)$ to $(X' \xrightarrow{f'} P' \to S')$ is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & S \arrow[d] \\ X' \arrow[r] & P' \arrow[r] & S' \end{tikzcd}$$

where both squares are cartesian and the left hand column is μ_r-equivariant.

In [2], the authors provide an explicit description of $\mathcal{H}_{\mathrm{sm}}(1,r,d)$ as a quotient stack.

Theorem 2.5 ([2], Thm. 4.1). — *The category $\mathcal{H}_{\mathrm{sm}}(1,r,d)$ is isomorphic to the quotient stack*

$$[\mathbb{A}_{sm}(N)/(GL_2/\mu_d)],$$

where $\mathbb{A}_{sm}(N)$ is the space of degree N smooth (that is, with distinct roots in the algebraic closure of k) binary forms. The group of d-th roots of unity μ_d is embedded diagonally into GL_2 and the action of GL_2/μ_d on $\mathbb{A}_{sm}(N)$ is given by $[A]\cdot f(x) = f(A^{-1}\cdot x)$.

In particular, it follows from the above result that $\mathcal{H}_{\mathrm{sm}}(1,r,d)$ is an irreducible smooth Deligne-Mumford stack of finite type over k of dimension $rd-3$. Analyzing the structure of the algebraic group GL_2/μ_d, one can rewrite the above isomorphism of stacks as follows.

Lemma 2.6 ([2], Cor. 4.6).

(i) *If d is odd, using the isomorphism $GL_2/\mu_d \to GL_2$ given by the map $[A]\mapsto \det(A)^{\frac{d-1}{2}}A$, the stack $\mathcal{H}_{\mathrm{sm}}(1,r,d)$ can be described as*

$$\mathcal{H}_{\mathrm{sm}}(1,r,d) = [\mathbb{A}_{sm}(N)/GL_2],$$

with the action given by $A\cdot f(x) = \det(A)^{\frac{r(d-1)}{2}} f(A^{-1}x)$.

(ii) *If d is even, using the isomorphism $GL_2/\mu_d \to \mathbb{G}_m \times PGL_2$ given by the map $[A] \mapsto (\det(A)^{\frac{d}{2}}, [A])$, the stack $\mathcal{H}_{\mathrm{sm}}(1,r,d)$ can be described as*

$$\mathcal{H}_{\mathrm{sm}}(1,r,d) = \left[\mathbb{A}_{sm}(N)/(\mathbb{G}_m\times PGL_2)\right],$$

with the action given by $(\alpha,[A])\cdot f(x) = \alpha^{-r}\det(A)^{\frac{rd}{2}} f(A^{-1}x)$.

For every smooth scheme X over k endowed with an action of an algebraic group H, we will consider the equivariant Chow ring $A^*_H(X)$ as an algebra over the Chow ring $A^*_H := A^*_H(\mathrm{Spec}(k))$ of the classifying stack BH of H, via pull-back along the structure map $X \to \mathrm{Spec}(k)$. We refer to [6] for the definitions and the basic properties of the equivariant Chow rings.

For the remainder of the paper, we set $G := GL_2/\mu_d$. In virtue of Theorem 2.5 and the results of [6, Sec. 4], the Chow ring $A^*(\mathcal{H}_{\mathrm{sm}}(1,r,d))$ is isomorphic to the A^*_G-algebra $A^*_G(\mathbb{A}_{sm}(N))$. Using Lemma 2.6, the Chow ring $A^*_G := A^*_G(\mathrm{Spec}(k))$ of the classifying space of $G = GL_2/\mu_d$ is given as follows (see for example [12]).

Lemma 2.7. — *The Chow ring A^*_G of the classifying stack of $G = GL_2/\mu_d$ is equal to*

$$A^*_G = \begin{cases} \mathbb{Z}[c_1, c_2] & \text{if } d \text{ is odd,} \\ \mathbb{Z}[c_1, c_2, c_3]/(2c_3) & \text{if } d \text{ is even.} \end{cases}$$

We can describe the classes appearing in the above Lemma as follows. If d is odd, then c_1, c_2 are the Chern classes of the standard representation V of GL_2. If d is even, then c_1 is the Chern class of the natural representation of $\mathbb{G}_m$ and c_2, c_3 are the Chern classes of the representation $\mathrm{Sym}^2(V)$ of $PGL_2 = SL_2/\mu_2$, where V is the two dimensional standard k-representation of SL_2 (the first Chern class being trivial).

One last piece of notation: Given a set S of elements of $A^*_H(X)$ (for some smooth scheme X acted upon by an algebraic group H), we denote by $\langle S \rangle$ the ideal generated by S inside the ring $A^*_H(X)$.

3. First reductions

3.1. Projectivization

The first step of our proof consists in passing to the projectivization, as in [15]. In order to do this, we consider the following G-equivariant diagram

$$\begin{tikzcd} \mathbb{A}_{sm}(N) \arrow[r, hook] \arrow[d] & \mathbb{A}(N) \smallsetminus \{0\} \arrow[d] \\ \mathbb{B}_{sm}(N) \arrow[r, hook] & \mathbb{B}(N) \end{tikzcd} \tag{3.1}$$

where $\mathbb{A}(N)$ is the vector space of binary form of degree N, $\mathbb{B}(N) := \mathbb{P}(\mathbb{A}(N) \smallsetminus \{0\})$ is its projectivization, and $\mathbb{B}_{sm}(N) := \mathbb{P}(\mathbb{A}_{sm}(N))$ is the open subset of smooth forms. The vertical arrows of the above diagram are principal $\mathbb{G}_m$-bundles associated to a G-equivariant line bundle $\mathcal{D} \otimes \mathcal{O}(-1)$, where $\mathcal{O}(-1)$ is the tautological bundle on $\mathbb{B}(N)$ and $\mathcal{D}$ is a one dimensional representation of G which, using Lemmas 2.6 and 2.7, has first Chern class in A^*_G given by

$$c_1(\mathcal{D}) = \begin{cases} \frac{r(d-1)}{2} c_1 = \frac{N-r}{2} c_1 & \text{if } d \text{ is odd,} \\ -r c_1 & \text{if } d \text{ is even.} \end{cases} \tag{3.2}$$

From the above diagram (3.1), we deduce the following exact diagram of A_G^*-algebras

(3.3)
$$\begin{tikzcd}
 & \langle t-c_1(\mathcal{D})\rangle \arrow[d, hook] & \langle t-c_1(\mathcal{D})\rangle \arrow[d, hook] \\
A_*^G(\mathbb{P}(\Delta)) \arrow[r, "i_*"] \arrow[d] & A_G^*(\mathbb{B}(N)) \cong A_G^*[t]/p_N(t) \arrow[r, two heads] \arrow[d] & A_G^*(\mathbb{B}_{sm}(N)) \arrow[d] \\
A_*^G(\Delta \smallsetminus \{0\}) \arrow[r] & A_G^*(\mathbb{A}(N) \smallsetminus \{0\}) \arrow[r, two heads] & A_G^*(\mathbb{A}_{sm}(N))
\end{tikzcd}$$

where $\Delta = \mathbb{A}(N)\setminus\mathbb{A}_{sm}(N)$ is the discriminant hypersurface of binary forms having at least two coincident roots over the algebraic closure of k, t is the first G-equivariant Chern class of $\mathcal{O}_{\mathbb{B}(N)}(1)$ and $p_N(t)$ is a monic polynomial in t of degree $N+1$ with coefficients in A_G^*, whose roots are the opposite of the equivariant Chern roots of the G-representation $\mathbb{A}(N)$. From the above diagram we deduce that

$$A^*(\mathcal{H}_{\mathrm{sm}}(1,r,d)) = \frac{A_G^*[t]}{\langle t-c_1(\mathcal{D}), p_N(c_1(\mathcal{D})), \mathrm{Im}(i_*)\rangle}. \tag{3.4}$$

3.2. Stratifying the discriminant

In order to compute the image of the i_*, we observe (following [15]) that the discriminant hypersurface Δ has a decreasing filtration into closed subsets

$$\Delta = \Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_{[N/2]} \supset \Delta_{[N/2]+1} = \emptyset,$$

where Δ_s is the closed subset of $\mathbb{B}(N)$ corresponding to forms divisible by the square of a polynomial of degree s over some extension of the base field k. There is a natural morphism

$$\pi_s \colon \mathbb{B}(s) \times \mathbb{B}(N-2s) \to \mathbb{B}(N),$$

which sends $([f],[g])$ into $[f^2g]$. The image of π_s is, by definition, the closed subset Δ_s. Arguing as in [15, Lemma 3.2, 3.3] (see also [5, Prop. 11]), we conclude that

Lemma 3.1. — *If* $\mathrm{char}(k) = 0$ *or* $\mathrm{char}(k) > N$*, then the ideal* $\langle \mathrm{Im}(i_*)\rangle$ *is generated by the images of the pushforward maps* $\pi_{s*} \colon A_G^*(\mathbb{B}(s)\times\mathbb{B}(N-2s)) \to A_G^*(\mathbb{B}(N))$ *where* $s = 1, \dots, [N/2]$.

4. d odd

In the case d odd, we have that $G = GL_2$ and therefore we are in the same situation as in [5, Section 4], where they proved the following

Proposition 4.1. — [5, Thm. 19] *If d is odd, then*

$$\langle \mathrm{Im}(\pi_{s*})\rangle_{s\geqslant 1} = \langle \mathrm{Im}(\pi_{1*})\rangle = \big\langle 2(N-1)t - N(N-1)c_1, t^2 - c_1 t - N(N-2)c_2\big\rangle,$$

where, as usual, $t := c_1^{GL_2}(\mathcal{O}_{\mathbb{B}(N)}(1)) \in A^*_{GL_2}(\mathbb{B}(N))$.

From this, we can deduce the following

Theorem 4.2. — *If d is odd and $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > N$, then*

$$A^*(\mathcal{H}_{\mathrm{sm}}(1,r,d)) = \frac{\mathbb{Z}[c_1, c_2]}{\langle r(N-1)c_1, \frac{(N-r)(N-r-2)}{4}c_1^2 - N(N-2)c_2\rangle}.$$

Proof. — The assertion will follow combining (3.2), (3.4), Lemma 3.1 and Proposition 4.1 once we prove that

$$p_N\left(\frac{N-r}{2}c_1\right) \in \left\langle r(N-1)c_1, \frac{(N-r)(N-r-2)}{4}c_1^2 - N(N-2)c_2\right\rangle.$$

Indeed, let V be the standard representation of GL_2 and let l_1, l_2 be its Chern roots. We have $c_1 = l_1 + l_2$ and $c_2 = l_1 l_2$. We can write more explicitly

$$\mathbb{B}(N) = \mathbb{P}\big(\mathrm{Sym}^N(V^*)\big).$$

Now the Chern roots of $\mathrm{Sym}^N(V^*)$ are

$$\{-(N-i)l_1 - il_2\}_{i=0,\dots,N}$$

therefore we have

$$p_N(t) = \prod_{i=0}^{N}\big(t - (N-i)l_1 - il_2\big)$$

We now multiply the factors of $p_N\left(\frac{N-r}{2}c_1\right)$ corresponding to $i = 0, 1, N-1, N$ and we get the polynomial

$$Q = \left(-\frac{c_1^2}{4}(N-r)(N+r) + N^2 c_2\right)\left(-\frac{c_1^2}{4}(N-r-2)(N+r-2) + (N-2)^2 c_2\right).$$

Let us write $f = r(N-1)c_1$ and $g = \frac{(N-r)(N-r-2)}{4}c_1^2 - N(N-2)c_2$. We now conclude as in [5, Lemma 21]

$$Q = f^2 c_2 + fgc_1 + g^2.$$

□

5. d even

In the case d even, we have that $G = \mathbb{G}_m \times PGL_2$. Since $\mathbb{G}_m$ acts trivially on the spaces $\mathbb{B}(r)$, we can (and we will) work with the equivariant cohomology with respect to PGL_2.

Observe that $\mathbb{B}(s)$ is the projectivization of a PGL_2-representation, namely $\mathbb{A}(s) := \mathrm{Sym}^s(V^*)$, if and only if s is even. In this case, the PGL_2-equivariant Chow ring of $\mathbb{B}(s)$ is equal to

$$A^*_{PGL_2}(\mathbb{B}(s)) = A^*_{PGL_2}[\xi_s]/(p_s(t)),$$

where ξ_s is the PGL_2-equivariant first Chern class of $\mathcal{O}_{\mathbb{P}^1}(1)$ and p_s is a monic polynomial in ξ_r of degree $s+1$ whose roots are the opposite of the Chern roots of the PGL_2-representation $\mathbb{A}(s)$. Finally observe that N is even since $N = rd$ and we are assuming that d is even. In this case, we set $t := \xi_N$.

5.1. Computing $\mathrm{Im}(\pi_{1*})$

In this subsection, we compute the image of the map π_{1*}. We need the following

Lemma 5.1. — *The equivariant Chow ring $A^*_{PGL_2}(\mathbb{P}^1)$, considered as an algebra over $A^*_{PGL_2} = \mathbb{Z}[c_2, c_3]/(2c_3)$ is equal to*

$$A^*_{PGL_2}(\mathbb{P}^1) = \mathbb{Z}[c_2, c_3, \tau]/(c_3, \tau^2 + c_2) = \mathbb{Z}[\tau],$$

where τ is the PGL_2-equivariant first Chern class of the PGL_2-linearized line bundle $\mathcal{O}_{\mathbb{P}^1}(2)$.

Proof. — Since PGL_2 acts transitively on $\mathbb{P}^1$ with stabilizer group $E := \mathbb{G}_a \rtimes \mathbb{G}_m$, we have that

$$A^*_{PGL_2}(\mathbb{P}^1) \xrightarrow{\cong} A^*_E.$$

According to [14, Prop. 2.7], we have that

$$A^*_E \xrightarrow{\cong} A^*_{\mathbb{G}_m} = \mathbb{Z}[\tau].$$

Since $\mathrm{Pic}(B\mathbb{G}_m) = A^1_{\mathbb{G}_m} = \mathbb{Z}\cdot \tau$ and

$$\mathrm{Pic}^{PGL_2}(\mathbb{P}^1) = A^1_{PGL_2}(\mathbb{P}^1) = \mathbb{Z}\cdot c_1^{PGL_2}(\mathcal{O}_{\mathbb{P}^1}(2))$$

(see [10, p. 32-33]), we can assume that $\tau = c_1^{PGL_2}(\mathcal{O}_{\mathbb{P}^1}(2))$. Finally, in order to understand the structure of $A^*_{PGL_2}$-algebra of $A^*_{PGL_2}(\mathbb{P}^1)$, we have to determine the natural pull-back map

$$A^*_{PGL_2} = \mathbb{Z}[c_2, c_3]/(2c_3) \to A^*_{PGL_2}(\mathbb{P}^1) = \mathbb{Z}[\tau].$$

Clearly c_3 goes to 0 since $A^*_{PGL_2}(\mathbb{P}^1)$ is torsion-free, while c_2 goes to $\alpha\tau^2$, where $\alpha \in \mathbb{Z}$.

In order to determine α, we consider the following commutative diagram

$$\begin{tikzcd}
A^*_{PGL_2} = \mathbb{Z}[c_2, c_3]/(2c_3) \arrow[r] \arrow[d] & A^*_{PGL_2}(\mathbb{P}^1) = \mathbb{Z}[\tau] \arrow[d] \\
A^*_{SL_2} = \mathbb{Z}[c_2] \arrow[r, hook] & A^*_{SL_2}(\mathbb{P}^1) = \mathbb{Z}[c_2, t]/(t^2 + c_2),
\end{tikzcd}$$

where $t := c_1^{SL_2}(\mathcal{O}_{\mathbb{P}^1}(1))$. The right vertical map clearly sends τ into $2t$, while the left vertical map sends $c_2 := c_2^{PGL_2}(\mathrm{Sym}^2(V)) \in A^*_{PGL_2}$ into $c_2^{SL_2}(\mathrm{Sym}^2 V) = 4c_2^{SL_2}(V) := 4c_2 \in A^*_{SL_2}$, where V is the standard two dimensional representation of SL_2. Therefore, following the image of the element $c_2 \in A^*_{PGL_2}$ in the above commutative diagram, we get the equality $4c_2 = 4\alpha t^2 = -4\alpha c_2$ in $A^*_{SL_2}(\mathbb{P}^1)$, from which we deduce that $\alpha = -1$. ☐

Proposition 5.2. — *The ideal generated by the image of the push-forward map π_{1*} is equal to*

$$\langle \mathrm{Im}(\pi_{1*})\rangle = \left\langle 2(N-1)t, 2t^2 - \frac{N(N-2)}{2}c_2 \right\rangle.$$

Proof. — By definition of the map π_1, we have that

$$\pi_1^*(t) = \tau + \xi_{N-2},$$

where, with an abuse of notation, we denote with τ and ξ_{N-2} the pull-back to $A^*_{PGL_2}(\mathbb{B}(1) \times \mathbb{B}(N-2))$ of the corresponding classes on the two factors. Therefore, by the projection formula together with the fact that $\tau^2 = -c_2$ (see the above Lemma 5.1), we get that $\mathrm{Im}(\pi_{1*})$ is generated, as an ideal of $A^*_{PGL_2}(\mathbb{B}(N))$, by $\pi_{1*}(1)$ and $\pi_{1*}(\tau)$.

In order to compute the above two elements $\pi_{1*}(1)$ and $\pi_{1*}(\tau)$, we will adapt the proof of [15] for the case $g = 2$. Consider the following PGL_2-equivariant commutative diagram

$$\begin{tikzcd}
\mathbb{B}(1) \times \mathbb{B}(1)^{N-2} \arrow[r, "\delta"] \arrow[d, "\sigma"] & \mathbb{B}(1)^N \arrow[d, "\rho"] \\
\mathbb{B}(1) \times \mathbb{B}(N-2) \arrow[r, "\pi_1"] & \mathbb{B}(N)
\end{tikzcd}$$

where ρ and σ are induced by multiplication of the forms and the map δ sends $([f_1],[f_2],\dots,[f_{N-1}])$ into $([f_1],[f_1],[f_2],[f_3],\dots,[f_{N-1}])$. The maps σ and ρ are finite and flat of degree $(N-2)!$ and $N!$ respectively, and therefore we have the formulas $\sigma_*\sigma^* = (N-2)!$ and $\rho_*\rho^* = N!$.

Since the elements $\pi_{1*}(1)$ and $\pi_{1*}(\tau)$ have degree 1 and 2 and the Chow ring $A^*_{PGL_2}(\mathbb{B}(N))$ has no torsion in degree 1 and 2, we can work with $\mathbb{Q}$-coefficients. Using the formula $\sigma_*\sigma^* = (N-2)!$, we get (for $i = 0, 1$)

$$\pi_{1*}(\tau^i) = \frac{1}{(N-2)!}\rho_*\delta_*\sigma^*(\tau^i). \tag{5.1}$$

Call τ_j the pullback to $\mathbb{B}(1)^N$ of the PGL_2-equivariant first Chern class of the line bundle $\mathcal{O}_{\mathbb{B}(1)}(2)$ on the j-th factor (for $j = 1,\dots,N$). By definition of the maps σ and δ, we have that $\sigma^*(\tau^i) = \delta^*(\tau_1^i) = \delta^*(\tau_2^i)$ for $i = 0, 1$. Hence, using the fact that $\delta_*(2) = \tau_1 + \tau_2$, we obtain

$$\begin{cases} \delta_*\sigma^*(2) = \delta_*(2) = \tau_1 + \tau_2, \\ \delta_*\sigma^*(2\tau) = \delta_*\delta^*(2\tau_1) = \tau_1(\tau_1 + \tau_2). \end{cases} \tag{5.2}$$

From the definition of the map ρ and the Lemma 5.1, we get

$$\begin{cases} \rho^*(1) = 1, \\ \rho^*(2t) = \sum \tau_j, \\ \rho^*(4t^2) = \left(\sum \tau_j\right)^2 = -Nc_2 + 2\sum\limits_{j<k} \tau_j\tau_k, \end{cases} \tag{5.3}$$

where the indices appearing in the above summations range from 1 to N. By taking the pushforward of the above equations (5.3) and using the formula $\rho_*\rho^* = N!$ and the fact that the pushforwards $\rho_*(\tau_j)$ and $\rho_*(\tau_j\tau_k)$ do not depend on the indices j and k (by symmetry of the map ρ), we get

$$\begin{cases} \rho_*(1) = N!, \\ \rho_*(\tau_j) = 2(N-1)!\,t, \\ \rho_*(\tau_j\tau_k) = (N-2)!\,[4t^2 + Nc_2]. \end{cases} \tag{5.4}$$

Putting together the formulas (5.1), (5.2), (5.4), we get the desired conclusion. □

5.2. Computing $\langle \pi_{s*} \rangle$

The aim of this subsection is to prove the following

Proposition 5.3. — *If d is even, we have that*

$$\langle \mathrm{Im}(\pi_{s*})\rangle_{s\geqslant 2} \subset \langle \mathrm{Im}(\pi_{1*})\rangle.$$

Note that it is enough to prove the assertion first with coefficients in the ring $\mathbb{Z}[1/2]$ obtained by inverting 2 and then in the ring $\mathbb{Z}_{(2)}$ obtained by localizing at 2. We will treat the two cases separately. We begin with some preliminary Lemmas. The first one generalizes [14, Prop. 2.3].

Lemma 5.4. — *Let X be a smooth scheme on which PGL_n acts and consider the induced action of SL_n via the quotient map $SL_n \twoheadrightarrow PGL_n$. Then the kernel of the natural pull-back map $A^*_{PGL_n}(X) \to A^*_{SL_n}(X)$ is of n-torsion.*

Proof. — Recall (see [6, Prop. 19]) that given a group G acting on a smooth scheme X, the equivariant Chow ring $A^*_G(X)$ is isomorphic to the operational Chow ring $A^*([X/G])$ of the quotient stack $[X/G]$. More precisely, any element $c \in A^i_G(X)$ defines a compatible system of operations $c(S \to [X/G])\colon A_*(S) \to A_{*-i}(S)$ for any map $S \to [X/G]$. If S is also smooth then such an operation is induced by the cup product with an element of $A^i(S)$ which, by abuse of notation, we denote also by $c(S \to [X/G]) \in A^i(S)$.

Let $p\colon [X/SL_n] \to [X/PGL_n]$ the natural map of quotient stacks and let $p^*\colon A^*([X/PGL_n]) \to A^*([X/SL_n])$ the pull-back map. Fix an integer i and an element $c \in A^i([X/PGL_n])$ such that $p^*c = 0$. We want to prove that $0 = n \cdot c \in A^i([X/PGL_n])$. It is enough to prove that for any map $\alpha\colon S \to [X/PGL_n]$ with S smooth, we have $nc(S \xrightarrow{\alpha} [X/PGL_n]) = 0$.

Indeed, let V be a representation of PGL_n and $U \subseteq V$ an open subset on which PGL_n acts freely and whose complement has codimension higher than i. Now if c is an assignment which is 0 on smooth varieties then c is 0 on the torsor $X \times U \to (X \times U)/PGL_n$. But by definition (see [6, Definition-Proposition 1])

$$A^i((X \times U)/PGL_n) = A^i([X/PGL_n])$$

so the class must be 0 in $A^i[X/PGL_n]$.

Now, the map $\alpha\colon S \to [X/PGL_n]$ corresponds to a PGL_n-torsor $P \to S$ over S together with a PGL_n-equivariant map $P \to X$. Let $f\colon \overline{P} \to S$ the Severi-Brauer scheme over S corresponding to the above PGL_n-torsor. The map $f\colon \overline{P} \to S$ is smooth with all the geometric fibers isomorphic to $\mathbb{P}^{n-1}$. Consider the following modified push-forward

$$\begin{aligned} f_{\#}\colon A^i(\overline{P}) &\longrightarrow A^i(S) \\ \alpha &\longmapsto f_*(\alpha \cdot c_{n-1}(T_f)), \end{aligned}$$

where T_f is the relative tangent bundle of the map $f\colon \overline{P} \to S$. By the projection formula and the fact that $\deg c_{n-1}(T_{\mathbb{P}^{n-1}}) = n$, we get that

$$f_\# \circ f^* = n.$$

Using this formula, the proof will be complete if we show that

$$f^* c\Big(S \xrightarrow{\alpha} [X/PGL_n]r\Big) = 0. \tag{5.5}$$

By functoriality, we have that

$$f^* c\Big(S \xrightarrow{\alpha} [X/PGL_n]\Big) = c\Big(\overline{P} \xrightarrow{\alpha' := f\circ\alpha} [X/PGL_n]\Big).$$

The map $\alpha'\colon \overline{P} \to [X/PGL_n]$ corresponds to the PGL_n-torsor $\overline{P} \times_S P \to \overline{P}$ and the PGL_n-equivariant map $\overline{P} \times_S P \to P \to X$. By construction, the PGL_n-torsor $\overline{P} \times_S P \to \overline{P}$ is trivial and therefore there exists a SL_n-torsor $E \to \overline{P}$ such that $E/\mu_n \cong \overline{P} \times_S P$. Moreover, since the group μ_n acts trivially on X, the PGL_n-equivariant map $\overline{P} \times_S P \to X$ extends to a SL_n-equivariant map $E \to X$. This data defines a morphism $\beta\colon \overline{P} \to [X/SL_n]$ such that $\alpha' = p \circ \beta$. Therefore, using the hypothesis that $p^* c = 0$, we get that

$$c\Big(\overline{P} \xrightarrow{\alpha'} [X/PGL_n]\Big) = (p^* c)\Big(\overline{P} \xrightarrow{\beta} [X/SL_n]\Big) = 0,$$

and this concludes the proof. □

LEMMA 5.5. — *If s is odd then*

$$\langle \mathrm{Im}(\pi_{s*})\rangle \otimes \mathbb{Z}_{(2)} \subset \langle \mathrm{Im}(\pi_{1*})\rangle \otimes \mathbb{Z}_{(2)}.$$

Proof. — Consider the following PGL_2-equivariant commutative diagram

$$\begin{tikzcd}
 & \mathbb{B}(1)\times\mathbb{B}(s-1)\times\mathbb{B}(N-2s) \arrow[dl, "\rho_s\times\mathrm{id}"] \arrow[dr, "\mathrm{id}\times\pi'_{s-1}"] & \\
\mathbb{B}(s)\times\mathbb{B}(N-2s) \arrow[dr, "\pi_s"] & & \mathbb{B}(1)\times\mathbb{B}(N-2) \arrow[dl, "\pi_1"] \\
 & \mathbb{B}(N) &
\end{tikzcd}$$

where ρ_s sends $([f],[g])$ to $[fg]$ and π'_{s-1} sends $([f],[g])$ to $[f^2 g]$. The map $\rho_s \times \mathrm{id}$ is finite and flat of degree s and therefore

$$(\rho_s \times \mathrm{id})_* \circ (\rho_s \times \mathrm{id})^* = s \cdot \mathrm{id}\,.$$

Using this and the commutativity of the above diagram, we get for any class $\alpha \in A^*_{PGL_2}(\mathbb{B}(s) \times \mathbb{B}(N-2s))$

$$s \cdot \pi_{s*}(\alpha) = \pi_{s*}(\rho_s \times \mathrm{id})_*(\rho_s \times \mathrm{id})^*(\alpha) = \pi_{1*}(\mathrm{id} \times \pi'_{s-1})_*(\rho_s \times \mathrm{id})^*(\alpha).$$

Since s is odd by hypothesis, and therefore invertible in $\mathbb{Z}_{(2)}$, we deduce that $\pi_{s*}(\alpha) \in \langle \mathrm{Im}(\pi_{1*}) \rangle \otimes \mathbb{Z}_{(2)}$. $\square$

Lemma 5.6. — *If s is even then $\langle \mathrm{Im}(\pi_{s*}) \rangle$ is 2-divisible in $A^*_{PGL_2}(\mathbb{B}(N))$.*

Proof. — We first prove the assertion in the case s is maximal, that is $s = N/2$ and N divisible by 4.

Consider an element $\alpha \in A^*_{PGL_2}(\mathbb{B}(s))$. Observe that the element $\pi_{s*}(\alpha)$ is 2-divisible in the ring $A^*_{PGL_2}(\mathbb{B}(N)) = A^*_{PGL_2}[t]/(p_N(t))$ if and only if the element $t \cdot \pi_{s*}(\alpha)$ is 2-divisible. The map $\pi_s \colon \mathbb{B}(s) \to \mathbb{B}(N)$, sending $[f]$ into $[f^2]$, verifies $\pi_s^*(t) = 2\xi_s$. Therefore, by the projection formula applied to the morphism π_s, we get that

$$t\pi_{s*}(\alpha) = \pi_{s*}(\pi_s^*(t) \cdot \alpha) = \pi_{s*}(2\xi_s \cdot \alpha) = 2\pi_{s*}(\xi_s \alpha),$$

from which we conclude in the case $s = N/2$. For the general case, observe that the map π_s factors as

$$\pi_s \colon \mathbb{B}(s) \times \mathbb{B}(N-2s) \xrightarrow{\pi'_s \times \mathrm{id}} \mathbb{B}(2s) \times \mathbb{B}(N-2s) \xrightarrow{m_s} \mathbb{B}(N)$$

where π'_s sends $[f]$ into $[f^2]$ and m_s sends $([f],[g])$ into $[fg]$. It is therefore enough to prove that $\langle \mathrm{Im}(\pi'_s \times \mathrm{id})_* \rangle$ is 2-divisible in $A^*_{PGL_2}(\mathbb{B}(s) \times \mathbb{B}(N-2s))$.

Observe that since s is even we have an isomorphism

$$A^*_{PGL_2}\big(\mathbb{B}(s) \times \mathbb{B}(N-2s)\big) = A^*_{PGL_2}(\mathbb{B}(N-2s)) \otimes_{A^*_{PGL_2}} A^*_{PGL_2}(\mathbb{B}(s)),$$

and similarly for $A^*_{PGL_2}(\mathbb{B}(2s) \times \mathbb{B}(N-2s))$. In particular, the ring $A^*_{PGL_2}(\mathbb{B}(s) \times \mathbb{B}(N-2s))$ (resp. $A^*_{PGL_2}(\mathbb{B}(2s) \times \mathbb{B}(N-2s))$) is a $A^*_{PGL_2}(\mathbb{B}(N-2s))$-module generated by the pull-back along the first projection of the class ξ_s (resp. ξ_{2s}). Moreover the push-forward $(\pi'_s \times \mathrm{id})_*$ is a morphism of $A^*_{PGL_2}(\mathbb{B}(N-2s))$-modules. Therefore we deduce the 2-divisibility of the image of the push-forward $(\pi'_s \times \mathrm{id})_*$ from the previous maximal case. $\square$

We are now ready to prove the Proposition 5.3.

Proof of Proposition 5.3 with coefficients in $\left[\frac{\mathbb{Z}}{1/2}\right]$. — Consider the following commutative diagram

$$\begin{tikzcd} A^*_{PGL_2}\big(\mathbb{B}(s)\times\mathbb{B}(N-2s)\big) \arrow[r, "\pi_{s*}"] \arrow[d] & A^*_{PGL_2}(\mathbb{B}(N)) \arrow[d] \\ A^*_{SL_2}\big(\mathbb{B}(s)\times\mathbb{B}(N-2s)\big) \arrow[r, "\pi_{s*}^{SL_2}"] & A^*_{SL_2}(\mathbb{B}(N)). \end{tikzcd}$$

According to Lemma 5.4, the two vertical arrows become injective after tensoring with $\mathbb{Z}[1/2]$. Therefore in order to prove the inclusion of Proposition 5.3 with $\mathbb{Z}[1/2]$-coefficients, it is enough to prove the analogous inclusion for the SL_2-equivariant Chow rings. But this is proved exactly as in the case GL_2 (see [5, Section 4]): The same proof works by simply putting $c_1 = 0$. □

Proof of Proposition 5.3 with coefficients in $\mathbb{Z}_{(2)}$. — The inclusion $\langle \mathrm{Im}(\pi_{s*})\rangle \subset \langle \mathrm{Im}(\pi_{1*})\rangle$ for s odd follows directly from Lemma 5.5. Let us fix an even $s \geqslant 2$. According to Lemma 5.6, we have that $\langle \mathrm{Im}(\pi_{s*})\rangle \subset \langle 2\rangle \subset A^*_{PGL_2}(\mathbb{B}(N))$. A direct check using Proposition 5.2 and the fact that N is even shows that we have also the inclusion $\langle \mathrm{Im}(\pi_{1*})\rangle \subset \langle 2\rangle$.

Consider the natural pull-back map $p^*\colon A^*_{PGL_2}(\mathbb{B}(N)) \to A^*_{SL_2}(\mathbb{B}(N))$. Since N is even, $\mathbb{B}(N)$ is a projective bundle over the classifying stack $BPGL_2$ and therefore the kernel of p^* is generated by the kernel of the natural pull-back map $A^*_{PGL_2} \to A^*_{SL_2}$, that is $\mathrm{Ker}(p^*) = \langle c_3\rangle$.

Using the relation $2c_3 = 0$ it is easy to check that $\langle 2\rangle \cap \langle c_3\rangle = 0$. Therefore the pull-back map p^* is injective on the ideal $\langle 2\rangle$. Since $\langle \mathrm{Im}(\pi_{s*})\rangle$ and $\langle \mathrm{Im}(\pi_{1*})\rangle$ are contained in the ideal $\langle 2\rangle$, in order to prove the inclusion $\langle \mathrm{Im}(\pi_{s*})\rangle \subset \langle \mathrm{Im}(\pi_{1*})\rangle$ it is enough to prove the similar inclusion in the ring $A^*_{SL_2}(\mathbb{B}(N))$. This is done exactly as in the case GL_2 (see [5, Section 4]): The same proof works by simply putting $c_1 = 0$. □

5.3. Conclusion

Now we put everything together to prove the following

Theorem 5.7. — *If d is even and* $\mathrm{char}(k) = 0$ *or* $\mathrm{char}(k) > N$, *we have if r is even*

$$A^*(\mathcal{H}_{\mathrm{sm}}(1,r,d)) = \frac{\mathbb{Z}[c_1,c_2,c_3]}{\left\langle 2c_3, 2r(N-1)c_1, 2r^2c_1^2 - \frac{N(N-2)}{2}c_2\right\rangle}.$$

while, if r is odd

$$A^*(\mathcal{H}_{\mathrm{sm}}(1,r,d)) = \frac{\mathbb{Z}[c_1,c_2,c_3]}{\left\langle p_N(-rc_1), 2c_3, 2r(N-1)c_1, 2r^2c_1^2 - \frac{N(N-2)}{2}c_2\right\rangle}.$$

Proof. — The assertion will follow combining (3.2), (3.4), Lemma 3.1, Propositions 5.2 and 5.3 once we prove that

(1) if r is even (Proposition 6.4)

$$p_N(-rc_1) \in \left\langle 2c_3, 2r(N-1)c_1, 2r^2c_1^2 - \frac{N(N-2)}{2}c_2 \right\rangle.$$

(2) if r is odd (Proposition 6.5)

$$p_N(-rc_1) \notin \left\langle 2c_3, 2r(N-1)c_1, 2r^2c_1^2 - \frac{N(N-2)}{2}c_2 \right\rangle.$$

An explicit form of $p_N(-rc_1)$ when r is odd is given in Proposition 6.5. □

Remark 5.8. — In the hyperelliptic case $\mathcal{H}_g = \mathcal{H}_{\mathrm{sm}}(1, 2, g+1)$, we get the following answer. If g is even (see also [5])

$$A^*(\mathcal{H}_g) = \frac{\mathbb{Z}[c_1, c_2]}{\langle 2(2g+1)c_1, g(g-1)c_1^2 - 4g(g+1)c_2 \rangle}.$$

If g is odd

$$A^*(\mathcal{H}_g) = \frac{\mathbb{Z}[c_1, c_2, c_3]}{\langle 2c_3, 4(2g+1)c_1, 8c_1^2 - 2g(g+1)c_2 \rangle}.$$

6. The polynomial $p_N(t)$

Throughout this section, we assume that d is even and we set $d = 2s$, so that $N = 2rs$. We want to compute the polynomial $p_N(t)$ in the ring

$$A^*_{PGL_2}[t] = \mathbb{Z}[c_2, c_3, t]/(2c_3).$$

Let V be the defining representation of GL_2. Let a, b, c be the Chern roots of $\mathrm{Sym}^2 V^*$, seen as a representation of PGL_2. We have

$$\begin{cases} a + b + c = 0, \\ ab + ac + bc = c_2, \\ abc = c_3, \\ 2abc = 0. \end{cases}$$

In the next Lemma, we determine the polynomial $p_N(t)$ modulo the ideal (2) of $\mathbb{Z}[c_2, c_3]/(2c_3)$.

Lemma 6.1. — *The polynomial $p_N(t)$ is equivalent modulo (2) to*

$$p_N(t) \equiv \begin{cases} t^{(N+4)/4}(t^3 + c_2t + c_3)^{N/4} & \bmod (2) \quad \textit{if } N \equiv 0 \mod 4, \\ t^{(N-2)/4}(t^3 + c_2t + c_3)^{(N+2)/4} & \bmod (2) \quad \textit{if } N \equiv 2 \mod 4. \end{cases}$$

Proof. — By the well-known plethysm formulas for PGL_2, we have that

$$\mathrm{Sym}^{2sr}(V^*) \oplus \mathrm{Sym}^{sr-2}(\mathrm{Sym}^2(V^*)) = \mathrm{Sym}^{sr}(\mathrm{Sym}^2(V^*)).$$

Therefore, from the Whitney's formula and the usual formulas for the Chern roots of a symmetric product of a representation, we get that

$$p_N(t) = \frac{\prod_{i,j,k\geqslant 0}^{i+j+k=rs}(t+ia+jb+kc)}{\prod_{i,j,k\geqslant 0}^{i+j+k=rs-2}(t+ia+jb+kc)}. \tag{6.1}$$

Consider the expression of $p_N(t)$ obtained in Lemma 6.1. Since, for all i, j, k such that $i + j + k = rs - 2$ we have

$$(t + ia + jb + kc) \equiv \big(t + ia + jb + (k + 2)c\big) \mod (2),$$

we can simplify the fraction in (6.1) modulo (2) and thus we get

$$p_N(t) \equiv \prod_{i=0}^{rs}\big(t + ia + (rs - i)b\big) \prod_{i=0}^{rs-1} \big(t + ia + (rs - 1 - i)b + c\big) \mod (2).$$

We compute separately the two products mod (2). In the first product, the coefficients of a and b have the same parity if rs is even and opposite parity if rs is odd, so that we get (using that $c \equiv a + b \mod (2)$):

$$(*)\quad \prod_{i=0}^{rs}(t+ia+(rs-i)b) \equiv \begin{cases} t^{\frac{rs}{2}+1}(t+c)^{\frac{rs}{2}} & \mod (2) \text{if } rs \text{ is even,}\\ (t+a)^{\frac{rs+1}{2}}(t+b)^{\frac{rs+1}{2}} & \mod (2) \text{if } rs \text{ is odd.}\end{cases}$$

A similar computation for the second product gives

$$(**)\quad \prod_{i=0}^{rs-1}(t+ia+(rs-1-i)b+c) \equiv \begin{cases} (t+b)^{\frac{rs}{2}}(t+a)^{\frac{rs}{2}} & \mod (2) \text{if } rs \text{ is even,}\\ t^{\frac{rs-1}{2}}(t+c)^{\frac{rs+1}{2}} & \mod (2) \text{if } rs \text{ is odd.}\end{cases}$$

We conclude by putting together (*) and (**), and using that $N = 2rs$ and $(t + a)(t + b)(t + c) = t^3 + c_2t + c_3$. □

We now determine the polynomial $p_N(t)$ modulo the ideal (c_3) of $\mathbb{Z}[c_2, c_3]/(2c_3)$.

Lemma 6.2. — *The polynomial $p_N(t)$ is equivalent modulo (c_3) to*

$$p_N(t) \equiv t \prod_{k=1}^{N/2}(t^2 + k^2c_2) \mod (c_3).$$

Proof. — We compare the PGL_2-equivariant Chow ring $A_{PGL_2}(\mathbb{B}(N))$ with the SL_2-equivariant Chow ring $A_{SL_2}(\mathbb{B}(N))$, in a similar way as we did in the proof of Lemma 5.1. To this aim, let us first compute the Chow ring $A_{SL_2}(\mathbb{B}(N))$. Clearly, we have that

$$A^*_{SL_2}(\mathbb{B}(N)) = A_{SL_2}[\tau]/(q_N(\tau)) = \mathbb{Z}[c_2, \tau]/(q_N(\tau)),$$

where $\tau = c_1^{SL_2}(\mathcal{O}_{\mathbb{B}(N)}(1))$ and $q_N(\tau)$ is a monic polynomial of degree $N+1$ in τ with coefficients in $\mathbb{Z}[c_2]$. Let α and β be the Chern roots of V^*, seen as a representation of SL_2. We have that

$$\begin{cases} \alpha+\beta = 0, \\ \alpha\beta = c_2 \in A^*_{SL_2}. \end{cases}$$

The Chern roots of the SL_2-representation $\mathrm{Sym}^N(V^*)$ are $\{i\alpha + (N-i)\beta\}_{i=0,\dots,N}$ and therefore we compute

$$\begin{aligned} q_N(\tau) &= \prod_{i=0}^{N} \big(\tau + i\alpha + (N-i)\beta\big) \\ &= \prod_{i=0}^{N/2-1} \big[(\tau + i\alpha + (N-i)\beta)\,(\tau + (N-i)\alpha + i\beta)\big] \cdot \left(\tau + \frac{N}{2}\alpha + \frac{N}{2}\beta\right) \\ &= \tau \prod_{i=0}^{N/2-1} \left[\tau^2 + \left(\frac{N}{2} - i\right)^2 4c_2\right] \\ &= \tau \prod_{k=1}^{N/2} \left[\tau^2 + k^2 4c_2\right]. \end{aligned}$$

Now consider the natural commutative diagram of rings (similar to the one considered in Lemma 5.1):

$$\begin{tikzcd} A^*_{PGL_2} = \mathbb{Z}[c_2, c_3]/(2c_3) \arrow[r] \arrow[d] & A^*_{PGL_2}(\mathbb{B}(N)) = \mathbb{Z}[c_2, c_3, t]/(2c_3, p_N(t)) \arrow[d] \\ A^*_{SL_2} = \mathbb{Z}[c_2] \arrow[r, hook] & A^*_{SL_2}(\mathbb{B}(N)) = \mathbb{Z}[c_2, \tau]/(q_N(\tau)). \end{tikzcd}$$

The left vertical maps sends c_3 to 0 and c_2 to $4c_2$ (see the proof of Lemma 5.1), while the right vertical map obviously sends $t = c_1^{PGL_2}(\mathcal{O}_{\mathbb{B}(N)}(1))$ into $\tau = c_1^{SL_2}(\mathcal{O}_{\mathbb{B}(N)}(1))$. This diagram tells us that the polynomial obtained from $p_N(t)$ by substituting t with τ, c_2 with $4c_2$ and c_3 with 0 should be equal to $q_N(\tau)$. From the above formula for $q_N(\tau)$, we get the conclusion. □

We can now put together the previous two Lemmas to get the following expression for $p_N(t) \in \mathbb{Z}[c_2, c_3, t]/(2c_3)$.

Corollary 6.3. — *If* $N \equiv 0 \mod 4$ *then we have*

$$p_N(t) = t \prod_{k=1}^{N/2} (t^2 + k^2 c_2) + t^{\frac{N}{4}+1} \sum_{k=1}^{N/4} \binom{\frac{N}{4}}{k} (t^3 + c_2 t)^{\frac{N}{4}-k} c_3^k,$$

while if $N \equiv 2 \mod 4$ *then*

$$p_N(t) = t\prod_{k=1}^{N/2}(t^2+k^2c_2) + t^{\frac{N-2}{4}}\sum_{k=1}^{(N+2)/4}\binom{\frac{N+2}{4}}{k}(t^3+c_2t)^{\frac{N+2}{4}-k}c_3^k.$$

Proof. — The polynomial $p_N(t)$ modulo (c_3) is given by Lemma 6.2. Since $2c_3 = 0$, the terms of the polynomial that are multiples of c_3 are given the corresponding terms in the expression of $p_N(t)$ modulo (2) (see Lemma 6.1). A straightforward computation allows to conclude. □

Now, we evaluate the class of $p_N(-rc_1)$ modulo the ideal

$$I := \left\langle 2r(N-1)c_1, 2r^2c_1^2 - \frac{N(N-2)}{2}c_2, 2c_3 \right\rangle \subset \mathbb{Z}[c_1, c_2, c_3].$$

The answer will depend upon the parity of r. Consider first the case where r is even.

Proposition 6.4. — *If r is even then $p_N(-rc_1)$ belongs to I.*

Proof. — Substituting $t = -rc_1$ into the first expression of Corollary 6.3 (note that $N = 2dr \equiv 0 \mod 4$) and using the fact that $2c_3 = 0$ and r is even, we get that

$$p_N(-rc_1) \equiv -rc_1\prod_{k=1}^{N/2}(r^2c_1^2 + k^2c_2) \mod I.$$

Consider the element

$$f := -rc_1\left(r^2c_1^2 + \frac{N^2}{4}c_2\right),$$

which appears as a factor of the above expression for $p_N(-rc_1)$. We will show that $f \in I$, which will conclude the proof. Since r is even (and therefore $N = 2rs \equiv 0 \mod 4$), we can write

$$\begin{aligned} f &= -\frac{r}{2}c_1\left(2r^2c_1^2 + \frac{N^2}{2}c_2\right) \equiv -\frac{r}{2}c_1\left(\frac{N(N-2)}{2}c_2 + \frac{N^2}{2}c_2\right) \\ &= -\frac{N}{4}c_2 \cdot 2r(N-1)c_1 \equiv 0 \mod I. \end{aligned}$$

□

Finally, we consider the case where r is odd.

Proposition 6.5. — *If r is odd then the expression of $p_N(-rc_1)$ modulo I is equal to*

$$p_N(-rc_1) \equiv c_1^{\frac{N}{4}+1}(c_1^3+c_1c_2+c_3)^{\frac{N}{4}} - c_1^{\frac{N}{2}-1}(c_1^2+c_2)^{\frac{N}{4}}\left[(r^2+1)c_1^2 + \frac{N^2}{4}c_2\right],$$

if $N \equiv 0 \mod 4$, *while if* $N \equiv 2 \mod 4$ *then we have*

$$p_N(-rc_1) \equiv c_1^{\frac{N-2}{4}}(c_1^3 + c_2c_1 + c_3)^{\frac{N+2}{4}} - c_1^{\frac{N}{2}}(c_1^2 + c_2)^{\frac{N-2}{4}}\left((r^2+1)c_1^2 + \frac{N^2+4}{4}c_2\right).$$

In both cases, $p_N(-rc_1) \notin I$.

Proof. — The first part follows by substituting $t = -rc_1$ into the formulas in Corollary 6.3 (and rearranging the terms), using the facts that r is odd, that $2c_3 = 0$ and that (see the proof of Proposition 6.4)

$$2 \cdot \left[-c_1\left(r^2c_1 + \frac{N^2}{4}c_2\right)\right] \in I.$$

To prove the last statement, it is enough to prove that $p_N(-rc_1) \notin (I, c_2, c_3)$. From the above formulas for $p_N(-rc_1)$, we get that (in both cases)

$$p_N(-rc_1) \equiv -r^2c_1^{N+1} \mod (I, c_2, c_3).$$

On the other hand, from the definition of the ideal I, we get the inclusion

$$(I, c_2, c_3) = (2r(N-1)c_1, 2r^2c_1^2, c_2, c_3) \subset (2c_1, c_2, c_3).$$

Now we conclude by observing that if r is odd then the element $-r^2c_1^{N+1}$ does not belong to the ideal $(2c_1, c_2, c_3)$ and hence, a fortiori, neither to the ideal (I, c_2, c_3). □

7. Some explicit computations

7.1. Explicit generators

In this section we give an explicit description for the generators of the Chow ring $A^*(\mathcal{H}_{\mathrm{sm}}(1, r, d))$, viewed as operational Chow ring (see [6, Prop. 17, 19]). We need first to fix some notation and recall two auxiliary stacks introduced in [2]. Consider a uniform μ_r-cover $\pi\colon \mathcal{F} \to S$ of a conic bundle $p\colon \mathcal{C} \to S$ such that the ramification divisor $W \subset \mathcal{F}$ and the branch divisor $D \subset \mathcal{C}$ are both finite and étale over S of degree N. By the classical theory of cyclic covers and the Hurwitz formula, there exists an r-root $\mathcal{L}^{-1} \in \mathrm{Pic}(\mathcal{C})$ of $\mathcal{O}_{\mathcal{C}}(D)$ such that, called $f\colon \mathcal{F} \to \mathcal{C}$ the cyclic cover of degree r, we have

$$f^*(\mathcal{L}^{-1}) = \mathcal{O}_{\mathcal{F}}(W), \tag{7.1}$$

$$f_*(\mathcal{O}_{\mathcal{F}}) = \bigoplus_{i=0}^{r-1} \mathcal{L}^i, \tag{7.2}$$

$$\omega_{\mathcal{F}/S} = f^*(\omega_{\mathcal{C}/S}) \otimes \mathcal{O}_{\mathcal{F}}((r-1)W). \tag{7.3}$$

In the above notation, it is easy to check that $\mathcal{H}_{\mathrm{sm}}(1,r,d)$ is isomorphic to the stack $\mathcal{H}'_{\mathrm{sm}}(1,r,d)$ whose fiber over a k-scheme S is the groupoid of collections $(\mathcal{C} \xrightarrow{p} S, \mathcal{L}, \mathcal{L}^{\otimes r} \overset{i}{\hookrightarrow} \mathcal{O}_{\mathcal{C}})$, where the morphisms are natural Cartesian diagrams (see [2, section 2]). Moreover, we consider an auxiliary stack $\widetilde{\mathcal{H}}_{\mathrm{sm}}(1,r,d)$ whose fiber over a k-schemes S is the groupoid of collections

$$\widetilde{\mathcal{H}}_{\mathrm{sm}}(1,r,d)(S) = \left\{ \left(\mathcal{C} \xrightarrow{p} S, \mathcal{L}, \mathcal{L}^{\otimes r} \overset{i}{\hookrightarrow} \mathcal{O}_{\mathcal{C}}, \phi\colon (\mathcal{C},\mathcal{L}) \cong \left(\mathbb{P}^1_S, \mathcal{O}_{\mathbb{P}^1_S}(-d)\right)\right)\right\},$$

where the isomorphism ϕ consists of an isomorphisms of S-schemes $\phi_0\colon \mathcal{C} \cong \mathbb{P}^1_S$ plus an isomorphism of invertible sheaves $\phi_1\colon \mathcal{L} \cong \phi_0^*\mathcal{O}_{\mathbb{P}^1_S}(-d)$. In [2, Theo. 4.1], it is proved that $\widetilde{\mathcal{H}}_{\mathrm{sm}}(1,r,d) \cong \mathbb{A}_{sm}(N)$ and that the forgetful morphism $\widetilde{\mathcal{H}}_{\mathrm{sm}}(1,r,d) \to \mathcal{H}'_{\mathrm{sm}}(1,r,d) \cong \mathcal{H}_{\mathrm{sm}}(1,r,d)$ is a principal GL_2/μ_d-bundle.

THEOREM 7.1. — *Assume that d is odd. The Chow ring $A^*(\mathcal{H}_{\mathrm{sm}}(1,r,d))$ is generated by the first two Chern classes of the vector bundle of rank 2*

$$\mathcal{E}_1(\mathcal{F} \xrightarrow{\pi} S) := \pi_*\omega_{\mathcal{F}/S}^{\frac{d-1}{2}}\Big(\frac{1+r+d-N}{2}W\Big).$$

Proof. — The equivariant Chow ring $A^*_{GL_2/\mu_d}(\mathbb{A}_{sm}(N))$ is a quotient of the Chow ring of the group GL_2/μ_d. From the isomorphism of algebraic groups

$$\begin{aligned} GL_2/\mu_d &\xrightarrow{\cong} GL_2 \\ [A] &\longrightarrow (\det A)^{\frac{d-1}{2}} A, \end{aligned}$$

and the fact that $A^*_{GL_2} = \mathbb{Z}[c_1,c_2]$, where c_1 and c_2 are the Chern classes of the standard representation of GL_2, we deduce that $A^*_{GL_2/\mu_d}(\mathbb{A}_{sm}(N))$ is generated by the first two equivariant Chern classes of the vector bundle $\tilde{\mathcal{E}}_1$ that associates to a trivial family $\mathbb{P}^1(V_S) = \mathbb{P}^1_S \xrightarrow{p_S} S$ the vector bundle

$$\tilde{\mathcal{E}}_1(\mathbb{P}^1_S \xrightarrow{p_S} S) := (\det V_S)^{\frac{d-1}{2}} \otimes V_S,$$

where $V_S = V \times_k S$ and V is the two dimensional standard k-representation of GL_2. Clearly we have that $V_S = (p_S)_*(\mathcal{O}_{\mathbb{P}^1_S}(1))$. Moreover, from the Euler exact sequence applied to the trivial family $p_S\colon \mathbb{P}^1_S \to S$

$$0 \to \mathcal{O}_{\mathbb{P}^1_S} \to p_S^*(V_S^*)(1) \to \omega^{-1}_{\mathbb{P}^1_S/S} \to 0,$$

we deduce the GL_2-equivariant isomorphism $p_S^*(\det V_S) \cong \mathcal{O}_{\mathbb{P}^1_S}(2) \otimes \omega_{\mathbb{P}^1_S/S}$. Using the projection formula and the equality $(p_S)_*(\mathcal{O}_{\mathbb{P}^1_S}) = \mathcal{O}_S$, we get the GL_2-equivariant isomorphism

$$\det V_S = (p_S)_*\Big(\mathcal{O}_{\mathbb{P}^1_S}(2) \otimes \omega_{\mathbb{P}^1_S/S}\Big). \tag{7.4}$$

where we consider the canonical actions of GL_2 on $\mathbb{P}^1_S$ and on the invertible sheaves involved. Using these two equalities and the isomorphism

$$\phi\colon (\mathcal{C},\mathcal{L}) \cong \left(\mathbb{P}^1_S, \mathcal{O}_{\mathbb{P}^1_S}(-d)\right),$$

the GL_2/μ_d-equivariant vector bundle $\tilde{\mathcal{E}}_1$ on $\widetilde{\mathcal{H}}_{\rm sm}(1,r,d)$ descends on $\mathcal{H}'_{\rm sm}(1,r,d)$ to the vector bundle

$$\mathcal{E}_1'(\mathcal{C} \xrightarrow{p} S, \mathcal{L}) := p_*\left(\omega_{\mathcal{C}/S}^{\frac{d-1}{2}} \otimes \mathcal{L}^{-1}\right).$$

Consider now a μ_r-cover $f\colon \mathcal{F} \to \mathcal{C}$ with ramification divisor W as above. Using the formulas (7.1), (7.3), we have that

$$f^*(\omega_{\mathcal{C}/S}^{(d-1)/2} \otimes \mathcal{L}^{-1}) = \omega_{\mathcal{F}/S}^{(d-1)/2}\Big(\frac{1+d+r-N}{2}W\Big).$$

Therefore, using the projection formula and the formula (7.2), we get

$$\begin{aligned}\pi_*\omega_{\mathcal{F}/S}^{\frac{d-1}{2}}\Big(\frac{1+d+r-N}{2}W\Big) &= p_*\left(\left(\omega_{\mathcal{C}/S}^{\frac{d-1}{2}} \otimes \mathcal{L}^{-1}\right) \otimes \bigoplus_{i=0}^{r-1}\mathcal{L}^i\right)\\ &= p_*\left(\omega_{\mathcal{C}/S}^{\frac{d-1}{2}} \otimes \mathcal{L}^{-1}\right),\end{aligned}$$

where in the last equality we used that $p_*(\omega_{\mathcal{C}/S}^{(d-1)/2} \otimes \mathcal{L}^j) = 0$ for $j \geqslant 0$, which follows from the fact that $\omega_{\mathcal{C}/S}^{(d-1)/2} \otimes \mathcal{L}^j$ has negative degree on the fibers of p, if $j \geqslant 0$. This show that under the above isomorphism of stacks $\mathcal{H}'_{\rm sm}(1,r,d) \cong \mathcal{H}_{\rm sm}(1,r,d)$, the vector bundle $\mathcal{E}_1'$ goes into the vector bundle $\mathcal{E}_1$. □

Theorem 7.2. — *Assume that d is even. The Chow ring $A^*(\mathcal{H}_g)$ is generated by the first Chern class of the line bundle*

$$\mathcal{G}(\mathcal{F} \xrightarrow{\pi} S) := \pi_*\omega_{\mathcal{F}/S}^{d/2}\Big(\frac{2+d-N}{2}W\Big),$$

and by the second and third Chern classes of the vector bundle of rank 3

$$\mathcal{E}_2(\mathcal{F} \xrightarrow{\pi} S) := \frac{\pi_*\omega_{\mathcal{F}/S}^{-1}((r-1)W)}{\pi_*\omega_{\mathcal{F}/S}^{-1}((r-2)W)}.$$

Moreover $\pi_*\omega_{\mathcal{F}/S}^{-1}((r-2)W) = 0$ if $d > 2$.

Proof. — The equivariant Chow ring $A^*_{GL_2/\mu_d}(\mathbb{A}_{sm}(N))$ is a quotient of the Chow ring of the group GL_2/μ_d which, for d even, is isomorphic to

$$\begin{aligned}GL_2/\mu_d &\xrightarrow{\cong} \mathbb{G}_m \times PGL_2\\ [A] &\longrightarrow ((\det A)^{d/2}, [A]).\end{aligned}$$

Recall that $A^*_{\mathbb{G}_m\times PGL_2} = \mathbb{Z}[c_1, c_2, c_3]/(2c_3)$, where c_1 is the first Chern class of the natural representation of $\mathbb{G}_m$ and c_2, c_3 are the second and third Chern classes of the representation $\mathrm{Sym}^2(V)$ of $PGL_2 = SL_2/\mu_2$, where V is the two dimensional standard k-representation of SL_2. Therefore we deduce that $A^*_{GL_2/\mu_d}(\mathbb{A}_{sm}(N))$ is generated by the first Chern class of the line bundle $\tilde{\mathcal{G}}$ that associates to a trivial family $\mathbb{P}^1(V_S) = \mathbb{P}^1(V\times_k S) = \mathbb{P}^1_S \xrightarrow{p_S} S$ the GL_2/μ_d-equivariant line bundle

$$\tilde{\mathcal{G}}(\mathbb{P}^1_S \xrightarrow{p_S} S) := (\det V_S)^{d/2},$$

and by the second and third Chern classes of the GL_2/μ_d-equivariant vector bundle

$$\tilde{\mathcal{E}}_2(\mathbb{P}^1_S \xrightarrow{p_S} S) = \mathrm{Sym}^2(V_S).$$

Clearly we have that $\mathrm{Sym}^2(V_S) = (p_S)_*(\omega^{-1}_{\mathbb{P}^1_S/S})$. Moreover, from the formula (7.4), we deduce that $(\det V_S)^{d/2} = (p_S)_*(\mathcal{O}_{\mathbb{P}^1_S}(d)\otimes\omega^{d/2}_{\mathbb{P}^1_S/S})$, where we consider the canonical actions of GL_2/μ_d on $\mathbb{P}^1_S$ and on the invertible sheaves involved. Using these two equalities and the isomorphism $\phi : (\mathcal{C},\mathcal{L}) \cong (\mathbb{P}^1_S, \mathcal{O}_{\mathbb{P}^1_S}(-d))$, the GL_2/μ_d-equivariant vector bundles $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{E}}_2$ on $\widetilde{\mathcal{H}}_{\mathrm{sm}}(1,r,d)$ descend on $\mathcal{H}'{}_{\mathrm{sm}}(1,r,d)$ to the vector bundles

$$\mathcal{G}'(\mathcal{C}\xrightarrow{p} S,\mathcal{L}) := p_*(\omega^{d/2}_{\mathcal{C}/S}\otimes\mathcal{L}^{-1}),$$
$$\mathcal{E}'_2(\mathcal{C}\xrightarrow{p} S,\mathcal{L}) := p_*(\omega^{-1}_{\mathcal{C}/S}).$$

Consider now the μ_r-cover $f\colon \mathcal{F}\to\mathcal{C}$ with ramification divisor W as above. Using the formulas (7.1), (7.3), we have that

$$\begin{cases} f^*(\omega^{d/2}_{\mathcal{C}/S}\otimes\mathcal{L}^{-1}) = \omega^{d/2}_{\mathcal{F}/S}\left(\dfrac{2+d-N}{2}W\right), \\ f^*(\omega^{-1}_{\mathcal{C}/S}) = \omega^{-1}_{\mathcal{F}/S}((r-1)W), \\ f^*(\omega^{-1}_{\mathcal{C}/S}\otimes\mathcal{L}) = \omega^{-1}_{\mathcal{F}/S}((r-2)W). \end{cases}$$

Therefore, using the projection formula, the formula (7.2) and the vanishings $p_*(\omega^{d/2}_{\mathcal{C}/S}\otimes\mathcal{L}^{-1}\otimes\mathcal{L}^i) = p_*(\omega^{-1}_{\mathcal{C}/S}\otimes\mathcal{L}^{i+1}) = 0$ for $i\geqslant 1$ (because these line bundles have negative degrees on the fibers of $p\colon\mathcal{C}\to S$), we get

$$\begin{cases} \pi_*\omega^{d/2}_{\mathcal{F}/S}\Big(\dfrac{2+d-N}{2}W\Big) = p_*\left(\left(\omega^{d/2}_{\mathcal{C}/S}\otimes\mathcal{L}^{-1}\right)\otimes\displaystyle\bigoplus_{i=0}^{r-1}\mathcal{L}^i\right) = p_*(\omega^{d/2}_{\mathcal{C}/S}\otimes\mathcal{L}^{-1}), \\ \pi_*\omega^{-1}_{\mathcal{F}/S}((r-1)W) = p_*\left(\omega^{-1}_{\mathcal{C}/S}\otimes\displaystyle\bigoplus_{i=0}^{r-1}\mathcal{L}^i\right) = p_*(\omega^{-1}_{\mathcal{C}/S})\oplus p_*(\omega^{-1}_{\mathcal{C}/S}\otimes\mathcal{L}), \\ \pi_*\omega^{-1}_{\mathcal{F}/S}((r-2)W) = p_*\left(\omega^{-1}_{\mathcal{C}/S}\otimes\mathcal{L}\otimes\displaystyle\bigoplus_{i=0}^{r-1}\mathcal{L}^i\right) = p_*(\omega^{-1}_{\mathcal{C}/S}\otimes\mathcal{L}). \end{cases}$$

This shows that, under the above isomorphism of stacks $\mathcal{H'}_{\rm sm}(1,r,d)\cong\mathcal{H}_{\rm sm}(1,r,d)$, the vector bundles $\mathcal{G}'$ and $\mathcal{E}_1'$ go respectively into $\mathcal{G}$ and $\mathcal{E}_1$. Moreover, if $d>2$ then $\omega_{\mathcal{C}/S}^{-1}\otimes\mathcal{L}$ has negative degree on the fibers of $p\colon\mathcal{C}\to S$ and therefore we get the vanishing

$$\pi_*\omega_{\mathcal{F}/S}^{-1}((r-2)W)=p_*\left(\omega_{\mathcal{C}/S}^{-1}\otimes\mathcal{L}\right)=0. \qquad \square$$

Remark 7.3. — In the hyperelliptic case. *i.e.* for $r=2$, one recovers the results of [8, Theo. 4.1] for the Picard group and of [5, Section 5.1] for the Chow ring in the case g even (*i.e.* $d=g+1$ odd).

7.2. λ-classes

In this last part of the section, we want to express the tautological λ classes of Mumford ([11]) in terms of the above explicit generators of $A^*(\mathcal{H}_{\rm sm}(1,r,d))$. Recall that the lambda classes are defined as Chern classes of the Hodge bundle:

$$\lambda_j(\mathcal{F}\xrightarrow{\pi}S):=c_j(\pi_*(\omega_{\mathcal{F}/S}))\text{ for }j=1,\dots,g.$$

Proposition 7.4.

(i) *Assume that d is odd. Then the polynomial expressing λ_j in terms of the above generators $c_1(\mathcal{E}_1)$ and $c_2(\mathcal{E}_2)$ is the same as the one expressing*

$$\bigoplus_{i=0}^{r-2}c_j\left((\det V)^{(r-1-i)\frac{1-d}{2}+1}\otimes\mathrm{Sym}^{(r-1-i)d-2}(V)\right)$$

in terms of the generators c_1 and c_2 of $A^*_{GL_2}=\mathbb{Z}[c_1,c_2]$, where V is the standard two dimensional representation of GL_2.

(ii) *Assume that d is even. Then the polynomial expressing λ_j in terms of the above generators $c_1(\mathcal{G})$, $c_2(\mathcal{E}_2)$ and $c_3(\mathcal{E}_2)$ is the same as the one expressing*

$$\bigoplus_{i=0}^{r-2}c_j\left(W^{(r-1-i)d}\otimes\mathrm{Sym}^{(r-1-i)d-2}(V)\right)$$

in terms of the generators c_1, c_2, and c_3 of $A^*_{\mathbb{G}_m\times PGL_2}=\mathbb{Z}[c_1,c_2,c_3]/(2c_3)$, where W is the standard one dimensional representation of $\mathbb{G}_m$ and V is the standard two dimensional representation of SL_2.

Proof. — Consider the Hodge vector bundle $\mathbb{H}(\mathcal{F} \xrightarrow{\pi} S) := \pi_*(\omega_{\mathcal{F}/S})$. Using the formulas (7.1), (7.2), (7.3) and the standard notations introduced in section 7.1, we compute

$$\begin{aligned}\pi_*(\omega_{\mathcal{F}/S}) &= p_* f_* f^*(\omega_{\mathcal{C}/S} \otimes \mathcal{L}^{-(r-1)}) \\ &= p_* \left(\bigoplus_{i=0}^{r-1} \omega_{\mathcal{C}/S} \otimes \mathcal{L}^{i-(r-1)} \right) \\ &= \bigoplus_{i=0}^{r-2} p_* \left(\omega_{\mathcal{C}/S} \otimes \mathcal{L}^{i-(r-1)} \right),\end{aligned}$$

where in the last equality we have used the vanishing $p_*(\omega_{\mathcal{C}/S}) = 0$. For a trivial family $(\mathcal{C} \xrightarrow{p} S, \mathcal{L}) \cong (\mathbb{P}^1_S \xrightarrow{p_S} S, \mathcal{O}_{\mathbb{P}^1_S}(-d))$, using the formula (7.4), we get the GL_2/μ_d-equivariant isomorphism

$$\begin{aligned}(p_S)_*(\omega_{\mathcal{C}/S} \otimes \mathcal{L}^{-1}) &= (p_S)_* \left(p_S^*(\det V_S) \otimes \mathcal{O}_{\mathbb{P}^1_S}((r-1-i)d-2) \right) \\ &= \det V_S \otimes \mathrm{Sym}^{(r-1-i)d-2}(V_S),\end{aligned}$$

where $V_S = V \times_k S$ and V is the standard representation of GL_2.

Suppose first that d is odd. Using the isomorphism of algebraic groups

$$\begin{aligned}GL_2 &\xrightarrow{\cong} GL_2/\mu_d \\ A &\longrightarrow \left[(\det A)^{\frac{-d+1}{2d}} A \right],\end{aligned}$$

the above GL_2/μ_d-representation $\det V_S \otimes \mathrm{Sym}^{(r-1-i)d-2}(V_S)$ becomes isomorphic to the GL_2-representation

$$\begin{aligned}&\left[(\det V)^{\otimes \frac{-d+1}{2d} 2} \otimes (\det V) \right] \otimes \left[(\det V)^{\frac{-d+1}{2d}[(r-1-i)d-2]} \otimes \mathrm{Sym}^{(r-1-i)d-2}(V) \right] \\ &\qquad = (\det V)^{(r-i-1)\frac{1-d}{2}+1}\, \mathrm{Sym}^{(r-1-i)d-2}(V),\end{aligned}$$

which gives the conclusion.

Finally, if d is even then, using the isomorphism of algebraic groups

$$\begin{aligned}\mathbb{G}_m \times PGL_2 &\xrightarrow{\cong} GL_2/\mu_d \\ (\alpha, [A]) &\longrightarrow \alpha^{\frac{1}{d}} (\det A)^{-1/2} A,\end{aligned}$$

the above GL_2/μ_d-representation $\det V_S \otimes \mathrm{Sym}^{(r-1-i)d-2}(V_S)$ becomes isomorphic to the $\mathbb{G}_m \times PGL_2$-representation

$$[W^{\frac{2}{d}}] \otimes \left[W^{\frac{(r-1-i)d-2}{d}} \otimes \mathrm{Sym}^{(r-1-i)d-2}(V) \right] = W^{(r-1-i)d} \otimes \mathrm{Sym}^{(r-1-i)d-2}(V),$$

which gives the conclusion. □

Consider now the natural representable map of stacks

$$\phi\colon \mathcal{H}_{\mathrm{sm}}(1,r,d) \to \mathcal{M}_g,$$

where $g = (r-1)(N-2)/2$. It induces a pull-back map $\phi^*\colon \operatorname{Pic}(\mathcal{M}_g) \to \operatorname{Pic}(\mathcal{H}_{\mathrm{sm}}(1,r,d))$. Recall (see [1]) that if $g \geqslant 2$ then $\operatorname{Pic}(\mathcal{M}_g)$ is cyclic generated (freely if $g \geqslant 3$) by λ_1.

Corollary 7.5. — *The class λ_1 in $\operatorname{Pic}(\mathcal{H}_{\mathrm{sm}}(1,r,d)) = \langle c_1\rangle$ is equal to*

$$\lambda_1 = \begin{cases} \sum_{j=0}^{r-1} \binom{dj}{2} c_1 & \text{if } d \text{ is odd,} \\ 2\sum_{j=0}^{r-1} \binom{dj}{2} c_1 & \text{if } d \text{ is even.} \end{cases}$$

Proof. — Assume first that d is odd. Consider the formula for λ_1 given in Proposition 7.4(i). Using the relations $c_1(\det(V)^m) = mc_1(\det(V)) = mc_1$ and $c_1(\operatorname{Sym}^m(V)) = \frac{(m+1)m}{2} c_1(V) = \frac{(m+1)m}{2} c_1$, we get

$$\begin{aligned} \lambda_1 &= \sum_{i=0}^{r-2} \Bigg\{ [(r-1-i)d-1] \left[(r-1-i)\frac{1-d}{2}+1\right] \\ &\qquad\qquad + \frac{[(r-1-i)d-2][(r-1-i)d-1]}{2} \Bigg\} c_1 \\ &= \sum_{i=0}^{r-2} \binom{(r-1-i)d}{2} c_1 = \sum_{j=0}^{r-1} \binom{jd}{2} c_1. \end{aligned}$$

Assume now that d is even. Consider the formula for λ_1 given in Proposition 7.4(ii). Using the relations $c_1(W^m) = mc_1(W) = mc_1$ and $c_1(\operatorname{Sym}^m(V)) = 0$, we get

$$\lambda_1 = \sum_{i=0}^{r-2} [(r-1-i)d-1][(r-1-i)d] c_1 = 2\sum_{j=0}^{r-1} \binom{jd}{2} c_1. \qquad \square$$

Remark 7.6. — In the hyperelliptic case $\mathcal{H}_g = \mathcal{H}_{\mathrm{sm}}(1,2,g+1)$, one recovers the result of [8, Corollary 4.4] since

$$\lambda_1 = \begin{cases} \binom{g+1}{2} c_1 \equiv \frac{g}{2} c_1 \mod 2(2g+1) & \text{if } g \text{ is even,} \\ 2\binom{g+1}{2} c_1 \equiv g\, c_1 \mod 4(2g+1) & \text{if } g \text{ is odd.} \end{cases}$$

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