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Compatible complex structures on twistor space


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COMPATIBLE COMPLEX STRUCTURES ON TWISTOR SPACE

by Guillaume DESCHAMPS (*)

Abstract. — Let \( M \) be a Riemannian 4-manifold. The associated twistor space is a bundle whose total space \( Z \) admits a natural metric. The aim of this article is to study properties of complex structures on \( Z \) which are compatible with the fibration and the metric. The results obtained enable us to translate some metric properties on \( M \) (scalar flat, scalar-flat Kähler...) in terms of complex properties of its twistor space \( Z \).

Résumé. — Soit \( M \) une 4-variété riemannienne. L’espace de twisteur associé est un fibré qui admet une métrique naturelle. Le but de cet article est d’étudier les structures complexes sur \( Z \) qui sont compatibles avec la fibration et la métrique. Les résultats obtenus permettent d’exprimer des propriétés métriques sur \( M \) (courbure scalaire nulle, Kähler à courbure scalaire nulle...) en termes de propriétés des structures complexes de l’espace de twisteur \( Z \).

Let \( (M, g) \) be a Riemannian 4-manifold. The twistor space \( Z \rightarrow M \) is a \( \mathbb{C}P^1 \)-bundle whose total space \( Z \) admits a natural metric \( \tilde{g} \). The aim of this article is to study properties of complex structures on \( (Z, \tilde{g}) \) which are compatible with the \( \mathbb{C}P^1 \)-fibration and the metric \( \tilde{g} \). The results obtained enable us to translate some metric properties on \( M \) in terms of complex properties on its twistor space \( Z \).

Introduction

Let \( (M, g) \) be an oriented 4-dimensional Riemannian manifold (not necessarily compact). Due to the Hodge-star operator *, we have a decomposition

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of the bivector bundle $\bigwedge^2 TM = \bigwedge^+ \oplus \bigwedge^-$. Here $\bigwedge^\pm$ is the eigen-subbundle for the eigenvalue $\pm 1$ of $\star$. The metric $g$ on $M$ induces a metric, denoted by $\langle , \rangle$, on the bundle $\bigwedge^2 TM$. Let $\pi : Z = \mathbb{S}(\bigwedge^+) \to M$ be the sphere bundle; the fiber over a point $m \in M$ parameterizes the complex structures on the tangent space $T_mM$ compatible with the orientation and the metric $g$. It is the twistor space of the manifold $(M, g)$. Since the structural group of $Z$ is $SO(3) \subset Aut(\mathbb{C}P^1)$, we can thus put the complex structure of $\mathbb{C}P^1$ on each fiber. On the other hand, the Levi-Civita connection on $(M, g)$ induces a splitting of the tangent bundle $TZ$ into the direct sum of the horizontal and vertical distributions: $TZ = H \oplus V$. Therefore, the twistor space $Z$ admits a natural metric $\tilde{g}$ defined by its restrictions to $H$ and $V$: we endow $V$ with the Fubini-Study metric and $H \simeq \pi^*TM$ with the pullback of the metric $g$.

In this article we study some aspects of almost complex structures on $(Z, \tilde{g})$ which are Hermitian and extend the complex structure of the fibers. These structures will be called compatible almost complex structures on $(Z, \tilde{g})$. In particular, the integrability of two such structures means that the metric $\tilde{g}$ is bihermitian [33], [4].

To each morphism respecting the twistor fibration

\[ Z \xleftarrow{\pi} M \xrightarrow{f} Z \xrightarrow{\pi} Z \]

we associate a compatible almost complex structure $J_f$ on $(Z, \tilde{g})$ in the following way. Let $z \in Z$ with $\pi(z) = m \in M$, and write $T_zZ = H_z \oplus V_z$. Here, $V_z$ is the tangent space to the fiber $\pi^{-1}(m) \simeq \mathbb{C}P^1$ and is therefore equipped with a complex structure. On the other hand, we endow $H_z \simeq T_mM$ with the complex structure associated to the point $f(z)$. Conversely, any compatible almost complex structure $J$ on $(Z, \tilde{g})$ defines a unique morphism $f : Z \to Z$ respecting the fibration such that $J_f = J$.

The almost complex structure $J_{Id}$ associated to the identity is the canonical twistor almost complex structure [6]. If $\sigma$ is the morphism of $Z$ whose restriction to each fiber of $\pi$ is the antipodal map of $\mathbb{S}^2$, we denote by $J_\sigma$ the almost complex structure associated to $\sigma$. That is the opposite of the almost complex structure $J_2$ defined in [17] which is know to be never integrable. Now, an almost complex manifold $(M, g, J_M)$ such that $J_M$ is compatible with the orientation and the metric $g$ defines a tautological section of $Z \to M$. This section can be taken as the infinity section

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and we can therefore consider the constant morphism \( f = \infty \). The associated almost complex structure will be denoted by \( \mathbb{J}_\infty \). Let \( \lambda \in \mathbb{C}^\ast \) and consider the morphism \( f = \lambda Id \) acting as \( \lambda Id \) in each fiber minus infinity (i.e. \( \mathbb{C}P^1 - \{\infty}\) \( \simeq \mathbb{C} \)) and preserving infinity. We denote by \( \mathbb{J}_{\lambda Id} \) the corresponding almost complex structure on \( Z \).

The integrability of the structures \( \mathbb{J}_{Id}, \mathbb{J}_\infty, \mathbb{J}_{\lambda Id} \) are related to the curvature of the metric \( g \) on \( M \). Let \( R : \bigwedge^2 TM \rightarrow \bigwedge^2 TM \) be the curvature operator. The decomposition \( \bigwedge^2 TM = \bigwedge^+ \oplus \bigwedge^- \) allows us to write \( R \) in block matrix form as follows

\[
R = \begin{pmatrix}
A & iB \\
B & C
\end{pmatrix},
\]

where \( A = W^+ + \frac{s}{12}Id, C = W^- + \frac{s}{12}Id, W^+ \) (resp. \( W^- \)) is the selfdual (resp. anti-selfdual) Weyl tensor, \( s \) is the scalar curvature and \( B \) the trace-free Ricci curvature \([11]\).

The main result of this article is the following:

**THEOREM 1.** — Let \((M, g)\) be an oriented Riemannian 4-manifold.

A) The complex structure \( \mathbb{J}_{Id} \) is integrable if, and only if, \( g \) anti-selfdual (i.e. \( A \) is a homothety) \([6]\).

B) Let \( J_M \) be an almost complex structure on \( M \) compatible with the metric \( g \) and the orientation. The complex structure \( \mathbb{J}_\infty \) is integrable if, and only if:

i) \( J_M \) is integrable;

ii) the kernel of \( A \) contains the plane \( J_M^+ \subset \bigwedge^+ \) orthogonal to the line generated by \( J_M \).

C) Let \((M, g, J_M)\) be a Kählerian surface. If \( \lambda \notin \{0, 1\} \), the complex structure \( \mathbb{J}_{\lambda Id} \) is integrable if, and only if, \((M, g, J_M)\) is scalar-flat Kähler (i.e. \( A=0 \)).

D) Let \((M, g)\) be an anti-selfdual Riemannian manifold. Its scalar curvature is zero if, and only if, any \( m \in M \) has an open neighborhood \( U \) such that, over \( U \), \((Z, \tilde{g})\) admits a compatible complex structure different from \( \mathbb{J}_{Id} \).

The conditions i) & ii) of part B in the previous theorem are satisfied as soon as \((M, g, J_M)\) is Kähler. We show in section B that this Kählerian property is equivalent to the integrability of \( \mathbb{J}_\infty \) in the compact case. For a scalar-flat Kähler surface \((M, g, J_M)\), the complex structures \( \mathbb{J}_{Id} \) \([19]\), \( \mathbb{J}_\infty \) and \( \mathbb{J}_{\lambda Id} \) are integrable and compatible with the metric \( \tilde{g} \) on \( Z \). This gives us a huge family of real 6-dimensional manifolds admitting a bihermitian metric.
Recall that the Penrose correspondence gives a dictionary between holomorphic properties of the twistor space $Z$ and properties of the Riemannian manifold $(M, g)$. The above result can be viewed as a new paragraph of that dictionary. In particular, we deduce from it some new characterizations of Kähler metrics, anti-selfdual scalar-flat metrics and scalar-flat Kähler metrics, in terms of twistor spaces.

The proof of Theorem 1 is split into four theorems, Theorem A, ... D, the proof of each being given in the corresponding labelled section. In section E we study more precisely the set of all compatible complex structures on the twistor space of a locally conformally Kähler surfaces. Whereas on section F we will study the case of bielliptic surfaces.

We conclude the paper by giving a generalisation of this theorem to quaternionic Kähler manifolds of dimension $4n$ for $n > 1$.

**Notation**

We will use Einstein summation convention over repeated indices. The fiber of $\pi : Z \to M$ over $m \in M$ will be freely identified with $\mathbb{S}^2$, $\mathbb{C}P^1$ or $SO(4)/U(2)$, the set of all complex structure on $T_m M$. The bundle of bivectors $\bigwedge^2 T M$ will be identified with the bundle of skew-symmetric endomorphisms of $T M$, or to the bundle of 2-forms.

Let $(\theta^*_1, \theta^*_2, \theta^*_3, \theta^*_4)$ be an oriented $g$-orthonormal frame defined over an open set $U$ of $(M, g)$. Define three linear operators $I, J, K \in \text{End}(T M)$, over $U$, by their matrix in the basis $(\theta^*_1, \ldots, \theta^*_4)$:

\[
I = \begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{bmatrix},
J = \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix},
K = \begin{bmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}.
\]

Then, $(I, J, K)$ gives an oriented orthonormal basis over $U$ of $\bigwedge^+$ and therefore defines a trivialization of the twistor space $\pi : Z \to M$ over $U$:

\[
\pi^{-1}(U) \simeq U \times SO(4)/U(2).
\]

Let $(\theta^*_1, \ldots, \theta^*_4)$ be the local coframe dual to $(\theta_1, \ldots, \theta_4)$. Locally, the covariant derivative $\nabla$ (on $M$) defined by the Levi-Civita connection of the metric $g$ writes $\nabla \theta_j = \Gamma^k_{ij} \theta^*_i \otimes \theta_k$. The $\Gamma^k_{ij}$ are the Christoffel symbols of the connection $\nabla$; they satisfy $\Gamma^k_{ij} = -\Gamma^j_{ik}$.

Let $z \simeq (m, Q) \in \pi^{-1}(U)$ be a point of $Z$ and write the tangent space as the direct sum of the horizontal and vertical tangent spaces: $T_z Z = V_z \oplus H_z$. 

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Denote by $\hat{\theta} \in H_z \simeq T_m M$ the horizontal lift of $\theta \in T_m M$. We then have [8]:

\[
\begin{align*}
V_z &= \left\{ X \frac{\partial}{\partial Q} \mid X \in \text{End}(T_m M), \quad ^tX = -X \text{ and } QX = -XQ \right\} \\
H_z &= \text{Vect} \left( \hat{\theta}_1(z), \ldots, \hat{\theta}_4(z) \right)
\end{align*}
\]

with

\[
\begin{align*}
\hat{\theta}_i(z) &= \theta_i(m) - [\Gamma_i, Q] \frac{\partial}{\partial Q} \\
[\Gamma_i, Q] \frac{\partial}{\partial Q} &= \left( \Gamma_i(m)Q - Q\Gamma_i(m) \right) \frac{\partial}{\partial Q} \in V_z.
\end{align*}
\]

**Remark.** — The complex structure of rational curves on the fiber $\pi^{-1}(m) \simeq S^2$ at a point $z = (m, Q)$ is given by the application [8]:

\[
V_z \simeq T_Q S^2 \quad \longrightarrow \quad V_z \simeq T_Q S^2 \\
X \frac{\partial}{\partial Q} \quad \longmapsto \quad QX \frac{\partial}{\partial Q}.
\]

For all $A \in \text{so}(4) = \{ A \in \text{End}(TM) \mid ^tA = -A \}$ we can define the vertical vector field $\tilde{A} = [A, Q] \frac{\partial}{\partial Q}$. These vector fields will be called basic.

**General results**

In this section $(M, g)$ will be an oriented Riemannian 4-manifold. Results – and proofs – given here in dimension 4, can be easily adapted to quaternionic Kähler $4n$-manifolds and will be used in the last section of the paper.

To study the integrability of the almost complex structure $J_f$ we need to compute the Nijenhuis tensor $N$ of $J_f$ [28]:

\[
N(X, Y) = [J_f X, J_f Y] - J_f [J_f X, Y] - J_f [X, J_f Y] - [X, Y] \quad \forall (X, Y) \in T_z Z.
\]

The first necessary condition for the integrability of $J_f$ appears in the next proposition.

**Proposition 1.** — For any morphism $f$ we have:

i) $N(X, Y) = 0$ for all $X, Y \in V_z$;

ii) let $X, \theta \in V_z \times H_z$, then

- the vertical component of $N(X, \theta)$ is zero
- the horizontal component of $N(X, \theta)$ is zero if and only if the restriction of $f$ to each fiber is holomorphic.

As $\sigma$ is an anti-holomorphic involution on fibers we easily recover the result from [17]:

**Corollary 1.** — The almost complex structure $J_\sigma$ is never integrable.
Proof of Proposition 1. For any morphism $f$, each fiber of $\pi : Z \to M$ has the structure of $\mathbb{C}P^1$. It follows immediately from [28] that $N(X,Y) = 0$ for all $X,Y \in V_z$.

Let $\tilde{X}$ be a basic vertical vector field and $\pi^{-1}(m)$ be a fixed fiber. The restriction to that fiber of the application $f$ is:

$$f|_{\pi^{-1}(m)} : \mathbb{S}^2 \simeq \pi^{-1}(m) \quad \to \quad \mathbb{S}^2 \simeq \pi^{-1}(m)$$

$$Q \quad \mapsto \quad f(Q)$$

Observe that $[\tilde{X}, \hat{\theta}_i]$ is vertical when $\tilde{X}$ is. Since the action of the complex structure $\mathbb{J}_f$ on the fiber is equal to the rational curve structure, it does not depend on the fiber. We then have: $[\mathbb{J}_f \tilde{X}, \hat{\theta}_i] = [Q \tilde{X}, \hat{\theta}_i] = Q[\tilde{X}, \hat{\theta}_i] = \mathbb{J}_f[\tilde{X}, \hat{\theta}_i]$. This implies that, for $i \in \{1, \ldots, 4\}$:

$$N(\tilde{X}, \hat{\theta}_i) = [Q \tilde{X}, f(Q)\hat{\theta}_i] - Q[Q \tilde{X}, \hat{\theta}_i] + \mathbb{J}_f[\tilde{X}, f(Q)\hat{\theta}_i] - [\tilde{X}, \hat{\theta}_i]$$

$$= \left( (Q \tilde{X}).f(Q) - f(Q)(\tilde{X}.f(Q)) \right) \hat{\theta}_i$$

$$= \left( d_Q f(Q \tilde{X}) - f(Q)d_Q f(\tilde{X}) \right) \hat{\theta}_i$$

where $d_Q f$ is the differential of $f$ at $Q \in \mathbb{S}^2$. The horizontal component of $N(X, \theta)$ vanishes for all $(X, \theta) \in V_z \times H_z$ if and only if the restrictions of $f$ to the fibers are holomorphic. \(\square\)

In the trivialization of $Z \to M$ over an open set $U$, the morphism $f$ can be written:

$$f|_{\pi^{-1}(U)} : U \times \mathbb{S}^2 \quad \to \quad U \times \mathbb{S}^2$$

$$(x, Q) \quad \mapsto \quad (x, f(x, Q))$$

In order to simplify the notation we set $P = f(x, Q)$ and $[P_i^j]$ denotes the matrix, in the basis $(\theta_1, \ldots, \theta_4)$, of the operator $P$ viewed as an endomorphism of $TM$.

Proposition 2. — Let $f$ be any morphism and $(m, Q) \in Z$. Then, for all $i, j \in \{1, \ldots, 4\}$ one has:

i) the horizontal component of $N(\hat{\theta}_i, \hat{\theta}_j)$ can be written as $E(\theta_i, \theta_j) + F_{ij}$

ii) the vertical component of $N(\hat{\theta}_i, \hat{\theta}_j)$ can be written as $G(\theta_i, \theta_j)\frac{\partial}{\partial Q}$.
where
\[
E(\theta_i, \theta_j) \text{ is the Nijenhuis tensor of the almost complex structure } P_0 \\
on TM \text{ defined by } f(\ast, Q) \text{ over the open set } \mathcal{U} \text{ (where } Q \text{ is fixed);}
\]
\[
F_{ij} = -P_r[\Gamma_{ij}^r, Q] \frac{\partial}{\partial Q} P_j^r \hat{\theta}_r + P_j^r[\Gamma_{ij}^r, Q] \frac{\partial}{\partial Q} P_i^r \hat{\theta}_l \\
- P\left(\left[\Gamma_{ij}^r, Q\right] \frac{\partial}{\partial Q} P_i^l \hat{\theta}_l - \left[\Gamma_i^r, Q\right] \frac{\partial}{\partial Q} P_j^r \hat{\theta}_r \right);
\]
\[
G(\theta_i, \theta_j) = \left[ R(\theta_i \wedge \theta_j - P\theta_i \wedge P\theta_j) + Q R\left( P\theta_i \wedge \theta_j + \theta_i \wedge P\theta_j \right), Q \right].
\]

Proof. — The curvature tensor is
\[
\Gamma(\theta_i, \theta_j) = \nabla_{\theta_i} \nabla_{\theta_j} - \nabla_{\theta_j} \nabla_{\theta_i} - \nabla_{[\theta_i, \theta_j]} = R^l_{kij} \theta^*_k \otimes \theta_l,
\]
with \(R^l_{kij} = g\left( R(\theta_i, \theta_j) \theta_k, \theta_l \right)\). Hence,
\[
\Gamma(\theta_i, \theta_j) \theta_k = \nabla_{\theta_i} (\Gamma_{jk}^m \theta_m) - \nabla_{\theta_j} (\Gamma_{ik}^m \theta_m) - \nabla_{(\Gamma_{ij}^m - \Gamma_{ij}^m) \theta_m} \theta_k
\]
yields
\[
R_{ijk}^l = \theta_i (\Gamma_{jk}^l) - \theta_j (\Gamma_{ik}^l) + [\Gamma_i^l, \Gamma_j^l]_k - (\Gamma_i^l - \Gamma_j^l) \Gamma_{ik}^l.
\]
To finish the proof of the proposition we need the following lemma.

Lemma 1. — The Lie bracket of \(\hat{\theta}_i\) with \(\hat{\theta}_j\) satisfies:
\[
[\hat{\theta}_i, \hat{\theta}_j] = \left[ \theta_i, \theta_j \right] - \left[ R^*_{ij}, Q \right] \frac{\partial}{\partial Q}.
\]

Proof of Lemma 1. From \(\hat{\theta}_i = \theta_i - [\Gamma_i^r, Q] \frac{\partial}{\partial Q}\) we can deduce that:
\[
[\hat{\theta}_i, \hat{\theta}_j] = \left[ \theta_i - [\Gamma_i^r, Q] \frac{\partial}{\partial Q}, \theta_j - [\Gamma_j^r, Q] \frac{\partial}{\partial Q} \right]
\]
\[
= \left[ \theta_i, \theta_j \right] - [\theta_i (\Gamma_{ji}^r), Q] \frac{\partial}{\partial Q} + [\theta_j (\Gamma_{ij}^r), Q] \frac{\partial}{\partial Q} - [\Gamma_i^r, \Gamma_j^r], Q \right] \frac{\partial}{\partial Q}
\]
\[
= \left( \Gamma_{ij}^m - \Gamma_{ji}^m \right) \theta_m - \left[ \theta_i (\Gamma_{ji}^r), \theta_j - (\Gamma_{ij}^r) \right] \frac{\partial}{\partial Q}
\]
\[
= \left( \Gamma_{ij}^m - \Gamma_{ji}^m \right) \theta_m - \left[ R_{ij}^m, Q \right] \frac{\partial}{\partial Q}
\]
\[
= \left[ \theta_i, \theta_j \right] - \left[ R^*_{ij}, Q \right] \frac{\partial}{\partial Q}.
\]

We can now complete the proof of Proposition 1. The Nijenhuis tensor is given by
\[
N(\hat{\theta}_i, \hat{\theta}_j) = [\| f \hat{\theta}_i, \| f \hat{\theta}_j] - \| f \left( [\| f \hat{\theta}_i, \hat{\theta}_j] + [\| f \hat{\theta}_i, \| f \hat{\theta}_j] \right) \] - [\hat{\theta}_i, \hat{\theta}_j],
\]
where:

\[ \mathbb{J}_f \hat{\theta}_i, \mathbb{J}_f \hat{\theta}_j = \left[ P_i^\dagger \hat{\theta}_i, P_j^\dagger \hat{\theta}_j \right] = \hat{P}_i \hat{\theta}_i, (P_j^\dagger) \hat{\theta}_j - \hat{P}_j \hat{\theta}_j, (P_i^\dagger) \hat{\theta}_i + P_i^\dagger P_j^\dagger [\hat{\theta}_i, \hat{\theta}_j] \]

\[ \mathbb{J}_f \hat{\theta}_i, \mathbb{J}_f \hat{\theta}_j + \hat{\theta}_i, \mathbb{J}_f \hat{\theta}_j = [P_i^\dagger \hat{\theta}_i, \hat{\theta}_j] + [P_j^\dagger \hat{\theta}_j, \hat{\theta}_i] + \hat{\theta}_i, (P_j^\dagger) \hat{\theta}_r + P_j^\dagger [\hat{\theta}_i, \hat{\theta}_r]. \]

By Lemma 1 the horizontal component of the Nijenhuis tensor is:

\[ \mathcal{H}_N(\hat{\theta}_i, \hat{\theta}_j) = \hat{P}_i \hat{\theta}_i, (P_j^\dagger) \hat{\theta}_r - \hat{P}_j \hat{\theta}_j, (P_i^\dagger) \hat{\theta}_i + P_i^\dagger P_j^\dagger [\hat{\theta}_i, \hat{\theta}_r] \]

Fix \( Q \) and denote by \( P_0 \) the almost complex structure on \( TM \), over \( \mathcal{U} \), defined by \( P_0(m) = f(m, Q) \). Then:

\[ \mathcal{H}_N(\hat{\theta}_i, \hat{\theta}_j) = \left[ P_0 \hat{\theta}_i, P_0 \hat{\theta}_j \right] - P_0 \left[ [P_0 \hat{\theta}_i, \hat{\theta}_j] + [\hat{\theta}_i, P_0 \hat{\theta}_j] \right] - [\hat{\theta}_i, \hat{\theta}_j] \]

\[ -P_i^\dagger [\Gamma^i_{\cdot 0}, Q] \frac{\partial}{\partial Q} P_j^\dagger \hat{\theta}_r + P_j^\dagger [\Gamma^i_{\cdot 0}, Q] \frac{\partial}{\partial Q} P_i^\dagger \hat{\theta}_i \]

\[ -P \left( -\hat{\theta}_j, (P_i^\dagger) \hat{\theta}_i + P_i^\dagger [\hat{\theta}_i, \hat{\theta}_j] + \hat{\theta}_i, (P_j^\dagger) \hat{\theta}_r + P_j^\dagger [\hat{\theta}_r, \hat{\theta}_i] \right) - [\hat{\theta}_i, \hat{\theta}_j]. \]

The vertical component of the Nijenhuis tensor is:

\[ \mathcal{V}_N(\hat{\theta}_i, \hat{\theta}_j) = \left[ [R^i_{\cdot 0}, Q] - P_i^\dagger P_j^\dagger [R^i_{\cdot 0}, Q] - Q \left( -P_i^\dagger [R^i_{\cdot 0}, Q] - P_j^\dagger [R^i_{\cdot 0}, Q] \right) \right] \frac{\partial}{\partial Q} \]

\[ = \left[ R(\hat{\theta}_i \wedge \hat{\theta}_j - P\hat{\theta}_i \wedge P\hat{\theta}_j) + QR(P\hat{\theta}_i \wedge \hat{\theta}_j + \hat{\theta}_i \wedge P\hat{\theta}_j, Q) \right] \frac{\partial}{\partial Q} \]

\[ = G(\hat{\theta}_i, \hat{\theta}_j) \frac{\partial}{\partial Q}. \]

In order to prove Theorem 1 we need to study the tensor \( G \) and we set:

\[ \begin{cases} 
G_1(\hat{\theta}_i, \hat{\theta}_j, P) = \hat{\theta}_i \wedge \hat{\theta}_j - P\hat{\theta}_i \wedge P\hat{\theta}_j \\
G_2(\hat{\theta}_i, \hat{\theta}_j, P) = P\hat{\theta}_i \wedge \hat{\theta}_j + \hat{\theta}_i \wedge P\hat{\theta}_j.
\end{cases} \]

An easy computation gives the following lemma.

**Lemma 2.** — Let \((\theta_1, \ldots, \theta_4)\) be an oriented orthonormal frame over an open set \( \mathcal{U} \) and \((I, J, K)\) be the associated basis of \( \wedge^+ \). Then we have:

\[ I = G_1(\theta_1, \theta_2, J) = G_1(\theta_1, \theta_2, K) \]

\[ J = G_1(\theta_1, \theta_3, I) = G_1(\theta_1, \theta_3, K) \]

\[ K = G_1(\theta_1, \theta_4, I) = G_1(\theta_1, \theta_4, J) \]

\[ 0 = G_1(\theta_1, \theta_2, I) = G_1(\theta_1, \theta_3, J) = G_1(\theta_1, \theta_4, K) \]

\[ G_1(\theta_1, \theta_2, aI + bJ + cK) = (1 - a^2)I - abJ - acK \]

\[ G_2(\theta_1, \theta_2, P) = PG_1(\theta_1, \theta_2, P). \]
A) The case where f is the identity

In this section we give a proof of (the well known) part A of Theorem 1:

**Theorem A** [6]. — The complex structure $\mathcal{J}_{Id}$ is integrable if and only if $A$ is a homothety.

The fact that $A$ is a homothety is equivalent to saying that the selfdual Weyl tensor $W^+$ is zero. In that case the metric is said to be anti-selfdual.

**Proof.** — In the local trivialization $\pi^{-1}(U) \simeq U \times \mathbb{C}P^1$ of the previous section the morphism $f = Id$ when restricted to fibers is a holomorphic map, which only depends on the second variable. By Proposition 1 we know that it is sufficient to study $N(\hat{\theta}_i, \hat{\theta}_j)$. We have:

$$F_{ij} = -Q_i[\Gamma^r, Q]\frac{\partial}{\partial Q} Q^r_j \hat{\theta}_r + Q^r_j[\Gamma^r, Q] \frac{\partial}{\partial Q} Q^i_r \hat{\theta}_i$$

$$-Q([\Gamma^r, Q] \frac{\partial}{\partial Q} Q^i_r \hat{\theta}_i - [\Gamma^r, Q] \frac{\partial}{\partial Q} Q^r_j \hat{\theta}_r)$$

$$= -Q_i[\Gamma^r, Q]\hat{\theta}_j + Q^r_j[\Gamma^r, Q] \hat{\theta}_i - Q([\Gamma^r, Q] \hat{\theta}_i - [\Gamma^r, Q] \hat{\theta}_j).$$

Using $[\Gamma^r_i, Q] = [\nabla_{\theta_i}, Q]$ one gets:

$$d\pi(F_{ij}) = -[\nabla_{Q\theta_i}, Q] \theta_j + [\nabla_{Q\theta_j}, Q] \theta_i - Q([\nabla_{\theta_i}, Q] \theta_i - [\nabla_{\theta_i}, Q] \theta_j)$$

$$= -Q_{\nabla_{\theta_i}, Q} \theta_j + Q \nabla_{Q\theta_i} \theta_j + Q \nabla_{Q\theta_j} \theta_i - Q \nabla_{Q\theta_i} \theta_j$$

$$-Q \nabla_{\theta_j} \theta_i + Q \nabla_{\theta_i} \theta_j + Q \nabla_{\theta_j} \theta_i$$

$$= -E(\theta_i, \theta_j).$$

The horizontal component of $N(\hat{\theta}_i, \hat{\theta}_j)$ is then zero. The vertical component is:

$$G(\theta_i, \theta_j) = \left[ R(\theta_i \wedge \theta_j - Q\theta_i \wedge Q\theta_j) + QR(\theta_i \wedge Q\theta_j + Q\theta_i \wedge \theta_j), Q \right].$$

But $Q$ preserves the orientation, hence:

$$\begin{cases} \theta_i \wedge \theta_j - Q\theta_i \wedge Q\theta_j \in \wedge^+ T_m M \\ \theta_i \wedge Q\theta_j + Q\theta_i \wedge \theta_j \in \wedge^+ T_m M. \end{cases}$$

Recall that the matrix of the curvature operator $R$ has the following splitting:

$$R = \begin{pmatrix} A & B \\ B & C \end{pmatrix}.$$

Since the elements of $\wedge^+$ of $\wedge^-$ commute [6], the component $A$ in the matrix $R$ is the only one which matters in the computation of $G(\theta_i, \theta_j)$. By Lemma 2, one has the equality:

$$(\theta_i \wedge \theta_j - Q\theta_i \wedge Q\theta_j) + Q(\theta_i \wedge Q\theta_j + Q\theta_i \wedge \theta_j) = 0, \ \forall \theta_i, \theta_j \in T_m M.$$
Therefore, if the matrix $A$ is a homothety the Nijenhuis tensor of $J_{Id}$ is zero.

Conversely, assume that $J_{Id}$ is integrable. We have noticed that the orthonormal frame $(\theta_1, \ldots, \theta_4)$ over $\mathcal{U}$ defines an oriented orthonormal basis $(I, J, K)$ of $\bigwedge^+ \mathcal{U}$. Since $G(\theta_i, \theta_j) = 0$ for all $i, j \in \{1, \ldots, 4\}$, Lemma 2 implies:

- at the point $(m, I)$, $G(\theta_1, \theta_3) = [A(J) + IA(K), I] = 0$
- at the point $(m, J)$, $G(\theta_1, \theta_2) = [A(I) + JA(-K), J] = 0$
- at the point $(m, K)$, $G(\theta_1, \theta_2) = [A(I) + KA(J), K] = 0$.

Since $(I, J, K)$ is an oriented orthonormal basis, it follows from $IJ = -JI = K$ that relations of the following type hold:

$$[A(J), I] = 2 < A(J), K > J - 2 < A(J), J > K.$$

From the previous system we then deduce the following one:

$$\begin{align*}
<i, J> &= -<IA(K), J> = <A(K), K> \\
<i, K> &= -<IA(K), K> = -<A(K), J> \\
<i, I> &= -<JA(-K), I> = <A(K), K> \\
<k, K> &= -<JA(-K), K> = -<A(K), I> \\
<k, I> &= -<KA(J), I> = <A(J), J> \\
<k, J> &= -<KA(J), J> = -<A(J), I>
\end{align*}$$

But the matrix $A$ in the basis $(I, J, K)$ is symmetric, thus $A$ is a homothety. \hfill $\square$

**B) The case where $f$ is constant**

Integrability theorem

In this section we give a proof of part B of Theorem 1.

**Theorem B.** — Let $(M, g, J_M)$ be an almost complex manifold such that $J_M$ is compatible with the orientation and the metric. The complex structure $J_{\infty}$ is integrable if and only if:

i) $J_M$ is integrable;

ii) the kernel of $A$ contains the subspace $J_M^+ \subset \bigwedge^+ \text{ orthogonal to the line generated by } J_M$ (i.e. $J_M^+ \subset \ker(A)$).
Notice that the integrability condition is not conformal on $g$. Moreover, when $J_\infty$ is integrable, it gives to the twistor projection $\pi: (Z, J_\infty) \rightarrow (M, J_M)$ the structure of a holomorphic $\mathbb{CP}^1$-bundle.

For a complex manifold $(M, g, J_M)$ we have a decomposition $\mathbb{C} \otimes TM = T^{1,0} \oplus T^{0,1}$ into $\pm i$ eigenspaces of $J_M$. We then obtain:

$$\left\{ \begin{array}{l}
\mathbb{C} \otimes \Lambda^+ = \mathbb{C}J_M \oplus^\perp (\Lambda^{2,0} \oplus \Lambda^{2,0}) \\
\mathbb{C} \otimes \Lambda^- = \{ \psi \in \Lambda^{1,1} \mid \langle \psi, J_M \rangle = 0 \} 
\end{array} \right.$$  

where $$\left\{ \begin{array}{l}
\Lambda^{2,0} = T^{1,0} \wedge T^{1,0} \\
\Lambda^{1,1} = T^{1,0} \wedge T^{0,1} 
\end{array} \right.$$  

Condition ii) says that $(\Lambda^{2,0} \oplus \Lambda^{2,0}) \subset \ker(A)$. For a Kählerian manifold the curvature $R$ may be viewed as a symmetric endomorphism of $\Lambda^{1,1}$, so in some orthonormal basis compatible with these decompositions we have $A = \begin{bmatrix}
\frac{s}{4} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}$ and $W^+ = \begin{bmatrix}
\frac{s}{6} & 0 & 0 \\
0 & -\frac{s}{12} & 0 \\
0 & 0 & -\frac{s}{12}
\end{bmatrix}$. We then have the following result:

**Proposition 3.** — For any Kählerian surface $(M, g, J_M)$ the complex structure $J_\infty$ on $(Z, \tilde{g})$ is integrable. Furthermore, if $(M, g, J_M)$ is Kähler and the scalar curvature of $g$ is never zero, then $J_\infty$ and $J_{-\infty}$ (the compatible complex structure on $(Z, \tilde{g})$ associated to $-J_M$) are the only compatible complex structures on $(Z, \tilde{g})$.

In other terms, for a Kählerian manifold whose scalar curvature is non zero there are, even locally, only two compatible complex structures on its twistor space.

**Proof.** — The first part being a consequence of Theorem B, we only need to prove the second part of the proposition. Let $J_f$ be a compatible complex structure on $(Z, \tilde{g})$ and assume that the scalar curvature of $(M, g, J_M)$ is never zero. One can build an orthonormal basis $(I, J, K)$ of $\Lambda^+$ over an open set $U$ as follows. Setting $I = J_M$, pick any unitary vector $J$ orthonormal to $I$ and define $K = IJ$. For any $m \in U$, there exists $(a, b, c) \in S^2$ such that $f(m, J) = aI + bJ + cK$. But, as $(M, g, J_M)$ is Kähler, in this basis we have $A = \begin{bmatrix}
\frac{s}{4} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}$. Let $\theta_1$ be a unitary vector field defined over $U$; set $\theta_2 = I\theta_1$. As $J_f$ is integrable, $G(\theta_1, \theta_2)$ is identically zero on $U$. In particular, at the point $(m, J)$ we obtain:

$$G(\theta_1, \theta_2) = 0$$  

$$= [A((1-a^2)I - abJ - acK) + JA(cJ - bK), J] = 0$$  

$$= [(1-a^2)\frac{s}{4}I, J] = (1-a^2)\frac{s}{2} K.$$
Therefore $a = \pm 1$, that is $f(m,J) = \pm I$ for all $J$ orthonormal to $I$. Since $f$ must be holomorphic in the fibers we get that $f$ is constant, equal to $I$ or $-I$.

Proof of Theorem B. — By Proposition 1, it is sufficient to check that $N(\hat{\theta}_i,\hat{\theta}_j) = 0$. As $f$ is constant on fibers we always have $F_{ij} = 0$. Therefore: $\mathbb{J}_\infty$ integrable $\iff E(\theta_i,\theta_j) = G(\theta_i,\theta_j) = 0 \iff \{J_M$ integrable and $G(\theta_i,\theta_j) = 0\}$. But for all $\theta_i,\theta_j \in TM$ we have

\[\begin{align*}
\{ \theta_i \wedge \theta_j - J_M \theta_i \wedge J_M \theta_j \in J_M^\perp \} & \\
J_M \theta_i \wedge \theta_j + \theta_i \wedge J_M \theta_j \in J_M^\perp & .
\end{align*}\]

Consequently, if $J_M^\perp \subset \ker(A)$ we obtain $G(\theta_i,\theta_j) = 0$ for all $\theta_i,\theta_j \in TM$.

Conversely, suppose that $\mathbb{J}_\infty$ is integrable. Set $J_0 = J_M$. Locally over an open set $U$ one can complete $\{J_0\}$ to get an oriented orthonormal basis $(I_0, J_0, K_0)$ of $\mathbb{J}^\perp$. Let $\theta_1$ be a unitary vector field defined over $U$; set $\theta_2 = I_0 \theta_1$. If $G = 0$, then, for all $m \in U$ and $Q \in \pi^{-1}(m)$, Lemma 2 implies that at the point $(m,Q)$:

\[G(\theta_1,\theta_2) = [A(I_0) + QA(-K_0), Q] = 0.\]

In particular, for $Q = A(K_0)$, we have $[A(I_0), A(K_0)] = 0$ and it follows that $A(K_0) = cA(I_0)$ for some constant $c$. The former equation yields:

\[
\forall Q \in \pi^{-1}(m), \quad 0 = [A(I_0) + QA(-K_0), Q] = (Id - cQ)[A(I_0), Q] \implies A(I_0) = 0.
\]

Therefore $J_0^\perp = \text{Vect}(I_0, K_0) \subset \ker A$.

Recall that we have a characterization of an integrable almost complex structure $J_M$ on $M$ in terms of the twistor space and one of the Kählerian complex structures.

**Proposition** (see, for example, [37, 15]). — Let $J_M$ be a Hermitian almost complex structure on $(M,g)$. Then:

- $J_M$ is integrable if and only if the associated section of the twistor space, $s : (M,J_M) \longrightarrow (Z,\mathbb{J}Id)$, is almost holomorphic, that is: the differential $ds$ satisfies $ds \circ J_M = \mathbb{J}Id \circ ds$;
- $J_M$ is Kähler if and only if $s$ is an horizontal section, that is to say: the tangent space of the submanifold $s(M) \subset Z$ is included in the horizontal distribution.

It is well known that the existence of a Kähler metric on a compact complex surface $(M,J_M)$ is equivalent for the first Betti number $b_1$ to be even [27, 39, 25]. Theorem B gives a new characterization of compact Kählerian surfaces in terms of compatible complex structures on the associated twistor spaces.
Proposition 4. — A compact almost Hermitian 4-dimensional manifold \((M, g, J_M)\) is Kähler if and only if \(J_\infty\) is integrable.

In section D we will deduce from that proposition a characterisation of compact scalar-flat Kähler manifolds in terms of compatible complex structures on \((Z, \tilde{g})\) (cf. Proposition 8).

Proof. — Let \(\theta\) be the Lee form of \((M, g, J_M)\) defined by \(dJ_M = -2\theta \wedge J_M\), where \(J \in \bigwedge^+\) is viewed as a 2-form. Denote by \(\kappa\) the conformal scalar curvature, which is related to the scalar curvature \(s\) by \(\kappa = s + 6(\delta\theta - |\theta|^2)\). The condition \(J_M^+ \subset \ker A\) is equivalent to the following: the selfdual Weyl tensor \(W^+\) is degenerate (meaning that, in every point, two of the eigenvalues coincident) and the scalar curvature of \((M, g)\) is equal to the conformal scalar curvature \([3]\). This is also equivalent to \(\delta\theta = |\theta|^2\).

Integrating this expression over \(M\) gives \(\theta = 0\) by the Brochner-Grenn theorem. But \((M, g, J_M)\) is Kähler if and only if \(\theta\) vanishes identically. □

Corollary 2. — Assume that a compact 4-dimensional manifold \((M, g)\) admits two almost complex structures \(J_1 \neq \pm J_2\) compatible with the metric and the orientation. Then the associated compatible almost complex structures \(J_{\infty 1}, J_{\infty 2}\) on \((Z, \tilde{g})\) are integrable if and only if \(\{J_1, J_2\}\) spans a hyperkähler structure on \((M, g)\).

Proof. — By Proposition 4, \(J_{\infty 1}\) and \(J_{\infty 2}\) are integrable if and only if \(J_1\) and \(J_2\) are Kähler. As \(J_1 \neq \pm J_2\), then \(J_1\) is different from \(\pm J_2\) everywhere. The holonomy of \(g\) reduces to \(U(2)\) by \(J_1\) and further to \(SU(2)\) by \(J_2\). This says that \(g\) is hyperkähler. □

Study of the manifold \((Z, J_\infty)\)

Any scalar-flat Kähler surfaces \((M, g, J_M)\) is automatically anti-selfdual \([19]\). For such a manifold we can put two natural complex structures on its twistor space: \(J_{Id}\) and \(J_\infty\). The next proposition shows that these complex structures are never deformation of each other.

Proposition 5. — If \((M, g, J_M)\) is a scalar-flat Kähler surface, the complex structure \(J_\infty\) on \(Z\) is never a deformation of the complex structure \(J_{Id}\).

Proof. — It is sufficient to show that \((Z, J_{Id})\) and \((Z, J_\infty)\) do not have the same Chern classes. Let \(h\) be the generator of the second cohomology group \(H^2(\mathbb{C}P^1, \mathbb{Z}) \simeq \mathbb{Z}\). By Leray-Hirsch theorem’s \([12]\) the cohomology ring of \(Z\) is a \(H^*(M, \mathbb{R})\)-module generated by \(h\) with relation \(4h^2 = 3\tau + 2\chi\), where
\[ \tau \text{ and } \chi \text{ are the signature and the Euler characteristic of } M. \] Denote by \( c_1(J_M) \) the first Chern class of the manifold \((M, J_M)\). Under this notation we have:

\[
c(J_{Id}) = 1 + 4h + 3\tau + 3\chi + 2h\chi \quad [22]
\]

\[
c(J_\infty) = (1 + 2h)(1 + c_1(J_M) + \chi)
= 1 + 2h + c_1(J_M) + 2hc_1(J_M) + \chi + 2h\chi.
\]

If the complex structures were deformations of each other, they would have the same Chern numbers:

\[
c_1(J_{Id})^3 = 16(3\tau + 2\chi)h = c(J_\infty)^3 = 8(3\tau + 2\chi)h.
\]

This forces \( 3\tau + 2\chi = 0 \). Let \( \mu_g \) be the volume form on \( M \) associated to the metric \( g \); by the Gauss-Bonnet formula [2], [20]:

\[
3\tau + 2\chi = \frac{1}{4\pi^2} \int_M 2\|W^+\| + \frac{1}{24}s^2 - 2\|B\|^2 \mu_g = -\frac{1}{2\pi^2} \int_M \|B\|^2 \mu_g.
\]

Thus, \( 3\tau + 2\chi = 0 \) implies \( B = 0 \). As the scalar curvature of \((M, g)\) is supposed to be zero, the manifold \((M, g, J_M)\) would be Ricci-flat, hence \( c_1(J_M) = 0 \). Therefore the first Chern classes of \((Z, J_{Id})\) and of \((Z, J_\infty)\) are different and these two manifolds are never deformations of each other. \( \square \)

When \((M, g, J_M)\) is a complex spin surface, Hitchin has shown that there exists a holomorphic line bundle \( L \to M \) such that \( L \otimes L = K_M \) is the canonical line bundle [21]. Then, the twistor space \( Z \) can be identified, in a \( C^\infty \)-way, to the projectivization bundle \( \mathbb{P}(L \oplus L^*) \) [36]. By this construction we see that the manifold \( Z \simeq \mathbb{P}(L \oplus L^*) \) admits a natural complex structure denoted by \( \mathbb{I} \). When \((M, g, J_M)\) is not spin, but only complex, the bundle \( L \oplus L^* \) exists only locally. Nevertheless, the projectivization \( \mathbb{P}(L \oplus L^*) \) still exists globally, due to the fact that the transition functions on \( L \oplus L^* \) are well defined holomorphic maps up to sign. In general \( \mathbb{I} \) is not a compatible complex structure on \((Z, \bar{g})\).

Now, if \((M, g, J_M)\) satisfies the conditions of Theorem B, we can put another complex structure on its twistor space, namely \( \mathbb{J}_\infty \). The question is then to determine the relationship between the manifolds \((Z, \mathbb{I})\) and \((Z, \mathbb{J}_\infty)\). In that direction we have the following result.

**Proposition 6.** — Let \((M, g, J_M)\) be a manifold satisfying conditions of Theorem B (i.e. \( \mathbb{J}_\infty \) integrable). The complex structures \( \mathbb{I} \) and \( \mathbb{J}_\infty \) on \( Z \) are deformations of each other: there exists on \( Z \) a path of integrable complex structures \( \mathbb{J}_t, t \in [0, 1] \), connecting \( \mathbb{I} \) to \( \mathbb{J}_\infty \).

By combining this result and [41, Theorem 4.1] we obtain another proof of Proposition 5.
Proof. — In an appropriate local trivialization of the bundle $Z \to M$, the almost complex structure $I$ on $U \times S^2$ can be identified with the product structure $J_M \times J_{CP^1}$. Let $(\theta_1, \theta_2, \theta_3, \theta_4)$ be an oriented orthonormal frame defined over $U$ providing this trivialization. Set $\hat{\theta}_{i,t} = \theta_i - t[\Gamma^i, Q] \frac{\partial}{\partial Q}$ for $t \in [0,1]$. The subspace $H_t = \text{Vect}(\hat{\theta}_{1,t}, \ldots, \hat{\theta}_{4,t})$ is in direct sum with the vertical distribution $V_z$ and can be glued into a global distribution over $Z$. Define the almost complex structure $J_t$ on $\pi^{-1}U$ as follows: endow $V_z$ with the complex structure of the fibers (complex projective lines) and pull back on $H_t \simeq T_m M$ the complex structure $J_M$. Then, $J_t$ is a path of almost complex structures from $I$ to $J_\infty$. The integrability of $J_t$ is shown in the same way as that of $J_\infty$. □

C) The case where $f$ is a homothety

Integrability theorem

In this section we prove part C of Theorem 1.

Theorem C. — Let $(M, g, J_M)$ be a Kähler surface. For all complex $\lambda \notin \{0,1\}$ the almost complex structure $J_{\lambda Id}$ is integrable if and only if the scalar curvature of $g$ is zero.

The condition $A = 0$ is equivalent to saying that the metric $g$ is Hermitian scalar-flat and anti-selfdual. These metrics are called optimal by LeBrun because they are absolute minimizers of the functional $K(g) = \int_M |R|^2 d\text{vol}$ [26]. Let $(M, g, J_M)$ be a compact scalar-flat Kähler surface and $c_1(M)$ be the real first Chern class of $(M, J_M)$. Two possibilities may occur [24]. Either $c_1(M) = 0$ and $(M, g, J_M)$ is then finitely covered by a hyperkähler surface, i.e. a flat torus or a $K3$-surface with Ricci-flat Kähler metric [13], [29]. Or $c_1(M) < 0$, in which case $(M, g)$ is obtained by blowing up a ruled surface [23], i.e. $(M, g)$ is obtained by blowing up $m$ points on a $\mathbb{C}P^1$-bundle over a Riemann surface of genus $\gamma$. The condition $c_1(M) < 0$ gives a lower bound on the number of points $m$ to be blown up: namely $m \geq 9$ when $\gamma = 0$, $m \geq 1$ when $\gamma = 1$ and there is no restriction for $\gamma > 1$. Conversely we have:

Theorem [23]. — A ruled surface $M$ has a blow-up $\tilde{M}$ which is a scalar-flat Kähler surface. Moreover, any further blow up of $\tilde{M}$ admits a scalar-flat Kähler metric.

For simply connected manifold we have the following result:
THEOREM [34]. — A smooth compact simply connected 4-manifold \( M \) admits a scalar-flat Kähler structure if, and only if, \( M \) is diffeomorphic to a \( K3 \)-surface or to the connected sum \( \mathbb{C}P^2_k \mathbb{C} \mathbb{P}^2 \) for some \( k \geq 10 \).

Proof of Theorem C. — By Propositions 1 & 2, if \( A = 0 \) it is enough to show that \( E(\theta_i, \theta_j) + F_{ij} = 0 \) to get the integrability of \( \mathbb{J}_M^4 \). Let \((m, Q)\) be a point of \( Z \). There exists an orthonormal basis \((\theta_1, \ldots, \theta_4)\) over an open set \( \mathcal{U} \) such that \( I = J_M \) and \( Q \simeq a I + b J \), for some \((a, b) \in \mathbb{S}^1 \). As \( J_M \) is Kähler, we know that \( \Gamma^i_k = \nabla_{\theta_i} \) belongs to the commutator of \( I \), for all \( i \). Hence, \([\Gamma^i_k, Q] \frac{\partial}{\partial Q} = [\nabla_{\theta_i}, b J] \frac{\partial}{\partial Q}\) is in the subspace of \( T_Q \mathbb{S}^2 \) generated by \( K \). Viewing \( \mathbb{S}^2 \) as a subset of \( \mathbb{R} \times \mathbb{C} \), with coordinates \((a, z)\), the application \( f = \lambda Id \) has the following form:

\[
f: \quad \mathcal{U} \times \mathbb{S}^2 \quad \rightarrow \quad \mathcal{U} \times \mathbb{S}^2
\]

\[
\left( m, (a, z) \right) \quad \mapsto \quad \left( m, (f_1(a), f_2(a) \lambda z) \right)
\]

Where \( f_1, f_2 \) only depend on \(|\lambda|\). Thus \( df(K) = f_2(a) \lambda K \). According to these notations we have at the point \((m, Q)\):

\[
d\pi(F_{ij}) = -df([\nabla_{\theta_i}, Q]) \theta_j + df([\nabla_{\theta_j}, Q]) \theta_i - P \left( df([\nabla_{\theta_i}, Q]) \theta_i - df([\nabla_{\theta_j}, Q]) \theta_j \right)
\]

\[
= f_2(a) \lambda \left( - [\nabla_{\theta_i}, b J] \theta_j + [\nabla_{\theta_j}, b J] \theta_i - P \left( [\nabla_{\theta_i}, b J] \theta_i - [\nabla_{\theta_j}, b J] \theta_j \right) \right)
\]

\[
= - [\nabla_{\theta_i}, f_2(a) \lambda b J] \theta_j + [\nabla_{\theta_j}, f_2(a) \lambda b J] \theta_i - P \left( [\nabla_{\theta_i}, f_2(a) \lambda b J] \theta_i - [\nabla_{\theta_j}, f_2(a) \lambda b J] \theta_j \right)
\]

One can conclude as in section A that \( d\pi(F_{ij}) = -E(\theta_i, \theta_j) \) and \( \mathbb{J}_M^4 \) is integrable.

Conversely, assume that \( \mathbb{J}_M^4 \) is integrable. Proposition 3 implies that the scalar curvature is zero, hence \( A = 0 \).

Study of the manifold \((Z, \mathbb{J}_M^4)\)

When \((M, g, J_M)\) is Kähler, the tangent bundle admits a \( \mathbb{C} \)-action which commutes with the holonomy group of the metric \( g \). The action of any \( \lambda \in \mathbb{S}^1 \) lifts naturally to a smooth action on the total space \( Z \) inducing the identity on the base manifold \( M \). This lift coincides with the homothety \( \lambda^2 Id \). Therefore, \((Z, \mathbb{J}_M^4)\) is isomorphic to \((Z, J^4)\) for each \( \lambda \in \mathbb{S}^1 \). Using Theorem A&C we recover the result from [19]: for any Kählerian surfaces \((M, g, J_M)\), the metric \( g \) is anti-selfdual if, and only if, \( g \) is scalar-flat.
At least two cases may occur.

Firstly, all the \((Z, \mathbb{J}_{\lambda Id})\) are biholomorphic to \(\mathbb{J}_{Id}\). Thus there exists a 1-dimensional family of biholomorphisms of \((Z, \mathbb{J}_{Id})\). We will see in section F that this is the case for any bi-elliptic surface (quotient of a flat torus).

Secondly, there is no one complex-parameter family of automorphisms of \((Z, \mathbb{J}_{Id})\). Then, we have a 1-dimensional family of non isomorphic complex structures on \(Z\). For example, if one blows-up at least 10 points in \(\mathbb{C}P^2\), one gets \(\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}\) for some \(k \geq 10\). This manifold admits a scalar-flat Kähler metric \(g\) [34] but there is no non trivial conformal map from \((\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}, g)\) to itself. Thus, on its twistor space, there does not exist any 1-dimensional family of biholomorphisms. Therefore, the structures \((Z, \mathbb{J}_{\lambda Id}), \lambda \in \mathbb{C}^\star\), give a 1-dimensional family of non isomorphic complex structures.

D) Metric properties on \(M\) in terms of compatible complex structures on \((Z, \tilde{g})\)

We can use the almost complex structures \(\mathbb{J}_f\) to characterize some properties of the metric \(g\) on \(M\). Indeed, by (the well known) Theorem A we have that \(g\) is anti-selfdual if and only if \(\mathbb{J}_{Id}\) is integrable. We showed that a compact almost Hermitian manifold \((M, g, J_M)\) is Kähler if and only if \(\mathbb{J}_\infty\) is integrable; furthermore the integrability of \(\mathbb{J}_{Id}\) and \(\mathbb{J}_\infty\) is equivalent to \((M, g, J_M)\) scalar-flat Kähler (cf. Proposition 8).

When limiting to the case where \((M, g)\) is anti-selfdual, we can give a characterization of metrics which are scalar-flat in terms of compatible complex structures on \((Z, \tilde{g})\). According to the terminology of LeBrun these provide examples of optimal metrics, in compact case [26].

**THEOREM D.** — *Let \((M, g)\) be an anti-selfdual Riemannian manifold. The following are equivalent:

- the scalar curvature of \(g\) is flat;
- every \(m \in M\) has an open neighborhood \(U\) such that \(Z\) admits, over \(U\), an integrable compatible complex structure \(\mathbb{J}_f\) for at least one (and then infinitely many) morphism(s) \(f \neq Id\).*

In other words, if \((M, g)\) is an anti-selfdual metric with non zero scalar curvature then, even locally on \(Z\), the only integrable almost complex structure among the \(\mathbb{J}_f\)’s is \(\mathbb{J}_{Id}\). This result should be compared to the following result of Salamon:
Proposition [38] (see also [33]). — A metric $g$ on $M$ is anti-selfdual if, and only if, locally around each point $m \in M$ there exist infinitely many compatible complex structures on $(M, g)$.

In a similar direction, Pontecorvo gives a conformal characterization of scalar-flat Kähler surfaces among anti-selfdual Hermitian surfaces. Indeed, let $(M, g, J_M)$ be an anti-selfdual complex Hermitian manifold. The complex structure $J_M$ on $M$ defines a section $s : Z \to M$ [15], whose image will be noted $\Sigma = s(M)$. Similarly, the hypersurface $\Sigma = \sigma(\Sigma)$ of $Z$ corresponds to the conjugate complex structure $-J_M$. Let $X$ be the divisor $\Sigma + \Sigma$ in $Z$ and consider the holomorphic line bundle $[X]$. Denote by $K_Z$ the canonical line bundle of $(Z, \mathbb{J}_{Id})$.

Proposition [30]. — Let $(M, g, J_M)$ be a Hermitian anti-selfdual manifold. The line bundles $[X]$ and $-\frac{1}{2} K_Z$ are isomorphic if and only if $g$ is conformal to a scalar-flat Kähler metric.

Notice that Theorem 1 and Proposition 3&4 give a non conformal characterization of compact scalar-flat Kähler surfaces.

Proposition 8. — Let $(M, g, J_M)$ be a compact almost Hermitian manifold. The following are equivalent:

- the metric $g$ is scalar-flat Kähler;
- the compatible complex structures $\mathbb{J}_{Id}$ and $\mathbb{J}_\infty$ on $(Z, \tilde{g})$ are integrable;
- the compatible complex structures $\mathbb{J}_{\lambda Id}$ and $\mathbb{J}_\infty$ on $(Z, \tilde{g})$ are integrable.

Proof. — A Kählerian surface $(M, g, J_M)$ is scalar-flat if and only if $g$ is anti-selfdual [19]. Then, it follows from Proposition 3&4 and Theorem 1 that: $\{\mathbb{J}_\infty$ and $\mathbb{J}_{\lambda Id}$ are integrable\} $\iff \{g$ is scalar-flat Kähler\} $\iff \{(M, g, J_M)$ is anti-selfdual Kähler\} $\iff \{\mathbb{J}_\infty$ and $\mathbb{J}_{Id}$ are integrable\}. □

Proof of Theorem D. — Let $(M, g)$ be a scalar-flat anti-selfdual manifold, its twistor space is complex and $(M, g)$ admits, locally, at least one complex structure $J_M$ [38]. Then Theorem B ensures that the locally defined almost complex structure $\mathbb{J}_\infty$ on $Z$ is integrable. Actually, as soon as $(M, g)$ is anti-selfdual there are, locally, infinitely many integrable complex structures $J_M$ on $M$ and so, when $g$ is also scalar-flat, there are infinitely many integrable complex structures $\mathbb{J}_\infty$ on $Z$.

Conversely, let $(M, g)$ be a manifold with an anti-selfdual metric $g$ having non zero scalar curvature. Let $f : Z \to Z$ be a morphism such that $\mathbb{J}_f$ is integrable over an open set $U$. Let $(m, Q)$ be a point in $\pi^{-1}(U)$ and set...
\[ f(m,Q) = P. \] According to our notations, if \( \mathcal{U} \) is small enough we can build an orthonormal basis \( (\theta_1, \ldots, \theta_4) \) of vector fields on \( M \) such that \( P = J = \theta_1 \land \theta_3 - \theta_2 \land \theta_4 \). Then there exists \( (a,b,c) \in S^2 \) such that \( Q = aI + bJ + cK \).

As \( Jf \) is integrable, \( G(\theta_1, \theta_2) \) vanishes everywhere. In particular, at the point \( (m, Q) \) one obtains:

\[
G(\theta_1, \theta_2) = 0 = \frac{s}{12} [I - QK, Q] = \frac{2s}{12} (acI - c(1 - b)J + (b(1 - b) - a^2)K) \quad \implies \quad \begin{cases} 
ac = 0 \\
(c(1 - b) = 0 \\
b = a^2 + b^2 
\end{cases}
\]

Therefore we have \( Q = J = P \) for every \( (m, Q) \in \pi^{-1}(\mathcal{U}) \), that is to say \( f = Id \). \( \Box \)

E) Compatible complex structure on locally conformally Kähler surfaces

The aim of this section is to give a local description of the set \( \mathcal{I} \) of integrable compatible complex structures on the twistor space \( (Z, \tilde{g}) \) of a compact locally conformally Kähler (abbreviated in l.c.k.) surface \( (M, g, J_M) \). This condition is equivalent to \( W^+ \) being degenerate, which means that at each point of \( M \) at least two eigenvalues of \( W^+ \) coincide.

We start by recalling the main results about the l.c.k. surfaces.

A result by Tricerri, generalizing the analogous result in the Kähler case, shows that it is enough to understand minimal complex surfaces.

**Proposition [40].** — A complex surfaces \( (M, g, J_M) \) is l.c.k if and only if the blow-up of \( M \) at a point is l.c.k.

When the first Betti number \( b_1 \) is even, a l.c.k. surface is globally Kähler.

**Proposition [42].** — Every l.c.k. metric on a compact surface \( (M, J_M) \) with even first Betti number is globally conformally Kähler.

When the first Betti number is odd and the Euler characteristic is zero, we have a classification due to Belgun, Gauduchon-Ornea, Tricerri, Vaisman.

**Proposition [9].** — The complete list of compact minimal l.c.k. surfaces with odd first Betti number and zero Euler characteristic is:

i) the properly elliptic surfaces (i.e. surfaces with \( \text{Kod}(M) = 1 \) and \( b_1 \) odd);

ii) the Kodaira surfaces (i.e. surfaces with \( \text{Kod}(M) = 0 \) and \( b_1 \) odd);
iii) the Hopf surfaces;
iv) the Inoue-Bombieri surfaces different from $S_{n,u}^-$ with $u \not\in \mathbb{R}$ [40].

When the first Betti number is odd and the Euler characteristic is non zero, the only other possible case is that of surfaces of class VII with $0 < \chi = b_2$ [7], for which there is (yet) no classification. (For some existence results see [18].)

Let $\mathcal{J}$ be a compatible almost complex structure on $(Z, \tilde{g})$. We say that $\mathcal{J}$ is semi-integrable if the vertical component of the Nijenhuis tensor is zero.

Denote by $I_{\frac{1}{2}}$ (resp. $I$) the set of semi-integrable (resp. integrable) compatible complex structures on $(Z, \tilde{g})$. Propositions 1 and 2 give a necessary and sufficient condition for $\mathcal{J}$ to be semi-integrable, or integrable. The set $\mathcal{I}$ on a l.c.k. manifold $(M, g, J)$ depends on the spectrum of the operator $A = W^+ + \frac{s}{12}$. Let $\kappa$ be the conformal curvature defined in the proof of proposition 4. Then on an adapted basis we have:

$$A = W^+ + \frac{s}{12} I_d = \begin{bmatrix} 2\kappa & 0 & 0 \\ 0 & \frac{-s}{12} & 0 \\ 0 & 0 & \frac{-s}{12} \end{bmatrix} + \begin{bmatrix} \frac{s}{12} & 0 & 0 \\ 0 & \frac{s}{12} & 0 \\ 0 & 0 & \frac{s}{12} \end{bmatrix} = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{bmatrix}.$$

Moreover $J_M$ is actually an eigenvector of $W^+$ for the simple eigenvalue $\frac{s}{6}$.

**Theorem 2.** — Let $(M, g, J_M)$ be a compact surface l.c.k., if we don’t have $x = y = 0$ we note $\frac{x}{y} \in \mathbb{R} \cup \{\infty\}$. On an open set $\mathcal{U}$ of $M$ :

A) We have $x = y = 0$ if, and only if, on $\mathcal{U}$ one of the following equivalent conditions hold:

i) $(M, g, J_M)$ is scalar-flat Kähler.

ii) $g$ anti-selfdual scalar-flat.

iii) The compatible complex structures $\mathcal{J}_I$, $\mathcal{J}_\infty$ and $\mathcal{J}_{\lambda I}$ are integrable.

iv) The cardinal of $\mathcal{I}$ is infinity. This is the case globally if, and only if, $(M, g, J_M)$ is a flat torus (or a quotient), a $K3$-surface with a Calabi-Yau metric (or a quotient), a $\mathbb{C}P^1$-bundle over a Riemann surface $\Sigma_\gamma$ of genus $\gamma > 1$ with the conformally flat Kähler metric which locally is a product of the $(+1)$-curvature metric on $\mathbb{C}P^1$ and $(-1)$-curvature metric on $\Sigma_\gamma$ [14], [31].

B) We have $\frac{x}{y} = \infty$ if, and only if, on $\mathcal{U}$ one of the following equivalent conditions hold:

i) $(M, g, J_M)$ is Kähler with $s \neq 0$.

ii) $\mathcal{I} = I_{\frac{1}{2}} = \{\mathcal{J}_\infty, \mathcal{J}_\infty\}$. 

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This is the case globally on $M$ if $(M, g, J_M)$ is Kähler-Einstein not Ricci-flat (that is a Fano manifolds or a manifold where the canonical line bundle is ample).

C) We have $|\frac{x}{y}| \leq 1$ if, and only if, on $U: \mathcal{I}_\frac{1}{2} = \{J_{e^{\pm i \theta} I_d}\}$ where $\cos \theta = \frac{x}{y}$.

D) We have $\infty \neq |\frac{x}{y}| \geq 1$ if, and only if, on $U: \mathcal{I}_\frac{1}{2} = \{J_{u_1 I_d}, J_{u_2 I_d}\}$ where $u_1 = \frac{1 + \sin \theta}{\cos \theta}$, $u_2 = \frac{1 - \sin \theta}{\cos \theta}$ and $\cos \theta = (\frac{x}{y})^{-1}$.

Remark. — We have $\frac{x}{y} = 1$ if, and only if, $(M, g, J_M)$ is anti-selfdual with $s \neq 0$. If it is the case globally then $(M, J_M)$ must be in class VII [14]. We can find some global example of manifolds $(M, g, J_M)$ with arbitrary $\frac{x}{y}$ in [5].

Proof of A. — The multiplicity of the eigenvalue 0 of $A$ is equal to 3 if, and only if, $(M, J_M, g)$ scalar-flat Kähler if, and only if, $(M, J_M, g)$ anti-selfdual scalar-flat [14] if, and only if, $J_{Id}, J_{\infty}$ and $J_{\lambda Id}$ integrable by proposition 8.

The equivalence with condition iv) will be a consequence of (the rest of the proof of) the theorem. □

Proof of B. — The multiplicity of the eigenvalue 0 of $A$ is equal to 2 if, and only if, $(M, J_M, g)$ Kähler with $s \neq 0$ if, and only if, $I = \mathcal{I}_\frac{1}{2} = \{J_{\infty}, J_{-\infty}\}$ by Proposition 3.

Proof of C & D. — In those cases the matrix of $A$ in a basis adapted to the decomposition $\mathbb{C} \otimes \Lambda^+ = \mathbb{C}J_M \oplus \Lambda^{1,0} \oplus \Lambda^{0,1}$ is

$$
\begin{bmatrix}
x & 0 & 0 \\
0 & y & 0 \\
0 & 0 & y
\end{bmatrix}
$$

with $y \neq 0$. Let $f$ such that $J_f \in \mathcal{I}_\frac{1}{4}$, $(m, Q)$ be any point of $Z$ and $(\theta_1, \ldots, \theta_4)$ be a local frame such that $J_M = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4$. So there exist $(a, b), (\alpha, \beta, \gamma) \in \mathbb{S}^2$ such that $Q = aI + bJ$ and $P = f(Q) = \alpha I + \beta J + \gamma K$. In that case at the point $(m, Q)$ we have :

$$
G(\theta_1, \theta_2) = 0 = \left[(1 - \alpha^2)xI - \alpha \beta yJ - \alpha \gamma yK + Q(\gamma yJ - \beta yK), Q\right] = \left[(1 - \alpha^2)x - b\beta y\right]I + (a - \alpha)\beta yJ + (a - \alpha)\gamma yK, Q
$$

$$
\leftrightarrow \begin{cases}
(a - \alpha)\gamma y a = 0 \\
b(1 - \alpha^2)x - b\beta y = a(a - \alpha)\beta y \\
\gamma = 0 \\
bx = y(1 - a\alpha) \\
a^2 + \beta^2 = 1
\end{cases}
\leftrightarrow \begin{cases}
\alpha = a \\
\beta = \frac{x}{y}b \\
\beta^2 + \gamma^2 = b^2
\end{cases}
$$
The resolution of $G(\theta_1, \theta_3) = 0$ or $G(\theta_1, \theta_4) = 0$ gives the same system. Two cases can happen first $|\frac{\xi}{\gamma}| > 1$ then the second system doesn’t have any solution and the first one has two solutions. An easy computation enable us to verify that they correspond to $f_1 = u_1 \text{Id}$ or $f_2 = u_2 \text{Id}$.

On the other hand if $|\frac{\xi}{\gamma}| < 1$ then the second system gives two solutions which correspond to $f = e^{\pm i\theta} \text{Id}$, whereas the first system doesn’t have any solution:

$$1 - \alpha^2 = \beta^2 = \frac{\gamma^2}{b^2 x^2} (1 - a\alpha)^2 > \frac{(1 - a\alpha)^2}{b^2}$$

$$\Rightarrow b^2 - b^2 \alpha^2 > 1 + a^2 \alpha^2 - 2a\alpha$$

$$\Rightarrow 0 > (\alpha - a)^2.$$  

When $|\frac{\xi}{\gamma}| = 1$ both system give the same solutions. □

**F) Example**

Let $T$ be a torus which is a quotient of $\mathbb{C}$ by the lattice $\mathbb{Z} \oplus i\mathbb{Z}$. Define $(M, g, I)$ to be the quotient of the complex flat torus $T^2 = \mathbb{T} \times \mathbb{T}$ by the group $H = \mathbb{Z}/2\mathbb{Z}$ generated by an element $h$. If $(z_1, z_2) = (x_1 + ix_2, x_3 + ix_4)$ are the canonical coordinates on $\mathbb{C} \times \mathbb{C}$, we have:

$$h(z_1, z_2) = \left( z_1 + \frac{1}{2}, -z_2 \right).$$

The manifold $(M, g, I)$ is a bi-elliptic surface which is scalar-flat Kähler; denote by $Z \longrightarrow M$ its twistor space. In this section we will study in details this example, especially the integrability of $J_f$. Thanks to Theorem 1, one knows that $J_{\text{Id}}, J_{\infty}$ and $J_{\lambda \text{Id}}$ are integrable.

Let $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4})$ be the canonical basis of $\mathbb{C}^2$ identified with $\mathbb{R}^4$. This furnishes a basis of vector fields on $T^2$ and, consequently, a global trivialisation of its twistor space $Z_0 \simeq T^2 \times S^2$. Define another basis (on $T^2$) by:

$$\theta_1 + i\theta_2 = \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \quad \text{and} \quad \theta_3 + i\theta_4 = e^{2i\pi x_1}(\frac{\partial}{\partial x_3} + i \frac{\partial}{\partial x_4}).$$

Then, $(\theta_1, \theta_2, \theta_3, \theta_4)$ is a global basis on $T^2$ which goes down to a basis of $M$. This defines a new trivialisation of $Z_0$, denoted by $\tilde{M} \times S^2$. The manifold $Z$ is the quotient of $\tilde{M} \times S^2$ by the group $\tilde{H} \simeq \mathbb{Z}/2\mathbb{Z}$, generated by $\tilde{h}$ acting as follows:

$$\tilde{h}: \tilde{M} \times S^2 \longrightarrow \tilde{M} \times S^2$$

$$\quad (m, Q) \longmapsto (h(m), Q).$$
Viewing $\mathbb{S}^2$ as a subset of $\mathbb{R} \times \mathbb{C}$ with coordinates $(a, z)$, the identity map $\Psi$ of $Z_0$ has the following form in these trivialisations:

$$\Psi : Z_0 \simeq T^2 \times \mathbb{S}^2 \rightarrow Z_0 \simeq \tilde{M} \times \mathbb{S}^2$$

$$\xi \simeq (m, (a, z)) \mapsto \xi \simeq (m, (a, e^{-2\pi x_1} \cdot z)).$$

The matrix, in both basis $(\partial \partial x_1, \partial \partial x_2, \partial \partial x_3, \partial \partial x_4)$ and $(\theta_1, \theta_2, \theta_3, \theta_4)$, of the natural complex structure $I$ on $T^2$ is equal to

$$\begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{bmatrix}.$$  

According to our notation, this is the infinity section.

Endow $Z_0$ with the complex structure of twistor space $\mathbb{J}_I$. As $(T^2, I)$ is hyperkähler, the projection $pr_2 : Z_0 \simeq T^2 \times \mathbb{S}^2 \rightarrow \mathbb{C}P^1$ is a holomorphic submersion [14]. For $n \in \mathbb{Z}^*$ and $\lambda \in \mathbb{C}^*$, consider the application $f_n : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1$ equal to $\lambda z^n$. Then there exist two applications $f_1, f_2$ depending only on $|\lambda|$ such that:

$$\begin{array}{ccc}
\mathbb{S}^2 & \rightarrow & \mathbb{S}^2 \\
(a, z) & \rightarrow & (f_1(a), \lambda f_2(a)z^n) \\
\mathbb{C} \cup \{\infty\} & \rightarrow & \mathbb{C} \cup \{\infty\} \\
U = \frac{z}{1-a} & \rightarrow & \lambda U^n
\end{array}$$

Introduce now the pull back $Z_n = f_n^* Z_0$:

$$\begin{array}{ccc}
Z_n & \rightarrow & Z_0 \\
\mathbb{C}P^1 & \rightarrow & \mathbb{C}P^1 \\
pr_2 & \rightarrow &
\end{array}$$

Since the fibration $Z_0 \rightarrow \mathbb{C}P^1$ is topologically trivial, this is also the case for $Z_n \rightarrow \mathbb{C}P^1$. Therefore one can identify the manifold $Z_n$ with $T^2 \times \mathbb{S}^2$ equipped with a complex structure denoted by $J_n$. If one considers the morphism $\tilde{f}_n = Id \times f_n : T^2 \times \mathbb{S}^2 \rightarrow T^2 \times \mathbb{S}^2$, which respects the fibration, one has $J_n = \mathbb{J}_{f_n}$.

We were wondering whether this complex structure goes down to $Z$, i.e.: does it commute with the action of the group $\tilde{H}$? We need to study

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Thus, in the trivialisation of $Z_0 \simeq \tilde{M} \times S^2$ associated to $(\theta_1, \theta_2, \theta_3, \theta_4)$, the complex structure $J_n$ is $J_{\Psi \circ \tilde{f}_n \circ \Psi^{-1}}$. It commutes with $\tilde{H}$ if and only if $n$ is odd. Moreover, for $n=1$, $\tilde{f}_1$ is a biholomorphism. We have proved the following:

**Proposition 9.** — For all $\lambda \in \mathbb{C}^*$ the complex structures $J_{\lambda z}$ on $Z$ are biholomorphic. Furthermore, the compatible almost complex structures $J_{\lambda e^{2i\pi(n-1)x_1z^n}}$ are integrable for odd $n$ in $\mathbb{Z}^*$.

This proposition can be generalised to other bi-elliptic surfaces. A computation similar to the one made in Proposition 5 enables us to say that, for different integers $n$, these complex structures are not deformation of each other. This is consequence of the fact that they do not have the same Chern classes. Indeed, the first Chern class satisfies $c_1(J_{\lambda e^{2i\pi(n-1)x_1z^n}}) = 2(n+1)h$.

In [16], following an idea of LeBrun, we showed that for any hypercomplex manifold $M$ there exist infinitely many complex structures on its twistor space $Z \simeq M \times S^2$ which are not deformation of each other. Recall that the only compact hypercomplex surfaces are the torus, the $K3$-surfaces and the quaternionic Hopf surfaces [14]. The previous proposition can therefore be viewed as a generalisation of this result to bi-elliptic surfaces.

**G) Higher dimension**

The previous sections have focused on the 4-dimensional case. We now briefly give a generalization of Theorem 1 in higher dimension. Let $n > 1$ and $(M, g)$ be an oriented $4n$-dimensional Riemannian manifold, not necessarily compact. An almost hypercomplex structure on $(M, g)$ is a triple $(I, J, K)$ of almost complex structures compatible with the orientation and...
the metric, such that $IJ = -JI = K$. When $I, J, K$ are integrable one speaks about a hypercomplex structure. When they are Kähler one says that $(M, g)$ is hyperkähler.

An almost quaternionic structure $D$ on $(M, g)$ is a rank 3 subbundle $D \subset \text{End}(TM)$ which is locally spanned by an almost hypercomplex structure $H = (I, J, K)$; such a triple is called a local admissible basis. For $n > 1$, one says that $(M, g, D)$ is a quaternionic structure if there exists a torsion free connection $\nabla$ on $TM$ preserving $D$. If one can choose $\nabla$ to be the Levi-Civita connection, $(M, g, D)$ is called quaternionic Kähler. This is equivalent to saying that the holonomy group of $g$ is contained in $\text{Sp}(1)\text{Sp}(n)$ [11].

A compatible almost complex structure on $(M, g, D)$ is a section $J_M$ of $D \to M$ such that $J_M^2 = -\text{Id}$.

Let $(M, g, D)$ be a Riemannian almost quaternionic 4n-manifold. One can define a scalar product on $D$ by saying that a local admissible basis of $D$ is orthonormal. One can then define the twistor space $Z \to M$, which is the unit sphere bundle of $D$. This is a locally trivial bundle over $M$ with fiber $\mathbb{S}^2$ and structure group $SO(3)$. As in the introduction, one can define a natural metric $\tilde{g}$ and look for the compatible almost complex structures on $(Z, \tilde{g})$ which are integrable. When $(M, g, D, J_M)$ is quaternionic Kähler with a compatible almost complex structure $J_M$, its twistor space $(Z, \tilde{g})$ admits different compatible almost complex structures: $\mathbb{J}_\sigma, \mathbb{J}_\text{Id}, \mathbb{J}_\infty, \mathbb{J}_\lambda \text{Id}$, defined as previously. The main result of this section is the following, where no hypothesis of compactness is made.

**Theorem 3.** — Let $(M, g, D)$ be a quaternionic Kähler manifold.

A) The almost complex structure $\mathbb{J}_\sigma$ is never integrable.

B) The almost complex structure $\mathbb{J}_\text{Id}$ is always integrable [35].

C) If $(M, g, D, J_M)$ is a compatible almost complex quaternionic Kähler manifold the almost complex structure $\mathbb{J}_\infty$ is integrable if, and only if:

i) $J_M$ is integrable;

ii) $g$ is scalar-flat.

D) If $(M, g, D, J_M)$ is a quaternionic Kähler manifold with a compatible Kählerian complex structure $J_M$ then, for all $\lambda \notin \{0, 1\}$, the complex structure $\mathbb{J}_{\lambda \text{Id}}$ is integrable if, and only if, $g$ is scalar-flat.

E) Let $(M, g, D)$ be a quaternionic Kähler manifold. Then the scalar curvature is flat if, and only if, one (and then any) $m \in M$ has an open neighborhood $U$ such that $(Z, \tilde{g})$ admits over $U$ an integrable compatible complex structure different from $\mathbb{J}_\text{Id}$.
Any quaternionic Kähler manifold which is scalar-flat is locally hyperkähler [11]. Thus, part E of the previous theorem yields a characterization of locally hyperkähler manifolds among quaternionic Kähler’s in terms of twistor spaces.

It is possible to give a simpler version of that theorem in the compact case because of the following result.

**Proposition** [32]. — In the compact case any compatible complex structure $J_M$ on a quaternionic Kähler manifold $(M, g, D)$ is automatically scalar-flat Kähler.

In particular, in the compact case, Theorem 3 has the following corollary.

**Corollary 3.** — Let $(M, g, D, J_M)$ be a compact quaternionic Kähler manifold with a compatible almost complex structure. Then $J_M$ is integrable if, and only if, $J_\infty$ is integrable. In this case $J_M$ is integrable for all $\lambda \in \mathbb{C}^*$.

**Proof of Theorem 3.** — Proposition 1 and Proposition 2 remain true in dimension $4n$. Since $\sigma$ is an antiholomorphic involution when restricted to the fibers, part A can be easily proved.

The proof of part B is the same as the one given in dimension 4. Notice first that $d\pi F_{ij} = -E(\theta_i, \theta_j)$ for all $(i, j) \in \{1, \ldots, 4n\}$. It remains to show that $G(\theta_i, \theta_j) = 0$ for all $i, j \in \{1, \ldots, 4n\}$. To get that result we use the following lemma.

**Lemma 3** [11]. — Let $r(\ldots)$ be the Ricci tensor. For all $(X, Y) \in TM$ one has:

\[
\begin{align*}
[R(X, Y), I] &= \gamma(X, Y)J - \beta(X, Y)K \\
[R(X, Y), J] &= -\gamma(X, Y)I + \alpha(X, Y)K \\
[R(X, Y), K] &= \beta(X, Y)I - \alpha(X, Y)J
\end{align*}
\]

with

\[
\begin{align*}
\alpha(X, Y) &= \frac{2}{n+2} r(I X, X) \\
\beta(X, Y) &= \frac{2}{n+2} r(J X, X) \\
\gamma(X, Y) &= \frac{2}{n+2} r(K X, X)
\end{align*}
\]

Let $(m, I) \in Z$ and $(I, J, K)$ be a local admissible basis. Then Lemma 3 yields:

\[
G(\theta_i, \theta_j) = \left[ R\left( \theta_i \land \theta_j - I\theta_i \land I\theta_j \right) + I R\left( \theta_i \land I\theta_j + I\theta_i \land \theta_j \right), I \right]
\]

\[= \gamma(\theta_i, \theta_j) J - \beta(\theta_i, \theta_j) K - \gamma(I\theta_i, I\theta_j) J + \beta(I\theta_i, I\theta_j) K + \gamma(I\theta_i, \theta_j) K + \beta(\theta_i, I\theta_j) J]
\]
But any quaternionic Kähler manifold is Einstein [10], hence \( r = \frac{s}{4} g \), where \( s \) is the scalar curvature of \( g \). One then has, for all \((\theta_i, \theta_j)\):

\[
G(\theta_i, \theta_j) = \frac{2s}{4(n+2)} \left( (2g(K\theta_i, \theta_j) - 2g(K\theta_i, \theta_j))J + (2g(J\theta_i, \theta_j) - 2g(J\theta_i, \theta_j))K \right)
= 0.
\]

To prove part C observe that, as in dimension 4: \( \{ J_\infty \text{ integrable} \} \iff \{ E(\theta_i, \theta_j) = G(\theta_i, \theta_j) = 0 \} \iff \{ J_M \text{ integrable and } G(\theta_i, \theta_j) = 0 \} \). Since \((M, g, Q)\) is Einstein, \((M, g, Q)\) scalar-flat implies \((M, g, Q)\) Ricci-flat and \( G(\theta_i, \theta_j) = 0 \). The converse is a consequence of part E: if \( J_\infty \) integrable then \( s = 0 \).

To get part D we use the technique of dimension 4 to prove that \( d\pi(F_{ij}) = -E(\theta_i, \theta_j) \). So \( J_M Id \) is integrable as soon as \( s = 0 \). The converse is again a consequence of part E.

Proof of E: suppose that the scalar curvature \( s \) of \((M, g, D)\) is non zero. Let \( f : Z \rightarrow Z \) be a morphism such that \( \mathbb{J}_f \) is integrable over an open set \( \mathcal{U} \). Let \((m, Q)\) be a point in \( \pi^{-1}(\mathcal{U}) \) and set \( f(m, Q) = P \). If \( \mathcal{U} \) is small enough there exists an orthonormal basis \((\theta_1, \ldots, \theta_{4n})\) and a local admissible basis \((I, J, K)\) such that \( P = J \). Write \( Q = aI + bJ + cK \) with \((a, b, c) \in \mathbb{S}^2 \).

As \( \mathbb{J}_f \) is integrable we have \( G(\theta_1, \theta_2) = 0 \) everywhere. In particular at the point \((m, Q)\):

\[
G(\theta_1, \theta_2) = 0
= \left[ R(\theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4) - Q R(\theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3), Q \right]
= \frac{2s}{4(n+2)} (-2cJ + 2bK) - Q \left[ R(\theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3), Q \right]
= \frac{s}{n+2} \left( -cJ + bK - Q(-bI + aJ) \right)
= \frac{s}{n+2} \left( acI + c(b-1)J + (b-1)K \right)
\]

Hence \( Q = J = P \) for any \((m, Q) \in \pi^{-1}(\mathcal{U})\), that is \( f = Id \).

The converse is the same as the one given in section D.

Indeed, a quaternionic Kähler manifolds \((M, g, D)\) admits, locally, infinitely many compatible complex structures \( J_M \) (for example [1]). \( \square \)
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