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RIEMANNIAN MANIFOLDS NOT QUASI-ISOMETRIC TO LEAVES IN CODIMENSION ONE FOLIATIONS

by Paul A. SCHWEITZER (*)

ABSTRACT. — Every open manifold L of dimension greater than one has complete Riemannian metrics g with bounded geometry such that (L, g) is not quasi-isometric to a leaf of a codimension one foliation of a closed manifold. Hence no conditions on the local geometry of (L, g) suffice to make it quasi-isometric to a leaf of such a foliation. We introduce the 'bounded homology property', a semi-local property of (L, g) that is necessary for it to be a leaf in a compact manifold in codimension one, up to quasi-isometry. An essential step involves a partial generalization of the Novikov closed leaf theorem to higher dimensions.

RÉSUMÉ. — Chaque variété ouverte L de dimension plus grande que 1 possède des métriques Riemanniennes complètes g avec géométrie bornée telles que (L, g) n'est pas quasi-isométrique à une feuille d'un feuilletage de codimension un d'une variété fermée. Donc il n'y a pas de conditions sur la géométrie locale de (L, g) qui suffisent pour qu'elle soit quasi-isométrique à une feuille de tel feuilletage. Nous introduisons la « propriété d'homologie bornée », une propriété semi-locale de (L, g) qui est nécessaire pour qu'elle puisse être feuille d'un feuilletage de codimension 1 d'une variété compacte, à une quasi-isométrie près. Une étape essentielle de la démonstration utilise une généralisation partielle du théorème de la feuille fermée de Novikov aux dimensions plus grandes.

1. Introduction

The question of when an open (i.e. noncompact) connected manifold can be realized up to diffeomorphism as a leaf in a foliation of a compact differentiable manifold was first posed by Sondow [21] for surfaces in 3-manifolds. It was solved positively for all open surfaces by Cantwell and Conlon [5] (although Ghys [8] showed afterwards that only six surfaces occur generically).

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In the opposite direction, Ghys [7] and independently Inaba, Nishimori, Takamura and Tsuchiya [13] constructed an open 3-manifold (an infinite connected sum of lens spaces for all odd primes) that cannot be a leaf in a foliation of a compact 4-manifold. Attie and Hurder [3] gave an uncountable family of smooth simply connected 6-dimensional manifolds, all having the homotopy type of an infinite connected sum of copies of $S^2 \times S^4$, that are not diffeomorphic to leaves in a compact 7-manifold. It is still an open problem whether every smooth open manifold of dimension greater than 2 is diffeomorphic to a leaf of a codimension two (or higher) foliation.

In the related question of when an open Riemannian manifold can be realized up to quasi-isometry as a leaf in a foliation of a compact manifold, Attie and Hurder [3] also produced an uncountable family of quasi-isometry types of Riemannian metrics on the 6-manifold $S^3 \times S^2 \times \mathbb{R}$, each with bounded geometry, which cannot be leaves in any codimension one foliation of a compact 7-manifold. On these and other 6-manifolds they also defined Riemannian metrics that have positive 'entropy', and hence cannot be leaves in any $C^{2,0}$ codimension one foliation, or in any C^1 foliation of arbitrary codimension, on any compact manifold. Zeghib [24] adapted this result to surfaces with exponential growth. Attie and Hurder prove their various results using the bounded Pontryagin classes defined by Januskiewicz [14], who had already used them to construct open manifolds that could not be leaves in certain compact manifolds, extending earlier results of Phillips and Sullivan [16].

The Attie-Hurder results extend to codimension one foliations of dimensions greater than 6, but their Question 2 asks for examples in the lower dimensions $3, 4$, and 5. In this paper we respond to this question by showing that *every* open manifold of dimension at least 3 admits complete Riemannian metrics of bounded geometry, and of every possible growth type, that are not quasi-isometric to leaves of codimension one foliations of compact manifolds. The same result for surfaces had already been proven in [18] (where the present paper was announced). Thus no set of local bounds on the geometry of an open Riemannian p-manifold L with $p \geqslant 2$ can be sufficient to guarantee that it be quasi-isometric to a leaf of a $C^{2,0}$ codimension one foliation of a closed $(p+1)$-manifold.

We define a $C^{2,0}$ foliation to be one in which the leaves are smooth of class C^2, and their C^2 differentiable structure varies continuously in the transverse direction along the leaves of a transverse foliation $\mathcal{T}$ (Definition 2.5). Attie and Hurder [3] call such foliations C^0. A bound on the geometry of L is **local** if it only depends on the Riemannian geometry of the ϵ-balls around the points of L, for some constant $\epsilon > 0$. For example, bounds on the various curvatures and on the injectivity radius of L are local geometric bounds.

We define two Riemannian manifolds to be **quasi-isometric** if there is a diffeomorphism f from one to the other such that both f and f^{-1} produce only bounded distortion of the metrics, up to a constant D, as formulated precisely in Definition 2.1 below and in [3]. It is easy to see that a leaf in a foliation of a smooth compact manifold M, with any Riemannian metric induced by a metric on M, must have **bounded geometry**, in the sense that the sectional curvature is uniformly bounded above and below, and the injectivity radius has a uniform positive lower bound. Because of the term D in Definition 2.1, quasi-isometry does not always preserve this property, but in this paper we consider only Riemannian metrics with bounded geometry. Then the following theorem, which is our main result, involves a restriction on the global geometry of an open manifold for it to be quasi-isometric to a leaf.

Theorem 1.1. — *Every connected non-compact smooth p-manifold L with $p \geqslant 2$ possesses C^∞ complete Riemannian metrics g with bounded geometry that are not quasi-isometric to any leaf of a codimension one $C^{2,0}$ foliation on any compact differentiable $(p+1)$-manifold.*

Furthermore g can be chosen such that no end is quasi-isometric to an end of a leaf of such a foliation, and also to have any growth type compatible with bounded geometry. Hence there are uncountably many quasi-isometry classes of such metrics g on every such manifold L.

When $p = 2$, Theorem 1.1 was proven in [18], so in this paper we shall assume that $p \geqslant 3$. The proof for $p \geqslant 3$ depends on Theorem 2.6 below, which states that the leaves of a $C^{2,0}$ codimension one foliation of a compact manifold M of dimension greater than 3, with Riemannian metrics that vary continuously in the transverse direction, have a certain **bounded homology property**, a property which we define for open Riemannian manifolds of dimension at least 3. In Theorem 2.8 we show how to modify a given complete Riemannian metric on a connected smooth open manifold, without changing its growth type, so that the new metric does not have the bounded homology property. The construction involves introducing spherical 'balloons' of arbitrarily large size, but with 'necks' of uniformly bounded size, as in [18]. (See Figures 1.1 and 1.2.)

The proof of Theorem 2.6 (in Section 3) involves a Finiteness Lemma (Lemma 3.2, proved in Section 5) and a partial extension to higher dimensions of Novikov's theorem [15, 11, 4] that vanishing cycles only occur

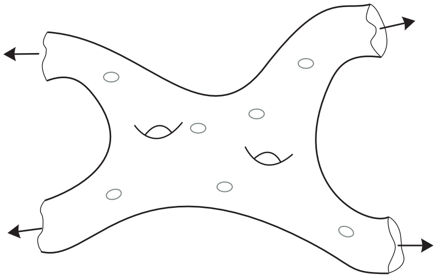

Figure 1.1. The manifold L with the original metric.

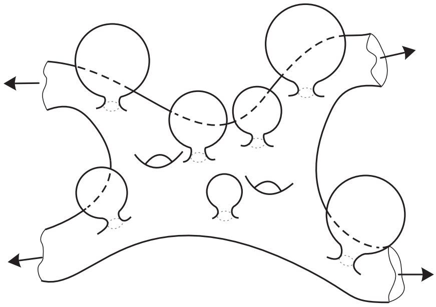

Figure 1.2. The manifold L with "balloons".

on the boundary of Reeb components (Theorem 2.13, proved in Section 7). Definitions and statements of results are given in Section 2 and the construction of Riemannian manifolds that cannot be leaves is given in Section 4.

The first three theorems of this paper constitute an extension to higher dimensions of three similar theorems for open surfaces in [18]. That paper, using a different bounded *homotopy* property for surfaces involving contractible loops rather than bounding submanifolds, showed that all open surfaces have complete Riemannian metrics with bounded geometry that cannot be leaves in $C^{2,0}$ foliations of compact 3-manifolds.

It is a pleasure to thank my student Fabio Silva de Souza for preparing the figures.

2. Definitions and statements of results

In this section we give several definitions leading up to the definition of the bounded homology property, and then we state Theorem 2.6 (which states that, under certain hypotheses, leaves have this property), Theorem 2.8, and Theorem 2.13 (a partial generalization of Novikov's Theorem on the existence of Reeb components). In this paper, all manifolds are smooth of class C^2, except that in the proof of Theorem 2.13 it suffices to assume that M and $\mathcal{F}$ are C^0.

Definition 2.1. — *A diffeomorphism $f : L \to L'$ between two Riemannian manifolds L and L' is a* **quasi-isometry** *if there exist constants $C, D > 0$ such that the distance functions d and d' on L and L' satisfy*

$$C^{-1}d'(f(x), f(y)) - D \leqslant d(x, y) \leqslant Cd'(f(x), f(y)) + D$$

for all points $x, y \in L$. When such a diffeomorphism exists we say that L and L' are **quasi-isometric**.

For example, any diffeomorphism between compact smooth Riemannian manifolds is a quasi-isometry. Note that quasi-isometry is an equivalence relation.

Let L be a p-dimensional Riemannian manifold, S a subset of L, and β a positive number. The open β-ball $V_\beta(x)$ at a point $x \in L$ is defined to be the set of points on L whose distance from x is less than β. Note that $V_\beta(x)$ may fail to be a topological ball, if β is greater than the injectivity radius at x.

Definition 2.2. — *The β-***volume** $\mathrm{Vol}_\beta(S) = \mathrm{Vol}_{\beta,L}(S) \in \mathbb{N} \cup \{\infty\}$ *of S on L is the smallest integer K such that S can be covered by K open β-balls in L, or ∞ if no such finite number exists.*

The β-**volume** of S depends on its embedding in L, but the ambient manifold L will be clear from the context, so it will not be made explicit in the notation.

Definition 2.3. — *Let C be a compact Riemannian p-manifold with boundary B and let β be a constant greater than 0. Then we define the* **Morse β-volume** $M(C, \beta)$ *to be the smallest positive integer M for which there is a Morse function $f : C \to [0, \infty)$ such that $f(B) = 0$ and each level set $f^{-1}(t)$ for $t \in [0, \infty)$ has $\mathrm{Vol}_\beta(f^{-1}(t)) \leqslant M$ on C. (See Figure 2.1, where various level sets are shown.)*

This definition makes sense since it is evident that the β-volumes of the level sets of a fixed Morse function f on a compact manifold are uniformly bounded, so that $M(C, \beta)$ is finite. For a set S contained in a Riemannian manifold L, we define the open β**-neighborhood** of S, $V_\beta(S)$, to be the set of all points in L whose distance from S, measured along geodesics in

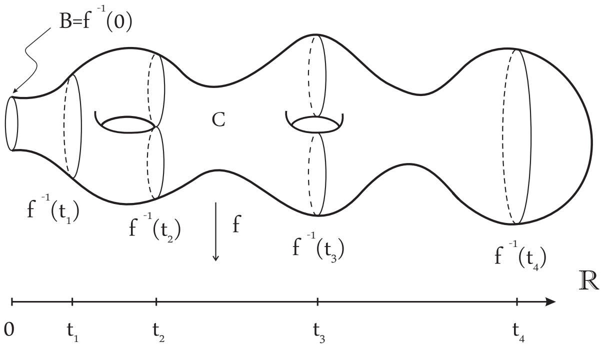

Figure 2.1. Morse β-volume of a compact manifold C with boundary B.

L, is less than β. We can now define the bounded homology property, the fundamental tool used in this paper.

Definition 2.4. — *A Riemannian p-manifold L ($p \geqslant 3$) has the* **bounded homology property** *if there exists a constant $\beta_0 \geqslant 0$ such that for every pair of constants $\beta > \beta_0$ and $k > 0$ there is an integer $K = K(\beta, k)$ such that if*

(1) *$B \subset L$ is a 1-connected smooth closed $(p-1)$-submanifold embedded in L,*
(2) *there is a closed neighborhood V of B that fibers over B as a smooth tubular neighborhood and contains the β-neighborhood $V_\beta(B)$ of B in L,*
(3) *B has β-volume $\mathrm{Vol}_\beta(B) \leqslant k$ on L, and*
(4) *B bounds a compact 1-connected region C on L,*

then the Morse β-volume of C satisfies $M(C, \beta) \leqslant K$.

Comments. A neighborhood V of B is a **tubular neighborhood** of B if there is a smooth retraction $V \to B$ which is a (locally trivial) fibration. In particular, it is well-known that for a smooth compact submanifold B of a Riemannian manifold $V_\beta(B)$ is a tubular neighborhood of B for every sufficiently small positive number β. On the other hand, because of the term D in Definition 2.1, we must require β to be 'sufficiently large', i.e., greater than some given β_0, in order for the bounded homology property to be invariant under quasi-isometry (see Proposition 2.10). We also require that B and C be 1-connected (i.e., connected and simply connected) so that when they are subsets of a leaf L of a foliation $\mathcal{F}$, they can be lifted in the transverse direction to leaves of $\mathcal{F}$ near to L. Recall that for any C^0 codimension one foliation $\mathcal{F}$ there always exists a 1-dimensional foliation $\mathcal{T}$ topologically transverse to $\mathcal{F}$ [12], [19].

Definition 2.5. — *A codimension one foliation $\mathcal{F}$ of a smooth manifold M is a $C^{2,0}$-foliation if the leaves are smooth submanifolds of class C^2 and the transverse foliation $\mathcal{T}$ can be chosen so that the C^2-structures on the leaves are preserved by the local homeomorphisms obtained by lifting open sets from one leaf to another along leaves of $\mathcal{T}$.*

Theorem 2.6. — *Let $\mathcal{F}$ be a $C^{2,0}$ codimension one foliation of dimension $p \geqslant 3$ on a closed smooth manifold M and let $\mathcal{T}$ be a foliation transverse to $\mathcal{F}$ chosen as in Definition 2.5. Then the leaves of $\mathcal{F}$ have the bounded homology property with $\beta_0 = 0$, and for every $k > 0$ and $\beta > 0$, the same integer $K = K(k, \beta) > 0$ can be chosen for all the leaves of $\mathcal{F}$.*

Corollary 2.7. — *If $\mathcal{F}$ is a $C^{2,0}$ codimension one foliation of dimension $p \geqslant 3$ of a closed smooth manifold M, then the leaves have the bounded homology property for any Riemannian metric induced on the leaves by a Riemannian metric on M.*

The proof of Theorem 2.6 is given in Section 3 using results proven in Sections 5 and 7. In Section 4 we show that every smooth open manifold of dimension at least 3 admits complete Riemannian metrics with bounded geometry such that the bounded homology property does not hold, thus giving the following result.

Theorem 2.8. — *Let L be a complete open connected Riemannian p-manifold of dimension $p \geqslant 3$ whose metric g_0 has bounded geometry. Then L has other complete Riemannian metrics g with bounded geometry and the same growth type as g_0 that do not possess the bounded homology property, and such that no end of L has the bounded homology property. Furthermore there are such metrics with uncountably many distinct growth types, and hence in uncountably many distinct quasi-isometry classes.*

We recall that the **growth function** $f : [0, \infty) \to [0, \infty)$ of a connected Riemannian p-manifold (L, g) with basepoint $x_0 \in L$ is defined to be $f(r) = \mathrm{Vol}(B(x_0, r))$, the p-dimensional volume of the ball of radius r centered at x_0. Given two increasing continuous functions $f_1, f_2 : [0, \infty) \to [0, \infty)$, we say that f_1 has **growth type** less than or equal to that of f_2 (denoted $f_1 \preceq f_2$) if there exist constants $A, B, C > 0$ such that for all $r \in [0, \infty)$, $f_1(r) \leqslant A f_2(Br + C)$ [11]. They have the same growth type if $f_1 \preceq f_2$ and $f_2 \preceq f_1$. We write $f_1 \prec f_2$ if $f_1 \preceq f_2$ but it is false that $f_2 \preceq f_1$. For example, $I \prec \exp$ where I is the identity function $I(r) = r$ and $\exp(r) = e^r$ on $[0, \infty)$. The growth type of (L, g) (i.e., of its growth function) is clearly invariant under quasi-isometry and change of the basepoint. Then the following observation proved in Section 4 establishes the last assertions of Theorems 1.1 and 2.8, since the growth types of the functions $f_k(r) = r^k$ for every real number $k > 1$ are distinct and can all be realized.

Proposition 2.9. — *Let $f : [0, \infty) \to [0, \infty)$ be any increasing continuous function of growth type greater than linear and at most exponential, i.e. $I \prec f \preceq \exp$. Then every smooth connected open manifold L of dimension at least two admits a complete Riemannian metric with bounded geometry whose growth type is the growth type of f.*

Theorem 1.1 follows immediately from Theorems 2.8 and 2.6 with the following fact.

Proposition 2.10. — *If L and L' are quasi-isometric complete Riemannian manifolds with bounded geometry and L has the bounded homology property, then so does L'.*

To prove this proposition we need the following result. Both will be proven in Section 6.

Proposition 2.11. — *Let L be a complete Riemannian manifold with bounded sectional curvature. Then given constants $0 < a < b$, there exists an integer $n > 0$ such that every open b-ball $V_b(x)$ on L centered at $x \in L$ can be covered by at most n open a-balls.*

As mentioned above, $V_b(x)$ may fail to be a topological ball.

One step in the proof of Theorem 2.6 (Proposition 3.3 below) will use a weak generalization of the second half of Novikov's theorem on the existence of Reeb components and a Corollary 2.14, which we shall state after the following definition.

Definition 2.12. — *A compact $(p + 1)$-dimensional manifold with a codimension one foliation $\mathcal{R}$ is a (generalized)* **Reeb component** *if the boundary ∂R is a nonempty finite union of leaves, the interior $\mathrm{Int}(R)$ fibers over the circle with the leaves as fibers, and there is a transverse orientation pointing inwards along all the components of ∂R.*

It is clear that the boundary leaves are compact and the interior leaves non-compact. In this paper, we shall usually consider Reeb components with connected boundary ∂R, and then the existence of the transverse orientation pointing inwards along ∂R is automatic.

Now let $\mathcal{F}$ be a p-dimensional foliation of a compact $(p+1)$-dimensional manifold M and let B be a compact connected $(p-1)$-dimensional manifold. The horizontal foliation $\mathcal{H}$ of $B \times [0,1]$ is given by the leaves $B \times \{t\}$ for $t \in [0,1]$. Consider a foliated map

$$h : (B \times [0,1], \mathcal{H}) \to (M, \mathcal{F})$$

and suppose that $h_0 : B \to L_0$ is an embedding, where for all t L_t is the leaf containing $h(B \times \{t\})$ and $h_t : B \to L_t$ is the map defined $h_t(b) = h(b,t)$. Note that now we are not supposing any differentiability, but there does exist a 1-dimensional foliation $\mathcal{T}$ of M topologically transverse to $\mathcal{F}$. Then we have the following weak generalization of the second half of Novikov's theorem. (A stronger generalization is given in [2]. The idea of using homological vanishing cycles is due to Sullivan [22].)

Theorem 2.13. — **An Extension of Novikov's Theorem.** *Suppose that one of the following conditions holds for every $t > 0$ sufficiently close to 0 but does not hold for $t = 0$:*

1. *$B_t = h_t(B)$ is the boundary of a compact 1-connected region $C_t \subset L_t$;*
2. *$B_t = h_t(B)$ is the boundary of a compact region $C_t \subset L_t$;*
3. *B is oriented and $0 = h_{t*}([B]) \in H_{p-1}(L_t)$ (where $[B]$ is the fundamental homology class of B); or*
4. *$0 = h_{t*}([B]) \in H_{p-1}(L_t; \mathbb{Z}_2)$ (where $[B]$ is the fundamental homology class of B with coefficients modulo 2).*

Then the leaf L_0 is the boundary of a Reeb component R whose interior $\mathrm{Int}(R)$ is the union of the leaves L_t for which $t > 0$. (See Figure 2.2.)

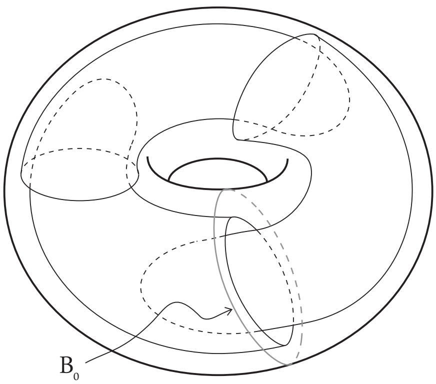

Figure 2.2. The classical Reeb component with vanishing cycle B_0.

Corollary 2.14. — *Assume the same hypotheses and also that $\mathcal{F}$ is a $C^{2,0}$ foliation. Then for any $\beta > 0$, and for any choice of Riemannian metrics on the leaves in the Reeb component that vary continuously along the transverse foliation $\mathcal{T}$, there is a constant $K > 0$ such that the Morse β-volume $M(C_t, \beta) \leqslant K$ for all $t \in (0, 1]$. (See Figure 2.3.)*

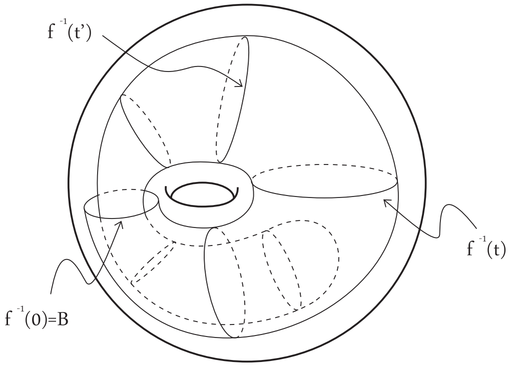

Figure 2.3. The Morse β-volume of a set C in a leaf of a Reeb component.

For the results of this paper, we only use condition (1) of Theorem 2.13, but the other three conditions are mentioned since they are of some interest and involve little extra work. In the Corollary we assume $\mathcal{F}$ to be smooth of class $C^{2,0}$ in order for the leaves to admit Riemannian metrics and Morse functions on the regions C_t. Note that as t approaches 0, the Riemannian volume of C_t and also its β-volume $\mathrm{Vol}_\beta(C_t)$ both tend to infinity, as is easily seen from the structure of the Reeb component. That is the reason for using the Morse β-volume $M(C_t, \beta)$, which is uniformly bounded according to the Corollary, rather than one of the other two volumes, which are not, in the Definition 2.4 of the bounded homology property.

The proofs of Theorem 2.13 and its Corollary are given in Section 7. We observe that the proof in this case, in which B is an *embedded* "vanishing cycle", is much easier than in the general case when B is only assumed to be *immersed*. (See Haefliger [10] or Camacho-Lins Neto [4] for good expositions of the proof of Novikov's original theorem [15] that treat the problem of double points of the immersion clearly, and [20] for the C^0 case.) In a paper in preparation [2], we give a stronger generalization of Novikov's theorem. In the course of the proof of Theorem 2.13, we find a construction of Reeb components that is used in the proof of the Corollary. In [1], where the structure of generalized Reeb components is studied in detail, it is shown that this construction actually produces all Reeb components, up to foliated homeomorphism.

3. Leaves have the bounded homology property

In this section we prove Theorem 2.6 using results that will be proven in Sections 5 (the Finiteness Lemma 3.2) and 7 (the extension of Novikov's Theorem 2.13 and Corollary 2.14). The idea of the proof is to consider a $(p-1)$-dimensional submanifold B that satisfies the hypotheses of Definition 2.4 and approximate it by a subcomplex X of a triangulation on the same leaf. Then we show that there is a finite set of "transverse families" of subcomplexes that suffice in this process (the Finiteness Lemma 3.2). Finally we show that there is a constant K that is an upper bound for all the resulting Morse β-volumes.

Let $\mathcal{F}$ be a codimension one foliation of a compact manifold M of dimension at least 4, so that the leaf dimension $p \geqslant 3$, and suppose that constants $k > 0$ and $\beta > 0$ are given. Suppose that the leaves are smooth (of class C^2) and their C^2 differentiable structure varies continuously in the transverse direction along leaves of a fixed transverse foliation $\mathcal{T}$, so that $\mathcal{F}$ is $C^{2,0}$, as in Definition 2.5. As a consequence, it is possible to choose Riemannian metrics on the leaves of $\mathcal{F}$ which vary continuously in the transverse direction along $\mathcal{T}$, and we fix such Riemannian metrics. Thus the hypotheses of Theorem 2.6 are satisfied. All these structures remain fixed throughout this section.

As usual, a **region** in a smooth manifold L is a compact connected submanifold with smooth boundary that is the closure of an open set in L. If L has a Riemannian metric, then the diameter of a smooth submanifold S in L (possibly with boundary and corners, such as a simplex) is the supremum of distances between pairs of points of S, as measured along geodesics in S. The **mesh** of a triangulation of a region is the maximum of the diameters of its simplexes. A triangulation is an **ϵ-triangulation** if its mesh is less than a positive number ϵ. All of the triangulations we shall consider will be β'-triangulations with $\beta' = \beta/4$.

The Simplicial Approximation Process (SAP). Let $\mathcal{B}$ be the set of all submanifolds B of leaves of $\mathcal{F}$ that satisfy the conditions (1), (2), and (3) of Definition 2.4 for the fixed constants k and β. Given $B \in \mathcal{B}$ on a leaf L of $\mathcal{F}$, choose a smooth β'-triangulation of a region in L that contains the tubular neighborhood V given by condition (2), and give B a smooth triangulation that is sufficiently fine so that the open star of each vertex v of B lies in the open star of some vertex, say $g(v)$, of the triangulation of L. Then the function g taking the vertices of B to those of L extends to a simplicial map $\bar{g} : B \to L$, which is a simplicial approximation to the inclusion $B \hookrightarrow L$. Note that the image $X_B = \bar{g}(B)$ is a $(p-1)$-dimensional complex contained in $V_{\beta'}(B)$. (See Figure 3.1, which shows B and its simplicial approximation $g(B)$ in the tubular neighborhood V which is contained in the region Ω.)

We say that X_B is obtained from B by the **SAP**, the simplicial approximation process.

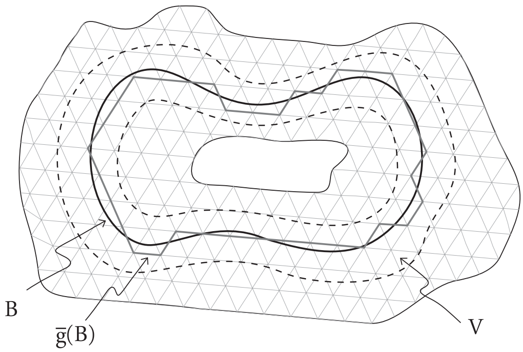

Figure 3.1. B and its simplicial approximation $\bar{g}(B)$ in the neighborhood V.

Transverse families and the Finiteness Lemma. Recall that a product manifold $X \times Y$ has two product foliations $\mathcal{H}$ and $\mathcal{V}$, the horizontal and vertical foliations, given respectively by the leaves $X \times \{y\}$ for $y \in Y$ and $\{x\} \times Y$ for $x \in X$. A map $f : (X \times Y, \mathcal{H}, \mathcal{V}) \to (M, \mathcal{F}, \mathcal{T})$ is **bifoliated** if it takes leaves of $\mathcal{H}$ and $\mathcal{V}$ into leaves of $\mathcal{F}$ and $\mathcal{T}$, respectively.

We recall that the foliations $\mathcal{F}$ and $\mathcal{T}$ and the constants k and β are fixed for the rest of this Section.

Definition 3.1. — *A triple (X, Ω, f) will be called a* **transverse family** *if Ω is a smoothly triangulated compact p-manifold with boundary, $X \subset \Omega$ is a $(p-1)$-dimensional subcomplex, and*

$$f : (\Omega \times [0,1], \mathcal{H}, \mathcal{V}) \to (M, \mathcal{F}, \mathcal{T})$$

is a smooth bifoliated embedding, such that if we set $\Omega_t = f(\Omega \times \{t\})$ and $X_t = f(X \times \{t\})$, then, for every $t \in [0,1]$, relative to the metric on the leaf L_t that contains Ω_t and X_t,

(1) *Ω_t contains $V_{2\beta'}(X_t)$, the $2\beta'$-neighborhood of X_t (with $\beta' = \beta/4$), and*
(2) *the triangulation induced on Ω_t by the triangulation on Ω is a smooth β'-triangulation.*

We say that a transverse family (X, Ω, f) is **good** if there is a submanifold $B \subset L_0$ with $B \in \mathcal{B}$ and a smooth bifoliated embedding

$$h : (B \times [0,1], \mathcal{H}, \mathcal{V}) \to (M, \mathcal{F}, \mathcal{T})$$

such that for every $t \in [0,1]$ the submanifold $B_t = h(B \times \{t\})$ is contained in L_t, $B_0 = B$, and X_t can be obtained from B_t by the SAP described above.

The following Finiteness Lemma will be proven in Section 5.

Lemma 3.2. — **(Finiteness Lemma).** *There is a finite set of good transverse families $(X_i, \Omega_i, f_i), 1 \leqslant i \leqslant \ell$, such that, for each $B \in \mathcal{B}$, the complex X_B obtained from B by the SAP can be chosen to be $X_{i,t} = f_i(X_i \times \{t\})$ for some $i \in \{1, \ldots, \ell\}$ and $t \in [0,1]$.*

For $1 \leqslant i \leqslant \ell$, let $\mathcal{B}_i$ be the set of $B \in \mathcal{B}$ for which there exists some $t \in [0,1]$ for such that the SAP can yield the complex $X_B = X_{i,t}$.

Proposition 3.3. — *For each $i, 1 \leqslant i \leqslant \ell$, there exists an integer K_i such that, for each $B \in \mathcal{B}_i$ that satisfies condition (4) of Definition 2.4, the region C with $\partial C = B$ given by condition (4) has Morse β-volume $M(C, \beta)$ less than or equal to K_i.*

Supposing Lemma 3.2 and Proposition 3.3, we can now prove Theorem 2.6.

Proof of Theorem 2.6. — By the Finiteness Lemma, we obtain a finite set of good transverse families $(X_i, \Omega_i, f_i), 1 \leqslant i \leqslant \ell$. The Proposition gives a common upper bound K_i for the Morse β-volumes of the regions C corresponding to all the submanifolds $B \in \mathcal{B}_i$. Since every $B \in \mathcal{B}$ belongs to some $\mathcal{B}_i$, $K = K(k, \beta) = \max_{1 \leqslant i \leqslant \ell} K_i$ is an upper bound for the Morse β-volumes $M(C, \beta)$ of the regions C corresponding to all submanifolds B that satisfy the hypotheses of the Theorem, as claimed. ☐

Proof of Proposition 3.3. —

First of all, we observe that if some $B \in \mathcal{B}$ bounds 1-connected regions on both sides in its leaf L, then the leaf L is compact, and by the Van Kampen Theorem it is also 1-connected. Then by the Reeb Stability Theorem for codimension one, all the leaves are compact and the foliation $\mathcal{F}$ fibers over the circle with the leaves as fibers. Thus there is a common upper bound for the β-volumes of the leaves, and hence for all the regions C, so that the conclusion of the Proposition holds. (In fact, the conclusion of Theorem 2.6 also holds.) Hence we may assume that each $B \in \mathcal{B}$ bounds a compact 1-connected region on at most one side.

Now fix an index i as in the Finiteness Lemma 3.2 and consider the good transverse family (X_i, Ω_i, f_i), which for simplicity we denote by (X, Ω, f) (without indicating the index i). Set

$$J = \{t \in [0,1] \mid \exists \text{ a } 1\text{−connected region } C_t \subset L_t \text{ such that } \partial C_t = B_t\}.$$

Thus the indices $t \in J$ are those for which $B_t = B_{i,t}$ satisfies condition (4) of Definition 2.4. It is clear that J is open in the interval $[0,1]$, since the region C_t is 1-connected by hypothesis, and therefore lifts along $\mathcal{T}$ to nearby leaves.

Observe that no connected component of J can be an open interval (a,b), for then the submanifolds B_a and B_b do not bound 1-connected regions (since $a, b \notin J$), but the leaves L_t for t in the interval (a,b) do. Then, by the extension of Novikov's Theorem, Theorem 2.13, the leaves L_a and L_b bound Reeb components, with interior leaves L_t for $t \in (a,1]$ in the first Reeb component, and L_t for $t \in [0,b)$ in the second one. But then L_a is compact as the boundary of the first Reeb component, but it is non-compact, since it is an interior leaf of the second one, a contradiction.

Consequently the only possibilities for the connected components of J are $[0,b)$, $(a,1]$, and $[0,1]$. In the first two cases, L_b or L_a is the boundary of a Reeb component with the leaves in the indicated interval as interior leaves, and then Corollary 2.14 gives a common upper bound for the Morse β-volumes of the corresponding regions C_t. Note that if for some $t_1 \in J$, $M(C_{t_1}, \beta) \leqslant K$ for a certain integer K, then $M(C_t, \beta) \leqslant K$ for all t in a small neighborhood of t_1, since the compact regions C_t vary continuously and C_{t_1} is covered by K open β-balls. Thus the Morse β-volumes $M(C_t, \beta)$ are locally bounded, and if $J = [0,1]$, the compactness of $[0,1]$ gives the desired common upper bound. If $J = \emptyset$, then there is nothing to be proven. Putting all these cases together, we get a common upper bound K'_i for all the C_t with boundary $B_t \in \mathcal{B}_i$, $t \in J \subset [0,1]$.

To complete the proof of the proposition we need the following lemma, whose proof is given in Section 5.

Lemma 3.4. — *There is an integer K_0 that depends on k, β, and the upper bound on the scalar curvature of the leaves of $\mathcal{F}$, such that if the same complex X_t can be obtained by the SAP from each of two submanifolds $B, B' \in \mathcal{B}$, and B satisfies condition (4) of Definition 2.4, i.e., there is* *a compact* 1*-connected region* C *on the leaf* L *containing* B *such that* $\partial C = B$*, then* B' *also satisfies condition (4), so there is a compact* 1*-connected region* $C' \subset L$ *whose boundary is* B'*, and we have*

$$|M(C',\beta) - M(C,\beta)| \leqslant K_0. \tag{3.1}$$

Every $B \in \mathcal{B}_i$ yields a complex $X_{i,t}$ under the SAP for some $t \in [0,1]$, and the submanifold $B_t = B_{i,t}$ yields the same complex, so the lemma shows that $K_i = K'_i + K_0$ is an upper bound for the Morse β-volumes of all the regions C with $\partial C = B \in \mathcal{B}_i$, as claimed. □

4. Modifying the metric on a manifold by inserting balloons.

In this section we prove Theorem 2.8 by showing how the Riemannian metric on any complete open Riemannian manifold L of dimension at least three with bounded geometry can be modified without changing the growth type, so that L with the new metric does not have the bounded homology property, and hence cannot be a leaf in a $C^{2,0}$ codimension one foliation of a closed manifold. The construction is an obvious adaptation of the construction for surfaces in Section 2 of [18], which we follow closely. We insert p-dimensional "balloons" of unbounded size with "necks" of uniformly bounded size into L. The balloons are widely spaced so that the original growth type of L does not change.

Proof of Theorem 2.8. — Let L be a smooth connected noncompact p-dimensional manifold ($p \geqslant 3$) and let g_0 be a complete Riemannian metric on L with globally bounded sectional curvature, with injectivity radius greater than some small constant $d > 0$, and with a given growth type. Hence the exponential map $\exp_x : B_x(0,d) \to L$ is injective for every point x in L, where $B_x(0,r)$ denotes the open ball of radius r centered at the origin in the tangent space at x. Consequently $\exp_x$ is a diffeomorphism from $B_x(0,d)$ onto the open ball $V_d(x)$ in L.

We fix a basepoint $x_0 \in L$ and consider a sequence of positive numbers $d_n, n = 1, \dots,$ with $d_n + 2d < d_{n+1}$ for each n. Choose a sequence of points $x_1, x_2, \dots$ such that $d(x_0, x_n) = d_n$ and consequently the balls $V_d(x_n)$ are disjoint. Since the metric g_0 has bounded geometry, there is no loss of generality in supposing, as we do, that the balls $V_d(x_n)$ are isometric to the balls $V_d(0)$ in Euclidean p-space $\mathbb{R}^p$. In fact, it suffices to modify g_0

slightly (perhaps replacing d by a smaller radius) with care to preserve the growth type and globally bounded geometry.

Choose an increasing sequence of positive numbers r_n such that $r_n \to \infty$ as $n \to \infty$. Let $S^p(r_n)$ be the the sphere of radius r_n in $\mathbb{R}^{p+1}$ centered at the origin and let $S = (0, \dots, 0, -r_n)$ be its basepoint at the south pole. Choose a diffeomorphism $\phi : V_d(x_n) \to S^p(r_n) \setminus \{S\}$ by setting

$$\phi(\exp_{x_n}(tv)) = \exp_S((d-t)h(v))$$

for $d/2 \leqslant t < d$ and every unit tangent vector v to L at x_n, where h is a linear isometry from the tangent space to L at x_n to the tangent space to $S^p(r_n)$ at S, and then extending ϕ as a diffeomorphism over the complementary disk $V_{d/2}(x_n)$. Define a new metric g on $V_d(x_n)$ by interpolating between the given metric g_0 and the metric g_1 obtained as the pullback under ϕ of the round metric on $S^p(r_n)$, so that g coincides with g_0 near the boundary of $V_d(x_n)$ and with g_1 on $V_{d/2}(x_n)$. This defines the new "balloon" metric on the balls $V_d(x_n)$, and outside these balls g is defined to coincide with the original metric g_0, as shown in Figure 1.2. The uniform manner of carrying out this construction as n varies ensures that (L, g) has globally bounded geometry. Note that the metric g on the closed ball $\overline{V_{d/2}(x_n)}$ of radius $d/2$ in the original metric g_0 is now the round metric of $S^p(r_n)$ from which a small neighborhood of S has been removed.

We claim that the new metric g does not have the bounded homology property. In fact, suppose that $\beta_0 \geqslant 0$ is given and fix a number $\beta > \beta_0$. Take n_0 sufficiently large so that $r_{n_0} > 2\beta + 2d$, and consider only the balloons for $n \geqslant n_0$. Let B_n be the $(p-1)$-sphere on this balloon with radius $\beta + d$ centered at the south pole S in the original metric on the sphere. Note that B_n has $(p-1)$-volume less than $a_{p-1}(\beta + d)^{p-1}$, where a_m denotes the m-volume of the unit m-sphere S^m in $\mathbb{R}^{m+1}$, and B_n is 1-connected since $p \geqslant 3$. Choose k to be sufficiently large so that S^{p-1} can be covered by k open balls of radius β; it follows that the same is true for B_n. Furthermore, it is clear that the closed β-neighborhood $V = \overline{V_\beta(B_n)}$ of B_n in the new metric is a tubular neighborhood fibered over B_n that contains $V_\beta(B_n)$. The closed complementary component C_n of B_n that contains the north pole $N = (0, \dots, r_n)$ is a compact 1-connected region on L with boundary $\partial C_n = B_n$. Hence the four conditions of Definition 2.4 are satisfied by B_n, but we shall see that the Morse volumes of the C_n are unbounded.

Let $f_n : C_n \to [0, \infty)$ be a Morse function with $f_n(B_n) = 0$. As t increases from 0, there is a 1-parameter family of closed complementary regions $A_n(t) = f_n^{-1}([t, \infty))$ of $f_n^{-1}(t)$, beginning with $A_n(0) = C_n$ and ending with

$A_n(t) = \emptyset$ for sufficiently large values of t. Note that $A_n(0)$ covers more than half of the balloon and the p-volume of the sets $A_n(t)$ varies continuously, so for some value of t, say t'_n, $A_n(t'_n)$ will have p-volume $a_p {r_n}^p/2$, i.e. $A_n(t'_n)$ will have half the p-volume of the round sphere of radius r_n, which is $a_p {r_n}^p$. Then, by the isoperimetric inequality on the sphere, the boundary $f_n^{-1}(t'_n) = \partial A_n(t'_n)$ must have $(p-1)$-volume at least equal to $a_{p-1}{r_n}^{p-1}$, the $(p-1)$-volume of the equator. (See Figure 4.1.) As n increases these $(p-1)$-volumes tend to infinity, so each β-volume $\mathrm{Vol}_\beta(f_n^{-1}(t'_n))$ is greater than some constant M_n such that $\lim_{n\to\infty} M_n = \infty$. It follows that the Morse β-volume of C_n satisfies $M(C_n, \beta) > M_n$, so (L, g) does not have the bounded homology property.

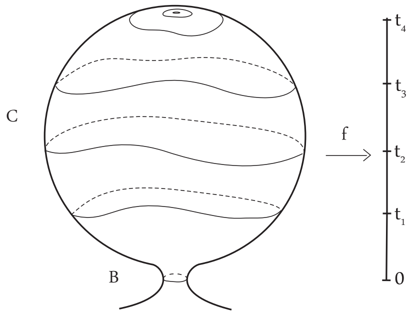

Figure 4.1. The Morse β-volume of a balloon C.

The points d_n can be chosen so that no end of L has the bounded homology property. In fact, if the points x_i have been chosen for $i \leqslant n$, let k be the number of connected components of the complement of $V_{d_n}(x_0)$ with noncompact closure, and choose $x_{n+1}, \dots, x_{n+k}$ so that one of them is in each of these components. Continue inductively, repeating this procedure with $n + k$ in place of n. This guarantees that each end will contain an infinite number of the points x_n and therefore does not have the bounded homology property.

We can choose the sequence $\{d_n\}$ to grow sufficiently fast, relative to the sequence $\{r_n\}$, so that the growth functions f_0 of g_0 and f of g satisfy $f_0(r) \leqslant f(r) \leqslant 2f_0(r)$. Hence g will have the same growth type as g_0, so the process of inserting balloons can be carried out so as to preserve the growth type.

The last conclusion, that there are uncountably many quasi-isometry classes of Riemannian metrics on L that do not have the bounded homology property, follows from Proposition 2.9, which realizes the distinct growth types $x \mapsto x^k$ for every $k > 1$. □

Proof of Proposition 2.9. — We outline the argument, leaving precise details to be filled in by the reader. Given a smooth connected non-compact manifold L of dimension $p \geqslant 2$ with basepoint x_0 and any continuous increasing function $f : [0,\infty) \to [0,\infty)$ such that $I \prec f \preceq \exp$, we shall find a metric g_2 on L such that the growth function is equivalent to f.

Let $q : L \to [0,\infty)$ be a proper smooth Morse function such that $q^{-1}(0) = \{x_0\}$. Choose Riemannian metrics on each level set $q^{-1}(t)$ such that they vary smoothly in t away from the singularities of q and the geometry of these level sets is uniformly bounded. We construct a Riemannian metric g_1 on L so that, away from the singularities of q, the level sets are totally geodesic, the gradient flow of $\phi \circ f$ (for a diffeomorphism $\phi : [0,\infty) \to [0,\infty)$ to be chosen later) is orthogonal to the level sets, and the distance between the level sets $q^{-1}(s)$ and $q^{-1}(t)$ is $|\phi(t) - \phi(s)|$; the metric must be adjusted in small neighborhoods of the singularities. If the function ϕ grows sufficiently rapidly, the level sets will be spread far apart in comparison with the growth of their volumes, and the growth function f_1 of the metric g_1 will satisfy $f_1 \preceq f$. Here we must use the hypothesis that f grows more than linearly, since the requirement of globally bounded geometry and the topology of the level sets $q^{-1}(t)$ may make it impossible to have a uniform upper bound on their volumes.

Now if $f_1 \prec f$ we can increase the growth type by inserting sufficiently many large balloons in balls $V_d(x_n)$ (and possibly in other disjoint balls $V_d(y)$ that are closer together, including balls on balloons already inserted) to change g_1 to a new metric g_2 so that the new growth function f_2 will have the same growth type as the given function f. Note that it is possible to get the exponential growth type by inserting so many balloons that their number grows exponentially as a function of the distance from the basepoint x_0, but since the geometry is required to have bounded sectional curvature, it is impossible to get any greater growth type. Clearly any intermediate growth type can be realized by choosing an appropriate distribution of inserted balloons. □

Remark 4.1. — It is an open question whether or not leaves of foliations of codimension greater than one on closed manifolds have the bounded homology property. Tsuboi's construction of a codimension two foliation whose 2-dimensional leaves do not have the bounded *homotopy* property (see the last section of [18]) uses the fact that certain loops on the leaves and are of unbounded length since they are connected and have unbounded diameter. That construction cannot be adapted to give leaves that do not have the bounded *homology* property, since the level sets of Morse functions need not be connected. In fact, the obvious adaptation of the 2-dimensional construction to higher dimensions gives leaves that do possess the bounded homology property.

5. Proof of two lemmas

In this Section we prove Lemmas 3.2 and 3.4. We assume all the conditions indicated in the first two paragraphs of Section 3.

Proof of the Finiteness Lemma 3.2. — Given a point $x \in M$ in a leaf L_x, let $D_x = \overline{B(x,\epsilon)}$ be the closed p-dimensional ball centered at x of radius ϵ for some positive ϵ less than the injectivity radius at x, so that D_x is actually homeomorphic to a closed ball. We may lift D_x along $\mathcal{T}$ to disks D_y on nearby leaves L_y for $y \in J$, where J is an open interval embedded in a leaf of $\mathcal{T}$ and containing x, thus obtaining a bifoliated map

$$h : (D_x \times J, \mathcal{H}, \mathcal{V}) \to (M, \mathcal{F}, \mathcal{T})$$

such that $h(x,y) = y \in L_y$. Fix a larger smooth compact region $E \subset L_x$ that contains $V_d(D_x)$, where $d = (2k+1)\beta$, and give E a smooth β'-triangulation, where as before $\beta' = \beta/4$.

If $B \in \mathcal{B}$ meets D_x, then $V_\beta(B) \subset E$ since B is connected and has β-volume less than or equal to k and therefore diameter less than $2k\beta$. Then the SAP applied to B will yield a $(p-1)$-dimensional subcomplex $X_B \subset E$. Let Ω_B be a smooth compact region containing $\overline{V_{2\beta'}(B)}$ in its interior and contained in $V_{3\beta'}(B)$, and give Ω_B a smooth β'-triangulation that agrees with the triangulation on E for all the simplices contained in $V_{2\beta'}(B)$. The closed tubular neighborhood V of B given by condition (2) of Definition 2.4 satisfies $\Omega_B \subset V_{3\beta'}(B) \subset V$ and is 1-connected, since by hypothesis B is. Therefore Ω_B can be lifted along $\mathcal{T}$ to $\Omega_{B,y}$ on every leaf L_y sufficiently close to L_x. Hence we get a bifoliated embedding

$$h_B : (\Omega_B \times J', \mathcal{H}, \mathcal{V}) \to (M, \mathcal{F}, \mathcal{T})$$

such that $h_B(x',y) = y \in L_y$ for every point y in a sufficiently small open subinterval $J' \subset J$ containing x and for some $x' \in B \cap D_x$.

Now since E is compact there are only finitely many $(p-1)$-dimensional subcomplexes of E; let $X_1, \dots X_m$ be those that are obtained by the SAP as X_B for some $B \in \mathcal{B}$ that meets D_x. Choose a submanifold B_i for each X_i, so that $X_i = X_{B_i}$, ie., X_i is obtained from B_i by the SAP. We may choose J' small enough so that for every $B \in \{B_1, \dots, B_m\}$ and $y \in J'$, the lifted triangulation on $\Omega_{B,y} = h_B(\Omega_B \times \{y\})$ has mesh less than β' and $\Omega_{B,y}$ contains the the $2\beta'$-neighborhood $V_{2\beta'}(X_{B,y})$ of the lifted complex $X_{B,y} = h_B(X_B \times \{y\})$ of X_B. Thus the conditions (1) and (2) of Definition 3.1 are satisfied for $\Omega_{B,y}$ and $X_{B,y}$ on their leaves, for every $y \in J'$. Now if we let $g : [0,1] \to J'$ be an embedding such that $x = g(t)$ for some $t \in (0,1)$, the triple (X_B, Ω_B, f_B), where $f_B(t,y) = h_B(g(t),y)$, will be a good transverse family. Thus we get m good transverse families corresponding to D_x. Every $B \in \mathcal{B}$ that meets D_y for $y \in g([0,1])$ will yield an $X_{B,y}$ under the SAP.

For each point $x \in M$ we get an open set $h(\mathrm{Int}(D_x) \times g(0,1))$ containing x. Such open sets cover M, so finitely many of them suffice to cover M, say for $x_1, \dots, x_n$. The union of the sets of triples (X_B, Ω_B, f_B) where B varies over all the $B_i's$ for all the points $x_1, \dots, x_n$, will be a finite set of good transverse families. It satisfies the conclusion of the Finiteness Lemma since every $B \in \mathcal{B}$ will meet one of the sets $h(\mathrm{Int}(D_x) \times g(0,1))$, for some $x \in \{x_1, \dots, x_n\}$, and so will yield one of the complexes X_{B_i} for that x. $\square$

Proof of Lemma 3.4. We suppose the hypotheses of the Lemma. Observe that the SAP moves points of each of B and B' a distance less than β', since the triangulation on the leaf has mesh less than β'. Hence $B \subset V_{\beta'}(X_t)$ and $X_t \subset V_{\beta'}(B)$, and similarly for B'. Thus $B' \subset V_{2\beta'}(B)$ and $B \subset V_{2\beta'}(B')$. Now each of the two connected submanifolds B and B' separates the leaf L into two connected components. For B they are the interior of C and $L \setminus C$. Since $B' \subset V_{2\beta'}(B)$, one of the two connected components of $L \setminus B'$ must be contained in $V_{2\beta'}(C) = C \cup V_{2\beta'}(B)$; call its closure C'. Since $B \subset V_{2\beta'}(B')$, we must have $C \subset V_{2\beta'}(C') = C' \cup V_{2\beta'}(B')$.

Clearly $\partial C' = B'$ and C' is connected. We must show that its fundamental group is trivial. By hypothesis, there is a closed tubular neighborhood V' of B' containing $V_\beta(B')$. Set $C'_- = C' \setminus \mathrm{Int}(V')$ and $C'_+ = C' \cup V'$, so that $C'_- \subset C' \subset C'_+$. These inclusions induce isomorphisms on the fundamental groups, since C' is the union of C'_- with a collar neighborhood of its boundary that is homeomorphic to $\partial C' \times I$ while C'_+ is the union of C' with a collar neighborhood of its boundary also homeomorphic to $\partial C' \times I$. Since $B \subset V_{2\beta'}(B')$, $(C' \setminus V_{2\beta'}(B')) \cap B = \emptyset$, so $C' \setminus V_{2\beta'}(B') \subset C$, and we have $C'_- \subset C' \setminus V_{2\beta'}(B') \subset C \subset C' \cup V_{2\beta'}(B') \subset C'_+$. Thus the inclusion $C'_- \subset C'_+$, which induces an isomorphism of fundamental groups, factors through C, which is 1-connected by hypothesis, so the homeomorphic sets C'_-, C', and C'_+ are also 1-connected.

Finally, we must show that there exists a constant K_0 such that $M(C',\beta) \leqslant M(C,\beta) + K_0$, which by symmetry will establish the second conclusion. Since the foliated manifold M is compact, there is an upper bound for the sectional curvature of the leaves of $\mathcal{F}$. By Proposition 2.11 there is a constant n_0, depending only on $\mathcal{F}$, k, and β, such that every open 2β-ball $V_{2\beta}(x)$ in a leaf of $\mathcal{F}$ can be covered by n_0 open β-balls in the leaf, and we set $K_0 = kn_0$. Now let $f : C \to [0,\infty)$ be a Morse function on C such that $f(B) = 0$ and every level set $f^{-1}(t)$ has β-volume $\mathrm{Vol}_\beta(f^{-1}(t)) \leqslant M(C,\beta)$. Since $C \smallsetminus V_{2\beta'}(B) \subset \mathrm{Int}(C')$, we may extend the Morse function $f+1$ restricted to $C \smallsetminus V_{2\beta'}(B)$ to a Morse function $f' : C' \to [0,\infty)$ such that $f'^{-1}(0) = B'$. By hypothesis there are k points $y_1,\dots,y_k \in L$ such that the union of the β-balls $V_\beta(y_j)$ covers B', and then the union of the 2β-balls $V_{2\beta}(y_j)$ covers $V_{2\beta'}(B')$. Each of these 2β-balls can be covered by at most n_0 β-balls, so the β-volume of $V_{2\beta'}(B')$ is at most $K_0 = kn_0$. Finally, each level set $f'^{-1}(t)$ is contained in the union $f^{-1}(t+1) \cup V_{2\beta'}(B')$, whose β-volume is at most $\mathrm{Vol}_\beta(f^{-1}(t+1)) + \mathrm{Vol}_\beta(V_{2\beta'}(B')) \leqslant M(C,\beta) + K_0$, showing that $M(C',\beta) \leqslant M(C,\beta) + K_0$, as claimed. □

6. Invariance under quasi-isometry

The goal of this section is to prove Proposition 2.10, which states that the bounded homology property is invariant under quasi-isometry. We shall use Proposition 2.11 and the following result, which is certainly well known.

PROPOSITION 6.1. — *Let L be a complete Riemannian manifold with sectional curvature between $-c$ and c for a constant $c \geqslant 1$ and let $v_1, v_2 \in S \subset T_xL$ be points on the unit sphere in the tangent plane at a point $x \in L$. Then the distance between the points $\exp_x tv_1$ and $\exp_x tv_2$ on L is at most e^{ct} times the distance between v_1 and v_2 on S.*

Proof. — If the distance on the unit sphere $S \subset T_xL$ from v_1 to v_2 is d, then there exists a geodesic $v : [0,d] \to S$ from v_1 to v_2 parametrized by arclength, so that $|v'(s)| \equiv 1$. Define $f : [0,\infty) \times [0,d] \to L$ to be $f(t,s) = \exp_x tv(s)$, where $\exp_x$ is the exponential map. Then for each s, $J_s(t) = \frac{\partial f}{\partial s}(t,s)$ is a Jacobi vector field (see do Carmo [6]) and satisfies the Jacobi equation

$$\frac{D^2 J}{dt^2} + R(\gamma_s'(t), J_s(t))\gamma_s'(t) = 0$$

where R is the curvature operator, $\frac{D}{dt}$ is the covariant derivative, and $\gamma_s(t) = \exp_x tv(s)$ is the geodesic starting at x with initial velocity $\gamma_s'(0) =$

$v(s)$. Fix a parallel orthonormal frame $e_1, \ldots, e_p$ along each γ_s, expand $J_s(t)$ as

$$J_s(t) = \sum_{i=1}^{p} f_{s,i}(t) e_i(t),$$

and set $a_{s,ij} = < R(\gamma_s'(t), e_i(t))\gamma_s'(t), e_j(t) >$, where $< \cdot, \cdot >$ denotes the Riemannian metric. Then the Jacobi equation translates to the system of p ordinary differential equations $f''_{s,j}(t) = -\sum_i a_{s,ij}(t) f_{s,i}(t)$ (see [6]) or a system of $2p$ first order differential equations $Y'(t) = B(t)Y(t)$, where $Y(t) = (y_1(t), \ldots, y_{2p}(t))$ with $y_i(t) = f_{s,i}(t)$ and $y_{i+p}(t) = f'_{s,i}(t)$ for $1 \leqslant i \leqslant p$, and $B(t)$ is a $2p \times 2p$ matrix of the form

$$B(t) = \begin{bmatrix} 0 & I \\ -A(t) & 0 \end{bmatrix}$$

where $A(t) = [a_{s,ij}(t)]$ and I is the identity matrix. Now $A(t)$ is symmetric (by the symmetry properties of the curvature R), and so we can diagonalize it to $\bar{A}(t)$ in a new orthonormal frame $\bar{e}_1, \ldots, \bar{e}_n$. Since R is multi-linear, each diagonal coefficient of the new diagonal matrix $\bar{A}(t)$ will be

$$\bar{a}_{s,ii} = < R(\gamma_s'(t), \bar{e}_i(t))\gamma_s'(t), \bar{e}_i(t) >$$

which is just the sectional curvature in the plane of $\gamma_s'(t)$ and $\bar{e}_i(t)$, so that $|\bar{a}_{s,ii}| \leqslant c$. Hence the matrix $A(t)$ has norm $|A(t)| = |\bar{A}(t)| \leqslant c$ and so $B(t)$ also has norm $|B(t)| \leqslant c$ (since $|I| = 1 \leqslant c$). Note that $J_s(0) = 0$ and $J_s'(0) = \frac{\partial^2 f}{\partial t \partial s}(0, s) = v'(s)$ with $|v'(s)| = 1$, so $|Y(0)| = 1$. Then Theorem 1.5.1 of [25] shows that $|J_s(t)| \leqslant |Y(t)| \leqslant \exp(\int_0^t |B(r)| dr) \leqslant e^{ct}$. We have shown that the velocity of the curve $s \mapsto \exp_x tv(s)$ is at most e^{ct} times that of the curve $v(s)$ on S, so the distance between the points $\exp_x tv_1$ and $\exp_x tv_2$ on L is at most $e^{ct}d$. $\square$

Proof of Proposition 2.11. — Let L be a complete Riemannian p-dimensional manifold with sectional curvature between $-c$ and c for some constant $c \geqslant 1$ and suppose that constants $0 < a < b$ are given. Take points $0 \leqslant t_1 < t_2 < \cdots < t_r \leqslant b$ so that every $t \in [0, b]$ lies within a distance less than $a/2$ of one of the points t_j. Let $c_0 = 2e^{cb}$ and choose a set $\{v_1, \ldots, v_m\} \subset S$ so that every point of the unit sphere $S \subset T_x L$ is at most at a distance a/c_0 on S from one of the points v_i. Then every point $y = \exp_x t'v$ in the open ball $B(x, b)$ of radius b centered at x (for some $v \in S$ and $t' \in [0, b]$) lies within a distance less than $e^{ct'} a/c_0 \leqslant a/2$ from a point $\exp_x(t'v_i)$ on a geodesic $t \mapsto \exp_x(tv_i)$, and this point is at a distance at most $a/2$ from one of the points $\exp_x(t_j v_i)$. Therefore y lies in one of the open balls $B(\exp_x(t_j v_i), a)$. Thus the open balls of radius a centered at the

mr points $\exp_x(t_j v_i), i = 1, \dots, m, j = 1, \dots, r$ cover the ball $B(x, b)$. The same argument applies with the same values m and r around any other point $x' \in L$, so every open b-ball on L can be covered by at most $n = mr$ a-balls, as claimed. □

Proof of Proposition 2.10. — Let $h : L' \to L$ be a quasi-isometry of two Riemannian manifolds with bounded geometry, as in Definition 2.1, so that there are constants $C_0 \geqslant 1$ and $D > 0$ such that the quasi-isometry inequalities

$$C_0^{-1} d(h(x'), h(y')) - D \leqslant d'(x', y') \leqslant C_0 d(h(x'), h(y')) + D$$

hold for all $x', y' \in L'$. Suppose that L has the bounded homology property, so there is a fixed $\beta_0 \geqslant 0$ such that for all $\beta > \beta_0$ and $k > 0$, there is an integer $K = K(k, \beta)$ such that any $B = \partial C \subset L$ satisfying the four conditions of Definition 2.4 must have $M(C, \beta) < K$. Let $\beta_0' = C_0\beta_0 + D$ and suppose that numbers $k' > 0$ and $\beta' > \beta_0'$ are given. We must find an integer $K' = K'(k', \beta')$ satisfying the bounded homology property on L'.

Set $\beta = C_0^{-1}(\beta' - D) > C_0^{-1}(\beta_0' - D) = \beta_0$, $k = n_1 k'$ and $K' = K$, where n_1 is a constant given by Proposition 2.11 such that on L every ball of radius $C_0(\beta' + D)$ is covered by at most n_1 balls of radius β.

Note that if $V_r(x)$ and $V_r'(x')$ are the open r-balls on L and L' centered at $x \in L$ and $x' \in L'$, then by the quasi-isometry inequalities

$$h(V'_{\beta'}(x')) \subset V_{C_0(\beta'+D)}(h(x'))$$

and

$$h^{-1}(V_\beta(h(x'))) \subset V'_{\beta'}(x')$$

since $d'(x', y') < \beta'$ implies that $d(h(x'), h(y')) < C_0(\beta' + D)$ and $d(h(x'), h(y')) < \beta$ implies that $d'(x', y') < C_0\beta + D = \beta'$.

Consider any $(p-1)$-submanifold B' of L' satisfying the four conditions of Definition 2.4 for the constants k' and β', so that there exists a tubular neighborhood V' of B' containing the β'-neighborhood $V'_{\beta'}(B')$ of B', B' has β'-volume $\mathrm{Vol}_{\beta'}(B') \leqslant k'$ on L', and there exists a compact 1-connected region C' in L' with $\partial C' = B'$. We shall show that $M(C', \beta') \leqslant K'$.

Set $B = h(B'), V = h(V')$, and $C = h(C')$. Note that $V_\beta(B)$ is contained in the tubular neighborhood V of B, for if $d(h(x'), h(y')) < \beta$ with $h(x') \in B = h(B')$, then by the quasi-isometry inequality $d'(x', y') < C_0\beta + D = \beta'$ so that $y' \in V_{\beta'}(B') \subset V'$ and $h(y') \in V = h(V')$. Also $\mathrm{Vol}_\beta(B) \leqslant k = n_1 k'$ since the image under h of a β'-ball on L' is contained in a $C_0(\beta' + D)$-ball which is covered by n_1 β-balls, and k' β'-balls cover B'. Thus B satisfies the four conditions of Definition 2.4.

By the bounded homology property for L, C has Morse β-volume $M(C,\beta)$ less than or equal to K, so that there exists a Morse function $f : C \to [0,\infty)$ on C whose level sets have β-volume at most K. Now $f \circ h$ is a Morse function on C' whose level sets are taken onto those of f by h. Since K balls of radius β on L suffice to cover each level set of f, their images under h^{-1} are contained in K balls of radius β' on L', and these balls cover the corresponding level set of $f \circ h$. In other words, $M(C',\beta') \leqslant K = K'$, as claimed. □

7. Novikov's Theorem for embedded vanishing cycles

In this Section we prove the extension of Novikov's Theorem (Theorem 2.13, on the existence of Reeb components) and the Corollary 2.14 which asserts that for every 1-parameter family of connected closed $(p-1)$-submanifolds embedded in leaves in the interior of a Reeb component, each bounding a compact region on its leaf, and for every $\beta > 0$, there is a common upper bound for the Morse β-volumes of the regions that they bound, relative to any fixed Riemannian metric on the Reeb component.

Throughout this section, we assume that $\mathcal{F}$ is a p-dimensional topological foliation of a compact $(p+1)$-dimensional manifold M (so no differentiability is assumed before Proposition 7, where we shall assume that $\mathcal{F}$ is $C^{2,0}$), B is a compact connected $(p-1)$-dimensional manifold, and there is a foliated map

$$h : (B \times [0,1], \mathcal{H}) \to (M, \mathcal{F})$$

where the horizontal and vertical foliations $\mathcal{H}$ and $\mathcal{V}$ of $B \times [0,1]$ are given by the leaves $B \times \{t\}$ for $t \in [0,1]$ and $\{x\} \times [0,1]$ for $x \in B$, respectively. Furthermore we assume that $h_0 : B \to L_0$ is an embedding, where for all $t \in [0,1]$, $h_t : B \to L_t$ is the map defined by $h_t(b) = h(b,t)$ and L_t is the leaf containing $B_t = h_t(B)$. Recall that a mapping is a topological immersion if it is a local embedding.

The principal step in the Proof of Theorem 2.13 is the following result. Let C be a compact connected p-dimensional manifold with boundary B.

Proposition 7.1. — *If $h : (C \times (0,1] \cup B \times [0,1], \mathcal{H}, \mathcal{V}) \to (M, \mathcal{F}, \mathcal{T})$ is a bifoliated topological immersion that cannot be extended over $C \times [0,1]$ and $h_0 : B \to M$ (defined by setting $h(x,0) = h_0(x)$) is an embedding into a leaf L_0, then the leaf L_0 is the boundary of a Reeb component whose interior is the union of the leaves meeting $h(B \times (0,1])$.*

Proof of Theorem 2.13 from Proposition 7.1. — In addition to the general hypotheses mentioned at the beginning of this section, we suppose that one of the following conditions holds for every $t > 0$ sufficiently close to 0 but not for $t = 0$:

(1) $B_t = h_t(B)$ is the boundary of a compact 1-connected region $C_t \subset L_t$;
(2) $B_t = h_t(B)$ is the boundary of a compact region $C_t \subset L_t$;
(3) $\mathcal{F}$ and B are oriented and $0 = h_{t*}([B]) \in H_{p-1}(L_t)$ (where $[B]$ is the fundamental homology class of B); or
(4) $0 = h_{t*}([B]) \in H_{p-1}(L_t; \mathbb{Z}_2)$ (where $[B]$ is the fundamental homology class of B with coefficients modulo 2),

and then we must show that the leaf L_0 is the boundary of a Reeb component R whose interior $\mathrm{Int}(R)$ is the union of the leaves L_t for which $t > 0$.

Fix a 1-dimensional foliation $\mathcal{T}$ topologically transverse to $\mathcal{F}$. We shall show that it is possible to modify h to a bifoliated topological immersion $h'' : (B \times [0,1], \mathcal{H}, \mathcal{V}) \to (M, \mathcal{F}, \mathcal{T})$ with $h''_0 = h_0$. First, observe that it is possible to modify h by a foliated homotopy, moving $h(B \times \{t\})$ in the leaf L_t, with the homotopy fixed on $B \times \{0\}$, so that the resulting mapping $h : (B \times [0,1], \mathcal{H}) \to (M, \mathcal{F})$ restricts to a bifoliated mapping $h' : (B \times [0,\epsilon], \mathcal{H}, \mathcal{V}) \to (M, \mathcal{F}, \mathcal{T})$ for some $\epsilon > 0$. Fix a basepoint $b_0 \in B$ and note that the points $a = h'(b_0, 0)$ and $b = h'(b_0, \epsilon)$ must be distinct, for otherwise the maps h'_0 and h'_ϵ would agree on all of B, which contradicts each of the conditions (1) through (4). Let f be a homeomorphism from $[0,1]$ onto the segment $[a,b]$ which is contained in a leaf of $\mathcal{T}$, say with $f(0) = a$. Define $h'' : B \times [0,1] \to M$ by setting $h''(b,t) = h'(b, g(t))$ where $g(t) = \min\{s \in [0,\epsilon] \mid h'(b_0, s) = f(t)\}$. In general, g will be monotone increasing but not continuous, but $t \mapsto h''(b_0, t) = f(t)$ is the homeomorphism f from $[0,1]$ onto $[a,b]$. Furthermore, since h'' is bifoliated, it follows from the local product structure given by $\mathcal{F}$ and $\mathcal{T}$ that

$$h'' : (B \times [0,1], \mathcal{H}, \mathcal{V}) \to (M, \mathcal{F}, \mathcal{T})$$

is a bifoliated topological immersion with $h''_0 = h_0$. It is clear that any of the conditions (1) through (4) that was satisfied by the original mapping h will still hold for the new bifoliated immersion, which for simplicity we shall still denote by h.

Next, we observe that each of the conditions (1), (3), and (4) implies condition (2). Suppose (1), so that B_t must be the boundary of a compact 1-connected region $C_t \subset L_t$ for every $t > 0$. If B_0 bounded a compact region C_0 on L_0, then by the Reeb Stability Theorem C_0 would have a product foliated neighborhood and therefore C_0 would be homeomorphic to C_t with $t > 0$ and hence 1-connected, contrary to (1). Thus (1) implies (2).

Condition (3) implies that for $t > 0$, B_t must be the boundary of a compact region C_t contained in the leaf L_t, since the $(p-1)$-dimensional cycle carried by B_t is a boundary on the p-dimensional leaf L_t. We are supposing in this case that the foliation $\mathcal{F}$ is oriented, so if B_0 bounds a compact region C_0 in L_0, then C_0 carries a relative homology class $[C_0] \in H_p(C_t, B_t)$ with $\partial[C_0] = [B_0]$, thus making $[B_0] = 0 \in H_{p-1}(L_t)$, contrary to hypothesis. This shows that (3) implies (2). The proof that (4) implies (2) is similar. Thus we assume only condition (2), which includes the other three cases.

Now if for some values of t, B_t bounds compact regions on both sides in L_t, then L_t will be a compact leaf. If there were such values t_n converging to 0, then L_0 would also be a compact leaf and for a sufficiently small t_n, the leaves L_0 and L_{t_n} would bound an I-bundle fibered by segments in the leaves of $\mathcal{T}$. Then C_{t_n} would project along $\mathcal{T}$ onto a homeomorphic region $C_0 \subset L_0$ so that $\partial C_0 = B_0$, contrary to hypothesis. Hence there is an $\epsilon > 0$ such that for each $t \in (0, \epsilon)$ B_t bounds a compact region on exactly one side in L_t. Let S_0 be the set of $t \in (0, \epsilon)$ for which B_t bounds a compact region C_t on the positive side in L_t, and S_1 the set for which B_t bounds a compact region C_t on the negative side, according to a coherent transverse orientation of B_t in L_t which we choose arbitrarily. Then $S_0 \cap S_1 = \emptyset$ and $S_0 \cup S_1 = (0, \epsilon)$.

If there exists $0 < \epsilon' \leqslant \epsilon$ such that $(0, \epsilon'] \subset S_0$ (or $(0, \epsilon'] \subset S_1$), then each of the corresponding compact regions C_t must have trivial holonomy and consequently each C_t will have a product neighborhood foliated as a product. Joining these regions C_t together we obtain a bifoliated immersion

$$h : (C \times (0, \epsilon'] \cup B \times [0, \epsilon'], \mathcal{H}, \mathcal{V}) \to (M, \mathcal{F}, \mathcal{T})$$

which does not extend over $C \times [0, \epsilon']$. Reparametrizing the interval and applying Proposition 7.1 shows that L_0 is the boundary of a Reeb component with the leaves L_t for $t \in (0, \epsilon']$ in its interior, as claimed. The remaining leaves containing $h(B \times \{t\}$ for the original mapping h must also be contained in the interior of the Reeb component, since the transverse orientation points inwards along the boundary L_0.

In the remaining case, 0 is a limit point of both S_0 and S_1, so we can find a strictly decreasing sequence $t_n \searrow 0$ with $t_n \in \overline{S}_0 \cap \overline{S}_1$. Suppose some $t_n \in S_0$ is a limit point of S_1 on the right. Considering the holonomy of

C_{t_n}, there must be an open interval $(t'_n, t'_n + \delta) \subset S_1$ for some $\delta > 0$ and some $t'_n \in S_0$ near to (and possibly equal to) t_n. Applying the Proposition as before we find that $L_{t'_n}$ bounds a Reeb component. The same conclusion holds if t_n is a limit point of S_1 on the left, and similarly if $t_n \in S_1$. Thus we find a sequence $t'_n \searrow 0$ such that every $L_{t'_n}$ bounds a Reeb component, which is impossible since the boundary leaf of a Reeb component is compact and the interior leaves are not. Hence this case does not occur, and the Theorem is proven. □

Proof of Proposition 7.1. — Suppose that

$$h : (C \times (0,1] \cup B \times [0,1], \mathcal{H}, \mathcal{V}) \to (M, \mathcal{F}, \mathcal{T})$$

is a bifoliated immersion that cannot be extended over $C \times [0,1]$ and its restriction $h|_{B\times\{0\}}$ is an embedding, as in the Proposition. There must be a point $x_0 \in C \smallsetminus B$ such that h does not extend to the point $(x_0, 0)$. (If not, h would extend uniquely and continuously over $C \times [0,1]$, contrary to hypothesis.) Note that $t \mapsto h(x_0, t)$ is an immersion of $(0,1]$ into a leaf of $\mathcal{T}$. Let $\{s_n\}$ be a sequence in $(0,1]$ converging to 0. Then, since M is compact, there exists a strictly decreasing subsequence $\{t_n\}$ such that the points $y_n = h(x_0, t_n)$ converge to some point y_0 of M as $n \to \infty$. Let V be a connected open neighborhood of y_0 on the leaf L of $\mathcal{F}$ that contains y_0.

Lemma 7.2. — *It is possible to choose the sequence $\{t_n\}$, the limit point y_0, its neighborhood $V \subset L$, and $\epsilon > 0$, so that*

(1) *the leaf L containing y_0 is distinct from L_0;*
(2) *for every n, $t_n < \epsilon$ and $y_n = h(x_0, t_n) \in V$; and*
(3) *V is disjoint from $h(B \times [0, \epsilon])$.*

Proof. — If it happens that $y_0 \in L_0$, by a small change of the values of the numbers t_n, we may move the points $y_n = h(x_0, t_n)$ along $\mathcal{T}$ so that the sequence y_n converges to another point (still denoted y_0) on another leaf L. Then since $y_0 \notin h(B \times \{0\})$, for a sufficiently small $\epsilon > 0$, y_0 will not lie on the set $h(B \times [0, \epsilon])$, and we may choose an open connected neighborhood V of y_0 on its leaf L whose closure $\bar{V}$ is disjoint from the compact set $h(B \times [0, \epsilon])$. Then, slightly changing the values of the numbers t_n (to move the points y_n along leaves of $\mathcal{T}$) and possibly passing to a subsequence, we may guarantee that $y_n = h(x_0, t_n) \in V$ and $t_n < \epsilon$. □

Now y_n lies on the intersection of V with the region $C(n) = h(C \times \{t_n\})$ on L, and V is disjoint from its boundary $B(n) = h(B \times \{t_n\}) = \partial C(n)$, so $V \subset C(n)$. The connected submanifolds $B(m) = \partial C(m))$ are pairwise disjoint and $y_0 \in C(n) \cap C(m)$, so either $C(n) \subset C(m)$ or $C(m) \subset C(n)$.

The sets $B(m)$ are separated by a positive distance on the leaf L, so only finitely many of the $C(m)$'s can be contained in the compact set $C(n)$. Thus for each n there exists some $n' > n$ such that $C(n) \subset C(n')$.

Lemma 7.3. — *The leaf L_0 is compact.*

Proof. — If L_0 is not compact, then there is a simple closed curve γ transverse to $\mathcal{F}$ in the positive direction that intersects L_0 in a single point $\gamma(0) = z$; we can choose z to be near to but not on the set B_0, and on the side of B_0 in L_0 on which each submanifold B_t bounds C_t. We may isotope γ slightly so that for some small positive numbers ϵ_1 and $\epsilon_2 \leqslant \epsilon$, the segment $\gamma([0,\epsilon_1])$ lies on a leaf of $\mathcal{T}$ and γ is disjoint from $h(B \times [0,\epsilon_2])$. Then for a sufficiently large index n, with $t_{n'} < t_n < \epsilon_2$, γ will enter into the immersed region $h(C \times [t_{n'}, t_n])$ at a point $\gamma(s) \in C(n')$ for some $s \in (0,\epsilon_1)$. Now γ can never exit from that region, whose boundary is contained in $C(n') \smallsetminus \operatorname{Int} C(n) \cup h(B \times [t_{n'}, t_n])$, for γ cannot exit along $C(n') \smallsetminus C(n)$ where the transverse orientation enters, and γ is disjoint from $h(B \times [t_{n'}, t_n]) \subset h(B, [0,\epsilon_2])$. This contradiction shows that L_0 must be compact. □

Let N be a positive one-sided tubular neighborhood of L_0 fibered by segments in leaves of $\mathcal{T}$ with projection map $p : N \to L_0$. Let $L_0' \subset N'$ be the result of cutting L_0 along B_0 and cutting N along $p^{-1}(B_0)$ to get N', so that $\partial L_0' = B_0^+ \cup B_0^-$, two disjoint copies of B_0, and N' is a positive one-sided tubular neighborhood of L_0'. The positive holonomy of the compact leaf L_0' must be trivial, for the leaves near to L_0' are contained in the compact sets C_t and thus are compact. Hence some smaller compact tubular neighborhood $N_0' \subset N'$ will be foliated as a product, say by leaves D_t with boundary $\partial D_t = B_t^+ \cup B_{f(t)}^-$, where f is a function defined on a small positive one-sided neighborhood of 0 in $[0,\epsilon_2)$ and $p(B_s^\pm) = B_0^\pm$ for every sufficiently small s. For definiteness we choose the notation so that $f(t) < t$ and consequently $C_t \subset C_{f(t)}$. This holds for all t less than or equal to some $t_0 \leqslant \epsilon_2$. As t varies in $[0,t_0]$ there is defined a continuous function $f : [0,t_0] \to [0,t_0]$ such that $f(t) < t$ for every $t > 0$, while in the limit $f(0) = 0$. Then it is clear that $C_t \cup D_t = C_{f(t)}$ with $C_t \cap D_t = B_t$.

Observe that the inclusion $i_t : C_t \subset C_{f(t)}$ defines an embedding $\phi = h_{f(t)}^{-1} \circ i_t \circ h_t : C \to C$ that does not depend on $t \in (0,t_0]$, since moving along the leaves of $\mathcal{T}$ produces the same result for each t. Thus $h : C \times (0,t_0] \to M$ passes to a quotient immersion $\bar{h} : R_0 \to M$, where $R_0 = C \times (0,t_0]/\{(x,t) \sim (\phi(x), f(t))\}$. For each $t \in (0,t_0]$, let $L(t)$ be the image of the set $\cup_{n=0}^{\infty} C \times \{f^n(t)\}$ in R_0. Then $\bar{h}|_{L(t)} : L(t) \to L_t$ is a homeomorphism, for the identifications correspond to the inclusions $C_t \subset C_{f(t)}$; there cannot be any further identifications, since the regions C_t were chosen to be embedded in leaves of $\mathcal{F}$ in M, and the union of the sets $C_{f^n(t)}$ exhausts the leaf L_t. No region C_s with $s > t$ that is not one of the sets $C_{f^n(t)}$ can meet the leaf L_t, for then there would be an index $n \geqslant 0$ such that $C_{f^n(t)} \subset C_s \subset C_{f^{n+1}(t)}$, which is impossible since $D_{f^n(t)}$ contains no B_s in its interior. Consequently $\bar{h} : R_0 \to \cup \{L_t |\ t \in (0, t_0)\}$ is a bijection, which is easily seen to be a homeomorphism.

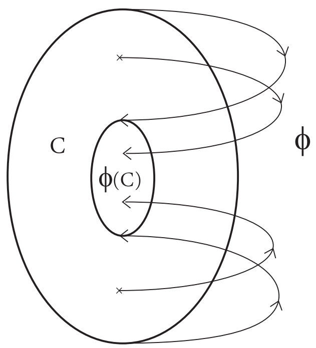

Figure 7.1. $\phi : C \hookrightarrow \operatorname{Int} C$.

The map $p : R_0 \to S^1 = [t, f(t)]/\{t \sim f(t)\}$, defined by setting $p(L_s) = s$ if $s \in [t, f(t)]$, is well defined, and it is a fibration whose local product structure is given by translations by holonomy mappings along leaves of $\mathcal{T}$. A small positive compact tubular neighborhood N_0 of L_0 will meet R_0 in $N_0 \smallsetminus L_0$. Hence $R = N_0 \cup R_0$ is compact, since it coincides with the union of the two compact sets N_0 and $h(C \times [t_1, f(t_1)])$ for some sufficiently small positive t_1; furthermore $R = L_0 \cup R_0$ and $\partial R = L_0$. Thus we have shown that R is a compact manifold with boundary L_0, and its interior R_0 fibers over the circle. Since the boundary is connected, there exists a transverse orientation pointing inwards, so R is a Reeb component. Finally, all the leaves L_t for $t \in (0, 1]$ in the original parametrization of the interval are contained in R_0 because the transverse orientation points inwards along ∂R; for any point $z \in B$ the curve $t \mapsto h(z, t)$ lies in R_0 for small values of t and as t increases it must be entirely contained in R_0. $\square$

Note that in the preceding proof the Reeb component R shown to exist was obtained from the map $\phi : C \to \operatorname{Int} C$ and the contraction $f : [0, t_0] \to [0, t_0]$, by a construction which we shall now describe. It is not difficult to show that such a construction gives **all** Reeb components, up to foliated homeomorphism (see [1] for the proof), although we shall not use that fact here.

Construction of Reeb components. Let C be a connected compact p-manifold C with connected boundary $\partial C = B$ and let $\phi : C \to \operatorname{Int} C$ be an embedding of C into its own interior. (See Figure 7.1.) This construction generalizes to the case in which B is not connected, but an extra condition is required, and we do not need this generalization here, so we omit it. Consider the product $C \times [0,1]$ with the product foliation whose leaves are $C \times \{t\}$.

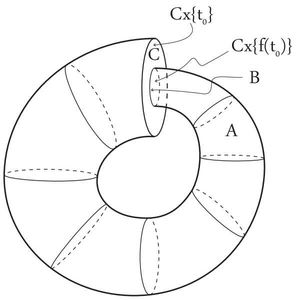

Figure 7.2. $C \times [f(t_0), t_0]$ with identifications.

On the submanifold

$$C' = C \times [0,1] \setminus \phi(\operatorname{Int} C) \times \{0\}$$

let $\sim$ be the equivalence relation generated by setting $(x,s) \sim (\phi(x), f(s))$ for every $(\phi(x), f(s)) \in C'$, where $f : [0,1] \to [0,1]$ is a continuous embedding such that $f(0) = 0$ and $f(s) < s$ for every $s > 0$. The compact set

$$(C \setminus \phi(\operatorname{Int} C)) \times [0,1] \ \cup \ C \times [f(1), 1]$$

projects onto the quotient $R = C'/\sim$, so R is compact. It is not difficult to check that R is a compact $(p+1)$-manifold with boundary endowed with a codimension one foliation $\mathcal{R}$ induced by the horizontal foliation on C', and that $(R, \mathcal{R})$ is a Reeb component. Figure 7.2 shows part of the Reeb component, the image of $C \times [f(t_0), t_0]$ with $C \times \{t_0\}$ identified with $\phi(C) \times \{f(t_0)\}$ by the equivalence relation $\sim$, foliated by the images of the sets $C \times \{t\}$. Figure 7.3 shows a diffeomorphic image of the same compact region with corners, but this view suggests how the Reeb component is built up as t_0 decreases to 0. The set $(C \setminus \phi(\operatorname{Int} C)) \times \{0\}$ projects onto the boundary of the Reeb component. We remark that the foliated homeomorphism type is independent of the choice of the embedding f since f is topologically conjugate to any other embedding with the same properties.

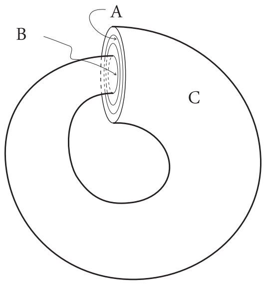

Figure 7.3. A diffeomorphic image of the same set showing part of the Reeb component.

Corollary 2.14 will follow from the following result, since the Reeb component constructed in the Proof of Theorem 2.13 satisfies its hypotheses, as long as $\mathcal{F}$ is assumed to be $C^{2,0}$, so that the embedding ϕ will be C^2.

Proposition 7.4. — *Let C be a compact connected p-manifold with connected boundary and let $\phi : C \to \mathrm{Int}(C)$ be a C^2 embedding into its interior. Endow the Reeb component $(R, \mathcal{R})$ obtained by the above construction with a Riemannian metric. Then for every $\beta > 0$ there is a constant $K > 0$ such that $M(C_t, \beta) \leqslant K$ for every region C_t appearing in the above construction.*

Proof. — Let $(R, \mathcal{R})$ be the Reeb component constructed from $\phi : C \to \mathrm{Int}(C)$, as above, where C and ϕ are smooth of class C^2. The boundary $L_0 = \partial R$ is diffeomorphic to $D/\{x \sim \phi(x)\}$, where $D = C \setminus \phi(\mathrm{Int}\ C)$. Let B_0 be the image of $\partial D = \partial C \cup \partial\phi(C)$ in L_0 under the identification. Clearly B_0 is two-sided in L_0, so by using a tubular neighborhood of B_0 in L_0 we may find a smooth map $g_0 : L_0 \to S^1$ with $1 \in S^1$ as a regular value and such that $g_0^{-1}(1) = B_0$. By a small perturbation we may suppose that g_0 is a smooth Morse function. (For convenience, we first consider Morse functions with values in S^1 rather than in $\mathbb{R}$.) Since L_0 is compact, there is an upper bound K_1 on the β-volumes $\mathrm{Vol}_\beta(g_0^{-1}(z))$ for all $z \in S^1$. Extend g_0 to $g : N \to S^1$, where N is a small (one-sided) tubular neighborhood of L_0 in R, by setting $g = g_0 \circ p$ where $p : N \to L_0$ is the projection along leaves of the transverse foliation $\mathcal{T}$ induced by the vertical foliation on $C' = C \times [0,1] \setminus \phi(\mathrm{Int}\ C) \times \{0\}$. Let $\tilde{N}$ be the cyclic cover of N

corresponding to the map $g_0 : N \to S^1 = \mathbb{R}/\mathbb{Z}$ and let $\tilde{g} : \tilde{N} \to \mathbb{R}$ be the natural lift of g to the cyclic cover. We let C_t, B_t, and D_t be the images in R of $C \times \{t\}$, $\partial C \times \{t\}$, and $(C \smallsetminus \partial C) \times \{f(t)\}$ under the identification. As in the proof of Theorem 7, $C_t \cup D_t = C_{f(t)}$ with $C_t \cap D_t = B_t$ and $\partial D_t = B_t \cup B_{f(t)}$. For each t, the region $C_t \cap N$ can be lifted to $\tilde{N}$ by lifting $D_t, D_{f(t)}$, etc., successively. For a sufficiently small tubular neighborhood N of L_0, $\tilde{g}$ will restrict to a Morse function $\tilde{g}_t$ on $C_t \cap N$, and K_1 will be an upper bound for the β-volume $\mathrm{Vol}_\beta(\tilde{g}_t^{-1}(r))$ of each level set $\tilde{g}_t^{-1}(r)$ for $r \in \mathbb{R}$, since $\tilde{g}_t^{-1}(r)$ is a compact set close to $g_0^{-1}(r \bmod 1)$ which is covered by at most K_1 open balls of radius β. Extend $\tilde{g}_t$ to a Morse function $\hat{g}_t : C_t \to \mathbb{R}$.

Next, let S_s be the image of $C \times [f(s), s]$ in $\mathrm{Int}(R)$. The sets S_s are nested and their interiors cover the compact set $R \smallsetminus \mathrm{Int}(N)$ (note that $L_0 \subset \mathrm{Int}(N)$) so for a sufficiently small $s > 0$ the union $\mathrm{Int}(N) \cup \mathrm{Int}(S_s)$ will be the whole manifold R. The images C_t of the sets $C \times \{t\}$ in the leaf L_t of $\mathcal{R}$ are compact and vary continuously, so there is a common upper bound K_2 for their β-volumes for all $t \in [f(s), s]$. Finally each level set $\hat{g}_t^{-1}(r)$ for $r \in \mathbb{R}$ is contained in the union of $\tilde{g}_t^{-1}(r) \subset \tilde{N}$ and S_t, so its β-volume is at most $K_1 + K_2$, a common upper bound for the β-volumes of the level sets $\hat{g}_t^{-1}(r)$ for all t and r. It follows that $K_1 + K_2$ is a common upper bound for the Morse volumes of the sets C_t, as claimed. □

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