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CONGRUENCES FOR SIEGEL MODULAR FORMS

by Dohoon CHOI, YoungJu CHOIE & Olav K. RICHTER (*)

Abstract. — We employ recent results on Jacobi forms to investigate congruences and filtrations of Siegel modular forms of degree 2. In particular, we determine when an analog of Atkin's $U(p)$-operator applied to a Siegel modular form of degree 2 is nonzero modulo a prime p. Furthermore, we discuss explicit examples to illustrate our results.

Résumé. — Nous utilisons des résultats récents sur les formes de Jacobi pour étudier des congruences et des filtrations des formes modulaires de Siegel de degré 2. En particulier, nous déterminons quand un analogue de l'opérateur $U(p)$ d'Atkin appliqué à une forme modulaire de Siegel du degré 2 est non nul modulo un nombre premier p. Nous donnons des exemples explicites pour illustrer ces résultats.

1. Introduction and statement of results

Fourier coefficients of modular forms display remarkable congruences. Of particular interest are congruences that involve Atkin's $U(p)$-operator. For example, Lehner's [12] $U(p)$-congruences of the modular j-function are esthetically pleasing, and, in addition, generalizations of these congruences appear in the context of class equations and supersingular j-invariants (see Ahlgren and Ono [1], Elkies, Ono, and Yang [5], and chapter 7 of Ono [16]). In this paper, we extend recent results on congruences and filtrations of Jacobi forms [19, 20] to Siegel modular forms of degree 2. Specifically, we

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introduce an analog of Atkin's $U(p)$-operator for Siegel modular forms of degree 2 and we explore $U(p)$-congruences of such forms.

Throughout, $Z := \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right)$ is a variable in the Siegel upper half space of degree 2, $q := e^{2\pi i\tau}$, $\zeta := e^{2\pi i z}$, $q' := e^{2\pi i\tau'}$, and $\mathbb{D} := (2\pi i)^{-2}\left(4\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tau'} - \frac{\partial^2}{\partial z^2}\right)$ is the generalized theta operator, which acts on Fourier expansions of Siegel modular forms as follows:

$$\mathbb{D}\left(\sum_{\substack{T={}^tT\geqslant 0\\ T \text{ even}}} a(T)e^{\pi i\,\mathrm{tr}(TZ)}\right) = \sum_{\substack{T={}^tT\geqslant 0\\ T \text{ even}}} \det(T)a(T)e^{\pi i\,\mathrm{tr}(TZ)},$$

where tr denotes the trace, and where the sum is over all symmetric, semi-positive definite, integral, and even 2×2 matrices. We now state our main result, which extends Tate's theory of theta cycles (see §7 of [10]) to Siegel modular forms of degree 2.

Theorem 1. — *Let*

$$F(Z) = \sum_{\substack{T={}^tT\geqslant 0\\ T \text{ even}}} a(T)e^{\pi i\,\mathrm{tr}(TZ)} = \sum_{\substack{n,r,m\in\mathbb{Z}\\ n,m,4nm-r^2\geqslant 0}} A(n,r,m)q^n\zeta^r q'^m$$

be a Siegel modular form of degree 2, even weight k, and with p-integral rational coefficients, where $p > k$ is a prime. Let

$$F(Z)\mid U(p) := \sum_{\substack{T={}^tT\geqslant 0\\ T \text{ even}\\ p\,\mid\,\det T}} a(T)e^{\pi i\,\mathrm{tr}(TZ)}$$

be the analog of Atkin's U-operator for Siegel modular forms. Assume that there exists an $A(n,r,m)$ with $p\nmid nm$ such that $A(n,r,m)\not\equiv 0 \pmod p$. If $p > 2k-5$, then $F\mid U(p)\not\equiv 0 \pmod p$. If $k < p < 2k-5$, then

$$\omega\left(\mathbb{D}^{\frac{3p+3}{2}-k}(F)\right) = \begin{cases} 3p-k+3 & \text{if} \quad F\mid U(p)\not\equiv 0 \pmod p\\ 2p-k+4 & \text{if} \quad F\mid U(p)\equiv 0 \pmod p),\end{cases}$$

where $\omega(\cdot)$ denotes the filtration modulo p of a Siegel modular form (for details, see Section 3).

In Section 2, we briefly review congruences and filtrations of Jacobi forms. In Section 3, we recall Nagaoka's [14] results on the structure of Siegel modular forms of degree 2 modulo p. In addition, we provide a result on the filtration of $\mathbb{D}(F)$ (where F is a Siegel modular form) which is a key tool in our proof of Theorem 1. Finally, in Section 4, we discuss $U(p)$-congruences for explicit examples of Siegel modular forms. If χ_{10} is the unique (normalized by $a\left(\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)\right)=1$) Siegel cusp form of degree 2 and weight 10, then the results on Jacobi forms in [19] imply that $\chi_{10}\mid U(p)\equiv 0 \pmod p$ for $p=5,11,13$, while $\chi_{10}\mid U(p)\not\equiv 0 \pmod p$ for all other primes $p>3$. However, the situation is more complicated for Siegel modular forms that are not in the image of the Saito-Kurokawa lifting, such as $(\chi_{10})^2$ for example. As an application of Theorem 1 we find that $(\chi_{10})^2\mid U(p)\equiv 0 \pmod p$ for $p=7,\ 11,\ 29$, while $(\chi_{10})^2\mid U(p)\not\equiv 0 \pmod p$ for all other primes $p>3$.

2. Congruences and filtrations of Jacobi forms

Let $J_{k,m}$ be the vector space of Jacobi forms of weight k and index m (for details on Jacobi forms, see Eichler and Zagier [4]). Throughout, let $p\geqslant 5$ be a prime and (for simplicity) assume throughout that k is even. Set

$$\widetilde{J}_{k,m}:=\left\{\phi \pmod p\,:\,\phi(\tau,z)\in J_{k,m}\cap\mathbb{Z}[\zeta,\zeta^{-1}]\,[[q]]\right\}.$$

If $\phi\in J_{k,m}$ has p-integral rational coefficients, then we denote its filtration modulo p by

$$\Omega\big(\phi\big):=\inf\left\{k\,:\,\phi \pmod p\in\widetilde{J}_{k,m}\right\}.$$

Recall the following facts:

Proposition 1 (Sofer [22]). — *Let $\phi(\tau,z)\in J_{k,m}\cap\mathbb{Z}[\zeta,\zeta^{-1}]\,[[q]]$ and $\psi(\tau,z)\in J_{k',m'}\cap\mathbb{Z}[\zeta,\zeta^{-1}]\,[[q]]$ such that $0\not\equiv\phi\equiv\psi \pmod p$. Then $m=m'$ and $k\equiv k' \pmod{p-1}$.*

Let $L_m:=(2\pi i)^{-2}\left(8\pi i m\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial z^2}\right)$ be the heat operator.

Proposition 2 ([20]). — *If $\phi(\tau,z)\in J_{k,m}\cap\mathbb{Z}[\zeta,\zeta^{-1}]\,[[q]]$, then $L_m(\phi) \pmod p$ is the reduction of a Jacobi form modulo p. Moreover, we have*

$$\Omega\big(L_m(\phi)\big)\leqslant\Omega\big(\phi\big)+p+1,$$

with equality if and only if $p\nmid\big(2\Omega\big(\phi\big)-1\big)\,m$.

Recall the analog of Atkin's U-operator for Jacobi forms:

Definition 1. — *For $\phi(\tau,z)=\sum\limits_{\substack{n,r\\4nm-r^2\geqslant 0}}c(n,r)q^n\zeta^r\in J_{k,m}$, we define*

$$\phi(\tau,z)\mid U_p:=\sum_{\substack{n,r\\4nm-r^2\geqslant 0\\p\,\mid\,(4nm-r^2)}}c(n,r)q^n\zeta^r.$$

Propositions 1 and 2 allow us to study heat cycles of Jacobi forms. Specifically, the argument in [19] applies also to Jacobi forms of higher index. We omit the details and only record the final result.

THEOREM 2. — *Let* $\phi(\tau, z) \in J_{k,m} \cap \mathbb{Z}[\zeta, \zeta^{-1}]\,[[q]]$ *such that* $\phi \not\equiv 0 \pmod{p}$ *and* $p \nmid m$. *If* $p > 2k-5$, *then* $\phi \mid U_p \not\equiv 0 \pmod{p}$. *If* $k < p < 2k-5$, *then*

$$\Omega\left(L_m^{\frac{3p+3}{2}-k}(\phi)\right) = \begin{cases} 3p-k+3 & \text{if} \quad \phi \mid U_p \not\equiv 0 \pmod{p} \\ 2p-k+4 & \text{if} \quad \phi \mid U_p \equiv 0 \pmod{p}. \end{cases}$$

We end this Section with the following Proposition, which is critical to Remark 2 of Section 3 and which also might be of independent interest. Note that there is no such result for integral weight (elliptic and Siegel) modular forms.

PROPOSITION 3. — *Let* $\phi(\tau, z) = \sum c(n,r) q^n \zeta^r \in J_{k,m} \cap \mathbb{Z}[\zeta, \zeta^{-1}]\,[[q]]$ *such that* $c(n,r) \equiv 0 \pmod{p}$ *for all but finitely many* n *and* r. *Then* $\phi \equiv 0 \pmod{p}$.

Proof. — We thank Michael Dewar for pointing out this short proof to us, which simplifies our original argument quite a bit. Recall that Theorem 2.2 of [4] asserts that $c(n,r)$ depends only on $4nm - r^2$ and on $r \pmod{2m}$. Consider an arbitrary coefficient $c(n,r)$. For any positive integer t set $n' := n + rt + mt^2$ and $r' := r + 2mt$. Then $4nm - r^2 = 4n'm - r'^2$ and $r = r' \pmod{2m}$, and hence $c(n,r) = c(n', r')$. Since almost all Fourier coefficients of ϕ are zero modulo p, one can choose t large enough such that $c(n', r') \equiv 0 \pmod{p}$, which implies that $c(n,r) \equiv 0 \pmod{p}$ for all n and r. $\square$

3. Siegel modular forms modulo p and the proof of Theorem 1

This section extends parts of Serre's [21] and Swinnerton-Dyer's [23] theory of modular forms modulo p to Siegel modular forms of degree 2. In particular, we establish an analog of Proposition 2 for Siegel modular forms of degree 2, which allows us to prove Theorem 1.

Let M_k denote the vector space of Siegel modular forms of degree 2 and even weight k and let S_k denote the space of cusp forms in M_k (for details on Siegel modular forms, see for example Freitag [6] or Klingen [11]). Let E_4, E_6, χ_{10}, and χ_{12} denote the usual generators of M_k of weights 4, 6, 10, and 12, respectively, where the Eisenstein series E_4 and E_6 are normalized by $a\left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right)=1$ and where the cusp forms χ_{10} and χ_{12} are normalized by $a\left(\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)\right)=1$. As before, let $p \geqslant 5$ be a prime. Guerzhoy [7], Nagaoka [14, 15], and Böcherer and Nagaoka [2] investigate Siegel modular forms modulo p. Set

$$\widetilde{M_k} := \left\{ F(\text{mod } p)\colon F(Z)=\sum a(T)e^{\pi i \operatorname{tr}(TZ)} \in M_k \text{ where } a(T)\in\mathbb{Z}_{(p)} \right\},$$

where $\mathbb{Z}_{(p)} := \mathbb{Z}_p \cap \mathbb{Q}$ denotes the local ring of p-integral rational numbers. If P is a polynomial with p-integral rational coefficients, then we also write $\widetilde{P}$ for its coefficient-wise reduction modulo p. Recall the following facts on the structure of Siegel modular forms of degree 2 modulo p:

Theorem 3 (Nagaoka [14]). — *If $F \in M_k$ with p-integral rational coefficients, then there exists a unique polynomial $P \in \mathbb{Z}_{(p)}[X_1,X_2,X_3,X_4]$ such that $F = P\left(E_4,E_6,\chi_{10},\chi_{12}\right)$.*

Theorem 4 (Nagaoka [14]). — *There exists an $E \in M_{p-1}$ with p-integral rational coefficients such that $E \equiv 1 \pmod p$ and such that $\phi(E)$ is the usual elliptic Eisenstein series of weight $p-1$, where ϕ is the Siegel ϕ-operator. Furthermore, for such an E, let $B \in \mathbb{Z}_{(p)}[X_1,X_2,X_3,X_4]$ be defined by $E = B\left(E_4,E_6,\chi_{10},\chi_{12}\right)$. Then the polynomial $\widetilde{B}-1$ is irreducible in $\mathbb{F}_p[X_1,X_2,X_3,X_4]$ and*

$$\widetilde{M_k} \simeq \mathbb{F}_p[X_1,X_2,X_3,X_4]/(\widetilde{B}-1). \tag{1}$$

Corollary 1 (Nagaoka [14]). — *Let $F_1 \in M_{k_1}$ and $F_2 \in M_{k_2}$ have p-integral rational coefficients and suppose that $0 \not\equiv F_1 \equiv F_2 \pmod p$. Then $k_1 \equiv k_2 \pmod{p-1}$.*

Since there are congruences among Siegel modular forms of different weights it is desirable to find the smallest weight in which the (coefficient-wise) reduction of a Siegel modular form modulo p exists. For a Siegel modular form F with p-integral rational coefficients, we define its filtrations modulo p by

$$\omega\big(F\big) := \inf\left\{k \,:\, F \pmod p \in \widetilde{M_k}\right\}.$$

Remark 1. — Let $F \in M_k$ with p-integral rational coefficients such that $F \not\equiv 0 \pmod p$. The isomorphism in Equation (1) shows that $\omega\big(F\big) < k$ if and only if $\widetilde{B}$ divides $\widetilde{P_F}$, where B is as in Theorem 4, and where $P_F \in \mathbb{Z}_{(p)}[X_1,X_2,X_3,X_4]$ is defined by $F = P_F\left(E_4,E_6,\chi_{10},\chi_{12}\right)$.

From the introduction, recall the generalized theta operator

$$\mathbb{D} := (2\pi i)^{-2}\left(4\frac{\partial}{\partial\,\tau}\,\frac{\partial}{\partial,\tau'} - \frac{\partial^2}{\partial\,z^2}\right).$$

Böcherer and Nagaoka [2] proved that if $F \in M_k$ has p-integral rational coefficients, then

$$\mathbb{D}(F) \in \widetilde{M}_{k+p+1}. \tag{2}$$

Our following result on the filtration of $\mathbb{D}(F)$ provides the analog of a classical result on elliptic modular forms (see Serre [21] and Swinnerton-Dyer [23]).

Proposition 4. — *Let $F \in M_k$ with p-integral rational coefficients and suppose that there is a Fourier-Jacobi coefficient ϕ_m of F with $p \nmid m$ and $\Omega\big(\phi_m\big) = \omega\big(F\big)$. Then*

$$\omega\big(\mathbb{D}(F)\big) \leqslant \omega\big(F\big) + p + 1,$$

with equality if and only if $p \nmid \big(2\omega\big(F\big) - 1\big)$.

Proof. — If $\omega\big(F\big) = k' < k$, then there exists a $G \in M_{k'}$ with $\omega\big(G\big) = k'$ such that $F \equiv G \pmod p$. We find that $\mathbb{D}(F) \equiv \mathbb{D}(G) \pmod p$, *i.e.*, $\omega\big(\mathbb{D}(F)\big) = \omega\big(\mathbb{D}(G)\big)$, and hence we may (and do) assume that $\omega\big(F\big) = k$.

Let $F(\tau, z, \tau') = \sum_{m=0}^{\infty} \phi_m(\tau, z) e^{2\pi i m \tau'}$ be the Fourier-Jacobi expansion of F, *i.e.*, ϕ_m is a Jacobi form of weight k and index m. Then

$$\mathbb{D}(F) = \sum_{m=0}^{\infty} L_m\left(\phi_m(\tau, z)\right) e^{2\pi i m \tau'}.$$

By assumption, there is a ϕ_m with $p \nmid m$ such that $\Omega\big(\phi_m\big) = k$. If $p \nmid (2k-1)$, then Proposition 2 implies that $\Omega\big(L_m(\phi_m)\big) = k+p+1$. Moreover, for each non-negative integer m we have

$$\Omega\big(L_m\left(\phi_m\right)\big) \leqslant \omega\big(\mathbb{D}(F)\big) \underset{(2)}{\leqslant} k + p + 1$$

and hence $\omega\big(\mathbb{D}(F)\big) = k + p + 1$.

Now assume that $p \mid (2k-1)$. Let $R \in M_{k'}$ with p-integral rational coefficients such that $\omega\big(R\big) = k'$ and $p \nmid k'(2k'-1)$. Choie and Eholzer [3] give an explicit formula for the nth Rankin-Cohen bracket of two Siegel modular forms of degree 2. In particular, for the first Rankin-Cohen bracket $[F, R]_1 \in M_{k+k'+2}$ of F and R, we have

$$\begin{aligned}
[F, R]_1 = &- \left(k' - \frac{1}{2}\right)\left(k + k' - \frac{1}{2}\right) \mathbb{D}(F) R \\
&- \left(k - \frac{1}{2}\right)\left(k + k' - \frac{1}{2}\right) F\, \mathbb{D}(R) \\
&+ \left(k - \frac{1}{2}\right)\left(k' - \frac{1}{2}\right) \mathbb{D}(FR).
\end{aligned}$$

Hence

$$[F,R]_1 \equiv -k'\left(k'-\frac{1}{2}\right)\cdot \mathbb{D}(F)R \pmod{p}.$$

If $\omega\big(\mathbb{D}(F)\big) = k+p+1$, then there exists a $G \in M_{k+p+1}$ with p-integral rational coefficients such that $\omega\big(G\big) = k+p+1$ and $\mathbb{D}(F) \equiv G \pmod{p}$. Let $P_G, P_R \in \mathbb{Z}_{(p)}[X_1,X_2,X_3,X_4]$ be defined by $G = P_G\left(E_4,E_6,\chi_{10},\chi_{12}\right)$ and $R = P_R\left(E_4,E_6,\chi_{10},\chi_{12}\right)$, respectively. Remark 1 shows that $\widetilde{P_G}\widetilde{P_R}$ is relatively prime to $\widetilde{B}$ in Theorem 4, *i.e.*, $\omega\big(GR\big) = k'+k+p+1$. We obtain the contradiction

$$k'+k+2 \geqslant \omega\big([F,R]_1\big) = \omega\big(\mathbb{D}(F)R\big) = \omega\big(GR\big) = k'+k+p+1.$$

Hence $\omega\big(\mathbb{D}(F)\big) < k+p+1$, which completes the proof. □

We are now in a position to prove Theorem 1.

Proof of Theorem 1. — We wish to apply Proposition 4, and we therefore need to show that there exists a Fourier-Jacobi coefficient ϕ_m of F with $p \nmid m$ such that $\Omega\big(\phi_m\big) = \omega\big(F\big)$. Suppose that for every m with $p \nmid m$, $\Omega\big(\phi_m\big) < \omega\big(F\big)$. We use that $p > k$; since $F \not\equiv A(0,0,0) \pmod{p}$, Corollary 1 implies that $\omega\big(F\big) = k$. Consequently, Proposition 1 implies that $\Omega\big(\phi_m\big) = 0$, *i.e.*, $\phi_m \equiv 0 \pmod{p}$ whenever $p \nmid m$. Hence if $p \nmid m$, then $A(n,r,m) \equiv 0 \pmod{p}$ and since $F(\tau,z,\tau') = F(\tau',z,\tau)$, *i.e.*, $A(n,r,m) = A(m,r,n)$, we have $A(n,r,m) \equiv 0 \pmod{p}$ for all n, m such that $p \nmid mn$, which contradicts the assumption of Theorem 1.

We conclude that Proposition 4 is applicable. We omit further details and only point out that an argument as in [19] (using Corollary 1 and Proposition 4) will finish the proof of Theorem 1. □

We end this Section with a remark on the compatibility of $U(p)$ and U_p.

Remark 2. — Let $F(Z) = \sum a(T)e^{\pi i\ \mathrm{tr}(TZ)} \in M_k$ with p-integral rational coefficients and with Fourier-Jacobi expansion

$$F(\tau,z,\tau') = \sum \phi_m(\tau,z)e^{2\pi i m\tau'}.$$

If $F \,\big|\, U(p) \equiv 0 \pmod{p}$, then $\phi_m \mid U_p \equiv 0 \pmod{p}$ for all $m \geqslant 0$. On the other hand, if $F \mid U(p) \not\equiv 0 \pmod{p}$, then $G := F - \mathbb{D}^{p-1}(F) \not\equiv 0 \pmod{p}$. Note that $G(\tau,z,\tau') = \sum \psi_m(\tau,z)e^{2\pi i m\tau'}$ with $\psi_m := \phi_m - L_m^{p-1}(\phi_m) \in \widetilde{J}_{k+(p-1)(p+1),m}$ and there exists an m such that $\phi_m \mid U_p \equiv \psi_m \not\equiv 0 \pmod{p}$. Proposition 3 implies that infinitely many Fourier coefficients of ψ_m are nonzero modulo p and since $G(\tau,z,\tau') \equiv G(\tau',z,\tau) \pmod{p}$, we find that there are infinitely many m such that $\psi_m \not\equiv 0 \pmod{p}$. We conclude that if $F \mid U(p) \not\equiv 0 \pmod{p}$, then there are infinitely many m such that $\phi_m \mid U_p \not\equiv 0 \pmod{p}$.

4. Examples

The U_p-congruences of Jacobi forms of index 1 in [19] in combination with Maass' lift [13] (see also Eichler and Zagier [4]) yield $U(p)$-congruences for Siegel modular forms in *Maass' Spezialschar*. For example, we find that $\chi_{10} \mid U(p) \equiv 0 \pmod{p}$ for $p = 5, 11, 13$, while $\chi_{10} \mid U(p) \not\equiv 0 \pmod{p}$ for all other primes $p > 3$. Similar $U(p)$-congruences hold also for other Maass Spezialscharformen. However, to obtain such results for Siegel modular forms that are not in the Maass Spezialschar we have to apply Theorem 1. We will now discuss $U(p)$-congruences for $(\chi_{10})^2 \in S_{20}$, which is not in Maass' Spezialschar. We are grateful to Cris Poor and David Yuen, who used Mathematica to generate the Fourier coefficients of Siegel modular forms needed for our example here.

Theorem 1 implies that $(\chi_{10})^2 \mid U(p) \not\equiv 0 \pmod{p}$ for $p > 35$. Furthermore, we have the following numerical data for the Fourier coefficients of $(\chi_{10})^2$:

$$\begin{aligned}
&a\left(\left(\begin{smallmatrix} 4 & 1 \\ 1 & 4 \end{smallmatrix}\right)\right) = -4 \not\equiv 0 \pmod{5}, && a\left(\left(\begin{smallmatrix} 6 & 3 \\ 3 & 8 \end{smallmatrix}\right)\right) = -11916 \not\equiv 0 \pmod{13},\\
&a\left(\left(\begin{smallmatrix} 6 & 3 \\ 3 & 10 \end{smallmatrix}\right)\right) = 73568 \not\equiv 0 \pmod{17}, && a\left(\left(\begin{smallmatrix} 8 & 2 \\ 2 & 10 \end{smallmatrix}\right)\right) = -2460288 \not\equiv 0 \pmod{19},\\
&a\left(\left(\begin{smallmatrix} 4 & 1 \\ 1 & 6 \end{smallmatrix}\right)\right) = 132 \not\equiv 0 \pmod{23}, && a\left(\left(\begin{smallmatrix} 4 & 1 \\ 1 & 8 \end{smallmatrix}\right)\right) = -1956 \not\equiv 0 \pmod{31},
\end{aligned}$$

which shows that $(\chi_{10})^2 \mid U(p) \not\equiv 0 \pmod{p}$ for $p =$ 5, 13, 17, 19, 23, 31. It remains to determine what happens for the cases $p =$ 7, 11, and 29. In each case, we use Theorem 3 together with the fact that S_k is the ideal generated by χ_{10} and χ_{12} in M_k to find a basis of the $\mathbb{Z}_{(p)}$-module

$$S_k^{(\mathbb{Z}_{(p)})} = \left\{F \in S_k \text{ with coefficients in } \mathbb{Z}_{(p)}\right\}.$$

Consider $p = 7$. We apply Proposition 4 to obtain $\omega\big(\mathbb{D}^6\left((\chi_{10})^2\right)\big) = 20$ or 32. We find that $F_1 := (E_4)^5\,\chi_{12}$, $F_2 := (E_4)^3\,(\chi_{10})^2$, $F_3 := (E_4)^2(\chi_{12})^2$, $F_4 := (\chi_{10})^2\chi_{12}$, $F_5 := (E_4)^4\,E_6\,\chi_{10}$, $F_6 := (E_4)^2\,(E_6)^2\,\chi_{12}$, $F_7 := E_4\,E_6\,\chi_{10}\,\chi_{12}$, $F_8 := E_4\,(E_6)^3\,\chi_{10}$, and $F_9 := (E_6)^2\,(\chi_{10})^2$ form a basis of $S_{32}^{(\mathbb{Z}_{(7)})}$. If $\omega\big(\mathbb{D}^6\left((\chi_{10})^2\right)\big) = 32$, then $F := \mathbb{D}^6\left((\chi_{10})^2\right)$ would be congruent modulo 7 to a linear combination of the nine basis elements of $S_{32}^{(\mathbb{Z}_{(7)})}$. The following table of Fourier coefficients modulo 7 shows that the only possible such linear combination is $\mathbb{D}^6\left((\chi_{10})^2\right) \equiv F_9 \pmod{7}$. However, $\omega\big(F_9\big) < 32$, since $E_6 \equiv 1 \pmod{7}$. Hence $\omega\big(\mathbb{D}^6\left((\chi_{10})^2\right)\big) = 20$, which implies that $(\chi_{10})^2 \mid U(7) \equiv 0 \pmod{7}$.

	$a\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right)$	$a\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right)$	$a\left(\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}\right)$	$a\left(\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\right)$	$a\left(\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\right)$	$a\left(\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\right)$	$a\left(\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\right)$	$a\left(\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}\right)$	$a\left(\begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}\right)$
F	0	0	0	0	$12^6 \equiv 1$	$-4 * 15^6 \equiv 3$	$6 * 16^6 \equiv 6$	$-24 * 20^6 \equiv 4$	$672 * 27^6 \equiv 0$
F_1	1	$10 \equiv 3$	$1112 \equiv 6$	$11868 \equiv 3$	$105984 \equiv 4$	$1757160 \equiv 6$	$12867200 \equiv 3$	$94258680 \equiv 5$	$4846256703 \equiv 1$
F_2	0	0	0	0	1	$-4 \equiv 3$	6	$696 \equiv 3$	$32352 \equiv 5$
F_3	0	0	0	0	1	$20 \equiv 6$	$102 \equiv 4$	$504 \equiv 0$	$55968 \equiv 3$
F_4	0	0	0	0	0	0	0	0	1
F_5	1	$-2 \equiv 5$	$440 \equiv 6$	$-876 \equiv 6$	$95616 \equiv 3$	$-199800 \equiv 1$	$207488 \equiv 1$	$55691880 \equiv 6$	$3020576319 \equiv 5$
F_6	1	$10 \equiv 3$	$-616 \equiv 0$	$-5412 \equiv 6$	$102528 \equiv 6$	$1288872 \equiv 4$	$874880 \equiv 6$	$-30765576 \equiv 5$	$4576937535 \equiv 2$
F_7	0	0	0	0	1	$8 \equiv 1$	$-18 \equiv 3$	$-408 \equiv 5$	$54528 \equiv 5$
F_8	1	$-2 \equiv 5$	$-1288 \equiv 0$	$2580 \equiv 4$	$154368 \equiv 4$	$1073736 \equiv 6$	$-2453632 \equiv 1$	$-124738968 \equiv 3$	$7476576831 \equiv 3$
F_9	0	0	0	0	1	$-4 \equiv 3$	6	$-1032 \equiv 4$	$101472 \equiv 0$

The cases $p = 11$ and $p = 29$ are treated analogously, but the computations are more involved. If $p = 11$, then Proposition 4 yields that $\omega\big(\mathbb{D}^{10}((\chi_{10})^2)\big) = 20$ or 50. The rank of $S_{50}^{(\mathbb{Z}_{(11)})}$ is 27 and if $\omega\big(\mathbb{D}^{10}((\chi_{10})^2)\big) = 50$, then $\mathbb{D}^{10}\big((\chi_{10})^2\big)$ would be congruent modulo 11 to a linear combination of the 27 basis elements of $S_{50}^{(\mathbb{Z}_{(11)})}$. It suffices to use Fourier coefficients $a(T)$ with $\det(T) \leqslant 75$ of the 27 canonical basis elements of $S_{50}^{(\mathbb{Z}_{(11)})}$ and of $\mathbb{D}^{10}\left((\chi_{10})^2\right)$ to verify (with Maple) that the only possible such linear combination would be

$$\mathbb{D}^{10}\left((\chi_{10})^2\right) \equiv (E_4E_6 + 2\chi_{10})^3(\chi_{10})^2 \pmod{11}.$$

However,

$$E_4E_6 + 2\chi_{10} \equiv E_4E_6 - \frac{2^{10}\cdot 3^5\cdot 5^2\cdot 7\cdot 53}{43867}\chi_{10} \underset{[8,\,9]}{=} E_{10} \equiv 1 \pmod{11},$$

where E_{10} is the Siegel Eisenstein series of weight 10 normalized by $a\left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right) = 1$. Thus, $\omega\big(\mathbb{D}^{10}\left((\chi_{10})^2\right)\big) = 20$ and hence $(\chi_{10})^2 \mid U(11) \equiv 0 \pmod{11}$.

Finally, if $p = 29$, then Theorem 1 implies that $\omega\big(\mathbb{D}^{25}((\chi_{10})^2)\big) = 42$ or 70. The rank of $S_{70}^{(\mathbb{Z}_{(29)})}$ is 67. In this case, it is sufficient to use Fourier coefficients $a(T)$ with $\det(T) \leqslant 147$ of the 67 canonical basis elements of $S_{70}^{(\mathbb{Z}_{(29)})}$ and of $\mathbb{D}^{25}\left((\chi_{10})^2\right)$ to show (with Maple) that if $\omega\big(\mathbb{D}^{25}((\chi_{10})^2)\big) = 70$, then

$$\mathbb{D}^{25}\left((\chi_{10})^2\right) \equiv EG \pmod{29}$$

where $G \in S_{42}$ is given by

$$\begin{aligned} G := {} & 19(E_4)^3(\chi_{10})^3 + 5(E_4)^5\chi_{10}\chi_{12} + 21(\chi_{10})^3\chi_{12} + 11(E_4)^2\chi_{10}(\chi_{12})^2 \\ & + 18(E_4)^4(E_6)(\chi_{10})^2 + 5E_4E_6(\chi_{10})^2\chi_{12} + 26(E_4)^3E_6(\chi_{12})^2 \\ & + 25E_6(\chi_{12})^3 + 7(E_6)^2(\chi_{10})^3 + 23(E_4)^2(E_6)^2\chi_{10}\chi_{12} \\ & + 18E_4(E_6)^3(\chi_{10})^2 + 9(E_6)^3(\chi_{12})^2 \end{aligned}$$

and where $E \in M_{28}$ is defined by

$$\begin{aligned} E := {} & 4(E_4)^7 + 18(E_4)^4(E_6)^2 + 8E_4(E_6)^4 + 27(E_4)^3E_6\chi_{10} \\ & + 26(E_4)^2(\chi_{10})^2 + 6(E_6)^3\chi_{10} + 21(E_4)^4\chi_{12} \\ & + 27E_4(E_6)^2\chi_{12} + 25E_6\chi_{10}\chi_{12} + 26E_4(\chi_{12})^2. \end{aligned} \tag{3}$$

Let $E_j = \sum a_j(T)e^{\pi i\,\mathrm{tr}(TZ)} \in M_k$ $(j = 1, 2)$ and suppose that for all T with dyadic trace up to $\frac{k}{3}$, one has that $a_j(T)$ are p-integral rational and $a_1(T) \equiv a_2(T) \pmod{p}$. Then Theorem 5.15 of Poor and Yuen [17] asserts that all $a_j(T)$ are p-integral rational and $E_1 \equiv E_2 \pmod p$. For more details on the dyadic trace, see Poor and Yuen [18]. For the Fourier coefficients of E in Equation (3), one finds that $a\left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\right) = 30 \equiv 1 \pmod{29}$, while if $T \neq \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$, then $a(T) \equiv 0 \pmod{29}$ for all T with dyadic trace up to $\frac{28}{3}$. Hence $E \equiv 1 \pmod{29}$ and $\omega\left(\mathbb{D}^{25}\left((\chi_{10})^2\right)\right) = \omega(G) < 70$. Theorem 1 gives that $(\chi_{10})^2 \mid U(29) \equiv 0 \pmod{29}$.

We conclude that $(\chi_{10})^2 \mid U(p) \equiv 0 \pmod p$ for $p = 7, 11, 29$, while $(\chi_{10})^2 \mid U(p) \not\equiv 0 \pmod p$ for all other primes $p > 3$.

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