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Simons Type Equation in $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$ and Applications


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SIMONS TYPE EQUATION IN $S^2 \times \mathbb{R}$ AND $H^2 \times \mathbb{R}$ AND APPLICATIONS

by Márcio Henrique BATISTA DA SILVA

Abstract. — Let $\Sigma^2$ be an immersed surface in $M^2(c) \times \mathbb{R}$ with constant mean curvature. We consider the traceless Weingarten operator $\phi$ associated to the second fundamental form of the surface, and we introduce a tensor $S$, related to the Abresch-Rosenberg quadratic differential form. We establish equations of Simons type for both $\phi$ and $S$. By using these equations, we characterize some immersions for which $|\phi|$ or $|S|$ is appropriately bounded.

Résumé. — Soit $\Sigma^2$ une surface immergée dans $M^2(c) \times \mathbb{R}$ avec une courbure moyenne constante. Nous considérons l’opérateur de Weingarten à trace nulle $\phi$ associé à la seconde forme fondamentale de la surface et nous introduisons un tenseur $S$, lié à la forme quadratique de Abresch-Rosenberg. Nous établissons les équations de type Simons pour $\phi$ et $S$. En utilisant ces équations, nous caractérisons les immersions pour lesquelles $|\phi|$ ou $|S|$ sont bornés.

1. Introduction

In 1994, using the traceless Weingarten operator $\phi = A - HI$ associated to an immersed hypersurface $M^n \hookrightarrow S^{n+1}$, H. Alencar and M. do Carmo, see [2], proved that

**Theorem.** — Let $M^n \hookrightarrow S^{n+1}$ be an immersed hypersurface. If $M^n$ is compact and orientable with constant mean curvature $H$ and

$$|\phi|^2 \leq B_H,$$

where $B_H$ is the square of the positive root of

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n(H^2 + 1).$$

Then:

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*Math. classification:* 53A10, 53C42.
Either $|\phi|^2 = 0$ (and $M^n$ is totally umbilic) or $|\phi|^2 = B_H$.

The $H(r)$-tori $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ with $r^2 \leq \frac{n-1}{n}$ are the only hypersurfaces with constant mean curvature $H$ and $|\phi|^2 = B_H$.

Motivated by this result we study this problem for surfaces in $M^2(c) \times \mathbb{R}$ with $c = \pm 1$, where $M^2(-1) = \mathbb{H}^2$ and $M^2(1) = \mathbb{S}^2$.

We begin by using the traceless Weingarten operator $\phi$ associated to an immersed surface $\Sigma^2 \hookrightarrow M^2(c) \times \mathbb{R}$.

In [1], the authors defined the quadratic differential form

$$ Q(X,Y) = 2H \langle AX, Y \rangle - c \langle X, \partial_t \rangle \langle Y, \partial_t \rangle, $$

and its $(2,0)$-part

$$ Q^{(2,0)}(X,Y) = \frac{1}{2}(Q(X,Y) - Q(JX,JY)) - \frac{1}{2}i(Q(JX,Y) + Q(X,JY)),$$

where $J$ is the standard counter-clockwise rotation operator.

Using this notation, Abresch and Rosenberg proved

**THEOREM.** (Thm. 1 in [1]) Let $\Sigma^2 \hookrightarrow M^2(c) \times \mathbb{R}$ be an immersed surface with constant mean curvature. Then its quadratic differential $Q^{(2,0)}$ is holomorphic on the surface $\Sigma^2$.

Inspired in the quadratic differential form $Q$ introduced by Abresch and Rosenberg, we study, in section 3, a special tensor $S$ defined by

$$ SX = 2HAX - c\langle X, T \rangle T + \frac{c}{2}(1 - \nu^2)X - 2H^2X, $$

where $X \in T_p \Sigma$, $A$ is the Weingarten operator associated to the second fundamental form, $H$ is the mean curvature, $T$ is the tangential component of the parallel field $\partial_t$, tangent to $\mathbb{R}$ in $M^2(c) \times \mathbb{R}$, and $\nu = \langle N, \partial_t \rangle$.

The tensor $S$ is the traceless tensor associated with the quadratic differential $Q$. In fact,

$$ \langle SX, Y \rangle = 2H \langle AX, Y \rangle - c\langle X, T \rangle \langle Y, T \rangle + \frac{c}{2}(1 - \nu^2)\langle X, Y \rangle - 2H^2\langle X, Y \rangle $$

$$ = Q(X,Y) - \frac{\text{tr}Q}{2} \langle X, Y \rangle. $$

We will prove that this operator satisfies Codazzi’s equation, provided $H$ is constant, with vanishing trace. Moreover, we remark that any surface with $|S| = 0$ and constant mean curvature is very interesting, because the $Q^{(2,0)}$ of these surfaces vanishes.

In [1], Theorem 3, p. 143, the authors described four distinct classes of complete, possibly immersed, constant mean curvature surfaces $\Sigma^2 \hookrightarrow M^2(c) \times \mathbb{R}$ with vanishing of their quadratic differential $Q^{(2,0)}$. 

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More precisely, the four classes are

(i) $\Sigma^2$ is an embedded rotationally invariant constant mean curvature sphere $S^2_H$;

(ii) $\Sigma^2$ is a convex rotationally invariant constant mean curvature graph $D^2_H$ over the horizontal leaf $M^2(c) \times \{t_0\}$;

(iii) $\Sigma^2$ is an embedded annulus, rotationally invariant constant mean curvature surface $C^2_H$ with two asymptotically conical ends;

(iv) $\Sigma^2$ is the embedded constant mean curvature surface $P^2_H$; it is an orbit under some two dimensional solvable subgroup of ambient isometries.

The surface in (i) was known to W.T. Hsiang and W.Y. Hsiang, in [6], and to R. Pedrosa and M. Ritoré, in [7]. We shall refer to $S^2_H$ as the embedded rotationally invariant constant mean curvature spheres. In this paper we will call the surfaces described in [1] by Abresch-Rosenberg surfaces.

Remark. 1. — In $\mathbb{S}^2 \times \mathbb{R}$ only the spheres $S^2_H$ occur.

We obtain an equation of Simons type for $S$ and apply it in some particular cases:

**Theorem 1.1.** — Let $\Sigma^2 \hookrightarrow M^2(c) \times \mathbb{R}$ be an immersed surface with non zero constant mean curvature $H$ and $S$ as defined in (1.1). Then,

$$\langle (\nabla^2 S)x, y \rangle = 2cv^2 \langle Sx, y \rangle + 2H \langle Ax, Sy \rangle - \langle A^2 x, Sy \rangle + \langle Ay, SAx \rangle - \langle Ax, y \rangle \text{tr}(AS)$$

and

$$\frac{1}{2} \Delta |S|^2 = |\nabla S|^2 - |S|^4 + |S|^2 \left( \frac{5cv^2}{2} - \frac{c}{2} + 2H^2 - \frac{c}{H} \langle ST, T \rangle \right) + c|ST|^2 - \frac{1}{4H^2} \langle ST, T \rangle^2.$$

Let us consider the polynomial $p_H(t) = -t^2 - \frac{1}{H} t + \left( \frac{4H^2 - 1}{2} \right)$. When $H$ is greater than one half there is a positive root for $p_H$. Let $L_H$ be this positive root. One has:

**Theorem 1.2.** — Let $\Sigma^2 \hookrightarrow \mathbb{S}^2 \times \mathbb{R}$ be an immersed surface with constant mean curvature $H$ greater than one half. If $\Sigma^2$ is complete and $\sup \Sigma |S| < L_H$, or $\Sigma^2$ is closed and $|S| \leq L_H$,
then $\Sigma^2 = S^2_H$, i.e, $\Sigma^2$ is an embedded rotationally invariant constant mean curvature sphere.

Remark 2. — The number $L_H$ is $\frac{\sqrt{2H(4H^2 - 1)}}{\sqrt{16H^4 - 4H^2 + 1} + 1}$.

Let us consider the polynomial

$$q_H(t) = -t^2 - \frac{1}{\sqrt{2H}}t + \left(\frac{8H^4 - 12H^2 - 1}{4H^2}\right).$$

When $H$ is greater than $\sqrt{\frac{3 + \sqrt{11}}{4}}$, there is a positive root for $q_H$. Let $M_H$ be this positive root.

**Theorem 1.3.** — Let $\Sigma^2 \hookrightarrow \mathbb{H}^2 \times \mathbb{R}$ be an immersed surface with constant mean curvature $H$ greater than $\sqrt{\frac{3 + \sqrt{11}}{4}} \approx 1.25664$. If $\Sigma^2$ is complete and $\sup_{\Sigma} |S| < M_H$,
or $$\Sigma^2$$ is closed and $|S| \leq M_H$,
then $\Sigma^2 = S^2_H$, i.e, $\Sigma^2$ is an embedded rotationally invariant constant mean curvature sphere.

Remark 3. — The number $M_H$ is $\frac{8H^4 - 12H^2 - 1}{\sqrt{2H(\sqrt{16H^4 - 24H^2 - 1} + 1)}}$.

Remark 4. — Besides Theorems 1.2 and 1.3, we obtain in section 4 further applications of Simons equation of Theorem 1.1.

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## 2. Preliminaries

Let $\Sigma^2 \hookrightarrow M^3$ be an immersed surface. Let $\nabla$ denote the Levi-Civita connection on $M^3$ and let $\nabla$ denote the Levi-Civita connection on $\Sigma$ for the induced metric.

Generally speaking, objects defined on $M^3$ will be denoted by the same symbols as the corresponding objects defined on $\Sigma$ plus a bar over the symbol.
The Riemannian metric extends to natural inner products on spaces of tensors and the above connections induce natural covariant derivatives of tensor fields. For example, for \( \{ e_1, e_2 \} \) a geodesic frame at \( p \in \Sigma^2 \) and a tensor \( \psi \) on \( \Sigma^2 \), we have

\[
\nabla^2 \psi(p) = \sum_{i=1}^{2} (\nabla_{e_i} \nabla_{e_i} \psi)(p).
\]

For more details about covariant derivatives of tensor fields see [8], sections 1 and 2.

We adopt the following convention for the curvature tensor: if \( x, y, z \in T_p \Sigma \), we define \( R_{x,y} z \) by

\[
R_{x,y} z = R(X,Y)Z(p) = (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z)(p),
\]

for any local vector fields which extend the given vectors \( x, y, z \).

The second fundamental form is defined by \( \alpha(X,Y) = (\nabla_X Y)_{\perp} \) and the associated Weingarten operator is given by \( A v = -(\nabla_v N)_{\perp} \), where \( N \) is a unit normal field on \( \Sigma^2 \). We use the Weingarten operator to define the following operators

\[
\langle \bar{R}(A)x,y \rangle := \sum_{i=1}^{2} \left( -\langle Ax, \bar{R}_{e_i,y} e_i \rangle - \langle Ay, \bar{R}_{e_i,x} e_i \rangle + \langle Ay, x \rangle \langle N, \bar{R}_{e_i,N} e_i \rangle - 2\langle Ae_i, \bar{R}_{e_i,x} y \rangle \right)
\]

and

\[
\langle \bar{R}' x, y \rangle := \sum_{i=1}^{2} \left\{ \langle (\nabla_x \bar{R})_{e_i,y} e_i, N \rangle + \langle (\nabla_{e_i} \bar{R})_{e_i,x} y, N \rangle \right\},
\]

where \( \{e_1, e_2\} \) is an orthonormal basis of \( T_p \Sigma \).

With this notation we have the following result:

THEOREM 2.1. — Let \( \Sigma^2 \hookrightarrow M^3 \) be an immersed surface with constant mean curvature \( H \). For any \( x, y \in T_p \Sigma \) we have

\[
\langle (\nabla^2 A) x, y \rangle = -|A|^2 \langle Ax, y \rangle + \langle \bar{R}(A)x, y \rangle + \langle \bar{R}' x, y \rangle + 2H \langle \bar{R}_{N,x} y, N \rangle + 2H \langle Ax, Ay \rangle.
\]

Proof. — See Theorem 2 in [3] and observe that the codimension is one.

\( \square \)

We will also use the result known as the Omori-Yau Maximum Principle whose proof can be found in [10], Theorem 1.
Theorem 2.2 (Omori-Yau Maximum Principle). — Let $M$ be a complete Riemannian manifold with Ricci curvature bounded from below. If $u \in C^\infty(M)$ is bounded from above, then there exists a sequence of points \( \{p_j\} \in M \) such that
\[
\lim_{j \to \infty} u(p_j) = \sup_M u, \quad |\nabla u|(p_j) < \frac{1}{j}, \text{ and } \Delta u(p_j) < \frac{1}{j}.
\]

Let us recall Gauss’ equation for $\Sigma^2$ in $M^2(c) \times \mathbb{R}$:
\[
R(Y, X)Z = \langle AX, Z\rangle Y - \langle AY, Z\rangle X + c(\langle X, Z\rangle Y - \langle Y, Z\rangle X) + (\langle X, T\rangle Y - \langle Y, T\rangle X),
\]
where $X, Y, Z$ in $T_p \Sigma$, $N$ is a unitary normal field on $\Sigma^2$ and $T$ is the tangential component of the parallel field $\partial_t$. For more details see [5].

3. Simons’ equation in $M^2(c) \times \mathbb{R}$

In this section we will obtain an equation of Simons type for the traceless Weingarten operator $\phi$ and for the tensor $S$ defined in (1.1).

Let $M^2(c) \times \mathbb{R}$, where $M^2(-1) = \mathbb{H}^2$ and $M^2(1) = \mathbb{S}^2$. In this case we have that $\bar{R}'=0$, because $M^2(c) \times \mathbb{R}$ is locally symmetric.

In Lemmas 3.1 and 3.2 we will consider an immersed surface $\Sigma^2 \hookrightarrow M^2(c) \times \mathbb{R}$ with constant mean curvature $H$ where $A$ is the Weingarten operator associated to the second fundamental form on $\Sigma^2$.

Lemma 3.1. — Denoting the identity by $I$, we have that
\[
\bar{R}(A) = c(5\nu^2 - 1)A - 4cH\nu^2 I.
\]

Proof. — Consider an orthonormal basis $\{e_1, e_2\}$ in $T_p \Sigma^2$ such that $Ae_i = k_ie_i$, $i = 1, 2$. Consider $x, y \in T_p \Sigma$. We have
\[
x = x_1e_1 + x_2e_2 \quad \text{and} \quad y = y_1e_1 + y_2e_2.
\]

Computing the first sum in (2.1)
\[
\sum_{i=1}^{2} \langle \bar{R}_{e_i}y e_i, Ax \rangle = k_2x_2y_2\langle \bar{R}_{e_1}e_2 e_1, e_2 \rangle + k_1x_1y_1\langle \bar{R}_{e_2}e_1 e_2, e_1 \rangle
\]
\[
= -\bar{K}_\Sigma (k_2x_2y_2 + k_1x_1y_1) = -\bar{K}_\Sigma \langle Ax, y \rangle,
\]
where $\bar{K}_\Sigma = \langle \bar{R}_{e_1}e_2 e_2, e_1 \rangle$. 

Hence,

\begin{equation}
\sum_{i=1}^{2} \langle \bar{R}_{e_i,y} e_i, Ax \rangle = -\bar{K}_\Sigma \langle Ax, y \rangle.
\end{equation}

It’s simple see that

\begin{equation}
\sum_{i=1}^{2} \langle \bar{R}_{e_i,x} e_i, Ay \rangle = -\bar{K}_\Sigma \langle Ax, y \rangle.
\end{equation}

In the third sum in (2.1) we have

\begin{align*}
\langle \bar{R}_{e_i,N} e_i, N \rangle &= -c\{(1 - \langle e_i, \partial_t \rangle)^2 (1 - \nu^2) - \nu^2 \langle e_i, \partial_t \rangle^2 \} \\
&= -c\{1 - \nu^2 - \langle e_i, \partial_t \rangle^2 \}.
\end{align*}

Therefore,

\begin{equation}
\sum_{i=1}^{2} \langle \bar{R}_{e_i,N} e_i, N \rangle = -c(1 - \nu^2).
\end{equation}

To finish, we computing the fourth sum.

\begin{align*}
\sum_{i=1}^{2} \langle \bar{R}_{e_i,x} y, Ae_i \rangle &= \bar{K}_\Sigma (k_1 x_2 y_2 + k_2 x_1 y_1) \\
&= \bar{K}_\Sigma (2H - k_2) x_2 y_2 + \bar{K}_\Sigma (2H - k_1) x_1 y_1) \\
&= \bar{K}_\Sigma (2H \langle x, y \rangle - \langle Ax, y \rangle),
\end{align*}

where we used that $2H = k_1 + k_2$.

Thus,

\begin{equation}
\sum_{i=1}^{2} \langle \bar{R}_{e_i,x} y, Ae_i \rangle = \bar{K}_\Sigma (2H \langle x, y \rangle - \langle Ax, y \rangle).
\end{equation}

Now, we need computing $\bar{K}_\Sigma$. Using the tensor of curvature in $M^2(c) \times \mathbb{R}$ we have:

\begin{align*}
\bar{K}_\Sigma &= \langle \bar{R}_{e_1, e_2} e_2, e_1 \rangle = c (1 - \langle e_1, T \rangle^2 - \langle e_2, T \rangle^2) = c(1 - |T|^2).
\end{align*}

Therefore,

\begin{equation}
\bar{K}_\Sigma = c \nu^2.
\end{equation}

Substituting (3.1), (3.2),(3.3) and (3.4) into (2.1), obtain

\begin{align*}
\langle \bar{R}(A)x, y \rangle &= 2\bar{K}_\Sigma \langle Ax, y \rangle - c(1 - \nu^2) \langle Ax, y \rangle - 2\bar{K}_\Sigma (2H \langle x, y \rangle - \langle Ax, y \rangle).
\end{align*}

Using (3.5) we obtain

\begin{align*}
\langle \bar{R}(A)x, y \rangle &= 5c \nu^2 \langle Ax, y \rangle - c \langle Ax, y \rangle - 4c \nu^2 H \langle x, y \rangle.
\end{align*}
Thus,
\[ \tilde{R}(A) = c(5\nu^2 - 1)A - 4cH\nu^2I. \]

**Lemma 3.2.** — \( \langle \tilde{R}_N, xy, N \rangle = -c\{ \langle x, T \rangle \langle y, T \rangle - \langle x, y \rangle \langle T, T \rangle \} \).

**Proof.** — We observe that
\[ \langle x^*, y^* \rangle = \langle x, y \rangle - \langle x, T \rangle \langle y, T \rangle, \]
and
\[ \langle x^*, N^* \rangle = \nu \langle x, T \rangle \]
and
\[ \langle N^*, N^* \rangle = 1 - \nu^2, \]
where we have used \( v^* = v - \langle v, \partial_t \rangle \partial_t \) for any \( v \in T_p(M^2(c) \times \mathbb{R}) \).

It follows that
\[ \langle \tilde{R}_N, xy, N \rangle = -c\{ \langle N^*, x^* \rangle \langle N^*, y^* \rangle - \langle N^*, N^* \rangle \langle x^*, y^* \rangle \} \]
\[ = -c\{ \langle x, T \rangle \langle y, T \rangle - \langle x, y \rangle \langle T, T \rangle \}. \]

This concludes the proof.

**Proposition 3.3.** — Let \( \Sigma^2 \hookrightarrow M^2(c) \times \mathbb{R} \) be an immersed surface with constant mean curvature \( H \) and let \( A \) be the Weingarten operator associated to the second fundamental form on \( \Sigma^2 \). Then,
\[ \langle (\nabla^2 A)x, y \rangle = -|A|^2\langle Ax, y \rangle + c(5\nu^2 - 1)\langle Ax, y \rangle - 4cH\nu^2\langle x, y \rangle + \]
\[ - 2cH\{ \langle x, T \rangle \langle y, T \rangle - \langle x, y \rangle \langle T, T \rangle \} + 2H\langle Ax, Ay \rangle, \]
where \( \nu = \langle N, \partial_t \rangle \).

**Proof.** — Consider equation (2.2)
\[ \langle (\nabla^2 A)x, y \rangle = -|A|^2\langle Ax, y \rangle + \langle \tilde{R}(A)x, y \rangle \]
\[ + \langle \tilde{R}'x, y \rangle + 2H\langle \tilde{R}_N, xy, N \rangle + 2H\langle Ax, Ay \rangle. \]

Now, we use Lemmas 3.1 and 3.2 and the fact that \( \tilde{R}' = 0 \) to obtain
\[ \langle (\nabla^2 A)x, y \rangle = -|A|^2\langle Ax, y \rangle + c(5\nu^2 - 1)\langle Ax, y \rangle - 4cH\nu^2\langle x, y \rangle + \]
\[ - 2cH\{ \langle x, T \rangle \langle y, T \rangle - \langle x, y \rangle \langle T, T \rangle \} + 2H\langle Ax, Ay \rangle. \]
Consider two tensors $V, W$ on $\Sigma^2$. We define the inner product $\langle V, W \rangle$ at $p \in \Sigma^2$ as
$$\langle V, W \rangle = \sum_{i=1}^{2} \langle Ve_i, We_i \rangle,$$
where $\{e_1, e_2\}$ is an orthonormal basis for $T_p \Sigma$.

**Corollary 3.4.** — Let $\Sigma^2 \hookrightarrow M^2(c) \times \mathbb{R}$ be an immersed surface with constant mean curvature and let $A$ be the Weingarten operator associated to the second fundamental form on $\Sigma^2$. Then,

(a) $\langle \nabla^2 A, I \rangle = 0$.

(b) $\langle \nabla^2 A, A \rangle = -|A|^4 + c(5\nu^2 - 1)|A|^2 - 8cH^2\nu^2 - 2cH \langle AT, T \rangle + + 4cH^2 |T|^2 + 2H \text{tr}(A^3)$.

**Proof.** — Consider $\{e_1, e_2\}$ an orthonormal basis of $T_p \Sigma$. We use the definition of the inner product between tensors and the expression in Proposition 3.3 to obtain

$$\langle \nabla^2 A, A \rangle = \sum_{i=1}^{2} \langle (\nabla^2 A)e_i, Ae_i \rangle = -|A|^2 \sum_{i=1}^{2} \langle Ae_i, Ae_i \rangle + + c(5\nu^2 - 1) \sum_{i=1}^{2} \langle Ae_i, Ae_i \rangle - 4cH\nu^2 \sum_{i=1}^{2} \langle Ae_i, e_i \rangle - 2cH \{\sum_{i=1}^{2} \langle AT, e_i \rangle \langle e_i, T \rangle + + \langle T, T \rangle \sum_{i=1}^{2} \langle Ae_i, e_i \rangle \} + 2H \sum_{i=1}^{2} \langle A^2 e_i, Ae_i \rangle.$$

Therefore,

$$\langle \nabla^2 A, A \rangle = -|A|^4 + c(5\nu^2 - 1)|A|^2 - 8cH^2\nu^2 - 2cH \langle AT, T \rangle + + 4cH^2 |T|^2 + 2H \text{tr}(A^3).$$

Using the definition of the inner product and Proposition 3.3 we obtain

$$\langle \nabla^2 A, I \rangle = \sum_{i=1}^{2} \langle (\nabla^2 A)e_i, e_i \rangle = -|A|^2 \sum_{i=1}^{2} \langle Ae_i, e_i \rangle + + c(5\nu^2 - 1) \sum_{i=1}^{2} \langle Ae_i, e_i \rangle - 8cH\nu^2 - 2cH \{\sum_{i=1}^{2} \langle T, e_i \rangle \langle e_i, T \rangle + + \langle T, T \rangle \} + 2H \sum_{i=1}^{2} \langle A^2 e_i, e_i \rangle.$$

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Therefore,
\[ \langle \nabla^2 A, I \rangle = -2H|A|^2 + c(5\nu^2 - 1)2H - 8cH\nu^2 + 2cH\langle T, T \rangle + 2H|A|^2 = 0, \]
where we have used that \( \nu^2 + |T|^2 = 1. \) \( \square \)

**Proposition 3.5.** — Let \( \Sigma \leftrightarrow M^2(c) \times \mathbb{R} \) be an immersed surface with constant mean curvature \( H \) and let \( \phi \) be the traceless Weingarten operator, then

(a) \( |\phi|^2 = |A|^2 - 2H^2. \)
(b) \( \nabla \phi = \nabla A. \)
(c) \( trA^3 = 3H|\phi|^2 + 2H^3. \)

**Proof.** — The proof of item (a) is:

\[ |\phi|^2 = \langle \phi, \phi \rangle = \langle A - HI, A - HI \rangle = \langle A, A \rangle - 2H\langle A, I \rangle + H^2\langle I, I \rangle \]
\[ = |A|^2 - 4H^2 + 2H^2 = |A|^2 - 2H^2, \]
where \( \langle A, I \rangle = 2H \) and \( \langle I, I \rangle = 2. \)

To prove item (b), we consider tangent fields \( X, Y. \) Then,

\[ (\nabla_X \phi)Y = (\nabla_X A)Y - (\nabla_X (HI))Y = (\nabla_X A)Y - \nabla_X HI(Y) + H\nabla_X Y \]
\[ = (\nabla_X A)Y - H\nabla_X Y - X(H)Y + H\nabla_X Y = (\nabla_X A)Y, \]
because \( H \) is constant.

Finally, the proof of item (c) is:

\[ tr(A^3) = \sum_{i=1}^{2} \langle A^3 e_i, e_i \rangle = \sum_{i=1}^{2} \langle (\phi + HI)^3 e_i, e_i \rangle \]
\[ = \sum_{i=1}^{2} \langle (\phi^3 + 3H\phi^2 + 3H^2\phi + H^3 I)e_i, e_i \rangle = 3H|\phi|^2 + 2H^3, \]
because \( tr\phi = tr\phi^3 = 0. \) \( \square \)

Next we shall derive an equation of Simons type for the traceless Weingarten operator \( \phi. \)

**Theorem 3.6.** — Let \( \Sigma \leftrightarrow M^2(c) \times \mathbb{R} \) be an immersed surface with constant mean curvature \( H \) and let \( \phi \) be the traceless Weingarten operator. Then

\[ \langle \nabla^2 \phi, \phi \rangle = -|\phi|^4 + (2H^2 + 5c\nu^2 - c)|\phi|^2 - 2cH\langle \phi T, T \rangle \]
and

\[ \frac{1}{2} \Delta |\phi|^2 = |\nabla \phi|^2 - |\phi|^4 + (2H^2 + 5c\nu^2 - c)|\phi|^2 - 2cH\langle \phi T, T \rangle. \]
Proof. — We use Proposition 3.5 to show that
\[ \langle \nabla^2 \phi, \phi \rangle = \langle \nabla^2 A, A - H I \rangle = \langle \nabla^2 A, A \rangle - H \langle \nabla^2 A, I \rangle. \]
Now, we use Corollary 3.4 to obtain
\[ \langle \nabla^2 \phi, \phi \rangle = -|A|^4 + c(5\nu^2 - 1)|A|^2 - 8cH^2\nu^2 + 2cH\langle AT, T \rangle + 4cH^2|T|^2 + 2H\text{tr}(A^3). \]
Therefore,
\[ \langle \nabla^2 \phi, \phi \rangle = -|\phi|^4 + 2H^2|\phi|^2 + c(5\nu^2 - 1)|\phi|^2 - 2cH\langle \phi T, T \rangle. \]
To finish, we use that \( \frac{1}{2} \Delta |\phi|^2 = |\nabla \phi|^2 + \langle \nabla^2 \phi, \phi \rangle \).

Now we evaluate the Laplacian of \( |S|^2 \) where \( S \) is defined by (1.1), i.e,
\[ S = 2HA - c\langle T, \cdot \rangle T + \frac{c}{2}(1 - \nu^2)I - 2H^2I. \]
We observe the fact that \( S \) is a traceless operator, i.e,
\[ \text{tr}(S) = 2H\text{tr}(A) - c|T|^2 + c(1 - \nu^2) - 4H^2 = 0, \]
where we used that \( |T|^2 + \nu^2 = 1 \) and \( \text{tr}(A) = 2H. \)

**Proposition 3.7 (Codazzi's Equation).** — Let \( \Sigma^2 \hookrightarrow M^2(c) \times \mathbb{R} \) be an immersed surface with constant mean curvature and the \( S \) be the tensor defined in (1.1). Then
\[ (\nabla_X S)Y = (\nabla_Y S)X, \]
for all tangent fields \( X, Y \) on \( \Sigma^2. \)

Proof. — We consider \( (u, v) \) isothermal parameters of the surface \( \Sigma^2. \) Now, we consider the complex parameter, \( z = u + iv. \) Let us set
\[ T_S(X, Y) := (\nabla_X S)Y - (\nabla_Y S)X = \nabla_X(\nabla_Y S) - \nabla_Y(\nabla_X S) - S[X, Y]. \]
We will prove that \( T_S \) is null. For this, consider the derivatives
\[ \partial_z = \frac{1}{2}(\partial_u - i\partial_v) \text{ and } \partial_{\bar{z}} = \frac{1}{2}(\partial_u + i\partial_v). \]
We will compute $T_S$ in the basis $\{\partial_z, \partial_{\bar{z}}\}$. First note that, 

$$
\langle T_S(\partial_z, \partial_{\bar{z}}), \partial_z \rangle = \partial_z \langle S \partial_{\bar{z}}, \partial_z \rangle - \langle S \partial_{\bar{z}}, \nabla_{\partial_z} \partial_z \rangle +
- \partial_{\bar{z}} \langle S \partial_z, \partial_z \rangle + \langle S \partial_z, \nabla_{\partial_{\bar{z}}} \partial_z \rangle
$$

$$
= -Q^{(2,0)}_{\bar{z}} = 0,
$$

because $Q^{(2,0)}$ is holomorphic, Theorem 1 in [1], and using the fact that 

$$\nabla_{\partial_z} \partial_{\bar{z}} = 0, \quad \nabla_{\partial_{\bar{z}}} \partial_z = \frac{\lambda}{\bar{\lambda}} \partial_z, \quad \langle S \partial_{\bar{z}}, \partial_z \rangle = Q^{(2,0)} \quad \text{and} \quad \langle S \partial_z, \partial_{\bar{z}} \rangle = 0,$$

where $\lambda = \langle \partial_z, \partial_{\bar{z}} \rangle$.

Next, 

$$
\langle T_S(\partial_z, \partial_{\bar{z}}), \partial_{\bar{z}} \rangle = -\partial_{\bar{z}} \langle \partial_z, S \partial_{\bar{z}} \rangle + \langle S \partial_z, \nabla_{\partial_{\bar{z}}} \partial_{\bar{z}} \rangle +
+ \partial_z \langle S \partial_{\bar{z}}, \partial_z \rangle - \langle S \partial_{\bar{z}}, \nabla_{\partial_z} \partial_{\bar{z}} \rangle
$$

$$
= Q^{(2,0)}_{\bar{z}} = 0,
$$

where we have used that $\nabla_{\partial_{\bar{z}}} \partial_z = \frac{\lambda}{\bar{\lambda}} \partial_z$ and $Q^{(2,0)}_{\bar{z}} = \overline{Q^{(2,0)}_z}$. It follows that $T_S = 0$. 

**Lemma 3.8.** — Let $Z$ be a symmetric operator satisfying Codazzi’s equation and $\text{tr}(Z) = 0$, then

\begin{equation}
(3.6) \quad \langle (\nabla^2 Z)x, y \rangle = \sum_{i=1}^2 \{-\langle Zy, R_{e_i,x}e_i \rangle - \langle Ze_i, R_{e_i,x}y \rangle\},
\end{equation}

where $\{e_1, e_2\}$ is an orthonormal basis of $T_p\Sigma$.

**Proof.** — See Lemma a. in [8], p. 81, adapted for codimension 1. 

Let us evaluate each summand in expression (3.6).

**Lemma 3.9.** — Let $Z$ be an operator as in Lemma 3.8. Then,

\begin{enumerate}
    \item \sum_{i=1}^2 \langle Zy, R_{e_i,x}e_i \rangle = -cv^2 \langle Zx, y \rangle - 2H \langle Ax, Zy \rangle + \langle A^2 x, Zy \rangle.
\end{enumerate}

and

\begin{enumerate}
    \item \sum_{i=1}^2 \langle Ze_i, R_{e_i,x}y \rangle = -cv^2 \langle Zx, y \rangle - \langle Ay, ZAx \rangle + \langle Ax, y \rangle \text{tr}(AZ).
\end{enumerate}
Proof. — Consider \{e_1, e_2\} an orthonormal basis of \( T_p\Sigma \). Using Gauss’ equation (2.3) we find

\[
\langle Zy, Re_{i}, xe_i \rangle = -c\{\langle x, Zy \rangle - \langle x, e_i \rangle \langle Zy, e_i \rangle - \langle x, T \rangle \langle Zy, T \rangle + \\
- \langle e_i, T \rangle^2 \langle x, Zy \rangle + \langle e_i, T \rangle \langle x, e_i \rangle \langle Zy, T \rangle + \\
+ \langle x, T \rangle \langle e_i, T \rangle \langle e_i, Zy \rangle \} - \langle Ae_i, e_i \rangle \langle Ax, Zy \rangle + \\
+ \langle Ax, e_i \rangle \langle Ae_i, Zy \rangle.
\]

Therefore,

\[
\sum_{i=1}^{2} \langle Zy, Re_{i}, xe_i \rangle = -c\{2\langle x, Zy \rangle - \sum_{i=1}^{2} \langle x, e_i \rangle \langle Zy, e_i \rangle + \cdots \\
\cdots - 2\langle x, T \rangle \langle Zy, T \rangle - \langle x, Zy \rangle \sum_{i=1}^{2} \langle e_i, T \rangle^2 + \\
+ \langle Zy, T \rangle \sum_{i=1}^{2} \langle e_i, T \rangle \langle x, e_i \rangle + \langle x, T \rangle \sum_{i=1}^{2} \langle e_i, T \rangle \langle e_i, Zy \rangle \} + \\
- \langle Ax, Zy \rangle \sum_{i=1}^{2} \langle Ae_i, e_i \rangle + \sum_{i=1}^{2} \langle Ax, e_i \rangle \langle Ae_i, Zy \rangle,
\]

which implies that

\[
\sum_{i=1}^{2} \langle Zy, Re_{i}, xe_i \rangle = -c\{2\langle x, Zy \rangle - \langle Zx, y \rangle - 2\langle x, T \rangle \langle Zy, T \rangle + \\
- \langle x, Zy \rangle |T|^2 + \langle Zy, T \rangle \langle x, T \rangle + \langle x, T \rangle \langle T, Zy \rangle \} + \\
- \langle Ax, Zy \rangle 2H + \langle Ax, AZy \rangle.
\]

Hence,

\[
\sum_{i=1}^{2} \langle Zy, Re_{i}, xe_i \rangle = -c\{1 - |T|^2\} \langle Zx, y \rangle - 2H \langle Ax, Zy \rangle + \langle A^2 x, Zy \rangle,
\]

which shows the validity of (i). Now, one may verify that

\[
\langle Ze_i, Re_{i}, yx \rangle = -c\{\langle e_i, y \rangle \langle Ze_i, x \rangle - \langle x, y \rangle \langle Ze_i, e_i \rangle + \\
- \langle x, T \rangle \langle Ze_i, T \rangle \langle e_i, y \rangle - \langle e_i, T \rangle \langle y, T \rangle \langle x, Ze_i \rangle + \\
+ \langle e_i, T \rangle \langle x, y \rangle \langle Ze_i, T \rangle + \langle x, T \rangle \langle y, T \rangle \langle e_i, Ze_i \rangle \} + \\
- \langle Ae_i, y \rangle \langle Ax, Ze_i \rangle + \langle Ax, y \rangle \langle Ae_i, Ze_i \rangle.
\]
Therefore
\[
2 \sum_{i=1}^{2} \langle Z e_i, R_{e_i, x} y \rangle = -c \{ 2 \sum_{i=1}^{2} \langle e_i, y \rangle \langle Z e_i, x \rangle - \langle x, y \rangle \sum_{i=1}^{2} \langle Z e_i, e_i \rangle + \\
- \langle x, T \rangle \sum_{i=1}^{2} \langle Z e_i, T \rangle \langle e_i, y \rangle - \langle y, T \rangle \sum_{i=1}^{2} \langle e_i, T \rangle \langle x, Z e_i \rangle + \cdots \\
\cdots + \langle x, y \rangle \sum_{i=1}^{2} \langle e_i, T \rangle \langle Z e_i, T \rangle + \langle x, T \rangle \sum_{i=1}^{2} \langle e_i, Z e_i \rangle \} + \\
- \sum_{i=1}^{2} \langle A e_i, y \rangle \langle A x, Z e_i \rangle + \langle A x, y \rangle \sum_{i=1}^{2} \langle A e_i, Z e_i \rangle.
\]

Therefore
\[
2 \sum_{i=1}^{2} \langle Z e_i, R_{e_i, x} y \rangle = -c \{ \langle Z x, y \rangle - \langle x, T \rangle \langle Z y, T \rangle - \langle y, T \rangle \langle Z x, T \rangle + \\
+ \langle Z T, T \rangle \langle x, y \rangle \} - \langle A y, Z A x \rangle + \langle A x, y \rangle tr(AZ),
\]
noting that \( trZ = 0 \).

Considering that
\[
- \langle x, T \rangle \langle Z y, T \rangle - \langle y, T \rangle \langle Z x, T \rangle + \langle Z T, T \rangle \langle x, y \rangle = -(1 - \nu^2) \langle Z x, y \rangle,
\]
we find
\[
2 \sum_{i=1}^{2} \langle Z e_i, R_{e_i, x} y \rangle = -c \nu^2 \langle Z x, y \rangle - \langle A y, Z A x \rangle + \langle A x, y \rangle tr(AZ),
\]
which demonstrates (ii).

\[\square\]

**Theorem 3.10.** — Let \( \Sigma^2 \hookrightarrow M^2(c) \times \mathbb{R} \) be an immersed surface with non zero constant mean curvature \( H \) and let \( Z \) be an operator on \( \Sigma^2 \) satisfying Codazzi’s equation with \( tr(Z) = 0 \). Then,
\[
\langle (\nabla^2 Z) x, y \rangle = 2c \nu^2 \langle Z x, y \rangle + 2H \langle A x, Z y \rangle - \langle A^2 x, Z y \rangle + \\
+ \langle A y, Z A x \rangle - \langle A x, y \rangle tr(AZ).
\]

**Proof.** — We use the expressions of Lemma 3.9 in equation (3.6) obtained in Lemma 3.8.

Next we derive an equation of Simons type for the operator \( S \) as defined in (1.1).
Theorem 3.11 (Thm 1.1 in Introduction). — Let $\Sigma^2 \hookrightarrow M^2(c) \times \mathbb{R}$ be an immersed surface with non zero constant mean curvature $H$ and $S$ as defined in (1.1). Then,

$$
\langle (\nabla^2 S)x, y \rangle = 2cv^2 \langle Sx, y \rangle + 2H \langle Ax, Sy \rangle - \langle A^2 x, Sy \rangle + \langle Ay, SAx \rangle - \langle Ax, y \rangle tr(AS),
$$

and

$$
\frac{1}{2} \Delta |S|^2 = |\nabla S|^2 - |S|^4 + |S|^2 \left( \frac{5cv^2}{2} - \frac{c}{2} + 2H^2 - \frac{c}{H} \langle ST, T \rangle \right) + c|ST|^2 - \frac{1}{4H^2} \langle ST, T \rangle^2.
$$

Proof. — First, since $S$ satisfies Proposition 3.7, we can use the Theorem 3.10 with $Z = S$.

Now, we know that $\frac{1}{2} \Delta |S|^2 = |\nabla S|^2 + \langle \nabla^2 S, S \rangle$. Furthermore, we find that

$$
\langle \nabla^2 S, S \rangle = 2cv^2 |S|^2 + 2H tr(AS^2) - [tr(AS)]^2.
$$

Now, we need to compute $tr(AS^2)$ and $tr(AS)$, as follows:

$$
tr(AS^2) = tr \{ S^2 (S + \frac{c}{2H} \langle T, \cdot \rangle T - \frac{c}{4H} (1 - \nu^2) I + HI) \} = trS^3 + \frac{c}{2H} tr(\langle T, S^2 \cdot \rangle T) - \left( \frac{c}{4H} (1 - \nu^2) - H \right) trS^2 = 0 + \frac{c}{2H} |ST|^2 - \left( \frac{c}{4H} (1 - \nu^2) - H \right) |S|^2
$$

and

$$
tr(AS) = tr \{ S (S + \frac{c}{2H} \langle T, \cdot \rangle T - \frac{c}{4H} (1 - \nu^2) I - HI) \} = trS^2 + \frac{c}{2H} tr(\langle T, S \cdot \rangle T) - \left( \frac{c}{4H} (1 - \nu^2) - H \right) tr S = |S|^2 + \frac{c}{2H} \langle ST, T \rangle - 0,
$$

noting that $trS = trS^3 = 0$, also that

$$
tr(\langle T, S \cdot \rangle T) = \sum_{i=1}^{2} \langle T, Se_i \rangle \langle T, e_i \rangle = \langle ST, T \rangle
$$

and that

$$
tr(\langle T, S^2 \cdot \rangle T) = \sum_{i=1}^{2} \langle T, S^2 e_i \rangle \langle T, e_i \rangle = \langle S^2 T, T \rangle.
$$
Therefore,
\[ \frac{1}{2} \Delta |S|^2 = |\nabla S|^2 + 2c\nu^2 |S|^2 + 2H \left( \frac{c}{2H} |ST|^2 - \left( \frac{c}{4H} (1 - \nu^2) - H \right) |S|^2 \right) + \]
\[ - \left( |S|^2 + \frac{c}{2H} \langle ST, T \rangle \right)^2 , \]
in this way,
\[ \frac{1}{2} \Delta |S|^2 = |\nabla S|^2 + 2c\nu^2 |S|^2 + c|ST|^2 - \left( \frac{c}{2}(1 - \nu^2) - 2H^2 \right) |S|^2 + \]
\[ - |S|^4 - \frac{c}{H} \langle ST, T \rangle |S|^2 - \frac{1}{4H^2} \langle ST, T \rangle^2 . \]
Rearranging terms, we obtain finally
\[ \frac{1}{2} \Delta |S|^2 = |\nabla S|^2 - |S|^4 + |S|^2 \left( \frac{5c\nu^2}{2} - \frac{c}{2} + 2H^2 - \frac{c}{H} \langle ST, T \rangle \right) + \]
\[ + c|ST|^2 - \frac{1}{4H^2} \langle ST, T \rangle^2 . \]

\[ \square \]

4. Applications

In this section, we will apply the results found in section 3 together with the Omori-Yau’s Theorem to classify some surfaces in \( M^2(c) \times \mathbb{R} \).

**Theorem 4.1.** — Let \( \Sigma^2 \hookrightarrow \mathbb{H}^2 \times \mathbb{R} \) be an oriented complete immersed minimal surface. Assume that
\[ \sup_{\Sigma} (|A|^2 + 5\nu^2) < 1 . \]
Then \( \Sigma^2 \) is a vertical plane \( \gamma \times \mathbb{R} \) for some geodesic \( \gamma \) in \( \mathbb{H}^2 \).

**Proof.** — Using Theorem 3.6 with \( H = 0 \) and \( c = -1 \), one finds
\[ \frac{1}{2} \Delta |A|^2 = |\nabla A|^2 - |A|^4 + (1 - 5\nu^2) |A|^2 \geq |A|^2 \left( -|A|^2 + 1 - 5\nu^2 \right) . \]
Let \( \frac{d}{2} := -\sup_{\Sigma} (|A|^2 + 5\nu^2) + 1 > 0 \). Therefore,
\[ \Delta |A|^2 \geq d \cdot |A|^2 . \]
Using Gauss’ equation (2.3) in \( \mathbb{H}^2 \times \mathbb{R} \) we have
\[ K_\Sigma = K_{ext} - \nu^2 = -\frac{|A|^2 + 5\nu^2}{2} + \frac{3\nu^2}{2} \geq -\frac{1}{2} . \]
Now we can use Theorem 2.2 with $u = |A|^2$, i.e., there exist $\{p_j\}$ in $\Sigma^2$ such that

$$\lim_{j \to \infty} |A|^2(p_j) = \sup_{\Sigma} |A|^2$$

and

$$\lim_{j \to \infty} \Delta |A|^2(p_j) \leq 0.$$

Next, we use inequality (4.1) to conclude that $\sup_{\Sigma} |A|^2 = 0$, i.e., $\Sigma^2$ is totally geodesic with $|\nu| < \sqrt{0.2}$.

Since $\Sigma^2$ is totally geodesic and $|\nu| < \sqrt{0.2}$ it cannot be a slice, it must be a vertical plane $\gamma \times \mathbb{R}$ for some geodesic $\gamma$ in $\mathbb{H}^2$.

**Theorem 4.2.** — Let $\Sigma^2 \hookrightarrow \mathbb{H}^2 \times \mathbb{R}$ be a complete immersed surface with constant mean curvature $H$. Assume that

$$\sup_{\Sigma} (|\phi|^2 + 5\nu^2) < 2H^2 + 1 \quad \text{and} \quad \langle \phi T, T \rangle \geq 0.$$

Then $\Sigma^2$ is a vertical plane $\gamma \times \mathbb{R}$ for some geodesic $\gamma$ in $\mathbb{H}^2$.

**Proof.** — We consider the expression in Theorem 3.6 for the particular case $c = -1$:

$$\frac{1}{2} \Delta |\phi|^2 = |\nabla \phi|^2 - |\phi|^4 + (2H^2 + 1 - 5\nu^2)|\phi|^2 + 2H\langle \phi T, T \rangle.$$

As $\langle \phi T, T \rangle \geq 0$, we find

$$\frac{1}{2} \Delta |\phi|^2 \geq -|\phi|^4 + (2H^2 + 1 - 5\nu^2)|\phi|^2.$$

Consider $\frac{d}{2} := 2H^2 + 1 - \sup_{\Sigma} (|\phi|^2 + 5\nu^2) > 0$. Then

$$\Delta |\phi|^2 \geq 2|\phi|^2(2H^2 + 1 - 5\nu^2 - |\phi|^2) \geq d|\phi|^2,$$

which implies,

$$\Delta |\phi|^2 \geq d|\phi|^2.$$

Using Gauss’ equation (2.3) in $\mathbb{H}^2 \times \mathbb{R}$ we have

$$K_\Sigma = K_{ext} - \nu^2 = -\frac{|\phi|^2 + 5\nu^2 - 2H^2}{2} + \frac{3\nu^2}{2} \geq -\frac{1}{2}.$$

Now we can use Theorem 2.2 with $u = |\phi|^2$, i.e., there exist $\{p_j\}$ in $\Sigma^2$ such that

$$\lim_{j \to \infty} |\phi|^2(p_j) = \sup_{\Sigma} |\phi|^2$$

and

$$\lim_{j \to \infty} \Delta |\phi|^2(p_j) \leq 0.$$

Furthermore, we use inequality (4.2) to conclude that $\sup_{\Sigma} |\phi|^2 = 0$, i.e., $\Sigma^2$ is totally umbilical.

Next, we use that if $\Sigma^2$ is totally umbilical with constant mean curvature in $\mathbb{H}^2 \times \mathbb{R}$ then $\Sigma^2$ is totally geodesic, which follows from [9] section 4.

Since $\Sigma^2$ is totally geodesic and $|\nu| < \sqrt{0.2}$ it must be a vertical plane $\gamma \times \mathbb{R}$ for some geodesic $\gamma$ in $\mathbb{H}^2$. This concludes the proof. □
We need the following result:

**Lemma 4.3.** — Let \( \Sigma^2 \hookrightarrow M^2(c) \times \mathbb{R} \) be a complete immersed surface with non zero constant mean curvature \( H \). Then \( |S| = 0 \) if and only if \( \Sigma^2 \) is an Abresch-Rosenberg surface.

**Proof.** — We consider \((u, v)\) isothermal parameters on the surface \( \Sigma^2 \). Now, we consider the complex parameter, \( z = u + iv \) and the \((2,0)\)-part of the Abresch-Rosenberg differential

\[
Q(x, y) = 2H \langle Ax, y \rangle - c \langle x, T \rangle \langle y, T \rangle.
\]

We can rewrite \( Q \) as

\[
Q(x, y) = \langle Sx, y \rangle - i \frac{\tilde{f}}{2},
\]

where \( \tilde{e} = \langle S\partial_u, \partial_u \rangle = -\langle S\partial_v, \partial_v \rangle = -\tilde{g} \) and \( \tilde{f} = \langle S\partial_u, \partial_v \rangle \). Therefore

\[
|Q^{(2,0)}| = \sqrt{\left(\frac{\tilde{e} - \tilde{g}}{4}\right)^2 + \frac{\tilde{f}^2}{4}} = \sqrt{\frac{\tilde{e}^2}{4} + \frac{\tilde{f}^2}{4}} = \frac{E^2}{2\sqrt{2}|S|},
\]

where \( E = |\partial_u| > 0 \). This concludes the proof. \( \square \)

Let us consider the polynomial \( p_H(t) = -t^2 - \frac{1}{\sqrt{2}H} t + \left(\frac{4H^2 - 1}{2}\right) \).

When \( H \) is greater than one half there is a positive root for \( p_H \). Let \( L_H \) be the positive root. One has:

**Theorem 4.4** (Thm 1.2 in Introduction). — Let \( \Sigma^2 \hookrightarrow S^2 \times \mathbb{R} \) be an immersed surface with constant mean curvature \( H \) greater than one half. If

\( \Sigma^2 \) is complete and \( \sup \Sigma |S| < L_H \)

or

\( \Sigma^2 \) is closed and \( |S| < L_H \),

then \( \Sigma^2 = S^2_H \), i.e, \( \Sigma^2 \) is an embedded rotationally invariant constant mean curvature sphere.

**Proof.** — Let consider two cases. First, \( \Sigma \) is complete and second, \( \Sigma \) is closed.

**First Case.** Consider the expression in Theorem 3.11 with \( c = 1 \):

\[
\frac{1}{2} \Delta |S|^2 = |\nabla S|^2 - |S|^4 + |S|^2 \left(\frac{5\nu^2}{2} - \frac{1}{2} + 2H^2 - \frac{1}{H} \langle ST, T \rangle \right) +
\]
$$+ |ST|^2 - \frac{1}{4H^2} \langle ST, T \rangle^2.$$ 

As $|\langle ST, T \rangle| \leq |ST| \leq \frac{1}{\sqrt{2}} |S|$, we have

$$\frac{1}{2} \Delta |S|^2 \geq -|S|^4 + |S|^2 \left( \frac{5\nu^2}{2} + \frac{4H^2 - 1}{2} - \frac{1}{\sqrt{2}H} |S| \right) + \left( \frac{4H^2 - 1}{4H^2} \right) \langle ST, T \rangle^2,$$

hence,

(4.3) \quad \frac{1}{2} \Delta |S|^2 \geq |S|^2 \left( \frac{4H^2 - 1}{2} - \frac{1}{\sqrt{2}H} |S| - |S|^2 \right) + \frac{5}{2} \nu^2 |S|^2,$$

because $H > \frac{1}{2}$.

Observe that

$$\frac{4H^2 - 1}{2} - \frac{1}{\sqrt{2}H} |S| - |S|^2 \geq p_H(\sup_{\Sigma} |S|) =: \frac{d}{2} > 0$$

and $\nu^2 |S|^2 \geq 0$. Therefore

(4.4) \quad \Delta |S|^2 \geq d |S|^2.

Now we estimate $|S|$.

$$|S| \geq 2H |A| - |\langle T, \cdot \rangle_T| - (1 - \nu^2) - 4H^2 \geq 2H |A| - 2(1 - \nu^2) - 4H^2,$$

that is,

$$L_H \geq |S| \geq 2H |A| - 2 - 4H^2.$$

Using Gauss’ equation (2.3) in $S^2 \times \mathbb{R}$ we find

$$K_{\Sigma} = K_{ext} + \nu^2 = -\frac{|A|^2}{2} + 2H^2 + \nu^2 \geq -\frac{1}{2} \left( \frac{L_H + 2 + 4H^2}{2H} \right)^2.$$

Now we can use Theorem 2.2 with $u = |S|^2$, i.e, there exists a $\{p_j\}$ in $\Sigma^2$ such that

$$\lim_{j \to \infty} |S|^2(p_j) = \sup_{\Sigma} |S|^2 \ and \ \lim_{j \to \infty} \Delta |S|^2(p_j) \leq 0.$$

By means of inequality (4.4) we conclude that $\sup_{\Sigma} |S|^2 = 0$, i.e, $|S| = 0$ in $\Sigma^2$. Using Lemma 4.3 and Remark 1 of the Introduction we conclude the proof.

Second case. Let us consider expression (4.3)

$$\frac{1}{2} \Delta |S|^2 \geq |S|^2 \left( \frac{4H^2 - 1}{2} - \frac{1}{\sqrt{2}H} |S| - |S|^2 \right) + \frac{5}{2} \nu^2 |S|^2.$$
As \(|S| \leq L_H\), we have \(\frac{4H^2 - 1}{2} - \frac{1}{\sqrt{2}H}|S| - |S|^2 \geq 0\). Hence,
\[
\frac{1}{2}\Delta|S|^2 \geq \frac{5}{2} \nu^2 |S|^2.
\]
Integrating and using Stokes’ Theorem we find
\[
0 \geq \frac{5}{2} \int_{\Sigma} \nu^2 |S|^2 d\Sigma \geq 0.
\]
It follows that
(4.5) \(|S| \cdot \nu = 0\).

Let \(\Theta = \{p \in \Sigma^2 : \nu(p) = 0\} = \nu^{-1}(0)\) be the nodal lines of \(\nu\). We know that
\[
\Delta \nu + (|A|^2 + Ric(N, N)) \nu = 0.
\]
Hence, we can apply Theorem 2.5 in [4], p. 49, to conclude that \(\Theta\) has empty interior. Thus, using (4.5), \(|S|\) vanishes in an open and dense set. By continuity, \(|S| = 0\) in \(\Sigma\).

Using Lemma 4.3 and Remark.1 of the Introduction we conclude the proof. \(\Box\)

**Theorem 4.5.** — There exists no \(\Sigma^2 \hookrightarrow \mathbb{S}^2 \times \mathbb{R}\) complete immersed surface with constant mean curvature greater than one half such that \(|S| = L_H\).

**Proof.** — Suppose that there exist \(\Sigma^2 \hookrightarrow \mathbb{S}^2 \times \mathbb{R}\) satisfying the condition of the theorem. Using expression (4.3)
\[
\frac{1}{2}\Delta|S|^2 \geq |S|^2 \left(\frac{4H^2 - 1}{2} - \frac{1}{\sqrt{2}H}|S| - |S|^2\right) + \frac{5}{2} \nu^2 |S|^2,
\]
with \(|S| = L_H\) one find that
\[
0 \geq 0 + \frac{5}{2} \nu^2 L_H^2 \geq 0.
\]
Hence \(\nu = 0\), i.e, \(\Sigma^2 \hookrightarrow \mathbb{S}^2 \times \mathbb{R}\) is a cylinder \(\gamma \times \mathbb{R}\) for some \(\gamma \in \mathbb{S}^2\) with constant curvature \(2H\).

On the other hand, for a cylinder \(\gamma \times \mathbb{R}\), where \(\gamma \in \mathbb{S}^2\) with constant curvature \(2H\), we may write
\[
S = \begin{pmatrix}
2H^2 + \frac{1}{2} & 0 \\
0 & -2H^2 - \frac{1}{2}
\end{pmatrix}.
\]
As \(|S| = \frac{\sqrt{2}}{2}(4H^2 + 1) > L_H\) we have a contradiction. \(\Box\)
In next theorem we need the following result:

**Lemma 4.6.** — Any Abresch-Rosenberg surface \( \Sigma^2 \hookrightarrow \mathbb{H}^2 \times \mathbb{R} \) with \( H > \frac{1}{2} \) is an embedded rotationally invariant constant mean curvature sphere.

**Proof.** — See Proposition 4.3 in [1], p. 159. □

Let us consider the polynomial

\[
q_H(t) = -t^2 - \frac{1}{\sqrt{2}H}t + \left( \frac{8H^4 - 12H^2 - 1}{4H^2} \right).
\]

When \( H \) is greater than a positive root of the polynomial \( r(x) = 8x^4 - 12x^2 - 1 \), i.e., \( H \) is greater than \( \sqrt{\frac{3 + \sqrt{11}}{4}} \), there is a positive root for \( q_H \). Let \( M_H \) be the positive root.

**Theorem 4.7** (Thm 1.3 in Introduction). — Let \( \Sigma^2 \hookrightarrow \mathbb{H}^2 \times \mathbb{R} \) be an immersed surface with constant mean curvature \( H \) greater than \( \sqrt{\frac{3 + \sqrt{11}}{4}} \approx 1.25664 \). If

\( \Sigma^2 \) is complete and \( \sup_\Sigma |S| < M_H \)

or

\( \Sigma^2 \) is closed and \( |S| \leq M_H \),

then \( \Sigma^2 = S^2_H \), i.e, \( \Sigma^2 \) is an embedded rotationally invariant constant mean curvature sphere.

**Proof.** — Let us consider two cases. First, \( \Sigma \) is complete and second, \( \Sigma \) is closed.

**First case.** Consider the expression in Theorem 3.11 with \( c = -1 \)

\[
\frac{1}{2} \Delta |S|^2 = |\nabla S|^2 - |S|^4 + |S|^2 \left( \frac{-5\nu^2}{2} + \frac{1}{2} + 2H^2 + \frac{1}{H} \langle ST, T \rangle \right) + |ST|^2 - \frac{1}{4H^2} \langle ST, T \rangle^2.
\]

As \( |\langle ST, T \rangle| \leq |ST| \leq \frac{1}{\sqrt{2}} |S| \), we may write

\[
\frac{1}{2} \Delta |S|^2 \geq -|S|^4 + |S|^2 \left( \frac{4H^2 + 1 - 5\nu^2}{2} - \frac{1}{\sqrt{2}H} |S| \right) - \left( \frac{4H^2 + 1}{4H^2} \right) |S|^2,
\]

i.e,

\[
\frac{1}{2} \Delta |S|^2 \geq |S|^2 \left( \frac{4H^2 - 4 + 5 - 5\nu^2}{2} - \frac{1}{\sqrt{2}H} |S| - \frac{4H^2 + 1}{4H^2} - |S|^2 \right).
\]
This may be rewritten as,
\begin{equation}
\frac{1}{2} \Delta |S|^2 \geq |S|^2 \left( \frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{\sqrt{2H}} |S| - |S|^2 \right) + \frac{5}{2} (1 - \nu^2)|S|^2.
\end{equation}

Observe that
\[ \frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{\sqrt{2H}} |S| - |S|^2 \geq q_H (\sup_{\Sigma} |S|) =: \frac{d}{2} > 0 \]
and 
\[ (1 - \nu^2)|S|^2 \geq 0. \]
Therefore,
\begin{equation}
\Delta |S|^2 \geq d|S|^2.
\end{equation}
Next we estimate \(|S|\).
\[ |S| \geq 2H|A| - |\langle T, \cdot \rangle T| - (1 - \nu^2) - 4H^2 \geq 2H|A| - 2(1 - \nu^2) - 4H^2, \]
i.e.,
\[ M_H \geq |S| \geq 2H|A| - 2 - 4H^2. \]
Using Gauss’ equation (2.3) in \(\mathbb{H}^2 \times \mathbb{R}\) we find
\[ K_{\Sigma} = K_{ext} - \nu^2 = -\frac{|A|^2}{2} + 2H^2 - \nu^2 \geq -\frac{1}{2} \left( \frac{M_H + 2 + 4H^2}{2H} \right)^2. \]
Now we can use Theorem 2.2 with \(u = |S|^2\), i.e, there exists a \(\{p_j\}\) in \(\Sigma^2\) such that
\[ \lim_{j \to \infty} |S|^2(p_j) = \sup_{\Sigma} |S|^2 and \lim_{j \to \infty} \Delta |S|^2(p_j) \leq 0. \]
Inequality (4.7) allows us conclude that \(\sup_{\Sigma} |S|^2 = 0\), i.e, \(|S| = 0\) in \(\Sigma^2\).
Then, by using Lemmas 4.3 and 4.6, we conclude the proof.

Second case. Let us consider expression (4.6)
\[ \frac{1}{2} \Delta |S|^2 \geq \left( \frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{\sqrt{2H}} |S| - |S|^2 \right) + \frac{5}{2} (1 - \nu^2)|S|^2. \]
As \(|S| \leq M_H\), we have that
\[ \frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{\sqrt{2H}} |S| - |S|^2 \geq 0. \] Hence,
\[ \frac{1}{2} \Delta |S|^2 \geq \frac{5}{2} (1 - \nu^2)|S|^2. \]
Integrating and using Stokes’ Theorem we write
\[ 0 \geq \frac{5}{2} \int_{\Sigma} (1 - \nu^2)|S|^2 d\Sigma \geq 0. \]
Moreover
\begin{equation}
(1 - \nu^2) \cdot |S|^2 = 0.
\end{equation}
Consider $\Theta = \{ p \in \Sigma^2; \nu^2(p) = 1 \} \subset \mathbb{H}^2 \times \{ t_0 \}$, for any $t_0$. Since $H$ is positive we have that $\Theta$ has empty interior. Thus, using (4.8), we conclude that $|S|$ vanishes in an open and dense set. By continuity, $|S| = 0$ in $\Sigma$. Using Lemma 4.3 and the fact that the only Abresch-Rosenberg closed surface is $S^2_H$ we conclude the proof. □

**Theorem 4.8. —** There exists no $\Sigma^2 \hookrightarrow \mathbb{H}^2 \times \mathbb{R}$ a complete immersed surface with constant mean curvature greater than $\sqrt{\frac{3 + \sqrt{11}}{4}} \approx 1.25664$ such that $|S| = M_H$.

**Proof. —** Suppose that there exists $\Sigma^2 \hookrightarrow \mathbb{H}^2 \times \mathbb{R}$ satisfying the condition of the theorem. Using expression (4.6)

$$\frac{1}{2} \Delta |S|^2 \geq |S|^2 \left( \frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{\sqrt{2}H} |S| - |S|^2 \right) + \frac{5}{2}(1 - \nu^2)|S|^2$$

with $|S| = M_H$ we obtain:

$$0 \geq 0 + \frac{5}{2}(1 - \nu^2)M_H^2 \geq 0.$$ 

Hence $\nu^2 = 1$, i.e, $\Sigma^2 \hookrightarrow \mathbb{H}^2 \times \mathbb{R}$ is a slice $\mathbb{H}^2 \times \{ t_0 \}$. But $\mathbb{H}^2 \times \{ t_0 \}$ has zero mean curvature, and this is impossible because $H$ is positive. □

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