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DIFFERENT METHODS FOR THE STUDY OF OBSTRUCTIONS IN THE SCHEMES OF JACOBI

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ABSTRACT. — In this paper the problem of obstructions in Lie algebra deformations is studied from four different points of view. First, we illustrate the method of local ring, an alternative to Gerstenhaber's method for Lie deformations. We draw parallels between both methods showing that an obstruction class corresponds to a nilpotent local parameter of a versal deformation of the law in the scheme of Jacobi. Then, an elimination process in the global ring, which defines the scheme, allows us to obtain nilpotent elements and to describe the global method. Finally, the obstruction problem is studied in the geometry defined by generators and relations. Under certain conditions, we prove that subschemes of grassmannians of \( T \)-invariant ideals of a free Lie algebra \( (T \) being a torus of derivations), after quotient by an action group, are the same as those defined from Jacobi polynomials after a similar quotient.

RÉSUMÉ. — Le problème des obstructions aux déformations d’algèbres de Lie est étudié de quatre points de vue différents. On illustrera d’abord la méthode de l’anneau local, une alternative à la méthode de Gerstenhaber. On compare les deux méthodes en montrant qu’une classe d’obstruction correspond à un paramètre local nilpotent d’une déformation verselle de la loi dans le schéma de Jacobi. Un procédé d’élimination dans l’anneau global permet ensuite d’obtenir des éléments nilpotents, constituant ainsi une méthode globale. Enfin, le problème des obstructions est traité dans la géométrie définie par générateurs et relations. Des sous-schémas de grassmanniennes constitués d’idéaux \( T \)-invariants d’une algèbre de Lie libre \( (T \) étant un tore bien choisi), après quotient par une action de groupe, sont égaux à ceux définis par les polynômes de Jacobi après passage à un quotient similaire.

Introduction

Let \( L_n \) be the scheme of laws of Lie algebras of dimension \( n \) over \( \mathbb{C} \) defined by antisymmetry and Jacobi identities and called “scheme of Jacobi”. If \( R \) is a completely reducible Lie subalgebra of the space \( C^1(\mathbb{C}^n, \mathbb{C}^n) \) of...
linear morphisms $\mathbb{C}^n \rightarrow \mathbb{C}^n$, then we can impose $R$-invariance conditions for the laws in $L_n$. Thus, we obtain a subscheme of $L_n$ denoted by $L_n^R$ and let $L_n^R(\mathbb{C})$ be the set of its points. This scheme was introduced in the study of algebraic Lie algebras admitting a Chevalley decomposition $g = R \oplus n$ with nilpotent part $n = (V, \Phi_0)$, for $V = \mathbb{C}^n$. It is well known that $L_n^R \cong \text{Spec}(\mathbb{C}[X_{ij}]/\text{Jac}_n)$, where $\text{Jac}_n$ is the ideal in the ring of polynomials $\mathbb{C}[X_{ij}]$, for $1 \leq i < j \leq n$, $1 \leq k \leq n$, generated by the antisymmetry, Jacobi identities and $R$-invariance conditions. We can deduce the local study of $g$ in the scheme $L_m$, with $m = n + \dim(R)$, from the local study of $n$ in $L_n^R$ under certain conditions on $R$. This type of result enters in the scope of the “Theorem of reduction” where a general statement is proposed in [6]. $R$ can be a torus $T$ (i.e., abelian and reducible) satisfying hypotheses of the reduction theorem. This allows us to work directly in Jacobi scheme $L_n^T$, and local results obtained for $n$ in $L_n^T$ are valid for $g$ in $L_m$. It suffices to choose a maximal $T$ for at least one law. According to Mostow, all maximal tori over a complex Lie algebra are conjugated by automorphisms. The schemes used in this paper are $T$-invariant but most of the results can be transferred to schemes $L_m$ thanks to the reduction theorem.

This paper is organized in five sections as follows:

1 – Section 1 deals with the classical theory of obstructions, which was initiated by M. Gerstenhaber [12] for associative laws within the framework of formal deformations. The fact that a vector $\Phi_1$ in the Zariski tangent space of $L_n^R$ at $\Phi_0$, $Z^2(n, n)^R$, cannot be lifted to a curve $\Phi_0 + \sum_{k \in \mathbb{N}} t^k \Phi_k$, but only to a “truncated deformation” up to an order $p \geq 1$, leads to the existence of a non null 3-class $\varpi_{p+1} \in H^3(n, n)^R$, called obstruction. If $\Phi_1$ doesn’t belong to the tangent space of the reduced scheme at $\Phi_0$, defined by the radical of the ideal $\text{Jac}_n$, $\sqrt{\text{Jac}_n}$, then it always presents an obstruction and the scheme $L_n^R$ is not reduced at $\Phi_0$. A certain number of technical difficulties are attached to this method, in particular the dependence on the choice of partial solutions $\Phi_2, \Phi_3, ..., \Phi_p$ in deformation equations.

The examples in [15] satisfying $\varpi_2 \neq 0$ are solved by Rauch rigidity criterion. This criterion is applied in [14] to Lie algebras $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}^n$, with $R = \mathfrak{sl}(2, \mathbb{C}) \neq T$. The first $T$-rigid examples known with $\varpi_p \neq 0$ for $p > 2$ are provided by filiform Lie algebras $\mathfrak{f}_n$ for $n \geq 12$ [1] and the obstruction appears at order 5. We illustrate this method sketching out cohomological calculations and using the fact that if $H^2(n, n)^R$ is equal to $\mathbb{C}$ then the choice of $\Phi_k (1 < k < p)$ for $1 < p \leq 4$ is irrelevant.
We prove a useful result which provides a link to the local ring method developed in section 2: \( \omega_{p+1} = 0 \) is equivalent to the existence of a parameter \( t \) in the maximal ideal of the local ring \( \mathcal{O} \) at \( \Phi_0 \), such that \( t^{p+1} \) doesn’t belong to an ideal \( \mathcal{B} \) whose quotient \( \mathcal{O} \to \mathcal{O}/\mathcal{B} \) defines a deformation \( \Phi_0 + \tilde{t}\Phi_1 + \cdots + \tilde{t}^{p+1}\Phi_{p+1} \) at order \( p+1 \). More concretely, in the study of \( f_n \) we obtain an obstruction at order 5, so \( \omega_5 \neq 0 \iff t^5 = 0 \ (t^4 \neq 0) \).

2 - The 1-parameter deformation method is not characteristic enough, as it gives only partial results. The right way is to introduce versal deformations, which describe the deformation question completely. This was done by Fialowski [9] and [10]. In [9] and later in [11] a straightforward method was given to construct a versal deformation. This construction starts with determining the universal infinitesimal deformation, and extending it step by step. In [5, 6] authors have developed a method giving versal deformations from the universal deformation constituted by the germs of coordinate functions at the point \( \Phi_0 \). The local ring \( \mathcal{O} \) at \( \Phi_0 \) in Jacobi scheme \( L_m \) or \( L_R \) gives maximal information about the local deformation problem. A deformation of \( \Phi_0 \) on a local ring \( \mathcal{A} \) is a local morphism from \( \mathcal{O} \) to \( \mathcal{A} \). In [6], versal deformations are obtained by reducing the number of parameters with a quotient of \( \mathcal{O} \) and the equivalence of two versal deformations as a solution of a universal problem is proved. A comparison with the Fialowski method (cf. [9] [10] [11]) is also made in [6]. The normalizing group \( H \) of a torus \( T \) in \( GL(n) \) acts canonically on \( L^T_n \) and its orbits are the isomorphism classes of laws in a good open set. We can define local charts for the space \( L^T_n / H_0 = L^T_n / G_0 \), where \( G_0 \) and \( H_0 \) are unit components of the groups \( GL(n)^T \) and \( H \) respectively. We fix the coordinates in \( L^T_n \) which are labelled with a certain choice of indices called an admisible set \( \mathcal{A} \). Under certain conditions, it defines a sub-scheme \( L^{T,A}_n \) of \( L^T_n (\mathbb{C}) \), called slice, which is transversal to each orbit in a certain open set of \( L^T_n (\mathbb{C}) \). A versal deformation in \( L^{T,A}_n \) at the point \( \Phi_0 \) can be seen as the canonical deformation in a slice at \( \Phi_0 \).

In section 2 this second method, called “local ring method”, will be illustrated by new examples showing the behavior of the slices. The schemes \( L^{T,A}_n \) associated with the torus \( T \), defined by the weights \( \alpha_k = k\alpha_1 \), \( 1 \leq k \leq 4 \) and \( \alpha_5 + k\alpha_1 \), \( k \geq 0 \), are studied by using the induction on central extensions. This allows us to study the relationship between the dimension and the number of essential parameters: we observe an increase and then a decrease in this number. We also present a new series of slices with a unique nilpotent parameter \( t^{p+1} = 0 \), \( t^p \neq 0 \), for each dimension \( n \geq 3p+6 \).
and each \( p > 0 \). These important examples give cohomological obstructions \( \omega_{p+1} \neq 0 \) for any \( p > 0 \) too.

3 – Section 3 develops the third method, which is entirely new and is attached to the global Jacobi scheme. Bearing in mind \( L^T_n = \text{Spec}(A_n) \) with \( A_n = \mathbb{C}[X^k_{ij}] / \text{Jac}_n \), the scheme is not reduced iff \( \sqrt{\text{Jac}_n} \neq \text{Jac}_n \), i.e., there are polynomials \( f \) such that \( f^{p+1} \in \text{Jac}_n \) and \( f^p \notin \text{Jac}_n \). We obtain a result on the existence and the determination of nilpotent elements in \( A_n \) corresponding to the nilpotent parameters found in local rings of the slice. The technique employed consists of applying an elimination procedure to certain coordinates in \( \mathbb{C}[X^k_{ij}] \) modulo the ideal \( \text{Jac}_n \). We proceed by reducing the dimension, in contrary to the local ring method. In the study of \( f_n \), using graded coordinates \( X_{ij} \), for each \( n \) we eliminate in Jacobi relations the coordinates \( X_{ij} \) with \( i + j = n \) except for \( X_{1,n-1} \), keeping the \( X_{hk} \) with \( h + k < n \) and so on. Finally, the remaining coordinates are those which correspond to one essential parameter \( X_{34} \) in the local ring method and a choice of orbital ones (\( X_{23} \) and the \( X_{1k} \)). It is very striking that by using this method we obtain polynomials with great factorizations (monomials in some cases) in the ideals \( \text{Jac}_n \) and \( \sqrt{\text{Jac}_n} \). We find nilpotent elements in \( A_n \) which are irreducible polynomials \( P \) in \( \sqrt{\text{Jac}_n} \). The number of factors in \( P \) minorates the number of irreducible components of the scheme. So, factorizations obtained by this method allow us to predict interesting properties in the scheme such as rigidity, non-reduced points and number of irreducible components, which is not otherwise possible. This global procedure can be completed profitably using the previous local method.

4 – The construction by generators and relations allows us to obtain Lie algebras (up to isomorphism) as quotient a free Lie algebra \( \mathfrak{L}_r \) with \( r \) generators by an ideal \( \mathfrak{J} \). G. Favre gave the first geometrical approach in this context [8]. We obtain a geometrization of the nilpotent quotient algebras \( n = \mathfrak{L}_r / \mathfrak{J} \) with the help of the subscheme \( J_n(\mathfrak{L}_r) \) of a grassmannian constituted by ideals \( \mathfrak{J} \) of codimension \( n \) in \( \mathfrak{L}_r \) containing \( \mathcal{C}^n(\mathfrak{L}_r) \). This subscheme structure, defined only by the simplest polynomial relations \( [x, \mathfrak{J}] \subset \mathfrak{J} \) for all \( x \in \mathfrak{L}_r \), is generally not reduced. The different tori \( T \) of maximal type give a finite number of subschemes \( J_{n}^{T}(\mathfrak{L}_r) \) (up to isomorphism) defined by adding \( T \)-invariance relations for ideals \( \mathfrak{J} \). In this space, we have the natural action of the normalizing group \( N \) of \( T \) in \( \text{Aut}(\mathfrak{L}_r) \). In section 4, we compare \( J_n^{T}(\mathfrak{L}_r) / N_0 \) with \( L_n^{T} / G_0 \) by using adapted slices, under certain conditions on \( T \). The most surprising result obtained here is the identity
between these two types of scheme structures given by the slices. In particular the two Zariski tangent spaces at \( n \) are given by \( H^2(n,n)^T \). These results are formulated in Theorems 4.4 and 4.5. Rigidity and obstruction studies in schemes \( J^T_n(\mathcal{L}_r) \) or \( L^T_n(\mathbb{C}) \) are the same problem. We obtain a different perspective of the same obstruction phenomenon and an original method.

5 – Section 5 focuses on some applications of the equivalence between rigidity in \( L^T_n(\mathbb{C}) \) and in the scheme of ideals. For instance, the rigidity of a “model” in [3], as the nilpotent part of a Borel algebra, follows immediately from Proposition 5.1 applied to the maximal rank case. We give new examples of rigid Lie algebras only defined by one relation and admitting a one dimensional torus \( T \). In a second example, we study the obstruction problem in this new formalism for algebras \( \mathfrak{a}_{4n}(t) \) where the ideal condition \( [x,\mathfrak{J}] \subset \mathfrak{J} \) involves the existence of a 2-order nilpotent parameter in the scheme of ideals for \( n \geq 9 \). The space \( H^2(n,n)^T \) parameterizes the essential local parameters in the two geometrical approaches.

1. Return to Gerstenhaber’s method of formal deformations. The integration of a 2-cocycle

Generalities on deformations in the schemes \( L^R_n \)

Let \( (e_i) \) be a basis of \( \mathbb{C}^m \), \( A \) be a commutative associative \( \mathbb{C} \)-algebra with unity 1 and \( L_m(A) \) be the set of laws of Lie \( A \)-algebras \( \Phi \) defined by their structure constants \( \Phi_{ij}^k \in A : \Phi(e_i, e_j) = \sum_{k=1}^m \Phi_{ij}^k e_k \). These structure constants satisfy the antisymmetry and Jacobi identities, i.e., \( \Phi_{ij}^k + \Phi_{jk}^i = 0 \) and \( \sum_{l} \Phi_{lk}^i \Phi_{kj}^l \Phi_{li}^p + \Phi_{ki}^l \Phi_{lj}^i = 0 \). A morphism of \( \mathbb{C} \)-algebras \( f : A \to B \) gives a map \( L_m(f) : L_m(A) \to L_m(B) \) defined by \( \Phi_{ij}^k \mapsto f(\Phi_{ij}^k) \). The scheme \( L_m \) is a functor from the category of commutative associative \( \mathbb{C} \)-algebras to the category of sets. We have \( L_m \simeq \text{Spec}(\mathbb{C}[X_{ij}^k]/J_m) \) where \( J_m \) is the ideal of the polynomial ring \( \mathbb{C}[X_{ij}^k], 1 \leq i,j,k \leq m \), generated by the antisymmetry and Jacobi polynomials, i.e., \( X_{ij}^k + X_{ji}^k \) and \( \sum_{l} X_{lj}^i X_{lk}^j + X_{jk}^l X_{lk}^i + X_{ki}^l X_{lj}^j \). A point \( \Phi_0 \in L_m(\mathbb{C}) \) can be identified with the ring \( \mathbb{C} \)-morphism \( \lambda : \mathbb{C}[X_{ij}^k]/J_m \to \mathbb{C} \) defined by \( \lambda(X_{ij}^k) = (\Phi_0)_{ij}^k \) or with the maximal ideal \( \text{Ker}(\lambda) \).

Let \( A \) be a local ring with maximal ideal \( m \) and residue field \( \mathbb{C} \). A deformation of a law \( \Phi_0 \in L_m(\mathbb{C}) \) with base \( A \) is a law \( \Phi \in L_m(A) \) such that \( \text{pr}(\Phi_{ij}^k) = (\Phi_0)_{ij}^k \) for all \( i,j,k \) where \( \text{pr} : A \to A/m \simeq \mathbb{C} \) is the quotient
mapping. If \( f : \mathbb{C}[X_{ij}] / J_m \to A \) is a ring morphism such that \( \text{pr} \circ f = \lambda \), it follows that \( f(\text{Ker}(\lambda)) \subseteq \mathfrak{m} \). Thus, using \( \mathcal{O}_{\Phi_0} \) to denote the localized ring of \( \mathbb{C}[X_{ij}] / J_m \) by \( \text{Ker}(\lambda) \), a deformation can be identified with a local ring morphism \( f : \mathcal{O}_{\Phi_0} \to A \). In particular, if \( A = \mathbb{C}[[t]] \) we obtain formal deformations [12].

Let \( \mathfrak{g} = R \oplus \mathfrak{n} \) be a Lie algebra of dimension \( m \) with reducible part \( R \) and nilpotent part \( \mathfrak{n} \) of dimension \( n \). Let \( L_n^R(\mathbb{C}) \) denote the set of laws \( \Phi \in L_n(\mathbb{C}) \) such that \( \delta \cdot \Phi = 0 \) for all \( \delta \in R \). Let \( \Delta_n \) be the ideal of \( \mathbb{C}[X_{ij}] \) generated by the polynomials \( (\delta \cdot \Phi)_{ij} \). We can consider \( L_n^R \) the sub-scheme of \( L_n \) isomorphic to \( \text{Spec}(\mathbb{C}[X_{ij}]/(J_n + \Delta_n)) \). We denote by \( \text{Jac}_n \) the ideal \( J_n + \Delta_n \) and by \( A_n \) the ring \( \mathbb{C}[X_{ij}]/(J_n + \Delta_n) \). A deformation of a law \( \Phi_0 \in L_n^R(\mathbb{C}) \) over a local ring \( A \) given as base, with canonical projection \( \text{pr} : A \to A / \mathfrak{m} \cong \mathbb{C} \), is a law \( \Phi \) of \( L_n^R(A) \) satisfying \( \text{pr}(a_{ij}^k) = (\Phi_0)_{ij}^k \) for all multi-indices. We obtain a deformation functor at \( \Phi_0 \), denoted by \( \text{Def}^R(\Phi_0, -) \), which may also be represented with the local ring \( \mathcal{O}_{\Phi_0}^R \) of the scheme at \( \Phi_0 \); thus \( \text{Def}^R(\Phi_0, A) = \text{Hom}_{\text{loc}}(\mathcal{O}_{\Phi_0}^R, A) \).

An interpretation of different notions attached to the usual schemes \( L_m \) can easily be formulated in the schemes \( L_n^R \). In particular, \( Z^2(\mathfrak{n}, \mathfrak{n})^R \) is the Zariski tangent space of \( L_n^R \) at \( \Phi_0 \), and \( B^2(\mathfrak{n}, \mathfrak{n})^R \) is the tangent space at \( \Phi_0 \) to the orbit of \( \Phi_0 \) by the neutral component \( G_0 = GL(n)_0^R \) under the classical action \( * \). Moreover, we obtain the analogous classical equivalence between the following two conditions: i) the orbit of \( \mathfrak{n} \) is open in \( L_n^R(\mathbb{C}) \) and the scheme is reduced at \( \mathfrak{n} \); ii) \( H^2(\mathfrak{n}, \mathfrak{n})^R = 0 \). [5]

Generalities on formal deformations in the schemes \( L_n^R \)

The classical theory of 1-parameter formal and analytic deformations developed by Gerstenhaber and Nijenhuis-Richardson [12, 13] is valid for the varieties \( L_n^R \) of \( R \)-invariant laws if \( R \) is completely reducible. The space \( C = C(V, V)^R = \bigoplus_p C^p(V, V)^R \), for \( C^p(V, V)^R \) the space of \( p \)-alternating \( R \)-invariant mappings \( f : \wedge^p V \to V \), is a graded Lie superalgebra for the bracket defined in [13] by \( [f, g] = f \circ g - (-1)^{(p-1)(q-1)} g \circ f \) for \( (f, g) \in C^p \times C^q \). The usual differential \( d \) of the Chevalley-Eilenberg’s cohomology on \( C \) satisfies \( df = (-1)^{p-1}[\Phi_0, f] \) for \( f \in C^p \). A deformation of a law \( \Phi_0 \in L_n^R(\mathbb{C}) \) is an analytic curve \( \Phi(t), (|t| < \varepsilon) \), with \( \Phi(0) = \Phi_0 \), contained in \( L_n^R(\mathbb{C}) \).

The expansion \( \Phi(t) = \Phi_0 + t\Phi_1 + \cdots + t^k\Phi_k + \cdots \) with \( \Phi_k \in C^2 \) is a deformation of \( \Phi_0 \) if \( \Phi(t) \in L_n^R(\mathbb{C}[[t]]) \) i.e., if we have \( [\Phi(t), \Phi(t)] = 0. \) Using
symmetry of the bracket for degree 2 and identifying formal developments, we obtain the following sequence of deformation equations:

\[(E_p) \quad d\Phi_p = \frac{1}{2} \sum_{p > k > 0} [\Phi_k, \Phi_{p-k}] =: \omega_p.\]

If we solve these equations successively, we get: for \( p = 1 \), \( d\Phi_1 = 0 \) (\( \Phi_1 \) is a tangent vector); \( p = 2, d\Phi_2 = \frac{1}{2}[\Phi_1, \Phi_1] = \Phi_1 \cdot \Phi_1 \); \( p = 3, d\Phi_3 = [\Phi_1, \Phi_2] \) and so on.

**Lemma 1.1.** — If the equations \((E_k)\) are solvable up to the order \( p - 1 \), i.e., if \( \Phi_0 + \sum_{k=1}^{p-1} t^k \Phi_k \) is a truncated solution or a solution modulo \( t^p \), then \( \omega_p \in Z^3(n,n)^R \). The class \( \bar{\omega}_p \in H^3(n,n)^R \) is called the obstruction to the deformation at order \( p \).

**Proof.** — Similar to the proof in [12] [13]. \( \square \)

A tangent vector in \( Z^2(n,n)^R \), called infinitesimal deformation, is integrable if we can solve all successive equations \((E_p)\) for all \( p > 0 \) or equivalently \( \bar{\omega}_p = 0 \) for all \( p > 0 \). Integrability of \( \Phi_1 \) only depends on its class \( \bar{\Phi}_1 \in H^2(n,n)^R \). Indeed, the action of the group generated by \( 1 + tC[[t]] \otimes C^1 \) on deformations sends the linear part \( \Phi_1 \) to \( \Phi_1 + df \) with \( f \in C^1 \). If \( H^3(n,n)^R = 0 \), then all obstructions are null and each \( \Phi_1 \) is integrable; in this case \( n \) is a simple point of \( L_n^R(C) \).

Consider the local ring \( \mathcal{O} \) at \( \Phi_0 \) in the scheme \( L_n^R \), \( m(\mathcal{O}) \) its maximal ideal and a sequence \( \Phi_k \in C^2(n,n)^R, k \geq 1 \). If \( \tau \) is a free variable, we have:

**Proposition 1.2.** — Let \( \Phi_1 \neq 0, \Phi_2, \cdots, \Phi_p \) be a series of solutions of \((E_k)\), i.e., \( \bar{\omega}_k = 0 \) for \( k \leq p \). There exist an ideal \( \mathfrak{A} \) of \( \mathcal{O} \) and \( t \in m(\mathcal{O}) \), such that \( t^p \notin \mathfrak{A}, t^{p+1} \in \mathfrak{A} \) and \( \mathcal{O}/\mathfrak{A} \simeq C[t]/(t^{p+1}) \simeq C[\tau]/(\tau^{p+1}) \). Moreover, the following conditions are equivalent:

i) \( \bar{\omega}_{p+1} = 0 \);

ii) \( t^{p+1} \neq 0 \) and there exists an ideal \( \mathfrak{B} \) of \( \mathcal{O} \) such that \( \mathfrak{A} = Ct^{p+1} + \mathfrak{B} \).

**Proof.** — Since \( \bar{\omega}_k = 0 \) for \( 1 \leq k \leq p \), we have a deformation \( \Phi_0 + \bar{\tau} \Phi_1 + \cdots \bar{\tau}^p \Phi_p \) with \( \bar{\tau} \equiv \tau \mod (\tau^{p+1}) \) or equivalently there exists a surjective local morphism \( f : \mathcal{O} \to C[\bar{\tau}], f(X_{ij}^k - (\Phi_0)^k_{ij}) = \bar{\tau}(\Phi_1)^k_{ij} + \cdots \). If \( \Phi_1 \neq 0 \), there are indices \( i,j,k \) with \( (\Phi_1)^k_{ij} \neq 0 \). There exist \( u \in \mathcal{O} \) with a triangular system \( f(u_l) = \bar{\tau}^l + \cdots \) for \( 1 \leq l \leq p \) and a linear combination \( t = \sum_{1 \leq l \leq p} \lambda_l u_l \) with \( f(t) = \bar{\tau} \). Then \( f \) is surjective and the ideal \( \mathfrak{A} = \text{Ker}(f) \) satisfies announced statements.

i)\( \Rightarrow \) ii): Condition \( \bar{\omega}_{p+1} = 0 \) means that we have a deformation \( g \) extending \( f \), i.e., we have a deformation \( \Phi_0 + \bar{\tau} \Phi_1 + \bar{\tau}^2 \Phi_2 + \cdots + \bar{\tau}^{p+1} \Phi_{p+1} \) with \( \bar{\tau} \equiv \tau \mod (\tau^{p+2}) \). We have \( \pi \circ g = f \) where \( \pi \) is the canonical projection.
\[ \mathbb{C}[\tau]/(\tau^{p+2}) \to \mathbb{C}[\tau]/(\tau^{p+1}) \] defined by \( \pi(\tilde{\tau}) = \tilde{\tau} \) and \( \text{Ker}(\pi) = \mathbb{C}_{p+1} \). Let \( u \) be such that \( g(u) = \tilde{\tau} \) as above. We have \( \pi(g(A)) = 0 \) and \( g(A) \subset \mathbb{C}_{p+1} \), but \( g(u^{p+1}) = \tilde{\tau}^{p+1} \neq 0 \). Equality \( f(u) = f(t) \) gives \( f(u^{p+1}) = f(t^{p+1}) = 0 \) and \( u^{p+1} \in A \). We obtain \( g(A) = \mathbb{C}_{p+1} \) and the kernel \( \mathcal{B} \) of \( g \) satisfies \( A = \mathbb{C}_{p+1} \oplus \mathcal{B} \). We have \( f(u - t) = 0 \) and then \( t \equiv u + \lambda u^{p+1} \mod (\mathcal{B}) \) with \( \lambda \in \mathbb{C} \), and \( t^{p+1} \equiv qu^{p+1} \mod (\mathcal{B}) \) with \( q = (1 + \lambda u)^{p+1} \) invertible in \( \mathcal{O} \). We have \( qA = A, q\mathcal{B} = \mathcal{B} \) and then \( A = \mathbb{C}_{p+1} \oplus \mathcal{B} \).

ii) \( \Rightarrow \) i): Due to ii), the \( t^k \) for \( k < p + 1 \) are linearly free, modulo \( \mathcal{B} \), and the quotient deformation \( h : \mathcal{O} \to \mathcal{O}/\mathcal{B} \) can be written as \( \Phi_0 + h(t)\Psi_1 + \cdots + h(t)^{p+1}\Psi_{p+1} \) with \( h(t)^{p+1} \not\equiv 0 \), \( \Psi_k \in \mathcal{C}^2 \). The quotient by \( A/B \) gives \( f : \mathcal{O} \to \mathcal{O}/A \), written as
\[
\Phi_0 + f(t)\Psi_1 + \cdots + f(t)^p\Psi_p = \Phi_0 + \tilde{\tau}\Phi_1 + \cdots + \tilde{\tau}^p\Phi_p.
\]
By identifying \( f(t) = \tilde{\tau} \), we obtain \( \Psi_k = \Phi_k \) for \( k < p \) and \( \Psi_{p+1} \) satisfies \( d\Psi_{p+1} = \omega_{p+1} \) where \( \omega_{p+1} \) is constructed with the \( \Phi_k \) for \( k < p \). Consequently \( \omega_{p+1} = 0 \).

The equivalence non (i) \( \Leftrightarrow \) non (ii) gives a correspondence between the cohomological and local formalisms in the obstruction problem.

### The obstruction \( \omega_2 \)

The bracket \( [\ ] \) stabilizes the cocycle subspace \( Z(V,V)^R \) and we can trivially define a bracket \( [\ ] \) on the quotient \( H(V,V)^R = Z(V,V)^R/B(V,V)^R \), \( [\ ] : H^p(V,V)^R \times H^q(V,V)^R \to H^{p+q-1}(V,V)^R \). We obtain \( \omega_2 = \frac{1}{2}[\mathcal{F}_1,\mathcal{F}_1] = S_q(\mathcal{F}_1) \) with the quadratic Rim mapping \( S_q \), giving a rigidity criterion [14]: if \( S_q^{-1}(0) = 0 \), then \( n \) is rigid. The filiform Lie algebras \( f_n \ abolishes non-null obstruction at order \( p > 2 \) for \( n \geq 12 \) and their rigidity study cannot be deduced from this criterion.

### Strong integrability for a 2-cocycle

The dependence on the choice of \( \Phi_2, \Phi_3 \ldots \) for solving equations (1.1) justifies the following definition.

**Definition 1.3.** — A 2-cocycle is called strongly integrable up to the order \( p \) (eventually \( \infty \)) if for each choice of partial solutions \( \Phi_k, 1 \leq k \leq p' \) of \( (E_k) \) up to \( p' \) with \( 1 < p' < p \), there are partial solutions for all equations \( (E_m) \) with \( p' < m \leq p \).
Integrability is generally not strong: if $\Phi_0 = 0$ is the abelian law with $R = 0$, we have $d = 0$ and $d\Phi_1 = 0$. For $\Phi_1 \in L_n(\mathbb{C})$, each $\Phi_2 \in C^2(V, V)$ is a solution of $d\Phi_2 = \frac{1}{2}[\Phi_1, \Phi_1] = 0$. Equation $d\Phi_3 = [\Phi_1, \Phi_2]$ admits a solution for $\Phi_2 = \Phi_1$ but no solution for $\Phi_2 \notin Z^2(\Phi_1, \Phi_1)$.

**Lemma 1.4.** — If $H^2(n, n)^R = \mathbb{C} \Phi_1 \simeq \mathbb{C}$ then the integration of $\Phi_1$ up to order 2, 3 or 4 is strong or impossible; obstructions $\varpi_2$, $\varpi_3$ or $\varpi_4$ only depend on $\Phi_1$.

**Proof.** — For $p = 2$ one sees that $\varpi_2$ only depends on $\Phi_1$. If $\Phi_2$ is a solution, another solution can be written as $\Phi'_2 = \Phi_2 + a\Phi_1 + df$ where $a \in \mathbb{C}$ and $f \in C^1$.

For $p = 3$ we have $\omega'_3 = [\Phi_1, \Phi'_2] = \omega_3 + d(2a\Phi_2 + [\Phi_1, f])$. If $\varpi_3 = 0$, the general solution at order 3 is $\Phi'_3 = \Phi_3 + 2a\Phi_2 + [\Phi_1, f] + b\Phi_1 + dg$ where $b \in \mathbb{C}$, $g \in C^1$, and $\Phi_3$ being a particular one.

For $p = 4$ and by using equality $d[\Phi, f] = -d[\Phi, f] + [\Phi, df]$ for $(f, \Phi) \in C^1 \times C^2$, we obtain:

$$\omega'_4 = [\Phi_1, \Phi'_3] + \frac{1}{2}[\Phi'_2, \Phi'_2] = \omega_4 + d(3a\Phi_3 + [\Phi_2, f]) + (2b + a^2)\Phi_2 + [\Phi_1, g + af] + \frac{1}{2}[df, f]).$$

The obstruction $\varpi_4$ only depends on $\Phi_1$ and the general solution at order 4 is $\Phi'_4 = \Phi_4 + 3a\Phi_3 + [\Phi_2, f] + (2b + a^2)\Phi_2 + [\Phi_1, g + af] + \frac{1}{2}[df, f] + c\Phi_1 + dh$ where $c \in \mathbb{C}$, $h \in C^1$, and $\Phi_4$ being a particular one. 

**Remark 1.5.** — The difference $\omega'_5 - \omega_5 = dl + \frac{1}{2}[[\Phi_1, [df, f]] - [f, d[\Phi_1, f]]]$ where $l \in C^2$ is not necessarily a coboundary.

**Application to the study of obstructions for $f_n$ ($n \geq 7$)**

If $T$ is the torus defined on $n$ by the weights $k\alpha_1$, for $1 \leq k \leq n$, a 2-cochain $\Phi \in C^2(n, n)^T$ is written $\Phi(e_i, e_j) = A_{ij}e_{i+j}$ for $i < j < i + j \leq n$, where $(e_k)_{1 \leq k \leq n}$ is a basis of $n$. The differential $d$ related to $n$ with structure constants $c_{ij}$ gives:

$$d\Phi_{ijk} = - c_{ij}A_{i+j,k} - c_{jk}A_{j+k,i} + c_{ik}A_{k+i,j}$$

$$+ c_{i,j+k}A_{jk} - c_{j,i+k}A_{ik} + c_{k,i+j}A_{ij}. \quad (1.2)$$

If $\Phi \in Z^2(f_n, f_n)^T$, then it satisfies $A_{3k} + A_{2,k+1} - A_{1,k+2} = A_{2k} - A_{1k}$ for $k \geq 3$ and $A_{jk} = A_{j+1,k} + A_{j,k+1} = A_{j+2,k} + A_{j,k+2}$ for $k > j \geq 3$.

We deduce from this the equalities $A_{pq} = 0$ for $3 < p < q$ and $A_{3q} = \ldots$
A_{34} for 3 < q < n - 2. Moreover, writing A_{1q} for q > 4 depending on A_{12}, A_{13}, A_{14}, A_{2k}, 2 < k < n - 1 and A_{34}, we also deduce that there are n linear independent parameters.

As the dimension of B^2(f_n, f_n)^T is n - 1 for n ≥ 7, we obtain H^2(f_n, f_n)^T ∼ ℂ. Moreover, a class of 2-cocycle is not null if and only if A_{34} ≠ 0 [1]. We choose Φ_1 by:

\begin{align*}
A_{1j} &= A_{23} = 0, \quad A_{2j} = 4 - j \quad (3 < j < n - 1), \\
A_{3j} &= 1 \quad (3 < j < n - 2), \quad A_{ij} = 0 \quad (3 < i < j).
\end{align*}

Having solved equations (E_k) with partial solutions Φ_2, Φ_3, Φ_4, we state:

**Proposition 1.6.**

1. For n ≥ 7, the Lie algebra f_n satisfies H^2(f_n, f_n)^T = ℂΦ_1 with the 2-cocycle Φ_1 defined in (1.3).
2. If 7 ≤ n ≤ 11, then Φ_1 is integrable and L_n^T(ℂ) is smooth at f_n; f_n is not rigid.
3. If n ≥ 12, then Φ_1 is strongly integrable up to the order 4 and the sequence of obstructions satisfies: \(\varpi_2 = \varpi_3 = \varpi_4 = 0, \varpi_5 ≠ 0\). The algebra f_n is rigid in L_n^T(ℂ).

We have the same results for the semi-direct product T ⊕ f_n in L_{n+1}(ℂ).

**Proof.** — A direct cohomological proof is obtained here with the help of Lemma 1.4. We can also apply Proposition 1.2 to the versal deformation Φ_0 + tΦ_1 + \cdots + t^4Φ_4 where t ∈ m(𝒪), obtained in [6]. Thus the condition non(ii) \(t^5 = 0\) and \(t^4 ≠ 0\), gives \(\varpi_5 ≠ 0\). □

### 2. The local ring method for studying the schemes L_n^T

**Generalities on the local ring method**

In this section, let T be a torus over ℂ^n with weights \(α_i > 0\) (i.e., \(α_i(t) > 0\) for \(t ∈ T, \forall i\)), let \(n(α_i)\) be the multiplicity of \(α_i\) and let \(Σ_n(T)\) be the set of laws on which T is maximal. Consequently, the T-invariant laws are nilpotent. In addition we suppose the following conditions: for each \(n ∈ Σ_n(T)\), the multiplicities of T-weights appearing in the quotient module \(n/[n, n]\) are 1. Thus, there are generators \(e_i\) of n belonging to a diagonalizing basis such that \(n(α_i) = 1, \forall i\).

This involves equality Der(n)^T = T and the dimension of the \(G_0\)-orbit, isomorphic to \(G_0/\text{Aut}(n)^T\), is \(\sum_i n(α_i)^2 - \text{dim} T\). We have the following properties, cf. [6]:

\[\text{Der}(n)^T = T, \quad \text{dim}(G_0/\text{Aut}(n)^T) = \sum_i n(α_i)^2 - \text{dim} T.\]
1. The isomorphic classes in $\Sigma_n(T)$ are the $H$-orbits, equal to finite unions of $G_0$-orbits;
2. $\Sigma_n(T)$ is the Zariski open set equal to the union of the $G_0$-orbits of maximal dimension. This dimension is $\sum_i n(\alpha_i)^2 - \dim(T)$;
3. $\Sigma_n(T)$ is the set of nilpotent laws $n \in L^T_n(\mathbb{C})$ for which the semidirect product $g = T \oplus n$ is complete (i.e., the derivations are inner and the center is zero). These Lie algebras satisfy the reduction theorem [6], so the local study of $g$ in $L_m(\mathbb{C})/GL(m)$ is equivalent to that of $n$ in $L^T_n(\mathbb{C})/G_0$. Assuming that multiplicities of all the weights $\alpha_i$ are 1, we can find slices in $L^T_n(\mathbb{C})$ playing the part of local charts in the quotient $\Sigma_n(T)/G_0 \subset L^T_n(\mathbb{C})/G_0$.

**Admissible part $\mathcal{A}$ associated with $\Phi_0$ and slice**

Since weights are distinct, it follows that the elements of $G_0$ are $s = (s_k)_{1 \leq k \leq n}$ with $s_k \in \mathbb{C}^*$ operating on $\Phi$ as

\[(s \ast \Phi)^k_{ij} = \frac{s_k}{s_i s_j} \Phi^k_{ij}.\]

By the unicity of the weights, $\Phi^k_{ij} \neq 0$ involves $\alpha_i + \alpha_j = \alpha_k$. Then, the triple indices giving non-null values $\Phi^k_{ij}$ can be defined by the pairs $(i < j)$. So we can write $\Phi_{ij}$ the coordinates instead of $\Phi^k_{ij}$ and let $C$ be the set of all pairs $(i < j)$.

**Definition 2.1.** — A subset $\mathcal{A} \subset C$ is called an admissible part associated with a law $\Phi_0 \in L^T_n(\mathbb{C})$ if the equation system $s \ast \Phi_0 = \Phi_0$ is equivalent to $(s \ast \Phi_0)^k_{ij} = (\Phi_0)^k_{ij}$, $(ij) \in \mathcal{A}$, and if $\mathcal{A}$ is minimal for this property [6].

One sees that all admissible sets $\mathcal{A}$ associated with $\Phi_0$ are contained in the set $\mathcal{I}(\Phi_0)$ of all pairs $(i < j)$ with $(\Phi_0)^k_{ij} \neq 0$. Using (2.1) we see that $\mathcal{A}$ can index a minimal system of equations equivalent to the following: $s_k = s_i s_j$, $(i < j) \in \mathcal{I}(\Phi_0)$.

In the same way, we can use the equations defining a derivation $\delta \in (\text{Der} \, n)^T = T$, i.e., $(\delta \Phi_0)^k_{ij} = (\delta^k_{ij} - \delta^i_j - \delta^j_i)(\Phi_0)^k_{ij} = 0$, $(i < j) \in \mathcal{I}(\Phi_0)$ where $\delta$ is given by its diagonal matrix $(\delta^i_i)$ over the basis $(e_i)$ for $1 \leq i \leq n$. If $T_n$ is the full torus on $\mathbb{C}^n$ diagonalized by the $e_i$ with weights $\varepsilon_i$ for $1 \leq i \leq n$, then we have $\delta^i_i = \varepsilon_i(\delta)$. The system is equivalent to

\[(2.2) \quad \varepsilon_k = \varepsilon_i + \varepsilon_j, \quad (i < j) \in \mathcal{A},\]

and defines the torus $T = \cap \text{Ker}(\varepsilon_k - \varepsilon_i - \varepsilon_j) \subset T_n$ with the weights $\alpha_i = \varepsilon_i |T$.
If $\mathcal{A}$ is an admissible part associated with $\Phi_0 \in L^T_n(\mathbb{C})$ we define the slice associated with $(\Phi_0, \mathcal{A})$ as the subscheme $L^T_{n, \Phi_0}$ of $L^T_n = \text{Spec}(A_n)$ defined by the quotient of $A_n$ by the ideal generated by the $X_{ij}^k = (\Phi_0)_i^k$, $(ij) \in A$, i.e., $L^T_{n, \Phi_0} \simeq \text{Spec}(\mathbb{C}[X_{ij}^k]/ \text{Jac}_n + (X_\alpha - (\Phi_0)_\alpha)_{\alpha \in A})$. Under the above hypotheses, if $\Phi_0 \in \Sigma_n(T)$ and if $\mathcal{A}$ is an admissible part associated with $\Phi_0$, the slice $L^T_{n, \Phi_0}$ satisfies the following properties, cf. [6]:

i) $\Phi \in L^T_n(\mathbb{C})$ admits $\mathcal{A}$ as an admissible part iff we have $\Phi^k_{ij} \neq 0$ for all $(ij) \in A$;

ii) all laws of $L^T_{n, \Phi_0}(\mathbb{C})$ admit $\mathcal{A}$ as an admissible part;

iii) $L^T_{n, \Phi_0}(\mathbb{C})$ is contained in $\Sigma_n(T)$ and its isomorphism classes are the traces of the $H/H_0$-orbits in $\Sigma_n(T)/H_0 = \Sigma_n(T)/G_0$;

iv) $\Phi$ admits $\mathcal{A}$ as an admissible part iff there is $s \in G_0$ such that $s \ast \Phi \in L^T_{n, \Phi_0}(\mathbb{C})$;

v) $H^2(\Phi_0, \Phi_0)^T$ is the Zariski tangent space of $L^T_{n, \Phi_0}$ at $\Phi_0$.

As a consequence, fixing $\mathcal{A}$ and $\Phi$ admitting $\mathcal{A}$ as an admissible part, all these schemes $L^T_{n, \Phi}$ are conjugated to the scheme $L^T_{n, \Phi}$ defined by conditions $X_\alpha = 1$ for $\alpha \in A$. Each admissible part $\mathcal{A}$ corresponds to an open set $\Omega_\alpha(A)$ defined by the points such that $X_\alpha \neq 0$ for all $\alpha \in \mathcal{A}$, where $X_\alpha$ is the residual class in $\mathbb{C}$ at the point and $\Sigma_n(T) = \bigcup A_\alpha \Omega_\alpha(A)$. Moreover, the schemes $L^T_{n, \Phi}(\mathbb{C}) \simeq \Omega_\alpha(A)/G_0$ constitute local affine charts of $\Sigma_n(T)/G_0$. The slices $L^T_{n, \Phi}(\mathbb{C})$ in $\Sigma_n(T)/G_0$ define continuous families of $L^T_n/G_0$ because the orbits are finite, the finite group $H/H_0$ is contained in the group of permutations of weights which are not the sum of two weights.

The local ring of the slice $L^T_{n, \Phi}$ at $\Phi_0$, denoted by $\mathcal{O}^T_{n, \Phi}$ or simply $\mathcal{O}$, can be directly constructed from antisymmetry, Jacobi and $T$-invariance relations and the fixation of the structure constants $(\Phi_0)_\alpha$ for $\alpha \in A$. We can choose the fixed values 1 for these coordinates. This local ring gives the universal deformation of $\Phi_0$ in the slice or equivalently the versal deformation associated with $\mathcal{A}$ in the scheme $L^T_n$, [6].

Weight paths and filiations $\mathcal{A}_n \longrightarrow \mathcal{A}_{n+1}$

The local study of laws of $L^T_n$ is made in relation to the construction of nilpotent Lie algebras by central extensions:

$$0 \longrightarrow \mathbb{C}e_{n+1} \longrightarrow (\mathbb{C}^{n+1}, \Phi_{n+1}) \longrightarrow (\mathbb{C}^n, \Phi_n) \longrightarrow 0.$$

We choose non-trivial extensions $\Phi_{n+1}$ of the law $\Phi_n$ with weights of multiplicities 1. This allows us to write the coordinates in the form $X_{ij}$. The
torus $T$ is extended on $\mathbb{C}^{n+1}$ by adding a weight $\alpha_{n+1} \in T^*$ appearing in the $T$-module structure of the two-homological group $H_2(\Phi_n)$, which is not null [7]. There are different possible choices of paths $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ called “paths of weights” associated with the various weights of the $T$-modules $H_2(\Phi_p) \neq 0$ for $p \leq n$. A path of weights is said to be simple if all weights are different. With this procedure, if $\Sigma_{n_0}(T) \neq \emptyset$ for the smallest integer $n_0$ called an initialization of the path, then it is possible to keep the same properties for all $n > n_0$. This enables us to construct the slices $L_n^T.A_n$ by induction on $n$: we add one dimension with a vector $e_{n+1}$ of weight $\alpha_{n+1}$ and the choice of a pair $(i_0 < j_0)$ such that $\alpha_{i_0} + \alpha_{j_0} = \alpha_{n+1}$ gives $A_{n+1} = A_n \cup\{(i_0, j_0)\}$. Hence, it appears:

- the new coordinates $X_{ij}$ with $\alpha_i + \alpha_j = \alpha_{n+1}$ ($X_{i_0, j_0}$ being fixed to 1);
- the new Jacobi polynomials $J^{n+1}_{ijk}$, $i < j < k$ with $\alpha_i + \alpha_j + \alpha_k = \alpha_{n+1}$.

An example of the induction process with a 3-order nilpotent parameter

We consider the following sequence of simple weights $\alpha_i$ defining the torus $T$:

$$(2.3) \quad \alpha_i = i\alpha_1 \quad \text{for} \ 1 \leq i \leq 4, \ \alpha_{5+k} = \alpha_5 + k\alpha_1 \quad \text{for} \ k \geq 0.$$  

The set $\Sigma_7(T)$ is the orbit of the Lie algebra $\mathfrak{a}_{5,7}$: $[e_1, e_i] = e_{i+1}$, $i = 2, 3, 5, 6$, $[e_2, e_5] = e_7$. We can take $A_7 = \{(12), (13), (15), (16), (25)\}$ and $n_0 = 7$ is the initialization of the central extension induction process defined by $A_{n+1} = A_n \cup\{(1n)\}$ for $n \geq n_0$.

For $n = 8$ we add the weight $\alpha_8 = \alpha_5 + 3\alpha_1$ and the coordinates $X_{17} = 1$, $X_{26}$ and $X_{35}$. Jacobi relation $J_{125}$ gives $X_{26} = -X_{35} + X_{25}$ and we have a free parameter $X_{35} = t$ and $X_{26} = 1 - t$. For $n = 9$ we add $\alpha_9 = \alpha_5 + 4\alpha_1$ and the coordinates $X_{18} = 1$, $X_{27}$, $X_{36}$ and $X_{45}$. Jacobi relations $J_{126}$ and $J_{135}$ give: $X_{36} = t - u$ and $X_{27} = 1 - 2t + u$, where $X_{45} = u$ is a new free parameter. For $n = 10$ we add $\alpha_5 + 5\alpha_1$ and the coordinates $X_{19} = 1$, $X_{28}$, $X_{37}$ and $X_{46}$. The Jacobi relations $J_{127}$, $J_{136}$ and $J_{145}$ give $X_{46} = u$, $X_{37} = t - 2u$, $X_{28} = 1 - 3t + 3u$. The last Jacobi relation $J_{235}$ gives:

$$(2.4) \quad u(2 + 3t) = 3t^2,$$

which is the equation of the hyperbola $xy = 1$ applying the affine change of variables $x = 9t/4 + 3/2$, $y = u - t + 2/3$. For $n = 11$ we add the weight $\alpha_5 +$
6α₁ and the coordinates X₁₁₀ = 1, X₂₂, X₃₈ and X₄₇. Relations J₁₂₈, J₁₃₇ and J₁₄₆ give X₄₇ = u, X₃₈ = t − 3u and X₂₂ = 1 − 4t + 6u. Relation J₂₃₆ gives 2ut = 3u² and J₂₄₅ doesn’t provide any new information. Projecting (2.4) in the residual field O/m(O) at each point of the slice, we obtain \( \bar{u}(2 + 3t) = 3t² \) and \( 2 + 3t \neq 0 \) in \( \mathbb{C} \). The scheme is contained in the principal open set defined by \( 2 + 3t \neq 0 \). In this open set, the slice is defined by the relations \( u = \frac{3t²}{2 + 3t} \) and \( t³(4 - 3t) = 0 \); it is the spectrum of \( \mathbb{C}[t, \frac{1}{2 + 3t}] / \langle t³(4 - 3t) \rangle \) isomorphic to \( \langle \mathbb{C}[t] / \langle t³ \rangle \rangle \oplus \mathbb{C} \). The slice has two points for \( t = 0 \) and \( \bar{t} = 4/3 \) which give the following T-rigid laws:

\[ \begin{align*}
(1) \quad & a_{5,11} \quad \text{for} \quad \bar{t} = 0, \quad \text{with nilpotent element} \quad t^3 = 0 \quad (\text{non-reduced case}); \\
(2) \quad & a'_{5,11} \quad \text{for} \quad \bar{t} = 4/3 \quad (u = 8/9, \text{regular case}).
\end{align*} \]

The scheme \( L^{n,A_{11}}_{11} \) consists of \( a_{5,11}(t) \) with \( t³ = 0 \) \((u = \frac{3}{2} t² ≠ 0)\) and \( a'_{5,11} \) for \( t = 4/3 \).

For \( n ≥ 12 \) the algebra \( a'_{5,11} \) doesn’t have central extensions in the slice defined by \( A_{12} = A_{11} \cup \{(1, 11)\} \) but the algebra \( a_{5,n} \) belongs to the slice associated with \( A_n \). We can state the following result, which can be proved by induction on \( n ≥ 11 \):

**Proposition 2.2.** — The slices \( L^{n,A_n}_{n} \) defined above with (2.3) and admissable part \( A_n = \{(25), (1i) \} \) for \( i ≠ 4, 1 < i < n \) for \( n ≥ 7 \), are affine schemes defined by the rings: \( \mathbb{C} \ (n = 7) \), \( \mathbb{C}[t] \ (n = 8) \), \( \mathbb{C}[t, u] \ (n = 9) \), \( \mathbb{C}[x, y] / \langle x y - 1 \rangle \ (n = 10) \), \( \mathbb{C}[t, x, \frac{1}{2 + 3t}] / \langle t³(4 - 3t) \rangle \ (n = 11) \) and \( \mathbb{C}[t] / \langle t³ \rangle \) \((n ≥ 12)\).

For \( n ≥ 11 \), this scheme is not reduced at \( a_{5,n} \). We have \( t³ = 0, \ t² ≠ 0 \) and the versal deformation at point \( \bar{t} = 0 \), associated with \( A_n \) in \( L^{n}_{n} \), is:

\[ \begin{align*}
X_{ij} = 1 \quad \text{for} \quad (ij) \in A_n; \quad X_{2,5+k} = 1 - kt + \frac{3}{4} k(k - 1) t² \quad \text{for} \quad n - 7 ≥ k ≥ 0; \\
X_{3,5+k} = t - \frac{3}{2} k t² \quad \text{for} \quad n - 8 ≥ k ≥ 0; \quad X_{4,5+k} = \frac{3}{2} k² \quad \text{for} \quad n - 9 ≥ k ≥ 0.
\end{align*} \]

**Other slices for (2.3)**

A complete study of \( \Sigma_n/G₀ \) gives the slices associated with all possible admissable sets. Some of them for \( n = 11 \) include:

- The slice associated with \( A'_{11} = \{(1j), (2k), j ≠ 4, 6, 8; k = 5, 8, 9\} \) isomorphic to \( \text{Spec}(\mathbb{C}[t] / \langle t³(1 - t)(4 - 3t) \rangle) \) where \( t = X_{35} \). It contains a new T-rigid law for \( t = 1 \).
- The slice associated with \( A''_{11} = \{(1j), (25), (35), \text{for} \ j ≠ 2, 4\} \) isomorphic to \( \text{Spec}(\mathbb{C}[u] / \langle u²(3u - 2) \rangle) \) where \( u = X_{45} \); \( u \) is 2-nilpotent at the new point \( X_{45} = X_{12} = 0 \).
• The slice associated with $\mathcal{A}_{11}^{'''} = \{(1j), (25), (35), (45), j > 4\}$ isomorphic to $\text{Spec}(\mathbb{C}[v]/(v(2 - 3v)))$ where $X_{12} = 2v$, $X_{13} = v$. For $\bar{v} = 0$ we have $X_{ij} = 1$ for $i \leq 4$ and $j > 4$ and 0 otherwise with $i < j$. This rigid law can be extended in the slices associated with $\mathcal{A}_n^{'''} (n \geq 11)$.

Remark 2.3. — a) Different slices containing a same point are generally not isomorphic as scheme and the number of their components can be different. By the universality of versal deformation [6], completed local rings of slices at this point are isomorphic.

b) The slice associated with $\mathcal{A}_{11}^{'''}$ gives rigid laws with different numbers of generators. If $v \neq 0$ we obtain a rigid law with 3 generators and if $v = 0$ the rigid law has 5 generators.

Nilpotent parameter at each order in local rings of some slices

We begin with the following result:

Lemma 2.4. — Let $\mathfrak{a}$ be a Lie algebra admitting a maximal torus with weights $\alpha_i$ on a diagonalizing basis $e_i$, $1 \leq i \leq k$. We suppose $e_1 \not\in [\mathfrak{a}, \mathfrak{a}]$ and let $V$ be a $\mathbb{C}$-space generated by vectors $e_{k+i}$ for $1 \leq i \leq p$, then:

i) We obtain a semi-direct Lie algebra product $g = \mathfrak{a} \oplus V$ of $\mathfrak{a}$ by the abelian one $V$ with the brackets: $[e_1, e_j] = e_{j+1}$ for $k < j < k + p$, $[e_i, e_j] = 0$ for $i > 1, j > k$.

ii) $g$ admits a maximal torus defined by the weights $\alpha_i$; for $1 \leq i \leq k$, $\beta + j\alpha_1$; for $0 \leq j < p$.

In this section we work on sequences of weights of rank two, generalizing (2.3):

(2.5) $\alpha, m_2\alpha, \cdots, m_r\alpha, \beta, \beta + \alpha, \beta + 2\alpha, \cdots, \beta + s\alpha$.

with integers $m_1 = 1 < m_2 < \cdots < m_r$.

Notations

We write $\alpha_i = i\alpha$ for $i \in \{m_1, \cdots, m_r\}$ and $\alpha_{b+k} = \beta + k\alpha$ where $b$ is an integer greater than $m_r$. The coordinates are $X_{ij}$ for $i, j, i + j \in \{m_1, \cdots, m_r\}$ and $X_{i,b+k}$ for $i \in \{m_1, \cdots, m_r\}$, $k + i \leq s$. The indices are
not consecutive but the additive writing is kept with \( X_{ij} = X_{ij}^{i+j} \). Jacobi relations for \( 1 \leq i < j \leq m_r \) and \( i + j + k \leq s \) are:

\[
J_{i,j,b+k} = X_{ij}X_{i,j,b+k} - X_{j,b+k}X_{i,b+j+k} + X_{i,b+k}X_{j,b+k+i} = 0.
\]

**Lemma 2.5.** — The sequence (2.5) is a simple path of weights for all \( s \geq m_r \).

**Proof.** — We observe that the law defined by \( X_{ij} = 0 \) for \( i < j \leq m_r \) and \( X_{i,b+k} = 1 \) for \( i = m_l \leq m_r \) and \( i + k \leq s \) satisfies (2.6). Any maximal torus on this algebra containing \( T \) defined by (2.5) commutes with \( T \) and has weights \( \alpha'_i \) on the same basis \( e_i \). If \( s \geq m_r \) we obtain \( ade_i(e_b) = (ade_1)^i(e_b) = e_{b+i} \) for any \( i \in \{ m_1, \ldots, m_r \} \) and the relations \( \beta' + \alpha'_i = \beta' + i\alpha' \) involve \( \alpha'_i = i\alpha' \). Consequently, we have a weight system of rank 2 equal to (2.5).

A particular case of special interest is provided by the sequence of weights:

\[
(2.7) \quad \alpha, \ 2\alpha, \ 3\alpha, \ 5\alpha, \ldots, (2p + 1)\alpha, \ \beta, \ \beta + \alpha, \ldots, \beta + s\alpha.
\]

All \( T \)-invariant laws \( n \) for (2.7) trivially satisfy:

1) The sum \( a = \mathbb{C}e_2 \oplus (\oplus_{0 \leq i \leq p} \mathbb{C}e_{2i+1}) \) is a subalgebra of \( n \) whose brackets are given by \( [e_2, e_{2i-1}] = c_i e_{2i+1}, 1 \leq i \leq p \). If \( c_i \neq 0 \) for \( 1 \leq i \leq p \), then \( a \) is isomorphic to the well-known filiform Lie algebra \( f^0_{p+2} : [x_1, x_i] = x_{i+1}, 1 < i < p + 2 \).

2) The Lie algebra \( n \) is the semi-direct product of \( a \) by the abelian ideal \( \oplus_{k=0}^s \mathbb{C}e_{b+k} \) (this is generally true for (2.5)).

3) The following \( n = p + 3 + s \) dimensional Lie algebra with \( a \simeq f^0_{p+2} \) is \( T \)-invariant:

\[
b_{p,n} = \begin{cases} 
[e_1, e_2] = e_3 \\
[e_{2i}, e_{2i-1}] = e_{2i+1} & 1 < i \leq p \\
[e_i, e_{b+k}] = e_{b+k+i} & i = 1, 2; k + i \leq s.
\end{cases}
\]

We can define by induction the following sets \( A_n^p \) for \( n \geq p + 5 \):

\[
A_{n+1}^p = \{(1, 2), (1, b), (1, b + 1), (2, b), (2, 2l - 1) \text{ for } 2 \leq l \leq p\},
\]

\[
A_{n+5}^p = \{(1, 2), (1, b), (1, b + 1), (2, b), (2, 2l - 1) \text{ for } 2 \leq l \leq p\},
\]

**Theorem 2.6.** — The sets \( A_n^p \) \( (p \geq 1) \) are admissible sets of \( b_{p,n} \) for \( n \geq p + 5 \) and the slices associated are the spectrum of the following rings with \( \tau \) a free variable: \( \mathbb{C} (n = p + 5) \), \( \mathbb{C}[\tau] (p + 6 \leq n \leq 3p + 5) \) and \( \mathbb{C}[\tau]/(\tau^{p+1}) (n \geq 3p + 6) \).

For \( n \geq p + 6 \), a versal deformation of \( b_{p,n} \) associated with \( A_n^p \) in \( L_n^T \) is given by:
\( X_{ij} = 1 \) for \((ij) \in A_n^p\), \(X_{2b+1,b+k} = 1 - kt\) \((0 \leq k \leq s - 2)\), \(X_{2l+1,b+k} = ct^l\) \((0 \leq l \leq p, 2l + 1 + k \leq s)\), where \(c_t\) is obtained in (2.15).

The local parameter \(t = X_{2b}\) is free for \(p + 6 \leq n \leq 3p + 5\) and nilpotent \(t^{p+1} = 0\), \(t^p \neq 0\) for \(n \geq 3p + 6\).

**Proof.** — For \(n \geq p + 5\), one checks that (2.7) defines a maximal torus on \(b_{p,n}\) and \(A_n^p\) is admissible. The coordinates of the scheme \(L_n^T\) are \(X_{12}, X_{2i}, i \in \{3, 5, \cdots, 2p - 1\}\) and \(X_{i,b+j}, i \in \{1, 2, 3, 5, \cdots, 2h + 1\}\) with \(h = \min(p, \lfloor \frac{s-1}{2} \rfloor)\) and \(i + j \leq s\). After fixing \(X_{ij} = 1\) for \((i, j) \in A_n^p\), the remaining parameters are \(X_{2b+j}\) for \(j \geq 1\) and \(X_{i,b+j}\) for \(i = 3, 5, \cdots, 2h + 1\) and \(i + j \leq s\).

If \(i = 1\), the equation (2.6) becomes:

\[
(2.9) \quad X_{1j}X_{j+1,b+k} = X_{j,b+k} - X_{j,b+k+1}
\]

where \(j\) is odd or equal to 2 and \(j + k + 1 \leq s\). This gives \(X_{j,b+k+1} = X_{j,b+k}\) for odd \(j \geq 3\) and we have:

\[
(2.10) \quad X_{j,b+k} = X_{j,b}, \quad j = 3, 5, \cdots, 2h + 1; \quad 0 \leq k \leq s - j.
\]

If \(j = 2\) in (2.9), using (2.10) we obtain the relation \(X_{2,b+k+1} = X_{2,b+k} - X_{3,b}\). By the repeated application of this latter relation, we obtain the following:

\[
(2.11) \quad X_{2,b+m} = 1 - mX_{3,b}, \quad 0 \leq m \leq s - 2.
\]

If \(i = 2\) in (2.6), from the relations (2.10) and (2.11) it follows that:

\[
(2.12) \quad X_{2,j}X_{j+2,b+k} = -jX_{j,b}X_{3,b}, \quad j = 3, 5, \cdots, 2p + 1; \quad j + 2 + k \leq s.
\]

Observe that \(X_{2,j} = 1\) if \(j + 2 \leq 2p + 1\) and \(X_{j+2,b} = 0\) if \(j + 2 > 2p + 1\). Using (2.10), the relation (2.12) becomes:

\[
(2.13) \quad X_{j+2,b} = -jX_{j,b}X_{3,b}, \quad j \leq 2p - 1;
\]

\[
(2.14) \quad 0 = -(2p + 1)X_{2p+1,b}X_{3,b}, \quad j = 2p + 1.
\]

In view of these equalities from \(j = 3\) to \(2m - 1\), we obtain for \(1 \leq m \leq p\):

\[
(2.15) \quad X_{2m+1,b} = (-1)^m-1 \frac{(2m-1)!}{2^{m-1}(m-1)!} (X_{3,b})^m.
\]

Thus, if \(s \leq 2p + 2\), we have \(2m + 1 \leq 2p + 2\) and formula (2.15) gives the components \(X_{2l+1,b}\) for \(5 \leq 2l + 1 \leq 2p + 1\). If \(s \geq 2p + 3\), (2.14) becomes:

\[
(2.16) \quad (X_{3,b})^{p+1} = 0.
\]

Finally if \(i > 2\) the equality (2.6) becomes \(X_{j,b+k}X_{i,b+j+k} = X_{i,b+k}X_{j,b+k+i}\), that is trivial with (2.10). \(\Box\)
Remark 2.7. — Case $p = 1$. We denote by $a_{4,n}$ for $n \geq 7$, the Lie algebra defined by the non null brackets $[e_i, e_j] = e_{i+j}$ for $i < j$ if $(i = 1, j \neq 3)$, $(i = 2, j \geq 4)$ or $(i = 3, j \geq 4)$. It is isomorphic to $b_{1,n}$ and its versal deformation $a_{4,n}(t)$ defined by $t = X_{34}$ is a continuous family for $n = 7, 8$. Condition $t^2 = 0$ for $n \geq 9$ corresponds to the non null quadratic Rim function calculated in [2].

Corollary 2.8. — The Lie algebra $b_{p,n}$ satisfies:

i) for $n \geq p + 6$, $H^2(b_{p,n}, b_{p,n})^T = \mathbb{C} \Phi_1$;

ii) for $n \geq 3p + 6$, we have a truncated deformation $\sum_{k=0}^{p} \bar{\tau}^k \Phi_k$ with an obstruction $\bar{\omega}_{p+1} \neq 0$ in $H^3(b_{p,n}, b_{p,n})^T$.

Proof. — Applying Proposition 1.2 to the slices, Theorem 2.6 gives deformations satisfying $t^{p+1} = 0$ and $t^p \neq 0$. Consequently, we obtain $\bar{\omega}_{p+1} \neq 0$. □

Algebras $g_{p,n} = T \oplus b_{p,n}$, defined as semi-direct products by the torus $T$ on $b_{p,n}$, satisfy hypotheses of the reduction theorem in [6]. Thus, the local study of $g_{p,n}$ in $L_{n+2}$ is equivalent to that of $b_{p,n}$ in $L_n^T$. We can summarize it in the following statement:

Corollary 2.9. — The group $H^2(g_{p,n}, g_{p,n})$ is null for $n = p + 5$ and equal to $\mathbb{C} \Phi_1 \neq 0$ for $n > p + 5$. Moreover,

i) for $n = p + 5$, $g_{p,n}$ is rigid and $L_{p+7}$ is reduced at this point;

ii) for $p + 6 \leq n \leq 3p + 5$, $\Phi_1$ is tangent to a continuous family in $L_{n+2}$;

iii) for $n > 3p + 5$, $g_{p,n}$ is rigid in $L_{n+2}$ and $\Phi_1$ is tangent to a versal deformation $\sum_{k \geq 0} t^k \Phi_k$ with $t^{p+1} = 0$, $t^p \neq 0$. This corresponds to an obstruction $\bar{\omega}_{p+1} \neq 0$ in $H^3(g_{p,n}, g_{p,n})$ and $\bar{\omega}_k = 0$ for $k \leq p$.

Remark 2.10. — Obstructions to deformation equations can appear at each order in schemes $L_n^T$ and $L_m$.

3. Elimination procedure in the search for nilpotent elements in global schemes $L_n^T$

An existence theorem

Let $\Phi_0$ be a law, $A$ be an admissible set associated with $\Phi_0$, $\mathcal{O}$ be its local ring in the slice $L_n^{T,A}$, $U \subset \mathbb{C}[X_{ij}]$ be the ideal generated by $X_n - 1$ for $\alpha \in A$ and $\pi$ be the projection of $A_n$ to the quotient $\overline{A}_n = A_n/U$ where $\overline{U} = U/\text{Jac}_n$. The image of $\sqrt{\mathcal{O}} \subset A_n$ by $\pi$ is contained in $\sqrt{\mathcal{O}} \subset \overline{A}_n$. If
Φ₀ is rigid, then \( m(O) = \sqrt{0} \) in \( O [6] \). Writing \( O = (A_n)_M \) as the localized ring by the maximal ideal \( M \) associated with \( Φ₀ \), each \( u ∈ m(O) \) is \( v/f \) where \( v ∈ M \) and \( f \not∈ M \); \( u \) and \( v \) have the same nilpotency order \( l : u^l = 0, u^{l-1} \neq 0 \). We are looking for \( P ∈ \mathbb{C}[X_{ij}] \) such that \( π(\bar{P}) = v \). Moreover, if \( P ∈ \sqrt{\text{Jac}_n} \), then \( π(\bar{P}^{l-1}) = v^{l-1} \neq 0 \) and the nilpotency order of \( \bar{P} \) in \( A_n \) is greater than or equal to \( l \). The existence of such a \( P \) (given by \( P = Q \cdot H \)) is assumed by the following Theorem 3.2. This result is also true for the schemes \( L_m \) and needs the following lemma:

**Lemma 3.1.** — Let \( K = \mathbb{C}[X_1, \ldots, X_r], W = \mathbb{C}[Y_1, \ldots, Y_s] \) and let \( Ψ_t : K \otimes_\mathbb{C} W \to W \) be the morphism defined by \( Ψ_t(f(X,Y)) = f(t,Y) \) for \( t ∈ \mathbb{C}^r \). If \( J \) is an ideal of \( K \otimes_\mathbb{C} W \), then \( Ψ_t(J) \) is an ideal of \( W \) and the ideal \( L = \cap_t Ψ_t(J) \), for \( t ∈ (\mathbb{C}^*)^r \), is \( \{ P ∈ W ; ∃ Q ∈ K, QP ∈ J \} \).

**Proof.** — If \( P ∈ L \), for each \( t \) there are polynomials \( g^t_i(X,Y) ∈ K \otimes_\mathbb{C} W \) such that:

\[
P(Y) = Ψ_t \left( \sum_i g^t_i(X,Y) f_i(X,Y) \right) = \sum_i g^t_i(t,Y) f_i(t,Y)
\]

where \( f_i(X,Y) \) are generators of \( J \). The polynomials \( g^t_i(X,Y) \) can be written as

\[
g^t_i(X,Y) = \sum_γ a^t_{iγ}(X)Y^γ
\]

where \( γ \) is a multi-index for the monomials \( Y^γ = Y_1^{γ_1} \cdots Y_s^{γ_s} \), and

\[
P(Y) = \sum_i \left( \sum_γ a^t_{iγ}(t)Y^γ \right) f_i(t,Y), \quad ∀t ∈ (\mathbb{C}^*)^r.
\]

If we consider \( f_i(X,Y) = \sum_β Q_{iβ}(X)Y^β \) and \( P(Y) = \sum_δ c_δ Y^δ \) with fixed polynomials \( Q_{iβ}(X) ∈ K \) and fixed \( c_δ ∈ \mathbb{C} \), it follows that

\[
P(Y) = \sum_δ c_δ Y^δ = \sum_δ \sum_β γ = δ \left( \sum_i a^t_{iγ}(t) Q_{iβ}(t) \right) Y^δ, \quad ∀t ∈ (\mathbb{C}^*)^r.
\]

Then (3.1) gives the set of the following equalities indexed over \( δ \):

\[
\sum_β γ = δ \left( \sum_i λ_{iγ}(X) Q_{iβ}(X) \right) = c_δ.
\]

This is a linear system of equations in \( λ_{iγ}(X) \) with coefficients \( Q_{iβ}(X) \) which can be solved over the field of fractions \( \mathbb{C}(X) \) by the pivot method. Writing the solutions \( λ_{iγ}(X) = h_{iγ}(X)/q_{iγ}(X) \), it suffices to take the polynomial \( Π_{iγ}q_{iγ} = Q \) and consequently we have \( PQ ∈ J \). □
Theorem 3.2. — Let \( u = v/f \) be a \( l \)-nilpotent element in \( \mathfrak{m}(O) \), with \( O \) the local ring in the local chart (or slice) associated with an admissible set \( \mathcal{A} \) at \( \Phi_0 \) for \( L_n^T/G_0 \). If \( H \in \mathbb{C}[X_\beta; \beta \notin \mathcal{A}] \) satisfies \( H = v \in \mathbb{A}_n \) and \( G_0 \ast H \subseteq \mathbb{C}^*H \), then \( H^l = 0 \) and there is \( Q \in \mathbb{C}[X_\alpha; \alpha \in \mathcal{A}] \) such that \( (QH)^l \) is in \( \text{Jac}_n \), i.e., \( QH \) is \( l \)-nilpotent in \( \mathbb{A}_n \).

Proof. — We have \( u = v/f \) with \( f \notin M \), \( v \in \mathbb{C}[X_{ij}]/(X_\alpha - 1, \alpha \in \mathcal{A}) + \text{Jac}_n \), \( v^l = 0 \), \( v^{l-1} \neq 0 \), i.e., there are polynomials \( g_\alpha \) such that
\[
H^l + \sum_{\alpha \in \mathcal{A}} (X_\alpha - 1)g_\alpha \in \text{Jac}_n.
\]
The action \( \ast \) of \( G_0 \) stabilizes \( \text{Jac}_n \) and \( \mathbb{C}^*H^l \). For \( t = (t_\alpha)_{\alpha \in \mathcal{A}} \), \( t_\alpha \in \mathbb{C}^* \), there is \( s \in G_0 \) with \( (s \ast X)_\alpha = X_\alpha/t_\alpha \) by definition of \( \mathcal{A} \) and we obtain:
\[
\lambda H^l + \sum_{\alpha} \left( \frac{X_\alpha - t_\alpha}{t_\alpha} \right) s \ast g_\alpha \in \text{Jac}_n
\]
with \( \lambda \in \mathbb{C}^* \). This relation involves \( H^l \in \Psi_l(\text{Jac}_n) \) for each \( t \). By applying Lemma 3.1, there exists \( Q \in \mathbb{C}[X_\alpha; \alpha \in \mathcal{A}] \) with \( QH^l \in \text{Jac}_n \) and thus \( (QH)^l \in \text{Jac}_n \). \( \square \)

Remark 3.3. — In the examples dealt with in this paper, \( H \) can be chosen as a simple coordinate and the condition \( G_0 \ast H \subseteq \mathbb{C}^*H \) is satisfied. If \( H \) is a homogeneous polynomial, we can choose \( Q \) to be homogeneous because the homogeneous parts of \( (HQ)^l \) are in \( \text{Jac}_n \) as well.

Although it is possible to solve the linear system (3.2) with \( P = H^l \), in practice this linear system is very laborious. For the examples proposed in this section, we prefer to use a different method in order to find global nilpotent elements. We proceed by eliminating coordinates \( X_\beta \) (\( \beta \notin \mathcal{A} \)) from the Jacobi polynomials \( J_k \), \( 1 \leq k \leq N \). If \( m \) is the number of distinct coordinates \( X_{ij} \) \( (i < j) \) and \( N \) the number of the Jacobi polynomials \( J_k \), then nilpotent elements can appear under the condition:
\[
N \geq m - |\mathcal{A}|.
\]

Elimination in \( F[X] \)

Let \( F \) be a factorial ring and \( F[X] \) be the polynomial ring in one variable \( X \). If \( f_1 = AX^p + P \) and \( f_2 = BX^q + Q \) are two polynomials in \( X \) with \( p \geq q \), \( \deg(P) < p \), and \( \deg(Q) < q \), we denote \( D \) a H.C.F. of \( A \) and \( B \), \( A = A'D, B = B'D \) and then we can obtain the polynomial
\[
f_3 = B'f_1 - A'X^{p-q}f_2 = B'P - A'X^{p-q}Q
\]
of degree smaller than $p$ in variable $X$. If $\deg(f_2) \geq \deg(f_3)$, we now consider the pair $(f_2, f_3)$; otherwise we consider the pair $(f_3, f_2)$. We can apply the same operation again to $f_2$ and $f_3$ to obtain $f_4$ and so on. Finally, we obtain a well-defined element of $F$ denoted by $\{f_1, f_2\}_X$. This element can be null.

In this work, we eliminate a coordinate $X_\beta$ where $\beta$ is an index $(ij)$ from two Jacobi polynomials $J_1$ and $J_2$, which depend on this coordinate. The polynomial obtained, denoted by $\{J_1, J_2\}_\beta$, is homogeneous and also belongs to the ideal $\text{Jac}_n$.

The method

It is relative to the choice of an admissible set $\mathcal{A}$. If $\beta \notin \mathcal{A}$ we consider the list constituted by all the $J_k$ depending on $X_\beta$. If we fix from this list one $J_i$ as a pivot, we calculate all the different polynomials $\{J_k, J_i\}_\beta$ with $k \neq i$. Then, we have a new list, in which we have replaced each $J_k$ by this new polynomial $\{J_k, J_i\}_\beta$ and where the pivot $J_i$, depending on $X_\beta$, doesn’t appear in it. We proceed by successively eliminating the coordinates indexed by $\beta_1, \cdots, \beta_l \notin \mathcal{A}$. In all examples encountered, the condition (3.3) is satisfied and we can choose pivots such that $\{J_k, J_i\}_\beta \neq 0$. This possibility allows us to obtain polynomials depending on the variables indexed over $\mathcal{A}$ and only one variable $X_\rho$ with $\rho \notin \mathcal{A}$. This is the best possibility on account of the following result:

**Lemma 3.4.** — If two polynomials $f$ and $g$ in $\text{Jac}_n$ only depend on the $X_\alpha$, $\alpha \in \mathcal{A}$, and $X_\rho$ for one unique $\rho \notin \mathcal{A}$, then $\{f, g\}_\rho = 0$.

**Proof.** — The polynomial $h = \{f, g\}_\rho$ belongs to $\text{Jac}_n$ and only depends on coordinates indexed by $\mathcal{A}$. By definition of admissible set $\mathcal{A}$ associated with a law $\Phi_0$, the parameters $(s \ast \Phi_0)_\alpha$, for $\alpha \in \mathcal{A}$ and $s \in G_0$, are independent and generate an open set $\Omega \neq \emptyset$ in $\mathbb{C}^{\vert \mathcal{A} \vert}$. The polynomial $h$, null on $L^T_n(\mathbb{C})$, depends on variables indexed by $\mathcal{A}$ only and then $h(G_0 \ast \Phi_0) = h(\Omega) = 0$; $h$ is null on the closure of $\Omega$, i.e., $h = 0$. $\square$

**Remark 3.5.** — a) The polynomials obtained after elimination generally depend on the choice in the order of the $\beta_i$ and the pivots.

b) If a non-null polynomial depends on variables indexed by $\mathcal{A}' \cup \{\rho\}$ with $\mathcal{A}' \subset \mathcal{A}$, then $\mathcal{A}' \cup \{\rho\}$ is not contained in an admissible set (proof of Lemma 3.4).
Irreducible polynomials in the radical of $\text{Jac}_n$

We say that a polynomial is irreducible in an ideal of $\mathbb{C}[X_{ij}]$ if it doesn’t have non-trivial factors in this ideal. In $\sqrt{\text{Jac}_n}$, such a polynomial can be written $P = P_1 P_2 \cdots P_r$ where the $P_i$ are irreducible in $\mathbb{C}[X_{ij}]$ and distinct up to a factor in $\mathbb{C}^*$. If we denote $\hat{P}_i$ the polynomial $\Pi_{k \neq i} P_k$, we have the following criterion:

**Proposition 3.6.** — A product of different irreducible polynomials $P = \Pi_{k=1}^r P_k$ is irreducible in $\sqrt{\text{Jac}_n}$ iff for each $i \in \{1, \ldots, r\}$, there is a law $\Phi \in L_n^T(\mathbb{C})$ with $\hat{P}_i(\Phi) \neq 0$.

**Proof.** — Notice that $P$ is irreducible in $\sqrt{\text{Jac}_n}$ iff $\hat{P}_i \notin \sqrt{\text{Jac}_n}$ for all $i$. Thanks to the Hilbert nullstellensatz, it means that $\hat{P}_i$ is not identically null on $L_n^T(\mathbb{C})$. $\square$

**Corollary 3.7.** — The number of irreducible components of $L_n^T(\mathbb{C})$ is bigger than the number of factors in any irreducible polynomial of $\sqrt{\text{Jac}_n}$.

Under the assumptions for path of weights, we can state:

**Lemma 3.8.** — If a polynomial is irreducible in $\sqrt{\text{Jac}_n}$, then it is irreducible in $\sqrt{\text{Jac}_m}$ for $m > n$.

**Proof.** — If $P = \Pi_{k=1}^r P_k$ is an irreducible polynomial in $\sqrt{\text{Jac}_n}$, for each $i \in \{1, \ldots, r\}$ there is $\Phi \in L_n^T(\mathbb{C})$ with $\hat{P}_i(\Phi) \neq 0$. The law $\Phi \times 0_{m-n}$, direct product of $\Phi$ by the abelian law $0_{m-n}$ on $\mathbb{C}^{m-n}$, belongs to $L_m^T(\mathbb{C})$ and satisfies $\hat{P}_i(\Phi \times 0_{m-n}) = \hat{P}_i(\Phi) \neq 0$. $\square$

For each irreducible polynomial $P$ in $\sqrt{\text{Jac}_n}$, we call nilpotency order of $P$ in the ring $A_n$ the unique number $\nu \geq 1$ such that $P^\nu \in \text{Jac}_n$ and $P^{\nu-1} \notin \text{Jac}_n$. The quotient $\bar{P}$ in $A_n$ satisfies $\bar{P}^\nu = 0$ and $\bar{P}^{\nu-1} \neq 0$. If $P = \Pi_{k=1}^r P_k$ is irreducible in $\sqrt{\text{Jac}_n}$ with nilpotency order $\nu$ in $A_n$, then each irreducible polynomial $Q$ in $\text{Jac}_n$ with factors $P_k$ for $1 \leq k \leq r$, satisfies $Q = P_1^{\nu_1} \cdots P_r^{\nu_r}$ and $\nu = \max_{1 \leq k \leq r}(\nu_k)$.

**The examples**

With the tori chosen in the following examples, we can adopt graded indexation for structure constants of laws $[e_i, e_j] = X_{ij} e_{i+j}$, $1 \leq i < j \leq n$, and for Jacobi polynomials:

\begin{equation}
J_{ijk} = X_{ij} X_{i+j,k} + X_{jk} X_{j+k,i} + X_{ki} X_{k+i,j}
\end{equation}

where $i + j + k \leq n$. 

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a) The Torus $\alpha_1, 2\alpha_1, 3\alpha_1, \alpha_4 + k\alpha_1$ ($k \geq 0$). We have 3 families of Jacobi polynomials for $p \geq 4$: $J_{12p}$ ($n = p + 3 \geq 7$), $J_{13p}$ ($n = p + 4 \geq 8$) and $J_{23p}$ ($n = p + 5 \geq 9$). Inequality (3.3) is satisfied for $n \geq 9$. The elimination process relative to the admissible set $\mathcal{A}_n = \{(1k), (24) \mid 1 < k < n, k \neq 3\}$ gives:

**Proposition 3.9.** — The monomial $X_{12}X_{17}X_{18}(X_{34})^2$ is irreducible in $\text{Jac}_n$ for $n \geq 9$. The monomial $X_{12}X_{17}X_{18}X_{34}$ is irreducible in $\sqrt{\text{Jac}_n}$ and two-nilpotent in $\mathcal{A}_n$.

**Proof.** — We obtain $X_{12}X_{17}X_{18}(X_{34})^2$ by eliminating the variables $X_{36}$, $X_{27}$, $X_{35}$, $X_{26}$ and $X_{25}$ from the 6 Jacobi polynomials for $n = 9$. According to Proposition 3.6, the monomial $X_{12}X_{17}X_{18}X_{34}$ is irreducible in $\sqrt{\text{Jac}_n}$, however it doesn’t belong to $\text{Jac}_n$ ($n \geq 9$) because in the local study of the scheme $L_n^{T,A}$ at point $a_{1,n}$ appears the condition $(\bar{X}_{34})^2 = 0$, $\bar{X}_{34} \neq 0$ (Remark 2.7). Thus, we deduce the irreducibility of $X_{12}X_{17}X_{18}(X_{34})^2$ in $\text{Jac}_n$.

b) The torus $\alpha_1, 2\alpha_1, 3\alpha_1, 4\alpha_1, \alpha_5 + k\alpha_1$ ($k \geq 0$). This torus corresponds to the example in Proposition 2.2. We have Jacobi polynomials $J_{ijk}$ for $1 \leq i < j \leq 4$ and $k \geq 5$.

For $n \geq 11$ we consider the admissible set $\mathcal{A}_n = \{(1k), (25) \mid 1 < k < n, k \neq 4\}$. Condition (3.3) is satisfied and we can consider the elimination procedure associated with $\mathcal{A}_n$. For $n = 11$, if we eliminate $X_{2k}$ for $k \geq 6$, $X_{3k}$ for $k \geq 6$ and $X_{4k}$ for $k \geq 5$ from the Jacobi polynomials, keeping only the variable $X_{35}$ not indexed by $\mathcal{A}_n$, we obtain the following polynomial:

$$I_1 = (X_{12})^3(X_{18})^2X_{19}X_{110}(3X_{12}X_{35} - 4X_{25}X_{17})(X_{35})^3 \in \text{Jac}_{11}.$$

Hence,

$$P = X_{12}X_{18}X_{19}X_{110}X_{35}(3X_{12}X_{35} - 4X_{17}X_{25}) \in \sqrt{\text{Jac}_{11}}.$$

**Proposition 3.10.** — If $T$ is defined by the weights $i\alpha_1$ for $1 \leq i \leq 4$ and $\alpha_5 + j\alpha_1$ for $j \geq 0$, we have for $n \geq 11$:

i) $P$, cf. (3.5), is irreducible in $\sqrt{\text{Jac}_n}$;

ii) $P$ gives a nilpotent element of order $\nu = 3$ in $\mathcal{A}_n$.

**Proof.** — i) According to the Lemma 3.8, it suffices to prove the irreducibility for $n = 11$. Thanks to Proposition 3.6, we are looking for a law $\Phi \in L_{11}^T(C)$ such that $\hat{P}_\alpha(\Phi) \neq 0$ for each irreducible factor of $P$ indexed by $\alpha$.

If $\alpha$ represents the factor $3X_{12}X_{35} - 4X_{17}X_{25}$, then we have $\hat{P}_\alpha(\Phi) \neq 0$ for the law $a_{511}'$ defined in section 2. If $\alpha = (35)$, we have $\hat{P}_\alpha(\Phi) \neq 0$ for
the law $a_{511}$. If $\alpha = (12)$, then the following Lie algebra law $\Phi$ satisfies $\tilde{P}_\alpha(\Phi) \neq 0$: $\Phi_{12} = \Phi_{13} = 0$, $\Phi_{ij} = 1$, for $(ij)$ different to (12) or (13), $i < j$.

If $a_k$ is given by a law of $L^T_k(A_k)(\mathbb{C})$, for $8 \leq k \leq 10$, with $\Phi_{35} \neq 0, 4/3$, we can construct a law $\Phi$ as in Lemma 2.4 giving the $\Phi_{ij}$ of $a_k$ for $i < j \leq k$, $\Phi_{1j} = 1$ for $k < j < 11$ and $\Phi_{ij} = 0$ for $i < j$ otherwise. We have $\tilde{P}_{(1k)}(\Phi) \neq 0$, concluding i).

ii) We project a power $P^k$ in the ring of slice associated with $A_n$, $n \geq 11$. We obtain the polynomial $(\bar{X}_{35})^k(3\bar{X}_{35} - 4)$ and the projection on the local ring at point $a_{5n}$ gives the polynomial: $-4(\bar{X}_{35})^k$ which is not null for $k = 2$. Then $\bar{P}^k$ is null in $A_n$ for $k \geq 3$, but it is not null for $k = 2$. \hfill $\square$

**c) The torus $k\alpha$ ($k \geq 1$).** We have the Jacobi (3.4) for $1 \leq i < j < k$. The inequality (3.3) is satisfied for $n \geq 12$. For $n = 12$, an elimination process associated with $A_n = \{(1k), (23), 1 < k < n\}$, shown later in the “calculation with a computer” section, gives the polynomial:

$$I = -36(X_{12})^5(X_{13})^4(X_{17})^3(X_{19})^2X_{110}X_{111}(X_{34})^5(X_{16}X_{24} - 10X_{12}X_{34}) \in \text{Jac}_{12}.$$ 

Hence, we obtain the polynomial:

$$f = X_{12}X_{13}X_{17}X_{18}X_{19}X_{110}X_{111}X_{34}(X_{16}X_{24} - 10X_{12}X_{34}) \in \sqrt{\text{Jac}_{12}}.$$ 

Taking into account the identity $X_{13}X_{16}X_{24} - X_{16}J_{123} = X_{15}X_{16}X_{23}$, the following polynomial in $\sqrt{\text{Jac}_{12}}$ with variables indexed on $A_{12} \cup \{34\}$ satisfies Proposition 3.11:

$$P = X_{12}X_{17}X_{18}X_{19}X_{110}X_{111}X_{34}(X_{15}X_{16}X_{23} - 10X_{12}X_{13}X_{34}).$$

**Proposition 3.11.** — If $T$ is defined by the weights $\alpha_k = k\alpha$ for $1 \leq k \leq n$, then for $n \geq 12$ we have:

i) $P$, cf. (3.6), is irreducible in $\sqrt{\text{Jac}_n}$;

ii) $P$ gives a nilpotent element of order $\nu = 5$ in the quotient ring $A_n$.

**Proof.** — i) It suffices to prove the irreducibility for $n = 12$ and we proceed as in proof of Proposition 3.10. If $\alpha$ represents the non-monomial factor of $P$, (3.6), then we have $\tilde{P}_\alpha(\Phi) \neq 0$ for the Witt algebra $w_{12}$: $\Phi_{ij} = i - j$. If $\alpha = (34)$, then we have $\tilde{P}_{34}(\Phi) \neq 0$ for $f_n$.

If $\alpha = (12)$, it suffices to check that the following law $\Phi$ belongs to $L^T_{12}(\mathbb{C})$ with $\tilde{P}_{12}(\Phi) \neq 0$: $\Phi_{12} = \Phi_{4k} = \Phi_{5k} = 0$ ($k > 4$), $\Phi_{ij} = 1$ for $i < j$ otherwise. If $a_k$ is a Lie algebra in $L^T_k(A_k)$ for $7 \leq k \leq 11$ with $\Phi_{34} \neq 0, 1/10$, we construct the law $\Phi$ as in Lemma 2.4 by giving the $\Phi_{ij}$.
of \( a_k \) for \( i < j \leq k \), \( \Phi_{1j} = 1 \) for \( k < j < 12 \) and otherwise \( \Phi_{ij} = 0 \) for \( i < j \). Also, we have \( \tilde{P}_{(1k)}(\Phi) \neq 0 \) for the other factors, concluding i).

For ii): Observe that \( P = f - QJ_{123} \) and that \( P^5 = f^5 + SJ_{123} \) belongs to \( \text{Jac}_n \) because \( f^5 \) is a multiple of \( I \). Hence, it gives \( \tilde{P}^5 = 0 \) in the quotient \( A_n \). If we project a power \( P^k \) in the ring defining the slice associated with the admissible set \( A_n \), then we obtain \( \tilde{X}_{ij} = 1 \) for \( (ij) \in A_n \) and \( \tilde{P}^k = (\tilde{X}_{34})^k(1-10\tilde{X}_{34})^k \). This expression is not null in the local ring associated with \( f_n \) for \( k \leq 4 \), thanks to the local study in Proposition 1.6. Then, \( \tilde{P}^4 \neq 0 \) and the nilpotency order of \( P \) in \( A_n \) is \( \nu = 5 \).

\[ \square \]

**Corollary 3.12.** — The scheme \( L^T_n \) has at least 8 irreducible components for \( n \geq 12 \).

### Calculation with a computer

Taking into account all the Jacobi polynomials (3.4) for \( n = 12 \), the polynomial \( I \) has been obtained from the polynomial \( J_{246} \) after applying the elimination method. In this process, the order of elimination of the variables and the election of the pivot associated with each variable are shown in the following table:

<table>
<thead>
<tr>
<th>order</th>
<th>variable</th>
<th>pivot</th>
<th>order</th>
<th>variable</th>
<th>pivot</th>
<th>order</th>
<th>variable</th>
<th>pivot</th>
</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td>( X_{57} )</td>
<td>( J_{156} )</td>
<td>7)</td>
<td>( X_{38} )</td>
<td>( J_{128} )</td>
<td>13)</td>
<td>( X_{36} )</td>
<td>( J_{126} )</td>
</tr>
<tr>
<td>2)</td>
<td>( X_{48} )</td>
<td>( J_{138} )</td>
<td>8)</td>
<td>( X_{29} )</td>
<td>( J^3_{147} )</td>
<td>14)</td>
<td>( X_{27} )</td>
<td>( J^2_{234} )</td>
</tr>
<tr>
<td>3)</td>
<td>( X_{39} )</td>
<td>( J_{129} )</td>
<td>9)</td>
<td>( X_{46} )</td>
<td>( J_{145} )</td>
<td>15)</td>
<td>( X_{35} )</td>
<td>( J_{134} )</td>
</tr>
<tr>
<td>4)</td>
<td>( X_{2.10} )</td>
<td>( J^3_{147} )</td>
<td>10)</td>
<td>( X_{37} )</td>
<td>( J_{127} )</td>
<td>16)</td>
<td>( X_{26} )</td>
<td>( J^1_{125} )</td>
</tr>
<tr>
<td>5)</td>
<td>( X_{56} )</td>
<td>( J_{146} )</td>
<td>11)</td>
<td>( X_{28} )</td>
<td>( J^2_{136} )</td>
<td>17)</td>
<td>( X_{25} )</td>
<td>( J_{124} )</td>
</tr>
<tr>
<td>6)</td>
<td>( X_{47} )</td>
<td>( J_{137} )</td>
<td>12)</td>
<td>( X_{45} )</td>
<td>( J_{135} )</td>
<td>18)</td>
<td>( X_{23} )</td>
<td>( J_{123} )</td>
</tr>
</tbody>
</table>

where the pivot \( J^3_{147} \) has been obtained from Jacobi polynomial \( J_{147} \) after eliminating, by this process, the variables \( X_{57}, X_{48} \) and \( X_{39} \). The pivot \( J^3_{236} \) is also obtained by eliminating the variables \( X_{56}, X_{47} \) and \( X_{38} \) in the polynomial \( J_{236} \). The polynomial \( J^2_{136} \), after the elimination of the variables \( X_{46} \) and \( X_{37} \) in \( J_{136} \). The pivot \( J^2_{234} \), by eliminating \( X_{45} \) and \( X_{36} \) in \( J_{234} \) and finally, \( J^1_{125} \) is obtained by elimination of the variable \( X_{35} \) in \( J_{125} \).

We have used the symbolic computational package MAPLE to execute this calculation.
4. A second geometry obtained with generators and relations: subschemes of ideals in Grassmannians

Generalities

A Lie algebra \( \mathfrak{g} \) with a finite number \( r \) of generators is built as the quotient of a free Lie algebra \( \mathfrak{L}_r \) to \( r \) generators by an ideal \( \mathfrak{J} \): \( 0 \rightarrow \mathfrak{J} \rightarrow \mathfrak{L}_r \rightarrow \mathfrak{g} \rightarrow 0 \). Two quotients of \( \mathfrak{L}_r \) are isomorphic iff the ideals are conjugated by the automorphism group \( \text{Aut}(\mathfrak{L}_r) \). If \( \mathcal{C}^n(\mathfrak{g}) \) is the central descending series of \( \mathfrak{g} \), a nilpotent Lie algebra \( \mathfrak{L}_r/\mathfrak{J} \) of dimension \( n \) satisfies \( \mathcal{C}^n(\mathfrak{L}_r/\mathfrak{J}) = \mathcal{C}^n(\mathfrak{L}_r)/\mathfrak{J} = 0 \), i.e., \( \mathcal{C}^n(\mathfrak{L}_r) \subseteq \mathfrak{J} \). Such an algebra is also the quotient of the finite dimensional Lie algebra \( \mathfrak{M} = \mathfrak{L}_r/\mathcal{C}^n(\mathfrak{L}_r) \) by the ideal \( \mathfrak{J}/\mathcal{C}^n(\mathfrak{L}_r) \). In this work we define \( J_n(\mathfrak{L}_r) \) as the set of ideals of codimension \( n \) in \( \mathfrak{L}_r \) containing \( \mathcal{C}^n(\mathfrak{L}_r) \); it is identified with a subscheme of the grassmannian \( Gr_{m-n}(\mathfrak{L}_r/\mathcal{C}^n(\mathfrak{L}_r)) \) for \( \dim(\mathfrak{M}) = m \). The nilpotent laws of dimension \( n \) are obtained for \( 2 \leq r \leq n \). For \( m > n \), the grassmannian \( Gr_{m-n}(\mathfrak{M}) \), with its natural reduced structure of scheme, contains as subscheme the set \( J_n(\mathfrak{L}_r) \) of \( n \)-codimensional ideals \( \mathfrak{J}/\mathcal{C}^n(\mathfrak{L}_r) \) of \( \mathfrak{M} \) defined by the simple polynomial relations \( [x, \mathfrak{J}] \subseteq \mathfrak{J} \) for \( x \in \mathfrak{L}_r \). This is the “minimal” definition of an ideal which is provided by its current algebraic characterization. Such a scheme is generally not reduced. Each point \( \{ \mathfrak{J} \} \) defines the Lie algebra quotient \( \mathfrak{n} = \mathfrak{L}_r/\mathfrak{J} \), hence giving a second geometry for the nilpotent laws.

Torus

A maximal torus \( T_r \) on \( \mathfrak{L}_r \), diagonalized by a family of generators \( e_i, \ 1 \leq i \leq r \), where \( t(e_i) = \varepsilon_i(t)e_i, \ t \in T_r \), is characterized by its weights \( \varepsilon = \sum_{i=1}^r n_i \varepsilon_i \) and the multiplicities given by the Witt formula:

\[
d\varepsilon = \dim(L_\varepsilon) = \frac{1}{|\varepsilon|} \sum_{k|\varepsilon} \mu(k) \binom{|\varepsilon|}{n_1|/k|!} \cdots (n_r|/k|!)
\]

where \( L_\varepsilon \) is the weight subspace of \( \mathfrak{L}_r \) associated with \( \varepsilon \), \( |\varepsilon| = \sum n_i \), and \( \mu \) the Möbius function. The \( T_r \)-module structure of \( \mathfrak{M} \) is given by the decomposition \( \bigoplus_{|\varepsilon|<n} L_\varepsilon \). Let \( T \) be a subtorus of \( T_r \); its weights on \( \mathfrak{L}_r \) are the restrictions \( \alpha = \varepsilon |_T = \sum_{i=1}^r n_i \alpha_i \) with \( \alpha_i = \varepsilon_i |_T \), and let \( \Pi \) be the set of this weights. We write \( \mathfrak{L}_r = \bigoplus_\alpha L_\alpha \) with \( L_\alpha = \bigoplus \{ L_\varepsilon; \alpha = \varepsilon |_T \} \). An ideal stable by \( T \) can be written as \( \mathfrak{J} = \bigoplus_\alpha J_\alpha \) with \( J_\alpha \subset L_\alpha \). The different tori \( T \) employed are of maximal type, i.e., maximal over one Lie algebra.
at least. We denote by $J^T_n(\mathfrak{L}_r)$ the subscheme of $J_n(\mathfrak{L}_r)$ constituted of all $T$-invariant ideals $\mathfrak{J}$ satisfying the additional polynomial relations $t(\mathfrak{J}) \subset \mathfrak{J}$ for $t \in T$. The torus $T$ operates on each quotient $n = \mathfrak{L}_r/\mathfrak{J}$. Let $\sigma_n$ or simply $\sigma$ be a sequence called weight system $\{(\alpha, n(\alpha)) ; \alpha \in \Pi, n(\alpha) \in \mathbb{N}\}$ such that $n(\alpha_i) = 1$ for $1 \leq i \leq r$ and $\sum n(\alpha) = n$ for $n \in \mathbb{N}$; let $j(\alpha) = \dim(L_\alpha) - n(\alpha)$ for each $\alpha \in \Pi$, and let $P$ denote the set of $\alpha \in \Pi$ with $n(\alpha) \neq 0$. Let $V^\sigma_r(T)$ be the set of ideals $\mathfrak{J} \in J^T_n(\mathfrak{L}_r)$ such that $\dim(J_\alpha) = j(\alpha)$, i.e., $\dim(n_\alpha) = n(\alpha)$ for $n_\alpha = L_\alpha/J_\alpha$; $V^\sigma_r(T)$ is a closed subscheme of a product of grassmannians

$$V^\sigma_r(T) = \{J_\alpha \in \Pi_\alpha Gr_{j(\alpha)}(L_\alpha), [\mathfrak{L}_r, J_\alpha] \subset J_\alpha \} \subset J^T_n(\mathfrak{L}_r).$$

In fact, it is a finite product of grassmannians because $n(\alpha) = 0$ involves $J_\alpha = L_\alpha$ and $Gr_{j(\alpha)}(L_\alpha)$ is trivial, so it can be omitted. The scheme $J^T_n(\mathfrak{L}_r)$ is a finite union of $V^\sigma_r(T)$. Let $W^\sigma_r(T)$ be the open set of $V^\sigma_r(T)$ consisting of the ideals on which $T$ is maximal.

**Action groups**

The normalizing subgroup $N$ of $T$ in $\text{Aut}(\mathfrak{L}_r)$, i.e., the set of $\theta \in \text{Aut}(\mathfrak{L}_r)$ with $\theta T \theta^{-1} = T$, stabilizes $V^\sigma_r(T)$ and $W^\sigma_r(T)$. The torus $T$ is maximal on $\mathfrak{L}_r/\mathfrak{J}$ iff it is maximal in the subalgebra of derivations of $\mathfrak{L}_r$ stabilizing $\mathfrak{J}$ [8]. We can state a lemma under the following hypothesis on $P$:

(H): Each linear automorphism of $T^*$ stabilizing $P$ and sending a base $B \subset \{\alpha_i\}_{1 \leq i \leq r}$ in $\{\alpha_i\}_{1 \leq i \leq r}$, stabilizes the part $\{\alpha_i\}_{1 \leq i \leq r}$ too.

**Lemma 4.1.** — Under conditions (H) and $n(\alpha_i) = 1$ ($1 \leq i \leq r$) for $\sigma$, two ideals in $W^\sigma_r(T)$ are conjugated by $N$ iff the corresponding quotient Lie algebras are isomorphic.

**Proof.** — The direct implication is obvious. Conversely, let $\mathfrak{J}_k$ for $k \in \{1, 2\}$ be two ideals of free $\mathfrak{L}_r$ stable by $T$ and such that the quotient algebras $\mathfrak{L}_r/\mathfrak{J}_k$ are isomorphic. The tori of derivations $T_k$ on $\mathfrak{L}_r/\mathfrak{J}_k$, deduced from $T$ by $t_kp_k = p_kt$ if $p_k$ is the canonical projection on $\mathfrak{L}_r/\mathfrak{J}_k$ and $t \in T$, being maximal, are conjugated by an isomorphism $h$ from $\mathfrak{L}_r/\mathfrak{J}_1$ to $\mathfrak{L}_r/\mathfrak{J}_2$ (Mostow’s theorem) and we have $T_2 = hT_1h^{-1}$. Under these hypotheses, we can find vectors $x_i$, $1 \leq i \leq r$, associated with $T$-weights $\alpha_i$ in $\mathfrak{L}_r$ such that $L_{\alpha_i} = \mathbb{C}x_i \oplus (\mathfrak{J}_1)_{\alpha_i}$. For each $t \in T$, there is a unique $t' \in T$ with $t' h = hT_1$. The transpose of the linear automorphism $t \rightarrow t'$ of $T$ is a linear automorphism $L$ of $T^*$: $(L\alpha)(t) = \alpha(t')$, $\alpha \in T^*$. It keeps $\sigma$ and $P$ because $h$ is a Lie algebra isomorphism. With an indexation such
that the weights $\alpha_i$ for $1 \leq i \leq s$ ($s \leq r$) satisfy $(\mathfrak{g})_{\alpha_i} \subset [\mathfrak{l}_r, \mathfrak{l}_r]_{\alpha_i}$, i.e., $p_1 e_i \notin [\mathfrak{l}_r; \mathfrak{g}]$, we have that $\{p_1 e_i\}_{1 \leq i \leq s}$ is a minimal generating family of $\mathfrak{l}_r/\mathfrak{g}$ as its image by $h$ in $\mathfrak{l}_r/\mathfrak{g}$. Each weight $L^{-1}(\alpha_i)$ of $h(p_1 e_i)$ for $1 \leq i \leq s$ is of the form $\alpha_j$, for $1 \leq j \leq r$. Notice that all weight vectors in $[\mathfrak{l}_r/\mathfrak{g}]$ are linear combinations of the $\alpha_i$, thus there is a base $B$ of $T^*$ contained in $\{\alpha_i; 1 \leq i \leq r\}$ as $L^{-1}(B)$. Applying hypothesis (H) to $L^{-1}$, we define a permutation $\zeta$ of $\{1, \cdots, r\}$ such that $L^{-1}(\alpha_i) = \alpha_{\zeta(i)}$, $1 \leq i \leq r$. We have $t_2^\alpha(h(p_1 x_i)) = h t_1(p_1 x_i) = \alpha_i(t) h(p_1 x_i)$, so $h(p_1 x_i) \neq 0$ is a weight vector for $T_2$. In the same way, we can choose a family $y_i$, $1 \leq i \leq r$, associated with $\zeta$, satisfying the same properties as $x_i$ and such that $h(p_1 x_i) = p_2(y_{\zeta(i)})$. If $p_1 x_i \notin [\mathfrak{l}_r/\mathfrak{g}]$, we take $x_i = e_i$, and $p_2 y_{\zeta(i)} \notin [\mathfrak{l}_r/\mathfrak{g}]$ gives $y_{\zeta(i)} = \lambda_i e_{\zeta(i)} + z_{\zeta(i)}$ with $\lambda_i \notin 0$ and $z_{\zeta(i)} \in [\mathfrak{l}_r, \mathfrak{l}_r]$. If $p_1 x_i \in [\mathfrak{l}_r/\mathfrak{g}]$, then $p_1 e_i$, $h(p_1 e_i)$ and $p_2 e_{\zeta(i)}$ are in their respective derived ideals and $h(p_1 e_i) - p_2 e_{\zeta(i)}$ can be written as $p_2 z_{\zeta(i)}$ with $z_{\zeta(i)} \in [\mathfrak{l}_r, \mathfrak{l}_r]$. Setting $\theta(e_i) = \lambda_i e_{\zeta(i)} + z_{\zeta(i)}$, with $\lambda_i = 1$ in the second case, we obtain an automorphism $\theta$ of $\mathfrak{l}_r$ defined on the generators $e_i$, satisfying $h p_1 = p_2 \theta$ and $\theta(\mathfrak{g}) \subset \mathfrak{g}$. With relations $\alpha_{\zeta(i)}(t') = \alpha_i(t)$, one checks equalities $t' \theta = \theta t$ for each $t \in T$ and $\theta \in N$.

This lemma allows us to treat Lie algebras that don’t have a fixed number of generators, which differs essentially from results of G. Favre in [8], where this number is given by $r$. Hypothesis (H) is satisfied by all examples studied in this paper.

The quotient space $W^\sigma_r(T)/N$ gives the isomorphic classes of the $\mathfrak{n} = \mathfrak{l}_r/\mathfrak{g}$. The neutral components $N_0$ of $N$ and of $\text{Aut}(\mathfrak{l}_r)^T$ are equal. The finite group $N/N_0$ operates on $W^\sigma_r(T)/N_0$, giving the isomorphic classes. If we compare $W^\sigma_r(T)$ to $\Sigma_\alpha(T)$ and $N_0$ to $G_0$ in the affine description of section 2, the problem is now to find good slices for $W^\sigma_r(T)/N_0$.

**Slices for $W^\sigma_r(T)/N_0$**

We impose the condition $n(\alpha_i) = 1$ for $1 \leq i \leq r$. Then, the $s \in N_0$ stabilize the weight subspaces $L_\alpha(\alpha \in \Pi)$ and are defined on the generators by $s(\alpha_i) = e_i e_i$, mod $[\mathfrak{l}_r, \mathfrak{l}_r]$ with $s_i \in \mathbb{C}^*$ for $1 \leq i \leq r$. If $e_I = [e_{i_1} e_{i_2} \cdots e_{i_p}]$ is a Lie product of generators $e_i$, where the brackets are omitted and $I = (i_1 i_2 \cdots i_p)$, we have for $s \in N_0$: $s(e_I) = s_{i_1} \cdots s_{i_p} e_I$ mod $(C^{p+1} \mathfrak{l}_r)_{\alpha}$, and $s(e_{\alpha}) = \sum c(I) s_{i_1} \cdots s_{i_p} e_I$ mod $(C^{p+1} \mathfrak{l}_r)_{\alpha}$ for $e_\alpha = \sum c(I) e_I \in L_\alpha$, $c(I) \in \mathbb{C}$.

We define a slice $\mathcal{F}$ of $W^\sigma_r(T)$ at point $\mathfrak{g}$ (or $\mathfrak{n} = \mathfrak{l}_r/\mathfrak{g}$), associated with the $N_0$-action group, as a subscheme of $W^\sigma_r(T)$ transversal to the orbit of
\[ \mathfrak{J} \] at \( \mathfrak{J} \), i.e., satisfying \( T_3 W^\sigma_r(T) = T_3 \mathcal{F} \oplus T_3(N_0, \mathfrak{J}) \) with reduced scheme structure on the orbit \( N_0, \mathfrak{J} \). A subscheme \( \mathcal{F} \) is a slice if it is a slice at each point. We can always construct such a subscheme by the “orbital parameters fixing” method developed in [6].

The diagonal subgroup \( D \subset \text{Aut}(\mathfrak{L}_r) \) defined by the \( s \) such that \( s(e_i) = s_i e_i, \ s_i \in \mathbb{C}^* \), \( 1 \leq i \leq r \), is contained in \( N_0 \). Condition \( n(\alpha_i) = 1 \) for \( 1 \leq i \leq r \) involves that each \( s \in N_0 \) can be written as \( s(e_i) \equiv s_i e_i \mod (\mathfrak{J}) \) with \( s_i \in \mathbb{C}^* \) for \( 1 \leq i \leq r \) if \( \mathfrak{J} \) is fixed in \( W^\sigma_r(T) \). The invariant subgroup \( U = \{ s \in N_0 ; s(e_i) \equiv e_i \mod (\mathfrak{J}) \} \) is contained in the stabilizer subgroup \( \text{Stab}(\mathfrak{J}) \) of \( \mathfrak{J} \) in \( N_0 \). We have \( N_0 / U \simeq D \) and the quotient \( \text{Stab}(\mathfrak{J}) / U \) is isomorphic to the neutral component \( D' \) of the group \( \text{Aut}(\mathfrak{L}_r / \mathfrak{J})^T \). Thanks to the hypothesis \( n(\alpha_i) = 1 \) for \( 1 \leq i \leq r \), \( D' \) can be identified with a subgroup of \( D \) whose Lie algebra is \( T \). The orbit \( \Omega(\mathfrak{J}) \) of \( \mathfrak{J} \) by the \( N_0 \) action can be identified with the space of classes

\[
(4.1) \quad \Omega(\mathfrak{J}) \simeq N_0 / \text{Stab}(\mathfrak{J}) \simeq D / D'
\]

and the tangent of \( \Omega(\mathfrak{J}) \) at \( \mathfrak{J} \) is \( T_r / T \). In practice, the orbits \( \Omega(\mathfrak{J}) \) are obtained by the natural action of the subgroup \( D \subset N_0 \) on ideals. Slices are obtained (cf. examples section 5) by fixing a minimal family of lines \( \mathbb{C} u_i \ (i \in I) \) in \( \mathfrak{J} \) which impose that \( T \) is a maximal torus, i.e., \( s(u_i) \subset \mathbb{C} u_i \ (i \in I) \) for \( s \in D \) involve \( s \in D' \). If \( T_r = T \) we have:

**Remark 4.2.** — The case of maximal rank \( T = T_r \) gives the slice \( \mathcal{F} = W^\sigma_r(T) \) simply.

**Schemes of ideals are Jacobi schemes**

We denote by a calligraphic letter the set of representatives \( f \) in \( \text{Hom}_\mathbb{C}(\mathfrak{L}_r, \mathbb{C}^n) \) whose kernel belongs to a grassmannian of \( \mathfrak{L}_r \). For example, \( J_n(\mathfrak{L}_r) \) is the set of \( \mathbb{C} \)-linear maps from \( \mathfrak{L}_r \) to \( \mathbb{C}^n \) such that \( \text{Ker}(f) \) is an ideal in \( J_n(\mathfrak{L}_r) \). Similarly, we define \( V^\sigma_r(T) = \{ f \in \text{Hom}(\mathfrak{L}_r, \mathbb{C}^n)^T ; \ \text{Ker}(f) \in V^\sigma_r(T) \} \) and so on. Such a \( f \in J_n(\mathfrak{L}_r) \) allows us to construct a Lie algebra bracket \( \Phi_f \) on \( \mathbb{C}^n \): \( \Phi_f(x, y) = f([f^{-1}(x), f^{-1}(y)]) \rightleftharpoons (x, y) \in (\mathbb{C}^n)^2 \), where \([ \ , \ ] \) is the bracket on \( \mathfrak{L}_r \). Notice that \( (\sigma, s) \in \text{Aut}(\mathfrak{L}_r) \times GL_n(\mathbb{C}) \) operates on \( f \in J_n(\mathfrak{L}_r) \) by \( s \circ f \circ \sigma^{-1} \), thus \( \Phi_{s f \sigma^{-1}} = s \ast \Phi_f \) and we can state the following:

**Lemma 4.3.** — The algebraic map \( h : f \rightleftharpoons \Phi_f \) from \( J_n(\mathfrak{L}_r) \) to \( L_n(\mathbb{C}) \) induces by quotient an injection on the classes:

\[ J_n(\mathfrak{L}_r) / \text{Aut}(\mathfrak{L}_r) \rightarrow L_n(\mathbb{C}) / GL_n(\mathbb{C}). \]
Proof. — The quotient of $J_n(\mathfrak{L}_r)$ by the left action of $GL_n(\mathbb{C})$ is identified with $J_n(\mathfrak{L}_r)$.

By restriction to $W^r_n(T)$, $h$ induces by quotient the injections $W^r_n(T)/N \rightarrow L^T_n(\mathbb{C})/H$ and $W^r_n(T)/N_0 \rightarrow L^T_n(\mathbb{C})/G_0$.

**Theorem 4.4.** — Slices of $W^r_n(T)$ associated with the action of $N_0$ in the scheme $V^r_n(T)$ can be identified with slices of $\Sigma_n(T) \subset L^T_n$ associated with the $G_0$ action in Jacobi schemes.

Proof. — Under hypotheses $n(\alpha_i) = 1$ for $1 \leq i \leq r$ and $T > 0$, there is $t \in T$ such that $\alpha(t) > 0$ for the weights and we have a partial order relation $\geq$ over the weights (in fact, total order) resulting from the order in the real numbers $\alpha(t)$. If $\delta \in P$ is maximal for $\geq$, we have $[n,n_\delta] = 0$ and $n_\delta$ is central in $n$. By induction on $\sigma = (\sigma', n(\delta))$, we construct

$$V^r_n(T) = \{ \mathfrak{J}' \times J_\delta \in V^r_n(T) \times Gr_j(\delta)(L_\delta); [\mathfrak{L}_r, \mathfrak{J}']_\delta \subset J_\delta \}.$$ 

If $\mathfrak{J}' \in V^r_n(T)$, we can consider the subspaces $J_\delta$ of codimension $n(\delta)$ in $L_\delta$ and containing $[\mathfrak{L}_r, \mathfrak{J}']_\delta$. These subspaces of codimension $n(\delta)$ are identified with their quotients $J_\delta$ in $E = L_\delta/[\mathfrak{L}_r, \mathfrak{J}']_\delta$. This space $E$ can be expressed with the help of the $T$-module $H_2(n')$ of homology of $n' = \mathfrak{L}_r/\mathfrak{J}'$ as:

$$E = \left( \frac{[\mathfrak{L}_r, \mathfrak{L}_r]}{[\mathfrak{L}_r, \mathfrak{J}']} \right)_\delta = [\mathfrak{n}', \mathfrak{n}']_\delta = H_2(n')_\delta.$$ 

If $\delta \neq \alpha_i$ for $1 \leq i \leq r$, we have $L_\delta = [\mathfrak{L}_r, \mathfrak{L}_r]_\delta$, but the quotient $\mathfrak{L}_r/[\mathfrak{L}_r, \mathfrak{J}']$ is an algebra $\tilde{\mathfrak{n}}'$, equal to the central extension of $n'$ by the kernel $H_2(n')$ defined in [3]. It is known that $H_2(n')_\delta$ can be identified with the quotient $(\wedge^2 \mathfrak{n}')_\delta/\Omega_\delta$, where $\Omega_\delta$ is the space generated by the vectors

$$\int_{(xyz)} x \wedge [y,z] = x \wedge [y,z] + y \wedge [z,x] + z \wedge [x,y], \, (x,y,z) \in n'_\alpha \times n'_\beta \times n'_\gamma,$$

with $\alpha + \beta + \gamma = \delta$. A subspace representative of codimension $n(\delta)$ in $E$ is a $\mathbb{C}$–morphism $f_\delta$ giving an exact sequence whose kernel contains $\Omega_\delta$:

$$0 \rightarrow \text{Ker}(f_\delta) \rightarrow (\wedge^2 \mathfrak{n}')_\delta \rightarrow \mathbb{C}^{n(\delta)} \rightarrow 0.$$ 

If $(x_i)_{1 \leq i \leq n'}$ is a basis of $n'$, and $(y_h)_{n' < h \leq n}$ a basis of $\mathbb{C}^{n(\delta)}$, we have

$$f_\delta(x_i \wedge x_j) = \sum_{h=n'+1}^n X_{ij}^h y_h$$

(4.2)
with variables $X_{ij}^k$. The condition $f_δ(Ω_δ) = 0$ can be expressed by:

$$f_δ \left( \int_{(ijk)} x_i \wedge [x_j, x_k] \right) = \int_{(ijk)} \sum_m c_{ijk}^m f_δ(x_i \wedge x_m) = \sum_h \left( \int_{(ijk)} \sum_m c_{ijk}^m X_{im}^h \right) y_h = 0,$$

i.e., $\int_{(ijk)} \sum_m c_{ijk}^m X_{im}^h = 0$ for each $(ijkh)$. These are the Jacobi relations associated with the weight $δ$, satisfied by Lie algebras $n$, where $c_{ijk}^m$ are structure constants of the quotient $n'$. Initialization of the induction is made on a weight $α_i$, $1 \leq i \leq r$, with $σ_1 = \{(α_i, 1)\}$. This gives trivial $V_r^{σ_1}(T)$ because the abelian Lie algebra $\mathbb{C}ξ_i$ is associated with the ideal $⊕_{α ≠ α_i} L_α$.

We have proved that $V_r^s(T)$ can be identified with the set of sequences $(f_{α_1}, \cdots, f_β, \cdots, f_δ)$ defined in (4.2) and seen as a linear morphism $f$. This is the set of variables $X_{ij}^k$ satisfying the Jacobi rules too. Observe that all quotients of $Σ_δ(s < r)$ are quotients of $Σ_r$ as well, thus the scheme $V_r^s(T)$ (respectively $W_r^s(T)$) can be identified with the open set of laws in $L_n^T$ (respectively $Σ_n(T)$) having less than $r$ generators. The scheme $V_r^s(T)$ is the set of ideals $\text{Ker}(f) = Π_β \text{Ker}(f_β)$ as quotient scheme of $V_r^s(T)$ by left action of $G_0 \simeq Π_β GL(n(β))$ and $W_r^s(T)$ is an open subscheme of $V_r^s(T)$.

The morphism $V_r^s(T) \rightarrow L_n^T(\mathbb{C})$ defined by $h$ is an injective morphism of Jacobi schemes. The subgroup $N_0$ of $\text{Aut}(Σ_r)$ operates right hand on $V_r^s(T)$ and $G_0$ operates canonically by $*$ on $L_n^T(\mathbb{C})$ and left hand on $V_r^s(T)$. These actions induce an action of $G_0 × N_0$ which is compatible with the morphism $h : f \rightarrow Φ_f$, $Φ_{sfu}^{-1} = s * Φ_f$, $(s, u) ∈ G_0 × N_0$. We obtain an injection on the quotients $V_r^s(T)/N_0 \rightarrow L_n^T(\mathbb{C})/G_0$, identifying the open set $W_r^s(T)/N_0$ with an open set of $Σ_n(T)/G_0$. Hence, $h$ identifies each (possible) slice of $W_r^s(T)/N_0$ with a slice of $Σ_n(T)/G_0$. □

With formula of Theorem 1.8 (iii) of [3] under hypothesis $n(α_i) = 1$, $1 \leq i \leq r$, we can state:

**Theorem 4.5.** — The Zariski tangent space of the scheme $W_r^s(T)$ at a point $n = Σ_r / 3$ defined by $3$ is equal to $\text{Hom}_\mathbb{C}(3, n)^{c + T}$. The Zariski tangent space to a slice of $W_r^s(T)/N_0$ is isomorphic to the second $T$-cohomological adjoint group $\text{Hom}_\mathbb{C}(3, n)^{c + T}/(T_r/T)$.

**Proof.** — Let $f_0 : Σ_r \rightarrow \mathbb{C}^n = n$ be a representative of the ideal $3 = \text{Ker}(f_0) ∈ W_r^s(T)$ and $f = f_0 + h$ be another representative in $W_r^s(T)$, with $h$ small. The ideal condition for $\text{Ker}(f)$ gives $f_0([y, x]) + h([y, x]) = 0$ for $(x, y) ∈ \text{Ker}(f) × Σ_r$, but $x$ can be written $x_0 + ξ$ with $x_0 ∈ 3$ and $ξ$.
small, thus we have:

\[ f_0([y, \xi]) + h([y, x_0]) + h([y, \xi]) = 0 \]  
\[ f(x) = f_0(\xi) + h(x_0) + h(\xi) = 0. \]

The \( L_r \)-module action \([y, f_0(\xi)]\) defined by \( f_0([y, \xi]) \) writes \(-h([y, x_0]) - h([y, \xi])\) with (4.3) and \(-[y, h(x_0)] - [y, h(\xi)]\). With (4.4) we have consequently at first order in \((h, \xi)\):

\[ h([y, x_0]) = [y, h(x_0)]; \]

it can be expressed by \(y\)-invariance of \(h\) restricted to \(\mathfrak{J}\). Similarly, the \(T\)-invariance of \(\text{Ker}(f)\) gives an equivalence to (4.3) for \(t \in T\):

\[ f(t(x)) = f_0(t\xi) + h(tx_0) + h(t\xi) = 0. \]

If \(C^n\) is endowed with a \(T\)-module structure by \(tf_0(x) = f_0(tx)\) for \((t, x) \in T \times L_r\), the term \(f_0(t\xi)\) can be written as \(tf_0(\xi)\). We obtain at the first order with (4.4) the equality \(h(tx_0) = t(h(x_0))\). We have shown the first assertion of the theorem. If \(F\) is a slice defined at point \(J\), then the \(N_0\)-orbit of \(J\) at this point admits a Zariski tangent space isomorphic to \(T_r/T\) with (4.1). The slice and the orbit are transversal at \(J\), and the tangent space of \(F\) is equal, as quotient of the tangent of \(W_\sigma^r(T)\) by \(T_r/T\), to \(H^2(n, n)^T\) [3].

The semi-continuous mapping \(n \to r = \dim(n/[n, n])\) involves a stratification \(\cup_{r \geq r_0} \Sigma^{(r_0)}_n(T)\) on \(\Sigma_n(T)\). The minimal value \(r_0\) gives an open stratum \(\Sigma^{(r_0)}_n(T)\). All quotients of \(\mathfrak{L}_\rho\) are quotients of \(\mathfrak{L}_r\) if \(\rho < r\), and from above we have isomorphisms, for each \(r\):

\[ W_\sigma^r(T)/N_0 \simeq \cup_{\rho \leq r} \Sigma^{(\rho)}_n(T)/G_0. \]

5. Study of the rigidity in varieties of ideals

**Proposition 5.1.** — If \(g = T \oplus n\) is a semi-direct product with maximal \(T > 0\), \(n(\alpha_i) = 1\) for \(1 \leq i \leq r\), then \(g\) is a complete Lie algebra and conditions i) ii) iii) are equivalent:

i) \(g\) is rigid in \(L_n(T)\);

ii) \(n\) is rigid in \(L_n^\sigma(T)\);

iii) \(n\) is rigid in \(V_\sigma^r(T)\) or \(W_\sigma^r(T)\).

Local rings of the different slices at \(n\) are isomorphic and the obstructions are the same.

**Proof.** — We have \(n(\alpha_i) = 1\) for the generators \(\bar{e}_i(1 \leq i \leq s)\) of \(n\), and then \(\text{Der}(n)^T = T\) and \(g\) is complete [4]. Equivalence i) \(\Leftrightarrow\) ii) results from reduction theorem [6] and ii) \(\Leftrightarrow\) iii) from Theorem 4.4. \(\square\)
Example 1: Series of rigid Lie algebras defined by one relation

Let $a, b > 0$ be two integer numbers with $(a, b) = 1$ and $r = a/b$. We consider the torus $T \subset T_2$ on $\mathfrak{L}_2$, $T = \text{Ker}(b\varepsilon_2 - a\varepsilon_1)$, $\alpha_2 = r\alpha_1$, with $\alpha_i = \varepsilon_i | T$ and $L_{m\alpha_1} = \oplus \{ L_{pe_1 + qe_2}; p + qr = m \}$ if $m \in \mathbb{Q}^+$. For $\nu = (a + 1 + r)\alpha_1$ we obtain the two-dimensional space $L_\nu = L_{(a+1)\varepsilon_1 + \varepsilon_2} \oplus L_{\varepsilon_1 + (b+1)\varepsilon_2}$. If we write a vector $u \in L_\nu$ as $u_1 + u_2$, according to this sum, and if $\langle u \rangle$ is the ideal generated by $u$ in $\mathfrak{L}_2$, then $u_1 \neq 0$, $u_2 \neq 0$ and $m > a + 1 + r$ involve that the ideal $J_{(m)} = \langle u \rangle + (\oplus_{k\geq m} L_{ka_1})$ is $T$-invariant and $T$ is maximal on $\mathfrak{L}_2/J_{(m)}$. The weight systems $\sigma_n$ of this quotients, associated with $T$, satisfy hypotheses of Proposition 5.1. The group $N_0$ is defined by $s(e_i) = s_i e_i$, $s_i \in \mathbb{C}^\ast$, on the generators $e_1$, $e_2$, and the orbit of $J_{(m)}$ is given by the action on $\mathbb{C}^\ast u$: $s(u) = s_i^{a+1} s_2 u_1 + s_1 s_2^{b+1} u_2$. It is an open set in the projective space $\mathbb{P}_1(L_\nu)$ of the lines of $L_\nu$. A slice is given by fixing $\mathbb{C}^\ast u$ in $L_\nu$. Thus, we obtain isolated points and the algebras $\mathfrak{L}_2/J_{(m)}$ are rigid in the schemes $V^\ast_2(u)(T)$ and $L^T/T_n$, according to Proposition 5.1. The two-$T$-cohomological group of these algebras, calculated with formula [3], is 0. If the dimension of $L_\nu$ is greater than or equal to 3, then we can obtain continuous families by this method.

The second example proposed here shows how an obstruction appears in this formalism.

Example 2: The local study of $\mathfrak{a}_{4,n}$ defined by generators and relations

Let $\mathfrak{L}_3$ be the free Lie algebra with 3 generators indexed by $e_1, e_2, e_4$, $T$ be the torus $\text{Ker}(2\varepsilon_2 - 2\varepsilon_1) \subset T_3$ with weights $\alpha_i$ satisfying $\alpha_2 = 2\alpha_1$ and $\mathfrak{L}_3 = \oplus L^m$ be the graduation defined by $L^m = \oplus \{ L_{pe_1 + qe_2 + re_4}; m = p + 2q + 4r \}$. We search for a sequence of $T$-invariant ideals $J_6 \supset J_7 \supset J_8 \cdots$ of $\mathfrak{L}_3$ such that the quotients are isomorphic to $\mathfrak{a}_{4,n}$ (Remark 2.7). The weights on $\mathfrak{a}_{4,n}$ are $\alpha_1, 2\alpha_1, 3\alpha_1, \alpha_4 + p\alpha_1$. These ideals contain the ideal $I$ generated by the subspaces $L_{pa_1 + qa_4}$ with $p > 3$ and $q = 0$, or $p \geq 0$ and $q > 1$. We have $L^1 = \mathbb{C}[e_1]$, $L^2 = \mathbb{C}[e_2]$, $L^3 = \mathbb{C}[e_1, e_2]$, $L^4 = \mathbb{C}[e_4] + \mathbb{C}[e_1, e_1, e_2]$, $L^5 = L_{5\alpha_1} \oplus L_{\alpha_4 + \alpha_1}$ and $J_n = I + \sum_{m>n} L^m$ for $n = 4, 5$.

For $n = 6$, we have $L^6 = L_{6\alpha_1} \oplus L_{\alpha_4 + 2\alpha_1}$, and we fix the line in $L_{\alpha_4 + 2\alpha_1} = \mathbb{C}(a[e_1])^2 e_4 + \mathbb{C}[e_2, e_4]$, $C$-generated by a vector $a[e_2, e_4] + b[e_1, e_1, e_2]$, $ab \neq 0$. It is stabilized by the subgroup of the $(s_1, s_2, s_4) \in (\mathbb{C}^\ast)^3$ such that $s_2 = (s_1)^2$. The choice $ab \neq 0$ breaks the $T_3$-invariance and $T$ becomes maximal as a torus over $J_6$ and the quotient as well. This corresponds to
the initialization in the induction process. All choices \( ab \neq 0 \) give the same
quotient, up to an isomorphism, and we choose \( u = [e_2, e_4] - [e_1, [e_1, e_4]] \),
\( J_9 = \mathbb{C}u + I + \sum_{m \geq 6} L^m \). For \( n = 7 \), we have \( L^7 = L_{7\alpha_1} + L_{\alpha_4+3\alpha_1} \), and
\[
L_{\alpha_4+3\alpha_1} = \mathbb{C}(ade_1)^3 e_4 + \mathbb{C}[e_1, [e_2, e_4]] + \mathbb{C}[e_2, [e_1, e_4]]
\]
contains \([e_1, u]\). The 2-dimensional spaces \( V, \mathbb{C}[e_1, u] \subset V \subset L_{\alpha_4+3\alpha_1} \), are
given by an additional vector
\[
v = \lambda[e_1, u] + x[e_1, [e_2, e_4]] + y[e_2, [e_1, e_4]] \notin \mathbb{C}[e_1, u],
\]
with \( x \neq 0 \) or \( y \neq 0 \). The ideals \( J_7 = \mathbb{C}v + \langle u \rangle + I + \sum_{m \geq 7} L^m \) define by
quotient the family \( a_{4,7}(t) \). For \( n = 8 \), we have \( L^8 = L_{8\alpha_1} + L_{\alpha_4+4\alpha_1} \) and
\( J_8 = \langle u \rangle + \langle v \rangle + I + \sum_{m \geq 8} L^m \). For \( n = 9 \), the ideal \( J_9 \) must contain the
ideal \( \langle u \rangle + \langle v \rangle + I + \sum_{m \geq 9} L^m \) and we study its codimension in \( \mathcal{L}_3 \) i.e., the
dimension of \( L_{\alpha_4+5\alpha_1}/(\langle u \rangle + \langle v \rangle + I)_{\alpha_4+5\alpha_1} \) depending on \( v \). This quotient
is isomorphic to \( E/(\langle u \rangle + \langle v \rangle)_{\alpha_4+5\alpha_1} \), where \( E \) is a \( T_3 \)-invariant complement
subspace of the intersection with \( I \) in \( L_{\alpha_4+5\alpha_1} \). We can generate \( E \) with
the following vectors:
\[
\mu = (ade_1)^5 e_4 \text{ in } L_{\varepsilon_4+5\varepsilon_1}, \nu = (ade_1)^3 [e_2, e_4] \text{ and } \rho = (ade_1)^2 ([e_4, [e_1, e_2]])
\]
in \( L_{\varepsilon_4+2\varepsilon_2+3\varepsilon_1} \); \( \sigma = ade_1 (ade_2)^2 e_4 \) and \( \delta = [[e_1, e_2], [e_2, e_4]] \) in \( L_{\varepsilon_4+2\varepsilon_2+\varepsilon_1} \).

We calculate the dimension of \( (\langle u \rangle + \langle v \rangle)_{\alpha_4+5\alpha_1} \), which is equal to the rank
of the system of the following vectors written over the basis \( \{\mu, \nu, \rho, \sigma, \delta\} \):
\[
(ade_1)^3 u, (ade_1)([e_2, u]), [[e_1, e_2], u], (ade_1)^2 v, [e_2, v].
\]
The dimension of \( E \) is equal to 5 and the dimension of \( (\langle u \rangle + \langle v \rangle)_{\alpha_4+5\alpha_1} \)
depending on \((x, y) \in \mathbb{C}^2 \) is given by one of the two following cases:

- If \( x + y \neq 0 \), the dimension is 5 and there is not possible extension for \( a_{4,8}(t), t \neq 0 \).
- If \( x+y = 0 \), the dimension is 4 and we have an extension corresponding to
  \( t = 0 \). Moreover, if \( y = -x \neq 0 \), the algebra corresponds to an isolated point
  \( \mathcal{J}_9 \), rigid in the variety \( W^p_1(T) \) or \( \Sigma_9(T)(\subset L^0_p) \). In this case we obtain a
  constraint between the vectors generating the weight space \( (\langle u \rangle + \langle v \rangle)_{\alpha_4+5\alpha_1} \)
given by:
\[
D(v) + D'(u) = (ade_2 - (ade_1)^2)v + (\lambda(ade_1)^3 + yade_1ade_2)
- (\lambda + x + 2y)ade_2ade_1)u \in I + \sum_{m>9} L^m.
\]
Calculation of $H^2(a_{4,n}, a_{4,n})^T$

We have:

$$H^2(a_{4,n}, a_{4,n})^T \simeq \{ f \in \text{Hom}_\mathbb{C}(\langle v \rangle, a_{4,n})^{2+T}; f(\langle u \rangle \cap \langle v \rangle) = 0 \}.$$  

We replace $J_n$ by $\langle u \rangle + \langle v \rangle$ in Theorem 4.5. The vector $f(u) = h \tilde{e}_6$ corresponds to $\alpha_4 + 2\alpha_1$. Observe that $T_3$ is embedded in natural way in $\text{Hom}(J_n, a_{4,n})^{2+T}$ with $T_3(u) \neq 0$, hence there is $\delta \in T_3$ such that $f_0 = f - h\delta$ is null on $\langle u \rangle$.

With this formula, the cohomological group is null for $n = 6$. For $n = 7$, we have a representative $f$ in the class defined by $f(v) = a\tilde{e}_7$ and $f(u) = f([e_1, u]) = 0$. For $n = 8$, we have $f([e_1, v]) = [e_1, f(v)] = a[e_1, \tilde{e}_7] = a\tilde{e}_8$ and $f(\langle u \rangle) = 0$. For $n = 9$, $f$ is compatible with the constraint expressed by:

$$f(Dv + D'u) = f(Dv) = Df(v) = a(ade_2 - (ade_1)^2)\tilde{e}_7 = 0.$$  

Thus, we have representatives $f \neq 0$ defined by $f(v) = a\tilde{e}_7$, $a \in \mathbb{C}^*$ for $n \geq 7$ and the second cohomological group is $\mathbb{C}$. Rigidity for $n \geq 9$ involves the existence of an obstruction. We calculate this obstruction, illustrating the last method.

**Remark 5.2. —** In the case where the second $T$-cohomological group becomes null in a central extension, compatibility of $f$ with the constraints is not satisfied and $f = 0$.

**Obtaining a nilpotent element in the scheme of ideals**

Theorem 4.4 allows us to obtain a nilpotent element in the slice of $W_3^T(T)$ by applying the simple ideal condition for $J_9$. It suffices to show here that the vector $(ade_1)^2 u$ belongs to the space generated by $(ade_1)^3 u$, $ade_1 ade_2(u)$, $(ad[e_1, e_2])u$ and $(ade_2)v$ modulo $I$. Thus, we have:

$$(ade_1)^2 v \equiv p(ade_1)^3 u + q(ad[ade_1 ade_2])u + r(ad[e_1, e_2])u + s(ade_2)v \mod (I)$$

where parameters $p, q, r, s, \lambda, x, y$ are chosen in the local ring of the scheme at the point $J_9$. Writing this equality on the basis vectors $\mu, \sigma, \delta, \nu$ and $\rho$, we obtain the following equalities respectively: (a) $p = \lambda$; (b) $q = -s(\lambda + x + y)$; (c) $r = s(\lambda + x + 2y)$; (d) $p - q - \lambda s = \lambda + x + y$; (e) $-2q + r - 3\lambda s = y$.

From (d) and (e), we deduce $(x + y)(s - 1) = 0$ and $(s - 1)y = -3s(x + y)$. Dividing by $y$ in the local ring because $\bar{y} \neq 0$ (second case $x + y = 0$ above),
we have \( s(x + y)^2 = 0 \) and \((s - 1)^2 = 0\). The parameter \( s \) can be inverted in the local ring and we obtain \((x + y)^2 = 0\).

Note that \((x, y) \in \mathbb{C}^2 - \{(0, 0)\}\) corresponds to a law satisfying, in the quotient by \(\mathcal{L}\): \(\hat{\vartheta} = x[e_1, e_2, e_4] + y[e_2, e_1, e_4] = 0\). If \(y \neq 0\), then we have \([e_2, e_1, e_4] = -\frac{x}{y}[e_1, e_2, e_4]\) and we can write \(-x/y = 1 - t\) with \(t\) given in Remark 2.7.

The case \(y = 0\) involves \([e_1, e_2, e_4] = 0\) with \(x \neq 0\) and defines the algebras:

\[
(A_7)\quad [e_1, e_2] = e_3, \quad [e_1, e_4] = e_5, \quad [e_1, e_5] = e_6,
\]

for \(n = 7\) and, by adding the following brackets:

\[
(A_8)\quad [e_1, e_7] = e_8, \quad [e_2, e_6] = 2e_8, \quad [e_3, e_5] = -e_8,
\]

for \(n = 8\). The nilpotent condition \((x + y)^2 = 0\) for \(n \geq 9\) gives \(t^2 = 0\) and we state:

**Proposition 5.3.** — If \(T\) is defined by the weights \(\alpha_1, 2\alpha_1, 3\alpha_1, \alpha_4 + k\alpha_1 (k \geq 0)\) with multiplicities one and if \(\mathcal{A} = \{(24), (1k)\} (1 < k < n, k \neq 3)\), a slice of the scheme \(W_3^\sigma(T)\) is given, up to isomorphism, by:

- The union of the scheme \(L_n^{T, A}\) and the point \(\{A_n\}\) for \(n = 7, 8\);
- \(L_n^{T, A} = \{a_{4n}(t)\}\) where \(t = \frac{x+y}{y}\) is a 2-nilpotent parameter for \(n \geq 9\).

**Case \(r > r_0\)**

If \(T\) is the torus defined in Proposition 5.3. then the set \(\mathcal{A}_0 = \{(1j), j \geq 4, (24)(34)\}\) is admissible in \(L_n^T(\mathbb{C})\) for \(n \geq 7\). The associated slice is given by \(X_{12} = t\), \(X_{1j} = 1(j \geq 4)\), \(X_{24} = 1\), \(X_{25} = 1 - t\), \(X_{34} = 1\) for \(n = 7\), adding \(X_{17} = 1\), \(X_{26} = 1 - 2t\) and \(X_{35} = 1\) for \(n = 8\). If \(n \geq 9\) we have \(t = 0\), thus we obtain the following rigid Lie algebra satisfying \(H^2(n_n^{(4)}, n_n^{(4)})^T = 0\):

\[
n_n^{(4)} : [x_i, x_j] = \begin{cases} 
  x_{i+j} & \text{for } 1 \leq i \leq 3, \ 4 \leq j \leq n - i \\
  0 & \text{otherwise } i < j.
\end{cases}
\]

This algebra is the unique 4-generated Lie algebra in \(\Sigma_n(T)\) for \(n \geq 7\) belonging, as quotient \(\mathcal{L}_4/\mathfrak{J}_n\), to \(W_4^\sigma(T)\). We have \(\dim(L_{3\alpha_1}) = 2\) and we obtain the algebras above as quotients of \(\mathcal{L}_4\) by 7-codimensional ideals \(\mathfrak{J}_7\) where the projection \((\mathfrak{J}_7)_{3\alpha_1}\) on \(L_{3\alpha_1} = L_{\varepsilon_3} \oplus L_{\varepsilon_1 + \varepsilon_2}\) is the line generated by \(te_3 - [e_1, e_2]\) for \(t \in \mathbb{C}\). If \(t \neq 0\), then the quotients \(\mathcal{L}_4/\mathfrak{J}_7\) are in fact quotients of \(\mathcal{L}_3\) describing the open set of 3-generated algebras. If \(t = 0\), then we obtain \(n_7^{(4)}\) with \(\mathfrak{J}_7 \subset [\mathcal{L}_4, \mathcal{L}_4]\).
Conclusion

Extrapolating this work, the idea that rigidity is a property which is not dependent on the particular choice of a geometry constitutes a valid new slant. Most generally, we can imagine a notion of continuous family attached to the category and not depending on a particular geometrical representation. Theorem 4.4 and Proposition 5.1 certainly move in this direction with two different geometrizations for an important class of nilpotent Lie algebras. This explains why different methods in classifications of nilpotent Lie algebras give the same continuous families, with different parameterizations depending only on the choice of a local chart.

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