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DIFFERENT METHODS FOR THE STUDY OF OBSTRUCTIONS IN THE SCHEMES OF JACOBI

by Roger CARLES & M. Carmen MÁRQUEZ

Abstract. — In this paper the problem of obstructions in Lie algebra deformations is studied from four different points of view. First, we illustrate the method of local ring, an alternative to Gerstenhaber's method for Lie deformations. We draw parallels between both methods showing that an obstruction class corresponds to a nilpotent local parameter of a versal deformation of the law in the scheme of Jacobi. Then, an elimination process in the global ring, which defines the scheme, allows us to obtain nilpotent elements and to describe the global method. Finally, the obstruction problem is studied in the geometry defined by generators and relations. Under certain conditions, we prove that subschemes of grassmannians of T-invariant ideals of a free Lie algebra (T being a torus of derivations), after quotient by an action group, are the same as those defined from Jacobi polynomials after a similar quotient.

Résumé. — Le problème des obstructions aux déformations d'algèbres de Lie est étudié de quatre points de vue différents. On illustrera d'abord la méthode de l'anneau local, une alternative à la méthode de Gerstenhaber. On compare les deux méthodes en montrant qu'une classe d'obstruction correspond à un paramètre local nilpotent d'une déformation verselle de la loi dans le schéma de Jacobi. Un procédé d'élimination dans l'anneau global permet ensuite d'obtenir des éléments nilpotents, constituant ainsi une méthode globale. Enfin, le problème des obstructions est traité dans la géométrie définie par générateurs et relations. Des sous-schémas de grassmanniennes constitués d'idéaux T-invariants d'une algèbre de Lie libre (T étant un tore bien choisi), après quotient par une action de groupe, sont égaux à ceux définis par les polynômes de Jacobi après passage à un quotient similaire.

Introduction

Let L_n be the scheme of laws of Lie algebras of dimension n over $\mathbb{C}$ defined by antisymmetry and Jacobi identities and called "scheme of Jacobi". If R is a completely reducible Lie subalgebra of the space $C^1(\mathbb{C}^n, \mathbb{C}^n)$ of

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linear morphisms $\mathbb{C}^n \to \mathbb{C}^n$, then we can impose R-invariance conditions for the laws in L_n. Thus, we obtain a subscheme of L_n denoted by L_n^R and let $L_n^R(\mathbb{C})$ be the set of its points. This scheme was introduced in the study of algebraic Lie algebras admitting a Chevalley decomposition $\mathfrak{g} = R \oplus \mathfrak{n}$ with nilpotent part $\mathfrak{n} = (V, \Phi_0)$, for $V = \mathbb{C}^n$. It is well known that $L_n^R \simeq \operatorname{Spec}(\mathbb{C}[X_{ij}^k]/\operatorname{Jac}_n)$, where Jac_n is the ideal in the ring of polynomials $\mathbb{C}[X_{ij}^k]$, for $1 \leqslant i < j \leqslant n$, $1 \leqslant k \leqslant n$, generated by the antisymmetry, Jacobi identities and R-invariance conditions. We can deduce the local study of $\mathfrak{g}$ in the scheme L_m, with $m = n + \dim(R)$, from the local study of $\mathfrak{n}$ in L_n^R under certain conditions on R. This type of result enters in the scope of the "Theorem of reduction" where a general statement is proposed in [6]. R can be a torus T (*i.e.,* abelian and reducible) satisfying hypotheses of the reduction theorem. This allows us to work directly in Jacobi scheme L_n^T and local results obtained for $\mathfrak{n}$ in L_n^T are valid for $\mathfrak{g}$ in L_m. It suffices to choose a maximal T for at least one law. According to Mostow, all maximal tori over a complex Lie algebra are conjugated by automorphisms. The schemes used in this paper are T-invariant but most of the results can be transferred to schemes L_m thanks to the reduction theorem.

This paper is organized in five sections as follows:

1 – Section 1 deals with the classical theory of obstructions, which was initiated by M.Gerstenhaber [12] for associative laws within the framework of formal deformations. The fact that a vector Φ_1 in the Zariski tangent space of L_n^R at Φ_0, $Z^2(\mathfrak{n}, \mathfrak{n})^R$, cannot be lifted to a curve $\Phi_0 + \sum_{k \in \mathbb{N}} t^k \Phi_k$, but only to a "truncated deformation" up to an order $p \geqslant 1$, leads to the existence of a non null 3-class $\overline{\omega}_{p+1} \in H^3(\mathfrak{n}, \mathfrak{n})^R$, called obstruction. If Φ_1 doesn't belong to the tangent space of the reduced scheme at Φ_0, defined by the radical of the ideal Jac_n, $\sqrt{\operatorname{Jac}_n}$, then it always presents an obstruction and the scheme L_n^R is not reduced at Φ_0. A certain number of technical difficulties are attached to this method, in particular the dependence on the choice of partial solutions $\Phi_2, \Phi_3, ..., \Phi_p$ in deformation equations.

The examples in [15] satisfying $\overline{\omega}_2 \neq 0$ are solved by Rauch rigidity criterion. This criterion is applied in [14] to Lie algebras $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}^n$, with $R = \mathfrak{sl}(2, \mathbb{C}) \neq T$. The first T-rigid examples known with $\overline{\omega}_p \neq 0$ for $p > 2$ are provided by filiform Lie algebras $\mathfrak{f}_n$ for $n \geqslant 12$ [1] and the obstruction appears at order 5. We illustrate this method sketching out cohomological calculations and using the fact that if $H^2(\mathfrak{n}, \mathfrak{n})^R$ is equal to $\mathbb{C}$ then the choice of $\Phi_k (1 < k < p)$ for $1 < p \leqslant 4$ is irrelevant.

We prove a useful result which provides a link to the local ring method developed in section 2: $\overline{\omega}_{p+1} = 0$ is equivalent to the existence of a parameter t in the maximal ideal of the local ring $\mathcal{O}$ at Φ_0, such that t^{p+1} doesn't belong to an ideal $\mathfrak{B}$ whose quotient $\mathcal{O} \to \mathcal{O}/\mathfrak{B}$ defines a deformation $\Phi_0 + \bar{t}\Phi_1 + \cdots + \bar{t}^{p+1}\Phi_{p+1}$ at order $p+1$. More concretely, in the study of $\mathfrak{f}_n$ we obtain an obstruction at order 5, so $\overline{\omega}_5 \neq 0 \Leftrightarrow t^5 = 0$ ($t^4 \neq 0$).

2 – The 1-parameter deformation method is not characteristic enough, as it gives only partial results. The right way is to introduce versal deformations, which describe the deformation question completely. This was done by Fialowski [9] and [10]. In [9] and later in [11] a straightforward method was given to construct a versal deformation. This construction starts with determining the universal infinitesimal deformation, and extending it step by step. In [5, 6] authors have developed a method giving versal deformations from the universal deformation constituted by the germs of coordinate functions at the point Φ_0. The local ring $\mathcal{O}$ at Φ_0 in Jacobi scheme L_m or L_n^R gives maximal information about the local deformation problem. A deformation of Φ_0 on a local ring A is a local morphism from $\mathcal{O}$ to A. In [6], versal deformations are obtained by reducing the number of parameters with a quotient of $\mathcal{O}$ and the equivalence of two versal deformations as a solution of a universal problem is proved. A comparison with the Fialowski method (*cf.* [9] [10] [11]) is also made in [6]. The normalizing group H of a torus T in $GL(\mathfrak{n})$ acts canonically on L_n^T and its orbits are the isomorphism classes of laws in a good open set. We can define local charts for the space $L_n^T/H_0 = L_n^T/G_0$, where G_0 and H_0 are unit components of the groups $GL(\mathfrak{n})^T$ and H respectively. We fix the coordinates in L_n^T which are labelled with a certain choice of indices called an admisible set $\mathcal{A}$. Under certain conditions, it defines a sub-scheme $L_n^{T,\mathcal{A}}$ of $L_n^T(\mathbb{C})$, called slice, which is transversal to each orbit in a certain open set of $L_n^T(\mathbb{C})$. A versal deformation in L_n^T at the point Φ_0 can be seen as the canonical deformation in a slice at Φ_0.

In section 2 this second method, called "local ring method", will be illustrated by new examples showing the behavior of the slices. The schemes L_n^T associated with the torus T, defined by the weights $\alpha_k = k\alpha_1$, $1 \leqslant k \leqslant 4$ and $\alpha_5 + k\alpha_1$, $k \geqslant 0$, are studied by using the induction on central extensions. This allows us to study the relationship between the dimension and the number of essential parameters: we observe an increase and then a decrease in this number. We also present a new series of slices with a unique nilpotent parameter $t^{p+1} = 0$, $t^p \neq 0$, for each dimension $n \geqslant 3p+6$

and each $p > 0$. These important examples give cohomological obstructions $\overline{\omega}_{p+1} \neq 0$ for any $p > 0$ too.

3 – Section 3 develops the third method, which is entirely new and is attached to the global Jacobi scheme. Bearing in mind $L_n^T = \operatorname{Spec}(A_n)$ with $A_n = \mathbb{C}[X_{ij}^k]/\operatorname{Jac}_n$, the scheme is not reduced iff $\sqrt{\operatorname{Jac}_n} \neq \operatorname{Jac}_n$, *i.e.*, there are polynomials f such that $f^{p+1} \in \operatorname{Jac}_n$ and $f^p \notin \operatorname{Jac}_n$. We obtain a result on the existence and the determination of nilpotent elements in A_n corresponding to the nilpotent parameters found in local rings of the slice. The technique employed consists of applying an elimination procedure to certain coordinates in $\mathbb{C}[X_{ij}^k]$ modulo the ideal Jac_n. We proceed by reducing the dimension, in contrary to the local ring method. In the study of $\mathfrak{f}_n$, using graded coordinates X_{ij}, for each n we eliminate in Jacobi relations the coordinates X_{ij}, $i+j=n$ except for $X_{1,n-1}$, keeping the X_{hk} with $h+k<n$ and so on. Finally, the remaining coordinates are those which correspond to one essential parameter X_{34} in the local ring method and a choice of orbital ones (X_{23} and the X_{1k}). It is very striking that by using this method we obtain polynomials with great factorizations (monomials in some cases) in the ideals Jac_n and $\sqrt{\operatorname{Jac}_n}$. We find nilpotent elements in A_n which are irreducible polynomials P in $\sqrt{\operatorname{Jac}_n}$. The number of factors in P minorates the number of irreducible components of the scheme. So, factorizations obtained by this method allow us to predict interesting properties in the scheme such as rigidity, non-reduced points and number of irreducible components, which is not otherwise possible. This global procedure can be completed profitably using the previous local method.

4 – The construction by generators and relations allows us to obtain Lie algebras (up to isomorphism) as quotient a free Lie algebra $\mathfrak{L}_r$ with r generators by an ideal $\mathfrak{J}$. G. Favre gave the first geometrical approach in this context [8]. We obtain a geometrization of the nilpotent quotient algebras $\mathfrak{n} = \mathfrak{L}_r/\mathfrak{J}$ with the help of the subscheme $J_n(\mathfrak{L}_r)$ of a grassmannian constituted by ideals $\mathfrak{J}$ of codimension n in $\mathfrak{L}_r$ containing $\mathcal{C}^n(\mathfrak{L}_r)$. This subscheme structure, defined only by the simplest polynomial relations $[x, \mathfrak{J}] \subset \mathfrak{J}$ for all $x \in \mathfrak{L}_r$, is generally not reduced. The different tori T of maximal type give a finite number of subschemes $J_n^T(\mathfrak{L}_r)$ (up to isomorphism) defined by adding T-invariance relations for ideals $\mathfrak{J}$. In this space, we have the natural action of the normalizing group N of T in $\operatorname{Aut}(\mathfrak{L}_r)$. In section 4, we compare $J_n^T(\mathfrak{L}_r)/N_0$ with L_n^T/G_0 by using adapted slices, under certain conditions on T. The most surprising result obtained here is the identity

between these two types of scheme structures given by the slices. In particular the two Zariski tangent spaces at $\mathfrak{n}$ are given by $H^2(\mathfrak{n},\mathfrak{n})^T$. These results are formulated in Theorems 4.4 and 4.5. Rigidity and obstruction studies in schemes $J_n^T(\mathfrak{L}_r)$ or $L_n^T(\mathbb{C})$ are the same problem. We obtain a different perspective of the same obstruction phenomenon and an original method.

5 – Section 5 focuses on some applications of the equivalence between rigidity in $L_n^T(\mathbb{C})$ and in the scheme of ideals. For instance, the rigidity of a "model" in [3], as the nilpotent part of a Borel algebra, follows immediately from Proposition 5.1 applied to the maximal rank case. We give new examples of rigid Lie algebras only defined by one relation and admitting a one dimensional torus T. In a second example, we study the obstruction problem in this new formalism for algebras $\mathfrak{a}_{4n}(t)$ where the ideal condition $[x,\mathfrak{J}]\subset\mathfrak{J}$ involves the existence of a 2-order nilpotent parameter in the scheme of ideals for $n\geqslant 9$. The space $H^2(\mathfrak{n},\mathfrak{n})^T$ parameterizes the essential local parameters in the two geometrical approaches.

1. Return to Gerstenhaber's method of formal deformations. The integration of a 2-cocycle

Generalities on deformations in the schemes L_n^R

Let (e_i) be a basis of $\mathbb{C}^m$, A be a commutative associative $\mathbb{C}$-algebra with unity 1 and $L_m(A)$ be the set of laws of Lie A-algebras Φ defined by their structure constants $\Phi_{ij}^k\in A:\Phi(e_i,e_j)=\sum_{k=1}^m\Phi_{ij}^k e_k$. These structure constants satisfy the antisymmetry and Jacobi identities, *i.e.*, $\Phi_{ij}^k+\Phi_{ji}^k=0$ and $\sum_l \Phi_{ij}^l\ \Phi_{lk}^p+\Phi_{jk}^l\ \Phi_{li}^p+\Phi_{ki}^l\ \Phi_{lj}^p=0$. A morphism of $\mathbb{C}$-algebras $f: A\to B$ gives a map $L_m(f):L_m(A)\to L_m(B)$ defined by $\Phi_{ij}^k\mapsto f(\Phi_{ij}^k)$. The scheme L_m is a functor from the category of commutative associative $\mathbb{C}$-algebras to the category of sets. We have $L_m\simeq \mathrm{Spec}(\mathbb{C}[X_{ij}^k]/J_m)$ where J_m is the ideal of the polynomial ring $\mathbb{C}[X_{ij}^k]$, $1\leqslant i,j,k\leqslant m$, generated by the antisymmetry and Jacobi polynomials, *i.e.*, $X_{ij}^k+X_{ji}^k$ and $\sum_l X_{ij}^l\ X_{lk}^p+X_{jk}^l\ X_{li}^p+X_{ki}^l\ X_{lj}^p$. A point $\Phi_0\in L_m(\mathbb{C})$ can be identified with the ring $\mathbb{C}$-morphism $\lambda:\mathbb{C}[X_{ij}^k]/J_m\to\mathbb{C}$ defined by $\lambda(\bar{X}_{ij}^k)=(\Phi_0)_{ij}^k$ or with the maximal ideal $\mathrm{Ker}(\lambda)$.

Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{C}$. A deformation of a law $\Phi_0\in L_m(\mathbb{C})$ with base A is a law $\Phi\in L_m(A)$ such that $\mathrm{pr}(\Phi_{ij}^k)=(\Phi_0)_{ij}^k$ for all i,j,k where $\mathrm{pr}:A\to A/\mathfrak{m}\simeq\mathbb{C}$ is the quotient mapping. If $f : \mathbb{C}[X_{ij}^k]/J_m \to A$ is a ring morphism such that $\mathrm{pr} \circ f = \lambda$, it follows that $f(\mathrm{Ker}(\lambda)) \subseteq \mathfrak{m}$. Thus, using $\mathcal{O}_{\Phi_0}$ to denote the localized ring of $\mathbb{C}[X_{ij}^k]/J_m$ by $\mathrm{Ker}(\lambda)$, a deformation can be identified with a local ring morphism $f : \mathcal{O}_{\Phi_0} \to A$. In particular, if $A = \mathbb{C}[[t]]$ we obtain formal deformations [12].

Let $\mathfrak{g} = R \oplus \mathfrak{n}$ be a Lie algebra of dimension m with reducible part R and nilpotent part $\mathfrak{n}$ of dimension n. Let $L_n^R(\mathbb{C})$ denote the set of laws $\Phi \in L_n(\mathbb{C})$ such that $\delta \cdot \Phi = 0$ for all $\delta \in R$. Let Δ_n be the ideal of $\mathbb{C}[X_{ij}^k]$ generated by the polynomials $(\delta \cdot \Phi)_{ij}^k$. We can consider L_n^R the sub-scheme of L_n isomorphic to $\mathrm{Spec}(\mathbb{C}[X_{ij}^k]/(J_n + \Delta_n))$. We denote by Jac_n the ideal $J_n + \Delta_n$ and by A_n the ring $\mathbb{C}[X_{ij}^k]/(J_n + \Delta_n)$. A deformation of a law $\Phi_0 \in L_n^R(\mathbb{C})$ over a local ring A given as base, with canonical projection $\mathrm{pr} : A \longrightarrow A/\mathfrak{m} \simeq \mathbb{C}$, is a law Φ of $L_n^R(A)$ satisfying $\mathrm{pr}(a_{ij}^k) = (\Phi_0)_{ij}^k$ for all multi-indices. We obtain a deformation functor at Φ_0, denoted by $\mathrm{Def}^R(\Phi_0, -)$, which may also be represented with the local ring $\mathcal{O}_{\Phi_0}^R$ of the scheme at Φ_0; thus $\mathrm{Def}^R(\Phi_0, A) = \mathrm{Hom}_{loc}(\mathcal{O}_{\Phi_0}^R, A)$.

An interpretation of different notions attached to the usual schemes L_m can easily be formulated in the schemes L_n^R. In particular, $Z^2(\mathfrak{n},\mathfrak{n})^R$ is the Zariski tangent space of L_n^R at Φ_0, and $B^2(\mathfrak{n},\mathfrak{n})^R$ is the tangent space at Φ_0 to the orbit of Φ_0 by the neutral component $G_0 = GL(n)_0^R$ under the classical action $\star$. Moreover, we obtain the analogous classical equivalence between the following two conditions: i) *the orbit of* $\mathfrak{n}$ *is open in* $L_n^R(\mathbb{C})$ *and the scheme is reduced at* $\mathfrak{n}$; ii) $H^2(\mathfrak{n},\mathfrak{n})^R = 0$. [5]

Generalities on formal deformations in the schemes L_n^R

The classical theory of 1-parameter formal and analytic deformations developed by Gerstenhaber and Nijenhuis-Richardson [12, 13] is valid for the varieties L_n^R of R-invariant laws if R is completely reducible. The space $C = C(V,V)^R = \oplus_p C^p(V,V)^R$, for $C^p(V,V)^R$ the space of p-alternating R-invariant mappings $f : \wedge^p V \to V$, is a graded Lie superalgebra for the bracket defined in [13] by $[f,g] = f \bullet g - (-1)^{(p-1)(q-1)} g \bullet f$ for $(f,g) \in C^p \times C^q$. The usual differential d of the Chevalley-Eilenberg's cohomology on C satisfies $df = (-1)^{p-1}[\Phi_0, f]$ for $f \in C^p$. A deformation of a law $\Phi_0 \in L_n^R(\mathbb{C})$ is an analytic curve $\Phi(t)$, $(|t| < \varepsilon)$, with $\Phi(0) = \Phi_0$, contained in $L_n^R(\mathbb{C})$.

The expansion $\Phi(t) = \Phi_0 + t\Phi_1 + \cdots + t^k \Phi_k + \cdots$ with $\Phi_k \in C^2$ is a deformation of Φ_0 if $\Phi(t) \in L_n^R(\mathbb{C}[[t]])$ *i.e.*, if we have $[\Phi(t), \Phi(t)] = 0$. Using symmetry of the bracket for degree 2 and identifying formal developments, we obtain the following sequence of deformation equations:

$$(1.1)\qquad (E_p) \qquad d\Phi_p = \frac{1}{2}\sum_{p>k>0} [\Phi_k, \Phi_{p-k}] =: \omega_p.$$

If we solve these equations successively, we get: for $p = 1$, $d\Phi_1 = 0$ (Φ_1 is a tangent vector); $p = 2$, $d\Phi_2 = \frac{1}{2}[\Phi_1, \Phi_1] = \Phi_1 \bullet \Phi_1$; $p = 3$, $d\Phi_3 = [\Phi_1, \Phi_2]$ and so on.

Lemma 1.1. — *If the equations (E_k) are solvable up to the order $p-1$, i.e., if $\Phi_0 + \sum_{k=1}^{p-1} t^k\Phi_k$ is a truncated solution or a solution modulo t^p, then $\omega_p \in Z^3(\mathfrak{n}, \mathfrak{n})^R$. The class $\overline{\omega}_p \in H^3(\mathfrak{n}, \mathfrak{n})^R$ is called the obstruction to the deformation at order p.*

Proof. — Similar to the proof in [12] [13]. □

A tangent vector in $Z^2(\mathfrak{n}, \mathfrak{n})^R$, called infinitesimal deformation, is integrable if we can solve all successive equations (E_p) for all $p > 0$ or equivalently $\overline{\omega}_p = 0$ for all $p > 0$. Integrability of Φ_1 only depends on its class $\overline{\Phi}_1 \in H^2(\mathfrak{n}, \mathfrak{n})^R$. Indeed, the action of the group generated by $1 + t\mathbb{C}[[t]] \otimes C^1$ on deformations sends the linear part Φ_1 to $\Phi_1 + df$ with $f \in C^1$. If $H^3(\mathfrak{n}, \mathfrak{n})^R = 0$, then all obstructions are null and each Φ_1 is integrable; in this case $\mathfrak{n}$ is a simple point of $L_n^R(\mathbb{C})$.

Consider the local ring $\mathcal{O}$ at Φ_0 in the scheme L_n^R, $\mathfrak{m}(\mathcal{O})$ its maximal ideal and a sequence $\Phi_k \in C^2(\mathfrak{n}, \mathfrak{n})^R$, $k \geqslant 1$. If τ is a free variable, we have:

Proposition 1.2. — *Let $\Phi_1 \neq 0$, $\Phi_2, \cdots, \Phi_p$ be a series of solutions of (E_k), i.e., $\overline{\omega}_k = 0$ for $k \leqslant p$. There exist an ideal $\mathfrak{A}$ of $\mathcal{O}$ and $t \in \mathfrak{m}(\mathcal{O})$, such that $t^p \notin \mathfrak{A}$, $t^{p+1} \in \mathfrak{A}$ and $\mathcal{O}/\mathfrak{A} \simeq \mathbb{C}[t]/(t^{p+1}) \simeq \mathbb{C}[\tau]/(\tau^{p+1})$. Moreover, the following conditions are equivalent:*

- i) $\overline{\omega}_{p+1} = 0$*;*
- ii) $t^{p+1} \neq 0$ *and there exists an ideal $\mathfrak{B}$ of $\mathcal{O}$ such that $\mathfrak{A} = \mathbb{C}t^{p+1} \oplus \mathfrak{B}$.*

Proof. — Since $\overline{\omega}_k = 0$ for $1 \leqslant k \leqslant p$, we have a deformation $\Phi_0 + \bar{\tau}\Phi_1 + \cdots \bar{\tau}^p\Phi_p$ with $\bar{\tau} \equiv \tau \mod (\tau^{p+1})$ or equivalently there exists a surjective local morphism $f : \mathcal{O} \to \mathbb{C}[\bar{\tau}]$, $f(X_{ij}^k - (\Phi_0)_{ij}^k) = \bar{\tau}(\Phi_1)_{ij}^k + \cdots$. If $\Phi_1 \neq 0$, there are indices i, j, k with $(\Phi_1)_{ij}^k \neq 0$. There exist $u \in \mathcal{O}$ with a triangular system $f(u^l) = \bar{\tau}^l + \cdots$ for $1 \leqslant l \leqslant p$ and a linear combination $t = \sum_{1\leqslant l\leqslant p} \lambda_l u^l$ with $f(t) = \bar{\tau}$. Then f is surjective and the ideal $\mathfrak{A} = \mathrm{Ker}(f)$ satisfies announced statements.

i)⇒ ii): Condition $\overline{\omega}_{p+1} = 0$ means that we have a deformation g extending f, *i.e.*, we have a deformation $\Phi_0 + \tilde{\tau}\Phi_1 + \tilde{\tau}^2\Phi_2 + \cdots + \tilde{\tau}^{p+1}\Phi_{p+1}$ with $\tilde{\tau} \equiv \tau \mod (\tau^{p+2})$. We have $\pi \circ g = f$ where π is the canonical projection

$\mathbb{C}[\tau]/(\tau^{p+2}) \to \mathbb{C}[\tau]/(\tau^{p+1})$ defined by $\pi(\tilde{\tau}) = \bar{\tau}$ and $\mathrm{Ker}(\pi) = \mathbb{C}\tilde{\tau}^{p+1}$. Let u be such that $g(u) = \tilde{\tau}$ as above. We have $\pi(g(\mathfrak{A})) = 0$ and $g(\mathfrak{A}) \subset \mathbb{C}\tilde{\tau}^{p+1}$, but $g(u^{p+1}) = \tilde{\tau}^{p+1} \neq 0$. Equality $f(u) = f(t)$ gives $f(u^{p+1}) = f(t^{p+1}) = 0$ and $u^{p+1} \in \mathfrak{A}$. We obtain $g(\mathfrak{A}) = \mathbb{C}\tilde{\tau}^{p+1}$ and the kernel $\mathfrak{B}$ of g satisfies $\mathfrak{A} = \mathbb{C}u^{p+1} \oplus \mathfrak{B}$. We have $f(u-t) = 0$ and then $t \equiv u + \lambda u^{p+1} \mod (\mathfrak{B})$ with $\lambda \in \mathbb{C}$, and $t^{p+1} \equiv qu^{p+1} \mod (\mathfrak{B})$ with $q = (1 + \lambda u^p)^{p+1}$ invertible in $\mathcal{O}$. We have $q\mathfrak{A} = \mathfrak{A}$, $q\mathfrak{B} = \mathfrak{B}$ and then $\mathfrak{A} = \mathbb{C}t^{p+1} \oplus \mathfrak{B}$.

ii)$\Rightarrow$ i): Due to ii), the t^k for $k \leqslant p+1$ are linearly free, modulo $\mathfrak{B}$, and the quotient deformation $h : \mathcal{O} \to \mathcal{O}/\mathfrak{B}$ can be written as $\Phi_0 + h(t)\Psi_1 + \cdots + h(t)^{p+1}\Psi_{p+1}$ with $h(t)^{p+1} \neq 0$, $\Psi_k \in C^2$. The quotient by $\mathfrak{A}/B$ gives $f :\ \mathcal{O} \to \mathcal{O}/\mathfrak{A}$, written as

$$\Phi_0 + f(t)\Psi_1 + \cdots + f(t)^p\Psi_p = \Phi_0 + \bar{\tau}\Phi_1 + \cdots + \bar{\tau}^p\Phi_p.$$

By identifying $f(t) = \bar{\tau}$, we obtain $\Psi_k = \Phi_k$ for $k \leqslant p$ and Ψ_{p+1} satisfies $d\Psi_{p+1} = \omega_{p+1}$ where ω_{p+1} is constructed with the Φ_k for $k \leqslant p$. Consequently $\overline{\omega}_{p+1} = 0$. □

The equivalence non (i)$\Leftrightarrow$ non (ii) gives a correspondence between the cohomological and local formalisms in the obstruction problem.

The obstruction $\overline{\omega}_2$

The bracket [ ] stabilizes the cocycle subspace $Z(V,V)^R$ and we can trivially define a bracket [ ] on the quotient $H(V,V)^R = Z(V,V)^R/B(V,V)^R$, $[\ ] : H^p(V,V)^R \times H^q(V,V)^R \longrightarrow H^{p+q-1}(V,V)^R$. We obtain $\overline{\omega}_2 = \frac{1}{2}[\overline{\Phi}_1, \overline{\Phi}_1] = Sq(\overline{\Phi}_1)$ with the quadratic Rim mapping Sq, giving a rigidity criterion [14] : *if* $Sq^{-1}(0) = 0$, *then* $\mathfrak{n}$ *is rigid.* The filiform Lie algebras $\mathfrak{f}_n\backslash([x_1, x_i] = x_{i+1}, [x_2, x_i] = x_{i+2})$ have a non-null obstruction at order $p > 2$ for $n \geqslant 12$ and their rigidity study cannot be deduced from this criterion.

Strong integrability for a 2-cocycle

The dependence on the choice of Φ_2, $\Phi_3 \cdots$ for solving equations (1.1) justifies the following definition.

Definition 1.3. — *A 2-cocycle is called strongly integrable up to the order p (eventually ∞) if for each choice of partial solutions Φ_k, $1 \leqslant k \leqslant p'$ of (E_k) up to p' with $1 < p' < p$, there are partial solutions for all equations (E_m) with $p' < m \leqslant p$.*

Integrability is generally not strong: if $\Phi_0 = 0$ is the abelian law with $R = 0$, we have $d = 0$ and $d\Phi_1 = 0$. For $\Phi_1 \in L_n(\mathbb{C})$, each $\Phi_2 \in C^2(V, V)$ is a solution of $d\Phi_2 = \frac{1}{2}[\Phi_1, \Phi_1] = 0$. Equation $d\Phi_3 = [\Phi_1, \Phi_2]$ admits a solution for $\Phi_2 = \Phi_1$ but no solution for $\Phi_2 \notin Z^2(\Phi_1, \Phi_1)$.

Lemma 1.4. — *If $H^2(\mathfrak{n}, \mathfrak{n})^R = \mathbb{C}\overline{\Phi}_1 \simeq \mathbb{C}$ then the integration of Φ_1 up to order 2, 3 or 4 is strong or impossible; obstructions $\overline{\omega}_2$, $\overline{\omega}_3$ or $\overline{\omega}_4$ only depend on Φ_1.*

Proof. — For $p = 2$ one sees that $\overline{\omega}_2$ only depends on $\overline{\Phi}_1$. If Φ_2 is a solution, another solution can be written as $\Phi_2' = \Phi_2 + a\Phi_1 + df$ where $a \in \mathbb{C}$ and $f \in C^1$.

For $p = 3$ we have $\omega_3' = [\Phi_1, \Phi_2'] = \omega_3 + d(2a\Phi_2 + [\Phi_1, f])$. If $\overline{\omega}_3 = 0$, the general solution at order 3 is $\Phi_3' = \Phi_3 + 2a\Phi_2 + [\Phi_1, f] + b\Phi_1 + dg$ where $b \in \mathbb{C}$, $g \in C^1$, and Φ_3 being a particular one.

For $p = 4$ and by using equality $d[\Phi, f] = -[d\Phi, f] + [\Phi, df]$ for $(f, \Phi) \in C^1 \times C^2$, we obtain:

$$\omega_4' = [\Phi_1, \Phi_3'] + \frac{1}{2}[\Phi_2', \Phi_2'] = \omega_4 + d(3a\Phi_3 + [\Phi_2, f] + (2b + a^2)\Phi_2 + [\Phi_1, g + af] + \frac{1}{2}[df, f]).$$

The obstruction $\overline{\omega}_4$ only depends on Φ_1 and the general solution at order 4 is $\Phi_4' = \Phi_4 + 3a\Phi_3 + [\Phi_2, f] + (2b + a^2)\Phi_2 + [\Phi_1, g + af] + \frac{1}{2}[df, f] + c\Phi_1 + dh$ where $c \in \mathbb{C}$, $h \in C^1$, and Φ_4 being a particular one. □

Remark 1.5. — The difference $\omega_5' - \omega_5 = dl + \frac{1}{2}[\Phi_1, [df, f]] - [f, d[\Phi_1, f]]$ where $l \in C^2$ is not necessarily a coboundary.

Application to the study of obstructions for $\mathfrak{f}_n$ ($n \geqslant 7$)

If T is the torus defined on $\mathfrak{n}$ by the weights $k\alpha_1$, for $1 \leqslant k \leqslant n$, a 2-cochain $\Phi \in C^2(\mathfrak{n}, \mathfrak{n})^T$ is written $\Phi(e_i, e_j) = A_{ij}e_{i+j}$ for $i < j < i + j \leqslant n$, where $(e_k)_{1 \leqslant k \leqslant n}$ is a basis of $\mathfrak{n}$. The differential d related to $\mathfrak{n}$ with structure constants c_{ij} gives:

$$(1.2)\qquad \begin{aligned}(d\Phi)_{ijk} = &- c_{ij}A_{i+j,k} - c_{jk}A_{j+k,i} + c_{ik}A_{k+i,j} \\ &+ c_{i,j+k}A_{jk} - c_{j,i+k}A_{ik} + c_{k,i+j}A_{ij}.\end{aligned}$$

If $\Phi \in Z^2(\mathfrak{f}_n, \mathfrak{f}_n)^T$, then it satisfies $A_{3k} + A_{2,k+1} - A_{1,k+2} = A_{2k} - A_{1k}$ for $k \geqslant 3$ and $A_{jk} = A_{j+1,k} + A_{j,k+1} = A_{j+2,k} + A_{j,k+2}$ for $k > j \geqslant 3$. We deduce from this the equalities $A_{pq} = 0$ for $3 < p < q$ and $A_{3q} =$

A_{34} for $3 < q < n-2$. Moreover, writing A_{1q} for $q > 4$ depending on A_{12}, A_{13}, A_{14}, A_{2k}, $2 < k < n-1$ and A_{34}, we also deduce that there are n linear independent parameters.

As the dimension of $B^2(\mathfrak{f}_n, \mathfrak{f}_n)^T$ is $n-1$ for $n \geqslant 7$, we obtain $H^2(\mathfrak{f}_n, \mathfrak{f}_n)^T \simeq \mathbb{C}$. Moreover, a class of 2-cocycle is not null if and only if $A_{34} \neq 0$ [1]. We choose Φ_1 by:

$$A_{1j} = A_{23} = 0,\ A_{2j} = 4 - j\ (3 < j < n-1),$$

$$A_{3j} = 1\ (3 < j < n-2),\ A_{ij} = 0\ (3 < i < j). \tag{1.3}$$

Having solved equations (E_k) with partial solutions Φ_2, Φ_3, Φ_4, we state:

Proposition 1.6.

(1) *For $n \geqslant 7$, the Lie algebra $\mathfrak{f}_n$ satisfies $H^2(\mathfrak{f}_n, \mathfrak{f}_n)^T = \mathbb{C}\overline{\Phi}_1$ with the 2-cocycle Φ_1 defined in (1.3).*
(2) *If $7 \leqslant n \leqslant 11$, then Φ_1 is integrable and $L_n^T(\mathbb{C})$ is smooth at $\mathfrak{f}_n$; $\mathfrak{f}_n$ is not rigid.*
(3) *If $n \geqslant 12$, then Φ_1 is strongly integrable up to the order 4 and the sequence of obstructions satisfies: $\overline{\omega}_2 = \overline{\omega}_3 = \overline{\omega}_4 = 0$, $\overline{\omega}_5 \neq 0$. The algebra $\mathfrak{f}_n$ is rigid in $L_n^T(\mathbb{C})$.*

We have the same results for the semi-direct product $T \oplus \mathfrak{f}_n$ in $L_{n+1}(\mathbb{C})$.

Proof. — A direct cohomological proof is obtained here with the help of Lemma 1.4. We can also apply Proposition 1.2 to the versal deformation $\Phi_0 + t\Phi_1 + \cdots + t^4\Phi_4$ where $t \in \mathfrak{m}(\mathcal{O})$, obtained in [6]. Thus the condition non(ii) $t^5 = 0$ and $t^4 \neq 0$, gives $\overline{\omega}_5 \neq 0$. □

2. The local ring method for studying the schemes L_n^T

Generalities on the local ring method

In this section, let T be a torus over $\mathbb{C}^n$ with weights $\alpha_i > 0$ (*i.e.*, $\alpha_i(t) > 0$ for $t \in T$, $\forall i$), let $n(\alpha_i)$ be the multiplicity of α_i and let $\Sigma_n(T)$ be the set of laws on which T is maximal. Consequently, the T-invariant laws are nilpotent. In addition we suppose the following conditions: for each $\mathfrak{n} \in \Sigma_n(T)$, the multiplicities of T-weights appearing in the quotient module $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ are 1. Thus, there are generators e_i of $\mathfrak{n}$ belonging to a diagonalizing basis such that $n(\alpha_i) = 1$, $\forall i$.

This involves equality $\mathrm{Der}(\mathfrak{n})^T = T$ and the dimension of the G_0-orbit, isomorphic to $G_0/\mathrm{Aut}(\mathfrak{n})_0^T$, is $\sum_i n(\alpha_i)^2 - \dim T$. We have the following properties, *cf.* [6]:

1. The isomorphic classes in $\Sigma_n(T)$ are the H-orbits, equal to finite unions of G_0-orbits;
2. $\Sigma_n(T)$ is the Zariski open set equal to the union of the G_0-orbits of maximal dimension. This dimension is $\sum_i n(\alpha_i)^2 - \dim(T)$;
3. $\Sigma_n(T)$ is the set of nilpotent laws $\mathfrak{n} \in L_n^T(\mathbb{C})$ for which the semi-direct product $\mathfrak{g} = T \oplus \mathfrak{n}$ is complete (*i.e.*, the derivations are inner and the center is zero). These Lie algebras satisfy the reduction theorem [6], so the local study of $\mathfrak{g}$ in $L_m(\mathbb{C})/GL(m)$ is equivalent to that of $\mathfrak{n}$ in $L_n^T(\mathbb{C})/G_0$. Assuming that multiplicities of all the weights α_i are 1, we can find slices in $L_n^T(\mathbb{C})$ playing the part of local charts in the quotient $\Sigma_n(T)/G_0 \subset L_n^T(\mathbb{C})/G_0$.

Admissible part $\mathcal{A}$ associated with Φ_0 and slice

Since weights are distinct, it follows that the elements of G_0 are $s = (s_k)_{1\leqslant k\leqslant n}$ with $s_k \in \mathbb{C}^*$ operating on Φ as

$$(s \star \Phi)_{ij}^k = \frac{s_k}{s_i s_j}\Phi_{ij}^k. \tag{2.1}$$

By the unicity of the weights, $\Phi_{ij}^k \neq 0$ involves $\alpha_i + \alpha_j = \alpha_k$. Then, the triple indices giving non-null values Φ_{ij}^k can be defined by the pairs $(i < j)$. So we can write Φ_{ij} the coordinates instead of Φ_{ij}^k and let $\mathcal{C}$ be the set of all pairs $(i < j)$.

Definition 2.1. — *A subset $\mathcal{A} \subset \mathcal{C}$ is called an admissible part associated with a law $\Phi_0 \in L_n^T(\mathbb{C})$ if the equation system $s\star\Phi_0 = \Phi_0$ is equivalent to $(s \star \Phi_0)_{ij}^k = (\Phi_0)_{ij}^k,\ (ij) \in \mathcal{A}$, and if $\mathcal{A}$ is minimal for this property* [6].

One sees that all admissible sets $\mathcal{A}$ associated with Φ_0 are contained in the set $\mathcal{I}(\Phi_0)$ of all pairs $(i < j)$ with $(\Phi_0)_{ij}^k \neq 0$. Using (2.1) we see that $\mathcal{A}$ can index a minimal system of equations equivalent to the following: $s_k = s_i s_j,\ (i < j) \in \mathcal{I}(\Phi_0)$.

In the same way, we can use the equations defining a derivation $\delta \in (\operatorname{Der}\mathfrak{n})^T = T$, *i.e.*, $(\delta.\Phi_0)_{ij}^k = (\delta_k^k - \delta_i^i - \delta_j^j)(\Phi_0)_{ij}^k = 0,\ (i < j) \in \mathcal{I}(\Phi_0)$ where δ is given by its diagonal matrix (δ_i^i) over the basis (e_i) for $1 \leqslant i \leqslant n$. If T_n is the full torus on $\mathbb{C}^n$ diagonalized by the e_i with weights ε_i for $1 \leqslant i \leqslant n$, then we have $\delta_i^i = \varepsilon_i(\delta)$. The system is equivalent to

$$\varepsilon_k = \varepsilon_i + \varepsilon_j,\ (i < j) \in \mathcal{A}, \tag{2.2}$$

and defines the torus $T = \cap \operatorname{Ker}(\varepsilon_k - \varepsilon_i - \varepsilon_j) \subset T_n$ with the weights $\alpha_i = \varepsilon_i\,|_T$.

If $\mathcal{A}$ is an admissible part associated with $\Phi_0 \in L_n^T(\mathbb{C})$ we define the slice associated with $(\Phi_0, \mathcal{A})$ as the subscheme $L_{n,\Phi_0}^{T,\mathcal{A}}$ of $L_n^T = \operatorname{Spec}(A_n)$ defined by the quotient of A_n by the ideal generated by the $X_{ij}^k - (\Phi_0)_{ij}^k$, $(ij) \in \mathcal{A}$, *i.e.*, $L_{n,\Phi_0}^{T,\mathcal{A}} \simeq \operatorname{Spec}(\mathbb{C}[X_{ij}^k]/\operatorname{Jac}_n + \langle X_\alpha - (\Phi_0)_\alpha \rangle_{\alpha\in\mathcal{A}})$. Under the above hypotheses, if $\Phi_0 \in \Sigma_n(T)$ and if $\mathcal{A}$ is an admissible part associated with Φ_0, the slice $L_{n,\Phi_0}^{T,\mathcal{A}}$ satisfies the following properties, *cf.* [6]:

i) $\Phi \in L_n^T(\mathbb{C})$ admits $\mathcal{A}$ as an admissible part iff we have $\Phi_{ij}^k \neq 0$ for all $(ij) \in \mathcal{A}$;
ii) all laws of $L_{n,\Phi_0}^{T,\mathcal{A}}(\mathbb{C})$ admit $\mathcal{A}$ as an admissible part;
iii) $L_{n,\Phi_0}^{T,\mathcal{A}}(\mathbb{C})$ is contained in $\Sigma_n(T)$ and its isomorphism classes are the traces of the H/H_0-orbits in $\Sigma_n(T)/H_0 = \Sigma_n(T)/G_0$;
iv) Φ admits $\mathcal{A}$ as an admissible part iff there is $s \in G_0$ such that $s \star \Phi \in L_{n,\Phi_0}^{T,\mathcal{A}}(\mathbb{C})$;
v) $H^2(\Phi_0, \Phi_0)^T$ is the Zariski tangent space of $L_{n,\Phi_0}^{T,\mathcal{A}}$ at Φ_0.

As a consequence, fixing $\mathcal{A}$ and Φ admitting $\mathcal{A}$ as an admissible part, all these schemes $L_{n,\Phi}^{T,\mathcal{A}}$ are conjugated to the scheme $L_n^{T,\mathcal{A}}$ defined by conditions $X_\alpha = 1$ for $\alpha \in \mathcal{A}$. Each admissible part $\mathcal{A}$ corresponds to an open set $\Omega_n(\mathcal{A})$ defined by the points such that $\bar{X}_\alpha \neq 0$ for all $\alpha \in \mathcal{A}$, where $\bar{X}_\alpha$ is the residual class in $\mathbb{C}$ at the point and $\Sigma_n(T) = \bigcup_{\mathcal{A}} \Omega_n(\mathcal{A})$. Moreover, the schemes $L_n^{T,\mathcal{A}}(\mathbb{C}) \simeq \Omega_n(\mathcal{A})/G_0$ constitute local affine charts of $\Sigma_n(T)/G_0$. The slices $L_{n,\Phi_0}^{T,\mathcal{A}}(\mathbb{C})$ in $\Sigma_n(T)/G_0$ define continuous families of L_n^T/G_0 because the orbits are finite, the finite group H/H_0 is contained in the group of permutations of weights which are not the sum of two weights.

The local ring of the slice $L_n^{T,\mathcal{A}}$ at Φ_0, denoted by $\mathcal{O}_{\Phi_0,\mathcal{A}}^T$ or simply $\mathcal{O}$, can be directly constructed from antisymmetry, Jacobi and T-invariance relations and the fixation of the structure constants $(\Phi_0)_\alpha$ for $\alpha \in \mathcal{A}$. We can choose the fixed values 1 for these coordinates. This local ring gives the universal deformation of Φ_0 in the slice or equivalently the versal deformation associated with $\mathcal{A}$ in the scheme L_n^T, [6].

Weight paths and filiations $\mathcal{A}_n \longrightarrow \mathcal{A}_{n+1}$

The local study of laws of L_n^T is made in relation to the construction of nilpotent Lie algebras by central extensions:

$$0 \longrightarrow \mathbb{C}e_{n+1} \longrightarrow (\mathbb{C}^{n+1}, \Phi_{n+1}) \longrightarrow (\mathbb{C}^n, \Phi_n) \longrightarrow 0.$$

We choose non-trivial extensions Φ_{n+1} of the law Φ_n with weights of multiplicities 1. This allows us to write the coordinates in the form X_{ij}. The torus T is extended on $\mathbb{C}^{n+1}$ by adding a weight $\alpha_{n+1} \in T^\star$ appearing in the T-module structure of the two-homological group $H_2(\Phi_n)$, which is not null [7]. There are different possible choices of paths $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$ called "paths of weights" associated with the various weights of the T-modules $H_2(\Phi_p) \neq 0$ for $p \leqslant n$. A path of weights is said to be simple if all weights are different. With this procedure, if $\Sigma_{n_0}(T) \neq \emptyset$ for the smallest integer n_0 called an initialization of the path, then it is possible to keep the same properties for all $n > n_0$. This enables us to construct the slices $L_n^{T,\mathcal{A}_n}$ by induction on n: we add one dimension with a vector e_{n+1} of weight α_{n+1} and the choice of a pair $(i_0 < j_0)$ such that $\alpha_{i_0} + \alpha_{j_0} = \alpha_{n+1}$ gives $\mathcal{A}_{n+1} = \mathcal{A}_n \cup \{(i_0, j_0)\}$. Hence, it appears:

- the new coordinates X_{ij} with $\alpha_i + \alpha_j = \alpha_{n+1}$ (X_{i_0,j_0} being fixed to 1);
- the new Jacobi polynomials J_{ijk}^{n+1}, $i < j < k$ with $\alpha_i + \alpha_j + \alpha_k = \alpha_{n+1}$.

An example of the induction process with a 3-order nilpotent parameter

We consider the following sequence of simple weights α_i defining the torus T:

$$\alpha_i = i\alpha_1 \text{ for } 1 \leqslant i \leqslant 4, \quad \alpha_{5+k} = \alpha_5 + k\alpha_1 \text{ for } k \geqslant 0. \tag{2.3}$$

The set $\Sigma_7(T)$ is the orbit of the Lie algebra $\mathfrak{a}_{5,7}$: $[e_1, e_i] = e_{i+1},\ i = 2, 3, 5, 6,\ [e_2, e_5] = e_7$. We can take $\mathcal{A}_7 = \{(12), (13), (15), (16), (25)\}$ and $n_0 = 7$ is the initialization of the central extension induction process defined by $\mathcal{A}_{n+1} = \mathcal{A}_n \cup \{(1n)\}$ for $n \geqslant n_0$.

For $n = 8$ we add the weight $\alpha_8 = \alpha_5 + 3\alpha_1$ and the coordinates $X_{17} = 1,\ X_{26}$ and X_{35}. Jacobi relation J_{125} gives $X_{26} = -X_{35} + X_{25}$ and we have a free parameter $X_{35} = t$ and $X_{26} = 1 - t$. For $n = 9$ we add $\alpha_9 = \alpha_5 + 4\alpha_1$ and the coordinates $X_{18} = 1,\ X_{27},\ X_{36}$ and X_{45}. Jacobi relations J_{126} and J_{135} give: $X_{36} = t - u$ and $X_{27} = 1 - 2t + u$, where $X_{45} = u$ is a new free parameter. For $n = 10$ we add $\alpha_5 + 5\alpha_1$ and the coordinates $X_{19} = 1,\ X_{28},\ X_{37}$ and X_{46}. The Jacobi relations $J_{127},\ J_{136}$ and J_{145} give $X_{46} = u,\ X_{37} = t - 2u,\ X_{28} = 1 - 3t + 3u$. The last Jacobi relation J_{235} gives:

$$u(2 + 3t) = 3t^2, \tag{2.4}$$

which is the equation of the hyperbola $xy = 1$ applying the affine change of variables $x = 9t/4 + 3/2,\ y = u - t + 2/3$. For $n = 11$ we add the weight $\alpha_5 +$

$6\alpha_1$ and the coordinates $X_{1,10}=1,\ X_{29},\ X_{38}$ and X_{47}. Relations J_{128}, J_{137} and J_{146} give $X_{47}=u,\ X_{38}=t-3u$ and $X_{29}=1-4t+6u$. Relation J_{236} gives $2ut=3u^2$ and J_{245} doesn't provide any new information. Projecting (2.4) in the residual field $\mathcal{O}/\mathfrak{m}(\mathcal{O})$ at each point of the slice, we obtain $\bar{u}(2+3\bar{t})=3\bar{t}^2$ and $2+3\bar{t}\neq 0$ in $\mathbb{C}$. The scheme is contained in the principal open set defined by $2+3t\neq 0$. In this open set, the slice is defined by the relations $u=\frac{3t^2}{2+3t}$ and $t^3(4-3t)=0$; it is the spectrum of $\mathbb{C}\left[t,\dfrac{1}{2+3t}\right]/\langle t^3(4-3t)\rangle$ isomorphic to $(\mathbb{C}[t]/\langle t^3\rangle)\oplus\mathbb{C}$. The slice has two points for $\bar{t}=0$ and $\bar{t}=4/3$ which give the following T-rigid laws:

(1) $\mathfrak{a}_{5,11}$ for $\bar{t}=0$, with nilpotent element $t^3=0$ (non-reduced case);
(2) $\mathfrak{a}\prime_{5,11}$ for $\bar{t}=4/3$ ($u=8/9$, regular case).

The scheme $L_{11}^{T,\mathcal{A}_{11}}$ consists of $\mathfrak{a}_{5,11}(t)$ with $t^3=0$ ($u=\frac{3}{2}t^2\neq 0$) and $\mathfrak{a}'_{5,11}$ for $t=4/3$.

For $n\geqslant 12$ the algebra $\mathfrak{a}'_{5,11}$ doesn't have central extensions in the slice defined by $\mathcal{A}_{12}=\mathcal{A}_{11}\cup\{(1,11)\}$ but the algebra $\mathfrak{a}_{5,n}$ belongs to the slice associated with $\mathcal{A}_n$. We can state the following result, which can be proved by induction on $n\geqslant 11$:

Proposition 2.2. — *The slices $L_n^{T,\mathcal{A}_n}$ defined above with (2.3) and admissible part $\mathcal{A}_n=\{(25),(1i)$ for $i\neq 4,\ 1<i<n\}$ for $n\geqslant 7$, are affine schemes defined by the rings: $\mathbb{C}$ ($n=7$), $\mathbb{C}[t]$ ($n=8$), $\mathbb{C}[t,u]$ ($n=9$), $\mathbb{C}[x,y]/\langle xy-1\rangle$ ($n=10$), $\mathbb{C}\left[t,\frac{1}{2+3t}\right]/\langle t^3(t-4/3)\rangle$ ($n=11$) and $\mathbb{C}[t]/\langle t^3\rangle$ ($n\geqslant 12$).*

For $n\geqslant 11$, this scheme is not reduced at $\mathfrak{a}_{5,n}$. We have $t^3=0,\ t^2\neq 0$ and the versal deformation at point $\bar{t}=0$, associated with $\mathcal{A}_n$ in L_n^T, is:
$X_{ij}=1$ for $(ij)\in\mathcal{A}_n$; $X_{2,5+k}=1-kt+\frac{3}{4}k(k-1)t^2$ for $n-7\geqslant k\geqslant 0$;
$X_{3,5+k}=t-\frac{3}{2}kt^2$ for $n-8\geqslant k\geqslant 0$; $X_{4,5+k}=\frac{3}{2}t^2$ for $n-9\geqslant k\geqslant 0$.

Other slices for (2.3)

A complete study of Σ_n/G_0 gives the slices associated with all possible admissible sets. Some of them for $n=11$ include:

- The slice associated with $\mathcal{A}'_{11}=\{(1j),(2k),j\neq 4,6,8;k=5,8,9\}$ isomorphic to $\mathrm{Spec}(\mathbb{C}[t]/\langle t^3(1-t)(4-3t)\rangle)$ where $t=X_{35}$. It contains a new T-rigid law for $t=1$.
- The slice associated with $\mathcal{A}''_{11}=\{(1j),(25),(35),\text{for } j\neq 2,4\}$ isomorphic to $\mathrm{Spec}(\mathbb{C}[u]/\langle u^2(3u-2)\rangle)$ where $u=X_{45}$; u is 2-nilpotent at the new point $\bar{X}_{45}=\bar{X}_{12}=0$.

- The slice associated with $\mathcal{A}'''_{11} = \{(1j),(25),(35),(45),\ j > 4\}$ isomorphic to $\mathrm{Spec}(\mathbb{C}[v]/\langle v(2-3v)\rangle)$ where $X_{12} = 2v$, $X_{13} = v$. For $\bar{v} = 0$ we have $X_{ij} = 1$ for $i \leqslant 4$ and $j > 4$ and 0 otherwise with $i < j$. This rigid law can be extended in the slices associated with $\mathcal{A}'''_n$ $(n \geqslant 11)$.

Remark 2.3. — a) Different slices containing a same point are generally not isomorphic as scheme and the number of their components can be different. By the universality of versal deformation [6], completed local rings of slices at this point are isomorphic.

b) The slice associated with $\mathcal{A}'''_{11}$ gives rigid laws with different numbers of generators. If $v \neq 0$ we obtain a rigid law with 3 generators and if $v = 0$ the rigid law has 5 generators.

Nilpotent parameter at each order in local rings of some slices

We begin with the following result:

LEMMA 2.4. — *Let $\mathfrak{a}$ be a Lie algebra admitting a maximal torus with weights α_i on a diagonalizing basis e_i, $1 \leqslant i \leqslant k$. We suppose $e_1 \notin [\mathfrak{a},\mathfrak{a}]$ and let V be a $\mathbb{C}$-space generated by vectors e_{k+i} for $1 \leqslant i \leqslant p$, then:*

i) *We obtain a semi-direct Lie algebra product $\mathfrak{g} = \mathfrak{a} \oplus V$ of $\mathfrak{a}$ by the abelian one V with the brackets: $[e_1, e_j] = e_{j+1}$ for $k < j < k+p$, $[e_i, e_j] = 0$ for $i > 1, j > k$.*

ii) *$\mathfrak{g}$ admits a maximal torus defined by the weights α_i; for $1 \leqslant i \leqslant k$, $\beta + j\alpha_1$; for $0 \leqslant j < p$.*

In this section we work on sequences of weights of rank two, generalizing (2.3):

$$\alpha,\ m_2\alpha, \cdots,\ m_r\alpha, \beta,\ \beta+\alpha,\ \beta+2\alpha, \cdots,\ \beta+s\alpha. \tag{2.5}$$

with integers $m_1 = 1 < m_2 < \cdots < m_r$.

Notations

We write $\alpha_i = i\alpha$ for $i \in \{m_1, \cdots, m_r\}$ and $\alpha_{b+k} = \beta + k\alpha$ where b is an integer greater than m_r. The coordinates are X_{ij} for $i, j, i+j \in \{m_1, \cdots, m_r\}$ and $X_{i,b+k}$ for $i \in \{m_1, \cdots, m_r\}$, $k + i \leqslant s$. The indices are not consecutive but the additive writing is kept with $X_{ij} = X_{ij}^{i+j}$. Jacobi relations for $1 \leqslant i < j \leqslant m_r$ and $i+j+k \leqslant s$ are:

$$J_{i,j,b+k} = X_{ij}X_{i+j,b+k} - X_{j,b+k}X_{i,b+j+k} + X_{i,b+k}X_{j,b+k+i} = 0. \tag{2.6}$$

Lemma 2.5. — *The sequence (2.5) is a simple path of weights for all $s \geqslant m_r$.*

Proof. — We observe that the law defined by $X_{ij} = 0$ for $i < j \leqslant m_r$ and $X_{i,b+k} = 1$ for $i = m_l \leqslant m_r$ and $i + k \leqslant s$ satisfies (2.6). Any maximal torus on this algebra containing T defined by (2.5) commutes with T and has weights α'_i on the same basis e_i. If $s \geqslant m_r$ we obtain $ade_i(e_b) = (ade_1)^i(e_b) = e_{b+i}$ for any $i \in \{m_1, \cdots, m_r\}$ and the relations $\beta' + \alpha'_i = \beta' + i\alpha' = \alpha'_{b+i}$ involve $\alpha'_i = i\alpha'$. Consequently, we have a weight system of rank 2 equal to (2.5). □

A particular case of special interest is provided by the sequence of weights:

$$\alpha,\ 2\alpha,\ 3\alpha,\ 5\alpha, \cdots, (2p+1)\alpha,\ \beta,\ \beta+\alpha, \cdots, \beta + s\alpha. \tag{2.7}$$

All T-invariant laws $\mathfrak{n}$ for (2.7) trivially satisfy:

1) The sum $\mathfrak{a} = \mathbb{C}e_2 \oplus (\oplus_{0 \leqslant l \leqslant p} \mathbb{C}e_{2l+1})$ is a subalgebra of $\mathfrak{n}$ whose brackets are given by $[e_2, e_{2i-1}] = c_i e_{2i+1}$, $1 \leqslant i \leqslant p$. If $c_i \neq 0$ for $1 \leqslant i \leqslant p$, then $\mathfrak{a}$ is isomorphic to the well-known filiform Lie algebra $\mathfrak{f}^0_{p+2}$: $[x_1, x_i] = x_{i+1}$, $1 < i < p+2$.
2) The Lie algebra $\mathfrak{n}$ is the semi-direct product of $\mathfrak{a}$ by the abelian ideal $\oplus_{k=0}^s \mathbb{C}e_{b+k}$ (this is generally true for (2.5)).
3) The following $n = p+3+s$ dimensional Lie algebra with $\mathfrak{a} \simeq \mathfrak{f}^0_{p+2}$ is T-invariant:

$$\mathfrak{b}_{p,n} : \begin{cases} [e_1, e_2] = e_3 & \\ [e_2, e_{2i-1}] = e_{2i+1} & 1 < i \leqslant p \\ [e_i, e_{b+k}] = e_{b+k+i} & i = 1, 2;\ k+i \leqslant s. \end{cases}$$

We can define by induction the following sets $\mathcal{A}^p_n$ for $n \geqslant p+5$:

$$\begin{aligned} &\mathcal{A}^p_{p+5} = \{(1,2), (1,b), (1,b+1), (2,b), (2,2l-1) \text{ for } 2 \leqslant l \leqslant p\}, \\ &\mathcal{A}^p_{n+1} = \mathcal{A}^p_n \cup \{(1, b+s)\}. \end{aligned} \tag{2.8}$$

Theorem 2.6. — *The sets $\mathcal{A}^p_n$ $(p \geqslant 1)$ are admissible sets of $\mathfrak{b}_{p,n}$ for $n \geqslant p+5$ and the slices associated are the spectrum of the following rings with τ a free variable: $\mathbb{C}$ $(n = p+5)$, $\mathbb{C}[\tau]$ $(p+6 \leqslant n \leqslant 3p+5)$ and $\mathbb{C}[\tau]/(\tau^{p+1})$ $(n \geqslant 3p+6)$.*

For $n \geqslant p+6$, a versal deformation of $\mathfrak{b}_{p,n}$ associated with $\mathcal{A}^p_n$ in L^T_n is given by:

$X_{ij} = 1$ for $(ij) \in \mathcal{A}_n^p$, $X_{2,b+k} = 1 - kt$ $(0 \leqslant k \leqslant s-2)$, $X_{2l+1,b+k} = c_l t^l$ $(0 \leqslant l \leqslant p,\ 2l+1+k \leqslant s)$, where c_l is obtained in (2.15).

The local parameter $t = X_{3b}$ is free for $p+6 \leqslant n \leqslant 3p+5$ and nilpotent $t^{p+1} = 0$, $t^p \neq 0$ for $n \geqslant 3p+6$.

Proof. — For $n \geqslant p+5$, one checks that (2.7) defines a maximal torus on $\mathfrak{b}_{p,n}$ and $\mathcal{A}_n^p$ is admissible. The coordinates of the scheme L_n^T are X_{12}, X_{2i}, $i \in \{3, 5, \cdots, 2p-1\}$ and $X_{i,b+j}$, $i \in \{1, 2, 3, 5, \cdots, 2h+1\}$ with $h = min(p, [\frac{s-1}{2}])$ and $i + j \leqslant s$. After fixing $X_{ij} = 1$ for $(i,j) \in \mathcal{A}_n^p$, the remaining parameters are $X_{2,b+j}$ for $j \geqslant 1$ and $X_{i,b+j}$ for $i = 3, 5, \cdots, 2h+1$ and $i + j \leqslant s$.

If $i = 1$, the equation (2.6) becomes:

$$X_{1j} X_{j+1,b+k} = X_{j,b+k} - X_{j,b+k+1} \tag{2.9}$$

where j is odd or equal to 2 and $j+k+1 \leqslant s$. This gives $X_{j,b+k+1} = X_{j,b+k}$ for odd $j \geqslant 3$ and we have:

$$X_{j,b+k} = X_{j,b}, \quad j = 3, 5, \cdots,\ 2h+1;\ 0 \leqslant k \leqslant s-j. \tag{2.10}$$

If $j = 2$ in (2.9), using (2.10) we obtain the relation $X_{2,b+k+1} = X_{2,b+k} - X_{3,b}$. By the repeated application of this latter relation, we obtain the following:

$$X_{2,b+m} = 1 - mX_{3,b}, \quad 0 \leqslant m \leqslant s-2. \tag{2.11}$$

If $i = 2$ in (2.6), from the relations (2.10) and (2.11) it follows that:

$$X_{2,j} X_{j+2,b+k} = -jX_{j,b} X_{3,b}, \quad j = 3, 5, \cdots,\ 2p+1;\ j+2+k \leqslant s. \tag{2.12}$$

Observe that $X_{2,j} = 1$ if $j + 2 \leqslant 2p+1$ and $X_{j+2,b} = 0$ if $j + 2 > 2p+1$. Using (2.10), the relation (2.12) becomes:

$$X_{j+2,b} = -jX_{j,b} X_{3,b}, \quad j \leqslant 2p-1; \tag{2.13}$$

$$0 = -(2p+1) X_{2p+1,b} X_{3,b}, \quad j = 2p+1. \tag{2.14}$$

In view of these equalities from $j = 3$ to $2m-1$, we obtain for $1 \leqslant m \leqslant p$:

$$X_{2m+1,b} = (-1)^{m-1} \frac{(2m-1)!}{2^{m-1}(m-1)!} (X_{3,b})^m. \tag{2.15}$$

Thus, if $s \leqslant 2p+2$, we have $2m+1 \leqslant 2p+2$ and formula (2.15) gives the components $X_{2l+1,b}$ for $5 \leqslant 2l+1 \leqslant 2p+1$. If $s \geqslant 2p+3$, (2.14) becomes:

$$(X_{3,b})^{p+1} = 0. \tag{2.16}$$

Finally if $i > 2$ the equality (2.6) becomes $X_{j,b+k} X_{i,b+j+k} = X_{i,b+k} X_{j,b+k+i}$, that is trivial with (2.10). □

Remark 2.7. — Case $p=1$. We denote by $\mathfrak{a}_{4,n}$ for $n \geqslant 7$, the Lie algebra defined by the non null brackets $[e_i, e_j] = e_{i+j}$ for $i < j$ if $(i=1, j \neq 3)$, $(i=2, j \geqslant 4)$ or $(i=3, j \geqslant 4)$. It is isomorphic to $\mathfrak{b}_{1,n}$ and its versal deformation $\mathfrak{a}_{4,n}(t)$ defined by $t = X_{34}$ is a continuous family for $n = 7, 8$. Condition $t^2 = 0$ for $n \geqslant 9$ corresponds to the non null quadratic Rim function calculated in [2].

Corollary 2.8. — *The Lie algebra $\mathfrak{b}_{p,n}$ satisfies:*

i) *for $n \geqslant p+6$, $H^2(\mathfrak{b}_{p,n}, \mathfrak{b}_{p,n})^T = \mathbb{C}\bar{\Phi}_1$;*

ii) *for $n \geqslant 3p+6$, we have a truncated deformation $\sum_{k=0}^{p} \bar{\tau}^k \Phi_k$ with an obstruction $\overline{\omega}_{p+1} \neq 0$ in $H^3(\mathfrak{b}_{p,n}, \mathfrak{b}_{p,n})^T$.*

Proof. — Applying Proposition 1.2 to the slices, Theorem 2.6 gives deformations satisfying $t^{p+1}=0$ and $t^p \neq 0$. Consequently, we obtain $\overline{\omega}_{p+1} \neq 0$. □

Algebras $\mathfrak{g}_{p,n} = T \oplus \mathfrak{b}_{p,n}$, defined as semi-direct products by the torus T on $\mathfrak{b}_{p,n}$, satisfy hypotheses of the reduction theorem in [6]. Thus, the local study of $\mathfrak{g}_{p,n}$ in L_{n+2} is equivalent to that of $\mathfrak{b}_{p,n}$ in L_n^T. We can summarize it in the following statement:

Corollary 2.9. — *The group $H^2(\mathfrak{g}_{p,n}, \mathfrak{g}_{p,n})$ is null for $n = p+5$ and equal to $\mathbb{C}\bar{\Phi}_1 \neq 0$ for $n > p+5$. Moreover,*

i) *for $n = p+5$, $\mathfrak{g}_{p,n}$ is rigid and L_{p+7} is reduced at this point;*

ii) *for $p+6 \leqslant n \leqslant 3p+5$, Φ_1 is tangent to a continuous family in L_{n+2};*

iii) *for $n > 3p+5$, $\mathfrak{g}_{p,n}$ is rigid in L_{n+2} and Φ_1 is tangent to a versal deformation $\sum_{k\geqslant 0} t^k \Phi_k$ with $t^{p+1} = 0$, $t^p \neq 0$. This corresponds to an obstruction $\overline{\omega}_{p+1} \neq 0$ in $H^3(\mathfrak{g}_{p,n}, \mathfrak{g}_{p,n})$ and $\overline{\omega}_k = 0$ for $k \leqslant p$.*

Remark 2.10. — Obstructions to deformation equations can appear at each order in schemes L_n^T and L_m.

3. Elimination procedure in the search for nilpotent elements in global schemes L_n^T

An existence theorem

Let Φ_0 be a law, $\mathcal{A}$ be an admissible set associated with Φ_0, $\mathcal{O}$ be its local ring in the slice $L_n^{T,\mathcal{A}}$, $U \subset \mathbb{C}[X_{ij}]$ be the ideal generated by $X_\alpha - 1$ for $\alpha \in \mathcal{A}$ and π be the projection of A_n to the quotient $\overline{A}_n = A_n/\overline{U}$ where $\overline{U} = U/\operatorname{Jac}_n$. The image of $\sqrt{0} \subset A_n$ by π is contained in $\sqrt{\bar{0}} \subset \overline{A}_n$. If

Φ_0 is rigid, then $\mathfrak{m}(\mathcal{O}) = \sqrt{0}$ in $\mathcal{O}$ [6]. Writing $\mathcal{O} = (\overline{A}_n)_M$ as the localized ring by the maximal ideal M associated with Φ_0, each $u \in \mathfrak{m}(\mathcal{O})$ is v/f where $v \in M$ and $f \notin M$; u and v have the same nilpotency order l: $u^l = 0$, $u^{l-1} \neq 0$. We are looking for $P \in \mathbb{C}[X_{ij}]$ such that $\pi(\bar{P}) = v$. Moreover, if $P \in \sqrt{\mathrm{Jac}_n}$, then $\pi(\bar{P}^{l-1}) = v^{l-1} \neq 0$ and the nilpotency order of $\bar{P}$ in A_n is greater than or equal to l. The existence of such a P (given by $P = Q \cdot H$) is assumed by the following Theorem 3.2. This result is also true for the schemes L_m and needs the following lemma:

Lemma 3.1. — *Let $K = \mathbb{C}[X_1, \dots, X_r]$, $W = \mathbb{C}[Y_1, \dots, Y_s]$ and let $\Psi_t : K \otimes_{\mathbb{C}} W \to W$ be the morphism defined by $\Psi_t(f(X,Y)) = f(t,Y)$ for $t \in \mathbb{C}^r$. If J is an ideal of $K \otimes_{\mathbb{C}} W$, then $\Psi_t(J)$ is an ideal of W and the ideal $L = \cap_t \Psi_t(J)$, for $t \in (\mathbb{C}^*)^r$, is $\{P \in W; \exists Q \in K, QP \in J\}$.*

Proof. — If $P \in L$, for each t there are polynomials $g_i^t(X,Y) \in K \otimes_{\mathbb{C}} W$ such that:

$$P(Y) = \Psi_t\Big(\sum_i g_i^t(X,Y) f_i(X,Y)\Big) = \sum_i g_i^t(t,Y) f_i(t,Y)$$

where $f_i(X,Y)$ are generators of J. The polynomials $g_i^t(X,Y)$ can be written as

$$g_i^t(X,Y) = \sum_\gamma a_{i\gamma}^t(X) Y^\gamma$$

where γ is a multi-index for the monomials $Y^\gamma = Y_1^{\gamma_1} \cdots Y_s^{\gamma_s}$, and

$$P(Y) = \sum_i \Big(\sum_\gamma a_{i\gamma}^t(t) Y^\gamma\Big) f_i(t,Y), \qquad \forall t \in (\mathbb{C}^*)^r.$$

If we consider $f_i(X,Y) = \sum_\beta Q_{i\beta}(X) Y^\beta$ and $P(Y) = \sum_\delta c_\delta Y^\delta$ with fixed polynomials $Q_{i\beta}(X) \in K$ and fixed $c_\delta \in \mathbb{C}$, it follows that

(3.1)
$$P(Y) = \sum_\delta c_\delta Y^\delta = \sum_\delta \sum_{\beta+\gamma=\delta} \Big(\sum_i a_{i\gamma}^t(t) Q_{i\beta}(t)\Big) Y^\delta, \qquad \forall t \in (\mathbb{C}^*)^r.$$

Then (3.1) gives the set of the following equalities indexed over δ :

$$\sum_{\beta+\gamma=\delta} \Big(\sum_i \lambda_{i\gamma}(X) Q_{i\beta}(X)\Big) = c_\delta. \tag{3.2}$$

This is a linear system of equations in $\lambda_{i\gamma}(X)$ with coefficients $Q_{i\beta}(X)$ which can be solved over the field of fractions $\mathbb{C}(X)$ by the pivot method. Writing the solutions $\lambda_{i\gamma}(X) = h_{i\gamma}(X)/q_{i\gamma}(X)$, it suffices to take the polynomial $\Pi_{i\gamma} q_{i\gamma} = Q$ and consequently we have $PQ \in J$. □

THEOREM 3.2. — *Let $u = v/f$ be a l-nilpotent element in $\mathfrak{m}(\mathcal{O})$, with $\mathcal{O}$ the local ring in the local chart (or slice) associated with an admissible set $\mathcal{A}$ at Φ_0 for L_n^T/G_0. If $H \in \mathbb{C}[X_\beta; \beta \notin \mathcal{A}]$ satisfies $\bar{H} = v \in \bar{A}_n$ and $G_0 \star H \subset \mathbb{C}^*.H$, then $\bar{H}^l = 0$ and there is $Q \in \mathbb{C}[X_\alpha; \alpha \in \mathcal{A}]$ such that $(Q.H)^l$ is in Jac_n, i.e., $Q.H$ is l-nilpotent in A_n.*

Proof. — We have $u = v/f$ with $f \notin M$, $v \in \mathbb{C}[X_{ij}]/(\langle X_\alpha - 1, \alpha \in \mathcal{A}\rangle + \mathrm{Jac}_n)$, $v^l = 0$, $v^{l-1} \neq 0$, *i.e.*, there are polynomials g_α such that

$$H^l + \sum_{\alpha \in \mathcal{A}} (X_\alpha - 1) g_\alpha \in \mathrm{Jac}_n\,.$$

The action $\star$ of G_0 stabilizes Jac_n and $\mathbb{C}^*.H^l$. For $t = (t_\alpha)_{\alpha \in \mathcal{A}}$, $t_\alpha \in \mathbb{C}^*$, there is $s \in G_0$ with $(s \star X)_\alpha = X_\alpha / t_\alpha$ by definition of $\mathcal{A}$ and we obtain:

$$\lambda H^l + \sum_\alpha \Big(\frac{X_\alpha - t_\alpha}{t_\alpha}\Big) s \star g_\alpha \in \mathrm{Jac}_n$$

with $\lambda \in \mathbb{C}^*$. This relation involves $H^l \in \Psi_t(\mathrm{Jac}_n)$ for each t. By applying Lemma 3.1, there exists $Q \in \mathbb{C}[X_\alpha; \alpha \in \mathcal{A}]$ with $Q.H^l \in \mathrm{Jac}_n$ and thus $(QH)^l \in \mathrm{Jac}_n$. □

Remark 3.3. — In the examples dealt with in this paper, H can be chosen as a simple coordinate and the condition $G_0 \star H \subset \mathbb{C}^* H$ is satisfied. If H is a homogeneous polynomial, we can choose Q to be homogeneous because the homogeneous parts of $(HQ)^l$ are in Jac_n as well.

Although it is possible to solve the linear system (3.2) with $P = H^l$, in practice this linear system is very laborious. For the examples proposed in this section, we prefer to use a different method in order to find global nilpotent elements. We proceed by eliminating coordinates X_β ($\beta \notin \mathcal{A}$) from the Jacobi polynomials J_k, $1 \leqslant k \leqslant N$. If m is the number of distinct coordinates X_{ij} ($i < j$) and N the number of the Jacobi polynomials J_k, then nilpotent elements can appear under the condition:

$$N \geqslant m - |\mathcal{A}|. \tag{3.3}$$

Elimination in $F[X]$

Let F be a factorial ring and $F[X]$ be the polynomial ring in one variable X. If $f_1 = AX^p + P$ and $f_2 = BX^q + Q$ are two polynomials in X with $p \geqslant q$, $\deg(P) < p$, and $\deg(Q) < q$, we denote D a H.C.F. of A and B, $A = A'D$, $B = B'D$ and then we can obtain the polynomial

$$f_3 = B' f_1 - A' X^{p-q} f_2 = B' P - A' X^{p-q} Q$$

of degree smaller than p in variable X. If $\deg(f_2) \geqslant \deg(f_3)$, we now consider the pair (f_2, f_3); otherwise we consider the pair (f_3, f_2). We can apply the same operation again to f_2 and f_3 to obtain f_4 and so on. Finally, we obtain a well-defined element of F denoted by $\{f_1, f_2\}_X$. This element can be null.

In this work, we eliminate a coordinate X_β where β is an index (ij) from two Jacobi polynomials J_1 and J_2, which depend on this coordinate. The polynomial obtained, denoted by $\{J_1, J_2\}_\beta$, is homogeneous and also belongs to the ideal Jac_n .

The method

It is relative to the choice of an admissible set $\mathcal{A}$. If $\beta \notin \mathcal{A}$ we consider the list constituted by all the J_k depending on X_β. If we fix from this list one J_i as a pivot, we calculate all the different polynomials $\{J_k, J_i\}_\beta$ with $k \neq i$. Then, we have a new list, in which we have replaced each J_k by this new polynomial $\{J_k, J_i\}_\beta$ and where the pivot J_i, depending on X_β, doesn't appear in it. We proceed by successively eliminating the coordinates indexed by $\beta_1, \cdots, \beta_l \notin \mathcal{A}$. In all examples encountered, the condition (3.3) is satisfied and we can choose pivots such that $\{J_k, J_i\}_\beta \neq 0$. This possibility allows us to obtain polynomials depending on the variables indexed over $\mathcal{A}$ and only one variable X_ρ with $\rho \notin \mathcal{A}$. This is the best possibility on account of the following result:

Lemma 3.4. — *If two polynomials f and g in* Jac_n *only depend on the X_α, $\alpha \in \mathcal{A}$, and X_ρ for one unique $\rho \notin \mathcal{A}$, then $\{f, g\}_\rho = 0$.*

Proof. — The polynomial $h = \{f, g\}_\rho$ belongs to Jac_n and only depends on coordinates indexed by $\mathcal{A}$. By definition of admissible set $\mathcal{A}$ associated with a law Φ_0, the parameters $(s \star \Phi_0)_\alpha$, for $\alpha \in \mathcal{A}$ and $s \in G_0$, are independent and generate an open set $\Omega \neq \emptyset$ in $\mathbb{C}^{|\mathcal{A}|}$. The polynomial h, null on $L_n^T(\mathbb{C})$, depends on variables indexed by $\mathcal{A}$ only and then $h(G_0 \star \Phi_0) = h(\Omega) = 0$; h is null on the closure of Ω, *i.e.*, $h = 0$. □

Remark 3.5. — a) The polynomials obtained after elimination generally depend on the choice in the order of the β_i and the pivots.

b) If a non-null polynomial depends on variables indexed by $\mathcal{A}' \cup \{\rho\}$ with $\mathcal{A}' \subset \mathcal{A}$, then $\mathcal{A}' \cup \{\rho\}$ is not contained in an admissible set (proof of Lemma 3.4).

Irreducible polynomials in the radical of Jac_n

We say that a polynomial is irreducible in an ideal of $\mathbb{C}[X_{ij}]$ if it doesn't have non-trivial factors in this ideal. In $\sqrt{\mathrm{Jac}_n}$, such a polynomial can be written $P = P_1P_2\cdots P_r$ where the P_i are irreducible in $\mathbb{C}[X_{ij}]$ and distinct up to a factor in $\mathbb{C}^*$. If we denote $\widehat{P}_i$ the polynomial $\Pi_{k\neq i}P_k$, we have the following criterion:

Proposition 3.6. — *A product of different irreducible polynomials $P = \Pi_{k=1}^r P_k$ is irreducible in $\sqrt{\mathrm{Jac}_n}$ iff for each $i \in \{1,\dots,r\}$, there is a law $\Phi \in L_n^T(\mathbb{C})$ with $\widehat{P}_i(\Phi) \neq 0$.*

Proof. — Notice that P is irreducible in $\sqrt{\mathrm{Jac}_n}$ iff $\widehat{P}_i \notin \sqrt{\mathrm{Jac}_n}$ for all i. Thanks to the Hilbert nullstellensatz, it means that $\widehat{P}_i$ is not identically null on $L_n^T(\mathbb{C})$. □

Corollary 3.7. — *The number of irreducible components of $L_n^T(\mathbb{C})$ is bigger than the number of factors in any irreducible polynomial of $\sqrt{\mathrm{Jac}_n}$.*

Under the assumptions for path of weights, we can state:

Lemma 3.8. — *If a polynomial is irreducible in $\sqrt{\mathrm{Jac}_n}$, then it is irreducible in $\sqrt{\mathrm{Jac}_m}$ for $m > n$.*

Proof. — If $P = \Pi_{k=1}^r P_k$ is an irreducible polynomial in $\sqrt{\mathrm{Jac}_n}$, for each $i \in \{1,\dots,r\}$ there is $\Phi \in L_n^T(\mathbb{C})$ with $\widehat{P}_i(\Phi) \neq 0$. The law $\Phi \times 0_{m-n}$, direct product of Φ by the abelian law 0_{m-n} on $\mathbb{C}^{m-n}$, belongs to $L_m^T(\mathbb{C})$ and satisfies $\widehat{P}_i(\Phi \times 0_{m-n}) = \widehat{P}_i(\Phi) \neq 0$. □

For each irreducible polynomial P in $\sqrt{\mathrm{Jac}_n}$, we call nilpotency order of P in the ring A_n the unique number $\nu \geqslant 1$ such that $P^\nu \in \mathrm{Jac}_n$ and $P^{\nu-1} \notin \mathrm{Jac}_n$. The quotient $\bar{P}$ in A_n satisfies $\bar{P}^\nu = 0$ and $\bar{P}^{\nu-1} \neq 0$. If $P = \Pi_{k=1}^r P_k$ is irreducible in $\sqrt{\mathrm{Jac}_n}$ with nilpotency order ν in A_n, then each irreducible polynomial Q in Jac_n with factors P_k for $1 \leqslant k \leqslant r$, satisfies $Q = P_1^{\nu_1}\cdots P_r^{\nu_r}$ and $\nu = \mathrm{Max}_{1\leqslant k\leqslant r}(\nu_k)$.

The examples

With the tori chosen in the following examples, we can adopt graded indexation for structure constants of laws $[e_i, e_j] = X_{ij}e_{i+j}$, $1 \leqslant i < j \leqslant n$, and for Jacobi polynomials:

$$J_{ijk} = X_{ij}X_{i+j,k} + X_{jk}X_{j+k,i} + X_{ki}X_{k+i,j} \tag{3.4}$$

where $i + j + k \leqslant n$.

a) The Torus $\alpha_1, 2\alpha_1, 3\alpha_1, \alpha_4 + k\alpha_1$ ($k \geqslant 0$). We have 3 families of Jacobi polynomials for $p \geqslant 4$: J_{12p} $(n = p+3 \geqslant 7)$, J_{13p} $(n = p+4 \geqslant 8)$ and J_{23p} $(n = p + 5 \geqslant 9)$. Inequality (3.3) is satisfied for $n \geqslant 9$. The elimination process relative to the admissible set $\mathcal{A}_n = \{(1k), (24) \text{ for } 1 < k < n, k \neq 3\}$ gives:

Proposition 3.9. — *The monomial $X_{12}X_{17}X_{18}(X_{34})^2$ is irreducible in Jac_n for $n \geqslant 9$. The monomial $X_{12}X_{17}X_{18}X_{34}$ is irreducible in $\sqrt{\mathrm{Jac}_n}$ and two-nilpotent in A_n.*

Proof. — We obtain $X_{12}X_{17}X_{18}(X_{34})^2$ by eliminating the variables X_{36}, X_{27}, X_{35}, X_{26} and X_{25} from the 6 Jacobi polynomials for $n = 9$. According to Proposition 3.6, the monomial $X_{12}X_{17}X_{18}X_{34}$ is irreducible in $\sqrt{\mathrm{Jac}_n}$, however it doesn't belong to Jac_n $(n \geqslant 9)$ because in the local study of the scheme $L_n^{T,\mathcal{A}}$ at point $\mathfrak{a}_{4,n}$ appears the condition $(\bar{X}_{34})^2 = 0$, $\bar{X}_{34} \neq 0$ (Remark 2.7). Thus, we deduce the irreducibility of $X_{12}X_{17}X_{18}(X_{34})^2$ in Jac_n . □

b) The torus α_1, $2\alpha_1$, $3\alpha_1$, $4\alpha_1$, $\alpha_5 + k\alpha_1$ ($k \geqslant 0$). This torus corresponds to the example in Proposition 2.2. We have Jacobi polynomials J_{ijk} for $1 \leqslant i < j \leqslant 4$ and $k \geqslant 5$.

For $n \geqslant 11$ we consider the admissible set $\mathcal{A}_n = \{(1k), (25) \text{ for } 1 < k < n, k \neq 4\}$. Condition (3.3) is satisfied and we can consider the elimination procedure associated with $\mathcal{A}_n$. For $n = 11$, if we eliminate X_{2k} for $k \geqslant 6$, X_{3k} for $k \geqslant 6$ and X_{4k} for $k \geqslant 5$ from the Jacobi polynomials, keeping only the variable X_{35} not indexed by $\mathcal{A}_n$, we obtain the following polynomial:

$$I_1 = (X_{12})^3(X_{18})^2X_{19}X_{110}(3X_{12}X_{35} - 4X_{25}X_{17})(X_{35})^3 \in \mathrm{Jac}_{11}\,.$$

Hence,

$$P = X_{12}X_{18}X_{19}X_{110}X_{35}(3X_{12}X_{35} - 4X_{17}X_{25}) \in \sqrt{\mathrm{Jac}_{11}}. \tag{3.5}$$

Proposition 3.10. — *If T is defined by the weights $i\alpha_1$ for $1 \leqslant i \leqslant 4$ and $\alpha_5 + j\alpha_1$ for $j \geqslant 0$, we have for $n \geqslant 11$:*

i) *P, cf. (3.5), is irreducible in $\sqrt{\mathrm{Jac}_n}$;*

ii) *P gives a nilpotent element of order $\nu = 3$ in A_n.*

Proof. — i) According to the Lemma 3.8, it suffices to prove the irreducibility for $n = 11$. Thanks to Proposition 3.6, we are looking for a law $\Phi \in L_{11}^T(\mathbb{C})$ such that $\widehat{P}_\alpha(\Phi) \neq 0$ for each irreducible factor of P indexed by α.

If α represents the factor $3X_{12}X_{35} - 4X_{17}X_{25}$, then we have $\widehat{P}_\alpha(\Phi) \neq 0$ for the law $\mathfrak{a}'_{511}$ defined in section 2. If $\alpha = (35)$, we have $\widehat{P}_\alpha(\Phi) \neq 0$ for the law $\mathfrak{a}_{511}$. If$\alpha = (12)$, then the following Lie algebra law Φ satisfies $\widehat{P}_\alpha(\Phi) \neq 0$: $\Phi_{12} = \Phi_{13} = 0$, $\Phi_{ij} = 1$, for (ij) different to (12) or (13), $i < j$.

If $\mathfrak{a}_k$ is given by a law of $L_k^{T,\mathcal{A}_k}(\mathbb{C})$, for $8 \leqslant k \leqslant 10$, with $\Phi_{35} \neq 0, 4/3$, we can construct a law Φ as in Lemma 2.4 giving the Φ_{ij} of $\mathfrak{a}_k$ for $i < j \leqslant k$, $\Phi_{1j} = 1$ for $k < j < 11$ and $\Phi_{ij} = 0$ for $i < j$ otherwise. We have $\widehat{P}_{(1k)}(\Phi) \neq 0$, concluding i).

ii) We project a power P^k in the ring of slice associated with $\mathcal{A}_n$, $n \geqslant 11$. We obtain the polynomial $(\bar{X}_{35})^k(3\bar{X}_{35} - 4)$ and the projection on the local ring at point $\mathfrak{a}_{5n}$ gives $-4(\bar{X}_{35})^k$ which is not null for $k = 2$. Then $\bar{P}^k$ is null in A_n for $k \geqslant 3$, but it is not null for $k = 2$. □

c) The torus $k\alpha$ $(k \geqslant 1)$. We have the Jacobi (3.4) for $1 \leqslant i < j < k$. The inequality (3.3) is satisfied for $n \geqslant 12$. For $n = 12$, an elimination process associated with $\mathcal{A}_n = \{(1k), (23), 1 < k < n\}$, shown later in the "calculation with a computer" section, gives the polynomial:

$$I = -36(X_{12})^5(X_{13})^4(X_{17})^3(X_{18})^3(X_{19})^2 X_{110} X_{111}(X_{34})^5 \\ (X_{16}X_{24} - 10X_{12}X_{34}) \in \mathrm{Jac}_{12}\,.$$

Hence, we obtain the polynomial:

$$f = X_{12}X_{13}X_{17}X_{18}X_{19}X_{110}X_{111}X_{34}(X_{16}X_{24} - 10X_{12}X_{34}) \in \sqrt{\mathrm{Jac}_{12}}.$$

Taking into account the identity $X_{13}X_{16}X_{24} - X_{16}J_{123} = X_{15}X_{16}X_{23}$, the following polynomial in $\sqrt{\mathrm{Jac}_{12}}$ with variables indexed on $\mathcal{A}_{12} \cup \{34\}$ satisfies Proposition 3.11:

$$P = X_{12}X_{17}X_{18}X_{19}X_{110}X_{111}X_{34}(X_{15}X_{16}X_{23} - 10X_{12}X_{13}X_{34}). \tag{3.6}$$

Proposition 3.11. — *If T is defined by the weights $\alpha_k = k\alpha$ for $1 \leqslant k \leqslant n$, then for $n \geqslant 12$ we have :*

i) *P, cf. (3.6), is irreducible in $\sqrt{\mathrm{Jac}_n}$;*
ii) *P gives a nilpotent element of order $\nu = 5$ in the quotient ring A_n.*

Proof. — i) It suffices to prove the irreducibility for $n = 12$ and we proceed as in proof of Proposition 3.10. If α represents the non-monomial factor of P, (3.6), then we have $\widehat{P}_\alpha(\Phi) \neq 0$ for the Witt algebra $\mathfrak{w}_{12}$: $\Phi_{ij} = i - j$. If $\alpha = (34)$, then we have $\widehat{P}_{34}(\Phi) \neq 0$ for $\mathfrak{f}_n$.

If $\alpha = (12)$, it suffices to check that the following law Φ belongs to $L_{12}^T(\mathbb{C})$ with $\widehat{P}_{12}(\Phi) \neq 0$: $\Phi_{12} = \Phi_{4k} = \Phi_{5k} = 0$ $(k > 4)$, $\Phi_{ij} = 1$ for $i < j$ otherwise. If $\mathfrak{a}_k$ is a Lie algebra in $L_k^{T,\mathcal{A}_k}(\mathbb{C})$ for $7 \leqslant k \leqslant 11$ with $\Phi_{34} \neq 0, 1/10$, we construct the law Φ as in Lemma 2.4 by giving the Φ_{ij}

of $\mathfrak{a}_k$ for $i < j \leqslant k$, $\Phi_{1j} = 1$ for $k < j < 12$ and otherwise $\Phi_{ij} = 0$ for $i < j$. Also, we have $\widehat{P}_{(1k)}(\Phi) \neq 0$ for the other factors, concluding i).

For ii): Observe that $P = f - QJ_{123}$ and that $P^5 = f^5 + SJ_{123}$ belongs to Jac_n because f^5 is a multiple of I. Hence, it gives $\bar{P}^5 = 0$ in the quotient A_n. If we project a power P^k in the ring defining the slice associated with the admissible set $\mathcal{A}_n$, then we obtain $\bar{X}_{ij} = 1$ for $(ij) \in \mathcal{A}_n$ and $\bar{P}^k = (\bar{X}_{34})^k(1 - 10\bar{X}_{34})^k$. This expression is not null in the local ring associated with $\mathfrak{f}_n$ for $k \leqslant 4$, thanks to the local study in Proposition 1.6. Then, $\bar{P}^4 \neq 0$ and the nilpotency order of P in A_n is $\nu = 5$. □

Corollary 3.12. — *The scheme L_n^T has at least 8 irreducible components for $n \geqslant 12$.*

Calculation with a computer

Taking into account all the Jacobi polynomials (3.4) for $n = 12$, the polynomial I has been obtained from the polynomial J_{246} after applying the elimination method. In this process, the order of elimination of the variables and the election of the pivot associated with each variable are shown in the following table:

order	variable	pivot	order	variable	pivot	order	variable	pivot
1)	X_{57}	J_{156}	7)	X_{38}	J_{128}	13)	X_{36}	J_{126}
2)	X_{48}	J_{138}	8)	X_{29}	J^3_{236}	14)	X_{27}	J^2_{234}
3)	X_{39}	J_{129}	9)	X_{46}	J_{145}	15)	X_{35}	J_{134}
4)	$X_{2,10}$	J^3_{147}	10)	X_{37}	J_{127}	16)	X_{26}	J^1_{125}
5)	X_{56}	J_{146}	11)	X_{28}	J^2_{136}	17)	X_{25}	J_{124}
6)	X_{47}	J_{137}	12)	X_{45}	J_{135}	18)	X_{23}	J_{123}

where the pivot J^3_{147} has been obtained from Jacobi polynomial J_{147} after eliminating, by this process, the variables $X_{57},\ X_{48}$ and X_{39}. The pivot J^3_{236} is also obtained by eliminating the variables $X_{56},\ X_{47}$ and X_{38} in the polynomial J_{236}. The polynomial J^2_{136}, after the elimination of the variables X_{46} and X_{37} in J_{136}. The pivot J^2_{234}, by eliminating X_{45} and X_{36} in J_{234} and finally, J^1_{125} is obtained by elimination of the variable X_{35} in J_{125}.

We have used the symbolic computational package MAPLE to execute this calculation.

4. A second geometry obtained with generators and relations: subschemes of ideals in Grassmannians

Generalities

A Lie algebra $\mathfrak{g}$ with a finite number r of generators is built as the quotient of a free Lie algebra $\mathfrak{L}_r$ to r generators by an ideal $\mathfrak{J}$: $0 \longrightarrow \mathfrak{J} \longrightarrow \mathfrak{L}_r \longrightarrow \mathfrak{g} \longrightarrow 0$. Two quotients of $\mathfrak{L}_r$ are isomorphic iff the ideals are conjugated by the automorphism group $\mathrm{Aut}(\mathfrak{L}_r)$. If $\mathcal{C}^p(\mathfrak{g})$ is the central descending series of $\mathfrak{g}$, a nilpotent Lie algebra $\mathfrak{L}_r/\mathfrak{J}$ of dimension n satisfies $\mathcal{C}^n(\mathfrak{L}_r/\mathfrak{J}) = \mathcal{C}^n(\mathfrak{L}_r)/\mathfrak{J} = 0$, *i.e.*, $\mathcal{C}^n(\mathfrak{L}_r) \subset \mathfrak{J}$. Such an algebra is also the quotient of the finite dimensional Lie algebra $\mathfrak{M} = \mathfrak{L}_r/\mathcal{C}^n(\mathfrak{L}_r)$ by the ideal $\mathfrak{J}/\mathcal{C}^n(\mathfrak{L}_r)$. In this work we define $J_n(\mathfrak{L}_r)$ as the set of ideals of codimension n in $\mathfrak{L}_r$ containing $C^n(\mathfrak{L}_r)$; it is identified with a subscheme of the grassmannian $Gr_{m-n}(\mathfrak{L}_r/C^n(\mathfrak{L}_r))$ for $\dim(\mathfrak{M}) = m$. The nilpotent laws of dimension n are obtained for $2 \leqslant r \leqslant n$. For $m > n$, the grassmannian $Gr_{m-n}(\mathfrak{M})$, with its natural reduced structure of scheme, contains as subscheme the set $J_n(\mathfrak{L}_r)$ of n-codimensional ideals $\mathfrak{J}/\mathcal{C}^n(\mathfrak{L}_r)$ of $\mathfrak{M}$ defined by the simple polynomial relations $[x, \mathfrak{J}] \subset \mathfrak{J}$ for $x \in \mathfrak{L}_r$. This is the "minimal" definition of an ideal which is provided by its current algebraic characterization. Such a scheme is generally not reduced. Each point $\{\mathfrak{J}\}$ defines the Lie algebra quotient $\mathfrak{n} = \mathfrak{L}_r/\mathfrak{J}$, hence giving a second geometry for the nilpotent laws.

Torus

A maximal torus T_r on $\mathfrak{L}_r$, diagonalized by a family of generators e_i, $1 \leqslant i \leqslant r$, where $t(e_i) = \varepsilon_i(t)e_i$, $t \in T_r$, is characterized by its weights $\varepsilon = \sum_{i=1}^r n_i\varepsilon_i$ and the multiplicities given by the Witt formula:

$$d\varepsilon = \dim(L_\varepsilon) = \frac{1}{\mid \varepsilon \mid} \sum_{k|n_i} \mu(k) \frac{(\mid \varepsilon \mid /k)!}{(n_1/k)! \cdots (n_r/k)!}$$

where L_ε is the weight subspace of $\mathfrak{L}_r$ associated with ε, $\mid \varepsilon \mid = \sum n_i$, and μ the Möbius function. The T_r-module structure of $\mathfrak{M}$ is given by the decomposition $\oplus_{|\varepsilon|<n} L_\varepsilon$. Let T be a subtorus of T_r; its weights on $\mathfrak{L}_r$ are the restrictions $\alpha = \varepsilon \mid_T = \sum_{i=1}^r n_i\alpha_i$ with $\alpha_i = \varepsilon_i \mid_T$, and let Π be the set of this weights. We write $\mathfrak{L}_r = \oplus_\alpha L_\alpha$ with $L_\alpha = \oplus\{L_\varepsilon; \alpha = \varepsilon \mid_T\}$. An ideal stable by T can be written as $\mathfrak{J} = \oplus_\alpha J_\alpha$ with $J_\alpha \subset L_\alpha$. The different tori T employed are of maximal type, *i.e.*, maximal over one Lie algebra

at least. We denote by $J_n^T(\mathfrak{L}_r)$ the subscheme of $J_n(\mathfrak{L}_r)$ constituted of all T-invariant ideals $\mathfrak{J}$ satisfying the additional polynomial relations $t(\mathfrak{J}) \subset \mathfrak{J}$ for $t \in T$. The torus T operates on each quotient $\mathfrak{n} = \mathfrak{L}_r/\mathfrak{J}$. Let σ_n or simply σ be a sequence called weight system $\{(\alpha, n(\alpha)); \alpha \in \Pi, n(\alpha) \in \mathbb{N}\}$ such that $n(\alpha_i) = 1$ for $1 \leqslant i \leqslant r$ and $\sum_\alpha n(\alpha) = n$ for $n \in \mathbb{N}$; let $j(\alpha) = \dim(L_\alpha) - n(\alpha)$ for each $\alpha \in \Pi$, and let P denote the set of $\alpha \in \Pi$ with $n(\alpha) \neq 0$. Let $V_r^\sigma(T)$ be the set of ideals $\mathfrak{J} \in J_n^T(\mathfrak{L}_r)$ such that $\dim(J_\alpha) = j(\alpha)$, *i.e.*, $\dim(\mathfrak{n}_\alpha) = n(\alpha)$ for $\mathfrak{n}_\alpha = L_\alpha/J_\alpha$; $V_r^\sigma(T)$ is a closed subscheme of a product of grassmannians

$$V_r^\sigma(T) = \{\mathfrak{J} = (J_\alpha) \in \Pi_\alpha Gr_{j_\alpha}(L_\alpha),\ [\mathfrak{L}_r, \mathfrak{J}]_\alpha \subset \mathfrak{J}_\alpha\} \subset J_n^T(\mathfrak{L}_r).$$

In fact, it is a finite product of grassmannians because $n(\alpha) = 0$ involves $J_\alpha = L_\alpha$ and $Gr_{j(\alpha)}(L_\alpha)$ is trivial, so it can be omitted. The scheme $J_n^T(\mathfrak{L}_r)$ is a finite union of $V_r^\sigma(T)$. Let $W_r^\sigma(T)$ be the open set of $V_r^\sigma(T)$ consisting of the ideals on which T is maximal.

Action groups

The normalizing subgroup N of T in $\mathrm{Aut}(\mathfrak{L}_r)$, *i.e.*, the set of $\theta \in \mathrm{Aut}(\mathfrak{L}_r)$ with $\theta T \theta^{-1} = T$, stabilizes $V_r^\sigma(T)$ and $W_r^\sigma(T)$. The torus T is maximal on $\mathfrak{L}_r/\mathfrak{J}$ iff it is maximal in the subalgebra of derivations of $\mathfrak{L}_r$ stabilizing $\mathfrak{J}$ [8]. We can state a lemma under the following hypothesis on P:

(H): Each linear automorphism of T^* stabilizing P and sending a base $B \subset \{\alpha_i\}_{1 \leqslant i \leqslant r}$ in $\{\alpha_i\}_{1 \leqslant i \leqslant r}$, stabilizes the part $\{\alpha_i\}_{1 \leqslant i \leqslant r}$ too.

Lemma 4.1. — *Under conditions (H) and $n(\alpha_i) = 1$ $(1 \leqslant i \leqslant r)$ for σ, two ideals in $W_r^\sigma(T)$ are conjugated by N iff the corresponding quotient Lie algebras are isomorphic.*

Proof. — The direct implication is obvious. Conversely, let $\mathfrak{J}_k$ for $k \in \{1, 2\}$ be two ideals of free $\mathfrak{L}_r$ stable by T and such that the quotient algebras $\mathfrak{L}_r/\mathfrak{J}_k$ are isomorphic. The tori of derivations T_k on $\mathfrak{L}_r/\mathfrak{J}_k$, deduced from T by $t_k p_k = p_k t$ if p_k is the canonical projection on $\mathfrak{L}_r/\mathfrak{J}_k$ and $t \in T$, being maximal, are conjugated by an isomorphism h from $\mathfrak{L}_r/\mathfrak{J}_1$ to $\mathfrak{L}_r/\mathfrak{J}_2$ (Mostow's theorem) and we have $T_2 = hT_1h^{-1}$. Under these hypotheses, we can find vectors x_i, $1 \leqslant i \leqslant r$, associated with T-weights α_i in $\mathfrak{L}_r$ such that $L_{\alpha_i} = \mathbb{C}x_i \oplus (\mathfrak{J}_1)_{\alpha_i}$. For each $t \in T$, there is a unique $t' \in T$ with $t'_2 h = ht_1$. The transpose of the linear automorphism $t \to t'$ of T is a linear automorphism L of T^*: $(L\alpha)(t) = \alpha(t')$, $\alpha \in T^*$. It keeps σ and P because h is a Lie algebra isomorphism. With an indexation such that the weights α_i for $1 \leqslant i \leqslant s$ $(s \leqslant r)$ satisfy $(\mathfrak{J}_1)_{\alpha_i} \subset [\mathfrak{L}_r, \mathfrak{L}_r]_{\alpha_i}$, *i.e.*, $p_1 e_i \notin [\mathfrak{L}_r/\mathfrak{J}_1, \mathfrak{L}_r/\mathfrak{J}_1]$, we have that $\{p_1 e_i\}_{1 \leqslant i \leqslant s}$ is a minimal generating family of $\mathfrak{L}_r/\mathfrak{J}_1$ as its image by h in $\mathfrak{L}_r/\mathfrak{J}_2$. Each weight $L^{-1}(\alpha_i)$ of $h(p_1 e_i)$ for $1 \leqslant i \leqslant s$ is of the form α_j, for $1 \leqslant j \leqslant r$. Notice that all weight vectors in $[\mathfrak{L}_r/\mathfrak{J}_1, \mathfrak{L}_r/\mathfrak{J}_1]$ are linear combinations of the α_i, thus there is a base B of T^* contained in $\{\alpha_i; 1 \leqslant i \leqslant r\}$ as $L^{-1}(B)$. Applying hypothesis (H) to L^{-1}, we define a permutation ζ of $\{1, \cdots, r\}$ such that $L^{-1}(\alpha_i) = \alpha_{\zeta(i)}$, $1 \leqslant i \leqslant r$. We have $t'_2(h(p_1 x_i)) = h t_1(p_1 x_i) = \alpha_i(t) h(p_1 x_i)$, so $h(p_1 x_i) \neq 0$ is a weight vector for T_2. In the same way, we can choose a family y_i, $1 \leqslant i \leqslant r$, associated with $\mathfrak{J}_2$, satisfying the same properties as x_i and such that $h(p_1 x_i) = p_2(y_{\zeta(i)})$. If $p_1 x_i \notin [\mathfrak{L}_r/\mathfrak{J}_1, \mathfrak{L}_r/\mathfrak{J}_1]$, we take $x_i = e_i$, and $p_2 y_{\zeta(i)} \notin [\mathfrak{L}_r/\mathfrak{J}_2, \mathfrak{L}_r/\mathfrak{J}_2]$ gives $y_{\zeta(i)} = \lambda_i e_{\zeta(i)} + z_{\zeta(i)}$ with $\lambda_i \neq 0$ and $z_{\zeta(i)} \in [\mathfrak{L}_r, \mathfrak{L}_r]$. If $p_1 x_i \in [\mathfrak{L}_r/\mathfrak{J}_1, \mathfrak{L}_r/\mathfrak{J}_1]$, then $p_1 e_i$, $h(p_1 e_i)$ and $p_2 e_{\zeta(i)}$ are in their respective derived ideals and $h(p_1 e_i) - p_2 e_{\zeta(i)}$ can be written as $p_2 z_{\zeta(i)}$ with $z_{\zeta(i)} \in [\mathfrak{L}_r, \mathfrak{L}_r]$. Setting $\theta(e_i) = \lambda_i e_{\zeta(i)} + z_{\zeta(i)}$, with $\lambda_i = 1$ in the second case, we obtain an automorphism θ of $\mathfrak{L}_r$ defined on the generators e_i, satisfying $hp_1 = p_2\theta$ and $\theta(\mathfrak{J}_1) \subset \mathfrak{J}_2$. With relations $\alpha_{\zeta(i)}(t') = \alpha_i(t)$, one checks equalities $t'\theta = \theta t$ for each $t \in T$ and $\theta \in N$. □

This lemma allows us to treat Lie algebras that don't have a fixed number of generators, which differs essentially from results of G. Favre in [8], where this number is given by r. Hypothesis (H) is satisfied by all examples studied in this paper.

The quotient space $W_r^\sigma(T)/N$ gives the isomorphic classes of the $\mathfrak{n} = \mathfrak{L}_r/\mathfrak{J}$. The neutral components N_0 of N and of $\mathrm{Aut}(\mathfrak{L}_r)^T$ are equal. The finite group N/N_0 operates on $W_r^\sigma(T)/N_0$, giving the isomorphic classes. If we compare $W_r^\sigma(T)$ to $\Sigma_n(T)$ and N_0 to G_0 in the affine description of section 2, the problem is now to find good slices for $W_r^\sigma(T)/N_0$.

Slices for $W_r^\sigma(T)/N_0$

We impose the condition $n(\alpha_i) = 1$ for $1 \leqslant i \leqslant r$. Then, the $s \in N_0$ stabilize the weight subspaces $L_\alpha (\alpha \in \Pi)$ and are defined on the generators by $s(e_i) \equiv s_i e_i \mod [\mathfrak{L}_r, \mathfrak{L}_r]$ with $s_i \in \mathbb{C}^*$ for $1 \leqslant i \leqslant r$. If $e_I = [e_{i_1} e_{i_2} \cdots e_{i_p}]$ is a Lie product of generators e_i where the brackets are omitted and $I = (i_1 i_2 \cdots i_p)$, we have for $s \in N_0$: $s(e_I) \equiv s_{i_1} \cdots s_{i_p} e_I \mod (\mathcal{C}^{p+1}\mathfrak{L}_r)_\alpha$, and $s(e_\alpha) \equiv \sum c(I) s_{i_1} \cdots s_{i_p} e_I \mod (\mathcal{C}^{p+1}\mathfrak{L}_r)_\alpha$ for $e_\alpha = \sum c(I) e_I \in L_\alpha$, $c(I) \in \mathbb{C}$.

We define a slice $\mathcal{F}$ of $W_r^\sigma(T)$ at point $\mathfrak{J}$ (or $\mathfrak{n} = \mathfrak{L}_r/\mathfrak{J}$), associated with the N_0-action group, as a subscheme of $W_r^\sigma(T)$ transversal to the orbit of

$\mathfrak{J}$ at $\mathfrak{J}$, *i.e.*, satisfying $T_{\mathfrak{J}} W_r^\sigma(T) = T_{\mathfrak{J}}\mathcal{F} \oplus T_{\mathfrak{J}}(N_0.\mathfrak{J})$ with reduced scheme structure on the orbit $N_0.\mathfrak{J}$. A subscheme $\mathcal{F}$ is a slice if it is a slice at each point. We can always construct such a subscheme by the "orbital parameters fixing" method developed in [6].

The diagonal subgroup $D \subset \operatorname{Aut}(\mathfrak{L}_r)$ defined by the s such that $s(e_i) = s_i e_i,\ s_i \in \mathbb{C}^*,\ 1 \leqslant i \leqslant r$, is contained in N_0. Condition $n(\alpha_i) = 1$ for $1 \leqslant i \leqslant r$ involves that each $s \in N_0$ can be written as $s(e_i) \equiv s_i e_i \mod (\mathfrak{J})$ with $s_i \in \mathbb{C}^*$ for $1 \leqslant i \leqslant r$ if $\mathfrak{J}$ is fixed in $W_r^\sigma(T)$. The invariant subgroup $U = \{s \in N_0;\ s(e_i) \equiv e_i \mod (\mathfrak{J})\}$ is contained in the stabilizer subgroup $\operatorname{Stab}(\mathfrak{J})$ of $\mathfrak{J}$ in N_0. We have $N_0/U \simeq D$ and the quotient $\operatorname{Stab}(\mathfrak{J})/U$ is isomorphic to the neutral component D' of the group $\operatorname{Aut}(\mathfrak{L}_r/\mathfrak{J})^T$. Thanks to the hypothesis $n(\alpha_i) = 1$ for $1 \leqslant i \leqslant r$, D' can be identified with a subgroup of D whose Lie algebra is T. The orbit $\Omega(\mathfrak{J})$ of $\mathfrak{J}$ by the N_0 action can be identified with the space of classes

$$\Omega(\mathfrak{J}) \simeq N_0/\operatorname{Stab}(\mathfrak{J}) \simeq D/D' \tag{4.1}$$

and the tangent of $\Omega(\mathfrak{J})$ at $\mathfrak{J}$ is T_r/T. In practice, the orbits $\Omega(\mathfrak{J})$ are obtained by the natural action of the subgroup $D \subset N_0$ on ideals. Slices are obtained (*cf.* examples section 5) by fixing a minimal family of lines $\mathbb{C}u_i$ $(i \in I)$ in $\mathfrak{J}$ which impose that T is a maximal torus, *i.e.*, $s(u_i) \subset \mathbb{C}u_i$ $(i \in I)$ for $s \in D$ involve $s \in D'$. If $T_r = T$ we have:

Remark 4.2. — The case of maximal rank $T = T_r$ gives the slice $\mathcal{F} = W_r^\sigma(T)$ simply.

Schemes of ideals are Jacobi schemes

We denote by a calligraphic letter the set of representatives f in $\operatorname{Hom}_{\mathbb{C}}(\mathfrak{L}_r, \mathbb{C}^n)$ whose kernel belongs to a grassmannian of $\mathfrak{L}_r$. For example, $\mathcal{J}_n(\mathfrak{L}_r)$ is the set of $\mathbb{C}$-linear maps from $\mathfrak{L}_r$ to $\mathbb{C}^n$ such that $\operatorname{Ker}(f)$ is an ideal in $J_n(\mathfrak{L}_r)$. Similarly, we define $\mathcal{V}_r^\sigma(T) = \{f \in \operatorname{Hom}(\mathfrak{L}_r, \mathbb{C}^n)^T;\ \operatorname{Ker}(f) \in V_r^\sigma(T)\}$ and so on. Such a $f \in \mathcal{J}_n(\mathfrak{L}_r)$ allows us to construct a Lie algebra bracket Φ_f on $\mathbb{C}^n$: $\Phi_f(x,y) = f([f^{-1}(x), f^{-1}(y)]),\ (x,y) \in (\mathbb{C}^n)^2$, where $[\ ,\ ]$ is the bracket on $\mathfrak{L}_r$. Notice that $(\sigma, s) \in \operatorname{Aut}(\mathfrak{L}_r) \times GL_n(\mathbb{C})$ operates on $f \in \mathcal{J}_n(\mathfrak{L}_r)$ by $s \circ f \circ \sigma^{-1}$, thus $\Phi_{sf\sigma^{-1}} = s \star \Phi_f$ and we can state the following:

Lemma 4.3. — *The algebraic map $h : f \longrightarrow \Phi_f$ from $\mathcal{J}_n(\mathfrak{L}_r)$ to $L_n(\mathbb{C})$ induces by quotient an injection on the classes:*

$$J_n(\mathfrak{L}_r)/\operatorname{Aut}(\mathfrak{L}_r) \longrightarrow L_n(\mathbb{C})/GL_n(\mathbb{C}).$$

Proof. — The quotient of $\mathcal{J}_n(\mathfrak{L}_r)$ by the left action of $GL_n(\mathbb{C})$ is identified with $J_n(\mathfrak{L}_r)$. □

By restriction to $\mathcal{W}_r^\sigma(T)$, h induces by quotient the injections $W_r^\sigma(T)/N \to L_n^T(\mathbb{C})/H$ and $W_r^\sigma(T)/N_0 \to L_n^T(\mathbb{C})/G_0$.

Theorem 4.4. — *Slices of $W_r^\sigma(T)$ associated with the action of N_0 in the scheme $V_r^\sigma(T)$ can be identified with slices of $\Sigma_n(T) \subset L_n^T$ associated with the G_0 action in Jacobi schemes.*

Proof. — Under hypotheses $n(\alpha_i) = 1$ for $1 \leqslant i \leqslant r$ and $T > 0$, there is $t \in T$ such that $\alpha(t) > 0$ for the weights and we have a partial order relation $\geqslant$ over the weights (in fact, total order) resulting from the order in the real numbers $\alpha(t)$. If $\delta \in P$ is maximal for $\geqslant$, we have $[\mathfrak{n}, \mathfrak{n}_\delta] = 0$ and $\mathfrak{n}_\delta$ is central in $\mathfrak{n}$. By induction on $\sigma = (\sigma', n(\delta))$, we construct

$$V_r^\sigma(T) = \{\mathfrak{J}' \times J_\delta \in V_r^{\sigma'}(T) \times Gr_{j(\delta)}(L_\delta); [\mathfrak{L}_r, \mathfrak{J}']_\delta \subset J_\delta\}.$$

If $\mathfrak{J}' \in V_r^{\sigma'}(T)$, we can consider the subspaces J_δ of codimension $n(\delta)$ in L_δ and containing $[\mathfrak{L}_r, \mathfrak{J}']_\delta$. These subspaces of codimension $n(\delta)$ are identified with their quotients $\bar{J}_\delta$ in $E = L_\delta/[\mathfrak{L}_r, \mathfrak{J}']_\delta$. This space E can be expressed with the help of the T-module $H_2(\mathfrak{n}')$ of homology of $\mathfrak{n}' = \mathfrak{L}_r/\mathfrak{J}'$ as:

$$E = \left(\frac{[\mathfrak{L}_r, \mathfrak{L}_r]}{[\mathfrak{L}_r, \mathfrak{J}']}\right)_\delta = [\hat{\mathfrak{n}}', \hat{\mathfrak{n}}']_\delta = H_2(\mathfrak{n}')_\delta.$$

If $\delta \neq \alpha_i$ for $1 \leqslant i \leqslant r$, we have $L_\delta = [\mathfrak{L}_r, \mathfrak{L}_r]_\delta$, but the quotient $\mathfrak{L}_r/[\mathfrak{L}_r, \mathfrak{J}']$ is an algebra $\hat{\mathfrak{n}}'$, equal to the central extension of $\mathfrak{n}'$ by the kernel $H_2(\mathfrak{n}')$ defined in [3]. It is known that $H_2(\mathfrak{n}')_\delta$ can be identified with the quotient $(\wedge^2\mathfrak{n}')_\delta/\Omega_\delta$, where Ω_δ is the space generated by the vectors

$$\int_{(xyz)} x \wedge [y, z] = x \wedge [y, z] + y \wedge [z, x] + z \wedge [x, y],\ (x, y, z) \in \mathfrak{n}'_\alpha \times \mathfrak{n}'_\beta \times \mathfrak{n}'_\gamma,$$

with $\alpha + \beta + \gamma = \delta$. A subspace representative of codimension $n(\delta)$ in E is a $\mathbb{C}-$morphism f_δ giving an exact sequence whose kernel contains Ω_δ:

$$0 \longrightarrow \operatorname{Ker}(f_\delta) \longrightarrow (\wedge^2\mathfrak{n}')_\delta \longrightarrow \mathbb{C}^{n(\delta)} \longrightarrow 0.$$

If $(x_i)_{1 \leqslant i \leqslant n'}$ is a basis of $\mathfrak{n}'$, and $(y_h)_{n' < h \leqslant n}$ a basis of $\mathbb{C}^{n(\delta)}$, we have

$$f_\delta(x_i \wedge x_j) = \sum_{h=n'+1}^{n} X_{ij}^h y_h \tag{4.2}$$

with variables X_{ij}^k. The condition $f_\delta(\Omega_\delta)=0$ can be expressed by:

$$f_\delta\Big(\int_{(ijk)} x_i\wedge[x_j,x_k]\Big)=\int_{(ijk)}\sum_m c_{jk}^m f_\delta(x_i\wedge x_m)$$
$$=\sum_h\Big(\int_{(ijk)}\sum_m c_{jk}^m X_{im}^h\Big)y_h=0,$$

i.e., $\int_{(ijk)}\sum_m c_{jk}^m X_{im}^h=0$ for each $(ijkh)$. These are the Jacobi relations associated with the weight δ, satisfied by Lie algebras $\mathfrak{n}$, where c_{jk}^m are structure constants of the quotient $\mathfrak{n}'$. Initialization of the induction is made on a weight $\alpha_i,\ 1\leqslant i\leqslant r$, with $\sigma_1=\{(\alpha_i,1)\}$. This gives trivial $V_r^{\sigma_1}(T)$ because the abelian Lie algebra $\mathbb{C}\bar{e}_i$ is associated with the ideal $\oplus_{\alpha\neq\alpha_i}L_\alpha$.

We have proved that $\mathcal{V}_r^\sigma(T)$ can be identified with the set of sequences $(f_{\alpha_1},\cdots,f_\beta,\cdots,f_\delta)$ defined in (4.2) and seen as a linear morphism f. This is the set of variables X_{ij}^k satisfying the Jacobi rules too. Observe that all quotients of $\mathfrak{L}_s(s\leqslant r)$ are quotients of $\mathfrak{L}_r$ as well, thus the scheme $\mathcal{V}_r^\sigma(T)$ (respectively $\mathcal{W}_r^\sigma(T)$) can be identified with the open set of laws in L_n^T (respectively $\Sigma_n(T)$) having less than r generators. The scheme $V_r^\sigma(T)$ is the set of ideals $\mathrm{Ker}(f)=\Pi_\beta\,\mathrm{Ker}(f_\beta)$ as quotient scheme of $\mathcal{V}_r^\sigma(T)$ by left action of $G_0\simeq\Pi_\beta GL(n(\beta))$ and $\mathcal{W}_r^\sigma(T)$ is an open subscheme of $\mathcal{V}_r^\sigma(T)$. The morphism $\mathcal{V}_r^\sigma(T)\longrightarrow L_n^T(\mathbb{C})$ defined by h is an injective morphism of Jacobi schemes. The subgroup N_0 of $\mathrm{Aut}(\mathfrak{L}_r)$ operates right hand on $\mathcal{V}_r^\sigma(T)$ and G_0 operates canonically by $\star$ on $L_n^T(\mathbb{C})$ and left hand on $\mathcal{V}_r^\sigma(T)$. These actions induce an action of $G_0\times N_0$ which is compatible with the morphism $h:f\longrightarrow\Phi_f$, $\Phi_{sfu^{-1}}=s\star\Phi_f,\ (s,u)\in G_0\times N_0$. We obtain an injection on the quotients $V_r^\sigma(T)/N_0\longrightarrow L_n^T(\mathbb{C})/G_0$, identifying the open set $W_r^\sigma(T)/N_0$ with an open set of $\Sigma_n(T)/G_0$. Hence, h identifies each (possible) slice of $W_r^\sigma(T)/N_0$ with a slice of $\Sigma_n(T)/G_0$. □

With formula of Theorem 1.8 (iii) of [3] under hypothesis $n(\alpha_i)=1$, $1\leqslant i\leqslant r$, we can state:

Theorem 4.5. — *The Zariski tangent space of the scheme* $W_r^\sigma(T)$ *at a point* $\mathfrak{n}=\mathfrak{L}_r/\mathfrak{J}$ *defined by* $\mathfrak{J}$ *is equal to* $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{J},\mathfrak{n})^{\mathfrak{L}+T}$. *The Zariski tangent space to a slice of* $W_r^\sigma(T)/N_0$ *is isomorphic to the second* T*-cohomological adjoint group* $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{J},\mathfrak{n})^{\mathfrak{L}+T}/(T_r/T)$.

Proof. — Let $f_0:\mathfrak{L}_r\longrightarrow\mathbb{C}^n\simeq\mathfrak{n}$ be a representative of the ideal $\mathfrak{J}=\mathrm{Ker}(f_0)\in W_r^\sigma(T)$ and $f=f_0+h$ be another representative in $\mathcal{W}_r^\sigma(T)$, with h small. The ideal condition for $\mathrm{Ker}(f)$ gives $f_0([y,x])+h([y,x])=0$ for $(x,y)\in\mathrm{Ker}(f)\times\mathfrak{L}_r$, but x can be written $x_0+\xi$ with $x_0\in\mathfrak{J}$ and ξ

small, thus we have :

$$f_0([y,\xi]) + h([y,x_0]) + h([y,\xi]) = 0 \tag{4.3}$$

$$f(x) = f_0(\xi) + h(x_0) + h(\xi) = 0. \tag{4.4}$$

The $\mathfrak{L}_r$-module action $[y, f_0(\xi)]$ defined by $f_0([y,\xi])$ writes $-h([y,x_0]) - h([y,\xi])$ with (4.3) and $-[y,h(x_0)] - [y,h(\xi)]$. With (4.4) we have consequently at first order in (h,ξ): $h([y,x_0]) = [y,h(x_0)]$; it can be expressed by $y-$invariance of h restricted to $\mathfrak{J}$. Similarly, the T-invariance of $\mathrm{Ker}(f)$ gives an equivalence to (4.3) for $t \in T$:

$$f(t(x)) = f_0(t\xi) + h(tx_0) + h(t\xi) = 0.$$

If $\mathbb{C}^n$ is endowed with a T-module structure by $tf_0(x) = f_0(tx)$ for $(t,x) \in T \times \mathfrak{L}_r$, the term $f_0(t\xi)$ can be written as $tf_0(\xi)$. We obtain at the first order with (4.4) the equality $h(tx_0) = t(h(x_0))$. We have shown the first assertion of the theorem. If $\mathcal{F}$ is a slice defined at point $\mathfrak{J}$, then the N_0-orbit of $\mathfrak{J}$ at this point admits a Zariski tangent space isomorphic to T_r/T with (4.1). The slice and the orbit are transversal at $\mathfrak{J}$, and the tangent space of $\mathcal{F}$ is equal, as quotient of the tangent of $W_r^\sigma(T)$ by T_r/T, to $H^2(\mathfrak{n},\mathfrak{n})^T$ [3]. □

The semi-continuous mapping $\mathfrak{n} \to r = \dim(\mathfrak{n}/[\mathfrak{n},\mathfrak{n}])$ involves a stratification $\cup_{r\geqslant r_0}\Sigma_n^{(r)}(T)$ on $\Sigma_n(T)$. The minimal value r_0 gives an open stratum $\Sigma_n^{(r_0)}(T)$. All quotients of $\mathfrak{L}_\rho$ are quotients of $\mathfrak{L}_r$ if $\rho < r$, and from above we have isomorphisms, for each r:

$$W_r^\sigma(T)/N_0 \simeq \cup_{\rho\leqslant r}\Sigma_n^{(\rho)}(T)/G_0.$$

5. Study of the rigidity in varieties of ideals

Proposition 5.1. — *If $\mathfrak{g} = T \oplus \mathfrak{n}$ is a semi-direct product with maximal $T > 0$, $n(\alpha_i) = 1$ for $1 \leqslant i \leqslant r$, then $\mathfrak{g}$ is a complete Lie algebra and conditions i) ii) iii) are equivalent:*

i) *$\mathfrak{g}$ is rigid in $L_m(\mathbb{C})$;*
ii) *$\mathfrak{n}$ is rigid in $L_n^T(\mathbb{C})$;*
iii) *$\mathfrak{n}$ is rigid in $V_r^\sigma(T)$ or $W_r^\sigma(T)$.*

Local rings of the different slices at $\mathfrak{n}$ are isomorphic and the obstructions are the same.

Proof. — We have $n(\alpha_i) = 1$ for the generators $\bar{e}_i (1 \leqslant i \leqslant s)$ of $\mathfrak{n}$, and then $Der(\mathfrak{n})^T = T$ and $\mathfrak{g}$ is complete [4]. Equivalence i)⇔ ii) results from reduction theorem [6] and ii)⇔ iii) from Theorem 4.4. □

Example 1: Series of rigid Lie algebras defined by one relation

Let a, $b > 0$ be two integer numbers with $(a, b) = 1$ and $r = a/b$. We consider the torus $T \subset T_2$ on $\mathfrak{L}_2$, $T = \mathrm{Ker}(b\varepsilon_2 - a\varepsilon_1)$, $\alpha_2 = r\alpha_1$, with $\alpha_i = \varepsilon_i|_T$ and $L_{m\alpha_1} = \oplus\{L_{p\varepsilon_1+q\varepsilon_2}; p+qr = m\}$ if $m \in \mathbb{Q}^+$. For $\nu = (a+1+r)\alpha_1$ we obtain the two-dimensional space $L_\nu = L_{(a+1)\varepsilon_1+\varepsilon_2} \oplus L_{\varepsilon_1+(b+1)\varepsilon_2}$. If we write a vector $u \in L_\nu$ as $u_1 + u_2$, according to this sum, and if $\langle u\rangle$ is the ideal generated by u in $\mathfrak{L}_2$, then $u_1 \neq 0$, $u_2 \neq 0$ and $m > a + 1 + r$ involve that the ideal $\mathfrak{J}_{(m)} = \langle u\rangle + (\oplus_{k\geqslant m} L_{k\alpha_1})$ is T-invariant and T is maximal on $\mathfrak{L}_2/\mathfrak{J}_{(m)}$. The weight systems σ_n of this quotients, associated with T, satisfy hypotheses of Proposition 5.1. The group N_0 is defined by $s(e_i) = s_i e_i$, $s_i \in \mathbb{C}^*$, on the generators e_1, e_2, and the orbit of $\mathfrak{J}_{(m)}$ is given by the action on $\mathbb{C}u$: $s(u) = s_1^{a+1} s_2 u_1 + s_1 s_2^{b+1} u_2$. It is an open set in the projective space $\mathbb{P}_1(L_\nu)$ of the lines of L_ν. A slice is given by fixing $\mathbb{C}u$ in L_ν. Thus, we obtain isolated points and the algebras $\mathfrak{L}_2/\mathfrak{J}_{(m)}$ are rigid in the schemes $V_2^{\sigma_n}(T)$ and L_n^T, according to Proposition 5.1. The two-T-cohomological group of these algebras, calculated with formula [3], is 0. If the dimension of L_ν is greater than or equal to 3, then we can obtain continuous families by this method.

The second example proposed here shows how an obstruction appears in this formalism.

Example 2: The local study of $\mathfrak{a}_{4,n}$ defined by generators and relations

Let $\mathfrak{L}_3$ be the free Lie algebra with 3 generators indexed by e_1, e_2, e_4, T be the torus $\mathrm{Ker}(\varepsilon_2 - 2\varepsilon_1) \subset T_3$ with weights α_i satisfying $\alpha_2 = 2\alpha_1$ and $\mathfrak{L}_3 = \oplus L^m$ be the graduation defined by $L^m = \oplus\{L_{p\varepsilon_1+q\varepsilon_2+r\varepsilon_4};\ m = p+2q+4r\}$. We search for a sequence of T-invariant ideals $\mathfrak{J}_6 \supset \mathfrak{J}_7 \supset \mathfrak{J}_8 \cdots$ of $\mathfrak{L}_3$ such that the quotients are isomorphic to $\mathfrak{a}_{4,n}$ (Remark 2.7). The weights on $\mathfrak{a}_{4,n}$ are $\alpha_1, 2\alpha_1, 3\alpha_1, \alpha_4 + p\alpha_1$. These ideals contain the ideal I generated by the subspaces $L_{p\alpha_1+q\alpha_4}$ with $p > 3$ and $q = 0$, or $p \geqslant 0$ and $q > 1$. We have $L^1 = \mathbb{C}e_1$, $L^2 = \mathbb{C}e_2$, $L^3 = \mathbb{C}[e_1, e_2]$, $L^4 = \mathbb{C}e_4 + \mathbb{C}[e_1, [e_1, e_2]]$, $L^5 = L_{5\alpha_1} \oplus L_{\alpha_4+\alpha_1}$ and $\mathfrak{J}_n = I + \sum_{m>n} L^m$ for $n = 4, 5$.

For $n = 6$, we have $L^6 = L_{6\alpha_1} \oplus L_{\alpha_4+2\alpha_1}$, and we fix the line in $L_{\alpha_4+2\alpha_1} = \mathbb{C}(ade_1)^2 e_4 + \mathbb{C}[e_2, e_4]$, $\mathbb{C}$-generated by a vector $a[e_2, e_4] + b[e_1, [e_1, e_4]]$, $ab \neq 0$. It is stabilized by the subgroup of the $(s_1, s_2, s_4) \in (\mathbb{C}^*)^3$ such that $s_2 = (s_1)^2$. The choice $ab \neq 0$ breaks the T_3-invariance and T becomes maximal as a torus over $\mathfrak{J}_6$ and the quotient as well. This corresponds to the initialization in the induction process. All choices $ab \neq 0$ give the same quotient, up to an isomorphism, and we choose $u = [e_2, e_4] - [e_1, [e_1, e_4]]$, $\mathfrak{J}_6 = \mathbb{C}u + I + \sum_{m>6} L^m$. For $n = 7$, we have $L^7 = L_{7\alpha_1} \oplus L_{\alpha_4+3\alpha_1}$, and

$$L_{\alpha_4+3\alpha_1} = \mathbb{C}(ade_1)^3 e_4 \oplus \mathbb{C}[e_1, [e_2, e_4]] \oplus \mathbb{C}[e_2, [e_1, e_4]]$$

contains $[e_1, u]$. The 2-dimensional spaces V, $\mathbb{C}[e_1, u] \subset V \subset L_{\alpha_4+3\alpha_1}$, are given by an additional vector

$$v = \lambda[e_1, u] + x[e_1, [e_2, e_4]] + y[e_2, [e_1, e_4]] \notin \mathbb{C}[e_1, u],$$

with $x \neq 0$ or $y \neq 0$. The ideals $\mathfrak{J}_7 = \mathbb{C}v + \langle u \rangle + I + \sum_{m>7} L^m$ define by quotient the family $\mathfrak{a}_{4,7}(t)$. For $n = 8$, we have $L^8 = L_{8\alpha_1} \oplus L_{\alpha_4+4\alpha_1}$ and $\mathfrak{J}_8 = \langle u \rangle + \langle v \rangle + I + \sum_{m>8} L^m$. For $n = 9$, the ideal $\mathfrak{J}_9$ must contain the ideal $\langle u \rangle + \langle v \rangle + I + \sum_{m>9} L^m$ and we study its codimension in $\mathfrak{L}_3$ *i.e.*, the dimension of $L_{\alpha_4+5\alpha_1}/(\langle u \rangle + \langle v \rangle + I)_{\alpha_4+5\alpha_1}$ depending on v. This quotient is isomorphic to $E/(\langle u \rangle + \langle v \rangle)_{\alpha_4+5\alpha_1}$, where E is a T_3-invariant complement subspace of the intersection with I in $L_{\alpha_4+5\alpha_1}$. We can generate E with the following vectors:

$\mu = (ade_1)^5 e_4$ in $L_{\varepsilon_4+5\varepsilon_1}$, $\nu = (ade_1)^3 [e_2, e_4]$ and $\rho = (ade_1)^2([e_4, [e_1, e_2]])$ in $L_{\varepsilon_4+\varepsilon_2+3\varepsilon_1}$; $\sigma = ade_1(ade_2)^2 e_4$ and $\delta = [[e_1, e_2], [e_2, e_4]]$ in $L_{\varepsilon_4+2\varepsilon_2+\varepsilon_1}$.

We calculate the dimension of $(\langle u \rangle + \langle v \rangle)_{\alpha_4+5\alpha_1}$, which is equal to the rank of the system of the following vectors written over the basis $\{\mu, \nu, \rho, \sigma, \delta\}$:

$$(ade_1)^3 u,\ (ade_1)([e_2, u]),\ [[e_1, e_2], u],\ (ade_1)^2 v,\ [e_2, v].$$

The dimension of E is equal to 5 and the dimension of $(\langle u \rangle + \langle v \rangle)_{\alpha_4+5\alpha_1}$ depending on $(x, y) \in \mathbb{C}^2$ is given by one of the two following cases:

If $x + y \neq 0$, the dimension is 5 and there is not possible extension for $\mathfrak{a}_{4,8}(t)$, $t \neq 0$.

If $x+y = 0$, the dimension is 4 and we have an extension corresponding to $t = 0$. Moreover, if $y = -x \neq 0$, the algebra corresponds to an isolated point $\mathfrak{J}_9$, rigid in the variety $W_r^\sigma(T)$ or $\Sigma_9(T)(\subset L_9^T)$. In this case we obtain a constraint between the vectors generating the weight space $(\langle u \rangle + \langle v \rangle)_{\alpha_4+5\alpha_1}$ given by:

$$D(v) + D'(u) = (ade_2 - (ade_1)^2)v + (\lambda(ade_1)^3 + yade_1ade_2 - (\lambda + x + 2y)ade_2ade_1)u \in I + \sum_{m>9} L^m.$$

Calculation of $H^2(\mathfrak{a}_{4,n}, \mathfrak{a}_{4,n})^T$

We have:

$$H^2(\mathfrak{a}_{4,n}, \mathfrak{a}_{4,n})^T \simeq \left\{f \in \mathrm{Hom}_{\mathbb{C}}(\langle v\rangle, \mathfrak{a}_{4,n})^{\mathfrak{L}+T};\ f(\langle u\rangle \cap \langle v\rangle) = 0\right\}.$$

We replace $\mathfrak{J}_n$ by $\langle u\rangle + \langle v\rangle$ in Theorem 4.5. The vector $f(u) = h\bar{e}_6$ corresponds to $\alpha_4 + 2\alpha_1$. Observe that T_3 is embedded in natural way in $\mathrm{Hom}(\mathfrak{J}_n, \mathfrak{a}_{4,n})^{\mathfrak{L}+T}$ with $T_3(u) \neq 0$, hence there is $\delta \in T_3$ such that $f_0 = f - h\delta$ is null on $\langle u\rangle$.

With this formula, the cohomological group is null for $n = 6$. For $n = 7$, we have a representative f in the class defined by $f(v) = a\bar{e}_7$ and $f(u) = f([e_1, u]) = 0$. For $n = 8$, we have $f([e_1, v]) = [e_1, f(v)] = a[e_1, \bar{e}_7] = a\bar{e}_8$ and $f(\langle u\rangle) = 0$. For $n = 9$, f is compatible with the constraint expressed by:

$$f(Dv + D'u) = f(Dv) = Df(v) = a(ade_2 - (ade_1)^2)\bar{e}_7 = 0.$$

Thus, we have representatives $f \neq 0$ defined by $f(v) = a\bar{e}_7,\ a \in \mathbb{C}^*$ for $n \geqslant 7$ and the second cohomological group is $\mathbb{C}$. Rigidity for $n \geqslant 9$ involves the existence of an obstruction. We calculate this obstruction, illustrating the last method.

Remark 5.2. — In the case where the second T-cohomological group becomes null in a central extension, compatibility of f with the constraints is not satisfied and $f = 0$.

Obtaining a nilpotent element in the scheme of ideals

Theorem 4.4 allows us to obtain a nilpotent element in the slice of $W_3^\sigma(T)$ by applying the simple ideal condition for $\mathfrak{J}_9$. It suffices to show here that the vector $(ade_1)^2v$ belongs to the space generated by $(ade_1)^3u$, $ade_1ade_2(u)$, $(ad[e_1, e_2])u$ and $(ade_2)v$ modulo I. Thus, we have:

$$(ade_1)^2v \equiv p(ade_1)^3u + q(ade_1ade_2)u + r(ad[e_1, e_2])u + s(ade_2)v, \quad \mod (I)$$

where parameters $p, q, r, s, \lambda, x, y$ are chosen in the local ring of the scheme at the point $\mathfrak{J}_9$. Writing this equality on the basis vectors μ, σ, δ, ν and ρ, we obtain the following equalities respectively: (a) $p = \lambda$; (b) $q = -s(\lambda+x+y)$; (c) $r = s(\lambda + x + 2y)$; (d) $p - q - \lambda s = \lambda + x + y$; (e) $-2q + r - 3\lambda s = y$.

From (d) and (e), we deduce $(x+y)(s-1) = 0$ and $(s-1)y = -3s(x+y)$. Dividing by y in the local ring because $\bar{y} \neq 0$ (second case $x+y = 0$ above), we have $s(x+y)^2 = 0$ and $(s-1)^2 = 0$. The parameter s can be inverted in the local ring and we obtain $(x+y)^2 = 0$.

Note that $(x,y) \in \mathbb{C}^2 - \{(0,0)\}$ corresponds to a law satisfying, in the quotient by $\mathfrak{J}_n$: $\bar{v} = x\overline{[e_1,[e_2,e_4]]} + y\overline{[e_2,[e_1,e_4]]} = 0$. If $y \neq 0$, then we have $\overline{[e_2,[e_1,e_4]]} = -\frac{x}{y}\overline{[e_1,[e_2,e_4]]}$ and we can write $-x/y = 1-t$ with t given in Remark 2.7.

The case $y = 0$ involves $\overline{[e_1,[e_2,e_4]]} = 0$ with $x \neq 0$ and defines the algebras:

$$(A_7) \qquad \begin{array}{l} [e_1,e_2] = e_3,\ [e_1,e_4] = e_5,\ [e_1,e_5] = e_6, \\ [e_2,e_4] = e_6,\ [e_2,e_5] = e_7,\ [e_3,e_4] = -e_7, \end{array}$$

for $n = 7$ and, by adding the following brackets:

$$(A_8) \qquad [e_1,e_7] = e_8,\ [e_2,e_6] = 2e_8,\ [e_3,e_5] = -e_8,$$

for $n = 8$. The nilpotent condition $(x+y)^2 = 0$ for $n \geqslant 9$ gives $t^2 = 0$ and we state:

Proposition 5.3. — *If T is defined by the weights $\alpha_1, 2\alpha_1, 3\alpha_1, \alpha_4 + k\alpha_1 (k \geqslant 0)$ with multiplicities one and if $\mathcal{A} = \{(24), (1k)$ for $1 < k < n, k \neq 3\}$, a slice of the scheme $W_3^\sigma(T)$ is given, up to isomorphism, by:*

- *The union of the scheme $L_n^{T,\mathcal{A}}$ and the point $\{A_n\}$ for $n = 7, 8$;*
- *$L_n^{T,\mathcal{A}} = \{\mathfrak{a}_{4,n}(t)\}$ where $t = \frac{x+y}{y}$ is a 2-nilpotent parameter for $n \geqslant 9$.*

Case $r > r_0$

If T is the torus defined in Proposition 5.3. then the set $\mathcal{A}'_n = \{(1j), j \geqslant 4, (24)(34)\}$ is admissible in $L_n^T(\mathbb{C})$ for $n \geqslant 7$. The associated slice is given by $X_{12} = t,\ X_{1j} = 1 (j \geqslant 4),\ X_{24} = 1,\ X_{25} = 1-t,\ X_{34} = 1$ for $n = 7$, adding $X_{17} = 1,\ X_{26} = 1-2t$ and $X_{35} = 1$ for $n = 8$. If $n \geqslant 9$ we have $t = 0$, thus we obtain the following rigid Lie algebra satisfying $H^2(\mathfrak{n}_n^{(4)}, \mathfrak{n}_n^{(4)})^T = 0$:

$$\mathfrak{n}_n^{(4)} : \ [x_i, x_j] = \begin{cases} x_{i+j} & \text{for } 1 \leqslant i \leqslant 3,\ 4 \leqslant j \leqslant n-i \\ 0 & \text{otherwise} \quad i < j. \end{cases}$$

This algebra is the unique 4-generated Lie algebra in $\Sigma_n(T)$ for $n \geqslant 7$ belonging, as quotient $\mathfrak{L}_4/\mathfrak{J}_n$, to $W_4^\sigma(T)$. We have $\dim(L_{3\alpha_1}) = 2$ and we obtain the algebras above as quotients of $\mathfrak{L}_4$ by 7-codimensional ideals $\mathfrak{J}_7$ where the projection $(\mathfrak{J}_7)_{3\alpha_1}$ on $L_{3\alpha_1} = L_{\varepsilon_3} \oplus L_{\varepsilon_1+\varepsilon_2}$ is the line generated by $te_3 - [e_1,e_2]$ for $t \in \mathbb{C}$. If $t \neq 0$, then the quotients $\mathfrak{L}_4/\mathfrak{J}_7$ are in fact quotients of $\mathfrak{L}_3$ describing the open set of 3-generated algebras. If $t = 0$, then we obtain $\mathfrak{n}_7^{(4)}$ with $\mathfrak{J}_7 \subset [\mathfrak{L}_4, \mathfrak{L}_4]$.

Conclusion

Extrapolating this work, the idea that rigidity is a property which is not dependent on the particular choice of a geometry constitutes a valid new slant. Most generally, we can imagine a notion of continuous family attached to the category and not depending on a particular geometrical representation. Theorem 4.4 and Proposition 5.1 certainly move in this direction with two different geometrizations for an important class of nilpotent Lie algebras. This explains why different methods in classifications of nilpotent Lie algebras give the same continuous families, with different parameterizations depending only on the choice of a local chart.

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