ANNALES

DE

L'INSTITUT FOURIER

Xavier BUFF & Arnaud CHÉRITAT

**A new proof of a conjecture of Yoccoz**

Tome 61, n° 1 (2011), p. 319-350.

<http://aif.cedram.org/item?id=AIF_2011__61_1_319_0>

Ann. Inst. Fourier, Grenoble
**61**, 1 (2011) 319-350

# A NEW PROOF OF A CONJECTURE OF YOCCOZ

by Xavier BUFF & Arnaud CHÉRITAT (*)

Abstract. — We give a new proof of the following conjecture of Yoccoz:

$$(\exists C \in \mathbb{R})\ (\forall \theta \in \mathbb{R} \setminus \mathbb{Q}) \quad \log \operatorname{rad} \Delta(Q_\theta) \leqslant -Y(\theta) + C,$$

where $Q_\theta(z) = \mathrm{e}^{2\pi i\theta} z + z^2$, $\Delta(Q_\theta)$ is its Siegel disk if $Q_\theta$ is linearizable (or $\varnothing$ otherwise), $\operatorname{rad}\Delta(Q_\theta)$ is the conformal radius of the Siegel disk of $Q_\theta$ (or 0 if there is none) and $Y(\theta)$ is Yoccoz's Brjuno function.

In a former article we obtained a first proof based on the control of parabolic explosion. Here, we present a more elementary proof based on Yoccoz's initial methods.

We then extend this result to some new families of polynomials such as $z^d + c$ with $d > 2$. We also show that the conjecture does not hold for $\mathrm{e}^{2\pi i\theta}(z + z^d)$ with $d > 2$.

Résumé. — Nous donnons une nouvelle preuve de la conjecture suivante de Yoccoz :

$$(\exists C \in \mathbb{R})\ (\forall \theta \in \mathbb{R} \setminus \mathbb{Q}) \quad \log \operatorname{rad} \Delta(Q_\theta) \leqslant -Y(\theta) + C,$$

où $Q_\theta(z) = \mathrm{e}^{2\pi i\theta} z + z^2$, $\Delta(Q_\theta)$ est son disque de Siegel si $Q_\theta$ est linéarisable (ou $\varnothing$ sinon), $\operatorname{rad}\Delta(Q_\theta)$ est le rayon conforme du disque de Siegel de $Q_\theta$ (ou 0 s'il n'y en a pas) et $Y(\theta)$ est la fonction de Brjuno de Yoccoz.

Dans un article précédent nous avons obtenu une première preuve basée sur le contrôle de l'explosion parabolique. Ici, nous présentons une preuve plus élémentaire basée sur les méthodes initiales de Yoccoz.

Nous étendons ce résultat à quelques nouvelles familles de polynômes telle que $z^d + c$ avec $d > 2$. Nous montrons également que la conjecture ne tient pas pour $\mathrm{e}^{2\pi i\theta}(z + z^d)$ avec $d > 2$.

In this article, the notation $\mathbb{N}$ stands for the set of non negative integers $\{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \{1, 2, \ldots\}$. We will use $m \wedge n$ to denote the greatest common divisor of $m$ and $n$. Let $D(z, r)$ stand for the disk of center $z$ and radius $r$ in $\mathbb{C}$ and let $\mathbb{D} = D(0, 1)$.

Many of our statements will concern the following set, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$:

$$\mathcal{S}_\theta = \left\{\text{univalent maps } f : \mathbb{D} \hookrightarrow \mathbb{C} \text{ with } f(z) = \mathrm{e}^{2\pi i\theta}\, z + \mathcal{O}(z^2)\right\}.$$

*Keywords:* Siegel disks, quadratic polynomials, harmonic and subharbonic functions, conformal radius, holomorphic motions.
*Math. classification:* 37F50.
(*) This research was partially funded by the grant *At the Boundary of Chaos* of the Agence Nationale de la Recherche, ANR–08–JCJC–0002.

# 1. Introduction and statements

## 1.1. Main theorem

Definition 1.1. — *The conformal radius (with respect to* $0$*) of a connected and simply connected open subset* $U$ *of* $\mathbb{C}$ *containing* $0$*, is the unique value of* $r \in (0, +\infty]$ *such that there exists a conformal map* $\psi : D(0, r) \to U$ *with* $\psi(0) = 0$ *and* $\psi'(0) = 1$*. It will be denoted by* $\operatorname{rad} U$ *in this article.*

*Remark 1.2.* — A consequence of Schwarz's lemma is that if $U \subset U'$ and $U'$ is also open connected and simply connected, then $\operatorname{rad} U' \geqslant \operatorname{rad} U$.

Let $f$ be a holomorphic map fixing the origin in $\mathbb{C}$. Assume that its differential at the origin is an aperiodic rotation: $f(z) = \mathrm{e}^{2\pi i\theta} z + \mathcal{O}(z^2)$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$. If $f$ is linearizable at the origin, the Siegel disk $\Delta(f)$ is the maximal domain containing $0$ on which $f$ is conjugated to a rotation. Then $f$ has a Siegel disk if and only if the radius of convergence of the normalized formal linearizing power series is positive, see Section 2.1. In this case the conformal radius of its Siegel disk is less than or equal to this convergence radius. If $f$ is not linearizable let us set $\Delta(f) = \varnothing$. Let us set by convention that

$$\operatorname{rad} \varnothing = 0.$$

Definition 1.3. — *Let* $Q_\theta : \mathbb{C} \to \mathbb{C}$ *be the quadratic polynomial defined by:*

$$Q_\theta(z) = \mathrm{e}^{2\pi i\theta} z + z^2.$$

In [18], Yoccoz used a technique of Ilyashenko and the polynomial-like map theory of Douady and Hubbard [7] to prove the following result.

Theorem A (Yoccoz). — *For all* $\varepsilon > 0$*, there exists a constant* $c(\varepsilon) > 0$ *such that the following holds. For all* $\theta \in \mathbb{R} \setminus \mathbb{Q}$ *and* $f \in \mathcal{S}_\theta$*, then:*

$$\operatorname{rad} \Delta(f) \geqslant c(\varepsilon) \cdot \big(\operatorname{rad} \Delta(Q_\theta)\big)^{1+\varepsilon}.$$

Our first result, whose proof takes its roots in the one of Yoccoz, asserts that one can choose a constant $c(\varepsilon)$ which does not depend on $\varepsilon$. This follows from the results obtained in [3] but there, the techniques are much more elaborate than the ones we present here.

Theorem 1.4 (main theorem). — *Assume* $f \in \mathcal{S}_\theta$ *with* $\theta \in \mathbb{R} \setminus \mathbb{Q}$*. Then*

$$\operatorname{rad} \Delta(f) \geqslant \frac{1}{10} \operatorname{rad} \Delta(Q_\theta).$$

We shall prove this theorem in Section 2. The constant $1/10$ is not optimal.

The main theorem can also be presented as follows.

Corollary 1.5. — *Let $\widetilde{Q}_\theta(z) = 2Q_\theta(z/2)$ (this affine conjugate of $Q_\theta$ is univalent on $\mathbb{D}$). For all $\theta \in \mathbb{R} \setminus \mathbb{Q}$,*

$$\frac{1}{20} \operatorname{rad} \Delta(\widetilde{Q}_\theta) \leqslant \inf_{f \in \mathcal{S}_\theta} \operatorname{rad} \Delta(f) \leqslant \operatorname{rad} \Delta(\widetilde{Q}_\theta).$$

Stated this way, the right hand inequality is trivial. The left hand follows from Theorem 1.4 and the following elementary formula expressing how $\operatorname{rad} \Delta(f)$ changes under a linear conjugacy: if $g(z) = \lambda f(z/\lambda)$ then $\operatorname{rad} \Delta(g) = |\lambda| \operatorname{rad} \Delta(f)$.

### 1.2. Yoccoz's work linking arithmetical properties of the rotation number with the size of Siegel disks

Definition 1.6. — *For $\theta \in \mathbb{R} \setminus \mathbb{Q}$ let*

$$Y(\theta) = \sum_{n=0}^{+\infty} \alpha_0 \cdots \alpha_{n-1} \log \frac{1}{\alpha_n}$$

*where $\alpha_0 = \operatorname{Frac}(\theta) = \theta - \lfloor \theta \rfloor$ and $\alpha_{n+1} = \operatorname{Frac}(1/\alpha_n)$.*

Definition 1.7. — *A Brjuno number is an irrational real number $\theta$ satisfying Brjuno's condition $Y(\theta) < +\infty$. We denote by $\mathcal{B}$ the set of Brjuno numbers.*

The following results were proved by Yoccoz in [18].

Theorem B (Yoccoz). — *There exists a constant $C \in \mathbb{R}$ such that for all $\theta \in \mathcal{B}$, for all $f \in \mathcal{S}_\theta$, the Siegel disk of $f$ contains the disk $D(0, \mathrm{e}^{-Y(\theta)-C})$. In particular,*

$$\log \operatorname{rad} \Delta(f) \geqslant -Y(\theta) - C.$$

Corollary (Yoccoz). — *If $\theta$ is a Brjuno number,* $\operatorname{rad} \Delta(Q_\theta) > 0$ *and*

$$\log \operatorname{rad} \Delta(Q_\theta) \geqslant -Y(\theta) - C - \log 2.$$

The term $-\log 2$ comes from the fact $Q_\theta$ is univalent only on the disk $D(0, 1/2)$.

Theorem C (Yoccoz). — *There exists a constant* $C \in \mathbb{R}$ *such that for all* $\theta \in \mathbb{R} \setminus \mathbb{Q}$ *there exists* $f \in \mathcal{S}_\theta$*, such that*

$$\log \operatorname{rad} \Delta(f) \leqslant -Y(\theta) + C.$$

*This includes the case* $\theta \notin \mathcal{B}$ *(i.e.,* $Y(\theta) = +\infty$*) if we interpret the above inequality as* $\operatorname{rad} \Delta(f) = 0$*.*

Theorems B and C can be presented together as follows:

Corollary (Yoccoz).

$$\forall \theta \in \mathbb{R} \setminus \mathbb{Q}, \quad -Y(\theta) - C \leqslant \inf_{f \in \mathcal{S}_\theta} \log \operatorname{rad} \Delta(f) \leqslant -Y(\theta) + C.$$

Combining Theorem A and Theorem C, Yoccoz obtained the following corollaries.

Corollary (Yoccoz). — *For all* $\varepsilon > 0$*, there exists* $C_\varepsilon \in \mathbb{R}$ *(that a priori may tend to* $+\infty$ *as* $\varepsilon \longrightarrow 0$*) such that for all* $\theta \in \mathbb{R} \setminus \mathbb{Q}$*,*

$$\log \operatorname{rad} \Delta(Q_\theta) \leqslant -(1-\varepsilon) Y(\theta) + C_\varepsilon.$$

*In particular, if* $\theta$ *is not a Brjuno number, then* $\operatorname{rad} \Delta(Q_\theta) = 0$*.*

Corollary (Yoccoz). — $\operatorname{rad} \Delta(Q_\theta) > 0$ *if and only if* $\theta$ *is a Brjuno number.*

### 1.3. Consequence of the main theorem

The second author found an independent proof of "$\theta \in \mathbb{R} \setminus \mathcal{B} \implies \operatorname{rad} \Delta(Q_\theta) = 0$" in [6], working directly in the family $Q_\theta$. He looked at how parabolic points explode into cycles and how these cycles hinder each others. The control on parabolic explosion uses the combinatorics of quadratic polynomials, and the Yoccoz inequality on the limbs of the Mandelbrot set. The *relative Schwarz lemma* of the first author then enabled us to have a good enough control on conformal radii to prove the following result, conjectured by Yoccoz [18], that enhances his estimate.

Theorem 1.8. — *There exists a constant* $C \in \mathbb{R}$ *such that for all* $\theta \in \mathbb{R} \setminus \mathbb{Q}$*,*

$$\log \operatorname{rad} \Delta(Q_\theta) \leqslant -Y(\theta) + C.$$

This article gives a new proof of Theorem 1.8, as an immediate corollary of Yoccoz's Theorem C and the main theorem, Theorem 1.4.

Together with the corollary following Theorem B, this gives:

Corollary 1.9. — *There exists a constant $C \in \mathbb{R}$ such that for all $\theta \in \mathbb{R} \setminus \mathbb{Q}$,*

$$-Y(\theta) - C \leqslant \log \operatorname{rad} \Delta(Q_\theta) \leqslant -Y(\theta) + C.$$

Thus the function

$$\Upsilon : \mathcal{B} \to \mathbb{R} \quad \text{defined by} \quad \Upsilon(\theta) = \log \operatorname{rad} \Delta(Q_\theta) + Y(\theta)$$

is uniformly bounded. In [4], we proved the stronger statement that this function has a continuous extension to $\mathbb{R}$. This problem is not addressed in the present article.

### 1.4. Extension to some other families of polynomials and counterexamples

In Section 3, we show that our techniques extend to other families of polynomials. Let us define a class of well-behaved polynomials that was studied by Lukas Geyer in [9]. Given a polynomial $P : \mathbb{C} \to \mathbb{C}$, a critical orbit tail is an equivalence class in the set of forward critical orbits(1), with the relation $z \equiv z' \iff \exists m, n \in \mathbb{N}$ such that $P^{\circ n}(z) = P^{\circ m}(z')$. We say it is infinite if a point of the class (and therefore every point in the class) has infinite forward orbit. Every infinite critical orbit tail falls in one and only one of the following five categories: let

- $ti_0$ count the number of infinite critical orbit tails falling in a Siegel disk,
- $ti_1$ count those that fall in a super-attracting basin,
- $ti_2$ those that fall in an attracting basin, not super-attracting,
- $ti_3$ those that fall in a parabolic basin,
- $ti_4$ those that belong to $J$.

Let $ti$ count the total number of infinite critical orbit tails: $ti = ti_0 + \cdots + ti_4$. Let

- $n_1$ be the number of superattracting cycles,
- $n_2$ be the number of attracting cycles, not super-attracting,
- $n_3$ be the number of parabolic cycles,
- $n_3' \geqslant n_3$ be the number of cycles of parabolic petals,
- $n_4$ be the number of irrationally indifferent cycles.

By the Fatou-Shishikura inequality (see [7], [15], [8]) $n_2 \leqslant ti_2$, $n_3' \leqslant ti_3$, $n_4 \leqslant ti_4$. Geyer's condition is that $n_4 = ti_4$. Note that we always have $n_3 + n_4 \leqslant ti$.

(1) or, equivalently, in the set of critical *points*

Definition 1.10. — *A polynomial has property (G) if the number of infinite critical orbit tails is equal to the number of indifferent cycles, i.e.,* $n_3 + n_4 = ti$.

Thus, $P$ has property (G) if and only if $ti_4 = n_4$, $ti_3 = n_3' = n_3$, $ti_2 = n_2 = 0$, $ti_1 = 0$ and $ti_0 = 0$, *i.e.*, there is no attracting cycle, the basin of attraction of super-attracting cycles and Siegel disks contains no infinite critical orbit tail and the basin of attraction of each parabolic cycle contains at most one infinite critical orbit tail. It follows that each parabolic cycle has exactly one cycle of petals and is virtually repelling (see [5]). Property (G) is less general than Geyer's condition.

*Remark 1.11.* — If $P$ has property (G), its iterates do not necessarily. For instance, take a degree 2 polynomial with a period one Siegel disk. Then $n_3(P) = 0$, $n_4(P) = 1$ and $ti(P) = 1$. However $n_3(P^{\circ n}) = 0$ and $n_4(P^{\circ n}) = 1$ whereas $ti(P^{\circ n}) = n$.

Lukas Geyer proved optimality of Brjuno's condition for polynomials satisfying $n_4 = ti_4$ (*i.e.*, polynomials such that the number of infinite critical orbit tails in the Julia set is equal to the number of irrationally indifferent cycles; they are called *saturated* polynomials), by using the same method as Yoccoz. It is therefore natural that our new observation adapts in a similar yet slightly less general setting.

Definition 1.12. — *The critical orbits of a polynomial* $P : \mathbb{C} \to \mathbb{C}$ *are the sets* $\left\{P^{\circ k}(c)\right\}_{k \geqslant 0}$ *where* $c$ *is a critical point of* $P$. *A point* $z$ *in a critical orbit is said to be free*[2] *if for all critical point* $c'$, $\forall k \in \mathbb{N}$, $\forall \ell \in \mathbb{N}^*$,

$$\left(P^{\circ k}(c') = P^{\circ \ell}(z)\right) \quad \Longrightarrow \left(k \geqslant \ell \text{ and } P^{\circ(k-\ell)}(c') = z\right).$$

*We denote by* $Z_P$ *the set of non-free points of critical orbits.*

This strange definition has the following interest: $Z_P$ is the smallest subset of the union of all critical orbits of $P$, on which for all map $g : \mathbb{C} \to \mathbb{C}$ such that $g\big|_{Z_P} = P\big|_{Z_P}$, for all critical points $c$, $c'$ of $P$ and all integers $k \geqslant 0$, $k' \geqslant 0$, if $P^{\circ k}(c) = P^{\circ k'}(c')$ then $g^{\circ k}(c) = g^{\circ k'}(c')$.

When a critical point $c$ has a finite forward orbit (*i.e.*, when it is periodic or preperiodic), then $c$ and its iterates are non-free points. When it has an infinite forward orbit, then there can be only finitely many non-free points in its orbit: let $P^{\circ p}(c)$, $p \geqslant 0$, be the last point in the orbit of $c$ where a critical orbit joins the orbit of $c$. The non-free points in the orbit of $c$

(2) This is not a standard terminology.

are exactly the $P^{\circ n}(c)$ for $0 \leqslant n < p$. In particular, if no other critical point shares the same tail then every point in the orbit of $c$ is free. Another consequence is that $Z_P$ is finite.

Let $I_P$ denote the set of all indifferent periodic points of $P$.

DEFINITION 1.13. — *A polynomial $P$ with an indifferent fixed point at the origin has "property (G) with bound $N$ and margin $\varepsilon$" if in addition to having property (G), the cardinal of $I_P \cup Z_P$ is at most $N$* [(3)] *and if $\forall z \in I_P \cup Z_P$, either $z = 0$ or $|z| \geqslant \varepsilon$.*

THEOREM 1.14. — *Let $N \in \mathbb{N}$, $\varepsilon > 0$ and $\mathcal{C}$ be a compact set of degree $d$ polynomials $h$ fixing $0$ with indifferent multiplier $\mathrm{e}^{2\pi i\theta(h)}$, having property (G) with bound $N$ and margin $\varepsilon$. Let $\mathcal{C}_{\mathbb{R}\setminus\mathbb{Q}} = \{h \in \mathcal{C} \mid \theta(h) \in \mathbb{R} \setminus \mathbb{Q}\}$. Then $\exists C \in \mathbb{R}$ such that*

$$\forall h \in \mathcal{C}_{\mathbb{R}\setminus\mathbb{Q}}, \quad \log \operatorname{rad} \Delta(h) \leqslant \log \operatorname{rad} \Delta(Q_{\theta(h)}) + C.$$

The following lemma shows that it is sufficient to check the margin on only $Z_P$.

LEMMA 1.15. — *Any compact set $\mathcal{C}$ of degree $d$ polynomials $h$ fixing $0$ with indifferent multiplier $\mathrm{e}^{2\pi i\theta(h)}$, having property (G) and such that $\forall P \in \mathcal{C}$, $|I_P| \leqslant N$, must have a margin for $I_P$ (i.e., there must exist $\varepsilon > 0$ such that $\forall P \in \mathcal{C}$ and $\forall z \in I_P$, either $z = 0$ or $|z| \geqslant \varepsilon$).*

*Proof.* — Assume that this is not the case: then since $\mathcal{C}$ is compact, there would be a map $P \in \mathcal{C}$, a sequence $P_n \in \mathcal{C}$ and a sequence $u_n \in I_P \setminus \{0\}$ such that $u_n \longrightarrow 0$ and $P_n \longrightarrow P$. The point $u_n$ would belong to an indifferent cycle of $P_n$ with period bounded by $|I_P| \leqslant N$. So an iterate of $P$ would have multiple fixed point at 0, *i.e.,* $P$ would have a parabolic fixed point at 0, and $P_n$ would have an indifferent fixed point at the origin, and an indifferent cycle close to 0. Then either $P$ would have at least two cycles of petals at 0, or the parabolic fixed point at 0 would be virtually indifferent (see [2] for a definition). In both cases, the basin of attraction of 0 would contain at least two critical points (see [5] for a proof), contradicting the fact that $P$ has property (G). □

*Remark 1.16.* — We did not try to get the most general result possible. For instance, it is possible that the hypothesis that 0 has period 1 is not required.[(4)]

[(3)] In a family of polynomials with bounded degrees, it is equivalent to bound the cardinal of $Z_P$ and to bound the sum of local degrees at points in $Z_P$.

[(4)] But in this case, the margin on $I_P$ is not automatic anymore: the indifferent cycle containing 0 may collapse on itself.

Corollary 1.17. — *Under the assumptions of Theorem 1.14,* $\exists C \in \mathbb{R}$ *such that*

$$\forall h \in \mathcal{C}_{\mathbb{R}\setminus\mathbb{Q}}, \quad -Y(\theta(h)) - C \leqslant \log \operatorname{rad} \Delta(h) \leqslant -Y(\theta(h)) + C.$$

The lower bound follows from Yoccoz's Theorem B and the compactness of $\mathcal{C}$ which implies that there is a ball $D(0,r)$ on which all maps in $\mathcal{C}$ are univalent.

Corollary 1.18. — *Under the same assumptions as in Theorem 1.14, let* $\mathcal{C}_{\mathcal{B}} = \{h \in \mathcal{C} | \theta(h) \in \mathcal{B}\}$*. Then the function*

$$\Upsilon : \mathcal{C}_{\mathcal{B}} \to \mathbb{R} \quad \text{defined by} \quad \Upsilon(h) = \log \operatorname{rad} \Delta(h) + Y\big(\theta(h)\big)$$

*is uniformly bounded.*

Corollary 1.19. — *Let* $d \geqslant 2$ *be an integer, and* $\mathcal{C}$ *be the boundary of the central hyperbolic component of the family of unicritical polynomials* $z^d + c$*, i.e., the set of* $c \in \mathbb{C}$ *for which the polynomial* $z^d + c$ *has an indifferent fixed point (this is the only indifferent cycle by the Fatou-Shishikura inequality). Then the function* $\Upsilon$ *is bounded on* $\mathcal{C}_{\mathcal{B}}$*.*

*Proof.* — Conjugate the maps in $\mathcal{C}$ by a translation to put the fixed point at the origin. We still have a compact family. By the Fatou-Shishikura inequality, the number of indifferent cycles is at most the number of infinite critical orbit tails. Thus $Z_P$ is empty and $I_P$ is a singleton. So we may apply Corollary 1.18. □

Corollary 1.20. — *Let* $d \geqslant 2$ *be an integer, and* $\mathcal{C}$ *be the family*

$$\big\{\mathrm{e}^{2\pi i\theta}\, z(1-z)^{d-1}\big\}_{\theta\in\mathbb{R}}.$$

*Then the function* $\Upsilon$ *is bounded on* $\mathcal{C}_{\mathcal{B}}$*.*

*Proof.* — The critical points are $z = 1/d$ and $z = 1$ (with multiplicity $d-2$). The second critical point is mapped in one step on $z = 0$ which is fixed, thus has finite orbit. Therefore the first one has an infinite orbit because[5] $ti \geqslant n_3 + n_4 \geqslant 1$. Thus the two critical points have disjoint orbits, so the one with an infinite orbit is free. So $Z_P = \{1, 0\}$. Also, there is only one indifferent cycle because $n_3 + n_4 \leqslant ti = 1$. So $I_P = \{0\}$. Thus we may apply Corollary 1.18: the family has property (G) with bound $N = 2$ and margin $\varepsilon = 1$. □

[5] It also follows from Fatou and Mañé's theorems.

Corollary 1.21. — *For the family*

$$f_\theta(z) = \mathrm{e}^{2\pi i\theta}(z + z^d),$$

*the following holds:* $\exists C > 0$ *such that* $\forall\theta \in \mathbb{R} \setminus \mathbb{Q}$,

$$-\frac{Y\big((d-1)\theta\big)}{d-1} - C \leqslant \log\operatorname{rad}\Delta(f_\theta) \leqslant -\frac{Y\big((d-1)\theta\big)}{d-1} + C.$$

*Proof.* — The family $f_\theta$ is semi-conjugated to the previous family: more precisely let $\phi(z) = -z^{d-1}$, and $g_\theta(z) = \mathrm{e}^{2\pi i\theta}\, z(1-z)^{d-1}$. Then(6)

$$\phi \circ f_\theta = g_{(d-1)\theta} \circ \phi.$$

The Siegel disk of $f_\theta$ is thus the preimage by $z \mapsto z^{d-1}$ of the Siegel disk of $g_{(d-1)\theta}$. The claim follows.(7) □

In Section 4, Lemma 4.1, we will prove that for any integer $m \geqslant 2$, the function

$$\theta \in \mathcal{B} \mapsto Y(\theta) - \frac{Y(m\theta)}{m}$$

is unbounded on any interval. It follows that the conclusions in Corollary 1.18 do not hold for this family:

Corollary 1.22. — *For the family*

$$f_\theta(z) = \mathrm{e}^{2\pi i\theta}(z + z^d),$$

*the function*

$$\Upsilon : \theta \in \mathcal{B} \mapsto \log\operatorname{rad}\Delta(f_\theta) + Y(\theta)$$

*is unbounded on any interval.*

In fact, the family $f_\theta$ does not satisfy property (G), which is an hypothesis of Corollary 1.18. Indeed, there is only one indifferent cycle: the origin, whereas there are $n-1$ infinite critical orbit tails. This can be seen either by using the semi-conjugacy or the fact that $f_\theta$ commutes with the map $z \mapsto \mathrm{e}^{2\pi i/(d-1)}\, z$.

In Section 4, Lemma 4.2, we will prove that

$$(\exists C > 0)\ (\forall m \in \mathbb{N}^*)\ (\forall \theta \in \mathbb{R}) \quad Y(\theta) \leqslant Y(m\theta) + C \log m.$$

It follows that

$$\log\operatorname{rad}\Delta(f_\theta) \leqslant -\frac{Y(\theta)}{d-1} + C'.$$

This suggests the following conjecture.

(6) Note how the rotation number changed.

(7) If $U$ is a connected simply connected open subset of $\mathbb{C}$ containing the origin and $U'$ is the preimage of $U$ by $z \mapsto z^k$ then $\operatorname{rad} U' = \sqrt[k]{\operatorname{rad} U}$.

Conjecture 1.23. — *There exists a constant* $C = C(d) \in \mathbb{R}$ *such that for all polynomial* $P$ *of degree* $d$ *with an indifferent fixed point at the origin,*

$$\log \operatorname{rad} \Delta(P) \leqslant -\frac{Y(\theta)}{d-1} + \log \min |c_i| + C$$

*where the* $c_i$ *are the critical points of* $P$ *and* $\theta$ *is the rotation number at the origin.*

There are possible refinements according to how many recurrent critical points are associated to the indifferent fixed point.

## 2. Optimality of the quadratic polynomial

In this section, we prove Theorem 1.4. Our proof follows closely Yoccoz's proof of Theorem A. It uses the following notion.

### 2.1. The radius of convergence of the linearizing power series

This section defines the notion, relates it to the conformal radius of the Siegel disk and collects a few known results.

Definition 2.1. — *Assume* $\theta \in \mathbb{R} \setminus \mathbb{Q}$ *and* $f$ *is a holomorphic map defined in a neighborhood of the origin and such that*

$$f(z) = \mathrm{e}^{2\pi i\theta}\, z + \mathcal{O}(z^2).$$

*Then, there is a unique formal series*

$$\phi_f(Z) = Z + \sum_{n=2}^{+\infty} \widehat{\phi}_n Z^n$$

*in* $\mathbb{C}[[Z]]$ *such that*

$$\phi_f(\mathrm{e}^{2\pi i\theta}\, Z) = f \circ \phi_f(Z),$$

*called the linearizing series. We let* $R(f) \in [0, +\infty]$ *be its radius of convergence.*

*Remark 2.2.* — For a fixed $\theta$, the coefficient $\widehat{\phi}_n$ is a polynomial in the coefficients $\widehat{f}_2, \ldots, \widehat{f}_n$ of the power series expansion of $f$ at the origin: $f(z) = \mathrm{e}^{2\pi i\theta}\, z + \widehat{f}_2 z^2 + \widehat{f}_3 z^3 + \ldots$

Let $U$ be an open subset of $\mathbb{C}$ containing the origin and $f : U \to \mathbb{C}$ be a holomorphic map such that $f(z) = \mathrm{e}^{2\pi i\theta}\, z + \mathcal{O}(z^2)$, with $\theta$ an irrational real number. The Siegel disk $\Delta$ of $f$ is the maximal domain $\subset U$ on which $f$ is linearizable. There is a Siegel disk if and only if $R(f) > 0$. Note that $R(f)$ depends only on the germ[8] of $f$, whereas $\Delta$ depends on the representative of the germ. Let rad be the conformal radius of $\Delta$ with respect to 0, *i.e.*, the unique $r \in (0, +\infty]$ such that there exists a conformal bijection $\psi : D(0, r) \to \Delta$ mapping 0 to 0 and such that $\psi'(0) = 1$. Conjugating $f$ by $\psi$ gives a conformal self map of $D(0, r)$ fixing the origin with multiplier $\mathrm{e}^{2\pi i\theta}$. So it is the rotation:

$$\psi^{-1} \circ f \circ \psi(z) = \mathrm{e}^{2\pi i\theta}\, z.$$

By uniqueness of the linearizing formal power series:

- $R(f) \geqslant \mathrm{rad}$,
- $\forall z \in D(0, \mathrm{rad})$, $\psi(z) = \phi(z)$,
- $\Delta = \phi(D(0, \mathrm{rad})) \subset \phi(D(0, R(f)))$.

Let inner denote $\sup\{r > 0 | D(0, r) \subset \Delta\}$. By Schwarz's lemma, $\mathrm{rad} \geqslant \mathrm{inner}$. By Koebe's one quarter theorem, $\mathrm{inner} \geqslant \frac{1}{4}\,\mathrm{rad}$.

The radii $R(f)$ and rad are not necessarily equal. An obvious possibility would be that $f$ has an extension to a bigger domain, and that this extension has a bigger linearization domain. But this is not the only thing that can happen, since $\phi_f$ is not necessarily injective on its disk of convergence. In fact $\phi_f$ can be *any convergent power series* of the form $z + \mathcal{O}(z^2)$. Indeed, for such a $\phi$, we can set $f(z) = \phi\big(\mathrm{e}^{2\pi i\theta}\, \phi^{-1}(z)\big)$ near 0... For instance, $\phi(z) = \mathrm{e}^z - 1 = z + \cdots$ has infinite radius of convergence, and is not injective on $\mathbb{C}$. The map $\phi$ can also have critical points.

Let us recall the following fact.

Lemma 2.3. — *If for some $r \leqslant R(f)$, $\phi(D(0, r))$ is contained in the domain of definition $U$ of $f$ then $r \leqslant \mathrm{rad}$ and $\phi(D(0, r)) \subset \Delta$. Thus*

$$\mathrm{rad} = \sup\{r \leqslant R(f) | \phi(D(0, r)) \subset U\}.$$

*Proof.* — First note that the relation $f \circ \phi(z) = \phi(\mathrm{e}^{2\pi i\theta}\, z)$ holds by analytic continuation for all $z \in D(0, r)$. To prove the lemma, it is thus enough to prove that $\phi$ is injective on $D(0, r)$, for then $\phi(D(0, r))$ is a simply connected subset of $U$, it is invariant by $f$, and uniformizing it to a disk conjugates $f$ to an automorphism that fixes the origin so is a rotation.

(8) An analytic germ is an equivalence class of analytic maps defined in a neighborhood of 0, two maps being equivalent if and only if they coincide in some possibly smaller neighborhood of 0. Analytic germs are completely characterized by the power series expansion at the origin.

If injectivity did not hold, then there would be $a, b \in D(0,r)$ such that $a \neq b$ and $\phi(a) = \phi(b)$. Without loss of generality, let us assume that $|a| \leqslant |b|$ (so $b \neq 0$). By iterating $f$ we would get $\phi(\rho^n a) = \phi(\rho^n b)$ where $\rho = \mathrm{e}^{2\pi i\theta}$. Therefore the relation $\phi(z) = \phi((a/b)z)$ would hold on a set with a point of accumulation in $D(0,r)$. It therefore would hold everywhere on $D(0,r)$ (both hands of the equality are defined when $z \in D(0,r)$ because $|a| \leqslant |b|$). This would contradict the injectivity of $\phi$ near 0. □

Proposition 2.4. — *Let $U$ be the domain of definition of $f$. If $R(f) >$ rad then $\Delta$ is bounded and its boundary in $\mathbb{C}$ contains at least one point of $\partial U$.*

*As a corollary:*

- *if $\Delta \Subset U$ (this includes polynomials and polynomial-like maps[9]),*
- *or $f$ is entire (this includes polynomials too),*

*then*

$$R(f) = \mathrm{rad}\,.$$

*Proof.* — Since $\psi(z) = \phi(z)$ for all $z \in D(0, \mathrm{rad})$, the set $\Delta$ is bounded, because it is equal to $\psi(D(0, \mathrm{rad}))$ that is contained in the compact set $\phi(\overline{D}(0, \mathrm{rad}))$. Now if the boundary of $\Delta$ did not contain a point of $\partial U$ then $\Delta$ would be compactly contained in $U$. Then there would exist some $r >$ rad such that $\phi(D(0,r)) \subset U$. By Lemma 2.3, $r \leqslant$ rad: contradiction. □

*Remark 2.5.* — If $R(f) >$ rad, then $\partial\Delta$ is the image of a euclidean circle by a holomorphic map $\phi$. So it is smooth but at finitely many points, and it is locally connected. The map $\phi$ may however be non-injective on the circle and may also have critical points of order 2 (think of a rotation inside a cardioid).

Let us state the following extension of a lemma of Yoccoz [18].

Lemma 2.6. — *Let $U$ be the domain of definition of $f$. If there exist $z_0 \in \partial U$, $\varepsilon > 0$, a holomorphic map $g : B(z_0, \varepsilon) \to \mathbb{C}$ and a path in $U \cap B(z_0, \varepsilon)$ ending on $z_0$ such that $g = f$ on the path, we say that $f$ has a path extension (note that the point $z_0$ must be accessible from $U$). If $f$ has no path extension then*

$$R(f) = \mathrm{rad}\,.$$

(9) The definition of polynomial-like maps is recalled in Definition 2.12. Note that polynomials have polynomial-like restrictions but are not polynomial-like by themselves, because $\mathbb{C}$ is not a compact subset of $\mathbb{C}$.

*Proof.* — Assume that $R(f) > \mathrm{rad}$. We have seen that, then, $\exists z_0 \in \partial\Delta \cap \partial U$ and that $\partial\Delta = \phi(\partial D(0, |\,\mathrm{rad}\,|))$. Thus $\exists a \in \mathbb{C}$ such that $|a| = \mathrm{rad}$ and $z_0 = \phi(a)$. If $a$ is not a critical point of $\phi$ then $\phi$ is invertible near $a$ and it is possible to find a path extension of $f$ at $z_0$ using $\phi$: $g(z) = \phi(\mathrm{e}^{2\pi i\theta}\,\phi^{-1}(z))$, the path being for instance the internal ray of $\Delta$: $\phi(ta)$ for $t < 1$ and close enough to 1. If $a$ is a critical point of $\phi$, then the function $g$ is multivalued with only $z_0$ as a singularity, so $z_0$ cannot be an isolated point of $\partial U$ for otherwise $g$ would have to coincide with $f$ in a neighborhood of $z_0$, so $g$ would be single-valued; then there exists an accessible $z_1 \in \partial U$ near $z_0$, at which $f$ has a path extension. □

Lemma 2.7. — *Given a non empty connected open subset $U$ of $\mathbb{C}$, the set of holomorphic functions $f : U \to \mathbb{C}$ that have no path extension is dense for the notion of uniform convergence on compact subsets of $U$.*

*Proof.* — If $\partial U = \varnothing$ there is nothing to prove. Otherwise let $f_0 : U \to \mathbb{C}$ be holomorphic. Let $K$ be a compact subset of $U$ and $\varepsilon > 0$. Let $u_n \in U$ be a dense sequence. For each $u_n \in U$ choose some $v_n \in \partial U$ such that $|v_n - u_n| = \mathrm{dist}(u_n, \partial U)$. There exists a sequence $\varepsilon_n$ such that

$$h(z) := \sum_n \frac{\varepsilon_n}{z - v_n}$$

is normally convergent on every compact subset of $U$ and such that $|h(z)| < \varepsilon$ on $K$ and such that for all $n$, $f = f_0 + h$ is unbounded on the segment $[u_n, v_n]$. If such a function $f$ had a path extension $g$, the same $g$ would yield by analytic continuation of equalities a path extension of $f$ along $[u_n, v_n]$ for some $n$. Contradiction. □

Lemma 2.8. — *If $f$ and $f_n$ are holomorphic functions defined on a common open subset $U$ of $\mathbb{C}$ containing $0$, with $f(z) = \mathrm{e}^{2\pi i\theta}\,z + \mathcal{O}(z^2)$ and $f_n(z) = \mathrm{e}^{2\pi i\theta_n}\,z + \mathcal{O}(z^2)$, if moreover $f_n$ has a Siegel disk for all $n$ and $f_n$ tends to $f$ on every compact subset of $U$, then*

$$\mathrm{rad}\,\Delta(f) \geqslant \limsup_{n\to+\infty} \mathrm{rad}\,\Delta(f_n).$$

*Proof.* — Let $r = \limsup_{n\to+\infty} \mathrm{rad}\,\Delta(f_n)$. If $r = 0$ there is nothing to prove. Otherwise let $\theta_n$ and $\psi_n$ be to $f_n$ what $\theta$ and $\psi$ are to $f$. By compactness of the set of Schlicht functions, extracting a subsequence one can assume that $\psi_n$ converges on every compact subset of $D(0, r)$ to some injective map $\zeta$ with $\zeta(0) = 0$ and $\zeta'(0) = 1$. One can assume that $\theta_n$ converges to some $\theta'$. Then the relation $f_n \circ \psi_n(z) = \psi_n(\mathrm{e}^{2\pi i\theta_n}\,z)$ passes to the limit: $f \circ \zeta(z) = \zeta(\mathrm{e}^{2\pi i\theta'}\,z)$ for all $z \in B(0, r)$. By computing the derivative at the origin, one gets $\theta' = \theta \mod 2\pi$. Therefore $\zeta(B(0, r))$

is a linearization domain, thus contained in $\Delta(f)$. By Schwarz's lemma, $r \leqslant \operatorname{rad}\Delta(f)$. $\square$

The following result can be found in [18], page 20.

Corollary 2.9 (Yoccoz).

$$\inf_{f\in\mathcal{S}_\theta} R(f) = \inf_{f\in\mathcal{S}_\theta} \operatorname{rad}\Delta(f).$$

*Proof.* — The following argument is very similar that of [18]. First, from $R(f) \geqslant \operatorname{rad}\Delta(f)$ it follows at once that $\inf_{f\in\mathcal{S}_\theta} R(f) \geqslant \inf_{f\in\mathcal{S}_\theta} \operatorname{rad}\Delta(f)$. To prove the reverse inequality, fix a function $h : \mathbb{D} \to \mathbb{C}$ having no path extension, provided for instance by Lemma 2.7. Given some $f \in \mathcal{S}_\theta$, consider the sequence of maps

$$f_n : \mathbb{D} \to \mathbb{C},\ z \mapsto \frac{1}{n} z^2 h(z) + \frac{f\big((1-\frac{1}{n})z\big)}{1-\frac{1}{n}}$$

defined on $\mathbb{D}$: they fix the origin and have multiplier $\mathrm{e}^{2\pi i\theta}$. They have no path extension. They also tend to $f$ uniformly on compact subsets of $\mathbb{D}$. By Lemma 2.8,

$$\limsup \operatorname{rad}\Delta(f_n) \leqslant \operatorname{rad}\Delta(f).$$

By Lemma 2.6,

$$R(f_n) = \operatorname{rad}\Delta(f_n).$$

There exists a sequence $\varepsilon_n \longrightarrow 0$ such that $f_n$ is injective on $D(0, 1-\varepsilon_n)$. Therefore the restriction $u_n$ to $\mathbb{D}$ of $z \mapsto u_n((1-\varepsilon_n)z)/(1-\varepsilon_n)$ belongs to $\mathcal{S}_\theta$. So $R(u_n) \geqslant \inf_{u\in\mathcal{S}_\theta} R(u)$. Also, $R(f_n) = (1-\varepsilon_n)R(u_n)$. Putting it all together we get:

$$\operatorname{rad}\Delta(f) \geqslant \inf_{u\in\mathcal{S}_\theta} R(u).$$

Since this holds for all $f \in \mathcal{S}_\theta$, this proves the corollary. $\square$

The following rule is elementary: if $g(z) = \lambda f(z/\lambda)$ holds in a neighborhood of 0 then

$$\phi_g(X) = \lambda\phi_f(X/\lambda).$$

In particular, $R(g) = |\lambda| R(f)$.

### 2.2. Holomorphic motions

The following notion is defined in [11].

Definition 2.10. — *A holomorphic motion of a subset $X$ of the Riemann sphere $\mathbb{S}$ with parameter in a complex manifold $\Lambda$ and base parameter $\lambda_0 \in \Lambda$ is an analytic map $h : \Lambda \times X \to \mathbb{S}$ such that*

- $\forall z \in X$, $h(\lambda_0, z) = z$,
- $\forall z \in X$, *the function* $\lambda \mapsto h(\lambda, z)$ *is holomorphic,*
- $\forall \lambda \in \Lambda$, *the function* $z \mapsto h(\lambda, z)$ *is injective.*

Then it can be proved, see [11], that $h$ is continuous and in fact $\forall \lambda \in \Lambda$, the map $z \mapsto h(\lambda, z)$ extends to a (non unique) quasiconformal homeomorphism of the Riemann sphere. The $\lambda$-lemma of Mañé, Sad and Sullivan in [11] states that a holomorphic motion of $X$ has a unique extension to a holomorphic motion of the closure of $X$ in $\mathbb{S}$.

A family of sets $X_\lambda$ is said to *undergo a holomorphic motion* if there exists a holomorphic motion such that $\forall \lambda$, $h(\lambda, X_{\lambda_0}) = X_\lambda$. The base parameter can then easily be forced to be any element of $\Lambda$.

Let $B$ be a complex manifold. In the present article, an *analytic family* parameterized by $B$ will refer to a family of maps $g_b$, with $b \in B$, such that the map $(b, z) \mapsto g_b(z)$ is defined on an open subset of $B \times \mathbb{C}$, is analytic, and takes values in $\mathbb{C}$.

An analytic family $g_b$ is said to *undergo a parabolic bifurcation* if there exists $b_0$ and $n \geqslant 1$ such that $z \mapsto g_b^{\circ n}(z) - z$ has a root that splits when $b$ varies away from $b_0$. Such a root is necessarily multiple and thus is a parabolic point of $g_{b_0}$. But a parabolic point does not necessarily bifurcate.

The following lemma was formulated by Mañé, Sad and Sullivan in [11] for rational maps, but it is known to work alike for polynomial-like maps.

Lemma 2.11. — *Let $B$ be a simply connected complex manifold. Assume that $(g_b)_{b \in B}$ is an analytic family of polynomial-like maps. Assume that this family does not undergo a parabolic bifurcation. Then the Julia set of $g_b$ undergoes a holomorphic motion.*

*Proof.* — The Julia set is the closure of the set of repelling periodic points. Because the maps $g_b$ are polynomial-like of the same degree, the projection $(b, z) \mapsto b$ is proper from the set $\mathcal{P}_n = \{(b, z) | g_b^{\circ n}(z) = z\}$ to $B$. The assumption that there is no parabolic bifurcation implies that roots of $g_b^{\circ n}(z) - z$ can be locally followed as a continuous functions of $b$. Properness implies that they can be followed as global functions over the universal cover of $B$. Since $B$ is simply connected, it is its own universal cover. These functions are in fact holomorphic as solutions of one dimensional analytic equations. So $\mathcal{P}_n$ consists in the disjoint union of finitely many graphs of holomorphic functions from $B$ to $\mathbb{C}$. Let $\mathcal{C}$ be one of these graphs, let

$(b,z) \in \mathcal{C}$ and $\mu = (g_b^n)'(z)$ be the corresponding multiplier. If $\mu = 1$, then the multiplier must be constant over $\mathcal{C}$ itself, for otherwise there would be a parabolic bifurcation. If $\mu$ is a $k$-th root of unity, it is also the case, since $\mathcal{C}$ is also one of the graphs composing $\mathcal{P}_{kn}$. If $\mu$ is irrationnally indifferent is it also the case, for otherwise a nearby parameter would be a root of unity. From all this, it follows that if $|\mu| > 1$ then the cycle is repelling on all of $\mathcal{C}$. The repelling points can therefore be followed holomorphically on $B$ an stay repelling. Moreover two different repelling periodic point cannot collide as $b$ varies in $B$, otherwise the iterate of $g_b$ of order the GCD of their periods would have a multiple fixed point that splits. So the set of repelling periodic points undergoes a holomorphic motion. By the $\lambda$-lemma of [11], its closure also undergoes a holomorphic motion. □

## 2.3. Proof of the main theorem

We will first compare the radius of convergence of the linearizing series of a univalent function $f$ on $\mathbb{D}$ with the corresponding radius of $Q_\theta$ (which is univalent on $D(0,1/2)$).

Let us assume that $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $R(Q_\theta) > 0$, since otherwise, there is nothing to prove. Consider a univalent function $f : \mathbb{D} \hookrightarrow \mathbb{C}$ fixing 0 with multiplier $\mathrm{e}^{2\pi i\theta}$. Following Ilyashenko and Yoccoz, consider the one-parameter families of maps

$$\big\{f_a : D(0,1) \to \mathbb{C}\big\}_{a\in\mathbb{C}} \quad \text{and} \quad \big\{\widetilde{g}_b : D\big(0,1/|b|\big) \to \mathbb{C}\big\}_{b\in\mathbb{C}}$$

defined by:

$$f_a(z) = f(z) + az^2 \quad \text{and} \quad \widetilde{g}_b(w) = \frac{1}{b} f_{1/b}(bw) = \frac{1}{b} f(bw) + w^2.$$

The family $\widetilde{g}_b$ extends analytically at $b = 0$ by $\widetilde{g}_0 = Q_\theta$. We have:

$$(\forall b \in \mathbb{C}^*) \quad R(\widetilde{g}_b) = \frac{1}{|b|} R(f_{1/b}). \tag{2.1}$$

The following notion is defined in [7].

Definition 2.12. — *A polynomial-like map is a proper holomorphic map (and thus a ramified covering) $f : U \to V$ of degree at least 2, between two simply connected domains $U \Subset V$ of $\mathbb{C}$. When the degree is 2, it is also called quadratic-like.*

The following observation is essentially due to Yoccoz [18].

Lemma 2.13. — *If $|b| < 2/19$, the map $\widetilde{g}_b$ has a quadratic-like restriction $g_b : U_b \to V$ with*

$$U_b = \{z \in D(0,4) | \widetilde{g}_b(z) \in D(0, 492/121)\} \quad \textit{and} \quad V = D(0, 492/121).$$

*Proof.* — Let $b_1 = 2/19$, $w_1 = 4$ and $\zeta_1 = 492/121$. These number satisfy:[10]

$$b_1 w_1 < 1 \tag{2.2}$$

$$\frac{w_1}{(1 - b_1 w_1)^2} \leqslant w_1^2 - \zeta_1 \tag{2.3}$$

$$w_1 < \zeta_1. \tag{2.4}$$

Since $f$ is univalent, we have, for all $z \in \mathbb{D}$:

$$\big|f(z)\big| \leqslant \frac{|z|}{\big(1 - |z|\big)^2}$$

(the Koebe function is extremal with this respect). It follows that when $|b| < b_1$, $|w| = w_1$ and $|\zeta| < \zeta_1$, then $|bw| < 1$ and

$$\big|(\widetilde{g}_b(w) - \zeta) - (w^2 - \zeta)\big| = \left|\frac{1}{b} f(bw)\right| \leqslant \frac{w_1}{(1 - b_1 w_1)^2} \leqslant w_1^2 - \zeta_1 < |w^2 - \zeta|.$$

Thus by Rouché's theorem, every $\zeta \in D(0, \zeta_1) = V$ has exactly two preimages by $\widetilde{g}_b$ in $D(0, w_1)$, counted with multiplicity. Therefore $g_b$ is proper holomorphic of degree 2.[11] If $U_b$ were not connected, the component of $U_b$ containing 0 would be mapped biholomorphically to $V$, which, by Schwarz's lemma, is not possible since $\big|\widetilde{g}_b'(0)\big| = 1$. Last, $U_b$ is compactly contained in $V$ because $w_1 < \zeta_1$. □

We saw in Section 2.1 that $R(f)$ is equal to the conformal radius of the Siegel disk when $f$ is polynomial-like.

---

(10) The constants are not optimal, but this is not really important: for the purpose of this article, we need only the existence of positive solutions to the three equations. We chose $b_1$ very close to its maximal possible value. For $b_1$ close to 0, it is very easy to find solutions.

(11) A holomorphic map $f : U \to V$ between open sets, such that every point has $d$ preimages counted with multiplicity, is proper (and has degree $d$ by definition). Indeed, let $K$ be a compact subset of $V$ and $L = f^{-1}(K)$. Let $u_n \in L$ be a sequence: $f(u_n) \in K$. Passing to a subsequence we can assume that $f(u_n)$ converges to some $v \in V$. The point $v$ has $d$ preimages in $U$ counted with multiplicity. For all such preimage $u$ (call $m$ its multiplicity), let $U' \subset U$ be a compact neighborhood of $u$. Then by Rouché, there exists a neighborhood $V'$ of $v$ such that for all $v' \in V' \setminus \{v\}$, $U'$ contains at least $m$ preimages of $v'$ counted with multiplicity. Thus near the preimages of $v$, one still finds all preimages of points near enough to $v$. Therefore, there must be a subsequence of $u_n$ converging in $U$. Its limit $u$ satisfies $f(u) = v$ thus belongs to $L$. Thus $L$ is compact. Thus $f$ is proper.

Proposition 2.14. — *Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let $g_b : U_b \to \mathbb{C}$ be an analytic family of maps*[12] *of the form $g_b(z) = \mathrm{e}^{2\pi i\theta}\, z + \mathcal{O}(z^2)$ (so the rotation number is independent of $b$). Assume that they all have a Siegel disk $\Delta(g_b) \subset U_b$, and that its boundary in $\mathbb{C}$ is not empty and undergoes a holomorphic motion (we do not require that the holomorphic motion commutes with the dynamics, nor even that $\Delta(g_b) \Subset U_b$). Let $\operatorname{rad}\Delta(g_b)$ be the conformal radius of $\Delta(g_b)$ with respect to $0$. Then the function $b \mapsto -\log\operatorname{rad}\Delta(g_b)$ is harmonic.*

*Proof.* — The following proof of this lemma of Sullivan was communicated to us by Saeed Zakeri. First, note that the conformal radius varies continuously. Indeed, the holomorphic motion implies that the function $b \mapsto \partial\Delta(g_b)$ is continuous for the Hausdorff topology on compact subsets of $\mathbb{C}$. It implies the sequential continuity of $b \mapsto (\Delta(g_b), 0)$ for the notion of Carathéodory convergence. The latter implies the continuity of the conformal radius. Then, consider an extension[13] of the holomorphic motion to a holomorphic motion of all the plane, but which does not necessarily commute with the dynamics. Let $b_0$ be any parameter and $w_n$ be any sequence in the Siegel disk of parameter $b_0$, converging to a point in the boundary of $\Delta(g_{b_0})$. For $b$ close to $b_0$ let $w_n(b)$ be the point to which $w_n$ is transported by the motion. Now look at

$$u_n(b) = \psi_{g_b}^{-1}\big(w_n(b)\big)$$

where $\psi_{g_b}$ is the unique conformal map from $D(0, \operatorname{rad}\Delta(g_b)) \to \Delta(g_b)$ that maps $0$ to $0$ and has derivative $1$ there. For each $b$, the sequence $\big(w_n(b)\big)$ converges to a point in the boundary of the Siegel disk $\Delta(g_b)$. Thus, $\big|u_n(b)\big| \leqslant \operatorname{rad}\Delta(g_b)$ converges to $\operatorname{rad}\Delta(g_b)$. Recall that $R(g_b) \geqslant \operatorname{rad}\Delta(g_b)$ and that the map $\psi_{g_b}$ and $\phi_{g_b}$ must coincide on $D(0, \operatorname{rad}\Delta(g_b))$ (see Section 2.1). As a corollary, the map $(b, w) \mapsto \psi_{g_b}(w)$ is analytic: indeed, the coefficients $\widehat{\phi}_n(g_b)$ are polynomials of the coefficients of $g_b$, thus vary analytically with $b$. Thus the map

$$(b, w) \mapsto \big(b, \psi_{g_b}(w)\big)$$

is bi-analytic (its inverse is $(b, w) \mapsto \big(b, \psi_{g_b}^{-1}(w)\big)$), so $b \mapsto u_n(b)$ is analytic. Therefore, $b \mapsto \log|u_n(b)|$ is harmonic (it does not vanish). Now, the map $b \mapsto \log\operatorname{rad}\Delta(g_b)$ is the pointwise limit of this sequence of harmonic functions. This sequence is bounded from above by the continuous function $b \mapsto \log\operatorname{rad}\Delta(g_b)$, therefore there is, locally, a uniform upper bound.

(12) that we do not assume polynomial-like

(13) Slodkowsky's theorem [16] provides one, but we can also use the Bers-Royden [1] or the Sullivan-Thurston [17] version since this argument is local in terms of the parameter.

We can thus apply Harnack's theorem: the limit is locally uniform and harmonic. □

Let us go back to the specific family $g_b$ we were studying.

PROPOSITION 2.15. — *The map* $b \mapsto \log \operatorname{rad} \Delta(g_b)$ *is well defined and harmonic in* $D(0, 2/19)$.

*Proof.* — Let $r_0 = 2/19$. By Lemma 2.13, $g_b : U_b \to V$ is quadratic-like for all $b \in \overline{D}(0, r_0)$. The radius of convergence of $\phi_{g_b}$ coincides with the conformal radius of the Siegel disk $\Delta(g_b)$ by Proposition 2.4. The maps $g_b$ all have an indifferent fixed point. This is the only non repelling cycle of the quadratic-like map $g_b$ (there can be at most one, see [7]). So there is never a parabolic periodic point. So by Lemma 2.11, the Julia set of $g_b$ undergoes a holomorphic motion as $b$ varies in $D(0, r_0)$. The Siegel disk of $g_b$ is the connected component containing 0 of the complement of its Julia set. Thus the boundary of the Siegel disk undergoes a holomorphic motion too. So by Proposition 2.14, the map $b \mapsto \log \operatorname{rad}(\Delta(g_b))$ is harmonic. □

DEFINITION 2.16. — *Let* $\underset{|z|=r}{\operatorname{avg}}\, m(z)$ *denote the average of the function* $m(z)$ *on the circle* $|z| = r$ *(with respect to the Lebesgue measure on the circle).*

As an immediate consequence of Proposition 2.15, we have the following equality:

$$\log \operatorname{rad} \Delta(Q_\theta) = \underset{|b|=1/10}{\operatorname{avg}} \log \operatorname{rad} \Delta(g_b).$$

And since for $|b| = 1/10$ the map $g_b$ is polynomial-like,

$$\operatorname{rad} \Delta(g_b) = R(g_b).$$

By equation (2.1):

$$|b| = 1/10 \implies R(g_b) = 10R(f_{1/b}).$$

Thus

$$\log \operatorname{rad} \Delta(Q_\theta) = \log 10 + \underset{|a|=10}{\operatorname{avg}} \log R(f_a). \tag{2.5}$$

PROPOSITION 2.17. — *We have* $\log R(f) \geqslant \underset{|a|=10}{\operatorname{avg}} \log R(f_a)$.

*Proof.* — Look at the formal linearizing power series of $f_a$:

$$\phi_{f_a}(Z) = Z + \sum_{n=2}^{+\infty} \widehat{\phi}_n(a) Z^n.$$

By Hadamard's theorem,

$$\frac{1}{R(f_a)} = \limsup_{n\to+\infty} \sqrt[n]{|\widehat{\phi}_n(a)|}.$$

By looking at the formal equation defining $\phi$, one can see that the coefficients $\widehat{\phi}_n(a)$ are polynomials in $a$ (see the lemma p. 59 in [18]). Therefore the maps $a \mapsto \log|\widehat{\phi}_n(a)|$ are subharmonic, so:

$$\frac{1}{n}\log\big|\widehat{\phi}_n(0)\big| \leqslant \operatorname*{avg}_{|a|=10} \frac{1}{n}\log\big|\widehat{\phi}_n(a)\big|.$$

By Lemma 2.13, for $|a| = 10$, the map $f_a$ has a quadratic-like restriction. In that case, the linearizing map $\phi_{f_a}$ takes its values in $D(0, 4/|a|) \subset \mathbb{D}$ and it follows from the Cauchy inequalities that

$$\big|\widehat{\phi}_n(a)\big| \leqslant \frac{1}{\big(R(f_a)\big)^n}.$$

By Proposition 2.15 $b \mapsto R(g_b)$ is, in particular, a continuous non vanishing function on the circle $|b| = 1/10$. Thus, when $|a| = 10$, $R(f_a) = R(g_b)/10$ reaches a minimum $c > 0$ and

$$\frac{1}{n}\log\big|\widehat{\phi}_n(a)\big| \leqslant \log\frac{1}{R(f_a)} \leqslant \log\frac{1}{c}.$$

This uniform upper bound allows us to apply Fatou's lemma:

$$\begin{aligned} -\log R(f) &= \limsup_{n\to+\infty} \frac{1}{n}\log|\widehat{\phi}_n(0)| \\ &\leqslant \operatorname*{avg}_{|a|=10} \limsup_{n\to+\infty} \frac{1}{n}\log|\widehat{\phi}_n(a)| = -\operatorname*{avg}_{|a|=10} \log R(f_a). \end{aligned}$$

□

Equality (2.5) and Proposition 2.17 yield:

$$\log R(f) \geqslant \operatorname*{avg}_{|a|=10} \log R(f_a) = \log \operatorname{rad} \Delta(Q_\theta) - \log 10,$$

whence

$$R(f) \geqslant \frac{1}{10} \operatorname{rad} \Delta(Q_\theta).$$

Therefore, $\inf R(f) \geqslant \frac{1}{10}\operatorname{rad}\Delta(Q_\theta)$, with the infimum taken over $\mathcal{S}_\theta$. By Corollary 2.9, $\inf \operatorname{rad}\Delta(f) = \inf R(f)$. Q.E.D.

## 3. Other families of polynomials

In this section, we shall first prove Theorem 1.14. Assume $N \in \mathbb{N}$ and $\mathcal{C}$ is a compact set of degree $d$ polynomials $P$ fixing 0 with indifferent multiplier $e^{2\pi i\theta(P)}$, having property (G) with bound $N$ and margin $\varepsilon$. Let $Z_P$ be the set of non-free points of critical orbits (see Definition 1.12). Let $I_P$ be the set of indifferent periodic points of $P$: $0 \in I_P$. Let $C_P$ be the set of critical points of $P$. Set

$$G_P(z) = \prod_{w \in Z_P \cup I_P \cup C_P} (z-w)^{1+\deg_P(w)}$$

where $\deg_P(w) =$ the local degree of $P$ at $w$. For $b \in \mathbb{C}$ let

$$\widetilde{g}_{P,b} = P + bG_P.$$

First, by compactness of $\mathcal{C}$, by the bound $N$, and by the definition of $G_P$, we see that the set $\{G_P | P \in \mathcal{C}\} \subset \mathbb{C}[X]$ is bounded in the sense that it has bounded degree and bounded coefficients. Therefore, there exists $r > 0$ and $R > 0$, independent of $P$, such that $|b| < r \implies \widetilde{g}_{P,b}(z)$ has a polynomial-like restriction $g_{P,b}$ of degree $d$ from the component of $\widetilde{g}_{P,b}^{-1}\big(D(0,R)\big)$ contained in $D(0,R)$ to $D(0,R)$.

Lemma 3.1. — *Fix any $P \in \mathcal{C}$. As $b$ varies in $D(0,r)$, the polynomial-like map $g_{P,b}$ cannot undergo a parabolic bifurcation.*

*Proof.* — The particular form of $G$ has the following consequences. First, every critical point of $P$ is a critical point of $g_{P,b}$ with the same local degree. Since $g_{P,b}$ is polynomial-like of degree $d$, it has $d-1$ critical points counted with multiplicity, which is the same as for $P$. Therefore, the critical points of $P$ and $g_{P,b}$ are the same and have the same degree. Recall that $Z_P$ contains the finite critical orbits. Thus for all point $x$ in a finite critical orbit of $P$, for all $b$, $g_{P,b}(x) = P(x)$. In particular, a critical point of $P$ with a finite orbit is a critical point of $g_{P,b}$ with a finite orbit. For all non-free point of a critical orbit of $P$, $g_{P,b}(x) = P(x)$. As a consequence, any two critical point in the same orbit tail of $P$ are two critical points in the same orbit tail of $g_{P,b}$. Therefore the number of infinite critical orbit tails of $g_{P,b}$ is at most that of $P$, and the latter is equal to the number of indifferent cycles of $P$ by the assumption (G). So $g_{P,b}$ can not have more indifferent cycles than $P$. But the particular form of $G$ implies also that all indifferent cycle of $P$ is also a cycle of $g_{P,b}$, with moreover the same multiplier. Therefore, $g_{P,b}$ must have exactly the same indifferent periodic points as $P$. Thus there can be no parabolic bifurcation. $\square$

Therefore by Lemma 2.11 the Julia set of $g_{P,b}$ undergoes a holomorphic motion. So, the same analysis as in Proposition 2.15 holds, with the triple $(P, g_{P,b}, r/2)$ playing the role of $(Q_\theta, g_b, 1/10)$, and we can thus write:

$$\forall P \in \mathcal{C}, \quad \log R(P) = \operatorname*{avg}_{|b|=r/2} \log R(g_{P,b}).$$

We have not yet used the margin hypothesis: the points in $(Z_P \cup I_P) \setminus \{0\}$ stay bounded away from 0 when $P$ varies in $\mathcal{C}$. It is also the case for the points of $C_P$ for otherwise there would exist a $P$ with $0 \in C_P$ contradicting the fact that $|P'(0)| = 1$. Now $0 \in I_P$ thus $G''_P(0) = 2 \prod (-w)^{1+\deg_P(w)}$ with the product taken over $w \in (Z_P \cup I_P \cup C_P) \setminus \{0\}$. As a corollary, $G''_P(0)$ is bounded away from 0 when $P$ varies in $\mathcal{C}$.

Let $a = 1/b$ and $f_{P,a}(z) = b g_{P,b}(b^{-1}z) = a^{-1} g_{P,1/a}(az)$. Then, as $a \longrightarrow 0$, $f_{P,a}$ tends (pointwise) to the degree 2 polynomial

$$f_P(z) = \mathrm{e}^{2\pi i \theta(P)}\, z + \frac{G''_P(0)}{2} z^2$$

The same analysis as in Proposition 2.17 also holds, with $(f_P, f_{P,a}, 2/r)$ playing the role of $(f, f_a, 10)$, and yields

$$\log R(f_P) \geqslant \operatorname*{avg}_{|a|=2/r} \log R(f_{P,a}).$$

Now since $f_P$ and $Q_{\theta(P)}$ are related by a linear conjugacy:

$$\log R(f_P) = \log R(Q_{\theta(P)}) - \log \frac{|G''_P(0)|}{2}$$

and since $f_{P,a}$ and $g_{P,b}$ are related by a linear conjugacy:

$$\log R(f_{P,a}) = \log R(g_{P,b}) + \log |b|.$$

Putting it all together, we get

$$\log R(P) \leqslant -\log r/2 + \log R(Q_{\theta(P)}) - \log \frac{|G''_P(0)|}{2}.$$

Since $G''_P(0)$ is bounded away from 0, we get Theorem 1.14. Q.E.D.

## 4. Estimates on Yoccoz's Brjuno function

Lemma 4.1. — *For any integer $m \geqslant 2$, the function*

$$\theta \in \mathcal{B} \mapsto Y(\theta) - \frac{Y(m\theta)}{m}$$

*is unbounded on any interval.*

*Proof.* — The proof relies on the following fact. For any rational number $p/q$ with $p$ and $q$ coprime, any integer $k \geqslant 1$, and any Brjuno number $\theta$,

$$Y\left(\frac{p}{q}+\frac{k}{N+\theta}\right) \underset{N\to+\infty}{=} \frac{\log N}{q}+\mathcal{O}(1) \tag{4.1}$$

where $N$ denotes an integer. Let us first show how this enables us to conclude. Assume $p/q$ is a rational number with $p$ and $q$ coprime and assume $q$ and $m$ are coprime. Choose a Brjuno number $\theta$ and set

$$\theta_N = \frac{p}{q}+\frac{1}{N+\theta}.$$

Note that

$$m\theta_N = \frac{mp}{q}+\frac{m}{N+\theta} \quad \text{with } mp \text{ and } q \text{ coprime.}$$

Then,

$$Y(\theta_N) \underset{N\to+\infty}{=} \frac{\log N}{q}+\mathcal{O}(1) \quad \text{and} \quad Y(m\theta_N) \underset{N\to+\infty}{=} \frac{\log N}{q}+\mathcal{O}(1).$$

Thus,

$$Y(\theta_N)-\frac{Y(m\theta_N)}{m} \underset{N\to+\infty}{=} \frac{m-1}{m}\cdot\frac{\log N}{q}+\mathcal{O}(1) \underset{N\to+\infty}{\longrightarrow} +\infty.$$

It follows that the function

$$\theta\in\mathcal{B}\mapsto Y(\theta)-\frac{Y(m\theta)}{m}$$

is unbounded in any neighborhood of $p/q$. This implies our lemma since the set of rational numbers $p/q$ with $q$ and $m$ coprime is dense in $\mathbb{R}$.

Let us now prove estimate (4.1). We will use the continued fraction notation:

$$[a_0,a_1,\dots,a_n] = a_0+\cfrac{1}{a_1+\cfrac{1}{\ddots+\cfrac{1}{a_n}}}.$$

Set $\theta_N = \dfrac{p}{q}+\dfrac{k}{N+\theta}$. Recall that a rational has two finite continued fraction expansions, one with $n$ odd, equal to the initial segment of close enough reals $x < p/q$ and another with $n$ even, equal to the initial segment of close enough reals $x > p/q$. An approximant of a real number is a partial quotient of its continued fraction expansion. It is also called a convergent. Let us recall the following property: if $\theta\in\mathbb{R}$ and $|\theta-p/q| < 1/2q^2$ then $p/q$ is one of the approximants of $\theta$ (Theorem 184 p. 196 in [10]). If $N$ is large enough, $p/q$ is an approximant of $\theta_N > p/q$: there exists $n$ even and $N'$ such that for all $N > N'$:

$$\frac{p}{q} = [a_0, a_1, \ldots, a_n] \quad \text{and} \quad \theta_N = [a_0, a_1, \ldots, a_n + \alpha_n] \quad \text{with} \quad \alpha_n \in \left]0,1\right[.$$

Set

$$\frac{p'}{q'} = [a_0, a_1, \ldots, a_{n-1}]$$

and for $m < n$, set

$$\alpha_m = [0, a_{m+1}, \ldots, a_n + \alpha_n].$$

Then (Theorem 150 p. 167 in [10]) $p'q - q'p = (-1)^n = 1$ because $n$ is even, and (see [18] Section 1.2 p. 12)

$$\alpha_0\alpha_1\cdots\alpha_{n-1} = |q'\theta_N - p'| = \left|\frac{q'p - p'q}{q} + \frac{kq'}{N+\theta}\right| \underset{N\to+\infty}{\longrightarrow} \frac{1}{q},$$

$$\alpha_0\alpha_1\cdots\alpha_n = |q\theta_N - p| = \frac{kq}{N+\theta} \underset{N\to+\infty}{\longrightarrow} 0$$

and

$$\frac{1}{\alpha_n} = -\frac{q'\theta_N - p'}{q\theta_N - p} = \frac{N+\theta}{kq^2} - \frac{q'}{q} = \frac{\theta}{kq^2} \bmod \frac{1}{kq^2}.$$

In particular, $\alpha_{n+1}$, which is the fractional part of $1/\alpha_n$, can take only $kq^2$ values which all are Brjuno numbers. It follows that as $N \to +\infty$,

$$\begin{aligned} Y(\theta_N) = {} & \underbrace{\log\frac{1}{\alpha_0} + \ldots + \alpha_0\alpha_1\cdots\alpha_{n-2}\log\frac{1}{\alpha_{n-1}}}_{\mathcal{O}(1)} \\ & + \underbrace{\alpha_0\alpha_1\cdots\alpha_{n-1}\log\frac{1}{\alpha_n}}_{q^{-1}\log N + \mathcal{O}(1)} + \underbrace{\alpha_0\alpha_1\cdots\alpha_n Y(\alpha_{n+1})}_{o(1)} = \frac{\log N}{q} + \mathcal{O}(1). \end{aligned}$$

□

Lemma 4.2. — $\exists C > 0$, $\forall m \in \mathbb{N}^*$, $\forall \theta \in \mathbb{R}$,

$$Y(\theta) \leqslant Y(m\theta) + C\log m.$$

*Proof.* — For $\theta \in \mathbb{Q}$ it reads: $-\infty \leqslant -\infty$, which holds. Let us now assume $\theta$ irrational. For $m = 1$ it reads: $Y(\theta) \leqslant Y(\theta)$ which is trivial. So we now also assume that $m \geqslant 2$.

We will use the Brjuno sum:

$$B(\theta) = \sum_{n\in\mathbb{N}} \frac{\log q_{n+1}}{q_n}$$

where $p_n/q_n$ are the approximants of $\theta$. We have the following arithmetical property (c.f. [18], page 14):

$$\big|B(\theta) - Y(\theta)\big| \text{ is bounded.}$$

We recall that

(a) if $p_n/q_n$ are the approximants of $\alpha$ then(14)

$$\frac{1}{2q_nq_{n+1}} < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_nq_{n+1}},$$

and also that

(b) $q_n \geqslant F_n$ where $F_n$ is the $n$-th Fibonacci number(15). Last,

(c) if $|\alpha - p/q| < 1/2q^2$, then $p/q$ is an approximant of $\alpha$. (Theorem 184 p. 196 in [10])

Now, for every approximant $p_n/q_n$ of $\theta$, note $mp_n/q_n = p'/q'$ with $q' = q_n/(m \wedge q_n)$.

Case 1: $p'/q'$ is itself an approximant of $m\theta$ in which case if we note $p''/q''$ the next approximant of $m\theta$, then

$$\frac{1}{2q'q''} < \left|m\theta - \frac{p'}{q'}\right| = m\left|\theta - \frac{p_n}{q_n}\right| < \frac{m}{q_nq_{n+1}}$$

whence

$$q'' > \frac{q_{n+1}q_n}{2mq'} \geqslant \frac{q_{n+1}}{2m}$$

and thus

$$\frac{\log q''}{q'} \geqslant \frac{\log q_{n+1}}{q'} - \frac{\log 2m}{q'} \geqslant \frac{\log q_{n+1}}{q_n} - \frac{\log 2m}{q'}.$$

Case 2: $mp_n/q_n = p'/q'$ is not an approximant of $m\theta$, which means that

$$\left|m\theta - \frac{p'}{q'}\right| \geqslant \frac{1}{2q'^2},$$

and thus

$$\frac{1}{q_nq_{n+1}} \geqslant \left|\theta - \frac{p_n}{q_n}\right| \geqslant \frac{1}{2mq'^2}$$

whence

$$q_{n+1} \leqslant \frac{2mq'^2}{q_n} \leqslant 2mq_n$$

and thus

$$\frac{\log q_{n+1}}{q_n} \leqslant \frac{\log q_n}{q_n} + \frac{\log 2m}{q_n}.$$

(14) By [18] p. 12, $\alpha - \frac{p_n}{q_n} = (-1)^n/q_n(q_{n+1} + \alpha_{n+1}q_n)$. Since $\alpha_{n+1} \in ]0,1[$ and $q_n \leqslant q_{n+1}$, $q_{n+1} < q_{n+1} + \alpha_{n+1}q_n < 2q_{n+1}$.

(15) It follows by induction from $q_0 = 1$, $q_1 = a_1 \geqslant 1$, and $q_n = a_nq_{n-1} + q_{n-2}$, 10.2.2 p. 166 of [10].

Finally,

$$\begin{aligned} B(\theta) &= \sum_{\text{case } 1} \frac{\log q_{n+1}}{q_n} + \sum_{\text{case } 2} \frac{\log q_{n+1}}{q_n} \\ &\leqslant \sum \frac{\log q''}{q'} + \log(2m) \sum \frac{1}{q'} + \sum{}' \frac{\log F_n}{F_n} + \log(2m) \sum \frac{1}{F_n}. \end{aligned}$$

The prime in the sum means the summand needs to be replaced by the smallest non increasing sequence greater or equal to the sequence $\log F_n / F_n$. For different values of $n$, the approximants $p'/q'$ of $m\theta$ are different since $p'/q' = mp_n/q_n$ and thus

$$B(\theta) \leqslant B(m\theta) + \log(2m) \sum \frac{1}{F_n} + \sum{}' \frac{\log F_n}{F_n} + \log(2m) \sum \frac{1}{F_n}.$$

Since $F_n$ is exponentially increasing, the sums (independent of $\theta$) they are involved in are finite. We get

$$\begin{aligned} Y(\theta) &\leqslant Y(m\theta) + C_1 \log m + C_2 \\ &\leqslant Y(m\theta) + C_3 \log m \end{aligned}$$

with $C_1 = 2 \sum \frac{1}{F_n}$, $C_2 = \sum{}' \frac{\log F_n}{F_n} + 2 \sum \frac{1}{F_n} \log(2) + 2\|B - Y\|_\infty$ and $C_3 = (C_1 + C_2 / \log 2) < +\infty$. □

## Appendix A. Remarks

This section does not claim to bring new results. It is just a discussion of probably known and hopefully useful facts.

### A.1. Subharmonicity

If $\theta$ is a Brjuno number, then for all analytic family $f_a(z) = \mathrm{e}^{2\pi i\theta} z + \mathcal{O}(z^2)$ of analytic maps, $a \in A$ (notice that the rotation number does not vary with $a$), the function $a \mapsto -\log R(f_a)$ is the lim sup of subharmonic functions (see Definition 2.1):

$$u : a \mapsto -\log R(f_a) = \limsup_{n \to +\infty} \frac{1}{n} \log |\widehat{\phi}_n(a)|.$$

Lemma A.1. — *If a closed disk $\overline{B}$ is contained $A$, then the sequence of functions $f_n(a) = \frac{1}{n} \log |\widehat{\phi}_n(a)|$ is uniformly bounded from above on $\overline{B}$.*

*Proof.* — Choose $r > 0$, such that $\forall a \in \overline{B}$, $f_a$ is defined and injective on $D(0,r)$. By Yoccoz's Theorem B, for all $a \in \overline{B}$, $f_a$ has a (not necessarily maximal) rotation domain contained in $D(0,r)$ and of conformal radius $\geqslant r_0 := r\,\mathrm{e}^{-Y(\theta)-C}$. It implies that $\phi_{f_a}$ is convergent on a disc containing $D(0,r_0)$, and maps this disc in $D(0,r)$. Using Cauchy's inequalities, $\widehat{\phi}_n(a) \leqslant r/r_0^n$ whence $n^{-1}\log|\widehat{\phi}_n(a)| \leqslant \log(r)/n + \log(1/r_0)$, which is bounded. □

By Fatou's lemma,[16] the function $a \mapsto -\log R(f_a)$ is therefore (everywhere) below its average on circles. But we can say more: by the Brelot-Cartan theorem (see [13], Theorem 3.4.3, p. 64), if we note

$$u^*(a) = \limsup_{a' \to a} u(a'),$$

then $u^*$ is subharmonic and $u = u^*$ except on a polar set.

We however cannot say that $u$ itself is subharmonic (iff $u = u^*$) because it is not necessarily upper semicontinuous, as the following counterexample shows. Let $f_0 = \mathrm{e}^{2\pi i\theta} z + \mathcal{O}(z^2)$ be the restriction to $\mathbb{D}$ of a map $\widetilde{f}$ defined on an open set $\Omega$ containing $\overline{\mathbb{D}}$, and such that its Siegel disk $\Delta(\widetilde{f})$ in $\Omega$ goes beyond the edge of $\mathbb{D}$, *i.e.*, is not contained in $\mathbb{D}$ (for instance $\widetilde{f}(z) = \mathrm{e}^{2\pi i\theta} z$ on $\Omega = \mathbb{C}$). Then $\Delta(f_0)$ is the biggest $\widetilde{f}$-invariant subdisk of $\Delta(\widetilde{f})$ that is contained in $\mathbb{D}$. Since $\Delta(f_0) \subsetneq \Delta(\widetilde{f})$,

$$\operatorname{rad}\Delta(f_0) < \operatorname{rad}\Delta(\widetilde{f}).$$

Recall that (see Section 2.1)

$$\operatorname{rad}\Delta(\widetilde{f}) \leqslant R(\widetilde{f}) = R(f_0).$$

Thus:

$$\operatorname{rad}\Delta(f_0) < R(f_0).$$

Let $f_a = f_0 + az^2 g(z)$ for $a \in \mathbb{C}$, where $g(z)$ is any analytic function on $\mathbb{D}$ that is singular on all of $\partial\mathbb{D}$ (no path extension). Then by Lemma 2.6,

$$R(f_a) = \operatorname{rad}\Delta(f_a).$$

By Lemma 2.8, $\limsup_{\substack{a \to 0 \\ \neq}} \operatorname{rad}\Delta(f_a) \leqslant \operatorname{rad}\Delta(f_0)$. Thus

$$\limsup_{\substack{a \to 0 \\ \neq}} R(f_a) < R(f_0).$$

[16] Fatou's lemma states that if $f_n$ is a sequence of non-negative measurable functions on a measure space, then $\int \liminf f_n \leqslant \liminf \int f_n$. As a corollary, if $f_n$ is a sequence of functions on the circle, uniformly bounded from above, then $\int \limsup f_n \geqslant \limsup \int f_n$ where the integral is for Lebesgue's measure on the circle.

Thus $a \mapsto R(f_a)$ is not upper semi-continuous at $a = 0$.

Now, still assuming $\theta \in \mathcal{B}$, the upper semicontinuity holds if, instead of considering the map $a \mapsto -\log R(f_a)$ we consider $a \mapsto -\log \operatorname{rad}(\Delta(f_a|_{\mathbb{D}}))$ and if all $f_a$ are defined on $\mathbb{D}$ and $f_a \longrightarrow f_0$ for the compact open topology on $\mathbb{D}$. This is a corollary of the work of Risler [14]. Lower semicontinuity also holds, this time for an elementary reason: see Lemma 2.8. We have thus proved:

Proposition A.2. — *Given $\theta \in \mathcal{B}$, let $H_\theta(\mathbb{D})$ be the set of analytic functions $f : \mathbb{D} \to \mathbb{C}$ fixing $0$ with multiplier $\mathrm{e}^{2\pi i\theta}$, equipped with the compact open topology.*(17)

*The map* $\begin{pmatrix} H_\theta(\mathbb{D}) & \to & (0,1] \\ f & \mapsto & \operatorname{rad}\Delta(f) \end{pmatrix}$ *is continuous.*

This is not true for $\theta \notin \mathcal{B}$: for the rotation $z \mapsto \mathrm{e}^{2\pi i\theta} z$ restricted to $\mathbb{D}$, we have $\Delta = \mathbb{D}$, so $\operatorname{rad} = 1$, but for the nearby map $\mathrm{e}^{2\pi i\theta} z + \varepsilon z^2$, we have $\operatorname{rad} = 0$.

Proposition A.3. — *If $\theta \in \mathcal{B}$ and $A$ is a one complex dimensional parameter space and*

$$(a, z) \in A \times \mathbb{D} \mapsto f_a(z) = \mathrm{e}^{2\pi i\theta} z + \mathcal{O}(z^2)$$

*is analytic, then the map*

$$a \mapsto -\log \operatorname{rad}\Delta(f_a)$$

*is continuous and subharmonic.*

*Proof.* — We already mentioned the continuity.

Now, the same trick(18) as before yields subharmonicity with little effort: consider a function $g$ as in the discussion above, *i.e.*, holomorphic on $\mathbb{D}$ and with singularities at all points of $\partial\mathbb{D}$. Consider the sequence of families

$$(a, z) \mapsto \tau_n^{-1} f_a(\tau_n z) + \frac{1}{n} z^2 g(z) \quad \text{with} \quad \tau_n = 1 - \frac{1}{n}.$$

They all satisfy $-\log R = -\log \operatorname{rad}\Delta$ as above, whence all these are (continuous) subharmonic functions of $a$. By the previous proposition, these functions tend (locally uniformly as $A$ is locally compact) to $-\log \operatorname{rad}\Delta(f_a)$. $\square$

(17) uniform convergence on compact subsets of $\mathbb{D}$

(18) It would be nice to have a more satisfactory (no power series) proof. Also, it could be true that subharmonicity still holds if the domain of definition of $f$ undergoes a holomorphic motion.

## A.2. Holomorphic motions

Proposition A.4. — *Let $(U_a)$ be simply connected open subsets of $\mathbb{C}$ whose boundaries move holomorphically with respect to $a$. Let $c_a$ be a holomorphically varying point in $U_a$ and $r(a)$ be the conformal radius of $U_a$ with respect to $c_a$. Then, $a \mapsto -\log r(a)$ is a subharmonic function.*

*Proof.* — Let $V_a$ be the image of $U_a$ by the inversion $z \mapsto 1/(z-c_a)$. The set $V_a$ is unbounded and undergoes a holomorphic motion of its boundary. The conformal radius of $U_a$ is the inverse of (see [12] Corollary 11.1) the capacity radius of $\mathbb{C} \setminus V_a$, which is itself expressible by an energy minimization[19] as follows (see [13]):

$$-\log r(a) = \log \text{capacity radius} = -\inf_{\mu} E(\mu)$$

where $\mu$ varies in the set of non-atomic probability measures on $\partial V_a$ (together with its Borel $\sigma$-algebra) and $E(\mu)$ (the energy) is defined by

$$E(\mu) = \iint_{\partial V_a \times \partial V_a} -\log \left|u - v\right| d\mu(u) d\mu(v).$$

Since $\mu$ is non-atomic, the mass of the diagonal $\{(u,v) | u = v\}$ is null. The integrand $-\log \left|u - v\right|$ is bounded from below by $-\log \operatorname{diam}(\partial V_a)$, and the measure $\mu \otimes \mu$ is finite, therefore $E(\mu)$ is well defined and belongs to $(-\infty, +\infty]$. Choose a basepoint $a_0$ and let $\xi_a(z) : \partial V_{a_0} \to \partial V_a$ be the holomorphic motion. Then, for all probability measure $\mu$ on $\partial V_{a_0}$,

$$E\big((\xi_a)_* \mu\big) = \iint_{\partial V_{a_0} \times \partial V_{a_0}} -\log \left|\xi_a(u) - \xi_a(v)\right| d\mu(u) d\mu(v).$$

Again, the integrand $-\log \left|\xi_a(u) - \xi_a(v)\right|$ has a lower bound, uniform over $(u,v)$, when $a$ remains in a compact set, because $-\log \operatorname{diam}(\partial V_a)$ has. Thus using Harnack's inequality, we get that $E\big((\xi_a)_* \mu\big)$ is either a harmonic function of $a$ or the constant function $+\infty$. Now, $-\log r(a) = -\inf E = \sup -E$ so it is the supremum of a set of harmonic functions that are locally bounded from above. This yields a continuous subharmonic function. □

## A.3. Harmonicity

Let us recall Proposition 2.14, whose proof was communicated to us by Saeed Zakeri.

[19] As a variant of this, one could instead use the transfinite diameter.

Proposition A.5. — *Assume that $f_a : U_a \to \mathbb{C}$ is an analytic family of maps of the form $f(z) = \mathrm{e}^{2\pi i\theta}\, z + \mathcal{O}(z^2)$, so that the rotation number is independent of $a$. Assume they all have a Siegel disk $\Delta(f_a)$, and that $\Delta(f_a) \neq \mathbb{C}$. Assume that the boundary in $\mathbb{C}$ of the Siegel disk $\partial\Delta(f_a)$ undergoes a holomorphic motion (we do not require $\Delta(f_a) \Subset U_a$ and we do not require the motion to commute with the dynamics). Then the function $a \mapsto -\log\operatorname{rad}\Delta(f_a)$ is harmonic.*

This is kind of surprising: let $A$ denote the fact that a simply connected domain undergoes a holomorphic motion (of its boundary), and $B$ denote the fact that this domain is a Siegel disk of an analytically varying family of analytic maps (with fixed rotation number) in $\mathbb{D}$. Then

$$\begin{aligned} A &\implies -\log\operatorname{rad} \text{ is subharmonic,}\\ B &\implies -\log\operatorname{rad} \text{ is subharmonic,}\\ (A \text{ and } B) &\implies -\log\operatorname{rad} \text{ is harmonic}\ldots \end{aligned}$$

Is it fair that when a number has two reasons to be negative, then it is null?

## A.4. Other radii of interest

We have

$$R(f) = \text{ the radius of convergence of } \phi_f$$

and

$$\operatorname{rad}\Delta(f) = \text{ the biggest radius } \leqslant R \text{ below which } \phi_f \text{ maps in } \Delta(f).$$

Here are a few other "natural" radii that one could study

$A =$ the biggest radius $\leqslant R$ on which $\phi_f$ is injective,

$B =$ the biggest radius $\leqslant R$ on which $\phi_f$ has no critical point,

$C =$ the biggest radius $\geqslant R$ on which $\phi_f$ has a meromorphic extension $\widetilde{\phi}_f$,

$D =$ the biggest radius $\leqslant C$ on which $\widetilde{\phi}_f$ is injective,

$E =$ the biggest radius $\leqslant C$ on which $\widetilde{\phi}_f$ has no critical point.

## A.5. Thanks

We would like to thank the Referees for their very useful suggestions, that greatly enhance the readability of the present work, and allowed us to correct mistakes in the original manuscript.

# BIBLIOGRAPHY

[1] L. Bers & H. L. Royden, "Holomorphic families of injections", *Acta Math.* **157** (1986), no. 3-4, p. 259-286.

[2] X. Buff, "Virtually repelling fixed points", *Publ. Mat.* **47** (2003), no. 1, p. 195-209.

[3] X. Buff & A. Chéritat, "Upper bound for the size of quadratic Siegel disks", *Invent. Math.* **156** (2004), no. 1, p. 1-24.

[4] ———, "The Brjuno function continuously estimates the size of quadratic Siegel disks", *Ann. of Math. (2)* **164** (2006), no. 1, p. 265-312.

[5] X. Buff & A. L. Epstein, "A parabolic Pommerenke-Levin-Yoccoz inequality", *Fund. Math.* **172** (2002), no. 3, p. 249-289.

[6] A. Chéritat, "Recherche d'ensembles de Julia de mesure de Lebesgue positive", thèse de doctorat, Université Paris-Sud, décembre 2001.

[7] A. Douady & J. H. Hubbard, "On the dynamics of polynomial-like mappings", *Ann. Sci. École Norm. Sup. (4)* **18** (1985), no. 2, p. 287-343.

[8] A. L. Epstein, "Infinitesimal Thurston rigidity and the Fatou-Shishikura inequality", Stony Brook IMS Preprint, 1991.

[9] L. Geyer, "Linearization of structurally stable polynomials", in *Progress in holomorphic dynamics*, Pitman Res. Notes Math. Ser., vol. 387, Longman, Harlow, 1998, p. 27-30.

[10] G. H. Hardy & E. M. Wright, *An introduction to the theory of numbers*, sixth ed., Oxford University Press, Oxford, 2008, Revised by D. R. Heath-Brown and J. H. Silverman, xxii+621 pages.

[11] R. Mañé, P. Sad & D. Sullivan, "On the dynamics of rational maps", *Ann. Sci. École Norm. Sup. (4)* **16** (1983), no. 2, p. 193-217.

[12] C. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975, With a chapter on quadratic differentials by Gerd Jensen, Studia Mathematica/Mathematische Lehrbücher, Band XXV, 376 pages.

[13] T. Ransford, *Potential theory in the complex plane*, London Mathematical Society Student Texts, vol. 28, Cambridge University Press, Cambridge, 1995, x+232 pages.

[14] E. Risler, "Linéarisation des perturbations holomorphes des rotations et applications", *Mém. Soc. Math. Fr. (N.S.)* (1999), no. 77, p. viii+102.

[15] M. Shishikura, "On the quasiconformal surgery of rational functions", *Ann. Sci. École Norm. Sup. (4)* **20** (1987), no. 1, p. 1-29.

[16] Z. Slodkowski, "Holomorphic motions and polynomial hulls", *Proc. Amer. Math. Soc.* **111** (1991), no. 2, p. 347-355.

[17] D. P. Sullivan & W. P. Thurston, "Extending holomorphic motions", *Acta Math.* **157** (1986), no. 3-4, p. 243-257.

[18] J.-C. Yoccoz, “Théorème de Siegel, nombres de Bruno et polynômes quadratiques”, *Astérisque* (1995), no. 231, p. 3-88, Petits diviseurs en dimension 1.

Manuscrit reçu le 23 mars 2007,
révisé le 4 janvier 2010,
accepté le 8 juin 2010.

Xavier BUFF & Arnaud CHÉRITAT
Université Paul Sabatier
Institut de Mathématiques de Toulouse
118, route de Narbonne
31062 Toulouse Cedex 9 (France)
xavier.buff@math.univ-toulouse.fr
arnaud.cheritat@math.univ-toulouse.fr