ANNALES

DE

L'INSTITUT FOURIER

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Tome 61, nº 1 (2011), p. 261-318.

<http://aif.cedram.org/item?id=AIF_2011__61_1_261_0>

Ann. Inst. Fourier, Grenoble
**61**, 1 (2011) 261-318

# AN EXPLICIT FORMULA FOR THE HILBERT SYMBOL OF A FORMAL GROUP

by Floric TAVARES RIBEIRO

Abstract. — A Brückner-Vostokov formula for the Hilbert symbol of a formal group was established by Abrashkin under the assumption that roots of unity belong to the base field. The main motivation of this work is to remove this hypothesis. It is obtained by combining methods of $(\varphi,\Gamma)$-modules and a cohomological interpretation of Abrashkin's technique. To do this, we build $(\varphi,\Gamma)$-modules adapted to the false Tate curve extension and generalize some related tools like the Herr complex with explicit formulas for the cup-product and the Kummer map.

Résumé. — Abrashkin a établi une formule de Brueckner-Vostokov pour le symbole de Hilbert d'un groupe formel sous la condition d'appartenance de racines de l'unité au corps de base. La motivation première de ce travail réside en la suppression de cette hypothèse. On l'obtient en combinant des méthodes de $(\varphi,\Gamma)$-modules et une interprétation cohomologique des techniques d'Abrashkin. Pour cela, on construit des $(\varphi,\Gamma)$-modules adaptés à l'extension dite de la fausse courbe de Tate et on généralise des outils tels que le complexe de Herr avec des formules explicites pour le cup-produit et l'application de Kummer.

## Introduction

### 0.1. $(\varphi,\Gamma)$-modules

Let $p$ be a prime number and $K$ a finite extension of $\mathbb{Q}_p$ with residue field $k$. Fix $\overline{K}$ an algebraic closure of $K$ and note $G_K=\mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of $K$. Let us furthermore introduce $K_\infty=\cup_n K(\zeta_{p^n})$ the cyclotomic extension of $K$ and $\Gamma_K=\mathrm{Gal}(K_\infty/K)$.

The context of this work is the theory of $p$-adic representations of the Galois group of a local field, here $G_K$. We are particularly interested in $\mathbb{Z}_p$-adic representations of $G_K$, *i.e.* $\mathbb{Z}_p$-modules of finite type endowed with a

*Keywords:* $p$-adic representations, $(\phi,\Gamma)$-modules, formal groups, explicit reciprocity law.
*Math. classification:* 11F80, 11S25, 14L05, 11S31, 11S23, 14F30.

linear and continuous action of $G_K$. In [13], Fontaine defined an equivalence of categories between the category of $\mathbb{Z}_p$-adic representations of $G_K$ and the one of étale $(\varphi, \Gamma_K)$-modules over a ring $\mathbf{A}_K$. A $(\varphi, \Gamma_K)$-module on $\mathbf{A}_K$ is a module of finite type over $\mathbf{A}_K$ endowed with commuting semi-linear actions of $\varphi$ and $\Gamma_K$.

Berger, in [5], showed how to recover the de Rham, semi-stable or crystalline module of Fontaine's theory from the $(\varphi, \Gamma_K)$-module associated with the representation. For absolutely unramified crystalline representations, Wach furnished in [27] another powerful construction which permits to recover the crystalline module in the associated $(\varphi, \Gamma_K)$-module. This construction was studied in details and made more precise by Berger ([7]). $(\varphi, \Gamma_K)$-modules are also intimately linked to Iwasawa theory as was shown in works by Cherbonnier and Colmez ([9]), Benois ([4]) or Berger ([6]).

## 0.2. The false Tate curve extension

The construction of $(\varphi, \Gamma_K)$-modules lies on the use of the cyclotomic tower and shows its fundamental role in the study of $p$-adic representations. But another extension appears as particularly significant. Fix $\pi$ a uniformizer of $K$ and $\pi_n$ a system of $p^n$-th roots of $\pi$: $\pi_0 = \pi$ and for all $n \in \mathbb{N}$, $\pi_{n+1}^p = \pi_n$. It is the behavior in the extension $K_\pi = \cup_n K(\pi_n)$ which makes the difference between a crystalline and a semi-stable representation. It is then natural to introduce $(\varphi, \Gamma)$-modules where the cyclotomic extension $K_\infty$ is replaced by $K_\pi$. However $K_\pi/K$ is not Galois and we only get $\varphi$-modules (also studied by Fontaine in [13]). These $\varphi$-modules were used by Breuil ([8]) or Kisin ([19]) to study $p$-adic representations and Abrashkin made use of the field of norms of $K_\pi/K$ in [2] and [1].

In order to recover the whole action of $G_K$, let us then consider the Galois closure $L$ of $K_\pi$ which is nothing more than the compositum of $K_\pi$ and $K_\infty$, a metabelian extension of $K$, *the false Tate curve extension*. What we lose here is the explicit description of the field of norms of this extension. Note $G_\infty = \mathrm{Gal}(L/K)$. Our first result can, for $\mathbf{A}' = \mathbf{A}$ or $\tilde{\mathbf{A}}$, and $\mathbf{A}'_L = \mathbf{A}'^{G_L}$ (where $\mathbf{A}$ and $\tilde{\mathbf{A}}$ are Fontaine rings defined in Paragraph 1.2), be expressed as:

Theorem 0.1. — *The functor $V \mapsto D_L(V) = (V \otimes_{\mathbb{Z}_p} \mathbf{A}')^{G_L}$ induces an equivalence of categories*

$$\{\mathbb{Z}_p - \textit{adic representations of } G_K\} \to \{\textit{étale } (\varphi, G_\infty) - \textit{modules over } \mathbf{A}'_L\}$$

In fact we show that the $(\varphi, G_\infty)$-module $D_L(V)$ is nothing but the scalar extension of the usual $(\varphi, \Gamma_K)$-module $D(V)$ from $\mathbf{A}_K$ to $\mathbf{A}'_L$.

### 0.3. Galois cohomology

Recall that in the case of $(\varphi, \Gamma_K)$-modules, Herr [16] showed that the homology of the complex $0 \longrightarrow D(V) \xrightarrow{f_1} D(V) \oplus D(V) \xrightarrow{f_2} D(V) \longrightarrow 0$ with maps $f_1 = \begin{pmatrix} \varphi - 1 \\ \gamma - 1 \end{pmatrix}$ and $f_2 = (\gamma - 1, 1 - \varphi)$ computes the Galois cohomology of $V$.

We introduce now a complex in $D_L(V)$ which computes the cohomology of $V$. Since the group $G_\infty$ has dimension 2, the corresponding complex loses some simplicity. Let $\tau$ be a topological generator of the subgroup $\mathrm{Gal}(L/K_\infty)$ and $\gamma$ a topological generator of $\mathrm{Gal}(L/K_\pi)$ satisfying $\gamma\tau\gamma^{-1} = \tau^{\chi(\gamma)}$, it can be described as:

Theorem 0.2. — *Let $V$ be a $\mathbb{Z}_p$-adic representation of $G_K$ and $D$ its $(\varphi, G_\infty)$-module. The homology of the complex*

$$0 \longrightarrow D \xrightarrow{\alpha} D \oplus D \oplus D \xrightarrow{\beta} D \oplus D \oplus D \xrightarrow{\eta} D \longrightarrow 0$$

*where*

$$\alpha = \begin{pmatrix} \varphi - 1 \\ \gamma - 1 \\ \tau - 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} \gamma - 1 & 1 - \varphi & 0 \\ \tau - 1 & 0 & 1 - \varphi \\ 0 & \tau^{\chi(\gamma)} - 1 & \delta - \gamma \end{pmatrix},$$

$$\eta = \left(\tau^{\chi(\gamma)} - 1, \delta - \gamma, \varphi - 1\right)$$

*with $\delta = (\tau^{\chi(\gamma)} - 1)(\tau - 1)^{-1} \in \mathbb{Z}_p[[\tau - 1]]$, identifies canonically and functorially with the continuous Galois cohomology of $V$.*

In fact, we get explicit isomorphisms. In particular for the first cohomology group, let $(x, y, z) \in \ker \beta$, let $b$ be a solution in $V \otimes \mathbf{A}'$ of $(\varphi - 1)b = x$, then the above theorem associates with the class of the triple $(x, y, z)$ the class of the cocycle:

$$c \;:\; \sigma \mapsto c_\sigma = -(\sigma - 1)b + \gamma^n \frac{\tau^m - 1}{\tau - 1} z + \frac{\gamma^n - 1}{\gamma - 1} y$$

where $m$ and $n$ are defined by the relation $\sigma_{|G_\infty} = \gamma^n \tau^m$.

Like Herr in [17], we also furnish explicit formulas for the cup-product.

### 0.4. Explicit formulas for the Hilbert symbol

The Hilbert symbol, for a field $K$ containing the group $\mu_{p^n}$ of $p^n$-th roots of unity is defined, for $r_K : K^* \to G_K^{\mathrm{ab}}$ the reciprocity map of class field theory, as the pairing

$$(a,b) \in K^* \times K^* \mapsto (a,b)_{p^n} := \left(\sqrt[p^n]{b}\right)^{r_K(a)-1} \in \mu_{p^n}.$$

Since 1858 and Kummer, many explicit formulas have been given for the Hilbert symbol. Let us cite Coleman's one ([10]): suppose that $K = K_0(\zeta_{p^n})$ where $K_0$ is a finite unramified extension of $\mathbb{Q}_p$ and $\zeta_{p^n}$ a fixed primitive $p^n$-th root of unity. Note $W$ the ring of integers of $K_0$. If $F \in 1 + (p, X) \subset W[[X]]$, then $F(\zeta_{p^n} - 1)$ is a principal unit. Extend the absolute Frobenius $\varphi$ from $W$ to $W[[X]]$ by putting $\varphi(X) = (1+X)^p - 1$. Denote for $F \in W[[X]]$

$$\mathscr{L}(F) = \frac{1}{p} \log \frac{F(X)^p}{\varphi(F(X))} \in W[[X]].$$

So for $F \in 1 + (p, X)$, $\mathscr{L}(F) = \left(1 - \frac{\varphi}{p}\right) \log F(X)$. Coleman's formula is:

Theorem 0.3 (Coleman). — *Let $F, G \in 1 + (p, X) \subset W[[X]]$, then*

$$(F(\zeta_{p^n} - 1), G(\zeta_{p^n} - 1))_{p^n} = \zeta_{p^n}^{[F,G]_n}, \text{ where}$$

$$[F,G]_n = \mathrm{Tr}_{K_0/\mathbb{Q}_p} \circ \mathrm{Res}_X \frac{1}{\varphi^n(X)} \left( \mathscr{L}(G) d\log F - \frac{1}{p} \mathscr{L}(F) d\log G^{\varphi} \right).$$

Brückner-Vostokov's formula is very similar to Coleman's one: suppose $p \neq 2$, let $\zeta_{p^n} \in K$, let $W$ be the ring of integers of $K_0$, the maximal unramified extension of $K/\mathbb{Q}_p$. Extend the Frobenius $\varphi$ from $W$ to $W[[Y]][1/Y]$ via $\varphi(Y) = Y^p$. Fix moreover $\pi$ a uniformizer of $K$.

Theorem 0.4 (Brückner-Vostokov). — *Let $F, G \in (W[[Y]][1/Y])^{\times}$, and $s \in W[[Y]]$ such that $s(\pi) = \zeta_{p^n}$, then*

$$(F(\pi), G(\pi))_{p^n} = \zeta_{p^n}^{[F,G]_n}, \text{ where}$$

$$[F,G]_n = \mathrm{Tr}_{K_0/\mathbb{Q}_p} \circ \mathrm{Res}_Y \frac{1}{s^{p^n} - 1} \left( \mathscr{L}(G) d\log F - \frac{1}{p} \mathscr{L}(F) d\log G^{\varphi} \right).$$

In the second part of this work, we show a generalization of this formula to formal groups.

Remark that there are other types of formulas, in particular the one of Sen ([20]), generalized to formal groups by Benois in [3]. We refer interested readers to Vostokov's [25] which provides a comprehensive background on such formulas.

### 0.5. An explicit formula for formal groups

We suppose now $p > 2$.[(1)]

Let $G$ be a connected smooth formal group of dimension $d$ and finite height $h$ over $W = W(k)$ the ring of Witt vectors with coefficients in a finite field $k^{(2)}$. Let $K$ be a finite extension of $K_0 = \mathrm{Frac}W$ containing the $p^M$-torsion $G[p^M]$ of $G$. Define the Hilbert symbol of $G$ as

$$(x, \beta) \in K^* \times G(\mathfrak{m}_K) \mapsto (x, \beta)_{G,M} := r_K(x)(\beta_1) -_G \beta_1 \in G[p^M]$$

where $r_K : K^* \to G_K^{\mathrm{ab}}$ is the reciprocity map and $\beta_1$ satisfies $p^M \mathrm{id}_G \beta_1 = \beta$.

Fix a basis of logarithms of $G$ under the form of a vectorial logarithm $l_G \in K_0[[\mathbf{X}]]^d$ where $\mathbf{X} = (X_1, \dots, X_d)$ so that one has the formal identity $l_G(\mathbf{X} +_G \mathbf{Y}) = l_G(\mathbf{X}) + l_G(\mathbf{Y})$. Complete $l_G$ with almost-logarithms $m_G \in K_0[[\mathbf{X}]]^{h-d}$ in a basis $\begin{pmatrix} l_G \\ m_G \end{pmatrix}$ of the Dieudonné module of $G$. Fontaine defined in [12] (see also [11] for an explicit description) a pairing between the Dieudonné module and the Tate module of $G$, $T(G) = \varprojlim G[p^n]$.

Honda showed in [18] the existence of a formal power series of the form $\mathcal{A} = \sum_{n \geqslant 1} F_n \varphi^n$ with $F_n \in M_d(W)$ such that $\left(1 - \frac{\mathcal{A}}{p}\right) \circ l_G(\mathbf{X}) \in M_d(W[[\mathbf{X}]])$.

Let us introduce the approximated period matrix. Fix $(o^1, \dots, o^h)$ a basis of $T(G)$ where $o^i = (o^i_n)_{n \geqslant 1}$ with $p\mathrm{id}_G o^i_n = o^i_{n-1}$. Then $(o^1_M, \dots, o^h_M)$ is a basis of $G[p^M]$. For all $i$, choose $\hat{o}^i_M \in F(YW[[Y]])$ such that $\hat{o}^i_M(\pi) = o^i_M$. We define

$$\mathcal{V}_Y = \begin{pmatrix} p^M l_G(\hat{o}^1_M) & \dots & p^M l_G(\hat{o}^h_M) \\ p^M m_G(\hat{o}^1_M) & \dots & p^M m_G(\hat{o}^h_M) \end{pmatrix}.$$

Now we can state the reciprocity law which generalizes Brückner-Vostokov's one and constitutes the goal of the second part of this work:

Theorem 0.5. — *Let $\alpha \in (W[[Y]][\frac{1}{Y}])^\times$ and $\beta \in G(YW[[Y]])$. Write* Tr *for* $\mathrm{Tr}_{W/\mathbb{Z}_p}$. *The coordinates of the Hilbert symbol $(\alpha(\pi), \beta(\pi))_{G,M}$ in the basis $(o^1_M, \dots, o^h_M)$ are*

$$(\mathrm{Tr} \circ \mathrm{Res}_Y) \mathcal{V}_Y^{-1} \left( \begin{pmatrix} (1 - \frac{\mathcal{A}}{p}) \circ l_G(\beta) \\ 0 \end{pmatrix} d_{\log}\alpha(Y) - \mathscr{L}(\alpha) \frac{d}{dY} \begin{pmatrix} \frac{\mathcal{A}}{p} \circ l_G(\beta) \\ m_G(\beta) \end{pmatrix} \right)$$

(1) The computations of the Kummer maps require 2 to be invertible (cf. Proposition 1.14 and the proof of Lemma 2.15 below).

(2) $W$ is then unramified over $\mathbb{Z}_p$. The ramified case seems much more complicated since we don't have the theory of Honda's systems [18, Theorem 4].

This formula was shown by Abrashkin in [2] under the assumption that $K$ contains $p^M$-th roots of unity. Vostokov and Demchenko proved it in [26] without any condition on $K$ for formal groups of dimension 1.

## 0.6. The strategy

The main idea is due to Benois who carried it out in [4] to show Coleman's reciprocity law. The point is to see the Hilbert symbol as a cup-product via the commutative diagram

$$\begin{array}{ccc} K^* \times K^* & \xrightarrow{(,)_{p^n}} & \mu_{p^n} \\ \kappa\times\kappa \downarrow & & \uparrow \mathrm{inv}_K \\ H^1(K,\mu_{p^n}) \times H^1(K,\mu_{p^n}) & \xrightarrow{\cup} & H^2(K,\mu_{p^n}^{\otimes 2}) \end{array}$$

where $\kappa$ is Kummer's map. He first explicitly computed $\kappa$ in terms of the Herr complex associated with the representation $\mathbb{Z}_p(1)$, then he used Herr's cup-product explicit formulas and he finally computed the image of the couple he obtained via the isomorphism $\mathrm{inv}_K$.

For a formal group, the situation is rather similar, we get the diagram

$$\begin{array}{ccc} K^* \times G(\mathfrak{m}_K) & \xrightarrow{(,)_{G,M}} & G[p^M] \\ \kappa\times\kappa_G \downarrow & & \uparrow \mathrm{inv}_K \\ H^1(K,\mu_{p^M}) \times H^1(K,G[p^M]) & \xrightarrow{\cup} & H^2(K,\mu_{p^M}\otimes G[p^M]) \end{array}$$

with identifications $H^2(K,\mu_{p^M}\otimes G[p^M]) \simeq H^2(K,\mathbb{Z}/p^M\mathbb{Z}(1))\otimes G[p^M]$ and $G[p^M] \simeq (\mathbb{Z}/p^M\mathbb{Z})^h$.

The formulas for the Kummer map and the cup-product are shown in the section on $(\varphi,\Gamma)$-modules. The computation of the explicit formula for the map $\kappa_G : G(\mathfrak{m}_K) \to H^1(K,G[p^M])$ constitutes the technical axis of this work. Abrashkin made use of the Witt symbol, and to conclude via the field of norms of extension $K_\pi/K$, he used the compatibility of the reciprocity map between the field of norms of an extension and the basis field. With the help of the four terms complex above, we give a cohomological interpretation of his method and carry his computations to the higher order to calculate $\kappa_G$.

Let us finish with some technical remarks on the remainder assumption that $K$ contains the $p^M$-torsion. Without this hypothesis, the Hilbert symbol is not well defined, but we can't even compute the Kummer map. Indeed the formula involves an approximation of the period matrix, which is built by approximating the basis of the Tate module by elements of $G(K)$.

However, the assumption that $K$ contains the $p^M$-torsion implies, because of Weil's pairing and in the case where the formal group comes from an abelian variety, that $K$ actually contains the $p^M$-th roots of unity so that we don't get any improvement in this case.

### 0.7. Organization of the paper

This work splits in two parts. First, we introduce $(\varphi, G_\infty)$-modules and give the associated Herr complex with explicit formulas between its homology and the cohomology of the representation. Then we provide explicit formulas for the cup-product and the Kummer map.

The second part is devoted to the proof of the Brückner-Vostokov formula for formal groups. The main difficulty lies in the fact that the period matrix does not live in the right place: we introduce an approximated period matrix and show that it enjoys similar properties as the original matrix modulo suitable rings. Then, we carry out the computation of the Hilbert symbol in terms of the Herr complex.

ACKNOWLEDGMENT. — *This work is based on my PhD thesis under the supervision of Denis Benois. I wish to thank him for the precious ideas he shared with me and the time and energy he offered me. I am also very grateful to Laurent Berger. He carefully read an earlier version of this paper, some of his remarks allowed me to improve it.*

## 1. $(\varphi, \Gamma)$-modules and cohomology

### 1.1. Notation

Let $p$ be a prime. Let us recall (cf. [21]) that if $\mathbb{K}$ is a perfect field of characteristic $p$, the ring of Witt vectors $W(\mathbb{K})$ over $\mathbb{K}$ is a strict $p$-ring with residue field $\mathbb{K}$. If $R$ is a subring of $\mathbb{K}$, we still denote by $W(R)$ the Witt vectors with coefficients in $R$. It is a subring of $W(\mathbb{K})$.

Fix $K$ a finite extension of $\mathbb{Q}_p$ with residue field $k$. Denote $W = W(k)$ the ring of Witt vectors over $k$. Then $K_0 = W \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ identifies with the maximal unramified sub-extension of $\mathbb{Q}_p$ in $K$. Fix $\overline{K}$ an algebraic closure of $K$ and denote $G_K = \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of $K$ and

$\mathbb{C}_p$ the $p$-adic completion of $\overline{K}$. Endow $\mathbb{C}_p$ with the $p$-adic valuation $v_p$ normalized by $v_p(p) = 1$. Recall that the action of $G_K$ on $\overline{K}$ extends by continuity to $\mathbb{C}_p$.

Let us fix $\varepsilon = (\zeta_{p^n})_{n\geqslant 0}$ a coherent system of $p^n$-th roots of unity, *i.e.* $\zeta_{p^n}^p = \zeta_{p^{n-1}}$ for all $n$, $\zeta_1 = 1$ and $\zeta_p \neq 1$. Then $K_\infty := \bigcup_{n\in\mathbb{N}} K(\zeta_{p^n})$ is the cyclotomic extension of $K$. Denote $G_{K_\infty} = \mathrm{Gal}(\overline{K}/K_\infty)$ its absolute Galois group and $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ the quotient.

Fix as well $\pi$ a uniformizer of $K$ and $\rho = (\pi_{p^n})_{n\geqslant 0}$ a coherent system of $p^n$-th roots of $\pi$. Denote $K_\pi = \bigcup_{n\geqslant 0} K(\pi_{p^n})$. The extension $K_\pi/K$ is not Galois, so put $L = \bigcup_{n\geqslant 0} K(\zeta_{p^n}, \pi_{p^n})$ its Galois closure. It is the compositum of $K_\pi$ and $K_\infty$. Denote $G_L = \mathrm{Gal}(\overline{K}/L)$ its absolute Galois group and $G_\infty = \mathrm{Gal}(L/K)$ the quotient. The cyclotomic character $\chi : G_K \to \mathbb{Z}_p^*$ factorizes through $G_\infty$ (even through $\Gamma_K$) ; it is also true for the map $\psi : G_K \to \mathbb{Z}_p$ defined by

$$\forall g \in G_K \quad g(\pi_{p^n}) = \pi_{p^n}\zeta_{p^n}^{\psi(g)}.$$

Moreover, the group $G_\infty$ identifies with the semi-direct product $\mathbb{Z}_p \rtimes \Gamma_K$. So if $p$ is odd $G_\infty$ is topologically generated by two elements, $\gamma$ and $\tau$ satisfying $\gamma\tau\gamma^{-1} = \tau^{\chi(\gamma)}$. Let us fix $\gamma$ and choose $\tau$ such that $\psi(\tau) = 1$, *i.e.* with $\tau(\rho) = \rho\varepsilon$.

We adopt the convention that complexes have their first term in degree $-1$ if this term is 0, and otherwise in degree 0.

*Remark 1.1.* — The group $G_\infty$ is a $p$-adic Lie group so that the extension $L/K$ is arithmetically profinite (cf [28, 24]).

### 1.2. The field $\tilde{\mathbf{E}}$, the ring $\tilde{\mathbf{A}}$ and some of their subrings.

We refer to [13] for results of this section. However we adopt Colmez' notation. Rings $R$, $W(\mathrm{Frac}R)$ or $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ become $\tilde{\mathbf{E}}^+$, $\tilde{\mathbf{A}}$ and $\mathbf{A}$.

Define $\tilde{\mathbf{E}}$ as the inverse limit $\tilde{\mathbf{E}} = \varprojlim \mathbb{C}_p$ where transition maps are exponentiation to the power $p$. An element of $\tilde{\mathbf{E}}$ is then a sequence $x = (x^{(n)})_{n\in\mathbb{N}}$ satisfying $(x^{(n+1)})^p = x^{(n)}$ for all $n \in \mathbb{N}$. Endow $\tilde{\mathbf{E}}$ with the addition $x + y = s$ where $s^{(n)} = \lim_{m\to+\infty} (x^{(n+m)} + y^{(n+m)})^{p^m}$ and the product $x.y = t$ where $t^{(n)} = x^{(n)}.y^{(n)}$. These operations make $\tilde{\mathbf{E}}$ into a field of characteristic $p$, algebraically closed and complete for the valuation $v_{\mathbf{E}}(x) := v_p(x^{(0)})$. The ring of integers of $\tilde{\mathbf{E}}$, denoted by $\tilde{\mathbf{E}}^+$, identifies then with the inverse limit $\varprojlim \mathcal{O}_{\mathbb{C}_p}$. It is a local ring whose maximal ideal, denoted by $\mathfrak{m}_{\tilde{\mathbf{E}}}$, identifies with $\varprojlim \mathfrak{m}_{\mathbb{C}_p}$ and with residue field isomorphic to

$\overline{k}$. The field $\tilde{\mathbf{E}}$, as well as its ring of integers $\tilde{\mathbf{E}}^+$, still has a natural action of $G_K$ continuous with respect to the $v_{\mathbf{E}}$-adic topology. Define the Frobenius $\varphi : x \mapsto x^p$ which acts continuously, commutes with the action of $G_K$ and stabilizes $\tilde{\mathbf{E}}^+$.

Let $\tilde{\mathbf{A}} = W(\tilde{\mathbf{E}})$ be the ring of Witt vectors on $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{A}}^+ = W(\tilde{\mathbf{E}}^+)$. Any element of $\tilde{\mathbf{A}}$ (resp. $\tilde{\mathbf{A}}^+$) can be written uniquely as $\sum_{n\in\mathbb{N}} p^n[x_n]$ where $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements in $\tilde{\mathbf{E}}$ (resp. in $\tilde{\mathbf{E}}^+$). The topology on $\tilde{\mathbf{A}}$ comes from the product topology on $W(\tilde{\mathbf{E}}) = \tilde{\mathbf{E}}^{\mathbb{N}}$. This topology is compatible with the ring structure on $\tilde{\mathbf{A}}$. It is weaker than the $p$-adic topology.

Let us remark that the sequences $\varepsilon$ and $\rho$ define elements in $\tilde{\mathbf{E}}^+$. Denote $X = [\varepsilon] - 1$ and $Y = [\rho]$. These are elements of $\tilde{\mathbf{A}}^+$ and even of $W(\mathfrak{m}_{\tilde{\mathbf{E}}})$. They are topologically nilpotent. We also have bases of neighborhoods of 0 in $\tilde{\mathbf{A}}$:

$$\{p^n\tilde{\mathbf{A}} + X^m\tilde{\mathbf{A}}^+\}_{(n,m)\in\mathbb{N}^2} \text{ and } \{p^n\tilde{\mathbf{A}} + Y^m\tilde{\mathbf{A}}^+\}_{(n,m)\in\mathbb{N}^2}.$$

Let us remark moreover that if $P$ is a polynomial with coefficients in $W[[X]]$ then $\tau(P) = P$ so that $Y$ cannot be a root of $P$ since otherwise $\tau^n(Y) = Y(1+X)^n$ would be another one, for any $n \in \mathbb{Z}$. Thus, $X$ and $Y$ are algebraically independent.

Let $W[[X,Y]]$ denote the subring of $\tilde{\mathbf{A}}^+$ consisting in sequences in $X$ and $Y$ ; it is stable under the actions of $G_K$ and $\varphi$ which are given by:

$$g(1+X) = (1+X)^{\chi(g)} \quad \text{and} \quad g(Y) = Y(1+X)^{\psi(g)}$$
$$\varphi(X) = (1+X)^p - 1 \quad \text{and} \quad \varphi(Y) = Y^p.$$

Let $\mathbf{A}_{\mathbb{Q}_p}$ denote the $p$-adic completion of $\mathbb{Z}_p[[X]][\frac{1}{X}]$, it consists in the set

$$\mathbf{A}_{\mathbb{Q}_p} = \left\{ \sum_{n\in\mathbb{Z}} a_n X^n |\ \forall n \in \mathbb{Z},\ a_n \in \mathbb{Z}_p \text{ and } a_n \underset{n\to-\infty}{\longrightarrow} 0 \right\}.$$

It is a local $p$-adic, complete subring of $\tilde{\mathbf{A}}$, with residue field $\mathbb{F}_p((\varepsilon-1))$. Define $\mathbf{A}$ the $p$-adic completion of the maximal unramified extension of $\mathbf{A}_{\mathbb{Q}_p}$ in $\tilde{\mathbf{A}}$. Its residue field is then the separable closure of $\mathbb{F}_p((\varepsilon-1))$ in $\tilde{\mathbf{E}}$. Denote this field by $\mathbf{E}$. It is a dense subfield of $\tilde{\mathbf{E}}$.

### 1.3. $p$-adic periods.

We refer to Fontaine's [14] for further details on these rings. The map

$$\theta : \sum_{n\geqslant 0} p^n[r_n] \in \tilde{\mathbf{A}}^+ \mapsto \sum_{n\geqslant 0} p^n r_n^{(0)} \in \mathcal{O}_{\mathbb{C}_p}$$

is onto, with kernel $W^1(\tilde{\mathbf{E}}^+)$ a principal ideal of $\tilde{\mathbf{A}}^+$ generated, for instance, by $\omega = X/\varphi^{-1}(X)$. Define

$$B_{dR}^+ = \varprojlim_n (\tilde{\mathbf{A}}^+ \otimes \mathbb{Q}_p)/(W^1(\tilde{\mathbf{E}}^+) \otimes \mathbb{Q}_p)^n$$

the completion of $\tilde{\mathbf{A}}^+ \otimes \mathbb{Q}_p$ with respect to the $W^1(\tilde{\mathbf{E}}^+)$-adic topology. The action of $G_K$ on $\tilde{\mathbf{A}}^+$ extends by continuity to $B_{dR}^+$. The sequence $\log[\varepsilon] = \sum_{n\geqslant 1}(-1)^{n+1}\frac{X^n}{n}$ converges in $B_{dR}^+$ towards an element denoted by $t$. Define $B_{dR} = B_{dR}^+[1/t]$. It is the fraction field of $B_{dR}^+$. It is still endowed with an action of $G_K$ for which $B_{dR}^{G_K} = K$ and with a compatible, decreasing, exhaustive filtration $\mathrm{Fil}^k B_{dR} = t^k B_{dR}^+$.

Define $A_{crys}$ to be the $p$-adic completion of the divided powers envelop of $\tilde{\mathbf{A}}^+$ with respect to $W^1(\tilde{\mathbf{E}}^+)$. It consists in the sequences $\sum_{n\geqslant 0} a_n \frac{\omega^n}{n!}$ with $a_n \in \tilde{\mathbf{A}}^+$ and $a_n \to 0$ $p$-adically. It is naturally a subring of $B_{dR}$. The sequence defining $t$ converges in $A_{crys}$, set $B_{crys}^+ = A_{crys} \otimes \mathbb{Q}_p$ and $B_{crys} = B_{crys}^+[1/t] = A_{crys}[1/t]$.

These rings, endowed with their $p$-adic topology, come with a continuous action of $G_K$, the filtration induced by the one on $B_{dR}$, and a Frobenius $\varphi$ extending by continuity the one on $\tilde{\mathbf{A}}^+$. Note that $B_{crys}^{G_K} = K_0$.

We call a *$\mathbb{Z}_p$-adic representation of $G_K$* any finitely generated $\mathbb{Z}_p$-module with a linear, continuous action of $G_K$ and a *$p$-adic representation of $G_K$* any finite dimensional $\mathbb{Q}_p$-vector space with a linear, continuous action of $G_K$. A $\mathbb{Z}_p$-adic representation is then turned into a $p$-adic representation by tensorizing by $\mathbb{Q}_p$.

Let $V$ be a $p$-adic representation of $G_K$. Let us introduce $D_{crys}(V) := (V \otimes_{\mathbb{Q}_p} B_{crys})^{G_K}$. It is a $K_0$-vector space of dimension lower or equal to the dimension of $V$ on $\mathbb{Q}_p$. The representation $V$ is said to be *crystalline* when these dimensions are equal. We say as well that a $\mathbb{Z}_p$-adic representation $V$, free over $\mathbb{Z}_p$, is crystalline when so is the $p$-adic representation $V \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

## 1.4. Fontaine's theory

Let $R$ be a topological ring with a linear, continuous action of some group $\Gamma$ and a continuous Frobenius $\varphi$ commuting with the action of $\Gamma$. Call a *$(\varphi, \Gamma)$-module on $R$* any finitely generated $R$-module $M$ with commuting semi-linear actions of $\Gamma$ and $\varphi$. A $(\varphi, \Gamma)$-module on $R$ is moreover said *étale* if the image of $\varphi$ generates $M$ as an $R$-module: $R\varphi(M) = M$.

### 1.4.1. The classical case

Let us recall the theory of $(\varphi,\Gamma)$-modules introduced by Fontaine in [13]. Set $\mathbf{A}_K = \mathbf{A}^{G_{K_\infty}}$. Define functors

$$D : V \mapsto D(V) = (\mathbf{A} \otimes_{\mathbb{Z}_p} V)^{G_{K_\infty}}$$

from the category of $\mathbb{Z}_p$-adic representations of $G_K$ to the one of $(\varphi,\Gamma_K)$-modules on $\mathbf{A}_K$ and

$$V : M \mapsto V(M) = (\mathbf{A} \otimes_{\mathbf{A}_K} M)^{\varphi=1}$$

from the category of étale $(\varphi,\Gamma_K)$-modules on $\mathbf{A}_K$ to the one of $\mathbb{Z}_p$-adic representations of $G_K$. The following theorem was shown by Fontaine ([13]):

Theorem 1.2. — *The following natural maps are isomorphisms*

$$\begin{aligned} \mathbf{A} \otimes_{\mathbf{A}_K} D(V) &\rightarrow \mathbf{A} \otimes_{\mathbb{Z}_p} V \\ \mathbf{A} \otimes_{\mathbb{Z}_p} V(M) &\rightarrow \mathbf{A} \otimes_{\mathbf{A}_K} M. \end{aligned}$$

*In particular, $D$ and $V$ are quasi-inverse equivalences of categories between the category of $\mathbb{Z}_p$-adic representations of $G_K$ and the one of étale $(\varphi,\Gamma_K)$-modules on $\mathbf{A}_K$.*

**Example.** Let us define the false Tate curve (or Tate's representation) by $V_{Tate} = \mathbb{Z}_p e_1 + \mathbb{Z}_p e_2$ with the action of $G_K$ given for all $g \in G_K$ by $g(e_1) = \chi(g)e_1$ and $g(e_2) = \psi(g)e_1 + e_2$ where $\chi$ is the cyclotomic character and $\psi$ is defined in Paragraph 1.1. The name "false Tate curve" comes from the similarity of this module with the Tate module of an elliptic curve with split multiplicative reduction at $p$.

The $(\varphi,\Gamma_K)$-module of the false Tate curve admits a basis $(1\otimes e_1, b\otimes e_1 + 1\otimes e_2)$ where $b \in \mathbf{A}_L$ satisfies $(\tau - 1)b = -1$. However $V_{Tate}$ is not potentially crystalline, and then, because of the main result of [27], not *of finite height*, which means $b \notin \mathbf{A}_L^+ = \mathbf{A}_L \bigcap \tilde{\mathbf{A}}^+$.

We want to build a $(\varphi,\Gamma)$-module which furnishes more information (which will then be redundant but easier to use) on the behavior of the associated representation in the extension $K_\pi/K$ or in its Galois closure $L/K$. For this, we want $\Gamma = G_\infty$.

### 1.4.2. The metabelian case

Suppose $\mathbf{A}' = \mathbf{A}$ or $\mathbf{A}' = \tilde{\mathbf{A}}$. Then, $\mathbf{A}'$ is a complete $p$-adic valuation ring, stable under both $G_K$ and $\varphi$. Its residue field $\mathbf{E}' = \mathbf{E}$ or $\tilde{\mathbf{E}}$ is separably closed. Set $\mathbf{A}'_L = \mathbf{A}'^{G_L}$ ; if $\mathbf{E}'_L = \mathbf{E}'^{G_L}$ then $\mathbf{A}'_L$ is a complete $p$-adic valuation ring with residue field $\mathbf{E}'_L$. For any $\mathbb{Z}_p$-adic representation $V$ of $G_K$, define

$$D'_L(V) = (\mathbf{A}' \otimes_{\mathbb{Z}_p} V)^{G_L}$$

and for any $(\varphi, G_\infty)$-module $D$, étale over $\mathbf{A}'_L$,

$$V'_L(D) = (\mathbf{A}' \otimes_{\mathbf{A}'_L} D)^{\varphi=1}.$$

Denote these functors by $D_L$ and $V_L$ when $\mathbf{A}' = \mathbf{A}$ and by $\widetilde{D}_L$ and $\widetilde{V}_L$ when $\mathbf{A}' = \tilde{\mathbf{A}}$. Remark that $D'_L(V)$ and $D(V) \otimes_{\mathbf{A}_K} \mathbf{A}'_L$ are $(\varphi, G_\infty)$-modules over $\mathbf{A}'_L$, the latter being étale. The following theorem shows that they are indeed isomorphic and assures that $D'_L$ is a good equivalent for $D$ in the metabelian case.

Theorem 1.3. —

(1) *The natural map $\iota : D(V) \otimes_{\mathbf{A}_K} \mathbf{A}'_L \to D'_L(V)$ is an isomorphism of $(\varphi, G_\infty)$-modules étale over $\mathbf{A}'_L$.*
(2) *Functors $D'_L$ and $V'_L$ are quasi-inverse equivalences of categories between the category of $\mathbb{Z}_p$-adic representations of $G_K$ and the one of étale $(\varphi, G_\infty)$-modules on $\mathbf{A}'_L$.*

*Proof.* — Because of Theorem 1.2 and after extending scalars, the natural map $D(V) \otimes_{\mathbf{A}_K} \mathbf{A}' \to V \otimes_{\mathbb{Z}_p} \mathbf{A}'$ is an isomorphism. Taking Galois invariants, we get an isomorphism

$$D(V) \otimes_{\mathbf{A}_K} \mathbf{A}'_L = (D(V) \otimes_{\mathbf{A}_K} \mathbf{A}')^{G_L} \xrightarrow{\sim} (V \otimes_{\mathbb{Z}_p} \mathbf{A}')^{G_L} = D'_L(V).$$

The functor $D'_L$ is then the composite of $D$ with the scalar extension $\otimes_{\mathbf{A}_K} \mathbf{A}'_L$. Theorem 1.2 then shows that it is fully faithful. Fontaine's computation (cf. [13, Proposition 1.2.6.]) applies and shows that $D'_L$ is essentially surjective on the category of étale $(\varphi, G_\infty)$-modules over $\mathbf{A}'_L$. The fact that $V'_L$ is a quasi-inverse of $D'_L$ still follows from Theorem 1.2. □

Corollary 1.4. — *The scalar-extension functor $D \mapsto D \otimes_{\mathbf{A}_K} \mathbf{A}'_L$ induces an equivalence*

$$\{\textit{étale}(\varphi, \Gamma_K) - \textit{modules over } \mathbf{A}_K\} \to \{\textit{étale}(\varphi, G_\infty) - \textit{modules over } \mathbf{A}'_L\}$$

**Example.** The $(\varphi, G_\infty)$-module of Tate's representation admits a trivial basis $(1 \otimes e_1, 1 \otimes e_2)$.

## 1.5. Galois Cohomology

### 1.5.1. Statement of the theorem

We suppose from now on $p$ odd.

Recall the classical case. Let $D(V)$ be the $(\varphi, \Gamma_K)$-module on $\mathbf{A}_K$ associated with a representation $V$. Fix $\gamma$ a topological generator of $\Gamma_K$. Herr introduced in [16] the complex

$$0 \longrightarrow D(V) \xrightarrow{f_1} D(V) \oplus D(V) \xrightarrow{f_2} D(V) \longrightarrow 0$$

with maps $f_1 = \begin{pmatrix} \varphi - 1 \\ \gamma - 1 \end{pmatrix}$ and $f_2 = (\gamma - 1, 1 - \varphi)$. He showed that the homology of this complex canonically and functorially identifies with the Galois cohomology of the representation $V$. This identification was explicitly given in [9] and [4] for the first cohomology group by associating with the class of a pair $(x, y)$ of elements in $D(V)$ satisfying $(\gamma - 1)x = (\varphi - 1)y$ the class of the cocycle

$$\sigma \mapsto -(\sigma - 1)b + \frac{\gamma^n - 1}{\gamma - 1} y$$

where $b \in V \otimes_{\mathbb{Z}_p} \mathbf{A}$ is a solution of $(\varphi - 1)b = x$ and $\sigma_{|\Gamma_K} = \gamma^n$ for some $n \in \mathbb{Z}_p$.

There still exists such a complex in the metabelian case. Since $G_\infty$ has dimension 2, it will be a bit longer. Let $M$ be a given étale $(\varphi, G_\infty)$-module over $\mathbf{A}'_L$. Let us associate with $M$ the four terms complex $C_{\varphi,\gamma,\tau}(M)$:

$$0 \longrightarrow M \xrightarrow{\alpha} M \oplus M \oplus M \xrightarrow{\beta} M \oplus M \oplus M \xrightarrow{\eta} M \longrightarrow 0$$

where

$$\alpha = \begin{pmatrix} \varphi - 1 \\ \gamma - 1 \\ \tau - 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} \gamma - 1 & 1 - \varphi & 0 \\ \tau - 1 & 0 & 1 - \varphi \\ 0 & \tau^{\chi(\gamma)} - 1 & \delta - \gamma \end{pmatrix},$$

$$\eta = (\tau^{\chi(\gamma)} - 1, \delta - \gamma, \varphi - 1)$$

with $\delta = (\tau^{\chi(\gamma)} - 1)(\tau - 1)^{-1} \in \mathbb{Z}_p[[\tau - 1]]$ defined as follows: set

$$\binom{u}{n} = \frac{u.(u-1)...(u-n+1)}{n!} \in \mathbb{Z}_p \text{ for all } u \in \mathbb{Z}_p \text{ and all } n \in \mathbb{N}.$$

Then $\tau^{\chi(\gamma)} = \sum_{n \geqslant 0} \binom{\chi(\gamma)}{n} (\tau - 1)^n$ since $\tau^{p^n}$ converges to 1 in $G_\infty$, and thus $\tau - 1$ is topologically nilpotent in $\mathbb{Z}_p[[G_\infty]]$. So

$$\delta = \frac{\tau^{\chi(\gamma)} - 1}{\tau - 1} = \sum_{n \geqslant 1} \binom{\chi(\gamma)}{n} (\tau - 1)^{n-1}.$$

The purpose of this paragraph is to show

Theorem 1.5. — *Let $V$ be a $\mathbb{Z}_p$-adic representation of $G_K$.*

i) *The homology of the complex $C_{\varphi,\gamma,\tau}(D_L(V))$ canonically and functorially identifies with the continuous Galois cohomology of $V$.*

ii) *Explicitly, let $(x,y,z) \in Z^1(C_{\varphi,\gamma,\tau}(D_L(V)))$, let $b$ be a solution in $V \otimes \mathbf{A}'$ of $(\varphi - 1)b = x$, then the identification above associates with the class of the triple $(x,y,z)$ the class of the cocycle:*

$$c \ : \ \sigma \mapsto c_\sigma = -(\sigma-1)b + \gamma^n \frac{\tau^m - 1}{\tau - 1} z + \frac{\gamma^n - 1}{\gamma - 1} y, \qquad \text{if } \sigma_{|G_\infty} = \gamma^n \tau^m.$$

### 1.5.2. Proof of Theorem 1.5 *i)*

The functor $F^\bullet$ which associates with a $\mathbb{Z}_p$-adic representation $V$ the homology of the complex $C_{\varphi,\gamma,\tau}(D_L(V))$ is a cohomological functor coinciding in degree 0 with the continuous Galois cohomology of $V$:

$$H^0(C_{\varphi,\gamma,\tau}(D_L(V)) = D_L(V)_{\varphi=1,\gamma=1,\tau=1} = V^{G_K}.$$

The proof consists then in showing that it is effaceable. In order to do that, we would like to work with a category with sufficiently many injectives and to see $V$ as a submodule of an explicit injective, its induced module, which is known to be cohomologically trivial. But the category of $\mathbb{Z}_p$-adic representations of $G_K$ doesn't admit induced modules. We will then work modulo $p^r$ for a fixed $r$, and even in the category of direct limits of $p^r$-torsion representations and deduce the result by passing to the limit. We have to show that the homology of the complex associated with an induced module concentrates in degree 0, which shows *a fortiori* the effaceability of $F^\bullet$. We will yet write this explicitly, which will let us get the second part of the theorem, and, then, an explicit description of the cup-product in terms of the complex.

Let $M_{G_K,p^r-tor}$ be the category of discrete $p^r$-torsion $G_K$-modules, it is also the category of direct limits of finite $p^r$-torsion $G_K$-modules or also the one of discrete $\mathbb{Z}/p^r\mathbb{Z}[[G_K]]$-modules. Let us remark that the functor $D_L$ extends to an equivalence of categories from this category to the one of direct limits of $p^r$-torsion étale $(\varphi, G_\infty)$-modules over $\mathbf{A}'_L$. Note finally that this category is stable under passing to the induced module:

Lemma 1.6. — *Let $V$ be an object of $M_{G_K,p^r-tor}$, define the induced module associated with $V$ by $\mathrm{Ind}_{G_K}(V) := \mathcal{F}_{cont}(G_K, V)$ the set of all continuous maps from $G_K$ to $V$. Endow $\mathrm{Ind}_{G_K}(V)$ with the discrete topology* *and the action of* $G_K$*:*

$$\begin{array}{rcl} G_K \times \mathrm{Ind}_{G_K}(V) & \to & \mathrm{Ind}_{G_K}(V) \\ g.\eta & = & [x \mapsto \eta(x.g)]. \end{array}$$

*Then* $\mathrm{Ind}_{G_K}(V)$ *is an object of* $M_{G_K, p^r-tor}$ *and* $V$ *canonically injects in* $\mathrm{Ind}_{G_K}(V)$*.*

*Proof.* — The first part is well-known. See [23] for details. The injection is given by sending $v \in V$ on $\eta_v \in \mathrm{Ind}_{G_K}(V)$ such that $\forall g \in G_K, \eta_v(g) = g(v)$. □

Let $F^i$ denote the functor $H^i(C_{\varphi,\gamma,\tau}(D_L(-)))$. The snake lemma gives for any short exact sequence $0 \to V \to V'' \to V' \to 0$ in $M_{G_K, p^r-tor}$ a long exact sequence

$$0 \to F^0(V) \to F^0(V'') \to F^0(V') \to F^1(V) \to F^1(V'') \to \cdots$$

which shows that $F^\bullet$ is a cohomological functor. Let us show that it coincides with the long exact cohomology sequence when $V'' = \mathrm{Ind}_{G_K}(V)$. We use the following

Proposition 1.7. — *Let* $U = \mathrm{Ind}_{G_K}(V)$ *be an induced module in the category* $M_{G_K, p^r-tor}$*, then* $F^i(U) = H^i(K, U) = 0$ *for all* $i > 0$*.*

Point $i$) of the theorem follows from this result: the commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & F^0(V) & \longrightarrow & F^0(\mathrm{Ind}_{G_K}(V)) & \longrightarrow & F^0(V') & \longrightarrow & F^1(V) & \longrightarrow 0 \\ & \| & & \| & & \| & & & \\ 0 \longrightarrow & H^0(K,V) & \longrightarrow & H^0(K, \mathrm{Ind}_{G_K}(V)) & \longrightarrow & H^0(K,V') & \longrightarrow & H^1(K,V) & \longrightarrow 0 \end{array}$$

shows that $H^1(K,V) \simeq F^1(V)$. And in higher dimension the vanishing of $F^i(\mathrm{Ind}_{G_K}(V))$ and $H^i(K, \mathrm{Ind}_{G_K}(V))$ proves both that $F^k(V') = F^{k+1}(V)$ and $H^k(K,V') = H^{k+1}(K,V)$. Thus, by induction, $F^i(V) = H^i(K,V)$ holds for all $i \in \mathbb{N}$ and for any module $V$ in $M_{G_K, p^r-tor}$.

*Proof of the proposition.* — The Galois cohomology part is classical (cf. [21, VII, Proposition 1], or [22, I.2.5]). For the second part, let us begin with a lemma.

Lemma 1.8. — *The map* $\varphi - 1 : \mathbf{A}' \to \mathbf{A}'$ *admits a continuous section.*

*Proof of the lemma.* — First, remark that $\varphi$ is topologically nilpotent on $\mathfrak{m}_{\tilde{\mathbf{E}}}$, so that $\varphi - 1$ is there invertible with inverse $\psi = -\sum_{n \geqslant 0} \varphi^n$. Let us deduce that there exists a continuous section to $\varphi - 1$ on $\tilde{\mathbf{E}}$: write

$$\tilde{\mathbf{E}} = \bigcup_{i \in I} i + \mathfrak{m}_{\tilde{\mathbf{E}}}$$

where $I$ is a set of representatives of $\tilde{\mathbf{E}}/\mathfrak{m}_{\tilde{\mathbf{E}}}$. Choose for any $i \in I$ a $y_i \in \tilde{\mathbf{E}}$ such that $(\varphi - 1)y_i = i$. Then $\overline{s} : \tilde{\mathbf{E}} \to \tilde{\mathbf{E}}$ defined by

$$\overline{s}(i + u) = y_i + \psi(u), i \in I, u \in \mathfrak{m}_{\tilde{\mathbf{E}}}$$

is a continuous section of $\varphi - 1$.

Now we have to lift this section mod $p$ to a section on $\tilde{\mathbf{A}}$. We will do it by successive approximations mod $p^n$. First consider $s_1 : \tilde{\mathbf{A}} \to \tilde{\mathbf{A}}$ defined by $s_1(x) = [\overline{s}(\overline{x})]$. Suppose now that we have built such a continuous map $s_n : \tilde{\mathbf{A}} \to \tilde{\mathbf{A}}$ satisfying for any $x \in \tilde{\mathbf{A}}$, $(\varphi - 1) \circ s_n(x) \equiv x \mod p^n$. Then there is a continuous $f_n : \tilde{\mathbf{A}} \to \tilde{\mathbf{A}}$ such that

$$\forall x \in \tilde{\mathbf{A}},\ (\varphi - 1) \circ s_n(x) \equiv x + p^n f_n(x) \mod p^{n+1}.$$

Then $s_{n+1}$ can be defined as $s_{n+1} = s_n + p^n s_1 \circ f_n$ so that the series $(s_n)$ converges to the desired section $s$.

Finally, the restriction of $s$ to $\mathbf{A}$ is obviously still a continuous section of $\varphi - 1$. □

Lemma 1.9. — *For any $V$ in $M_{G_K, p^r-tor}$ and $\alpha \in \mathbb{Z}_p^*$, there are short exact sequences:*

$$0 \longrightarrow \mathrm{Ind}_{G_\infty}(V) \longrightarrow D_L(\mathrm{Ind}_{G_K}(V)) \xrightarrow{\varphi-1} D_L(\mathrm{Ind}_{G_K}(V)) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Ind}_{\Gamma_K}(V) \longrightarrow \mathrm{Ind}_{G_\infty}(V) \xrightarrow{\tau^\alpha - 1} \mathrm{Ind}_{G_\infty}(V) \longrightarrow 0$$

$$0 \longrightarrow V^{G_K} \longrightarrow \mathrm{Ind}_{\Gamma_K}(V) \xrightarrow{\gamma-1} \mathrm{Ind}_{\Gamma_K}(V) \longrightarrow 0.$$

*Proof of the lemma.* — Tensorize with $\mathrm{Ind}_{G_K}(V)$ the short exact sequence $0 \to \mathbb{Z}_p \to \mathbf{A}' \xrightarrow{\varphi-1} \mathbf{A}' \to 0$. The existence of a continuous section of $\varphi - 1$ permits, taking Galois invariants, to get a long exact sequence beginning with

$$0 \longrightarrow \mathrm{Ind}_{G_K}(V)^{G_L} \longrightarrow \mathcal{D} \xrightarrow{\varphi-1} \mathcal{D} \longrightarrow H^1(L, \mathrm{Ind}_{G_K}(V))$$

where $\mathcal{D} = D_L(\mathrm{Ind}_{G_K}(V))$. The kernel is $\mathrm{Ind}_{G_K}(V)^{G_L} = \mathrm{Ind}_{G_\infty}(V)$. It remains to show the vanishing of $H^1(G_L, \mathrm{Ind}_{G_K}(V))$. But it is the direct limit $\lim H^1(G_M, \mathrm{Ind}_{G_K}(V))$ taken over the set of all finite Galois sub-extensions $M$ of $L/K$ (cf.[22, Chapitre I, Proposition 8]). Indeed, the sub-Galois groups $G_M$ of $G_K$ form, for inclusion, a projective system with limit $\bigcap G_M = G_L$ and this system is compatible with the inductive system formed by the $G_M$-modules by restriction $\mathrm{Ind}_{G_K}(V)$ whose limit is the $G_L$-module by restriction $\mathrm{Ind}_{G_K}(V)$.

To prove the lemma, it suffices then to show for any finite Galois extension $M/K$ included in $L$ the vanishing of $H^1(G_M, \mathrm{Ind}_{G_K}(V))$. But, $G_M$ being open in $G_K$, we have the finite decomposition $G_K = \bigcup_{\overline{g}\in \mathrm{Gal}(M/K)} gG_M$ from which we deduce that, as a $G_M$-module, $\mathrm{Ind}_{G_K}(V)$ admits a decomposition

$$\mathrm{Ind}_{G_K}(V) = \bigoplus_{\overline{g}\in \mathrm{Gal}(M/K)} \mathcal{F}_{cont}(gG_M, V) \simeq \bigoplus_{\mathrm{Gal}(M/K)} \mathrm{Ind}_{G_M}(V).$$

Thus $H^1(G_M, \mathrm{Ind}_{G_K}(V)) \simeq \bigoplus_{\mathrm{Gal}(M/K)} H^1(G_M, \mathrm{Ind}_{G_M}(V))$ and the summands of the right-hand side are zero because of the first part of the proposition. On the other hand, $\tau^\alpha$ topologically generates $\mathrm{Gal}(L/K_\infty)$, thus the complex $\mathrm{Ind}_{G_\infty}(V) \xrightarrow{\tau^\alpha-1} \mathrm{Ind}_{G_\infty}(V)$ computes $H^\bullet(\mathrm{Gal}(L/K_\infty), \mathrm{Ind}_{G_\infty}(V))$. We get the kernel $\mathrm{Ind}_{G_\infty}(V)^{\mathrm{Gal}(L/K_\infty)} \simeq \mathrm{Ind}_{\Gamma_K}(V)$. And the vanishing of $H^1(\mathrm{Gal}(L/K_\infty), \mathrm{Ind}_{G_\infty}(V))$ is shown as the one of $H^1(L, \mathrm{Ind}_{G_K}(V))$ above.

Finally, the complex $\mathrm{Ind}_{\Gamma_K}(V) \xrightarrow{\gamma-1} \mathrm{Ind}_{\Gamma_K}(V)$ computes the cohomology $H^\bullet(\Gamma_K, \mathrm{Ind}_{\Gamma_K}(V))$. The surjectivity of $\gamma - 1$ still comes from the nullity of $H^1(\Gamma_K, \mathrm{Ind}_{\Gamma_K}(V))$, proved as before. □

The surjectivity of $(\varphi - 1)$ on $D_L(U)$ proves that $F^3(U) = 0$ and gives:

$$\mathrm{Ker}\,\eta = \{(x,y,z); x,y \in D_L(U)\ ,\ z \in (1-\varphi)^{-1}((\tau^{\chi(\gamma)}-1)(x)+(\delta-\gamma)(y))\}.$$

Let $x, y \in D_L(U)$ and fix $x', y' \in D_L(U)$ such that $(1-\varphi)(x') = x$ and $(1-\varphi)(y') = y$ ; proving that $F^2(U) = 0$ consists then in proving

$$\forall u \in \mathrm{Ind}_{G_\infty}(V),\ (x, y, (\tau^{\chi(\gamma)} - 1)(x') + (\delta - \gamma)(y') + u \otimes 1) \in \mathrm{Im}\ \beta.$$

But $(\tau^{\chi(\gamma)} - 1)$ is surjective on $\mathrm{Ind}_{G_\infty}(V)$, thus it suffices to consider $\beta(0, x' + u', y')$ with $u'$ chosen so that $(\tau^{\chi(\gamma)} - 1)(u') = u$.

Let $(u, v, w) \in \mathrm{Ker}(\beta)$, *i.e.* satisfying:

$$\begin{cases} (\gamma - 1)u = (\varphi - 1)v \\ (\tau - 1)u = (\varphi - 1)w \\ (\tau^{\chi(\gamma)} - 1)v = (\gamma - \delta)w \end{cases}$$

Fix $x_0 \in D_L(U)$ such that $(\varphi - 1)x_0 = u$. The first two relations show that $v_0 := v - (\gamma - 1)x_0$ and $w_0 := w - (\tau - 1)x_0$ lie in the kernel of $\varphi - 1$ thus in $\mathrm{Ind}_{G_\infty}(V)$, and satisfy furthermore $(\tau^{\chi(\gamma)} - 1)v_0 = (\gamma - \delta)w_0$. Choose now $\eta \in \mathrm{Ind}_{G_\infty}(V)$ such that $(\tau - 1)\eta = w_0$. Then

$$(\tau^{\chi(\gamma)} - 1)(\gamma - 1)\eta = (\gamma - \delta)(\tau - 1)\eta = (\tau^{\chi(\gamma)} - 1)v_0$$

so $v_0 - (\gamma-1)\eta \in \mathrm{Ind}_{\Gamma_K}(V)$ and $\exists\varepsilon \in \mathrm{Ind}_{\Gamma_K}(V)$ with $(\gamma-1)\varepsilon = v_0 - (\gamma-1)\eta$ thus $(\gamma-1)(\eta+\varepsilon) = v_0$ and $(\tau-1)(\eta+\varepsilon) = w_0$. Define then $x := x_0 + \eta + \varepsilon$, we have:

$$\begin{array}{rclclcl}(\varphi-1)x &=& (\varphi-1)x_0+(\varphi-1)(\eta+\varepsilon) &=& (\varphi-1)x_0 &=& u\\ (\gamma-1)x &=& (\gamma-1)x_0+(\gamma-1)(\eta+\varepsilon) &=& v-v_0+v_0 &=& v\\ (\tau-1)x &=& (\tau-1)x_0+(\tau-1)(\eta+\varepsilon) &=& w-w_0+w_0 &=& w\end{array}$$

so that $\alpha(x)=(u,v,w)$ which proves the proposition. □

### 1.5.3. Explicit Formulas

**Proof of Theorem 1.5** *ii*)**.** In order to make the isomorphism explicit, it suffices to do a diagram chasing following the snake lemma: let $(x,y,z)\in Z^1(C_{\varphi,\gamma,\tau}(D_L(V)))$, through the injection $D_L(V)\hookrightarrow D_L(\mathrm{Ind}_{G_K}(V))$, we can see $(x,y,z)$ as an element of $Z^1(C_{\varphi,\gamma,\tau}(D_L(\mathrm{Ind}_{G_K}(V))))$. The vanishing of $H^1(C_{\varphi,\gamma,\tau}(D_L(\mathrm{Ind}_{G_K}(V))))$ implies the existence of an element $b'\in D_L(\mathrm{Ind}_{G_K}(V))$ with $\alpha(b')=(x,y,z)$. Consider $\overline{b'}\in D_L(\mathrm{Ind}_{G_K}(V)/V)$ the reduction of $b'$ modulo $D_L(V)$,

$$\overline{b'}\in H^0(C_{\varphi,\gamma,\tau}(D_L(\mathrm{Ind}_{G_K}(V)/V)))=(\mathrm{Ind}_{G_K}(V)/V)^{G_K}.$$

Thus, if $\tilde b\in\mathrm{Ind}_{G_K}(V)$ lifts $\overline{b'}$, the image of $(x,y,z)$ in $H^1(K,V)$ is the class of the cocycle $c:\sigma\mapsto c_\sigma=(\sigma-1)\tilde b$. But we can choose $\tilde b=b'-b$ since $(\varphi-1)(b'-b)=x-x=0$ so that $b'-b\in\mathrm{Ind}_{G_K}(V)$ and then $b'-b$ lifts $\overline{b'}$. So if $\sigma_{|G_\infty}=\gamma^n\tau^m$, write

$$\begin{array}{rcl}c_\sigma &=& (\sigma-1)(b'-b)=-(\sigma-1)(b)+(\gamma^n\tau^m-1)b'\\ &=& -(\sigma-1)b+\gamma^n\dfrac{\tau^m-1}{\tau-1}z+\dfrac{\gamma^n-1}{\gamma-1}y.\end{array}$$

Let us finally show how to pass to the limit in order to get the result for a representation which is not necessarily torsion. Let $V$ be a $\mathbb{Z}_p$-adic representation of $G_K$. For all $r\geqslant 1$, $V_r=V\otimes\mathbb{Z}/p^r\mathbb{Z}$ is a $p^r$-torsion representation such that $V=\lim_{\leftarrow}V_r$. Then we know that the continuous cohomology of $V$ can be expressed as the limit:

$$\forall i\geqslant 0,\ H^i(K,V)=\varprojlim H^i(K,V_r)=\varprojlim F^i(V_r).$$

It suffices to show that for all $i\geqslant 0$, $F^i(V)=\lim_{\leftarrow}F^i(V_r)$. For short, let $H^i_r$ (resp. $B^i_r$, $Z^i_r$) denote $H^i(C_{\varphi,\gamma,\tau}(D_L(V_r)))$ (resp. $B^i(C_{\varphi,\gamma,\tau}(D_L(V_r)))$, $Z^i(C_{\varphi,\gamma,\tau}(D_L(V_r)))$). The maps in the Herr complex are $\mathbb{Z}_p$-linear so that in the category of $\mathbb{Z}_p$-modules there is an exact sequence $0\to B^i_r\to Z^i_r\to H^i_r\to 0$ from which is obtained the exact sequence

$$0\to\varprojlim B^i_r\to\varprojlim Z^i_r\to\varprojlim H^i_r\to\varprojlim{}^1 B^i_r$$

where $\varprojlim^1$ is the first derived functor of the functor $\varprojlim$. But for all $r$,

$$B_r^i \simeq B^i(C_{\varphi,\gamma,\tau}(D_L(V))) \otimes \mathbb{Z}/p^r\mathbb{Z}$$

so that the transition maps in the projective system $(B_r^i)$ are surjective, and then this system satisfies Mittag-Leffler conditions. Thus $\varprojlim^1 B_r^i = 0$ shows that the homology of the inverse limit is equal to the inverse limit of the homology, as desired.

**The explicit formula for $H^2$.**

Proposition 1.10. — *The identification between the homology of the complex $C_{\varphi,\gamma,\tau}(D_L(V))$ and the Galois cohomology of $V$ associates with $(a,b,c) \in Z^2(C_{\varphi,\gamma,\tau}(D_L(V)))$ the class of the 2-cocycle:*

$$(g,h) \mapsto s_g - s_{gh} + g s_h + \gamma^{n_1} \frac{\tau^{m_1}-1}{\tau-1} \frac{(\delta^{-1}\gamma)^{n_2}-1}{\delta^{-1}\gamma-1} \delta^{-1} c$$

*where $g_{|G_\infty} = \gamma^{n_1}\tau^{m_1}$, $h_{|G_\infty} = \gamma^{n_2}\tau^{m_2}$ and $s : G_K \to \mathbf{A}' \otimes V$ is if $\sigma_{|G_\infty} = \gamma^n \tau^m$, such that $s_\sigma = \phi\left(\frac{\gamma^n-1}{\gamma-1} a + \gamma^n \frac{\tau^m-1}{\tau-1} b\right)$ where $\phi$ is a continuous section of $\varphi - 1$.*

*Proof.* — The proof is, mutatis mutandis, the same as above. Let $\alpha = (a,b,c) \in Z^2(C_{\varphi,\gamma,\tau}(D_L(V)))$. Because of the injection $V \hookrightarrow \mathrm{Ind}_{G_K}(V)$, we can consider $\alpha \in Z^2(C_{\varphi,\gamma,\tau}(D_L(\mathrm{Ind}_{G_K}(V))))$.

The vanishing of $H^2(C_{\varphi,\gamma,\tau}(D_L(\mathrm{Ind}_{G_K}(V))))$ shows that $\alpha$ is a coboundary, *i.e.*, there is $\beta = (\eta_x, \eta_y, \eta_z) \in D_L(\mathrm{Ind}_{G_K}(V))^3$ such that

$$\begin{cases} a = (\gamma-1)\eta_x - (\varphi-1)\eta_y \\ b = (\tau-1)\eta_x - (\varphi-1)\eta_z \\ c = (\tau^{\chi(\gamma)}-1)\eta_x - (\gamma-\delta)\eta_y \end{cases}$$

It corresponds to the class of the reduction $\overline{\beta}$ of $\beta$ in $D_L(\mathrm{Ind}_{G_K}(V)/V)^3$ an element of $H^1(K, \mathrm{Ind}_{G_K}(V)/V)$. Its image in $H^2(K,V)$ is the element corresponding to $\alpha$. Let us compute it.

Let $\eta_b \in A \otimes \mathrm{Ind}_{G_K}(V)$ be such that $(\varphi-1)\eta_b = \eta_x$, then for $\sigma_{|G_\infty} = \gamma^n\tau^m$,

$$c_\sigma = -(\sigma-1)\eta_b + \gamma^n \frac{\tau^m-1}{\tau-1}\eta_z + \frac{\gamma^n-1}{\gamma-1}\eta_y$$

is a cocycle with values in $\mathrm{Ind}_{G_K}(V) + \mathbf{A}' \otimes V$ which reduction modulo $\mathbf{A}' \otimes V$ is a cocycle corresponding to $\overline{\beta}$. Let us fix $\sigma$ and calculate

$$\begin{aligned}(\varphi-1)r_\sigma &= -(\sigma-1)(\varphi-1)\eta_b + \gamma^n\frac{\tau^m-1}{\tau-1}(\varphi-1)\eta_z + \frac{\gamma^n-1}{\gamma-1}(\varphi-1)\eta_y \\ &= -(\gamma^n\tau^m-1)\eta_x + \gamma^n\frac{\tau^m-1}{\tau-1}((\tau-1)\eta_x - b) + \\ &\qquad + \frac{\gamma^n-1}{\gamma-1}((\gamma-1)\eta_x - a) \\ &= -\frac{\gamma^n-1}{\gamma-1}a - \gamma^n\frac{\tau^m-1}{\tau-1}b =: -\tilde{s}_\sigma\end{aligned}$$

Let us choose now a section $\phi$ of $\varphi-1$ and define a map $s : G_K \to A\otimes V$ via $s = \phi\circ\tilde{s}$. Then, $(\varphi-1)s = \tilde{s}$. The choice is unique modulo $\mathcal{F}_{cont}(G_K, V)$. Therefore, $r+s : G_K \to \mathrm{Ind}_{G_K}(V)$ is a lift of a cocycle corresponding to $\overline{\beta}$. Its image through the coboundary operator takes values in $V$, it is the desired 2-cocycle. It is written as

$$d(r+s)(g,h) = r_g + s_g - r_{gh} - s_{gh} + gr_h + gs_h$$

Let us compute the $r$ part. Obviously, for $g_{|G_\infty} = \gamma^{n_1}\tau^{m_1}$ and $h_{|G_\infty} = \gamma^{n_2}\tau^{m_2}$ we can write $r_g - r_{gh} + gr_h$ as

$$\gamma^{n_1}(\tau^{m_1}-1)\frac{\gamma^{n_2}-1}{\gamma-1}\eta_y + \gamma^{n_1}\left(\frac{\tau^{m_1}-1}{\tau-1} - \gamma^{n_2}\frac{\tau^{\chi(\gamma)^{-n_2}m_1}-1}{\tau-1}\right)\eta_z$$

Remark on the one hand that:

$$(\tau-1)\frac{\gamma^{n_2}-1}{\gamma-1} = \frac{(\delta^{-1}\gamma)^{n_2}-1}{\delta^{-1}\gamma-1}(\tau-1)$$

and on the other hand:

$$\gamma^{n_2}\frac{\tau^{\chi(\gamma)^{-n_2}m_1}-1}{\tau-1} = \frac{\tau^{m_1}-1}{\tau-1}(\delta^{-1}\gamma)^{n_2}$$

so that

$$\begin{aligned}r_g - r_{gh} + gr_h &= \gamma^{n_1}\frac{\tau^{m_1}-1}{\tau-1}\frac{(\delta^{-1}\gamma)^{n_2}-1}{\delta^{-1}\gamma-1}((\tau-1)\eta_y + \delta^{-1}(\delta-\gamma)\eta_z) \\ &= \gamma^{n_1}\frac{\tau^{m_1}-1}{\tau-1}\frac{(\delta^{-1}\gamma)^{n_2}-1}{\delta^{-1}\gamma-1}\delta^{-1}c.\end{aligned}$$

□

*Remark 1.11.* — In the classical case, with the class of $a$ is associated the class of the 2-cocycle:

$$(g_1, g_2) \mapsto \tilde{\gamma}^{n_1}(h-1)\frac{\tilde{\gamma}^{n_2}-1}{\tilde{\gamma}-1}\tilde{a}$$

where $(\varphi-1)\tilde{a} = a$, $\tilde{\gamma}$ lifts $\gamma$ in $G_K$, $g_1 = \tilde{\gamma}^{n_1}h$, $g_2 = \tilde{\gamma}^{n_2}h'$ with $h, h' \in G_{K_\infty}$ and $n_1, n_2 \in \mathbb{Z}_p$.

## 1.6. Explicit formulas for the cup-product

Herr gave in [17] explicit formulas for the cup-product in terms of his complex. The following theorem gives the formulas for the metabelian case:

THEOREM 1.12. — *Let $V$ and $V'$ be $\mathbb{Z}_p$-adic representations of $G_K$. The cup-product induces maps:*

(1) *Let $(a) \in H^0(C_{\varphi,\gamma,\tau}(D_L(V)))$ and $(a') \in H^0(C_{\varphi,\gamma,\tau}(D_L(V')))$,*

$$(a) \cup (a') = (a \otimes a') \in H^0(C_{\varphi,\gamma,\tau}(D_L(V \otimes V'))),$$

(2) *let $(x, y, z) \in H^1(C_{\varphi,\gamma,\tau}(D_L(V)))$ and $(a') \in H^0(C_{\varphi,\gamma,\tau}(D_L(V')))$,*

$$(x, y, z) \cup (a') = (x \otimes a', y \otimes a', z \otimes a') \in H^1(C_{\varphi,\gamma,\tau}(D_L(V \otimes V'))),$$

(3) *let $(a) \in H^0(C_{\varphi,\gamma,\tau}(D_L(V)))$ and $(x', y', z') \in H^1(C_{\varphi,\gamma,\tau}(D_L(V')))$,*

$$(a) \cup (x', y', z') = (a \otimes x', a \otimes y', a \otimes z') \in H^1(C_{\varphi,\gamma,\tau}(D_L(V \otimes V')))$$

(4) *let $(x, y, z) \in H^1(C_{\varphi,\gamma,\tau}(D_L(V)))$, $(x', y', z') \in H^1(C_{\varphi,\gamma,\tau}(D_L(V')))$, $(x, y, z) \cup (x', y', z') \in H^2(C_{\varphi,\gamma,\tau}(D_L(V \otimes V')))$ can be written as:*

$$(y \otimes \gamma x' - x \otimes \varphi y' \ , \ z \otimes \tau x' - x \otimes \varphi z' \ , \ \delta z \otimes \tau^{\chi(\gamma)} y' - y \otimes \gamma z' + \Sigma_{z,z'})$$

*where* $\Sigma_{z,z'} = \sum_{n \geqslant 1} \binom{\chi(\gamma)}{n+1} \sum_{k=1}^{n} \binom{n}{k} (\tau-1)^{k-1} z \otimes \tau^k (\tau-1)^{n-k} z'$.

*Proof.* — The only non trivial identity is the last one. We will use the construction of the previous paragraph and we can then suppose that $V$ and $V'$ are objects of $M_{G_K, p^r-tor}$. We will use the cup-product property $da \cup b = d(a \cup b)$ and the exact sequences

$$0 \to V \to \mathrm{Ind}_{G_K}(V) \to V'' \to 0$$

$$0 \to F^0(V) \to F^0(\mathrm{Ind}_{G_K}(V)) \to F^0(V'') \to F^1(V) \to 0.$$

Fix indeed $(x,y,z)$ and $(x',y',z')$ as in the theorem. Then there exists $a \in D_L(\mathrm{Ind}_{G_K}(V))$ satisfying $\alpha(a)=(x,y,z)$ and $\overline{a} \in (\mathrm{Ind}_{G_K}(V)/V)^{G_K}$. Thus $(x,y,z)\cup(x',y',z')$ is equal to

$$\begin{aligned}\alpha(a)\cup(x',y',z') &= d(\overline{a}\otimes x',\overline{a}\otimes y',\overline{a}\otimes z') \;=\; \beta(a\otimes x',a\otimes y',a\otimes z')\\ &= ((\gamma-1)(a\otimes x')-(\varphi-1)(a\otimes y'),\\ &\quad (\tau-1)(a\otimes x')-(\varphi-1)(a\otimes z'),\\ &\quad (\tau^{\chi(\gamma)}-1)(a\otimes y')-(\gamma-\delta)(a\otimes z'))\end{aligned}$$

Now we use the formal identity

$$(\sigma-1)(a\otimes b)=(\sigma-1)a\otimes\sigma b+a\otimes(\sigma-1)b. \tag{1.1}$$

The first term $(\gamma-1)a\otimes x'-(\varphi-1)a\otimes y'$ can be written as

$$(\gamma-1)a\otimes\gamma x'+a\otimes(\gamma-1)x'-(\varphi-1)a\otimes y'-a\otimes(\varphi-1)y'=y\otimes\gamma x'-x\otimes y'.$$

From a similar computation, we get for the second one

$$(\tau-1)(a\otimes x')-(\varphi-1)(a\otimes z')=z\otimes\gamma x'-x\otimes z'.$$

Let us finally write the computation of the third term. First, using (1.1), we get

$$(\tau^{\chi(\gamma)}-1)a\otimes y'=\delta z\otimes\tau^{\chi(\gamma)}y'+a\otimes(\gamma-\delta)z'$$

and

$$(\gamma-1)a\otimes z'=y\otimes\gamma z'+a\otimes(\gamma-1)z'.$$

It remains to compute $\delta(a\otimes z')$. Recall $\delta=\frac{\tau^{\chi(\gamma)}-1}{\tau-1}=\sum_{n\geqslant 1}\binom{\chi(\gamma)}{n}(\tau-1)^{n-1}$. Moreover, iterating (1.1), we get by induction:

$$(\sigma-1)^n(a\otimes b)=\sum_{k=0}^{n}\binom{n}{k}(\sigma-1)^k a\otimes\sigma^k(\sigma-1)^{n-k}b.$$

So

$$\delta(a\otimes z')=\sum_{n\geqslant 1}\binom{\chi(\gamma)}{n}\sum_{k=0}^{n-1}\binom{n-1}{k}(\tau-1)^k a\otimes\tau^k(\tau-1)^{n-1-k}z'$$

$$\delta(a\otimes z')=a\otimes\delta z'+\sum_{n\geqslant 1}\binom{\chi(\gamma)}{n}\sum_{k=1}^{n-1}\binom{n-1}{k}(\tau-1)^{k-1}z\otimes\tau^k(\tau-1)^{n-1-k}z'$$

whence the result. □

### 1.7. Kummer's map

In this paragraph, we suppose $\mathbf{A}' = \tilde{\mathbf{A}}$.

The purpose is to compute, in terms of the Herr complex, Kummer's map $\kappa : K^* \to H^1(K, \mathbb{Z}_p(1))$. More precisely, let $F(Y) \in \left(W[[Y]][\frac{1}{Y}]\right)^\times$, we will compute a triple $(x, y, z) \in Z^1(C_{\varphi,\gamma,\tau}(\tilde{\mathbf{A}}_L(1)))$ corresponding to the image $\kappa o \theta(F(Y))$ of $\theta(F(Y)) = F(\pi) \in K^*$. Remark that there exist $d \in \mathbb{Z}$ and $G(Y) \in (W[[Y]])^\times$ such that $F(Y) = Y^d G(Y)$. In fact $G(Y)$ can be written as the product of a $((\sharp k) - 1)$th root of unity (which doesn't play any role) and a series in $1 + (p) \subset W[[Y]]$.

Denote $\alpha = \theta(F(Y)) \in K^*$ and choose $\tilde{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots) \in \tilde{\mathbf{E}}$ such that $\alpha_0 = \alpha$. Then $\frac{\tilde{\alpha}}{\rho^d} \in \tilde{\mathbf{E}}^+$ thus $\frac{[\tilde{\alpha}]}{Y^d} \in \tilde{\mathbf{A}}^+$ and for all $\sigma \in G_K$, there exists $\psi_\alpha(\sigma) \in \mathbb{Z}_p$ such that

$$\sigma(\tilde{\alpha}) = \tilde{\alpha}\varepsilon^{\psi_\alpha(\sigma)}.$$

The map $\sigma \mapsto \varepsilon^{\psi_\alpha(\sigma)}$ is in fact a cocycle computing $\kappa(\alpha)$. So

$$\sigma([\tilde{\alpha}]) = [\tilde{\alpha}](1+X)^{\psi_\alpha(\sigma)} \text{ where } \kappa(\alpha) = \varepsilon^{\psi_\alpha} \in H^1(K, \mathbb{Z}_p(1)).$$

Since $\frac{[\tilde{\alpha}]}{F(Y)} \in \tilde{\mathbf{A}}^+$ and $\theta\left(\frac{[\tilde{\alpha}]}{F(Y)}\right) = 1$, the series defining $\log \frac{[\tilde{\alpha}]}{F(Y)}$ converges in $\mathrm{Fil}^1 B_{crys}$.

For all $h \in G_L$, $(h-1)\log \frac{[\tilde{\alpha}]}{F(Y)} = \psi_\alpha(h)t$ where $t = \log(1+X)$. Set $\tilde{b} = \left(\log \frac{[\tilde{\alpha}]}{F(Y)}\right)/t \in \mathrm{Fil}^0 B_{crys}$. Then

$$\psi_\alpha(h) = (h-1)(\tilde{b}) \ \forall h \in G_L.$$

And $(\varphi - 1)(\tilde{b}) = \frac{1}{t} f(Y)$ where $f(Y) = \mathscr{L}(F) = \frac{1}{p}\log \frac{F(Y)^p}{\varphi(F(Y))} \in W[[Y]]$.

Choose $b_1 \in \tilde{\mathbf{A}}$ a solution of $(\varphi - 1)b_1 = -\frac{f(Y)}{X}$. Let $X_1 = \varphi^{-1}(X) = [\varepsilon^{\frac{1}{p}}] - 1$, and $\omega = \frac{X}{X_1} \in \tilde{\mathbf{A}}^+$ then $(\varphi - \omega)(b_1 X_1) = -f(Y)$. Write $b_1 X_1 = \sum_{n \geqslant 0} p^n [a_n]$. Reducing modulo $p$ the previous identity yields to an equation of the form $a_0^p - \overline{\omega} a_0 = -\overline{f(Y)}$ and since $\tilde{\mathbf{E}}^+$ is integrally closed, $a_0 \in \tilde{\mathbf{E}}^+$. Let us deduce that $a_n \in \tilde{\mathbf{E}}^+$ for any $n \in \mathbb{N}$ by induction. We have the identity

$$\sum_{n \in \mathbb{N}} p^n [a_n^p] - \omega \sum_{n \in \mathbb{N}} p^n [a_n] = -f(Y) = \sum_{n \in \mathbb{N}} p^n [b_n].$$

Suppose that there is $u_n \in \tilde{\mathbf{A}}^+$ with

$$[a_n^p] - \omega[a_n] \equiv u_n \mod p.$$

Then $a_n$ still belongs to $\tilde{\mathbf{E}}^+$ because it is integrally closed and

$$p[a_{n+1}^p] - \omega p[a_{n+1}] \equiv u_n - [a_n^p] + \omega[a_n] + p[b_n] \mod p^2.$$

Thus $u_{n+1} = \frac{u_n - [a_n^p] + \omega[a_n]}{p} + [b_n] \in \tilde{\mathbf{A}}^+$.

Finally, it comes $b_1 X_1 \in \tilde{\mathbf{A}}^+$. But $\frac{1}{X_1} \in \mathrm{Fil}^0 B_{crys}$, namely the series

$$\frac{t}{X_1} = \sum_{n>0} (-1)^{n+1} \frac{\omega X^{n-1}}{n} = \sum_{n>0} (-1)^{n+1} \frac{\omega^n X_1^{n-1}}{n}$$

converges in $\mathrm{Fil}^1 A_{crys}$, and thus $\frac{1}{X_1} = \frac{t}{X_1}\frac{1}{t} \in \mathrm{Fil}^0 B_{crys}$. So $b_1 = (b_1 X_1).\frac{1}{X_1}$ lies in $\mathrm{Fil}^0 B_{crys}$. Moreover, $(\varphi - 1) b_2 = -\frac{f(Y)}{2}$ admits a solution $b_2$ in $\tilde{\mathbf{A}}^+$, so that if we set $x = -\frac{f(Y)}{X} - \frac{f(Y)}{2} \in \tilde{\mathbf{A}}_L$ and choose a solution $b \in \tilde{\mathbf{A}}$ of $(\varphi - 1) b = x$, then $b \in \mathrm{Fil}^0 B_{crys}$.

So $\tilde{b} + b \in \mathrm{Fil}^0 B_{crys}$ and $(\varphi - 1)(\tilde{b} + b) = (\frac{1}{t} - \frac{1}{X} - \frac{1}{2}) f(Y)$. And we have the following

Lemma 1.13. — *Solutions of the equation*

$$(\varphi - 1)(\mu) = \left(\frac{1}{t} - \frac{1}{X} - \frac{1}{2}\right) f(Y) \tag{1.2}$$

*in* $\mathrm{Fil}^0 B_{crys}$ *lie in* $\mathbb{Q}_p + \mathrm{Fil}^1 B_{crys}$ *and are invariant under the action of* $G_L$.

*Proof of the lemma.* — Consider

$$\begin{aligned} u &= t\left(\frac{1}{t} - \frac{1}{X} - \frac{1}{2}\right) f(Y) = \left(1 - \frac{t}{X} - \frac{t}{2}\right) f(Y) \\ &= -\sum_{n \geqslant 2} \frac{(-X)^n}{n+1} f(Y) + \sum_{n \geqslant 2} \frac{(-X)^n}{2n} f(Y) \end{aligned}$$

then letting $\mu' = t\mu$, Equation (1.2) becomes

$$\left(\frac{\varphi}{p} - 1\right)(\mu') = u \tag{1.3}$$

but the sequences $\frac{(-X)^n}{n+1} f(Y)$ and $\frac{(-X)^n}{2n} f(Y)$ converge to 0 in $B_{crys}$ and

$$\left(\frac{\varphi}{p}\right)^k \left(\frac{X^n}{n+1}\right) = \frac{((1+X)^{p^k} - 1)^n}{(n+1)p^k}$$

but

$$((1+X)^{p^k} - 1) = \sum_{1 \leqslant r \leqslant p^k} \frac{p^k!}{(p^k - r)!} \frac{X^r}{r!} \in p^k A_{crys}$$

so $\left(\frac{\varphi}{p}\right)^k \left(\frac{X^n}{n+1}\right) \in \frac{p^{k(n-1)}}{n+1} A_{crys}$ converges to 0 uniformly in $n$ in $B_{crys}$. The same holds for $\left(\frac{\varphi}{p}\right)^k \left(\frac{X^n}{n+1}\right)$. We get a solution $-\sum_{n \geqslant 0} \left(\frac{\varphi}{p}\right)^n u$ of (1.3) in $(\mathrm{Fil}^2 B_{crys})^{G_L}$ thus a solution of Equation (1.2) in $(\mathrm{Fil}^1 B_{crys})^{G_L}$. And the lemma follows from $(\mathrm{Fil}^0 B_{crys})_{\varphi = 1} = \mathbb{Q}_p$. □

So $b+\tilde{b}\in(\mathrm{Fil}^0B_{crys})^{G_L}$, thus, for all $h\in G_L$,

$$(h-1)(-b)=(h-1)\tilde{b}=\psi_\alpha(h).$$

We conclude that there exist a unique $z\in\tilde{\mathbf{A}}_L(1)$ and $y\in\tilde{\mathbf{A}}_L(1)$ unique modulo $(\gamma-1)\mathbb{Z}_p(1)$ such that $\kappa(\alpha)$ is the image in $H^1(K,\mathbb{Z}_p(1))$ of the triple $(x,y,z)\in Z^1(C_{\varphi,\gamma,\tau}(\tilde{\mathbf{A}}_L(1)))$ where $x=-(\frac{1}{X}+\frac{1}{2})f(Y)\otimes\varepsilon$. Namely, we know that there exists such a triple $(x',y',z')$, and $x'-x\in(\varphi-1)\tilde{\mathbf{A}}_L(1)$ which shows the existence, and $x$ being fixed, the unicity modulo $\alpha(\mathbb{Z}_p)$ (where $\alpha$ is the first map in the Herr complex $C_{\varphi,\gamma,\tau}(M)$, cf. section 1.5).

We get the more precise result:

Proposition 1.14. — *Let $F(Y)\in\left(W[[Y]][\frac{1}{Y}]\right)^\times$. Then the image of $F(\pi)$ by Kummer's map corresponds to the class of a triple*

$$\left(-f(Y)\left(\frac{1}{X}+\frac{1}{2}\right),y,z\right)\otimes\varepsilon$$

*with $y,z\in W[[X,Y]]$. This triple is congruent modulo $XYW[[X,Y]]$ to*

$$\left(-\frac{f(Y)}{X}-\frac{f(Y)}{2},\ 0,\ Y d_{\log}F(Y)\right)\otimes\varepsilon.$$

*Proof.* — We have to show the congruences. Remark that

$$\begin{aligned}\gamma\left(\frac{1\otimes\varepsilon}{X}\right) &= \frac{\chi(\gamma)\otimes\varepsilon}{\chi(\gamma)X+\frac{\chi(\gamma)(\chi(\gamma)-1)}{2}X^2+X^3u(X)}\\ &= \left(\frac{1}{X}-\frac{(\chi(\gamma)-1)}{2}+Xv(X)\right)\otimes\varepsilon\end{aligned}$$

so that $(\gamma-1)x\in XYW[[X,Y]](1)$ where $\varphi^n$ is topologically nilpotent, thus $\varphi-1$ invertible. Because $(\gamma-1)x=(\varphi-1)y$, it comes

$$y\in\ker(\varphi-1)+XYW[[X,Y]](1)=\mathbb{Z}_p(1)+XYW[[X,Y]](1).$$

Moreover, let $\tilde{\gamma}$ lift $\gamma$ in $G_K$, we still have

$$(\tilde{\gamma}-1)(\tilde{b}\otimes\varepsilon)=\psi_\alpha(\tilde{\gamma})$$

where, because of *ii*) of Theorem 1.5 on the one hand, and Lemma 1.13 above on the other hand,

$$(\tilde{\gamma}-1)(\tilde{b}\otimes\varepsilon+b\otimes\varepsilon)=\psi_\alpha(\tilde{\gamma})+(\tilde{\gamma}-1)(b\otimes\varepsilon)=y\in\mathrm{Fil}^1B_{crys}(1)$$

which shows that $y\in XYW[[X,Y]](1)$. We proceed as well for $z$:

$$\begin{aligned}(\tau-1)f(Y)=f(Y+XY)-f(Y) &= \sum_{n\geqslant 1}\frac{(XY)^n}{n!}f^{(n)}(Y)\\ &\equiv XYf'(Y)\mod (XY)^2.\end{aligned}$$

Remark moreover $\left(Y\frac{d}{dY}\right)\circ\frac{\varphi}{p}=\varphi\circ\left(Y\frac{d}{dY}\right)$ so that

$$(\tau-1)f(Y)\equiv X(1-\varphi)\left(Yd_{\log}F(Y)\right)\mod (XY)^2$$

and thus $(\tau-1)x\equiv(\varphi-1)(Yd_{\log}F(Y)\otimes\varepsilon)\mod XYW[[X,Y]](1)$ which shows

$$z\in Yd_{\log}F(Y)\otimes\varepsilon+\mathbb{Z}_p(1)+XYW[[X,Y]](1). \tag{1.4}$$

And if $\tilde\tau$ lifts $\tau$ in $G_K$,

$$\begin{aligned}(\tilde\tau-1)(\tilde b+b) &= \psi_\alpha(\tilde\tau)-\log\frac{F(Y(1+X))}{F(Y)}/t+(\tilde\tau-1)b\in \mathrm{Fil}^1B_{crys}\\ z &= \psi_\alpha(\tilde\tau)+(\tilde\tau-1)b\in\log\frac{F(Y(1+X))}{F(Y)}/t+\mathrm{Fil}^1B_{crys}\end{aligned}$$

which, combined with (1.4), proves the desired result. □

## 2. Formal Groups

We still suppose that $p$ is an odd prime.

In this section, we will prove the Brückner-Vostokov explicit formula for formal groups. In [2], Abrashkin showed it under the condition that the $p^M$-th roots of unity belong to the base field, which turns out not to be necessary. To remove this assumption, we will explicitly compute the Kummer map linked to the Hilbert symbol of a formal group in terms of its $(\varphi,\Gamma)$-module, then compute the cup-product with the usual Kummer map and the image of this cup-product through the reciprocity isomorphism, which gives the desired formula.

### 2.1. Notation and background on formal groups

#### 2.1.1. Formal groups

We fix from now on an integer $M\in\mathbb{N}$.

Consider $G$ a $d$-dimensional commutative connected smooth formal group over $W=W(k)$, the ring of Witt vectors with coefficients in the finite field $k$. Denote by $K_0$ the fraction field of $W$ and $K$ a totally ramified extension of $K_0$. Under these hypotheses, $G$ is determined by a formal group law

$$\mathbf{F}(\mathbf{X},\mathbf{Y})=(F_i(X_1,\dots,X_d,Y_1,\dots,Y_d))_{1\leqslant i\leqslant d}\in(W[[\mathbf{X},\mathbf{Y}]])^d$$

where $\mathbf{X} = (X_1, \ldots, X_d)$, $\mathbf{Y} = (Y_1, \ldots, Y_d)$ (cf. [12]). We note the group law by $+_G$. Suppose moreover that $G$ has finite height $h$, that is the isogeny $p\mathrm{id}_G : G \to G$ is finite and flat over $W$ of degree $p^h$. Define $G[p^n] = \ker(p^n \mathrm{id}_G : G \to G)$ the sub-formal group of $p^n$-torsion points of $G$ and denote the Tate module of $G$ by $T(G) = \lim_{\leftarrow} G[p^n](\overline{K})$.

Suppose moreover $G[p^M](\overline{K}) = G[p^M](K)$, that is, $p^M$-torsion points of $G$ lie in $K$. Then $T(G)$ is a free $\mathbb{Z}_p$-module of rank $h$ and $G[p^M](\overline{K}) = G[p^M](K)$ is isomorphic as a $\mathbb{Z}_p$-adic representation of $G_K$ to $(\mathbb{Z}/p^M\mathbb{Z})^h$.

The space of pseudo-logarithms of $G$ (on $K_0$) is defined as the quotient of $\left\{F \in K_0[[\mathbf{X}]],\ F(\mathbf{X} +_G \mathbf{Y}) - F(\mathbf{X}) - F(\mathbf{Y}) \in \mathcal{O}_{K_0}[[\mathbf{X}, \mathbf{Y}]] \otimes \mathbb{Q}_p\right\}$ by $\mathcal{O}_{K_0}[[\mathbf{X}]] \otimes \mathbb{Q}_p$. Denote it by $H^1(G)$. It is a $K_0$-vector space of dimension $h$. The space of logarithms of $G$ is

$$\Omega(G) = \{F \in K_0[[\mathbf{X}]] \mid F(\mathbf{X} +_G \mathbf{Y}) = F(\mathbf{X}) + F(\mathbf{Y})\}.$$

It is a sub-$K_0$-vector space of $H^1(G)$ of dimension $d$. Moreover, $H^1(G)$ admits the filtration

$$\mathrm{Fil}^0(H^1(G)) = H^1(G), \quad \mathrm{Fil}^1(H^1(G)) = \Omega(G), \quad \mathrm{Fil}^2(H^1(G)) = 0.$$

With the Frobenius $\varphi : F(\mathbf{X}) \mapsto F^{\varphi}(\mathbf{X}^p)$, $H^1(G)$ is called the *Dieudonné module* of $G$.

### 2.1.2. $p$-adic periods

Fontaine defined in [12, Chapitre V, Proposition 1.2 ] a pairing $H^1(G) \times T(G) \to B^+_{crys}$ explicitly described by Colmez in [11, §3]. It is defined as follows: let $\overline{F} \in H^1(G)$, and $o = (o_s)_{s \geqslant 0} \in T(G)$ ; choose for all $s$ a lift $\hat{o}_s \in W(\mathfrak{m}_{\tilde{\mathbf{E}}})^d$ of $o_s$, *i.e.* satisfying $\theta(\hat{o}_s) = o_s$. The sequence $p^s F(\hat{o}_s)$ converges to an element $\int_o d\overline{F}$ in $B^+_{crys}$ independent of the choice of lifts $\hat{o}_s$ and $F$. This pairing is compatible with actions of Galois and $\varphi$ and with filtrations: if $F$ is a logarithm, then $\int_o dF \in \mathrm{Fil}^1 B^+_{crys}$.

This pairing permits (cf. [12, Chapitre V, Proposition 1.2 ]) to identify $H^1(G)$ with $\mathrm{Hom}_{G_{K_0}}(T(G), B^+_{crys})$ with the filtration induced by the one of $B^+_{crys}$. In order to work at an entire level, let us introduce a lattice of $H^1(G)$, the $W$-module $D^*_{crys}(G) = \mathrm{Hom}_{G_{K_0}}(T(G), A_{crys})$ endowed with the filtration and the Frobenius $\varphi$ induced by those on $A_{crys}$. The functor $D^*_{crys}$ is a contravariant version of the crystalline functor of Fontaine's theory. The filtration has length 1 (cf. [11, Proposition 3.1]) and we denote

$$\begin{aligned} D^0(G) &= D^*_{crys}(G) = \mathrm{Hom}_{G_{K_0}}(T(G), A_{crys}) \\ D^1(G) &= \mathrm{Fil}^1 D^*_{crys}(G) = \mathrm{Hom}_{G_{K_0}}(T(G), \mathrm{Fil}^1 A_{crys}). \end{aligned}$$

So $D^1(G)$ is a direct factor of $D^0(G)$ of rank $d$. Fix then a basis $\{l_1,\dots,l_d\}$ of $D^1(G)$ completed into a basis $\{l_1,\dots,l_d,m_1,\dots,m_{h-d}\}$ of $D^0(G)$.

For all $1\leqslant i\leqslant d$, $\varphi(l_i)$ takes values in $\varphi(\mathrm{Fil}^1 A_{crys})^d\subset(pA_{crys})^d$ so, $\frac{\varphi}{p}(l_i)$ belongs to $D^0(G)$. Moreover, [12, Chapitre III, Proposition 6.1] and [15, §9.7] show on the one hand that $\varphi$ is topologically nilpotent on $D^0(G)$ (because $G$ is connected) and on the other hand that the filtered module $D^0(G)$ satisfies $D^0(G)=\varphi D^0(G)+\frac{\varphi}{p}D^1(G)$. Thus, we define $\tilde{\varphi}$ an endomorphism of $D^0$ by

$$\tilde{\varphi}(l_i)=\frac{\varphi}{p}(l_i)\ \forall\ 1\leqslant i\leqslant d,\quad\text{and}\quad\tilde{\varphi}(m_i)=\varphi(m_i)\ \forall\ 1\leqslant i\leqslant h-d.$$

Its matrix $\mathcal{E}$ lies in $GL_h(W)$. Let $\mathbf{l}={}^t(l_1,\dots,l_n)$ and $\mathbf{m}={}^t(m_1,\dots,m_{h-n})$, then

$$\begin{pmatrix}\frac{\varphi}{p}(\mathbf{l})\\ \varphi(\mathbf{m})\end{pmatrix}=\mathcal{E}\begin{pmatrix}\mathbf{l}\\ \mathbf{m}\end{pmatrix}.$$

So, we can write a block decomposition $\mathcal{E}^{-1}=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ so that $\mathbf{l}=A\frac{\varphi}{p}(\mathbf{l})+B\varphi(\mathbf{m})$ and $\mathbf{m}=C\frac{\varphi}{p}(\mathbf{l})+D\varphi(\mathbf{m})$. But $\varphi$ is topologically nilpotent on $D^0(G)$, and we can write

$$\mathbf{l}=\sum_{u\geqslant 1}F_u\frac{\varphi^u(\mathbf{l})}{p},\quad \mathbf{m}=\sum_{u\geqslant 1}F'_u\frac{\varphi^u(\mathbf{l})}{p}\tag{2.1}$$

where

$$F_1=A,\ F_2=B\varphi(C),\ F_u=B\left(\prod_{1\leqslant k\leqslant u-2}\varphi^k(D)\right)\varphi^{u-1}(C)\ \text{for } u>2,$$

$$F'_1=C,\ F'_2=D\varphi(C),\ F'_u=\left(\prod_{0\leqslant k\leqslant u-2}\varphi^k(D)\right)\varphi^{u-1}(C).$$

Define a $\mathbb{Z}_p$-linear operator $\mathcal{A}=\sum_{u\geqslant 1}F_u\varphi^u$ on $K_0[[\mathbf{X}]]^d$. The vectorial formal power series

$$l_{\mathcal{A}}(\mathbf{X})=\mathbf{X}+\sum_{m\geqslant 1}\frac{\mathcal{A}^m(\mathbf{X})}{p^m}$$

gives then (cf. [18, Theorem 4]) the vectorial logarithm of a formal group $F$ over $W$ from which we can recover the formal group law $\mathbf{F}$ by

$$\mathbf{F}(\mathbf{X},\mathbf{Y})=l_{\mathcal{A}}^{-1}\left(l_{\mathcal{A}}(\mathbf{X})+l_{\mathcal{A}}(\mathbf{Y})\right).$$

In [18], Honda introduced the *type* of a logarithm. A logarithm log is of type $u\in M_d(W)[[\varphi]]$ if $u$ is special, i.e. $u\equiv pI_d \mod \varphi$ and if $u(\log)\equiv 0 \mod p$. We remark that $pI_d-\mathcal{A}$ is special and that, by construction, $l_{\mathcal{A}}$ is of type $pI_d - \mathcal{A}$. Moreover, $\mathbf{l}$ is also of type $pI_d - \mathcal{A}$ because of Equation (2.1).

Furthermore, Honda showed in [18, Theorem 4] that two formal groups with vectorial logarithms of the same type are isomorphic over $W$. Thus, we can replace the study of the formal group $G$ by the one of $F$, which is easier because we know an explicit expression of its logarithms, which gives us a control on denominators.

### 2.2. Properties of the formal group $F$

In this section, the reader can refer to [2] from which we recall principal constructions.

Let us first describe the Dieudonné module of $F$. We already know a basis of the logarithms, the coordinate power series of the vectorial series $l_{\mathcal{A}}(\mathbf{X}) = \mathbf{X} + \sum_{m\geqslant 1} \frac{\mathcal{A}^m(\mathbf{X})}{p^m}$. Complete it into a basis of $H^1(F)$ by putting (cf. [2, §1.5.2])

$$m_{\mathcal{A}}(\mathbf{X}) = \sum_{u\geqslant 1} F'_u \frac{\varphi^u(l_{\mathcal{A}}(\mathbf{X}))}{p}.$$

Let $o = (o_s)_{s\geqslant 0} \in T(F)$. For all $s \geqslant 0$, choose a lift $\hat{o}_s \in W(\mathfrak{m}_{\tilde{\mathbf{E}}})^d$ of $o_s$, that is, with $\theta(\hat{o}_s) = o_s$. Then the following lemma says that the sequence $p^s \mathrm{id}_F \hat{o}_s$ converges in $W^1(\mathfrak{m}_{\tilde{\mathbf{E}}})^d$ towards an element $j(o)$ independent of the choice of lifts:

Lemma 2.1. —

(1) *The series $l_{\mathcal{A}}$ defines a continuous one-to-one morphism of $G_K$-modules*

$$l_{\mathcal{A}} \ : \ F(W(\mathfrak{m}_{\tilde{\mathbf{E}}})) \to A^d_{crys} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

*Its restriction to $F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))$ takes values in $(\mathrm{Fil}^1 A_{crys})^d$.*

(2) *The endomorphism $p\mathrm{id}_F$ of $F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))$ is topologically nilpotent. The convergence of $p\mathrm{id}_F$ to zero is uniform on $F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))$.*

(3) *The map $j \ : \ T(F) \to W^1(\mathfrak{m}_{\tilde{\mathbf{E}}})^d$ is well defined and provides a continuous one-to-one homomorphism of $G_K$-modules $j \ : \ T(F) \to F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))$.*

*Proof.* — Point 1. is Lemma 1.5.1 of [2].

Point 2. follows from that $W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}) = \omega W(\mathfrak{m}_{\tilde{\mathbf{E}}})$ with $\omega = X/\varphi^{-1}(X) \in W(\mathfrak{m}_{\tilde{\mathbf{E}}}) + p\tilde{\mathbf{A}}^+$ and that the series corresponding to $p\mathrm{id}_F$ can be written $p\mathrm{id}_F\mathbf{X} = p\mathbf{X} +$ higher degrees. Let us recall briefly the proof of Point 3.

For all $s \geqslant 0$, $\theta(p^s \mathrm{id}_F \hat{o}_s) = o_0 = 0$ so that $p^s \mathrm{id}_F \hat{o}_s \in F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))$. On the other hand, for all $s \geqslant 0$, $p\mathrm{id}_F \hat{o}_{s+1} \equiv \hat{o}_s \mod F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))$ thus

$$p^{s+1}\mathrm{id}_F \hat{o}_{s+1} \equiv p^s \mathrm{id}_F \hat{o}_s \mod p^s \mathrm{id}_F \left(F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))\right)$$

And Point 2. provides the convergence of the sequence $(p^s \mathrm{id}_F \hat{o}_s)_s$.

The fact that the convergence is given without compatibility condition on the lifts shows the independence of the limit with respect to the choice of these lifts. Namely, let $(\hat{o}_s)_{s\geqslant 0}$ and $(\hat{o}'_s)_{s\geqslant 0}$ be two given lifts of $(o_s)_{s\geqslant 0}$, then for any lift $(\hat{o}''_s)_{s\geqslant 0}$ where $\forall s \geqslant 0$, $\hat{o}''_s = \hat{o}_s$ or $\hat{o}'_s$, we still have the convergence of $(p^s \mathrm{id}_F \hat{o}''_s)_s$, from which we deduce that the limits are the same. The remainder is straightforward. □

Composing the vectorial logarithm $l_{\mathcal{A}}$ with $j$ gives a $G_K$-equivariant injection that we will denote by $\mathbf{l}$ from $T(F)$ into $(\mathrm{Fil}^1 A_{crys})^d$. This map satisfies then for any $o$ in $T(F)$:

$$\mathbf{l}(o) = l_{\mathcal{A}}(\lim_{s\to\infty} p^s \mathrm{id}_F \hat{o}_s) = \lim_{s\to\infty} p^s l_{\mathcal{A}}(\hat{o}_s).$$

Put now $\mathbf{m} = \sum_{u\geqslant 1} F'_u \dfrac{\varphi^u(\mathbf{l})}{p}$, then $\begin{pmatrix}\mathbf{l}\\ \mathbf{m}\end{pmatrix}$ provides a basis of $D^0(F)$ with $\mathbf{l}$ a basis of $D^1(F)$. The map $\begin{pmatrix}\mathbf{l}\\ \mathbf{m}\end{pmatrix} : T(F) \to A^h_{crys}$ then factorizes through

$$\begin{pmatrix} l_{\mathcal{A}} \\ m_{\mathcal{A}} \end{pmatrix} \ : \ F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}})) \to A^h_{crys}.$$

Recall (cf. [2, Remark 1.7.5]) that this map takes values in $\tilde{\mathbf{A}}^+[[X^{p-1}/p]]$. It is also a consequence of Wach's computation for potentially crystalline representations (cf. [27]).

Fix now a basis $(o^1, \ldots, o^h)$ of $T(F)$. We can then introduce the period matrix $\mathcal{V} = \begin{pmatrix} \mathbf{l}(o^1) & \ldots & \mathbf{l}(o^h) \\ \mathbf{m}(o^1) & \ldots & \mathbf{m}(o^h)\end{pmatrix}$ which lies in

$$M_h(\tilde{\mathbf{A}}^+[[X^{p-1}/p]]) \cap GL_h(\mathrm{Frac}\tilde{\mathbf{A}}^+[[X^{p-1}/p]]).$$

It satisfies

$$\begin{pmatrix} I_d \frac{\varphi}{p} & 0 \\ 0 & I_{h-d}\varphi \end{pmatrix} \mathcal{V} = \mathcal{E}\mathcal{V}.$$

Remark that the inverse of $\mathcal{V}$ is then the change of basis matrix from the basis $(o^1, \ldots, o^h)$ to a basis of $D_{crys}(T(F)) = (T(F) \otimes_{\mathbb{Z}_p} A_{crys})^{G_K}$, the covariant version of the crystalline module of Fontaine's theory associated with $T(F)$.

Let $u \in T(F) \otimes A_{crys}$ and let $U$ denote the coordinate vector of $u$ in $(o^1, \ldots, o^h)\mathcal{V}^{-1}$. The coordinates of $\varphi(u) = (o^1, \ldots, o^h)\varphi(\mathcal{V}^{-1})\varphi(U)$ are now computed. We know that

$$\varphi(\mathcal{V}) = \begin{pmatrix} pI_d & 0 \\ 0 & I_{h-d} \end{pmatrix} \begin{pmatrix} I_d \frac{\varphi}{p} & 0 \\ 0 & I_{h-d}\varphi \end{pmatrix} \mathcal{V} = \begin{pmatrix} pI_d & 0 \\ 0 & I_{h-d} \end{pmatrix} \mathcal{E}\mathcal{V}$$

whence

$$\varphi(\mathcal{V}^{-1}) = \mathcal{V}^{-1}\mathcal{E}^{-1} \begin{pmatrix} p^{-1}I_d & 0 \\ 0 & I_{h-d} \end{pmatrix}$$

and coordinates of $\varphi(y)$ in $(o^1, \ldots, o^h)\mathcal{V}^{-1}$ are then $\mathcal{E}^{-1} \begin{pmatrix} I_d \frac{\varphi}{p} & 0 \\ 0 & I_{h-d}\varphi \end{pmatrix} U$.

Keeping this in mind, the following lemma shows that $\begin{pmatrix} \frac{\mathcal{A}}{p} & 0 \\ 0 & I_{h-d} \end{pmatrix}$ acts as the Frobenius on $D_{crys}(T(F))$.

Lemma 2.2. — *One has:* $\mathcal{E}^{-1} \begin{pmatrix} \frac{\varphi}{p} \circ l_{\mathcal{A}} \\ \varphi \circ m_{\mathcal{A}} \end{pmatrix} = \begin{pmatrix} \frac{\mathcal{A}}{p} \circ l_{\mathcal{A}} \\ m_{\mathcal{A}} \end{pmatrix}$.

*Proof.* — Compute:

$$A\frac{\varphi}{p}(l_{\mathcal{A}}) + B\varphi(m_{\mathcal{A}}) = A\frac{\varphi}{p}(l_{\mathcal{A}}) + \sum_{u \geqslant 1} B\varphi F'_u \frac{\varphi^u(l_{\mathcal{A}})}{p} = \frac{\mathcal{A}}{p}(l_{\mathcal{A}})$$

for $B\varphi F'_u = F_{u+1}$ for all $u \geqslant 1$. And:

$$C\frac{\varphi}{p}(l_{\mathcal{A}}) + D\varphi(m_{\mathcal{A}}) = C\frac{\varphi}{p}(l_{\mathcal{A}}) + \sum_{u \geqslant 1} D\varphi F'_u \frac{\varphi^u(l_{\mathcal{A}})}{p} = m_{\mathcal{A}}$$

since $D\varphi F'_u = F'_{u+1}$ for all $u \geqslant 1$. □

Abrashkin ([2, Proposition 2.1.]) computed the cokernel of injection $j$ :

Proposition 2.3. — *There is an equality* $(\mathcal{A} - p) \circ l_{\mathcal{A}}(F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))) = (\mathcal{A} - p) \circ l_{\mathcal{A}}(F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}})))$ *and the following sequence is exact:*

$$0 \longrightarrow T(F) \xrightarrow{j} F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}})) \xrightarrow{(\frac{\mathcal{A}}{p}-1)\circ l_{\mathcal{A}}} W(\mathfrak{m}_{\tilde{\mathbf{E}}})^d \longrightarrow 0$$

*Remark 2.4.* — Beware that if $x \in F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))$, $\varphi(l_{\mathcal{A}})(x) = \varphi(l_{\mathcal{A}}(x))$, and then $\mathcal{A}(l_{\mathcal{A}})(x) = \mathcal{A}(l_{\mathcal{A}}(x))$ hold if $\varphi(x) = x^p$ (e.g. when $x$ is a Teichmüller representative) but not in general ! On the left side, $\varphi$ and $\mathcal{A}$ act on $W[[\mathbf{X}]]$, whereas they act on $A_{crys}$ on the right side.

Abrashkin showed furthermore (cf. [2, Lemma 1.6.2.])

Lemma 2.5. — $F(\mathfrak{m}_{\tilde{\mathbf{E}}})$ *is uniquely* $p$*-divisible.*

This provides a continuous one-to-one $G_K$-equivariant morphism

$$\delta \;:\; F(\mathfrak{m}_{\tilde{\mathbf{E}}}) \to F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))^{(\mathcal{A}-p)\circ l_{\mathcal{A}}=0}$$

defined as follows: let $x \in F(\mathfrak{m}_{\tilde{\mathbf{E}}})$, then because of the lemma, for all $s \geqslant 0$ there exists a unique $x_s \in F(\mathfrak{m}_{\tilde{\mathbf{E}}})$ such that $p^s \mathrm{id}_F x_s = x$. Thus the sequence $(p^s \mathrm{id}_F[x_s])_s$ converges to an element $\delta(x)$ in $F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))$. The map $\delta$ is a morphism since

$$\delta(x +_F y) = \lim_s p^s \mathrm{id}_F[x_s +_F y_s] = \lim_s p^s \mathrm{id}_F([x_s] +_F [y_s] +_F u_s)$$

with $u_s \in pW(\mathfrak{m}_{\tilde{\mathbf{E}}})$ where the convergence of $p^s \mathrm{id}_F$ towards zero is uniform. Moreover, since $\mathcal{A} \circ l_{\mathcal{A}}$ coincides with $\mathcal{A}(l_{\mathcal{A}})$ on Teichmüller representatives, we get:

$$(\mathcal{A} - p) \circ l_{\mathcal{A}}(\delta(x)) = (\mathcal{A} - p)(l_{\mathcal{A}})(\delta(x)) = 0.$$

Finally, $\theta(\delta(x)) = \theta([x])$. Namely, $\forall s \geqslant 0$, $\theta(p^s \mathrm{id}_F[x_s]) = p^s \mathrm{id}_F \theta([x_s]) = \theta([x])$.

## 2.3. The ring $\mathcal{G}_{[b,a]}$ and some subrings.

### 2.3.1. Introducing the objects

Fix $e$ the absolute ramification index of $K$.

In [5], Berger introduced for $s \geqslant r \geqslant 0$ the ring $\tilde{\mathbf{A}}_{[s,r]}$, the $p$-adic completion of the ring $\tilde{\mathbf{A}}^+ \left[\frac{p}{Y^{rep/(p-1)}}, \frac{Y^{sep/(p-1)}}{p}\right]$. Let us then introduce for $a > b \geqslant 0$, the ring

$$\mathcal{G}_{[b,a]} := \tilde{\mathbf{A}}^+ \left[\left[\frac{Y^{ae}}{p}, \frac{p}{Y^{be}}\right]\right]$$

which for integers $a$ and $b$ admits the description

$$\mathcal{G}_{[b,a]} = \left\{ \sum_{n \in \mathbb{Z}} a_n Y^n \mid a_n \in \tilde{\mathbf{A}}^+ \left[\frac{1}{p}\right], \begin{array}{ll} aev_p(a_n) + n \geqslant 0 & \text{for } n \geqslant 0 \\ bev_p(a_n) + n \geqslant 0 & \text{for } n \leqslant 0 \end{array} \right\}.$$

Note that the expression $\sum_{n \in \mathbb{Z}} a_n Y^n$ for an element of $\mathcal{G}_{[b,a]}$ is not unique. The ring $\mathcal{G}_{[b,a]}$ is, for $a > \alpha \geqslant \beta > b$ a subring of $\tilde{\mathbf{A}}_{[\alpha(p-1)/p, \beta(p-1)/p]}$. We even have inclusions

$$\tilde{\mathbf{A}}_{[a(p-1)/p, b(p-1)/p]} \subset \mathcal{G}_{[b,a]} \subset \tilde{\mathbf{A}}_{[\alpha(p-1)/p, \beta(p-1)/p]}.$$

Let us fix such an $\alpha = ps/(p-1)$ and $\beta = pr/(p-1)$ and endow $\mathcal{G}_{[b,a]}$ with the induced topology which is compatible with the ring structure.

We shall prove that

$$V_{N,k} := \left\{ \sum_{n>N} a_n \left(\frac{Y^{ae}}{p}\right)^n + \sum_{n>N} b_n \left(\frac{p}{Y^{be}}\right)^n ; a_n, b_n \in \tilde{\mathbf{A}}^+ \right\} + p^k \mathcal{G}_{[b,a]}$$

for $N, k \in \mathbb{N}$ form a basis of neighborhoods of zero.

First, let us prove that for any $k, N$ there is $m \in \mathbb{N}$ such that

$$V_{k,N} \supset p^m \tilde{\mathbf{A}}_{[s,r]} \bigcap \mathcal{G}_{[b,a]}.$$

Let $x \in \tilde{\mathbf{A}}_{[s,r]}$. To say that $p^m x \in \mathcal{G}_{[b,a]}$ means that one can write

$$p^m x = \sum_{n \geqslant 0} a_n \frac{Y^{aen}}{p^n} + \sum_{n>0} b_n \frac{p^n}{Y^{ben}}$$

and that

$$x = \sum_{n \geqslant 0} a_n \frac{Y^{aen}}{p^{n+m}} + \sum_{n>0} b_n \frac{p^{n-m}}{Y^{ben}}$$

makes sense in $\tilde{\mathbf{A}}_{[s,r]}$. Consequently, it remains to prove that for $m$ large enough

$$\sum_{n \leqslant N} a_n \frac{Y^{aen}}{p^n} + \sum_{n \leqslant N} b_n \frac{p^n}{Y^{ben}} \in p^k \mathcal{G}_{[b,a]}.$$

But there are $n_0 \leqslant N + m$ and $a'_{n_0}, a''_{n_0} \in \tilde{\mathbf{A}}^+$ such that

$$\begin{aligned} a_n \frac{Y^{aen}}{p^n} &= p^m a'_{n_0} \frac{Y^{\alpha e n_0}}{p^{n_0}} \\ &= p^{m - \left\lfloor \frac{n_0 \alpha}{a} \right\rfloor} a''_{n_0} \frac{Y^{ae \left\lfloor \frac{n_0 \alpha}{a} \right\rfloor}}{p^{\left\lfloor \frac{n_0 \alpha}{a} \right\rfloor}} \end{aligned}$$

and a direct computation shows that $m - \left\lfloor \frac{n_0 \alpha}{a} \right\rfloor \geqslant k$ if $m$ is large enough, say $m \geqslant \frac{ak + \alpha N}{a - \alpha}$.

On the other hand, let us now fix $m \in \mathbb{N}$ and prove that for $N \gg m$, $V_{m,N} \subset p^m \tilde{\mathbf{A}}_{[s,r]}$. One has

$$\left(\frac{Y^{ae}}{p}\right)^n = p^m \frac{Y^{\alpha e(n+m)}}{p^{n+m}} Y^{(a-\alpha)en - \alpha em}$$

and $Y^{(a-\alpha)en-\alpha em} \in \tilde{\mathbf{A}}^+$ as soon as $n \geqslant \frac{\alpha}{a-\alpha} m$. The same computation works for $\left(\frac{p}{Y^{be}}\right)^n$ which proves that for $N$ large enough, we have the desired inclusion.

As a byproduct, we get that the topology does not depend on the choices of $\alpha$ or $\beta$.

For $a_1 > a_2 > b_2 > b_1 \geqslant 0$ we still have continuous injections $\mathcal{G}_{[a_1,b_1]} \hookrightarrow \mathcal{G}_{[a_2,b_2]}$. Define then $\mathcal{G}_{[b,a[}$ for $a \geqslant b \geqslant 0$ to be the $p$-adic completion of $\bigcup_{\alpha>a} \mathcal{G}_{[b,\alpha]}$. For integers $a$ and $b$,

$$\mathcal{G}_{[b,a[} = \left\{ \sum_{n\in\mathbb{Z}} a_n Y^n ; a_n \in \tilde{\mathbf{A}}^+ \left[\frac{1}{p}\right], \begin{array}{ll} aev_p(a_n) + n > 0 & \text{if } n > 0 \\ aev_p(a_n) + n \underset{n\to+\infty}{\longrightarrow} +\infty, & \\ bev_p(a_n) + n \geqslant 0 & \text{if } n \leqslant 0 \end{array} \right\}$$

Moreover, the inclusion

$$\bigcup_{\alpha>a} \mathcal{G}_{[b,\alpha]} \hookrightarrow \bigcup_{\alpha>a,\beta>b} \tilde{\mathbf{A}}_{[\alpha(p-1)/p,\beta(p-1)/p]}$$

permits to regard the ring $\mathcal{G}_{[b,a[}$ as a subring of the $p$-adic completion of $\bigcup_{\alpha>a,\beta>b} \tilde{\mathbf{A}}_{[\alpha(p-1)/p,\beta(p-1)/p]}$. Let us endow this last ring with the $p$-adic topology and $\mathcal{G}_{[b,a[}$ with the induced topology. Let us also introduce for $b \geqslant 0$,

$$\mathcal{G}_{[b,\infty[} := \bigcap_{a>b} \mathcal{G}_{[b,a]} = \tilde{\mathbf{A}}^+ \left[\left[\frac{p}{Y^{eb}}\right]\right] \subset \tilde{\mathbf{A}}$$

and this inclusion is continuous since the preimage of a neighborhood $Y^N\tilde{\mathbf{A}}^+ + p^{2k}\tilde{\mathbf{A}}$ is

$$Y^N\tilde{\mathbf{A}}^+ + \left\{ \sum_{n=1}^{2k} a_n \left(\frac{p}{Y^{eb}}\right)^n, v_p(a_n) \geqslant 2k-n \geqslant \right\} + \left\{ \sum_{n>2k} a_n \left(\frac{p}{Y^{eb}}\right)^n \right\}$$

which contains $p^k\mathcal{G}_{[b,\infty[} + \left\{ \sum_{n>k} a_n \left(\frac{p}{Y^{eb}}\right)^n + \sum_{n\geqslant N} b_n Y^n, a_n, b_n \in \tilde{\mathbf{A}}^+ \right\}$ which is a neighborhood of $\mathcal{G}_{[b,\infty[}$ for the topology induced by any of the $\mathcal{G}_{[b,a]}$, $a > b$.

Moreover, for $b$ integer,

$$\mathcal{G}_{[b,\infty[} = \left\{ \sum_{n\leqslant 0} a_n Y^n ; a_n \in \tilde{\mathbf{A}}^+, bev_p(a_n) + n \geqslant 0 \text{ for } n \leqslant 0 \right\}.$$

Remark that the Frobenius

$$\varphi_{\mathcal{G}} \left( \sum_{n<0} a_n Y^{aen} + \sum_{n\geqslant 0} a_n Y^{ben} \right) = \sum_{n<0} \varphi(a_n) Y^{paen} + \sum_{n\geqslant 0} \varphi(a_n) Y^{pben}$$

defines a one-to-one morphism from $\mathcal{G}_{[b,a]}$ (resp. $\mathcal{G}_{[b,a[}$) into $\mathcal{G}_{[pb,pa]}$ (resp. $\mathcal{G}_{[pb,pa[}$).

Introduce for integers $a$ and $b$ the subring of $\mathcal{G}_{[b,a]}$

$$\begin{aligned}\mathcal{G}_{Y,[b,a]} &:= W[[Y]]\left[\left[\frac{Y^{ae}}{p}, \frac{p}{Y^{be}}\right]\right] \\ &= \left\{\sum_{n\in\mathbb{Z}} a_nY^n \ ;\ a_n \in K_0,\ \begin{array}{ll} aev_p(a_n)+n \geqslant 0 & \text{if } n \geqslant 0 \\ bev_p(a_n)+n \geqslant 0 & \text{if } n \leqslant 0 \end{array}\right\}\end{aligned}$$

and $\mathcal{G}_{Y,[b,a[}$ the subring of $\mathcal{G}_{[b,a[}$ admitting the description

$$\mathcal{G}_{Y,[b,a[} = \left\{\sum_{n\in\mathbb{Z}} a_nY^n \ ;\ a_n \in K_0,\ \begin{array}{ll} aev_p(a_n)+n > 0 & \text{if } n > 0 \\ aev_p(a_n)+n \underset{n\to+\infty}{\longrightarrow} +\infty, & \\ bev_p(a_n)+n \geqslant 0 & \text{if } n \leqslant 0 \end{array}\right\}.$$

Finally, for $b \geqslant 0$,

$$\mathcal{G}_{Y,[b,\infty[} := \bigcap_{a>b} \mathcal{G}_{Y,[b,a]} = W[[Y]]\left[\left[\frac{p}{Y^{eb}}\right]\right] = \left\{\sum_{n\in\mathbb{Z}} a_nY^n;\ \begin{array}{l} a_n \in W, \\ bev_p(a_n)+n \geqslant 0 \end{array}\right\}$$

Contrary to the above situation, the expression $\sum_{n\in\mathbb{Z}} a_nY^n$ is unique as is shown in the

Lemma 2.6. — (1) *In* $\mathcal{G}_{[b,a]}\left[\frac{1}{p}\right]$, *one has*

$$\mathcal{G}_{[0,a]}\left[\frac{1}{p}\right] \bigcap \mathcal{G}_{[b,\infty[}\left[\frac{1}{Y}\right] = \tilde{\mathbf{A}}^+.$$

(2) *Every element of* $\mathcal{G}_{Y,[b,a]}$ *or* $\mathcal{G}_{Y,[b,a[}$ *can be written in a unique way as* $\sum_{n\in\mathbb{Z}} a_nY^n$ *with* $a_n \in K_0$.

(3) *Let* $a \geqslant \alpha \geqslant \beta \geqslant b$, *and* $)$ *designate* $]$ *or* $[$, *then* $\mathcal{G}_{Y,[\beta,\alpha]}\left[\frac{1}{p}\right] \bigcap \mathcal{G}_{[b,a)} = \mathcal{G}_{Y,[b,a)}$.

*Proof.* — The first point can be shown in Berger's rings $\tilde{\mathbf{A}}_{[s,r]}$, in fact in the ring $\tilde{\mathbf{A}}_{[s,\infty[}\left[\frac{1}{Y}\right] + \tilde{\mathbf{A}}_{[0,r]}\left[\frac{1}{p}\right]$. Any element of this ring is of the form $\sum_{n\in\mathbb{N}} p^n(\frac{x_n}{Y^k} - \frac{y_n}{p^l})$ with $x_n \in \tilde{\mathbf{A}}^+\left[\frac{p}{Y^{rep/(p-1)}}\right]$ and $y_n \in \tilde{\mathbf{A}}^+\left[\frac{Y^{sep/(p-1)}}{p}\right]$. Such an element is zero when

$$p^l \sum_{n\in\mathbb{N}} p^n x_n = Y^k \sum_{n\in\mathbb{N}} p^n y_n \in \tilde{\mathbf{A}}_{[s,\infty[} \bigcap \tilde{\mathbf{A}}_{[0,r]}.$$

The condition is that for all $N \in \mathbb{N}$, $\sum_{n<N} p^n(p^l x_n - Y^k y_n) \in p^N \tilde{\mathbf{A}}_{[s,r]}$. That is $p^l \sum_{n<N} p^n x_n$ belongs to $\tilde{\mathbf{A}}^+ + p^N \tilde{\mathbf{A}}^+\left[\frac{p}{Y^{rep/(p-1)}}\right]$ and then

$$\sum_{n<N} p^n x_n \in \tilde{\mathbf{A}}^+ + p^{N-l}\tilde{\mathbf{A}}^+\left[\frac{p}{Y^{rep/(p-1)}}\right],$$

and similarly

$$\sum_{n<N} p^n y_n \in \tilde{\mathbf{A}}^+ + p^{N-l/se}\tilde{\mathbf{A}}^+\left[\frac{Y^{sep/(p-1)}}{p}\right].$$

The limit $p^l\sum_{n\in\mathbb{N}} p^n x_n = Y^k\sum_{n\in\mathbb{N}} p^n y_n$ lies then in $p^l\tilde{\mathbf{A}}^+\bigcap Y^k\tilde{\mathbf{A}}^+ = p^lY^k\tilde{\mathbf{A}}^+$, hence, as claimed,

$$\sum_{n\in\mathbb{N}} p^n\frac{x_n}{Y^k} = \sum_{n\in\mathbb{N}} p^n\frac{y_n}{p^l} \in \tilde{\mathbf{A}}^+.$$

Because of the first point, it is enough to prove the second one for $\sum_{n<0} a_nY^n$ and $\sum_{n>0} a_nY^n$. It is to prove that such a series converges to zero if and only if all the $a_n$ actually are zero. For the first case, recall that there is a continuous injection $\mathcal{G}_{[\beta,\infty[} \to \tilde{\mathbf{A}}$ so that it is sufficient to prove it in $\tilde{\mathbf{A}}$.

Consider there a series $\sum_{n\geqslant 0}\frac{a_n}{Y^n}$ converging to 0 with $a_n \in W$ and $a_n$ also converging to 0. Let us prove by induction that $p^k$ divides all the $a_n$ in $W$. The case $k=0$ being obvious, let us suppose it true for a given $k$. Since $(a_n)$ converges to 0, there is an $M\in\mathbb{N}$ such that $p^{k+1}$ divides all the $a_n$ with $n>M$. Write then

$$0 = Y^M\sum_{n\geqslant 0}\frac{a_n}{p^kY^n} \equiv \sum_{n=0}^{M}\frac{a_n}{p^k}Y^{M-n} \mod p.$$

An obvious induction using successive reductions modulo $Y^k$ in $\tilde{\mathbf{E}}^+$ then shows that all the $\frac{a_n}{p^k}$ are 0 in $k$ so that all the $a_n$ are divisible by $p^{k+1}$, whence the result.

On the other side, $\mathcal{G}_{Y,[0,\alpha)}$ is naturally a subring of the separable completion of $\tilde{\mathbf{A}}\left[\frac{1}{p}\right]$ for the $Y$-adic topology. The result then follows similarly from successive reductions modulo $Y^k$.

We will proceed in a similar way to show the last point. Because of the first one, it suffices to prove both $\mathcal{G}_{Y,[\beta,\infty[}\left[\frac{1}{p}\right]\bigcap\mathcal{G}_{[b,\infty[} = \mathcal{G}_{Y,[b,\infty[}$ and $\mathcal{G}_{Y,[0,\alpha]}\left[\frac{1}{p}\right]\bigcap\mathcal{G}_{[0,a)} = \mathcal{G}_{Y,[0,a)}$. First consider then

$$x = \sum_{n\leqslant 0} a_nY^n \in \frac{1}{p^\lambda}\mathcal{G}_{Y,[\beta,\infty[} \text{ with } \beta ev_p(a_n)+n+\lambda \geqslant 0, \forall n\leqslant 0.$$

We suppose furthermore that $x$ belongs to $\mathcal{G}_{[b,\infty[}$, that is, it can be written as $\sum_{n\in\mathbb{N}} b_n\frac{p^n}{Y^{ebn}}$ with $b_n\in\tilde{\mathbf{A}}^+$. The identity $\sum_{n\leqslant 0} a_nY^n = \sum_{n\in\mathbb{N}} b_n\frac{p^n}{Y^{ebn}}$ makes sense in $\frac{1}{p^\lambda}\mathcal{G}_{[\beta,\infty[}$, thus in $\tilde{\mathbf{A}}$. Denote by $n_0$ the highest integer, supposing it exists, satisfying $bev_p(a_{n_0})+n_0<0$. We can then suppose the identity above of the form $\sum_{n\leqslant n_0} a_nY^n = \sum_{n\in\mathbb{N}} b_n\frac{p^n}{Y^{ebn}}$. Multiplying by $Y^{bev_p(a_{n_0})}$ and reducing modulo $p^{v_p(a_{n_0})}$ in $\tilde{\mathbf{A}}$ yields then to

$$\sum_{n=n_0'}^{n_0} a_nY^{n+bev_p(a_{n_0})} \equiv \sum_{n=0}^{v_p(a_{n_0})} b_np^nY^{eb(v_p(a_{n_0})-n)} \mod p^{v_p(a_{n_0})}$$

but the right term belongs to $\tilde{\mathbf{A}}^+$ and not the left one, whence a contradiction.

Consider as before an identity of the form $\sum_{n\geqslant 0} a_nY^n = \sum_{n\in\mathbb{N}} b_n\frac{Y^{ean}}{p^n}$ and denote by $n_0$ the lowest integer satisfying $aev_p(a_{n_0}) + n_0 < 0$. It can be reduced to an identity of the form $\sum_{n\geqslant n_0} a_nY^n = \sum_{n\in\mathbb{N}} b_n\frac{Y^{ean}}{p^n}$. Multiplying by $p^{-v_p(a_{n_0})}$ and reducing modulo $Y^{n_0+1}$ yields to

$$p^{-v_p(a_{n_0})}a_{n_0}Y^{n_0} \equiv \sum_{0\leqslant n\leqslant \frac{n_0}{ea}} b_np^{-v_p(a_{n_0})-n}Y^{ean} \mod Y^{n_0+1}$$

and the contradiction comes from the inequality $n \leqslant \frac{n_0}{ea} < -v_p(a_{n_0})$ hence the right term is divisible by $p$, and not the left one.

The case of $\mathcal{G}_{Y,[0,\alpha]}\bigcap\mathcal{G}_{[0,a[} = \mathcal{G}_{Y,[0,a[}$ follows from a similar argument. □

*Remark 2.7.* — As said before, periods of formal groups belong to $\tilde{\mathbf{A}}^+[[X^{p-1}/p]] = \tilde{\mathbf{A}}^+[[Y^{pe}/p]]$, that is $\mathcal{G}_{[0,p]}$. We can also recover $\tilde{\mathbf{A}}^+$ as $\mathcal{G}_{[0,\infty[}$.

### 2.3.2. Some topological precisions

Lemma 2.8. —

(1) *The set of finite sums*

$$\left\{\sum_{n=0}^{N} a_n\left(\frac{Y^{ea}}{p}\right)^n + b_n\left(\frac{p}{Y^{eb}}\right)^n; a_n, b_n \in \tilde{\mathbf{A}}^+, N\in\mathbb{N}\right\}$$

*is a dense subset of $\mathcal{G}_{[b,a]}$. The same holds for the sub-algebra*

$$\mathcal{G}_{[b,a]}\bigcap\tilde{\mathbf{A}}\left[\frac{1}{p}\right] = \left\{\sum_{n=0}^{N} a_n\left(\frac{Y^{ea}}{p}\right)^n + \sum_{n\in\mathbb{N}} b_n\left(\frac{p}{Y^{eb}}\right)^n; a_n, b_n \in \tilde{\mathbf{A}}^+, N\in\mathbb{N}\right\}$$

(2) *The topology of $\mathcal{G}_{[b,a]}$ is weaker than the $p$-adic topology.*

(3) *$\mathcal{G}_{[b,a]}$ is Hausdorff and complete.*

(4) *The ring $\mathcal{G}_{[b,a]}$ is local with residue field $\overline{k}$ and maximal ideal*

$$\mathfrak{m}_{[b,a]} = \left\{\sum_{n\geqslant 1} a_n\left(\frac{Y^{ea}}{p}\right)^n + b_n\left(\frac{p}{Y^{eb}}\right)^n; a_n, b_n \in \tilde{\mathbf{A}}^+\right\} + W(\mathfrak{m}_{\tilde{\mathbf{E}}}).$$

(5) *Any element of* $\mathfrak{m}_{[b,a]}$ *is topologically nilpotent.*
(6) *Powers of the ideal*

$$\mathfrak{m}^1_{[b,a]} = \left\{ \sum_{n\geqslant 1} a_n \left(\frac{Y^{ea}}{p}\right)^n + b_n \left(\frac{p}{Y^{eb}}\right)^n ; a_n, b_n \in \tilde{\mathbf{A}}^+ \right\} + Y^{e(a-b)}\tilde{\mathbf{A}}^+$$

*form a basis of neighborhoods of* 0 *consisting in ideals of* $\mathcal{G}_{[b,a]}$.
(7) *The ring* $\mathcal{G}_{[b,a[}$ *is local with maximal ideal* $\mathfrak{m}_{[b,a[}$ *the* $p$*-adic completion of* $\bigcup_{\alpha>a}\mathfrak{m}_{[b,\alpha]}$ *and with residue field* $\overline{k}$.
(8) *Any element of* $\mathfrak{m}_{[b,a[}$ *is topologically nilpotent.*

*Proof.* — Let us introduce the notation

$$\mathcal{G}^{>N}_{[b,a]} = \left\{ \sum_{n>N} a_n \left(\frac{Y^{ae}}{p}\right)^n + \sum_{n>N} b_n \left(\frac{p}{Y^{be}}\right)^n ; a_n, b_n \in \tilde{\mathbf{A}}^+ \right\} \subset \mathcal{G}_{[b,a]}.$$

Recall that $\left\{\mathcal{G}^{>N}_{[b,a]} + p^k\mathcal{G}_{[b,a]};\ N,k\in\mathbb{N}\right\}$ is a basis of neighborhoods of zero in $\mathcal{G}_{[b,a]}$. This shows the first two points. The fact that $\mathcal{G}_{[b,a]}$ is Hausdorff follows from that $\tilde{\mathbf{A}}_{[s,r]}$ is (cf. [5]). The following shows that the topology on $\mathcal{G}_{[b,a]}$ is metrizable, and one can immediately see from the form of neighborhoods of zero that any series with a general term going to 0 converges. This shows that $\mathcal{G}_{[b,a]}$ is complete.

We will prove Points 4., 5. et 6. simultaneously: we first show $\mathfrak{m}_{[b,a]}$ is an ideal, then that any element of $\mathfrak{m}_{[b,a]}$ has a power in $\mathfrak{m}^1_{[b,a]}$ and we make powers of $\mathfrak{m}^1_{[b,a]}$ explicit, which allows to conclude. Let

$$x = \sum_{n<0} a_n \left(\frac{Y^{ea}}{p}\right)^{-n} + \sum_{n\geqslant 0} a_n \left(\frac{p}{Y^{eb}}\right)^n$$

we say that $x$ is the element of $\mathcal{G}_{[b,a]}$ associated with the sequence $(a_n)_{n\in\mathbb{Z}} \in \left(\tilde{\mathbf{A}}^+\right)^{\mathbb{Z}}$. Let $y$ be the element associated with another sequence $(b_n)_{n\in\mathbb{Z}}$, then write the product of two elements $x$ and $y$ of $\mathcal{G}_{[b,a]}$:

$$xy = \sum_{n<0} c_n \left(\frac{Y^{ea}}{p}\right)^{-n} + \sum_{n\geqslant 0} c_n \left(\frac{p}{Y^{eb}}\right)^n$$

is associated with the sequence

(2.2)
$$c_n = \begin{cases} \displaystyle\sum_{k>0} Y^{e(a-b)k}(a_{k+n}b_{-k} + a_{-k}b_{k+n}) + \sum_{k=0}^{n} a_k b_{n-k} & \text{if } n \geqslant 0 \\ \displaystyle\sum_{k>0} Y^{e(a-b)k}(a_k b_{n-k} + a_{n-k}b_k) + \sum_{k=0}^{-n} a_{-k} b_{n+k} & \text{if } n \leqslant 0. \end{cases}$$

This shows that $\mathfrak{m}_{[b,a]}$ is an ideal because of

$$c_0 = \sum_{n\in\mathbb{Z}} Y^{e(a-b)|n|} a_n b_{-n}. \tag{2.3}$$

Suppose $x \in \mathfrak{m}_{[b,a]}$. Because of the previous computation, one can define for all $k \in \mathbb{N}$ a sequence $(c_{n,k})_{n\in\mathbb{Z}}$ such that $x^k$ is associated with $(c_{n,k})_{n\in\mathbb{Z}}$. The fact that there exists a $k$ such that $x^k \in \mathfrak{m}^1_{[b,a]}$ is equivalent to that the rest $\overline{c}_{0,k} \in \tilde{\mathbf{E}}^+$ of $c_{0,k}$ modulo $p$ has a valuation greater or equal to $a-b$. But because of Equality (2.3), $v_{\mathbf{E}}(\overline{c}_{0,k}) \geqslant \min(a-b, kv_{\mathbf{E}}(\overline{a}_0))$ which shows $x^k \in \mathfrak{m}^1_{[b,a]}$ for $k$ large enough.

Let us show now that $\mathfrak{m}^k_{[b,a]} = \left(\mathfrak{m}^1_{[b,a]}\right)^k$ consists in elements associated with sequences $(a_n)_{n\in\mathbb{Z}}$ such that $\forall n \in \mathbb{Z},\ v_{\mathbf{E}}(\overline{a}_n) \geqslant g^k_{a,b}(n)$ where

$$g^k_{a,b}(n) = \left\lfloor \frac{(k-|n|+1)_+}{2} \right\rfloor (a-b) = \begin{cases} \left\lfloor \frac{k-|n|+1}{2} \right\rfloor (a-b) & \text{if } |n| \leqslant k \\ 0 & \text{otherwise} \end{cases}$$

satisfying the induction relation

$$g^{k+1}_{a,b}(n) = \begin{cases} g^k_{a,b}(n-1) + a - b & \text{if } -k-1 \leqslant n \leqslant 0 \\ g^k_{a,b}(n+1) + a - b & \text{if } 0 \leqslant n \leqslant k+1 \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

or equivalently

$$g^{k+1}_{a,b}(n) = \begin{cases} g^k_{a,b}(n+1) & \text{if } n < 0 \\ g^k_{a,b}(n-1) & \text{if } n > 0 \end{cases}. \tag{2.5}$$

Remark also that $g^k_{a,b}$ is even and decreasing on $\mathbb{N}$.

Let then $x \in \mathfrak{m}^k_{[b,a]}$ be associated with a sequence $(a_n)_{n\in\mathbb{Z}}$ satisfying the previous induction relation, let $y \in \mathfrak{m}^k_{[b,a]}$ be associated with $(b_n)_{n\in\mathbb{Z}}$ and $xy \in \mathfrak{m}^{k+1}_{[b,a]}$ be associated with $(c_n)_{n\in\mathbb{Z}}$ which we compute as before. Equations (2.2) show the relation for $n \geqslant 0$ (case $n < 0$ provides the same computation):

$$v_{\mathbf{E}}(\overline{c}_n) \geqslant \inf \left\{ \begin{array}{ll} (a-b)r + g^k_{a,b}(n+r), & \text{for } r > 0, \\ (a-b)r + g^k_{a,b}(-r), & \text{for } r > 0, \\ g^k_{a,b}(r), & \text{for } 0 \leqslant r < n, \\ g^k_{a,b}(n) + a - b & \end{array} \right\}$$

which gives because $g^k_{a,b}$ is even and decreasing

$$v_{\mathbf{E}}(\overline{c}_n) \geqslant \inf \left\{ \begin{array}{ll} (a-b)r + g^k_{a,b}(n+r), & \text{for } r > 0, \\ g^k_{a,b}(n-1) & \\ g^k_{a,b}(n) + a - b & \end{array} \right\}.$$

But

$$(a-b)r+g_{a,b}^k(n+r)=(a-b)\left(r+\left\lfloor\frac{(k-|n+r|+1)_+}{2}\right\rfloor\right)$$

is strictly increasing in $r$ and $(a-b)+g_{a,b}^k(n+1)\geqslant g_{a,b}^{k+1}(n)$ because of (2.4). Likewise,

$$g_{a,b}^k(n)+a-b\geqslant g_{a,b}^{k+1}(n-1)\geqslant g_{a,b}^{k+1}(n)$$

and finally, according to (2.5), $g_{a,b}^k(n-1)=g_{a,b}^{k+1}(n)$. The minimum is then equal to $g_{a,b}^{k+1}(n)$, which lets us conclude on the description of $\mathfrak{m}_{[b,a]}^k$.

Point 6. follows from this description, and proves 5. Point 4. follows because any $x\in\mathcal{G}_{[b,a]}$ can be written as $x=a_0-u$ with $u\in\mathfrak{m}_{[b,a]}, a_0\in\tilde{\mathbf{A}}^+$ and since $W(\mathfrak{m}_{\tilde{\mathbf{E}}})\subset\mathfrak{m}_{[b,a]}$ we can even choose $a_0=[\overline{a_0}]$ with $\overline{a_0}\in\overline{k}$. Then either $\overline{a_0}=0$ and $x\in\mathfrak{m}_{[b,a]}$ or $x$ is invertible with

$$x^{-1}=[a_0^{-1}]\sum_{n\geqslant 0}\left([a_0^{-1}]u\right)^n.$$

Let us prove now 8. Remark that any $x\in\mathfrak{m}_{[b,a[}$ can be written as $x=x_0+px_1$ ; $x_0\in\mathfrak{m}_{[b,\alpha]}, x_1\in\mathcal{G}_{[b,a[}$ for some $a>\alpha>b$. Write $x^m=\sum_{n+k=m}\binom{m}{k}p^kx_0^nx_1^k$. We have to show $p^kx_0^nx_1^k\underset{k,n\to+\infty}{\longrightarrow}0$. When $k$ goes to infinity, it is clear. When $n$ goes to infinity, remark that the convergence of $x_0^n$ to 0 in $\mathcal{G}_{[b,\alpha]}$ implies for any $N$ and $t$ in $\mathbb{N}$, $x_0^n$ can be written, for $n$ large enough as $\sum_{s>N}a_s\left(\frac{Y^{\alpha e}}{p}\right)^s+\sum_{s>N}b_s\left(\frac{p}{Y^{be}}\right)^s+p^tu$ with $u\in\mathcal{G}_{[b,a[}$ and $a_s,b_s$ in $\tilde{\mathbf{A}}^+$. Fix then a $t$, and let us remark that if we choose $a>\alpha>\alpha'\geqslant\beta'>\beta>b$,

$$\left(\frac{Y^{\alpha e}}{p}\right)^s=p^t\frac{Y^{\alpha es}}{p^{s+t}}=p^t\frac{Y^{\alpha' e\frac{\alpha s}{\alpha}'}}{p^{s+t}}$$

belongs to $\tilde{\mathbf{A}}_{[\alpha'(p-1)/p,\beta'(p-1)/p]}$ for $N>\frac{\alpha' t}{\alpha-\alpha'}$. The same computation for $\left(\frac{p}{Y^{be}}\right)^s$ shows that if $N$ is chosen large enough, that is, for $n$ large enough, $x_0^n\in p^t\tilde{\mathbf{A}}_{[\alpha'(p-1)/p,\beta'(p-1)/p]}$, whence 8.

At last, 7. is a consequence of 8. Namely, any $x\in\mathcal{G}_{[b,a[}$ can be written as $x=x_0+pu$ with $x_0\in\mathcal{G}_{[b,\alpha]}$ for some $a>\alpha>b$ and $u\in\mathcal{G}_{[b,a[}$, so that $pu\in\mathfrak{m}_{[b,a[}$. We deduce that $x=[\overline{x_0}]+v$ with $\overline{x_0}\in\overline{k}$ and $v\in\mathfrak{m}_{[b,\alpha]}+\mathfrak{m}_{[b,a[}=\mathfrak{m}_{[b,a[}$. So that, just like before, $x$ either belongs to $\mathfrak{m}_{[b,a[}$ or is invertible in $\mathcal{G}_{[b,a[}$. □

*Remark 2.9.* — The preceding lemma makes $\mathcal{G}_{[b,a)}$ into a complete valuation ring with the valuation given by $v_{[b,a)}(x)=\lim_{n\to\infty}\frac{k_n}{n}$ where $k_n=\sup\{k\in\mathbb{N},x^n\in\mathfrak{m}_{[b,a)}^k\}$.

The following lemma provides a link between algebras $\mathcal{G}_{[b,a]}$ and $A_{crys}$.

LEMMA 2.10. —

(1) *$\mathcal{G}_{[0,p]}$ injects continuously in $A_{crys}$.*
(2) *Frobenius $\varphi$ of $A_{crys}$ and $\varphi_{\mathcal{G}}$ coincide on $\mathcal{G}_{[0,p]}$.*
(3) *Any non zero element of $\mathcal{G}_{Y,[0,p]}$ is invertible in $\mathcal{G}_{Y,[p-1,p-1[} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*
(4) *The series defining $t/X$ converges in $\mathcal{G}_{[0,p]}$ where it is invertible.*

*Proof.* — The first point consists in showing that $\frac{Y^{pen}}{p^n} \in A_{crys}$ for all $n$ and converges to 0. Let $E_\pi$ be an Eisenstein polynomial for $\pi$, it has degree $e$ and $E_\pi(Y)$ generates $W^1(\tilde{\mathbf{E}}^+)$ so that $A_{crys}$ is the $p$-adic completion of $\tilde{\mathbf{A}}^+[\frac{E_\pi(Y)^n}{n!}]$ and it is obvious that $\frac{Y^{pen}}{p^n}$ lies in this ring and $p$-adically converges to 0. The second point is an immediate consequence of the first one.

Now let $x \in \mathcal{G}_{Y,[0,p]}$, then there exists a sequence $(a_n)_{n\in\mathbb{N}} \in \left(\tilde{\mathbf{A}}^+\left[\frac{1}{p}\right]\right)^{\mathbb{N}}$ such that $x = \sum_{n\in\mathbb{N}} a_n Y^n$ with $\forall n \in \mathbb{N}$ ; $epv_p(a_n) + n \geqslant 0$. Then, $\forall n \in \mathbb{N}$ ; $e(p-1)v_p(a_n) + n \geqslant \frac{n}{p}$ and for $x$ non zero, $e(p-1)v_p(a_n) + n$ goes to $+\infty$ when $n \to +\infty$, it reaches its minimum $K$ a finite number of times and we fix $n_0$ the greater integer with $K = e(p-1)v_p(a_{n_0}) + n_0$, so that

$$e(p-1)v_p(a_n/a_{n_0}) + n - n_0 \geqslant 0 \quad \text{if } n \leqslant n_0 \tag{2.6}$$

$$e(p-1)v_p(a_n/a_{n_0}) + n - n_0 > 0 \quad \text{if } n > n_0 \tag{2.7}$$

and $\forall n > n_0$ ; $e(p-1)v_p(a_n/a_{n_0}) + n - n_0 > \frac{n}{p} - K$ hence it comes

$$\liminf_{n\to\infty} \frac{e(p-1)v_p(a_n/a_{n_0}) + n - n_0}{n - n_0} \geqslant \frac{1}{p},$$

which, combined with (2.7), shows the existence of some $0 < \lambda < 1$ with

$$\begin{aligned} e(p-1)v_p(a_n/a_{n_0}) + n - n_0 &\geqslant \lambda(n - n_0) \\ e\frac{p-1}{1-\lambda}v_p(a_n/a_{n_0}) + n - n_0 &\geqslant 0. \end{aligned}$$

This shows that for $a = \frac{p-1}{1-\lambda} > p-1$, $\sum_{n>n_0} \frac{a_n}{a_{n_0}} Y^{n-n_0} \in \mathfrak{m}_{[0,a]}$. Inequality (2.6) shows furthermore that $\sum_{n=0}^{n_0-1} \frac{a_n}{a_{n_0}} Y^{n-n_0}$ lies in $\mathfrak{m}_{[p-1,\infty[}$ and finally $\sum_{n\neq n_0} \frac{a_n}{a_{n_0}} Y^{n-n_0} \in \mathfrak{m}_{[p-1,a]}$. Then,

$$x = a_{n_0} Y^{n_0}(1+\epsilon) \ ; \ \epsilon \in \mathfrak{m}_{[p-1,a]}$$

is invertible in $\mathcal{G}_{Y,[p-1,a]} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \subset \mathcal{G}_{Y,[p-1,p-1[} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Remark finally

$$X = [\varepsilon - 1] + pv = Y^{ep/(p-1)}u + pv \ ; \ u, v \in \tilde{\mathbf{A}}^+$$

so that $X^{p-1} = Y^{ep}u' + pv',\ u',v' \in \tilde{\mathbf{A}}^+$ from which we deduce for $s$ prime to $p$,

$$\begin{aligned}\frac{X^{p^r s-1}}{p^r s} = \frac{X^{p^r(s-1)}}{s}\frac{X^{p^r-1}}{p^r} &= X^{p^r(s-1)}\sum_{k=0}^{p^r-1}\left(\frac{Y^{pek}}{p}\right)^{\frac{p^r-1}{p-1}-k} p^r u_k \\ &= X^{p^r(s-1)}\sum_{k=0}^{p^r-1}\left(\frac{Y^{pek}}{p}\right)^{\frac{p^r-1}{p-1}-r} p^k u_k\end{aligned}$$

where $u_k \in \tilde{\mathbf{A}}^+$. But $\frac{p^r-1}{p-1} \geqslant r$ so that for all $n \geqslant 1$, $X^{n-1}/n \in \mathcal{G}_{[0,p]}$ and $\frac{p^r-1}{p-1} - r$ goes to $+\infty$ with $r \to \infty$, which shows that $X^{n-1}/n$ converges $p$-adically to 0 in $\mathcal{G}_{[0,p]}$. □

## 2.4. The Hilbert symbol of a formal group

### 2.4.1. The pairing associated with the Hilbert symbol

In this paragraph we express the Hilbert symbol of $F$ in terms of the Herr complex attached to $F[p^M]$. Let us recall that the Hilbert symbol of a formal group is defined as the pairing:

$$(\alpha,\beta) \in K^* \times F(\mathfrak{m}_K) \mapsto (\alpha,\beta)_{F,M} = r(\alpha)(\beta_1) -_F \beta_1 \in F[p^M]$$

where $\beta_1 \in F(\mathfrak{m}_{\mathbb{C}_p})$ satisfies $p^M \mathrm{id}_F \beta_1 = \beta$ and $r : K^* \to G_K^{\mathrm{ab}}$ is the reciprocity map of local class field theory. In fact, we will be interested in the pairing

$$(\beta, g) \in F(\mathfrak{m}_K) \times G_K \mapsto (\beta, g]_{F,M} = g\beta_1 -_F \beta_1 \in F[p^M]$$

where $\beta_1 \in F(m_{\mathbb{C}_p})$ satisfies $p^M \mathrm{id}_F \beta_1 = \beta$. Then $(\beta, r(\alpha)]_{F,M} = (\alpha,\beta)_{F,M}$. Put

$$\mathcal{R}(F) = \left\{(x_i)_{i\geqslant 0} \in F(\mathfrak{m}_{\mathbb{C}_p})\ ;\ x_0 \in F(\mathfrak{m}_K) \text{ and } (p\mathrm{id}_F)x_{i+1} = x_i\ \forall i \geqslant 0\right\}$$

then the Hilbert symbol is a mod $p$ reduction of the pairing

$$(x,g) \in \mathcal{R}(F) \times G_K \mapsto (x,g]_{\mathcal{R}(F)} = (gx_i -_F x_i)_i \in T(F)$$

with $((x,g]_{\mathcal{R}(F)})_M = (x_0, g]_{F,M}$ for any $x = (x_i) \in \mathcal{R}(F)$.

This pairing is linked with the connecting map $F(\mathfrak{m}_K) \to H^1(K, T(F))$ in the long exact sequence associated with the short exact one:

$$0 \to T(F) \to \varprojlim F(\mathfrak{m}_{\mathbb{C}_p}) \to F(\mathfrak{m}_{\mathbb{C}_p}) \to 0$$

where transition maps in the inverse limit are $p\mathrm{id}_F$ and the last map is the projection on the first component. The ring $\mathcal{R}(F)$ is then the preimage of $F(\mathfrak{m}_K)$ by $\varprojlim F(\mathfrak{m}_{\mathbb{C}_p}) \to F(\mathfrak{m}_{\mathbb{C}_p})$.

Let now $x \in F(\mathfrak{m}_{\tilde{\mathbf{E}}})$ be such that $\theta([x]) \in F(\mathfrak{m}_K)$. Then for all $g \in G_K$,

$$(g-1)\delta(x) \in F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))^{(\mathcal{A}-p)\circ l_{\mathcal{A}}=0} \simeq T(F)$$

with $\delta$ defined at the end of §2.2. The following diagram is commutative

$$\begin{array}{ccc}
F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))_K^{(\mathcal{A}-p)\circ l_{\mathcal{A}}=0} \times G_K & \longrightarrow & F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))^{(\mathcal{A}-p)\circ l_{\mathcal{A}}=0} \\
{\scriptstyle \delta\times\mathrm{id}}\uparrow & & \\
F(\mathfrak{m}_{\tilde{\mathbf{E}}})_K \times G_K & & \uparrow {\scriptstyle j} \\
{\scriptstyle \iota\times\mathrm{id}}\downarrow & & \\
\mathcal{R}(F) \times G_K & \longrightarrow & T(F)
\end{array}$$

where $\iota(x) = (\theta \circ \delta(p^{-s}\mathrm{id}_F(x)))_s = (\theta([p^{-s}\mathrm{id}_F(x))])_s$ and we denote by $F(\mathfrak{m}_{\tilde{\mathbf{E}}})_K$ (resp. $F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))_K$) the set of $x \in F(\mathfrak{m}_{\tilde{\mathbf{E}}})$ (resp. $F(W(\mathfrak{m}_{\tilde{\mathbf{E}}})))$ with $\theta([x]) \in K$ (resp. $\theta(x) \in K$) and where the first pairing is simply $(u,g) \mapsto (g-1)u$.

Fix now $\alpha \in F(\mathfrak{m}_K)$ and a lift $\xi$ of $\alpha$ in $F(\mathfrak{m}_{\tilde{\mathbf{E}}})$ which then satisfies $\theta([\xi]) = \alpha$. We get

$$j((\iota(\xi), g]_{\mathcal{R}(F)}) = (g-1)\delta(\xi)$$

for all $g \in G_K$. Choose now $\beta \in F(YW[[Y]])$ such that $\theta(\beta) = \alpha = \theta([\xi])$. Then

$$\forall h \in G_L,\ (h-1)(\delta(\xi) -_F \beta) = j((\iota(\xi), h]_{\mathcal{R}(F)}).$$

Moreover, $\delta(\xi) -_F \beta \in F(W^1(\mathfrak{m}_{\tilde{\mathbf{E}}}))$ thus $l_{\mathcal{A}}(\delta(\xi) -_F \beta) \in (\mathrm{Fil}^1 A_{crys})^d$ and

$$m_{\mathcal{A}}(\delta(\xi) -_F \beta) = \sum_{u\geqslant 1} F'_u \frac{\varphi^u(l_{\mathcal{A}}(\delta(\xi) -_F \beta))}{p}$$

converge in $A_{crys}^{h-n}$. Put now $\Lambda = \mathcal{V}^{-1}\begin{pmatrix} l_{\mathcal{A}}(\delta(\xi) -_F \beta) \\ m_{\mathcal{A}}(\delta(\xi) -_F \beta) \end{pmatrix} \in A_{crys}^h$. These are coordinates of a $\lambda \in D_{crys}(T(F)) \otimes A_{crys}$ in the basis $(o^1, \ldots, o^h)$. And,

$$\forall h \in G_L,\ (h-1)\lambda = (\iota(\xi), h]_{\mathcal{R}(F)}. \tag{2.8}$$

### 2.4.2. The approximated period matrix

Now we explicitly compute the Hilbert symbol of $F$, *i.e.* the image of $\iota(\xi)$ in $H^1(K, F[p^M])$ which coincides with the one of $\alpha$. For that, we have to give a triple in the first homology group of the Herr complex of $F[p^M]$

corresponding to a cocycle representing the image of $\iota(\xi)$. Recall that if such a triple is written as $(x, y, z)$, then the associated cocycle is

$$g \mapsto (g-1)(-b) + \gamma^n \frac{\tau^m - 1}{\tau - 1} z + \frac{\gamma^n - 1}{\gamma - 1} y$$

where $g_{|\Gamma} = \gamma^n \tau^m$ and $b \in F[p^M] \otimes \tilde{\mathbf{A}}$ is a solution of $(\varphi - 1)b = x$. In particular, the image of $h \in G_L$ through this cocycle is $(h-1)(-b)$. Let us start with finding $b \in T(F) \otimes \tilde{\mathbf{A}}$ such that for all $h \in G_L$, $(h-1)b \equiv -(\iota(\xi), h]_{\mathcal{R}(F)}$ mod $p^M$. Equality (2.8) incites to build $b$ as an approximation of $-\lambda$. In fact, we will build $x$ by approximating $(\varphi - 1)(-\lambda)$, whose coordinates in the basis $(o^1, \dots, o^h)$ are $\mathcal{V}^{-1} \begin{pmatrix} (\frac{\mathcal{A}}{p} - 1) \circ l_{\mathcal{A}}(\beta) \\ 0 \end{pmatrix}$.

Indeed, Lemma 2.2 shows that the action of $\varphi$ is written in the basis $(o^1, \dots, o^h)\mathcal{V}^{-1}$ as $\begin{pmatrix} \frac{\mathcal{A}}{p} & 0 \\ 0 & I_{h-d} \end{pmatrix}$. Because $(o^1 = (o^1_n)_n, \dots, o^h)$ is the fixed basis of $T(F)$, $(o^1_M, \dots, o^h_M)$ is a basis of $F[p^M]$ and we further fix $\hat{o}^1_M, \dots,$ $\hat{o}^h_M$ elements in $F(YW[[Y]])$ such that for all $i$, $\theta(\hat{o}^i_M) = \hat{o}^i_M(\pi) = o^i_M$. Define then the matrix

$$\mathcal{V}_Y = \begin{pmatrix} p^M l_{\mathcal{A}}(\hat{o}^1_M) & \dots & p^M l_{\mathcal{A}}(\hat{o}^h_M) \\ p^M m_{\mathcal{A}}(\hat{o}^1_M) & \dots & p^M m_{\mathcal{A}}(\hat{o}^h_M) \end{pmatrix}$$

whose coefficients belong to $A_{crys}$, and more precisely to $W[[Y]]\left[\left[\frac{Y^{pe}}{p}\right]\right] = \mathcal{G}_{Y,[0,p]}$. From Lemma 2.10, $\mathcal{V}_Y$ is invertible in $\mathcal{G}_{Y,[p-1,p-1[} \otimes \mathbb{Q}_p$.

Lemma 2.11. —

(1) *The matrix* $X\mathcal{V}_Y^{-1}$ *has coefficients in* $\mathcal{G}_{[0,p]} + \frac{p^M}{Y^{e/(p-1)}} \mathfrak{m}_{[\frac{1}{p-1}, \infty[} \subset \mathcal{G}_{[\frac{1}{p-1}, p]}$ *and thus* $\varphi_{\mathcal{G}}(X\mathcal{V}_Y^{-1}) \in \mathcal{G}_{[p/(p-1), p]}$.

(2) *Coefficients of* $\mathcal{V}_Y^{-1}$ *lie in* $\frac{1}{Y^{ep/(p-1)}} \mathcal{G}_{[1,p]}$, *thus in* $\frac{1}{Y^{\lceil ep/(p-1) \rceil}} \mathcal{G}_{Y,[1,p]}$ *and*

$$\mathcal{V}_Y^{-1} \equiv \mathcal{V}^{-1} \mod \frac{p^M}{Y^{e(p+1)/(p-1)}} \mathfrak{m}_{[1,p]}.$$

(3) *The principal part* $\mathcal{V}_Y^{(-1)}$ *of* $\mathcal{V}_Y^{-1}$ *has p-entire coefficients and its derivative* $\frac{d}{dY}\mathcal{V}_Y^{(-1)}$ *has coefficients in* $p^M \tilde{\mathbf{A}}$.

(4) *The matrix* $X\mathcal{V}_Y^{(-1)}$ *has coefficients in* $\tilde{\mathbf{A}}^+ + p^M \tilde{\mathbf{A}}$.

*Proof.* — We use the strategy of [2, Paragraph 3.4.]. Let us recall that Abrashkin there showed

$$\begin{aligned} p^M l_{\mathcal{A}}(\hat{o}^i_M) &\in \left(E_\pi(Y)YW[[Y]] + \frac{E_\pi(Y)^p}{p}W[[Y]]\left[\left[\frac{Y^{ep}}{p}\right]\right]\right)^n \\ p^M m_{\mathcal{A}}(\hat{o}^i_M) &\in \left(YW[[Y]] + \frac{Y^{ep}}{p}W[[Y]]\left[\left[\frac{Y^{ep}}{p}\right]\right]\right)^{h-n} \end{aligned}$$

and

$$\begin{aligned} \mathbf{l}(o^i) - p^M l_{\mathcal{A}}(\hat{o}^i_M) &\in p^M\left(E_\pi(Y)W(\mathfrak{m}_{\tilde{\mathbf{E}}}) + \frac{E_\pi(Y)^p}{p}\tilde{\mathbf{A}}^+\left[\left[\frac{Y^{ep}}{p}\right]\right]\right)^n \\ \mathbf{m}(o^i) - p^M m_{\mathcal{A}}(\hat{o}^i_M) &\in p^M\left(W(\mathfrak{m}_{\tilde{\mathbf{E}}}) + \frac{Y^{ep}}{p}\tilde{\mathbf{A}}^+\left[\left[\frac{Y^{ep}}{p}\right]\right]\right)^{h-n}. \end{aligned}$$

Let $\mathcal{V}^D$ be the matrix of the group dual to $F$. It satisfies the relation ${}^t\mathcal{V}^D\ \mathcal{V} = tI_h$. And one can then write

$$\begin{aligned} {}^t\mathcal{V}^D\ \mathcal{V}_Y &\equiv tI_h \mod p^M\left(E_\pi(Y)W(\mathfrak{m}_{\tilde{\mathbf{E}}}) + \frac{E_\pi(Y)^p}{p}\tilde{\mathbf{A}}^+\left[\left[\frac{Y^{ep}}{p}\right]\right]\right) \\ {}^t\mathcal{V}^D\ \mathcal{V}_Y &\equiv tI_h \mod p^M\left(Y^eW(\mathfrak{m}_{\tilde{\mathbf{E}}}) + \frac{Y^{ep}}{p}\tilde{\mathbf{A}}^+\left[\left[\frac{Y^{ep}}{p}\right]\right]\right). \end{aligned}$$

Remark, because of Lemma 2.10, that the element $t/X$ converges in $\mathcal{G}^*_{[0,p]}$, and $X = \omega[\varepsilon^{1/p} - 1] = E_\pi(Y)Y^{e/(p-1)}v$ with $v \in \mathcal{G}^*_{[\frac{1}{p-1},\infty[}$, so that ${}^t\mathcal{V}^D\ \mathcal{V}_Y = t(I_h - p^M u)$ with

$$\begin{aligned} u \in \frac{E_\pi(Y)}{t}W(\mathfrak{m}_{\tilde{\mathbf{E}}}) + \frac{Y^{ep}}{pt}\mathcal{G}_{[0,p]} &\subset \frac{1}{Y^{e/(p-1)}}\mathfrak{m}_{[\frac{1}{p-1},p]} + \frac{Y^{e\frac{p^2-2p}{p-1}}}{p}\mathcal{G}_{[\frac{1}{p-1},p]} \\ &\subset \frac{1}{Y^{e/(p-1)}}\mathfrak{m}_{[\frac{1}{p-1},p]} \subset \frac{1}{p}\mathfrak{m}_{[\frac{1}{p-1},p]} \end{aligned}$$

thus $\mathcal{V}_Y^{-1} = \frac{1}{t}\left(\sum_{n\in\mathbb{N}} p^{Mn}u^n\right){}^t\mathcal{V}^D \in \frac{1}{t}\mathcal{G}_{[\frac{1}{p-1},p]}$ whence the first point ; and even

$$\mathcal{V}_Y^{-1} \equiv \mathcal{V}^{-1} \mod \frac{p^M}{tY^{e/(p-1)}}\mathfrak{m}_{[\frac{1}{p-1},p]} \text{ or } \mathcal{V}_Y^{-1} \in \frac{1}{t}\mathcal{G}_{[0,p]} + \frac{p^M}{Y^{e/(p-1)}}\mathfrak{m}_{[\frac{1}{p-1},p]}.$$

Recall $t = E_\pi(Y)\varphi^{-1}(X)u'$ ; $u' \in \mathcal{G}^*_{[0,p]}$ and remark that because $E_\pi$ is an Eisenstein polynomial, $E_\pi(Y)$ and $Y^e$ are associated in $\mathcal{G}_{[1,\infty[}$ ; finally, with the above computation, we deduce that $t$ and $Y^{ep/(p-1)}$ are associated in $\mathcal{G}_{[1,p]}$. Then

$$\mathcal{V}_Y^{-1} \in \frac{1}{Y^{ep/(p-1)}}\mathcal{G}_{[1,p]} + \frac{p^M}{Y^{e(p+1)/(p-1)}}\mathfrak{m}_{[1,p]} \subset \frac{1}{Y^{ep/(p-1)}}\mathcal{G}_{[1,p]}.$$

So, $Y^{\lceil ep/(p-1)\rceil}\mathcal{V}_Y^{-1}$ has coefficients in $\mathcal{G}_{Y,[p-1,p-1[}\left[\frac{1}{p}\right]\bigcap\mathcal{G}_{[1,p]}=\mathcal{G}_{Y,[1,p]}$ because of Lemma 2.6. Let us further deduce that $\mathcal{V}_Y^{(-1)}$ has $p$-entire coefficients. It is to show that any $x=\sum_{n\in\mathbb{Z}}a_nY^n\in\frac{1}{Y^{\lceil ep/(p-1)\rceil}}\mathcal{G}_{Y,[1,p]}$ satisfies $a_n\in W$ for all $n\leqslant 0$. But that means that

$$Y^{\lceil ep/(p-1)\rceil}x=\sum_{n\in\mathbb{Z}}a_nY^{n+\lceil ep/(p-1)\rceil}\in\mathcal{G}_{Y,[1,p]}$$

and thus if $v_p(a_n)\leqslant 1$, $n+ep/(p-1)\geqslant ep$ hence $n\geqslant\frac{(p-2)ep}{p-1}>0$ and $\mathcal{V}_Y^{(-1)}$ has $p$-entire coefficients.

For the third point, let us recall the argument of [2, Lemma 4.5.4]. Write

$$\frac{d}{dY}\mathcal{V}_Y^{(-1)}=-\mathcal{V}_Y^{(-1)}\left(\frac{d}{dY}\mathcal{V}_Y\right)\mathcal{V}_Y^{(-1)}$$

since differentials of $l_{\mathcal{A}}$ and $m_{\mathcal{A}}$ have coefficients in $W$, we get $\frac{d}{dY}\mathcal{V}_Y\in p^MM_h(W[[Y]])$ so

$$\frac{d}{dY}\mathcal{V}_Y^{(-1)}\in\mathcal{G}_{Y,[p-1,p-1[}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\bigcap\frac{1}{Y^{2ep/(p-1)}}\mathcal{G}_{[1,p]}$$

and the same argument as above permits to conclude (we get the inequality $n\geqslant\frac{(p-3)ep}{p-1}\geqslant 0$).

Finally, the proof of Point 4. is the same as the one of Proposition 3.7, Point *d)* in [2]. Let us write it in the following way: we know on the one hand that $\mathcal{V}_Y^{(-1)}$ and then also $X\mathcal{V}_Y^{(-1)}$ has $p$-entire coefficients, so they have coefficients in $\mathcal{G}_{[p-1,\infty[}\left[\frac{1}{Y}\right]$ and that $\mathcal{U}=X(\mathcal{V}_Y^{-1}-\mathcal{V}_Y^{(-1)})$ has coefficients in $\mathcal{G}_{[0,p-1[}\left[\frac{1}{p}\right]$. On the other hand, Lemma 2.11 tells

$$X\mathcal{V}_Y^{-1}\in M_h\left(\mathcal{G}_{[0,p]}+p^{M-1}\mathcal{G}_{[1/(p-1),\infty[}\right).$$

Remark

$$\mathcal{G}_{[1/(p-1),\infty[}=\tilde{\mathbf{A}}^+\left[\left[\frac{p}{Y^{e/(p-1)}}\right]\right]=\tilde{\mathbf{A}}^++\frac{p}{Y^{e/(p-1)}}\mathcal{G}_{[1/(p-1),\infty[}$$

Thus we can write $X\mathcal{V}_Y^{-1}=M_1+p^MM_2$ with $M_1$ having coefficients in $\mathcal{G}_{[0,p]}$ and $M_2$ in $\frac{1}{Y^{e/(p-1)}}\mathcal{G}_{[1/(p-1),\infty[}\subset\tilde{\mathbf{A}}$. Therefore the matrix $X\mathcal{V}_Y^{(-1)}-p^MM_2=M_1-\mathcal{U}$ has coefficients in $\mathcal{G}_{[p-1,\infty[}\left[\frac{1}{Y}\right]\bigcap\mathcal{G}_{[0,p-1[}\left[\frac{1}{p}\right]=\tilde{\mathbf{A}}^+$, as desired. □

Remark that $x \in F(W(\mathfrak{m}_{\tilde{\mathbf{E}}}))$ can be written as $x = [x_0] +_F u$ with $u \in F(pW(\mathfrak{m}_{\tilde{\mathbf{E}}}))$, thus

$$\begin{aligned}\left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(x) &= \left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}([x_0]) + \left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(u) \\ &= [x_0] + \left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(u) \in W(\mathfrak{m}_{\tilde{\mathbf{E}}})^d\end{aligned}$$

since $l_{\mathcal{A}}(u) \in pW(\mathfrak{m}_{\tilde{\mathbf{E}}})^d$. In particular $\left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(\beta) \in W(\mathfrak{m}_{\tilde{\mathbf{E}}})^d$, so that

$$\mathcal{V}_Y^{(-1)} \begin{pmatrix} \left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(\beta) \\ 0 \end{pmatrix} \in \tilde{\mathbf{A}}^h.$$

### 2.4.3. An explicit computation of the Hilbert symbol

We come to the proposition that explicitly gives the desired triple. The basic ingredient can be seen as a rewording of Proposition 3.8 of [2] which provides the $x$ coordinate of the triple and allows to prove that $y$ is zero. However, in order to get $z$, we have to carry Abrashkin's computations to the higher order. Indeed, we already know that $z$ belongs to $W(\mathfrak{m}_{\tilde{\mathbf{E}}})$, but we need its value modulo $XW(\mathfrak{m}_{\tilde{\mathbf{E}}})$.

Proposition 2.12. — *Denote by $U$ the principal part of the vector $\mathcal{V}_Y^{(-1)} \begin{pmatrix} \left(\frac{\mathcal{A}}{p} - 1\right) \circ l_{\mathcal{A}}(\beta) \\ 0 \end{pmatrix}$ and define $\hat{x} = (o^1, \dots, o^h)U$. Then*

(1) $U \in (W[[\frac{1}{Y}]] \cap \tilde{\mathbf{A}})^h$

(2) *Let $\hat{b} \in T(F) \otimes \tilde{\mathbf{A}}$ be a solution of $(\varphi - 1)\hat{b} = \hat{x}$ then for any $g \in G_K$,*

$$(g-1)\hat{b} \equiv (\beta(\pi), g]_{F,M} \mod p^M \tilde{\mathbf{A}} + W(\mathfrak{m}_{\tilde{\mathbf{E}}}).$$

*Proof.* — Point 1. can be shown like Point 3. of Lemma 2.11 above. The second point appears as a rewording of [2, Proposition 3.8]. Let us give another proof. Let us recall from Lemma 2.11:

$$\mathcal{V}_Y^{-1} \equiv \mathcal{V}^{-1} \mod \frac{p^M}{Y^{e(p+1)/(p-1)}} \mathfrak{m}_{[1,p]}$$

then there is $\delta \in \frac{p^M}{Y^{e(p+1)/(p-1)}} \mathfrak{m}_{[1,p]}$ with $\mathcal{V}_Y^{-1} = \mathcal{V}^{-1} + \delta$. Write $\delta = \delta_1 + \delta_2$ with $\delta_1 \in p^{M-1} Y^{e(p^2-2p-1)/(p-1)} \mathcal{G}_{[0,p]}$ and $\delta_2 \in \frac{p^M}{Y^{e(p+1)/(p-1)}} \mathfrak{m}_{[1,\infty[}$. Let us recall that we write $\mathcal{V}_Y^{-1} = \mathcal{V}_Y^{(-1)} + \mathcal{U}$ so that

$$X\mathcal{V}_Y^{(-1)} - X\delta_2 = X\mathcal{V}^{-1} + X\delta_1 - X\mathcal{U} \in \mathcal{G}_{[p-1,\infty[}\left[\frac{1}{Y}\right] \bigcap \mathcal{G}_{[0,p-1[}\left[\frac{1}{p}\right] = \tilde{\mathbf{A}}^+.$$

Then, if $\mathcal{B}$ is a matrix with coefficients in $\tilde{\mathbf{A}}$ such that

$$(\varphi-1)\mathcal{B} = \left(\mathcal{V}_Y^{(-1)} - \delta_2\right)\begin{pmatrix}\left(\frac{\mathcal{A}}{p}-1\right)\circ l_{\mathcal{A}}(\beta)\\ 0\end{pmatrix}, \tag{2.9}$$

as in Paragraph 1.7, $(\varphi-\omega)(X_1\mathcal{B}) = \left(X\mathcal{V}_Y^{(-1)} - X\delta_2\right)\begin{pmatrix}\left(\frac{\mathcal{A}}{p}-1\right)\circ l_{\mathcal{A}}(\beta)\\ 0\end{pmatrix}$ has coefficients in $\tilde{\mathbf{A}}^+$ so that, by successive approximations modulo $p^k$ and since $\tilde{\mathbf{E}}^+$ is integrally closed, we get $\mathcal{B}\in\frac{1}{X_1}\tilde{\mathbf{A}}^+\subset \mathrm{Fil}^0 B_{crys}$. Still write

$$\Lambda = \mathcal{V}^{-1}\begin{pmatrix} l_{\mathcal{A}}(\delta(\xi) -_F \beta)\\ m_{\mathcal{A}}(\delta(\xi) -_F \beta)\end{pmatrix} \in \left(\mathrm{Fil}^0 A_{crys}\right)^h.$$

We compute

$$(\varphi-1)\left(\mathcal{B}-\Lambda\right) = \left(\delta_1 - \mathcal{U}\right)\begin{pmatrix}\left(\frac{\mathcal{A}}{p}-1\right)\circ l_{\mathcal{A}}(\beta)\\ 0\end{pmatrix}$$

Since the coefficients of $\delta_1' := \delta_1\begin{pmatrix}\left(\frac{\mathcal{A}}{p}-1\right)\circ l_{\mathcal{A}}(\beta)\\ 0\end{pmatrix}$ lie in $Y\mathcal{G}_{[0,p]}$, the series $-\sum_{n\in\mathbb{N}}\varphi^n(\delta_1')$ converges to $\Delta_1\in Y\mathcal{G}_{[0,p]}$ with $(\varphi-1)(\Delta_1)=\delta_1'$. Likewise the coefficients of $\delta_2' = \mathcal{U}\begin{pmatrix}\left(\frac{\mathcal{A}}{p}-1\right)\circ l_{\mathcal{A}}(\beta)\\ 0\end{pmatrix}$ belong to $YW[[Y]] + \frac{Y^{ep-\lceil ep/(p-1)\rceil}}{p}\mathcal{G}_{Y,[0,p]}$ so that $-\sum_{n\in\mathbb{N}}\varphi^n(\delta_2')$ converges to a $\Delta_2$ with coefficients in $YW[[Y]] + \frac{Y^{ep-\lceil ep/(p-1)\rceil}}{p}\mathcal{G}_{Y,[0,p]}$ satisfying $(\varphi-1)(\Delta_2)=\delta_2'$. Finally,

$$(\varphi-1)\left(\mathcal{B}-\Lambda-\Delta_1+\Delta_2\right)=0$$

with $\mathcal{B}-\Lambda-\Delta_1+\Delta_2$ having coefficients in $\mathrm{Fil}^0 B_{crys}$. And the fact that $(\mathrm{Fil}^0 B_{crys})_{\varphi=1}=\mathbb{Q}_p$ shows $\mathcal{B}-\Lambda-\Delta_1+\Delta_2\in\mathbb{Q}_p$. Then, for $g\in G_K$,

$$(g-1)(o^1,\dots,o^h)\left(\mathcal{B}-\Lambda-\Delta_1+\Delta_2\right)\equiv 0 \mod p^M$$

so that

$$\begin{aligned}(g-1)(o^1,\dots,o^h)\left(\mathcal{B}\right) &= (g-1)(o^1,\dots,o^h)\left(\Lambda+\Delta_1-\Delta_2\right)\\ &= (\iota(\xi),g]_{\mathcal{R}(F)} + (g-1)(o^1,\dots,o^h)\left(\Delta_1-\Delta_2\right)\end{aligned}$$

And since $\Delta_1-\Delta_2$ has coefficients in $\frac{Y}{p}\mathcal{G}_{[0,p]}$, the same holds for the coordinates of $(g-1)(o^1,\dots,o^h)\left(\Delta_1-\Delta_2\right)$. We find that the coordinates of $(g-1)\left((o^1,\dots,o^h)\mathcal{B}\right) - (\iota(\xi),g]_{\mathcal{R}(F)}$ have coefficients in

$$\frac{1}{Y^{e(p+1)/(p-1)}}\mathcal{G}_{[\frac{1}{p-1},\infty[}\bigcap\frac{Y}{p}\mathcal{G}_{[0,p]} = Y\tilde{\mathbf{A}}^+\subset W(\mathfrak{m}_{\tilde{\mathbf{E}}}).$$

To finish, recall Equality (2.9): there exists $\delta_2 \in \frac{p^M}{Y^{e(p+1)/(p-1)}}\mathfrak{m}_{[1,\infty[} \subset p^M\tilde{\mathbf{A}}$ such that

$$(\varphi-1)\mathcal{B} = \left(\mathcal{V}_Y^{(-1)} - \delta_2\right)\begin{pmatrix}\left(\frac{\mathcal{A}}{p}-1\right)\circ l_{\mathcal{A}}(\beta)\\ 0\end{pmatrix}$$

And surjectivity of $\varphi-1$ on $\tilde{\mathbf{A}}$ permits to conclude. □

*Remark 2.13.* — It is possible to get rid of $A_{crys}$ here by studying the action of $(\varphi-1)$ on $\mathcal{G}_{[0,p]}\left[\frac{1}{Y}\right]$.

### 2.4.4. An explicit computation of the Kummer map

We will use the above result in the following specified form.

Proposition 2.14. — *Let $\alpha \in F(\mathfrak{m}_K)$ and $\beta \in F(YW[[Y]])$ be such that $\alpha = \theta(\beta) = \beta(\pi)$. Put*

$$x = (o^1,\dots,o^h)\mathcal{V}_Y^{(-1)}\begin{pmatrix}(\frac{\mathcal{A}}{p}-1)\circ l_{\mathcal{A}}(\beta)\\ 0\end{pmatrix} \in \tilde{D}_L(T(F)).$$

*There exists $z \in \tilde{D}_L(T(F)) \cap T(F)\otimes W(\mathfrak{m}_{\tilde{\mathbf{E}}})$ unique modulo $p^M$ such that the class of $(x,0,z)$ corresponds to the image of $\alpha$ by the Kummer map $F(\mathfrak{m}_K) \to H^1(K, F[p^M])$. Moreover,*

$$z \equiv XY\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix} \mod XW(\mathfrak{m}_{\tilde{\mathbf{E}}}).$$

*Proof.* — We use Proposition 2.12, and remark that

$$\hat{x} - x \in T(F)\otimes YW[[Y]] \subset (\varphi-1)(T(F)\otimes YW[[Y]]).$$

So, if $b \in T(F)\otimes\tilde{\mathbf{A}}$ satisfies $(\varphi-1)b = x$, then for any $g \in G_K$,

$$(g-1)b \equiv (\alpha, g]_{F,M} \mod p^M\tilde{\mathbf{A}} + W(\mathfrak{m}_{\tilde{\mathbf{E}}}).$$

Thus for any $h \in G_L$, since $(h-1)b \in \ker(\varphi-1) = T(F)$,

$$(h-1)b \equiv (\alpha, h]_{F,M} \mod p^M T(F).$$

We deduce there exist $y, z \in \tilde{D}_L(T(F))$ unique modulo $p^M$ such that the class of the triple $(x,y,z)$ corresponds to the image of $\alpha$ in $H^1(K, F[p^M])$; indeed let $(x_1,y_1,z_1)$ be such a triple, and $b_1 \in T(F)\otimes\tilde{\mathbf{A}}$ a solution of $(\varphi-1)b_1 = x_1$ then,

$$\forall h \in G_L,\ (h-1)(b_1-b) \equiv 0 \mod p^M,\ \text{ thus, }\ b_1 - b \in \tilde{D}_L(F[p^M]),$$

which shows that the class of $(x, y_1 + (\gamma-1)(b-b_1), z_1 + (\tau-1)(b-b_1))$ corresponds to the same class as $(x_1,y_1,z_1)$ and, if $x$ is fixed, this triple is unique.

Let us now determine $y$: let $\tilde{\gamma}$ lift $\gamma$ then

$$(\tilde{\gamma}-1)(-b)+y=(\alpha,\tilde{\gamma}]_{F,M}\equiv(\tilde{\gamma}-1)(-b)\mod p^M\tilde{\mathbf{A}}+W(\mathfrak{m}_{\tilde{\mathbf{E}}})$$

hence, since $(\tilde{\gamma}-1)(-b)\in T(F)$, $y\in T(F)\otimes W(\mathfrak{m}_{\tilde{\mathbf{E}}})\cap T(F)=\{0\}$.

Likewise, let $\tilde{\tau}$ lift $\tau$ then

$$(\tilde{\tau}-1)(-b)+z=(\alpha,\tilde{\tau}]_{F,M}\equiv(\tilde{\tau}-1)(-b)\mod p^M\tilde{\mathbf{A}}+W(\mathfrak{m}_{\tilde{\mathbf{E}}})$$

hence $z\in T(F)\otimes W(\mathfrak{m}_{\tilde{\mathbf{E}}})$. As $z$ satisfies moreover $(\tau-1)x=(\varphi-1)z$, this uniquely determines $z$ since $\varphi-1$ is injective on $T(F)\otimes W(\mathfrak{m}_{\tilde{\mathbf{E}}})$. In order to specify $z$, we need the

Lemma 2.15. —

(1) *For all $U\in W[[Y]]$, the following congruence holds*

$$(\tau-1)\mathcal{V}_Y^{(-1)}U\equiv XY\mathcal{V}_Y^{(-1)}\frac{dU}{dY}\mod XW(\mathfrak{m}_{\tilde{\mathbf{E}}})+p^M\tilde{\mathbf{A}}.$$

(2) *There exists $u\in\mathfrak{m}_{[p/(p-1),p]}$ such that*

$$\varphi_{\mathcal{G}}(X\mathcal{V}_Y^{-1})=(\varphi_{\mathcal{G}}(X)\mathcal{V}_Y^{-1}\mathcal{E}^{-1}+p^Mu)\begin{pmatrix}\frac{1}{p}I_d & 0\\ 0 & I_{h-d}\end{pmatrix}$$

*Proof of the lemma.* — We first specify $(\tau-1)\mathcal{V}_Y^{(-1)}$. Remark that if $f(Y)$ is a series in $W\{\{Y\}\}\cap\tilde{\mathbf{A}}$, $(\tau-1)f(Y)=\sum_{n\geqslant1}\frac{(XY)^n}{n!}f^{(n)}(Y)$. Thus:

$$(\tau-1)\mathcal{V}_Y^{(-1)}=XY\frac{d}{dY}\mathcal{V}_Y^{(-1)}+\frac{(XY)^2}{2}\frac{d^2}{dY^2}\mathcal{V}_Y^{(-1)}+\sum_{n\geqslant3}\frac{(XY)^n}{n!}\frac{d^n}{dY^n}\mathcal{V}_Y^{(-1)}$$

Let us estimate the summand $\frac{(XY)^n}{n!}\frac{d^n}{dY^n}\mathcal{V}_Y^{(-1)}$. Lemma 2.11 shows that $\frac{d}{dY}\mathcal{V}_Y^{-1}=p^M\mathcal{V}_Y^{-1}\widetilde{W}\mathcal{V}_Y^{-1}$ for some $\widetilde{W}$ with coefficients in $W[[Y]]$ and the principal part of $\mathcal{V}_Y^{-1}\widetilde{W}\mathcal{V}_Y^{-1}$ is entire. Thus, on the one hand

$$XY\frac{d}{dY}\mathcal{V}_Y^{(-1)}+\frac{(XY)^2}{2}\frac{d^2}{dY^2}\mathcal{V}_Y^{(-1)}\in p^M\tilde{\mathbf{A}}$$

and on the other hand one can write

$$\frac{d^n}{dY^n}\mathcal{V}_Y^{-1}=\sum_{k=1}^{n}p^{Mk}w_{n,k}$$

where the $w_{n,k}$ are sums of terms of the form $\mathcal{V}_Y^{-1}\widetilde{W}_{n,1}\mathcal{V}_Y^{-1}\widetilde{W}_{n,2}\ldots\widetilde{W}_{n,k}\mathcal{V}_Y^{-1}$, where $\widetilde{W}_{n,i}\in W[[Y]]$ are derivatives of $\widetilde{W}$. Recall that the coefficients of $\mathcal{V}_Y^{-1}$ belong to $\frac{1}{X}\left(\mathcal{G}_{[0,p]}+p^{M-1}\mathfrak{m}_{[1/(p-1),p]}\right)$, then the coefficients of $\mathcal{V}_Y^{-1}\widetilde{W}_{n,1}\ldots\widetilde{W}_{n,k}\mathcal{V}_Y^{-1}$ lie in $\frac{1}{X^{k+1}}\mathcal{G}_{[0,p]}+p^{M-1}\left(\frac{1}{X^{k+1}}\mathfrak{m}_{[1/(p-1),p]}\right)$.

Suppose $1 < k < n-1$. Since $v_p(n!) \leqslant \lfloor n/(p-1) \rfloor = n'$, there is $u \in \mathbb{Z}_p$ such that

$$p^{Mk} \frac{(XY)^n}{n!} = Y^n X^{k+2} u \frac{X^{n-k-2}}{p^{n'-Mk}}.$$

Since $p > 2$ and $k > 1$,

$$(n'-Mk)(p-1) \leqslant n-k-2 \text{ and } \frac{X^{n-k-2}}{p^{n'-Mk}} \in W\left[\left[\frac{X^{p-1}}{p}\right]\right] \subset \mathcal{G}_{[0,p]}.$$

Thus, $p^{Mk} \frac{(XY)^{n-1}}{n!} w_{n,k}$ lies in $\mathfrak{m}_{[0,p]} + p^{M-1}\mathfrak{m}_{[1/(p-1),p]}$.

Let now $k = 1$, write

$$w_{n,1} = \mathcal{V}_Y^{-1} \frac{d^{n-1}}{dY^{n-1}} \widetilde{W} \mathcal{V}_Y^{-1} = (n-1)! \mathcal{V}_Y^{-1} \widetilde{W}' \mathcal{V}_Y^{-1}$$

and

$$\frac{(XY)^{n-1}}{n!} p^M w_{n,1} = \frac{(XY)^{n-1}}{n} \mathcal{V}_Y^{-1} \widetilde{W}' \mathcal{V}_Y^{-1} = \frac{X^{n-p}}{n} Y^n X^{p-1} \mathcal{V}_Y^{-1} \widetilde{W}' \mathcal{V}_Y^{-1}$$

has coefficients in $\mathfrak{m}_{[0,p]} + p^{M-1}\mathfrak{m}_{[1/(p-1),p]}$ as before.

Let $k = n$, one has

$$w_{n,n} = n! \mathcal{V}_Y^{-1} \widetilde{W}_{n,1} \mathcal{V}_Y^{-1} \widetilde{W}_{n,2} \ldots \widetilde{W}_{n,n} \mathcal{V}_Y^{-1}, \ \forall i \ \widetilde{W}_{n,i} = \widetilde{W},$$

so that $\frac{(XY)^n}{n!} p^{Mn} w_{n,n}$ lies in $p^{Mn} \frac{1}{X} \left(\mathcal{G}_{[0,p]} + p^{M-1}\mathfrak{m}_{[1/(p-1),p]}\right)$.

Finally, for $k = n-1$, since $v_p(n!) \leqslant n/(p-1) \leqslant n-1$,

$$\frac{(XY)^n}{n!} p^{M(n-1)} w_{n,n-1} \in Y^n \left(\mathcal{G}_{[0,p]} + p^{M-1}\mathcal{G}_{[1/(p-1),p]}\right).$$

The same argument as for Point 1. above shows that for $n > 2$, the coefficients of $\frac{(XY)^n}{n!} \frac{d^n}{dY^n} \mathcal{V}_Y^{(-1)}$ lie in $XW(\mathfrak{m}_{\tilde{\mathbf{E}}}) + p^M \tilde{\mathbf{A}}$ hence the coefficients of $(\tau - 1)\mathcal{V}_Y^{(-1)}$ lie in $XW(\mathfrak{m}_{\tilde{\mathbf{E}}}) + p^M \tilde{\mathbf{A}}$. Point 2. then follows from

$$(\tau-1)\mathcal{V}_Y^{(-1)} U = \left((\tau-1)\mathcal{V}_Y^{(-1)}\right) \tau U + \mathcal{V}_Y^{(-1)} (\tau-1) U$$

and the congruence $(\tau - 1)U \equiv XY \frac{dU}{dY} \mod XW(\mathfrak{m}_{\tilde{\mathbf{E}}})$.

Now, let us carry on computations of Lemma 2.11: we write $X\mathcal{V}_Y^{-1} = \frac{X}{t}(I_h + p^{M-1}u_1)^t \mathcal{V}^D$ with $u_1 \in \mathfrak{m}_{[\frac{1}{p-1},p]}$. And since $\mathcal{V}^D$ has coefficients in

$\mathcal{G}_{[0,p]} \subset A_{crys}$ where $\varphi_{\mathcal{G}}$ and $\varphi$ coincide, the following holds in $\mathcal{G}_{[p/(p-1),p]}$:

$$\begin{aligned}
&\varphi_{\mathcal{G}}\left(X\mathcal{V}_Y^{-1}\right) = \varphi_{\mathcal{G}}\left(\frac{X}{t}\right)\left(I_h + p^M\varphi_{\mathcal{G}}(v_1)\right)\varphi_{\mathcal{G}}\left({}^t\mathcal{V}^D\right)\\
= \quad & \varphi_{\mathcal{G}}\left(\frac{X}{t}\right)\left(I_h + p^M\varphi_{\mathcal{G}}(v_1)\right) p\ {}^t\mathcal{V}^D\mathcal{E}^{-1}\begin{pmatrix}\frac{1}{p}I_d & 0\\ 0 & I_{h-d}\end{pmatrix}\\
= \quad & \varphi_{\mathcal{G}}\left(\frac{X}{t}\right)(I_h + p^M\varphi_{\mathcal{G}}(v_1))(I_h - p^M v)pt\mathcal{V}_Y^{-1}\mathcal{E}^{-1}\begin{pmatrix}\frac{1}{p}I_d & 0\\ 0 & I_{h-d}\end{pmatrix}\\
= \quad & \varphi_{\mathcal{G}}(X)\left(\mathcal{V}_Y^{-1}\mathcal{E}^{-1} + p^M\tilde{v}\right)\begin{pmatrix}\frac{1}{p}I_d & 0\\ 0 & I_{h-d}\end{pmatrix}
\end{aligned}$$

where $\tilde{v} = \left(\varphi_{\mathcal{G}}(v_1) - v - p^M\varphi_{\mathcal{G}}(v_1)v\right)\mathcal{V}_Y^{-1}\mathcal{E}^{-1}$.

Let us clarify this: $v, v_1 \in \frac{1}{p}\mathfrak{m}_{[1/(p-1),p]}$, thus $\varphi_{\mathcal{G}}(v_1)$ lies in $\frac{1}{p}\mathfrak{m}_{[p/(p-1),p]}$. Therefore $p^M v\varphi_{\mathcal{G}}(v_1) \in \frac{1}{p}\mathfrak{m}_{[p/(p-1),p]}$ and,

$$\varphi_{\mathcal{G}}(v_1) - v - p^M\varphi_{\mathcal{G}}(v_1)v \in \frac{1}{p}\mathfrak{m}_{[p/(p-1),p]}.$$

Hence, since $\mathcal{V}_Y^{-1} \in \frac{1}{Y^{ep/(p-1)}}\mathcal{G}_{[1,p]}$, $p\tilde{v}$ lies in $\frac{1}{Y^{ep/(p-1)}}\mathfrak{m}_{[p/(p-1),p]}$. The result follows then from $\varphi_{\mathcal{G}}(X) \in pX\mathcal{G}_{[0,p]}$ and $X \in Y^{ep/(p-1)}\mathcal{G}_{[p/(p-1),\infty[}$. □

Remark $\varphi\left(XY \circ \frac{d}{dY}\right) = \frac{\varphi(X)}{p}Y\frac{d}{dY} \circ \varphi$ and $u\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix} \in \mathfrak{m}_{[p/(p-1),p]}$ so that we compute modulo $p^M\mathfrak{m}_{[p/(p-1),p]}$:

$$\begin{aligned}
\varphi_{\mathcal{G}}\left(XY\mathcal{V}_Y^{-1}\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\right) &\equiv \frac{\varphi(X)}{p}Y\mathcal{V}_Y^{-1}\frac{d}{dY}\mathcal{E}^{-1}\begin{pmatrix}\frac{1}{p}I_d & 0\\ 0 & I_{h-d}\end{pmatrix}\varphi\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\\
&\equiv \frac{\varphi(X)}{p}Y\mathcal{V}_Y^{-1}\frac{d}{dY}\begin{pmatrix}\frac{\mathcal{A}}{p} \circ l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}.
\end{aligned}$$

This yields to

$$\begin{aligned}
&\varphi\left(XY\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\right)\\
= \quad & \varphi_{\mathcal{G}}\left(XY\mathcal{V}_Y^{-1}\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix} + XY\left(\mathcal{V}_Y^{(-1)} - \mathcal{V}_Y^{-1}\right)\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\right)\\
= \quad & \frac{\varphi(X)}{p}Y\mathcal{V}_Y^{-1}\frac{d}{dY}\begin{pmatrix}\frac{\mathcal{A}}{p} \circ l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix} + p^M u\\
& \qquad + \varphi_{\mathcal{G}}\left(XY\left(\mathcal{V}_Y^{(-1)} - \mathcal{V}_Y^{-1}\right)\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\right)
\end{aligned}$$

with $u \in \mathfrak{m}_{[p/(p-1),p]}$. Write $u = u_1 + u_2$ with $u_1 \in \frac{X^{p-1}}{p}\mathcal{G}_{[0,p]}$, thus $p^M u_1 \in X\mathfrak{m}_{[0,p]}$ and $u_2 \in \mathfrak{m}_{[p/(p-1),\infty[}$. In addition, $\mathcal{V}_Y^{(-1)} - \mathcal{V}_Y^{-1} \in \mathcal{G}_{[0,p]} \otimes \mathbb{Q}_p$ hence

$$\varphi_{\mathcal{G}} \left( XY \left( \mathcal{V}_Y^{(-1)} - \mathcal{V}_Y^{-1} \right) \frac{d}{dY} \begin{pmatrix} l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta) \end{pmatrix} \right) \in X\mathcal{G}_{[0,p]} \otimes \mathbb{Q}_p.$$

Write moreover

$$\frac{\varphi(X)}{p} Y\mathcal{V}_Y^{-1} \frac{d}{dY} \begin{pmatrix} \frac{\mathcal{A}}{p} \circ l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta) \end{pmatrix} = XY\mathcal{V}_Y^{(-1)} \frac{d}{dY} \begin{pmatrix} \frac{\mathcal{A}}{p} \circ l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta) \end{pmatrix} + \Xi_1 + \Xi_2,$$

with

$$\Xi_1 = XY \left( \mathcal{V}_Y^{-1} - \mathcal{V}_Y^{(-1)} \right) \frac{d}{dY} \begin{pmatrix} \frac{\mathcal{A}}{p} \circ l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta) \end{pmatrix} \in \left( X\mathfrak{m}_{[0,p]} \otimes \mathbb{Q}_p \right)^h$$

$$\Xi_2 = \frac{\varphi(X) - pX}{p} Y\mathcal{V}_Y^{-1} \frac{d}{dY} \begin{pmatrix} \frac{\mathcal{A}}{p} \circ l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta) \end{pmatrix} \in X \left( \mathfrak{m}_{[0,p]} + p^{M-1}\mathcal{G}_{[p/(p-1),\infty[} \right)^h$$

It can then be written as $M_1 + M_2$ with $M_1 \in X\mathfrak{m}_{[0,p]}$ and $M_2 \in p^M \tilde{\mathbf{A}}$. Eventually,

$$\varphi \left( XY\mathcal{V}_Y^{(-1)} \frac{d}{dY} \begin{pmatrix} l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta) \end{pmatrix} \right) - XY\mathcal{V}_Y^{(-1)} \frac{d}{dY} \begin{pmatrix} \frac{\mathcal{A}}{p} \circ l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta) \end{pmatrix} - p^M M_0$$

lies in $M_h(X\mathcal{G}_{[0,p]} \otimes \mathbb{Q}_p)$ for some $M_0 \in \tilde{\mathbf{A}}$. Then, since $X\mathcal{G}_{[0,p]} \otimes \mathbb{Q}_p \cap \tilde{\mathbf{A}} = X\tilde{\mathbf{A}}^+$, we deduce the congruence modulo $X\tilde{\mathbf{A}}^+ + p^M\tilde{\mathbf{A}}$

$$\varphi \left( XY\mathcal{V}_Y^{(-1)} \frac{d}{dY} \begin{pmatrix} l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta) \end{pmatrix} \right) \equiv XY\mathcal{V}_Y^{(-1)} \frac{d}{dY} \begin{pmatrix} \frac{\mathcal{A}}{p} \circ l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta) \end{pmatrix}$$

which lets us prove the proposition since modulo $X\tilde{\mathbf{A}}^+ + p^M\tilde{\mathbf{A}}$

$$\begin{aligned} (\varphi - 1)XY\mathcal{V}_Y^{(-1)} \frac{d}{dY} \begin{pmatrix} l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta) \end{pmatrix} &\equiv XY\mathcal{V}_Y^{(-1)} \frac{d}{dY} \begin{pmatrix} \left( \frac{\mathcal{A}}{p} - 1 \right) \circ l_{\mathcal{A}}(\beta) \\ 0 \end{pmatrix} \\ &\equiv (\tau - 1)x \end{aligned}$$

and since the equation $(\varphi - 1)Z = \alpha \in X\tilde{\mathbf{A}}^+ + p^M\tilde{\mathbf{A}}$ admits a solution $Z \in X\tilde{\mathbf{A}}^+ + p^M\tilde{\mathbf{A}}$. □

### 2.5. The explicit formula

We come now to the proof of the main theorem, the explicit formula for the Hilbert symbol. Write Tr for the trace $\mathrm{Tr}_{W/\mathbb{Z}_p}$.

THEOREM 2.16. — *Let $\beta \in F(YW[[Y]])$ and $\alpha \in (W[[Y]][\frac{1}{Y}])^\times$. Denote*

$$\mathscr{L}(\alpha) = \left(1 - \frac{\varphi}{p}\right) \log \alpha(Y) = \frac{1}{p} \log \frac{\alpha(Y)^p}{\alpha^\varphi(Y^p)} \in W[[Y]].$$

*Then coefficients of the Hilbert symbol $(\alpha(\pi), \beta(\pi))_{F,M}$ in $(o_M^1, \dots, o_M^h)$ are*

$$(\mathrm{Tr} \circ \mathrm{Res}_Y)\mathcal{V}_Y^{-1}\left(\begin{pmatrix}\left(1 - \frac{\mathcal{A}}{p}\right) \circ l_{\mathcal{A}}(\beta) \\ 0\end{pmatrix} d_{\log}\alpha(Y) - \mathscr{L}(\alpha)\frac{d}{dY}\begin{pmatrix}\frac{\mathcal{A}}{p} \circ l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta)\end{pmatrix}\right)$$

*Proof.* — We use the fact that if $\eta \in H^1(K, \mathbb{Z}/p^M\mathbb{Z})$ and $r(x) \in G_K^{ab}$ is the image by the reciprocity isomorphism $x \in K$ then $\mathrm{inv}_K(\partial x \cup \eta) = \eta(r(x))$. From Proposition 1.14, $\partial\alpha(\pi)$ corresponds to a triple $(x, y, z)$ congruent modulo $XYW[[X, Y]]$ to

$$\left(-\frac{s(Y)}{X} - \frac{s(Y)}{2},\ 0,\ Y d_{\log}S(Y)\right) \otimes \varepsilon.$$

We compute its cup-product with the image $(x', 0, z')$ in $H^1(K, F[p^M])$ of $\theta(\beta)$ given by Proposition 2.14 where we recall that

$$\begin{aligned} x' &= \mathcal{V}_Y^{(-1)}\begin{pmatrix}(\frac{\mathcal{A}}{p} - 1) \circ l_{\mathcal{A}}(\beta) \\ 0\end{pmatrix} \\ z' &\equiv XY\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta) \\ m_{\mathcal{A}}(\beta)\end{pmatrix} \mod XW(\mathfrak{m}_{\tilde{\mathbf{E}}}) \end{aligned}$$

We get the triple $(a, b, c)$ where:

$$a = y\mathcal{V}_Y^{-1}\begin{pmatrix}(\frac{\mathcal{A}}{p} - 1) \circ l_{\mathcal{A}}(\beta) \\ 0\end{pmatrix} \in W(\mathfrak{m}_{\tilde{\mathbf{E}}})$$

because Proposition 1.14 says that $y \in XYW[[X, Y]]$ and Lemma 2.11 that $XY\mathcal{V}_Y^{(-1)}$ has coefficients in $W(\mathfrak{m}_{\tilde{\mathbf{E}}}) + p^M\tilde{\mathbf{A}}$. Moreover,

$$c = -y \otimes \gamma z' + \sum_{n \geqslant 1}\binom{\chi(\gamma)}{n}\sum_{k=1}^{n-1} C_{n-1}^k(\tau - 1)^{k-1}z \otimes \tau^k(\tau - 1)^{n-1-k}z'$$

lies in $W(\mathfrak{m}_{\tilde{\mathbf{E}}})$ because $y, z, z' \in W(\mathfrak{m}_{\tilde{\mathbf{E}}})$. Finally, $b = z \otimes \tau x' - x \otimes \varphi z'$ and

$$z \otimes \tau x' = (\tau - 1)(\log(S(Y))/t + \mu)\ \tau\left(\mathcal{V}_Y^{(-1)}\begin{pmatrix}(\frac{\mathcal{A}}{p} - 1) \circ l_{\mathcal{A}}(\beta) \\ 0\end{pmatrix}\right) \otimes \varepsilon.$$

On the one hand

$$(\tau - 1)(\log(S(Y))/t + \mu) \equiv Y d_{\log}F(Y) \mod XYW[[X, Y]]$$

and on the other hand, Lemma 2.15 says that $\tau\left(\mathcal{V}_Y^{(-1)}\begin{pmatrix}(\frac{\mathcal{A}}{p}-1)\circ l_{\mathcal{A}}(\beta)\\0\end{pmatrix}\right)$ is congruent modulo $XYW[[X,Y]]$ to

$$\mathcal{V}_Y^{(-1)}\begin{pmatrix}(\frac{\mathcal{A}}{p}-1)\circ l_{\mathcal{A}}(\beta)\\0\end{pmatrix}+XY\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix}(\frac{\mathcal{A}}{p}-1)\circ l_{\mathcal{A}}(\beta)\\0\end{pmatrix}.$$

Thus, since $XY\mathcal{V}_Y^{(-1)}$ has coefficients in $W(\mathfrak{m}_{\tilde{\mathbf{E}}})+p^M\tilde{\mathbf{A}}$,

$$z\otimes\tau x'\equiv Y\mathcal{V}_Y^{(-1)}\begin{pmatrix}(\frac{\mathcal{A}}{p}-1)\circ l_{\mathcal{A}}(\beta)\\0\end{pmatrix}d_{\log}S(Y)\mod W(\mathfrak{m}_{\tilde{\mathbf{E}}}).$$

Finally,

$$-x\otimes\varphi z'=\left(-\frac{s(Y)}{X}-\frac{s(Y)}{2}\right)z'\otimes\varepsilon$$

and since $z'\equiv XY\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\m_{\mathcal{A}}(\beta)\end{pmatrix}$ modulo $XW(\mathfrak{m}_{\tilde{\mathbf{E}}})$, we get the congruence

$$-x\otimes\varphi z'\equiv Ys(Y)\mathcal{V}_Y^{(-1)}\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\m_{\mathcal{A}}(\beta)\end{pmatrix}\mod W(\mathfrak{m}_{\tilde{\mathbf{E}}}).$$

Eventually, $(a,b,c)$ is congruent mod $W(\mathfrak{m}_{\tilde{\mathbf{E}}})$ to $(0,b',0)$ with $b'$ equal to

$$Y\mathcal{V}_Y^{(-1)}\left(\begin{pmatrix}(\frac{\mathcal{A}}{p}-1)\circ l_{\mathcal{A}}(\beta)\\0\end{pmatrix}d_{\log}S(Y)+\frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\m_{\mathcal{A}}(\beta)\end{pmatrix}\frac{1}{p}\log\frac{S(Y)^p}{S(Y^p)}\right)\otimes\varepsilon.$$

The theorem follows then from the lemma:

LEMMA 2.17. — *Let $C=C_{\varphi,\gamma,\tau}(\tilde{\mathbf{A}}_L(1))$ be the complex computing Galois cohomology of $\mathbb{Z}_p(1)$.*

(1) *Let $f(Y)=\sum_{n>0}\frac{a_n}{Y^n}\in M_h(\tilde{\mathbf{A}})$ be the principal part of $\mathcal{V}_Y^{(-1)}g(Y)$ with $g(Y)$ having coefficients in $W[[Y]]$. Then there exists a triple $(x_1,x_2,0)$ with coefficients in $W(\mathfrak{m}_{\tilde{\mathbf{E}}})$ such that $(x_1,x_2+f(Y)\otimes\varepsilon,0)\in B^2(C)$. In other words the image of $(x_1,x_2+f(Y)\otimes\varepsilon,0)$ in $H^2(K,\mathbb{Z}_p(1))$ is zero.*
(2) *Let $(x,y,z)\in Z^2(C)$ with $x,y,z\in W(\mathfrak{m}_{\tilde{\mathbf{E}}})(1)$ then $(x,y,z)\in B^2(C)$.*
(3) *Let $w\in W$ then $(0,w\otimes\varepsilon,0)\in Z^2(C)$ and its image through the reciprocity isomorphism is* $\mathrm{Tr}(w)$.

*Proof of the lemma.* — Put $w_n=\frac{1}{Y^n((1+X)^{-n}-1)}+\frac{1}{2Y^n}\in\tilde{\mathbf{A}}_L$. Then

$$(\tau-1)w_n=\frac{1}{Y^n}+\frac{1}{2}(\tau-1)\frac{1}{Y^n}\tag{2.10}$$

and

$$\gamma\left(\frac{\varepsilon}{Y^n\left((1+X)^{-n}-1\right)}\right) = \frac{\chi(\gamma)\varepsilon}{Y^n\left((1+X)^{-\chi(\gamma)n}-1\right)} = \chi(\gamma)\delta^{-1}\left(\frac{\varepsilon}{Y^n\left((1+X)^{-n}-1\right)}\right).$$

The Taylor expansion

$$\delta^{-1} = \chi(\gamma) - \frac{\chi(\gamma)(\chi(\gamma)-1)}{2}(\tau-1) + (\tau-1)^2 g(\tau-1)$$

where $g(\tau-1)$ is a power series in $\tau-1$ yields to the relation

$$(\gamma-1)w_n\otimes\varepsilon = g(\tau-1)(\tau-1)\frac{1}{Y^n}. \tag{2.11}$$

From Lemma 2.15, we know $(\tau-1)\mathcal{V}^{(-1)}U$ for $U\in W[[Y]]$ has coefficients in $W(\mathfrak{m}_{\tilde{\mathbf{E}}})$. Relation (2.10) then shows $(\tau-1)\sum_{n>0}a_nw_n = f(Y) \mod W(\mathfrak{m}_{\tilde{\mathbf{E}}})$ and Relation (2.11) that $(\gamma-1)\sum_{n>0}a_nw_n = 0 \mod W(\mathfrak{m}_{\tilde{\mathbf{E}}})$ which proves that the coboundary image of triple $(\sum_{n>0}a_nw_n,0,0)$ in $H^2(C)$ has the desired form, hence 1.

To show 2. we have to solve for $x,y,z\in W(\mathfrak{m}_{\tilde{\mathbf{E}}})(1)$ the system

$$\begin{aligned} x &= (\gamma-1)u+(1-\varphi)v\\ y &= (\tau-1)u+(1-\varphi)w\\ z &= (\tau^{\chi(\gamma)}-1)v+(\delta-\gamma)w. \end{aligned}$$

Consider therefore $v,\ w\in W(\mathfrak{m}_{\tilde{\mathbf{E}}})(1)$ solutions of $x=(\varphi-1)v$ and $y=(\varphi-1)w$ which exist, and are unique since $\varphi-1$ is bijective on $W(\mathfrak{m}_{\tilde{\mathbf{E}}})(1)$. Then, by combining these equations with the ones of the system, we get

$$(\varphi-1)((\tau^{\chi(\gamma)}-1)v+(\delta-\gamma)w) = -(\tau^{\chi(\gamma)}-1)x-(\delta-\gamma)y = (\varphi-1)z.$$

Since $z$ and $(\tau^{\chi(\gamma)}-1)v+(\delta-\gamma)w$ are elements of $W(\mathfrak{m}_{\tilde{\mathbf{E}}})(1)$ where $(\varphi-1)$ is injective, the equality $z=(\tau^{\chi(\gamma)}-1)v+(\delta-\gamma)w$ holds ; $(x,y,z)$ is then a coboundary, image of $(0,v,w)$.

Finally, for Point 3., remark that $(0,w\otimes\varepsilon,0)=(0,0,1\otimes\varepsilon)\cup(w,0,0)$. Proposition 1.14 says $(0,0,1\otimes\varepsilon)$ is the image through the Kummer map of $\pi$ a uniformizer of $K$. (To see this, take $F(Y)=Y$.) In addition $(0,w\otimes\varepsilon,0)$ corresponds from Theorem 1.5 to the character $\eta$ of $G_K$ defined in the following way: choose $b\in\tilde{\mathbf{A}}$ such that $(\varphi-1)b=w$, then for all $g\in G_K$, $\eta(g)=(1-g)b$. Remark that since $w\in W$, we can choose $b\in W^{nr}$ and that the image through the Kummer map of a uniformizer is the Frobenius $\mathrm{Frob}_K$, thus the image through reciprocity isomorphism of $(0,w\otimes\varepsilon,0)$ is

$$(1-\mathrm{Frob}_K)b = (1-\varphi^{f_K})b = (1+\varphi+\cdots+\varphi^{f_K-1})w = \mathrm{Tr}w$$

where $f_K = f(K/\mathbb{Q}_p)$, which proves the lemma. □

We prove then the theorem by remarking, from the congruence shown above, that the triple $(a, b, c)$ can be written as a sum of a triple $(0, g(Y), 0)$ where $g$ is the negative part of a vector series in $Y$ and then is zero in $H^2(K, \mathbb{Z}/p^M\mathbb{Z})$, of a triple with coefficients in $W(\mathfrak{m}_{\tilde{\mathbf{E}}})(1)$, then also a coboundary because of the lemma above and finally a triple $(0, w \otimes \varepsilon, 0)$ where $w$ is the constant term of the vector series

$$Y\mathcal{V}_Y^{(-1)}\left(\begin{pmatrix}(\frac{\mathcal{A}}{p}-1)\circ l_{\mathcal{A}}(\beta)\\ 0\end{pmatrix} d_{\log}\alpha(Y) + \frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\frac{1}{p}\log\frac{\alpha(Y)^p}{\alpha(Y^p)}\right)$$

hence the residue of

$$\mathcal{V}_Y^{-1}\left(\begin{pmatrix}\left(\frac{\mathcal{A}}{p}-1\right)\circ l_{\mathcal{A}}(\beta)\\ 0\end{pmatrix} d_{\log}\alpha(Y) + \frac{d}{dY}\begin{pmatrix}l_{\mathcal{A}}(\beta)\\ m_{\mathcal{A}}(\beta)\end{pmatrix}\frac{1}{p}\log\frac{\alpha(Y)^p}{\alpha(Y^p)}\right).$$

The only term with a non zero contribution is then the residue, and that contribution is, according to the lemma, given by the trace. □

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Manuscrit reçu le 17 mars 2009,
révisé le 6 novembre 2009,
accepté le 23 novembre 2009.

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