Geertrui Klara IMMINK

Accelero-summation of the formal solutions of nonlinear difference equations


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ACCELERO-SUMMATION OF THE FORMAL
SOLUTIONS OF NONLINEAR DIFFERENCE
EQUATIONS

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ABSTRACT. — In 1996, Braaksma and Faber established the multi-summability, on suitable multi-intervals, of formal power series solutions of locally analytic, nonlinear difference equations, in the absence of “level 1+”. Combining their approach, which is based on the study of corresponding convolution equations, with recent results on the existence of flat (quasi-function) solutions in a particular type of domains, we prove that, under very general conditions, the formal solution is accelero-summable. Its sum is an analytic solution of the equation, represented asymptotically by the formal solution in a certain unbounded domain.

1. Introduction

We consider nonlinear difference equations of the form

\[ y(z + 1) = z^{\lambda/p} F(z^{1/p}, y(z)) \]

where \( p \in \mathbb{N} \) (the set of positive integers), \( \lambda \in \mathbb{Z} \) and \( F \) is a \( \mathbb{C}^n \)-valued function, analytic in a neighbourhood of \((\infty, y_0)\) for some \( y_0 \in \mathbb{C}^n \). We assume that the equation possesses a formal power series solution of the

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form \( \hat{f}(z) = \sum_{m=0}^{\infty} a_m z^{-m/p} \) where \( a_m \in \mathbb{C} \) for all \( m \in \mathbb{N} \), \( a_0 = y_0 \), and, furthermore (identifying \( D_2 \mathcal{F} \) with its Taylor series at \((\infty, y_0)\)), that

\[
(1.2) \quad \hat{A} := D_2 \mathcal{F}(z^{1/p}, \hat{f}(z)) \in GL(n; \mathbb{C}[[z^{-1/p}]][z^{1/p}])
\]

It is the purpose of this paper to lift the, usually divergent, formal solution \( \hat{f} \) to actual, analytic solutions, represented asymptotically by \( \hat{f} \) in certain unbounded domains of the complex plane and characterized by their asymptotic properties in some way. To this end we use the powerful tool of accelero-summation developed by Jean Ecalle.

In [3] Braaksma and Faber prove that, under some additional conditions and on appropriate multi-intervals, \( \hat{f} \) is multi-summable and its multi-sum satisfies the equation (1.1). In particular, they assume that the levels of a linear difference operator \( \Delta \), associated with (1.1), are \( \leq 1 \) (in general, the levels of a difference operator are nonnegative rational numbers \( \leq 1 \), or the so-called level \( 1^+ \)). Their approach is based on the study of corresponding convolution equations, one for each positive level \( k_j \) of \( \Delta \), obtained by applying a Borel transformation of order \( k_j \) to the original equation.

The present paper is concerned with the case that \( \Delta \) possesses an additional level \( 1^+ \). In that case, formal power series solutions are generally not multi-summable on any multi-interval. Combining some of the results and techniques from [3] with theorems on the existence of flat (quasi-function) solutions of nonlinear difference equations in [12, 14], we establish the accelero-summability of \( \hat{f} \) on appropriate multi-intervals. We restrict ourselves to domains of the complex plane that are invariant under a “forward shift” \( z \mapsto z + 1 \). However, if condition (1.2) is satisfied, analogous results can be proved for domains that are invariant under a “backward shift” \( z \mapsto z - 1 \).

The paper is arranged as follows. In section 2 we introduce notations and general definitions and recall a number of basic results. Furthermore, we present two simple examples of nonlinear difference equations with three distinct levels, including the level \( 1^+ \).

Section 3 deals with the relatively simple case that \( 0 \) is not a singular direction of level 1, or, equivalently, \(-\pi/2\) is not a Stokes direction of level 1. In that case, the formal solution is shown to be accelero-summable in the sense of [4], where the corresponding result for linear difference equations is proved. In this section we closely follow the method used in [3], except for the very last step in the summation procedure. The main result of this section is Theorem 3.8.

In section 4 we introduce a somewhat weaker notion of accelero-summability and prove that, according to this new definition, the formal
solution of (1.1) is accelero-summable, even if 0 is a singular direction of level 1. The main result of this paper is stated in § 4.2, Theorem 4.12.

2. Preliminaries

2.1. Levels and Stokes directions

We use the symbol $\tau$ to denote both the automorphism of $\mathbb{C}[[z^{-1/p}]]/(z^{1/p}]$ defined by $\tau(z^{1/p}) = z^{1/p} \sum_{h=0}^{\infty} \left( \frac{1}{h!} \right) z^{-h}$ and the shift operator $\tau y(z) := y(z+1)$. Two formal difference operators $\hat{\Delta}_1 := \hat{B}_1 \tau - \hat{A}_1$ and $\hat{\Delta}_2 := \hat{B}_2 \tau - \hat{A}_2$, where $\hat{A}_i$ and $\hat{B}_i \in \text{Gl}(n; \mathbb{C}[[z^{-1/p}]]/(z^{1/p}])$, for $i = 1, 2$, will be called equivalent if there exists $\hat{F} \in \text{Gl}(n; \mathbb{C}[[z^{-1/p}]]/(z^{1/p}])$ such that $(\tau \hat{F})^{-1} \hat{B}_1^{-1} \hat{\Delta}_1 \hat{F} = \hat{B}_2^{-1} \hat{\Delta}_2$, or equivalently, $(\tau \hat{F})^{-1} \hat{B}_1^{-1} \hat{A}_1 \hat{F} = \hat{B}_2^{-1} \hat{A}_2$. Any formal difference operator $\hat{\Delta} := \hat{B} \tau - \hat{A}$ and $\hat{B} \in \text{Gl}(n; \mathbb{C}[[z^{-1/p}]]/(z^{1/p}])$, is known to be equivalent to a canonical operator $\Delta^c$:

$$
(\tau \hat{F})^{-1} \hat{B}^{-1} \hat{\Delta} \hat{F} = \Delta^c = \tau - A^c
$$

where $A^c \in \text{Gl}(n; \mathbb{C}\{z^{-1/p}\}[[z^{1/p}]](1)$ and $A^c$ is a block-diagonal matrix of a particularly simple form (cf. [17, 8]). If the convergent series $A^c$ is identified with its sum, the canonical operator $\Delta^c$ can be viewed as an analytic difference operator, and the homogeneous equation $\Delta^c y = 0$ has a fundamental system of analytic solutions $\{y_j : j = 1, \ldots, n\}$ of the following form

$$
y^c_j(z) = z^{d_j z e q_j(z)} z^{\lambda_j} g_j(z), \ j = 1, \ldots, n,
$$

where $d_j \in \mathbb{Q}$, $q_j(z)$ is a polynomial in $z^{1/p}$ of degree at most $p$ and with vanishing constant term, $\lambda_j \in \mathbb{C}$ and $g_j \in \mathbb{C}^n \{\log z\}$, $j = 1, \ldots, n$. If $d_j = 0$, we denote the leading term of $q_j$ by $\mu_j z^{\kappa_j}$. If $d_j \neq 0$ we define $\kappa_j = 1$ and denote the term of order 1 by $\mu_j z$. When $\kappa_j = 1$, the number $\mu_j$ is determined up to a multiple of $2\pi i$. The numbers $\kappa_j$ with $j \in \{1, \ldots, n\}$ such that $d_j = 0$, are called levels of $\hat{\Delta}$. If there is a $j \in \{1, \ldots, n\}$ such that $d_j \neq 0$, $\hat{\Delta}$ is said to possess a level $1^+$. 

Remark 2.1.

(i) If we replace $\hat{F}$ in (2.1) by $z^{-\mu} \hat{F}$, the right-hand side of (2.1) continues to be a canonical difference operator, but with $\lambda_j$ in (2.2) replaced by $\lambda_j + \mu$

---

(1) In fact, it may be necessary to increase $p$ to some multiple $pq$ with $q \in \mathbb{N}$, but here we assume that $p$ has been chosen sufficiently large at the outset.
for \( j = 1, \ldots, n \). Thus, the numbers \( \lambda_j \) are determined by \( \hat{\Delta} \) up to a multiple of \( 1/p \).

(ii) If \( \hat{\Delta} \) has a level \( < 1 \), then it is considered to have a level 1 as well, even if there is no \( j \in \{1, \ldots, n\} \) with the property that \( d_j = 0 \) and \( \kappa_j = 1 \). This is related to the fact that \( y_j^c(z)e^{2l\pi iz} \) is a solution of the homogeneous equation \( \Delta^c y(z) = 0 \) for any \( l \in \mathbb{Z} \), with exponential growth or decay of order 1 at \( \infty \) if \( \kappa_j < 1 \) and \( l \neq 0 \).

Obviously, equivalent formal difference operators have the same canonical forms.

**Definition 2.2. (Stokes directions).** — Let \( 0 < \kappa < 1 \) be one of the positive levels of \( \hat{\Delta} \). We shall call singular directions of \( \hat{\Delta} \), of level \( \kappa \), the directions \( \frac{1}{\kappa} \left( \pi - \arg \mu_j \right) \), where \( j \in \{1, \ldots, n\} \) is such that \( \kappa_j = \kappa \) and \( \arg \mu_j \) is determined up to a multiple of \( 2\pi \). The singular directions of level 1 are the directions \( \pi - \arg (\mu_j + 2l\pi i) \), where \( l \in \mathbb{Z}, j \in \{1, \ldots, n\} \) such that \( d_j = 0 \) and \( \kappa_j = 1 \), and the directions \( \frac{\pi}{2} \mod \pi \) if there is a \( j \in \{1, \ldots, n\} \) such that \( d_j = 0 \) and \( \kappa_j < 1 \). If \( \alpha \) is a singular direction of level \( \kappa \in (0, 1] \), \( \alpha - \pi / (2\kappa) \) is called a Stokes direction and the pair \( \{\alpha - \pi / (2\kappa), \alpha + \pi / (2\kappa)\} \) a Stokes pair of \( \hat{\Delta} \), of level \( \kappa \).

The numbers \( \theta \in \mathbb{R} \) such that \( d_j \neq 0 \) and \( d_j \theta = \text{Im} \mu_j \) for some value of \( \text{Im} \mu_j \) (determined up to a multiple of \( 2\pi \)) will be called pseudo-Stokes directions of level \( \kappa \). The set of all pseudo-Stokes directions of level \( \kappa \) is denoted by \( \Theta(\hat{\Delta}) \).

**Remark 2.3.**

(i) Note that, for \( j \in \{1, \ldots, n\} \) such that \( d_j = 0 \) and \( \kappa_j > 0 \), \( y_j^c \) decreases exponentially of order \( \kappa_j \) as \( z \to \infty \) in a sector of the form: \( \frac{1}{\kappa_j} \left( \frac{\pi}{2} - \arg \mu_j \right) < \arg z < \frac{1}{\kappa_j} \left( \frac{\pi}{2} - \arg \mu_j + \pi \right) \) (bounded by a Stokes pair), uniformly on closed subsectors. The singular directions of \( \hat{\Delta} \) are the directions of maximal decrease at \( \infty \), of \( y_j^c(z)e^{2l\pi iz} \), where \( j \in \{1, \ldots, n\} \) is such that \( d_j = 0 \) and \( l \in \mathbb{Z}, l \neq 0 \) if \( \kappa_j = 0 \). (In some texts, cf. for instance [14], the singular directions have the opposite sign.)

(ii) If \( \hat{\Delta} \) has levels other than \( 1^+ \), \( \frac{\pi}{2} \mod \pi \) is either a singular direction or an accumulation point of singular directions of level 1.

With equation (1.1) we associate the formal difference operator \( \hat{\Delta} = \tau - \hat{A} \), where \( \hat{A} \) is defined by (1.2). We shall sometimes refer to the levels and Stokes directions of this operator as the levels and Stokes directions of the equation.
2.2. Asymptotic behaviour on sectors and multi-summability

In this section, we define classes of analytic functions with different types of asymptotic behaviour on sectors of the Riemann surface of the logarithm, to be denoted by \( \mathbb{C}^* \). We recall the definitions of multi-summability and \( k \)-precise quasi-function.

**Definition 2.4.** — Let \( I \) be an interval of \( \mathbb{R} \). By \( |I| \) we denote the length of \( I \), defined by \( |I| := \sup I - \inf I \). We use the following notations

\[
S(I) = \{ z \in \mathbb{C}^* : \arg z \in I \}, \quad S(I, R) = \{ z \in S(I) : |z| > R \}
\]

We write \( I' \prec I \) if \( I' \) is a relatively compact subinterval of \( I \) (i.e. \( I' \) is bounded and \( I' \subset I \)).

By \( \mathcal{A}_0^{ \leq 0} (I) \) we denote the set of functions \( f : S(I) \to \mathbb{C} \), with the property that, for every \( I' \prec I \), there exists a positive number \( r \) such that \( f \) is holomorphic and bounded on \( \{ z \in S(I') : |z| < r \} \). (More precisely, we consider equivalence classes of functions: two such functions are identified if, for every \( I' \prec I \), they coincide on \( \{ z \in S(I') : |z| < r' \} \) for some \( r' > 0 \).)

By \( \mathcal{A}_0 (I) \) we denote the set of functions \( f \in \mathcal{A}_0^{ \leq 0} (I) \) admitting an asymptotic power series expansion of the form \( \sum_{m=0}^{\infty} a_m z^{m/p} \), with \( p \in \mathbb{N} \), such that

\[
\sup_{z \in S(I')} |z^{-N/p} (f(z) - \sum_{m=0}^{N-1} a_m z^{m/p})| < \infty
\]

for any \( I' \prec I \) and some sufficiently small, positive \( r \) (depending on \( I' \)).

By \( \mathcal{A}^{ \leq 0} (I) \) we denote the set of (equivalence classes of) functions \( f : S(I) \to \mathbb{C} \), with the property that, for every \( I' \prec I \), there exists a positive number \( R \) such that \( f \) is holomorphic and bounded on \( S(I', R) \).

By \( \mathcal{A} (I) \) we denote the set of functions \( f \in \mathcal{A}^{ \leq 0} (I) \) admitting an asymptotic expansion \( \hat{f}(z) = \sum_{m=0}^{\infty} a_m z^{-m/p} \), with \( p \in \mathbb{N} \), such that

\[
\sup_{z \in S(I', R)} |z^{N/p} (f(z) - \sum_{m=0}^{N-1} a_m z^{-m/p})| < \infty
\]

for any \( I' \prec I \) and some sufficiently large \( R \) (depending on \( I' \)). For all \( N \in \mathbb{N} \), \( R_N(f; z) \) will denote the remainder: \( f(z) - \sum_{m=0}^{N-1} a_m z^{-m/p} \).

By \( \mathcal{A}^{ \leq -k} (I) \) we denote the set of \( f \in \mathcal{A} (I) \) with the property that, for any \( I' \prec I \), there exist positive constants \( R \) and \( c \) such that

\[
\sup_{z \in S(I', R)} e^{c|z|^k} |f(z)| < \infty
\]

(so \( \hat{f} = 0 \)).
By $\mathcal{A}^{\leq -1^+}(I)$ we denote the set of $f \in \mathcal{A}(I)$ with the property that, for any $I' \prec I$, there exist positive constants $R$ and $c$ such that
\[
\sup_{z \in S(I', R)} e^{c|z| \log |z|}|f(z)| < \infty
\]

$\mathcal{A}_0^\leq$ and $\mathcal{A}^{\leq 0}$ are sheaves on $\mathbb{R}$. $\mathcal{A}^{\leq -k}$ is a subsheaf of $\mathcal{A}$ for every $k > 0$.

**Definition 2.5 (multi-summability).** — Let $I_0 = \mathbb{R}$ and $I_h$, $h = 1, \ldots, q$, be open intervals of $\mathbb{R}$ such that
- $I_q \subset I_{q-1} \subset \cdots \subset I_1$.
- $|I_h| > \frac{\pi}{k_h}$ for $h = 1, \ldots, q$.

$\hat{f} \in \mathbb{C}[[z^{-1/p}]]$ is multi-summable on the multi-interval $(I_1, \ldots, I_q)$, with multi-sum $f_q \in A(I_q)$ if there exist $f_h \in (\mathcal{A}/\mathcal{A}^{\leq -k_h+1})(I_h)$, $h = 0, \ldots, q-1$ with asymptotic expansion $\hat{f}$, satisfying the following conditions:
- $f_0(z e^{2\pi i}) = f_0(z)$,
- $f_{h-1}|I_h = f_h \bmod \mathcal{A}^{\leq -k_h}$, $h = 1, \ldots, q$.

Any element of $\mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -k}(I)$ can be represented by a collection of functions $\{\phi_\nu : \nu \in \mathcal{N}\}$, where $\phi_\nu \in \mathcal{A}^{\leq 0}(I_\nu)$, $\{I_\nu : \nu \in \mathcal{N}\}$ is an open covering of $I$ and $\phi_\nu - \phi_\mu \in \mathcal{A}^{\leq -k}(I_\nu \cap I_\mu)$ for all $\mu$ and $\nu \in \mathcal{N}$. $\{\phi_\nu : \nu \in \mathcal{N}\}$ is called a $k$-precise quasi-function (cf. [18]).

Suppose that the formal difference operator $\hat{\Delta}$ associated with (1.1) has the positive levels $0 < k_1 < \cdots < k_q = 1$. With the formal power series solution $\hat{f}$ of (1.1) one can associate a unique global section $f_0$ of $(\mathcal{A}/\mathcal{A}^{\leq -k_1})^n$ with the property that $f_0(z e^{2\pi i}) = f_0(z)$ (this is a consequence of the Gevrey order of $\hat{f}$: $\hat{f} \in \mathbb{C}[[z^{-1/p}]]^n$, cf. [18, 12]). $f_0$ can be represented by a $k_1$-precise quasi-function $\{\phi_\nu : \nu \in \mathcal{N}\}$, where $\phi_\nu \in \mathcal{A}(I_\nu)$, $\{I_\nu : \nu \in \mathcal{N}\}$ is an open covering of $\mathbb{R}$, $\phi_\nu - \phi_\mu \in \mathcal{A}^{\leq -k}(I_\nu \cap I_\mu)$ for all $\mu$ and $\nu \in \mathcal{N}$ and $\phi_\nu$ is represented asymptotically by $\hat{f}$ as $z \to \infty$, $\arg z \in \mathcal{I}_\nu$, for all $\nu \in \mathcal{N}$.

Let $I_0 = \mathbb{R}$ and $I_h$, $h = 1, \ldots, q$, be open intervals of $\mathbb{R}$ with the following properties:
- $(-\frac{\pi}{2}, \frac{\pi}{2}) \subset I_q \subset I_{q-1} \subset \cdots \subset I_1$.
- $|I_h| > \frac{\pi}{k_h}$.
- $I_h$ does not contain a Stokes pair of level $k_h$.

In [3] it is proved that, under these conditions, (1.1) has solutions $f_h \in (\mathcal{A}/\mathcal{A}^{\leq -k_h+1})^n(I_h)$, $h = 1, \ldots, q-1$ with the property that
\[
f_{h-1}|I_h = f_h \bmod (\mathcal{A}^{\leq -k_h})^n
\]
Moreover, if $\hat{\Delta}$ doesn’t possess a level $1^+$, then (1.1) has a solution $f_q \in \mathcal{A}(I_q)^n$ with the property that $f_{q-1}|I_q = f_q \bmod (\mathcal{A}^{\leq -1})^n$. This implies
that \( \hat{f} \) is multi-summable on the multi-interval \((I_1, \ldots, I_q)\), with multi-sum \( f_q \).

### 2.3. A particular type of domains

In the case of difference equations without level \( 1^+ \), the study of the asymptotic behaviour of solutions on sectors suffices. This is no longer true for difference equations possessing a level \( 1^+ \), due to the complicated asymptotic behaviour of \( y_j^c \) if \( d_j \neq 0 \) (cf. (2.2)). For example, for all \( j \in \{1, \ldots, n\} \) such that \( d_j < 0 \), \( y_j^c \in (A^{\leq -1^+}(-\frac{\pi}{2}, \frac{\pi}{2}))^n \) and \( y_j^c(z) \) increases supra-exponentially as \( z \to \infty \) in any direction \( \in (\frac{\pi}{2}, \frac{3\pi}{2}) \) mod \( 2\pi \), regardless of the value of \( \mu_j \). As \( \mu_j \) is determined up to a multiple of \( 2\pi i \), this implies that, in some sense, \( \{-\frac{\pi}{2}, \frac{\pi}{2}\} \) may be viewed as a Stokes pair of level \( 1^+ \), of “infinite multiplicity”. However, looking more carefully into the asymptotic behaviour of \( y_j^c \) and noting that

\[
|e^{d_j z \log z + \mu_j z}| = e^{d_j (\Re(z (\log z + i \frac{\Im \mu_j}{d_j}))) e^{\Re \mu_j \Re z}}
\]

and

\[
\Re \left( z \left( \log z + i \frac{\Im \mu_j}{d_j} \right) \right) = \Re(z (\log z + i \theta)) - \left( \frac{\Im \mu_j}{d_j} - \theta \right) \Im z
\]

where \( \theta \) is a real number, we find that \( y_j^c \) decreases exponentially of order 1 as \( \Im z \to \infty \) on any curve of the form \( \Re(z (\log z + i \theta)) = c \) with \( c \in \mathbb{R} \) and \( \theta > \frac{\Im \mu_j}{d_j} \) and increases exponentially if \( \theta < \frac{\Im \mu_j}{d_j} \). Similarly, \( y_j^c \) decreases exponentially of order 1 as \( \Im z \to -\infty \) on a curve of this form if \( \theta < \frac{\Im \mu_j}{d_j} \) and increases exponentially if \( \theta > \frac{\Im \mu_j}{d_j} \). This is why, in order to characterize solutions of (1.1) by their asymptotic properties in the presence of a level \( 1^+ \), we need to consider domains bounded by curves of the type \( \Re(z (\log z + i \theta)) = c \).

**Definition 2.6.** — By \( S_+ \) we denote the sector \( S(-\pi, \pi) \). Let \( \theta \in \mathbb{R} \), \( z \in S_+ \) and

\[
\psi_\theta(z) := z(\log z + i \theta).
\]

By \( C_\theta(z) \) we denote the level set of \( \Re \psi_\theta \) containing \( z \):

\[
C_\theta(z) = \{ \zeta \in S_+: \Re \psi_\theta(\zeta) = \Re \psi_\theta(z) \}
\]

\( C^+_\theta(z) \) and \( C^-_\theta(z) \) are defined by

\[
C^+_\theta(z) = \{ \zeta \in C_\theta(z): \pm \Im(\zeta - z) > 0 \}
\]

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In previous papers we introduced two types of domains, denoted by $D_I(z)$ and $\tilde{D}_I(z)$ and arising rather naturally in the study of difference equations possessing a level $1^+$ (cf. [11], [12], [14] and the appendix below). $D_I(z)$ is an *intersection* of domains of the form \{ $\zeta \in S_+: \text{Re} \psi_\theta(\zeta) \geq \text{Re} \psi_\theta(z)$ \}, whereas $\tilde{D}_I(z)$ is a *union* of such domains. In problems involving the level $1^+$, these domains play a role similar to that of sectors of aperture $\leq \pi$ and $\geq \pi$, respectively, in problems of level 1. For notational convenience, we combine both types of domain into one, more general type of domain, somewhat similarly to the example of sectors. Roughly speaking, we ‘label’ $C_\theta^{-}(z)$ by a negative number and $C_\theta^{+}(z)$ by a positive number and define a domain $\hat{D}_{(a,b)}(R)$, bounded by two such ‘rays’, with ‘labels’ $a$ and $b$, respectively, and part of the ‘circle’ $|z| = R$. By providing a unified notation for the domains to be considered, this somewhat artificial construction considerably simplifies the study of asymptotic behaviour on these domains. In particular, it allows us to identify a class of “$1^+$-precise” quasi-functions with the representatives of sections of a (quotient-) sheaf on $\mathbb{R}$.

Let $\phi_-: \mathbb{R} \to (-\infty, 0)$ and $\phi_+: \mathbb{R} \to (0, \infty)$ be continuous, monotone decreasing and onto. By $\vartheta: \mathbb{R}^* \to \mathbb{R}$ we denote the mapping defined by

$$\vartheta(a) = \begin{cases} \phi_{-}^{-1}(a) & \text{if } a < 0 \\ \phi_{+}^{-1}(a) & \text{if } a > 0 \end{cases}.$$

For example, one might choose $\phi_{-}(\theta) := -e^\theta$ and $\phi_{+}(\theta) := e^{-\theta}$. In that case,

$$\vartheta(a) = \begin{cases} \log(-a) & \text{if } a < 0 \\ -\log a & \text{if } a > 0 \end{cases}.$$  

**Definition 2.7 (domains).** — Let $a, b \in \mathbb{R}$, $a < b$, and $R > 1$. If $a < 0 < b$,

$\hat{D}_{(a,0)}(R) = \{ z \in S_+: \text{arg} z \leq 0, |z| \geq R, \text{Re} \psi_\theta(a)(z) \geq 0 \}$

$\hat{D}_{(0,b)}(R) = \{ z \in S_+: \text{arg} z \geq 0, |z| \geq R, \text{Re} \psi_\theta(b)(z) \geq 0 \}$

and

$\hat{D}_{(a,b)}(R) = \hat{D}_{(a,0)}(R) \cup \hat{D}_{(0,b)}(R)$.

If $a < b < 0$,

$\hat{D}_{(a,b)}(R) = \{ z \in S_+: |z| \geq R, \text{Re} \psi_\theta(a)(z) \geq 0 \text{ and } \text{Re} \psi_\theta(b)(z) \leq 0 \}$

and if $0 < a < b$,

$\hat{D}_{(a,b)}(R) = \{ z \in S_+: |z| \geq R, \text{Re} \psi_\theta(a)(z) \leq 0 \text{ and } \text{Re} \psi_\theta(b)(z) \geq 0 \}$. 


Figure 2.1. This figure shows the curves $\tilde{C}_a(\theta)$ for various values of $a$.

For any finite interval $(a, b) \subset \mathbb{R}$ such that $a \neq 0 \neq b$, $\tilde{D}_{(a,b)}(R)$ is bounded by an arc of the ‘circle’ $|z| = R$, and the level curves of $\text{Re} \psi_{\theta(a)}$ and $\text{Re} \psi_{\theta(b)}$ with height 0. In order to determine the points of intersection of these curves with the circle $|z| = R$, we note that $\text{Re} \psi_{\theta}(z) = 0$ iff $\log|z| \cos \arg z = (\arg z + \theta) \sin \arg z$. For any $R > 1$ and $\theta \in \mathbb{R}$, the equation $G(\phi, \theta) := \log R \cos \phi - (\phi + \theta) \sin \phi = 0$ has unique solutions for $\phi$ in $(-\pi, 0)$ and in $(0, \pi)$. Noting that, for all $(\phi, \theta) \in G^{-1}(0)$, \(\frac{\partial G/\partial \theta}{\partial G/\partial \phi}(\phi, \theta) = \frac{\sin^2 \phi}{\log R + \sin^2 \phi}\), one easily verifies that both solutions are continuous, monotone decreasing functions of $\theta$, mapping $\mathbb{R}$ onto $(-\pi, 0)$ and $(0, \pi)$, respectively. Now, let $R > 1$ and $a \in \mathbb{R}$. For $a > 0$, let $\phi_R(a)$ denote the unique $\phi \in (0, \pi)$ such that

\begin{equation}
\log R \cos \phi = (\phi + \vartheta(a)) \sin \phi
\end{equation}

and, for $a < 0$, the unique $\phi \in (-\pi, 0)$ such that (2.3) holds. Putting $\phi_R(0) := 0$, we obtain a continuous, monotone increasing mapping $\phi_R$ from $\mathbb{R}$ onto $(-\pi, \pi)$. 
**Definition 2.8.** — Let $R > 1$. For all $a \in \mathbb{R}$ we define

$$z_a(R) := \text{Re}^{i\phi_R(a)},$$

$$\widehat{C}_a(R) := \{ z \in S_+: \text{Re} \psi_{\theta(a)}(z) = 0, \text{Im}(z - z_a(R)) \leq 0 \} \text{ if } a < 0,$$

$$\widehat{C}_a(R) := \{ z \in S_+: \text{Re} \psi_{\theta(a)}(z) = 0, \text{Im}(z - z_a(R)) \geq 0 \} \text{ if } a > 0,$$

and

$$\widehat{C}_0(R) := (R, \infty) \text{ (cf. Figure 2.1).}$$

If $a \neq 0$, $\text{Re} \psi_{\theta(a)}(z_a(R)) = 0$. Obviously, $\widehat{C}_a(R) = C_{\theta(a)}(z_a(R))$ if $a < 0$, $\widehat{C}_a(R) = C_{\theta(a)}^+(z_a(R))$ if $a > 0$. From (2.3) we deduce that $\phi_R(a) = \pm \frac{\pi}{2}$ iff $\psi(a) = \mp \frac{\pi}{2}$, so $\widehat{C}_{\phi_-(\frac{\pi}{2})}(R)$ is the half line $\{ z \in S_+: \arg z = -\frac{\pi}{2}, |z| \geq R \}$ and $\widehat{C}_{\phi_+(\frac{\pi}{2})}(R)$ is the half line $\{ z \in S_+: \arg z = \frac{\pi}{2}, |z| \geq R \}$.

Thus, for any finite interval $(a, b) \subset \mathbb{R}$, $\widehat{D}_{(a,b)}(R)$ is the closed domain in $S_+$ bounded by $\widehat{C}_a(R), \widehat{C}_b(R)$ and the arc of the circle $|z| = R$ between $z_a(R)$ and $z_b(R)$. In particular, $\widehat{D}_{(\phi_-(\frac{\pi}{2}),0)}(R) = S((-\frac{\pi}{2},0), R)$, $\widehat{D}_{(\phi_-(\frac{\pi}{2}),\phi_+(\frac{\pi}{2}))}(R) = S((-\frac{\pi}{2}, \frac{\pi}{2}), R)$. For every bounded interval $I = (a, b)$, we note

$$\theta_-(I) := \begin{cases} \infty & \text{if } a = 0 \\ \vartheta(a) & \text{otherwise} \end{cases}, \quad \theta_+(I) := \begin{cases} -\infty & \text{if } b = 0 \\ \vartheta(b) & \text{otherwise} \end{cases}.$$  

For every open interval $I \subset \mathbb{R}$ not containing $0$, we define

$$\widetilde{I} := \vartheta(I)$$

and for every interval $I = (a, b)$ containing $0$, such that $\vartheta(a) \neq \vartheta(b)$,

$$\widetilde{I} := \left( \min\{\vartheta(a), \vartheta(b)\}, \max\{\vartheta(a), \vartheta(b)\} \right).$$

We call $I = (a, b)$ a large interval if $0 \in I$ and $\theta_+(I) < \theta_-(I)$. If $I$ is a large interval, then $\widetilde{I} = (\vartheta(b), \vartheta(a)) = (\theta_+(I), \theta_-(I))$. If $\theta_-(I) = \theta_+(I) = \theta$, then $I = (\phi_-(\theta), \phi_+(\theta))$. In this case, $\widetilde{I}$ is not defined. If $\theta_-(I) < \theta_+(I)$, then $\widetilde{I} = (\vartheta(a), \vartheta(b)) = (\theta_-(I), \theta_+(I))$. Note that $\theta_-(I) < \theta_+(I)$ implies that $0 \in I$. If $I$ is a large interval, we have $a = \phi_-(\vartheta(a)) < \phi_-(\vartheta(b)) < 0 < \phi_+(\vartheta(a)) < \phi_+(\vartheta(b)) = b$. Hence $I' := (\phi_-(\vartheta(b)), \phi_+(\vartheta(a))) \subset I$ and $I' = I$. $\widehat{D}_{I'}(R)$ and $\widehat{D}_{I}(R)$ are bounded by different parts of the same level curves: $\text{Re} \psi_{\theta(a)}(z) = 0$ and $\text{Re} \psi_{\theta(b)}(z) = 0$ (cf. Figure 2.2). For each open interval $(\theta_1, \theta_2)$ there exist two intervals $I_1$ and $I_2$ such that $0 \in I_1 \cap I_2$ and $\widetilde{I}_1 = \widetilde{I}_2 = (\theta_1, \theta_2)$: the large interval $I_1 = (\phi_-(\theta_2), \phi_+(\theta_1))$ and the interval $I_2 = (\phi_-(\theta_1), \phi_+(\theta_2))$. We end this subsection with some properties of the domains $\widehat{D}_{I}(R)$ that will be needed later on. In this paper, we are mainly interested in the
The dark region is \( \hat{D}(\phi_-(\pi), \phi_+(\pi)) \) (6), the larger domain is \( \hat{D}(\phi_-(\pi), \phi_+(\pi)) \) (6). The interval \( (\phi_-(\pi), \phi_+(\pi)) \) is a large interval, whereas \( (\phi_-(\pi), \phi_+(\pi)) \) is not. In both cases, \( \hat{D}(\phi_-(\pi), \phi_+(\pi)) \) is bounded by the level curves \( \Re \psi_\pi(z) = 0 \) and \( \Re \psi_\pi(z) = 0 \), and \( \tilde{I} = (-\pi, -\pi^2) \).

case that \( 0 \in I \). From (2.3) it easily follows that \( \arg z \to \pm \frac{\pi}{2} \) as \( z \to \infty \) on \( \hat{C}_a(R) \) if \( \pm a > 0 \). Hence, if \( I' \) is an open interval containing \( [-\frac{\pi}{2}, \frac{\pi}{2}] \), \( \hat{D}(\phi_-(\pi), \phi_+(\pi)) \) for any bounded, open interval \( I \) and all sufficiently large \( R \). On the other hand, if \( I' \prec (-\frac{\pi}{2}, \frac{\pi}{2}) \), \( I \) is an interval containing 0 and \( R > 1 \), then \( S(I', R') \subset \hat{D}(\phi_-(\pi), \phi_+(\pi)) \) for all sufficiently large \( R' \).

Obviously, \( I' \subset I \) implies \( \hat{D}(\phi_-(\pi), \phi_+(\pi)) \subset \hat{D}(\phi_-(\pi), \phi_+(\pi)) \). In particular, \( I \subset (\phi_-(\pi), \phi_+(\pi)) \) implies that \( \hat{D}(\phi_-(\pi), \phi_+(\pi)) \) is contained in the right half plane \( S((-\frac{\pi}{2}, \frac{\pi}{2}), \hat{D}(\phi_-(\pi), \phi_+(\pi))) \). It can be shown that, for any interval \( I \prec (\phi_-(\pi), \phi_+(\pi)) \),

\[
\Re z \geq \delta \frac{|z|}{\log |z|} \quad \text{for all } z \in \hat{D}(\phi_-(\pi), \phi_+(\pi)) \]

(2.4)

where \( \delta > 0 \), provided \( R \) is sufficiently large (this is a particular case of (4.5) below).

Let \( I \) be an open interval such that \( \theta_1 := \theta_-(I) < \theta_2 := \theta_+(I) \). Noting that, for \( i \in \{1, 2\}, \theta \in (\theta_1, \theta_2) \) and all \( z \in \hat{D}(\phi_-(\pi), \phi_+(\pi)) \), \( \Re \psi_\theta(z) = \Re \psi_{\theta_i}(z) + \)
Let \((\theta_i - \theta) \Im z \geq (\theta_i - \theta) \Im z\), one easily verifies that
\[(2.5) \quad \Re \psi_{\theta}(z) \geq c|z| \text{ for all } z \in \hat{D}_I(R),\]
where \(c > 0\).

### 2.4. Sheaves of functions with particular asymptotic properties in the domains \(\hat{D}_I(R)\)

We shall now introduce sheaves on \(R\) of functions with different types of asymptotic behaviour in the domains \(\hat{D}_I(R)\).

**Definition 2.9.** — Let \(I\) be an interval of \(R\). By \(\hat{A}(I)\) we denote the set of (equivalence classes of) continuous functions \(f : S_+ \to \mathbb{C}\), holomorphic in \(\text{int } \hat{D}_I(R)\) and admitting an asymptotic expansion \(\hat{f} = \sum_{m=0}^{\infty} a_m z^{-m/p}\), with \(p \in \mathbb{N}\), uniformly on \(\hat{D}_I(R)\), for any open interval \(I' \prec I\) and some sufficiently large \(R\) (depending on \(I')\). By \(\hat{A}^{\leq -1}(I)\) we denote the set of all \(f \in \hat{A}(I)\) with the property that, for any open interval \(I' \prec I\), there exist positive constants \(R\) and \(c\) such that
\[\sup_{z \in \hat{D}_I'(R)} e^{\frac{|z|}{mR^{|1}|}} |f(z)| < \infty.\]

By \(\hat{A}^{\leq -1+}(I)\) we denote the set of all \(f \in \hat{A}(I)\) with the property that, for any open interval \(I' \prec I\), there exist positive constants \(R\) and \(c\) such that
\[\sup_{z \in \hat{D}_I'(R)} e^{c|z|} |f(z)| < \infty.\]

\(\hat{A}\) is a sheaf on \(R\) and \(\hat{A}^{\leq -1}\) and \(\hat{A}^{\leq -1+}\) are subsheaves. From the properties of the domains \(\hat{D}_I(R)\) discussed at the end of § 2.3 it follows, with a slight abuse of notation, that \(A([-\pi, \pi]) \subset \hat{A}(R)\) (strictly speaking, the elements of \(A([-\pi, \pi])\) should first be extended to continuous functions on \(S_+)\) and, for any open interval \(I\) containing 0, \(\hat{A}(I) \subset A(-\pi, \pi)\). Moreover, it can be shown that \(\hat{A}^{\leq -1}(I) \subset A^{\leq -1}(-\pi, \pi)\) (cf. Lemma 4.9 and remark 4.10) and from Lemma 2.12 below it follows that \(\hat{A}^{\leq -1+}(I) \subset A^{\leq -1+}(-\pi, \pi)\), for any open interval \(I\) containing 0.

**Example 2.10.**

(i) \((2.4)\) implies that, for any \(\mu < 0\), \(e^{\mu z} \in \hat{A}^{\leq -1}(I)\) iff \(I \subset (\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2}))\).

(ii) Let \(d < 0\) and \(\mu \in \mathbb{R}\). From \((2.5)\) we deduce that \(e^{dz \log z + \mu z} \in \hat{A}^{\leq -1+}(I)\) iff \(I \subset (\phi_-(\theta), \phi_+(\theta))\) with \(\theta = \text{Im } \mu/d\).
In § 3, where we consider (finite) Laplace transforms of functions admitting an asymptotic power series expansion in $t^{1/p}$ at the origin, we shall need the following lemma.

**Lemma 2.11.** — Let $r > 0$ and let $u$ be a continuous function on $(0, r)$, admitting an asymptotic power series expansion in $t^{1/p}$ as $t \to 0$. Then

$$
\int_0^r u(t)e^{-tz}dt \in \hat{A}(\phi_{-}(\frac{r}{2}), \phi_{+}(\frac{r}{2})).
$$

**Proof.** — Let $y_r(z) = \int_0^r u(t)e^{-tz}dt$. It is a known fact that $y_r \in \mathcal{A}(-\frac{r}{2}, \frac{r}{2})$. More precisely, there exist positive numbers $M_N$, $N \in \mathbb{N}$, such that, for all $z \in \mathbb{C}$ with the property that $\Re z \geq C > 0$ and all $N \in \mathbb{N}$, $|R_N(y_r; z)| \leq M_N(\Re z)^{-N/p}$. Let $I \prec \hat{A}(\phi_{-}(\frac{r}{2}), \phi_{+}(\frac{r}{2}))$. In view of (2.4) there exist positive numbers $R$ and $\delta$ such that, for all $z \in \hat{D}_I(R)$, $\Re z \geq \delta \frac{|z|}{\log |z|}$ and thus,

$$
|R_N(y_r; z)| \leq M_N\delta^{-N/p}\left(\frac{|z|}{\log |z|}\right)^{-N/p}, \quad N \in \mathbb{N}.
$$

The proof is completed by observing that

$$
|R_N(y_r; z)| \leq |R_{N+1}(y_r; z)| + C_N|z|^{-N/p}
$$

where $C_N > 0$ and, for all $z \in \hat{D}_I(R)$ and $h > 0$,

$$
(\log |z|)^h \leq \left(\frac{h}{\epsilon}\right)^h |z|.
$$

\[\square\]

**Lemma 2.12.**

1. Let $I$ be an open interval such that $\theta_1 := \theta_-(I) \leq \theta_2 := \theta_+(I)$ and let $\theta \in [\theta_1, \theta_2]$. $f \in \hat{A}_{\leq -1}^-(I)$ iff, for every interval $I' \prec I$ and some sufficiently large $R$, there exists a positive number $t$ such that

$$
\sup_{z \in \hat{D}_{I'}(R)} |e^{t\psi_\theta(z)}f(z)| < \infty.
$$

2. For every large interval $I$, $\hat{A}_{\leq -1}^-(I) = \{0\}$.

This lemma is easily deduced from [12, Lemma 0.15], with the aid of Lemma 5.3 in the appendix (§ 5).

We end this section with an important preliminary result. Two difference operators $B_1\tau - A_1$ and $B_2\tau - A_2$, where $A_i$ and $B_i \in \text{Gl}(n, \hat{A}(I)[z^{1/p}])$, admitting asymptotic expansions $\hat{A}_i$ and $\hat{B}_i$, for $i = 1, 2$, will be called formally equivalent if the formal operators $\hat{B}_1\tau - \hat{A}_1$ and $\hat{B}_2\tau - \hat{A}_2$ are equivalent in the sense of § 2.1.

**Theorem 2.13.** — Let $I$ be an open interval such that $\theta_-(I) \leq \theta_+(I)$ and let $A \in \text{Gl}(n, \hat{A}(I)[z^{1/p}])$. 

(i) The difference operator \( \Delta := \tau - A \) is formally equivalent to an operator \( \Delta^c \) of the form (2.1), and the homogeneous linear difference equation \( \Delta y(z) = 0 \) has a fundamental system of solutions of the form

\[
y_j(z) = z^{d_j} e^{g_j(z)} z^{\lambda_j} g_j(z), \quad j = 1, \ldots, n
\]

where \( g_j \in \hat{\mathcal{A}}(I)^n[\log z] \), \( d_j, q_j \) and \( \lambda_j \) are defined as in (2.2).

(ii) \( \ker(\Delta, (\hat{\mathcal{A}}^{\leq -1}(I))^n) \) is a linear space over \( \mathbb{C} \), spanned by all solutions of the form \( y_j(z) e^{2ltiz} \) with \( j \in \{1, \ldots, n\} \) and \( l \in \mathbb{Z} \) such that \( d_j < 0 \) and \( \frac{\ln \mu_j + 2lti}{d_j} \in [\theta_-(I), \theta_+(I)] \).

(iii) \( \ker(\Delta, (\hat{\mathcal{A}}^{\leq -1}(I))^n) = \ker(\Delta, (\hat{\mathcal{A}}^{\leq -1}(I))^n) \) if \( I \nsubseteq (\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2})) \). If \( I \subset (\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2})) \), then \( \ker(\Delta, (\hat{\mathcal{A}}^{\leq -1}(I))^n) \) is the linear space over \( \mathbb{C} \), spanned by all solutions of the form \( y_j(z) e^{2ltiz} \) with \( j \in \{1, \ldots, n\} \) and \( l \in \mathbb{Z} \), such that \( d_j < 0 \) and \( \frac{\ln \mu_j + 2lti}{d_j} \in [\theta_-(I), \theta_+(I)] \), or \( d_j = 0, k_j = 1, \arg \mu_j = \pi \) and \( l = 0 \).

\textbf{Proof.}

(i) \( A \) has an asymptotic expansion \( \hat{A} \in \text{Gl}(n; \mathbb{C}[[z^{-1/p}]][[z^{1/p}]] \) and there exists \( \hat{F} \in \text{Gl}(n; \mathbb{C}[[z^{-1/p}]][[z^{1/p}]] \) such that \( (\tau \hat{F})^{-1} \hat{A} \hat{F} = A^c \). Hence the difference equation

\[
Y(z + 1) = A(z)Y(z)A^c(z)^{-1}
\]

has the formal solution \( \hat{F} \). Consequently, it has an analytic solution \( F \in \text{Gl}(n, \hat{\mathcal{A}}(I)[z^{1/p}]) \) with asymptotic expansion \( \hat{F} \) and \( \{Fy_j^z : j = 1, \ldots, n\} \) is a fundamental system of solutions of the difference equation \( \Delta y = 0 \) (cf. [11, Theorem 1.2] and Remark 5.2 below).

(ii) Obviously,

\[
y \in \ker(\Delta, (\hat{\mathcal{A}}^{\leq -1}(I))^n) \text{ iff } u := F^{-1} y \in \ker(\Delta^c, (\hat{\mathcal{A}}^{\leq -1}(I))^n). \quad \text{Let } \theta \in [\theta_-(I), \theta_+(I)]. \quad \text{In view of Lemma 2.12, } u \in (\hat{\mathcal{A}}^{\leq -1}(I))^n \text{ iff, for every open interval } I' \prec I \text{ and some sufficiently large } R, \text{ there exists a positive number } t \text{ such that sup}_{z \in \hat{D}_I(R)} \left| e^{t\psi_0(z)} u(z) \right| < \infty. \quad \text{Let } I = (a, b), I' = (a', b') \text{ such that } a < a' < 0 < b' < b \text{ and } \hat{I}' = (\theta_1, \theta_2). \text{ Then } \theta_1 = \vartheta(a') < \vartheta(a) = \theta_-(I) \text{ and } \theta_2 = \vartheta(b') > \vartheta(b) = \theta_+(I). \text{ Without loss of generality, we may assume that } \hat{I}' \cap \Theta(\Delta^c) = [\theta_-(I), \theta_+(I)] \cap \Theta(\Delta^c) \text{ and that conditions (1) - (4) of Proposition 1.5 in [12] are satisfied (note that there a slightly different definition of } y_j^z \text{ is used and cf. Lemma 5.3 below for the relation between the domains } D_I(R) \text{ and } \hat{D}_I(R)). \text{ By that proposition, with } k = 1^+, \Delta^c \text{ has a}
right inverse $\Lambda^c$ with the property that $\Lambda^c\Delta^c u(z) = u(z) - \sum_{j=1}^n y^c_j(z)p_j(z)$, where $p_j \equiv 0$ unless $d_j < 0$ and
\[ p_j(z) = \sum_{l=s_j}^{l_j} p_{j,l} e^{2il\pi z} \]
if $d_j < 0$. Here, $p_{j,l} \in \mathbb{C}$, $s_j = \inf \{ l \in \mathbb{Z} : \text{Im} \mu_j + 2\pi l \in (\theta_1, \theta_2) \}$ and $l_j = \sup \{ l \in \mathbb{Z} : \text{Im} \mu_j + 2\pi l \in (\theta_1, \theta_2) \}$. Consequently, $y = Fu = \sum_{j=1}^n y_j p_j$, where $y_j = F y^c_j$. Conversely, it is easily verified that every function of this form belongs to $\text{Ker}(\Delta, (\hat{\Lambda}^{\leq -1}(I))^n)$.

(iii) Similarly, $y \in \text{Ker}(\Delta, (\hat{\Lambda}^{\leq -1}(I))^n)$ iff $u := F^{-1}y \in \text{Ker}(\Delta^c, (\hat{\Lambda}^{\leq -1}(I))^n)$. The last two statements follow again from Proposition 1.5 in [12] by observing that $I \subset (\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2}))$ iff $\theta_-(I) \leq \frac{\pi}{2}$ and $\theta_+(I) \geq -\frac{\pi}{2}$. 

COROLLARY 2.14. — Let $I$ be an open interval such that $\theta_-(I) \leq \theta_+(I)$, $A \in \text{Gl}(n, \hat{\Lambda}(I)[z^{1/p}])$ and $\Delta = \tau - A$. If $[\theta_-(I), \theta_+(I)] \cap \Theta(\Delta^c) = \emptyset$, then $\text{Ker}(\Delta, (\hat{\Lambda}^{\leq -1}(I))^n) = \{0\}$.

2.5. A prepared form of (1.1)

Following Braaksma and Faber in [3], we first transform (1.1) into a convenient ’prepared form’, consisting of a linear part in canonical form plus a “perturbation” (which may also contain linear terms).

Let $\hat{\Delta} = \tau - \hat{\Lambda}$ denote the formal difference operator associated with (1.1), where $\hat{\Lambda}$ is defined by (1.2) and let $\Delta^c$ be a canonical form of $\hat{\Delta}$. Thus, there exists $\hat{F} \in \text{Gl}(n; \mathbb{C}[z^{-1/p}][z^{1/p}])$ such that $(\tau \hat{F})^{-1}\hat{\Delta}\hat{F} = \Delta^c = \tau - A^c$ (cf. (2.1) and (2.2)). $A^c$ is a block-diagonal matrix of the form
\[ A^c(z) = \bigoplus_{h=0}^r \left\{ I_{n_h} + z^{k_h-1}(A_h + z^{-1/p}B_h(z)) \right\}. \]

Here, $r$ is a positive integer and, for each $h \in \{0, \ldots, r\}$, $n_h$ is a nonnegative integer and $k_h$ is a nonnegative multiple of $1/p$, such that $0 = k_0 < k_1 < \cdots < k_q = 1 \leq k_{q+1} \leq \cdots \leq k_r$, where $1 \leq q \leq r$. Moreover, $n_h > 0$ if $h \not\in \{0, q\}$ and $k_h > 1$ if $h > q + 1$. The rational numbers $k_1, \ldots, k_q$ correspond to the positive levels of $\hat{\Delta}$: for each $h \in \{0, \ldots, q\}$ such that $n_h > 0$, there exists an integer $j \in \{1, \ldots, n\}$ with the property that $d_j = 0$ and $k_j = k_h$. (The exceptional case that $q = 1$ and $n_0 = n_q = 0$ will not concern us here as it has been dealt with in [10].) $A_h$ is a constant
$n_h \times n_h$ matrix in Jordan normal form, nonsingular and diagonal if $h > 0$, and $B_h \in \text{End}(n_h; \mathbb{C}[z^{-1/p}])$. Here, we shall assume that $n_0 + n_1 > 0$, that $r > q$, that $A_q + I_{n_q}$ is nonsingular if $n_q > 0$, that $k_{q+1} = 1$ and $A_{q+1} = -I_{n_{q+1}}$. In this case the equation (1.1) possesses both a level $\leq 1$ and the level $1^+$. More precisely, $n_{q+1} = \#\{j \in \{1, \ldots, n\} : d_j < 0\}$ and $\sum_{q+1 < h \leq r} n_h = \#\{j \in \{1, \ldots, n\} : d_j > 0\}$. If $n_q > 0$, the eigenvalues of $A_q + I_{n_q}$ are the numbers $e^{\mu_j}$ with $j \in \{1, \ldots, n\}$ such that $d_j = 0$ and $\kappa_j = 1$. Note that, if $n_0 > 0$ or $q > 1$, $\hat{\Delta}$ has a level 1, even if $n_q = 0$ (cf. remark 2.1 (ii)).

Now, let $S \in \text{Gl}(n; \mathbb{C}[z^{-1/p}][z^{1/p}])$ and $P \in \mathbb{C}^n[z^{-1/p}]$ be obtained by truncating $\hat{F}$ and the formal solution $\hat{f}$ of (1.1), respectively, at some sufficiently large power $M$ of $z^{-1/p}$. By the substitution

$$y \mapsto Sy + P$$

(1.1) is transformed into an equation of the form:

$$\Delta y(z) = \varphi_0(z^{1/p}) + E(z^{1/p}, y(z)),$$

with formal solution $\hat{f}_N := S^{-1}(\hat{f} - P) \in z^{-N/p}\mathbb{C}^n[[z^{-1/p}]]$ for some $N \in \mathbb{N}$. $\Delta$ is a linear difference operator of the form

$$\Delta = \bigoplus_{h=0}^{r} z^{-1-k_h} I_{n_h} (\tau - 1) - \bigoplus_{h=0}^{r} A_h - z^{-1/p} B(z)$$

where $B \in \text{End}(n; \mathbb{C}[z^{-1/p}])$. $\varphi_0$ is an analytic function in a neighbourhood of $\infty$, which is $O(z^{-N'})$ as $z \to \infty$ for some positive integer $N'$, and $E$ is analytic in a neighbourhood $U$ of $(\infty, 0)$. Furthermore, $E(z^{1/p}, 0) = 0$ and $D_2 E(z^{1/p}, 0) = 0$ for all sufficiently large $z$. If $\hat{F} \in z^{-\mu} \text{Gl}(n; \mathbb{C}[[z^{-1/p}]][z^{1/p}])$ for some sufficiently large $\mu \in \mathbb{N}$, then $E(\infty, y) = 0$ as well. We shall assume that this is the case (cf. Remark 2.1 (i)). Moreover,

$$\bigoplus_{h=0}^{r} z^{k_h-1} I_{n_h} \Delta - \Delta^c \in z^{-M'/p} \text{End}(n; \mathbb{C}[z^{-1/p}])$$

where $M' > 0$. $N$, $N'$ and $M'$ can be made arbitrarily large by choosing a sufficiently large $M$. Therefore, since $A^c \in \text{Gl}(n; \mathbb{C}[z^{-1/p}][z^{1/p}])$,

$$\bigoplus_{h=0}^{r} (z^{-k_h} I_{n_h} + A_h) + z^{-1/p} B(z) \in \text{Gl}(n; \mathbb{C}[z^{-1/p}][z^{1/p}])$$

if $M$ is sufficiently large. We shall refer to (2.7) as a “prepared form of (1.1)”. It is easily seen that $\Delta$ and $\Delta^c$ are formally equivalent. Therefore, we also call $\Delta^c$ a canonical form of $\Delta$. 

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The following result is essentially due to Sibuya, who used a similar idea in the theory of differential equations (cf. [19]).

Proposition 2.15. — Let \( I_1 \) and \( I_2 \) be open intervals and, for \( i = 1, 2, \)\) let \( y_i \in (\hat{\mathcal{A}}(I_i))^n \) be a solution of the nonlinear difference equation (2.7). Then \( y_1 - y_2 \) satisfies a homogeneous linear difference equation of the form

\[
\Delta y(z) := \Delta y(z) - H(z)y(z) = 0
\]

where \( H \in \text{End}(n; \hat{\mathcal{A}}(I_1 \cap I_2)) \). There exists a positive constant \( K \), independent of \( y_1 \) and \( y_2 \), such that, for any \( I' \prec I_1 \cap I_2 \) and some sufficiently large number \( R \), depending on \( I' \), and for all \( z \in \hat{D}_{I'}(R) \),

\[
|E(z^{1\over p}, y_1(z)) - E(z^{1\over p}, y_2(z))| \leq K \max\{|y_1(z)|, |y_2(z)|\} y_1(z) - y_2(z)
\]

and

\[
|H(z)| \leq K \max\{|y_1(z)|, |y_2(z)|\}.
\]

Moreover, if \( N \) is sufficiently large, \( \tilde{\Delta} \) and \( \Delta \) have the same canonical form \( \Delta^c \).

Proof. — As both \( y_1 \) and \( y_2 \) are solutions of (2.7),

\[
\Delta(y_1 - y_2)(z) = E(z^{1\over p}, y_1(z)) - E(z^{1\over p}, y_2(z))
\]

\[
= \int_0^1 D_2 E(z^{1\over p}, ty_1(z) + (1-t)y_2(z)) dt(y_1 - y_2)(z).
\]

Thus, \( y_1 - y_2 \) satisfies (2.10), with \( H(z) = \int_0^1 D_2 E(z^{1\over p}, ty_1(z) + (1-t)y_2(z)) dt \). Obviously, \( t y_1 + (1-t)y_2 \in \hat{\mathcal{A}}(I_1 \cap I_2)^n \) for all \( t \in [0,1] \) and \((z^{1\over p}, ty_1(z) + (1-t)y_2(z)) \in U \) for all \( z \in \hat{D}_{I'}(R) \), provided \( I' \prec I_1 \cap I_2 \) and \( R \) is sufficiently large. \( E \) is analytic in \( U \), hence \( H \in \text{End}(n; \hat{\mathcal{A}}(I_1 \cap I_2)) \). As \( D_2 E(z^{1\over p}, 0) = 0 \), there exists a positive constant \( K \), independent of \( z \) and \( y \), such that \( |D_2 E(z^{1\over p}, y)| \leq K|y| \) for all \((z^{1\over p}, y) \in U \) and thus, (2.11) and (2.12) hold for all \( z \) such that \((z^{1\over p}, ty_1(z) + (1-t)y_2(z)) \in U \) and, consequently, for all \( z \in \hat{D}_{I'}(R) \), provided \( I' \prec I_1 \cap I_2 \) and \( R \) is sufficiently large. Due to the fact that \( \hat{f}_N \in z^{-N/p}\mathcal{C}[[z^{-1\over p}]]^n \), \( y_i \in z^{-N/p}(\hat{\mathcal{A}}(I_i))^n \) for \( i = 1, 2 \). This implies that \( |H(z)| \leq K'|z|^{-N/p} \) for all \( z \in \hat{D}_{I'}(R) \) and hence it can be deduced that \( \tilde{\Delta} \) and \( \Delta \) are formally equivalent if \( N \) is sufficiently large and thus have the same canonical form \( \Delta^c \).
2.6. Examples

In [14] we discuss in some detail the trivial, but instructive example of a system of two uncoupled difference equations, of level 1 and $1^+$, respectively. Below, we give two simple examples of equations with three distinct levels: $\frac{1}{2}$, 1 and $1^+$, both having analytic coefficients at $\infty$.

**Example 2.16.**

\[(2.13)\quad y(z + 1) = \begin{pmatrix} \frac{a_1}{z} & 1 & 0 \\ 1 & 1 + \frac{a_1}{z} & 0 \\ 0 & 0 & \frac{a_2}{z} \end{pmatrix} y(z) + z^{-2} f(z, y(z))\]

where $a_1$ and $a_2 \in \mathbb{C}^*$ and $f$ is a 3-dimensional vector function, polynomial in $y$ and analytic at $\infty$ in $z$. The substitution

\[y \mapsto S y\]

where

\[S(z) = \begin{pmatrix} 1 & 0 & 0 \\ a_1^{1/2} z^{-1/2} & -a_1^{1/2} z^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\]

changes (2.13) into an equation of the form:

\[(2.14)\quad \text{diag}\{z^{1/2}, z^{1/2}, 1\} (y(z + 1) - y(z)) = \text{diag}\{a_1^{1/2}, -a_1^{1/2}, -1\} y(z) + z^{-1/2} g(z^{1/2}, y(z))\]

where $g$ is a 3-dimensional vector function, analytic at $(\infty, 0)$. In this example, $r = 3$, $q = 2$, $k_1 = 1/2$, $n_0 = n_2 = 0$, $n_1 = 2$ and $n_3 = 1$. From [13, Theorem 2.7] one easily deduces the existence of a formal solution $\tilde{f} \in z^{-1/2} \mathbb{C}[[z^{-1/2}]]$ of (2.14) and hence the existence of a formal solution $\tilde{f} = S^{-1} f \in \mathbb{C}[[z^{-1/2}]]$ of (2.13). The formal difference operator $\tilde{\Delta}$ associated with (2.13) has a diagonal canonical form $\Delta^c$ and the homogeneous equation $\Delta^c y = 0$ has solutions of the form $y_1(z) = e^{2a_1^{1/2} z^{1/4}} e_1$, $y_2(z) = e^{-2a_1^{1/2} z^{1/4}} e_2$ and $y_3(z) = -z(z^{1/2})^{1/2} e_3$, where $e_i$ denotes the $i$-th unit vector of $\mathbb{C}^3$. Thus, the singular directions of level $\frac{1}{2}$ are given by $-\arg a_1 \mod 2\pi$, those of level 1 by $\frac{\pi}{2} \mod \pi$ and the pseudo-Stokes directions of level $1^+$ correspond to the different determinations of $-\arg a_2$.

**Example 2.17.**

\[(2.15)\quad y(z + 1) = \begin{pmatrix} \frac{a_1}{z} & 1 & 0 & 0 \\ 1 & 1 + \frac{a_1}{z} & 0 & 0 \\ 0 & 0 & 1 + a_2 & 0 \\ 0 & 0 & 0 & \frac{a_3}{z} \end{pmatrix} y(z) + z^{-2} f(z, y(z))\]
where \(a_1, a_2\) and \(a_3 \in \mathbb{C}^*, a_2 \neq -1\) and \(f\) is a 4-dimensional vector function, polynomial in \(y\) and analytic at \(\infty\) in \(z\). Similarly to (2.13), (2.15) can be transformed into an equation of the form:

\[
\text{diag}\left\{z^{1/2}, z^{1/2}, 1, 1\right\}(y(z + 1) - y(z)) = \text{diag}\left\{a_1^{1/2}, -a_1^{1/2}, a_2, -1\right\}y(z) + z^{-1/2}g(z^{1/2}, y(z))
\]

where \(g\) is a 4-dimensional vector function, analytic at \((\infty, 0)\). In this example, \(r = 3, q = 2, k_1 = 1/2, n_0 = 0, n_1 = 2,\) and \(n_2 = n_3 = 1\). (2.15) has a formal power series solution \(\in z^{-1/2}\mathbb{C}[[z^{-1/2}]]^4\). In this case, the singular directions of level \(1\) are \(\frac{\pi}{2} \mod \pi\) and \(\pi - \arg \{\log(1 + a_2) + 2l\pi i\}\), where \(l \in \mathbb{Z}\). \(0\) is a singular direction of level \(1\) iff \(a_2 \in (-1, 0)\).

3. The case that \(0\) is not a singular direction of level \(1\)

Throughout most of this section (with the exception of Theorem 3.8), we assume that the difference equation already is in a prepared form (2.7), where \(\Delta\) has the form (2.8), and (2.9) is satisfied for some sufficiently large \(M\). Thus, \(\hat{f}\) will denote the formal solution of (2.7) and, consequently, \(\hat{f} \in z^{-N/p}\mathbb{C}^n[[z^{-1/p}]]\), where \(N\) is some sufficiently large integer, depending on \(M\). Let \(I_0 = \mathbb{R}\), let \(f_0\) denote the unique global section of \((\mathcal{A}/\mathcal{A}^{\leq -k_1})^n\) with the property that \(f_0(ze^{2p\pi i}) = f_0(z)\), associated with \(\hat{f}\) (cf. §2.2), and let \(I_h, h = 1, \ldots, q\), be open intervals of \(\mathbb{R}\) with the following properties:

- \((-\frac{\pi}{2}, \frac{\pi}{2}) \subset I_q \subset I_{q-1} \subset \cdots \subset I_1\).
- \(|I_h| > \frac{\pi}{k_h}\).
- \(I_h\) does not contain a Stokes pair of level \(k_h\).

From [3] we know that (2.7) has solutions \(f_h \in z^{-N/p}(\mathcal{A}/\mathcal{A}^{\leq -k_{h+1}})^n(I_h)\) such that

\[
f_{h-1}|I_h = f_h \mod (\mathcal{A}^{\leq -k_h})^n, \ h = 1, \ldots, q - 1.
\]

The approach taken in [3] is based on a study of convolution equations. With (2.7) one can associate, for every \(h \in \{1, \ldots, q\}\), a convolution equation of the form \(T_h \eta = \eta\), obtained by applying a formal Borel transformation of order \(k_h\) to (2.7). The formal Borel transform of order \(k_1\) of \(f\) is a convergent power series, defining an analytic solution \(u_1\) of \(T_1 \eta = \eta\). In [3] it is shown that \(u_1\) is analytic in an infinite sector of the form \(S(\widehat{I}_1)\), where \(-\widehat{I}_1\) is an open interval not containing any singular direction of level \(k_1\), and that \(u_1\) satisfies a specific growth condition in this sector. By means of a so-called acceleration operator (an extension of a Laplace transformation
of order $k_1$ followed by a Borel transformation of order $k_2$) $u_1$ can be transformed into a solution $u_2$ of $T_2\eta = \eta$. $u_2$ is analytic in a sector of the form $S(I_2)$, where $-I_2$ is an open interval not containing any singular direction of level $k_2$, and, if $q > 2$, it can be transformed into a solution $u_3$ of $T_{3\eta} = \eta$, etc. Moreover, for $h = 1, \ldots, q$, $u_h$ coincides with a Borel transform of order $k_h$ of $f_{h-1}$. If $r = q$, then $u_q$ has at most exponential growth of order 1 and its Laplace transform $f_q$ is the $(k_1, \ldots, k_q)$-sum of $\hat{f}$ on $(I_1, \ldots, I_q)$. In the case that $r > q$, one has to deal with an additional convolution equation, corresponding to the level $1^+$, which, in general, doesn’t admit an analytic solution in a sector of the form $S(I)$ for any open interval $I$.

In this section it is assumed that 0 is not a singular direction of level 1, or, equivalently, that $-\frac{\pi}{2}$ is not a Stokes direction of level 1. Similarly to the case without level $1^+$, $u_q$ can be analytically continued to a sector of the form $S(I_q^*)$, where $-I_q^*$ is an open interval not containing any singular direction of level 1, but it may have supra-exponential growth. We show that it satisfies a particular growth condition, making it accelerable from level 1 to level $1^+$, by means of a so-called weak acceleration operator. From the growth property of $u_q$ we can deduce the existence of a particular representative of $f_{q-1}|_{I_q}$, which also represents a solution $f_q \in (\mathcal{A}/\mathcal{A}^{\leq -1})^n(I_q)$ of (2.7) (cf. proposition 3.9 below). In proposition 3.11 it is shown that $f_q$ has a representative defining a solution $f_{q+1} \in \hat{\mathcal{A}}(I_{q+1})^n$, for every large interval $I_{q+1} \subset (\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2}))$ such that $I_{q+1}$ doesn’t contain any pseudo-Stokes directions. $f_{q+1}$ is the $(k_1, \ldots, k_q, 1^+)$-sum or accelero-sum of $\hat{f}$ on $(I_1, \ldots, I_{q+1})$.

### 3.1. Growth properties of $u_q$

**Definition 3.1** (Laplace and Borel transformations). — For any open interval $I = (\alpha, \beta)$, with $\beta - \alpha > \pi$, $I^*$ is defined by

$$I^* := \left(-\beta + \frac{\pi}{2}, -\alpha - \frac{\pi}{2}\right).$$

By

$$\mathcal{L} : t^{1/p-1}\mathcal{A}_0^{\leq 0}(I^*) \to z^{-1/p}\mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -1}(I)$$

we denote the finite (or incomplete) Laplace transformation, defined as follows. Let $u \in t^{1/p-1}\mathcal{A}_0^{\leq 0}(I^*)$. Let $\{\alpha_\nu : \nu \in \mathcal{N}\} \subset I^*$ such that $\{(-\alpha_\nu - \frac{\pi}{2}, -\alpha_\nu + \frac{\pi}{2}) : \nu \in \mathcal{N}\}$ is a covering of $I$, and, for each $\nu \in \mathcal{N}$, let $r_\nu > 0$ such that $u$ is continuous on $(0, r_\nu e^{i\alpha_\nu}]$. Then $\mathcal{L}(u)$ is the element of $z^{-1/p}\mathcal{A}^{\leq 0}/\mathcal{A}^{\leq -1}(I)$ represented by $\int_0^{r_\nu e^{i\alpha_\nu}} u(t)e^{-tz}dt : \nu \in \mathcal{N}$. $\mathcal{L}$ is a bijection and its inverse $\mathcal{B}$ is the “ordinary” Borel transformation.
Remark 3.2. — It is well-known that \( \mathcal{L}(t^{N/p-1}A_0(I^*)) = z^{-N/p}A/A^{\leq -1}(I) \) for every \( N \in \mathbb{N} \). Note that \(-I^*\) doesn’t contain a singular direction of level 1 iff \( I \) doesn’t contain a Stokes pair of level 1.

If \( M \) is sufficiently large, the function \( u_q = \mathcal{B}(f_{q-1}) \) satisfies the convolution equation \( T\eta = \eta \), obtained by applying a formal Borel transformation to (2.7). Let \( \widetilde{E}(z,y(z)) = z^{-1/p}B(z)y(z) + \varphi_0(z^{1/p}) + E(z^{1/p}, y(z)) \) and \( y = \bigoplus_{h=0}^\infty y_h \), in the partition used in (2.8). Then we have

\[
y_h(z + 1) - y_h(z) = z^{k_h-1}\{A_hy_h(z) + \widetilde{E}(z,y(z))\} \quad \text{for } h < q,
\]

\[
y_q(z + 1) - (I_{n_q} + A_q)y_q(z) = \widetilde{E}(z,y(z))_q \quad ( \text{if } n_q > 0),
\]

\[
y_{q+1}(z + 1) = \widetilde{E}(z,y(z))_{q+1},
\]

\[
A_hy_h(z) = z^{1-k_h}(y_h(z + 1) - y_h(z)) - \widetilde{E}(z,y(z))_h \quad \text{for } h > q + 1.
\]

Hence we deduce the following form for \( T\eta \):

\[
(T\eta)_h = (e^{-t} - 1)^{-1}\left\{ \frac{t^{-k_h}}{\Gamma(1-k_h)} * (A_h\eta_h + \mathcal{E}(t,\eta)_h) \right\} \quad \text{for } h < q,
\]

\[
(T\eta)_q = \{ (e^{-t} - 1)I_{n_q} - A_q \}^{-1}\mathcal{E}(t,\eta)_q \quad ( \text{if } n_q > 0),
\]

\[
(T\eta)_{q+1} = e^t\mathcal{E}(t,\eta)_{q+1},
\]

\[
(T\eta)_h = A_h^{-1}\left\{ \frac{t^{k_h-2}}{\Gamma(k_h-1)} * (e^{-t} - 1)\eta_h - \mathcal{E}(t,\eta)_h \right\} \quad \text{for } h > q + 1.
\]

where \( * \) denotes the convolution product: \( u * v(t) = \int_0^t u(s)v(t-s)ds \) and \( \mathcal{E}(t,\mathcal{B}y(t)) = \mathcal{B}(\widetilde{E}(.,y(.)))(t) \). Thus, \( \mathcal{E}(t,\eta) = \sum_{m \in \mathbb{N}_0^q} \mathcal{E}_m * \eta^{*m}(t) \), where \( \mathbb{N}_0 \) denotes the set of nonnegative integers and each \( \mathcal{E}_m \) is analytic on the Riemann surface of \( t \), satisfying a condition of the form

\[
|\mathcal{E}_m(t)| \leq K|t|^{|m|}|t|^{1/p-1}e^{c_0|t|}
\]

uniformly on \( S(I^*_q) \), and \( \mathcal{E}_0(t)_h = O(t^{N/p-1}) \) for all \( h \). \( K, b \) and \( c_0 \) are positive constants. \( |m| \) denotes the 1-norm of the \( n \)-vector \( m \): \( |m| := \sum_{i=1}^n m_i \).

Lemma 3.3. — Assume that \( I^*_q \subset (-\frac{\pi}{2},\frac{\pi}{2}) \) and \(-I^*_q\) does not contain a singular direction of level 1. Then \( u_q \) can be analytically continued to \( S(I^*_q) \), and its analytic continuation, also denoted by \( u_q \), has the following property: for every interval \( I' \prec I^*_q \) there exist positive numbers \( B \) and \( C \) (depending on \( I' \)) such that, for all \( t \in S(I') \),

\[
|u_q(t)| \leq Ce^{B|t|}.
\]

Proof. — The proof of the first statement is analogous to the proof in the case without level \( 1^+ \), sketched in [3] (cf. also [1]). In order to prove
the growth property, proceeding as in [3], we introduce an operator $T : C([0, \infty)) \to C([0, \infty))$, defined by

$$(3.6) \quad T\psi(t) = Me^t(t^{1/p-1}e^{ct}) \ast \sum_{m=0}^{\infty} b^m \psi^m(t)$$

where $M$ is a sufficiently large positive number and $c_0' > c_0$. Let $I' < I_q^*$ and let $v \in C([0, \infty))$ be defined by

$$v(t) = \sup \{|u_q(s)| : \arg s \in I', |s| = t\} \quad \text{if } t > 0, \quad v(0) = 0$$

($v$ is continuous, due to the fact that $u_q \in t^{N/p-1}(A_0(I_q^*))$). As $I' < (-\pi/2, \pi/2)$, $e^{-t}$ and $(e^{-t} - 1)^{-1}$ are bounded on $S(I', 1)$. Due to the assumption that $-I_q^*$ does not contain a singular direction of level 1, $((e^{-t} - 1)I_n - A_q)^{-1}$ is bounded on $S(I', 1)$ as well (the eigenvalues of $(e^{-t} - 1)I_n - A_q$ are $e^{-t} - e^{\mu_j}$ with $j \in \{1, \ldots, n\}$ such that $d_j = 0$ and $\kappa_j = 1$, and $-\arg t \neq \pi - \arg(\mu_j + 2l\pi i)$ for all such $j$ and all $l \in \mathbb{Z}$). Furthermore, for all $t \in (0, \infty)$,

$$t^{-k} \ast (t^{1/p-1}e^{ct}) \leq B(1 - k, 1/p)t^{1/p-1}e^{ct} \leq C t^{1/p-1}e^{ct}$$

and $t^{-k} \leq Ct^{1/p-1}e^{ct}$ if $0 \leq k \leq 1 - 1/p$, and $t^{k-2} \leq Ct^{1/p-1}e^{ct}$ for all $t \in (0, \infty)$ if $k \geq 1/p + 1$. Here, $B$ denotes the Euler beta-function and $C$ is a positive constant. Hence it follows that $|u_q(t)| = |Tu_q(t)| \leq T\psi(|t|)$ for all $t \in S(I', 1)$, provided $M$ is sufficiently large. Let $M_0 > \max\{v(t) : t \leq 1\}$ and $\chi_0(t) = e^{Bt+e^{B't}}$, where $B$ and $B' > 0$. Then $|u_q(t)| < M_0\chi_0(|t|)$ for all $t \in S(I')$ such that $|t| \leq 1$. Suppose there exists $t_0 \in (0, \infty)$ such that

$$v(t) < M_0\chi_0(t) \quad \text{for all } t < t_0 \quad \text{and } v(t_0) = M_0\chi_0(t_0).$$

Obviously, $t_0 > 1$. Consequently, $v(t_0) = \sup\{|u_q(s)| : \arg s \in I', |s| = t_0\} \leq T\psi(t_0) < T(M_0\chi_0)(t_0)$, $T$ being a monotone operator. From Lemma 3.4(iii) below we deduce that $T\psi(t_0) \leq M_0\chi_0(t)$ for all $t \geq 1$, sufficiently large $B'$ and suitable values of $B$. This implies $v(t_0) < M_0\chi_0(t_0)$, contradictory to the assumption and thus we conclude that $|u_q(t)| < M_0\chi_0(|t|)$ for all $t \in S(I')$, provided $B'$ and $M$ are sufficiently large. Hence the result follows.

**Lemma 3.4.** — Let $p \in \mathbb{N}$, $B' > 0$ and $0 < B < eB'$, and let

$$\chi_0(t) = e^{Bt+e^{B't}}.$$

(i) $\chi_0^m(t) \leq \left(\frac{e^{2e}}{B'}\right)^{m-1}\chi_0(t)$ for all $t \in (0, \infty)$ and all $m \in \mathbb{N}$. 

**Proof.** 

\[\square\]
(ii) Let \( 1/p \leq a \leq 1, c \geq 0 \) and \( B \geq B' + c \). Then
\[
(t^{a-1} e^{ct}) \ast \chi_0(t) \leq (p + 1) e^{-B'at} \chi_0(t)
\]
for all \( t \geq 1 \), provided \( B' \) is sufficiently large.

(iii) For all sufficiently large values of \( B' \) and \( B' + c_0 < B < eB' \), there exists a positive constant \( K(B') \) with the property that \( K(B') \to 0 \) as \( B' \to \infty \), and
\[
\overline{T}(M_0 \chi_0)(t) \leq bMK(B')M_0 \chi_0(t)
\]
for all \( t \geq 1 \).

Proof.
(i) We have
\[
(\chi_0 \ast \chi_0)(t) = e^{Bt} \int_0^t e^{e^{B't}(t-r)+e^{B't}r} dr = 2te^{Bt} \int_0^{1/2} e^{e^{B't}(1-s)+e^{B't}s} ds.
\]
Due to the convexity of \( e^{B't}s \) w.r.t. \( s \),
\[
e^{B't(1-s)} + e^{B't}s \leq e^{B't} + 1 + 2s(2e^{B't/2} - e^{B't} - 1)
\]
for all \( s \in [0, 1/2] \), and thus
\[
(\chi_0 \ast \chi_0)(t) \leq 2te^{Bt+e^{B't}+1} \int_0^{1/2} e^{2s(2e^{B't/2} - e^{B't} - 1)} ds
\]
\[
= 2te^{Bt+e^{B't}+1} \int_0^{1/2} e^{-2(e^{B't/2}-1)^2} ds
\]
\[
= \frac{te(1 - e^{-(e^{B't/2}-1)^2})}{(e^{B't/2} - 1)^2} \chi_0(t).
\]
Hence we deduce that
\[
(\chi_0 \ast \chi_0)(t) \leq \frac{te}{(e^{B't/2} - 1)^2} \chi_0(t) \leq \frac{4e}{B'^2} \chi_0(t) \leq \frac{4e}{B'} \chi_0(t)
\]
for all \( t \geq 1/B' \). Furthermore, for all \( t \leq 1/B' \), we have
\[
(\chi_0 \ast \chi_0)(t) \leq \chi_0(1/B')(1 \ast \chi_0)(t) \leq \frac{e^{B'/B'}+e}{B'} \chi_0(t) \leq \frac{e^{2e}}{B'} \chi_0(t)
\]
provided \( B \leq eB' \). The first statement of the lemma follows easily by means of an inductive argument.
(ii) For all $t \geq 1$, $B' \geq 1$ and $B \geq B' + c$ we have

$$
(t^{a-1}e^{ct}) \ast \chi_0(t) = e^{ct} \int_0^t (t-\tau)^{a-1}e^{(B-c)\tau+e^{B'\tau}}d\tau + e^{ct} \int_t^{t-e^{-B't}} (t-\tau)^{a-1}e^{(B-c)\tau+e^{B'\tau}}d\tau
$$

$$
\leq e^{(B-B')t}e^{(1-a)B't} \int_0^t e^{B'\tau+e^{B'\tau}}d\tau + e^{Bt+e^{B't}} \int_t^{t-e^{-B't}} (t-\tau)^{a-1}d\tau
$$

$$
\leq \chi_0(t)(\frac{1}{B'} + \frac{1}{a})e^{-B'at} \leq (p+1)e^{-B'at}\chi_0(t).
$$

(iii) From (i) we deduce that, for all $t > 0$,

$$
e^{-tT}(M_0\chi_0)(t) \leq M\left((t^{1/p-1}e^{c_0't}) \ast 1 + (t^{1/p-1}e^{c_0't})
\ast \sum_{m=1}^{\infty} b\left(\frac{e^{2e}bM_0}{B'}\right)^{m-1}M_0\chi_0(t)\right)
\leq pMt^{1/p}e^{c_0't}\ast 2Mb(t^{1/p-1}e^{c_0't}) \ast M_0\chi_0(t)
$$

if $B' \geq 2e^{2e}bM_0$. In view of (ii) this implies that

$$
e^{-tT}(M_0\chi_0)(t) \leq pMt^{1/p}e^{c_0't}\ast 2Mb(t^{1/p-1}e^{c_0't}) \ast M_0\chi_0(t)
$$

for all $t \geq 1$, provided $B' + c_0' < B < eB'$ and hence the result follows, with $K(B') = 3(p+1)e^{1-B'/p}$, provided $bM_0 \geq 1$, $B' > p$ and $B' + c_0' < B < eB'$.

\[\square\]

3.2. Accelero-summability of $\hat{f}$

We begin by giving the definition of accelero-summability used in [4], which suits our present purpose. The accelero-sum of the formal solution of (1.1) will be an element of $(\hat{A}(I_{q+1}))^n$, where $I_{q+1}$ is an appropriate interval. The main result for the case that 0 is not a singular direction of level 1 is stated in Theorem 3.8.

**DEFINITION 3.5** (accelero-summability, first version). — Let $0 = k_0 < k_1 < \cdots < k_q = 1$, $I_0 = \mathbb{R}$ and let $I_h$, $h = 1, \ldots, q + 1$, be open intervals of $\mathbb{R}$ with the following properties:

- $[-\frac{\pi}{2}, \frac{\pi}{2}] \subset I_q \subset \cdots \subset I_1$.
- $|I_h| > \frac{\pi}{kh}$ for $h = 1, \ldots, q$ and $I_{q+1}$ is a large interval.
\( \hat{f} \in \mathbb{C}[[z^{-1/p}]] \) is called \((k_1, \ldots, k_\xi, 1^+)\)-summable on \((I_1, \ldots, I_{q+1})\) with \((k_1, \ldots, k_\xi, 1^+)\)-sum \(f_{q+1}\). If there exist \(f_h \in \mathcal{A}/\mathcal{A}^{\leq -k_h+1}(I_h)\), \(h = 0, \ldots, q-1\), \(f_q \in \mathcal{A}/\mathcal{A}^{\leq -1+}(I_q)\) and \(f_{q+1} \in \hat{\mathcal{A}}(I_{q+1})\), with asymptotic expansion \(\hat{f}\), such that

- \(f_0(ze^{2p\pi i}) = f_0(z)\),
- \(f_{h-1}1 = f_h \mod \mathcal{A}^{\leq -k_h}, \quad h = 1, \ldots, q\)

and any representative \(\{\phi_{\nu} : \nu \in \mathcal{N}\} \) of \(f_q\), where \(\phi_{\nu} \in \mathcal{A}(I_{q, \nu})\) and \(\{I_{q, \nu} : \nu \in \mathcal{N}\}\) is an open covering of \(I_q\), has the following property: for any interval \(I'_{q, \nu} \varsubsetneq I_{q, \nu}\) and any interval \(I' < I_{q, 1}\), there exist positive constants \(R, c\) and \(C\) such that

\[
|f_{q+1}(z) - \phi_{\nu}(z)| \leq Ce^{-c|z|} \text{ for all } z \in \hat{D}_{I'}(R) \cap S(I'_{q, \nu}).
\]

**Remark 3.6.**

(i) If \(f_q\) has one representative \(\{\phi_{\nu} : \nu \in \mathcal{N}\}\) such that (3.7) holds, then all representatives have this property.

(ii) In § 4.1 (Lemma 4.6) it is shown that any \(f \in \mathcal{A}/\mathcal{A}^{\leq -1+}([-\pi/2, \pi])\) defines an element \(\hat{f}^+ \in \hat{\mathcal{A}}/\hat{\mathcal{A}}^{\leq -1+}(\mathbb{R})\). Thus, (3.7) can be replaced by the condition

\[
\hat{f}_q^+ |_{I_{q+1}} = f_{q+1} \mod \hat{\mathcal{A}}^{\leq -1+}.
\]

Lemma 2.12 shows that \(f_{q+1}\) is determined uniquely by \(f_q\). The uniqueness of \(f_q\) can be deduced from the following lemma:

**Lemma 3.7** ("relative Watson Lemma" I). — *Let \(I\) be an open interval of \(\mathbb{R}\) such that \(|I| > \pi\). Then \(\mathcal{A}^{\leq -1}/\mathcal{A}^{\leq -1+}(I) = \{0\}\).*

The proof of this lemma is analogous to the one given in [15, 16], cf. also Lemma 4.14 below.

**Theorem 3.8.** — *Let \(F\) be a \(\mathbb{C}^n\)-valued function, analytic in a neighbourhood of \((\infty, y_0)\) for some \(y_0 \in \mathbb{C}^n\). Suppose that (1.1) has a formal solution \(\hat{f} \in \mathbb{C}^n[[z^{-1/p}]]\), with constant term \(y_0\), such that (1.2) holds, and that the corresponding difference operator \(\hat{\Delta} = \tau - \hat{A}\) has positive levels \(k_1 < \cdots < k_\xi = 1^+\). Let \(I_h, h = 1, \ldots, q+1\), be open intervals of \(\mathbb{R}\) with the following properties:

- \(\bar{I}_{q+1} \subset [-\pi/2, \pi/2] \subset I_q \subset \cdots \subset I_1\).
- \(|I_h| > \frac{\pi}{k_h} \text{ for } h = 1, \ldots, q\) and \(I_{q+1}\) is a large interval.
- \(I_h\) does not contain a Stokes pair of level \(k_h\) for \(h = 1, \ldots, q\).
- \(I_{q+1} \cap \Theta(\hat{\Delta}) = \emptyset\).*
Then $\hat{f}$ is $(k_1, \ldots, k_q, 1^+)$-summable on $(I_1, \ldots, I_q, I_{q+1})$ and its sum is a solution of (1.1).

This theorem can be derived from propositions 3.9 and 3.11 below. The condition $\hat{I}_{q+1} \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$, implying that $f_{q+1}$ is defined on a subset of $S(([-\frac{\pi}{2}, \frac{\pi}{2}]))$, will be lifted in section 4, Theorem 4.12, I.

As before, let us assume that (1.1) is in prepared form (2.7), let $f_h \in z^{-N/p}(\mathcal{A}/\mathcal{A}^{\leq -k_{h+1}})^n(I_h)$ be solutions of (2.7), satisfying (3.1) for $h = 1, \ldots, q - 1$, and let $u_q = B(f_{q-1})$. The conditions on $I_q$ imply that $0 \in I_q^*$ and $-I_q^*$ does not contain a singular direction of level 1 (cf. Remark 3.2). Hence, in view of Remark 2.3(ii), $I_q^* \subset (-\frac{\pi}{2}, \frac{\pi}{2})$.

For all $\alpha \in I_q^*$, $\theta \in \mathbb{R}$ and $z \in S((-\pi, \pi))$ such that $\alpha + \arg \log(z e^{i\theta}) \in I_q^*$, we define

$$\phi_\alpha^\theta(z) := \int_0^{r(z)} e^{i\alpha \log(z e^{i\theta})/B_\alpha^\theta} u_q(s) e^{-sz} ds.$$  

(The idea of using finite Laplace integrals of the form $\int_0^{r(z)} u(s) e^{-sz} ds$, where $r(z) \to \infty$ as $z \to \infty$ is due to Braaksma, cf. [2].)

**Proposition 3.9.** — Let $(-\frac{\pi}{2}, \frac{\pi}{2}) \subset I_q \subset I_{q-1}$ such that $|I_q| > \pi$, $I_q^* \subset (-\frac{\pi}{2}, \frac{\pi}{2})$ and $-I_q^*$ does not contain a singular direction of level 1. Let $\mathcal{I} \subset \mathbb{R}$.

Then $\{\phi_\alpha^\theta : \alpha \in I_q^*, \theta \in \mathcal{I}\}$ represents a solution $f_q$ of (2.7) in $(\mathcal{A}/\mathcal{A}^{\leq -1^+}(I_q))^n$ with the property that $f_q \bmod (\mathcal{A}^{\leq -1})^n = f_{q-1}|_{I_q}$.

**Proof.** — By Lemma 3.3, for every $\alpha \in I_q^*$ and $\delta_\alpha > 0$ such that $[\alpha - \delta_\alpha, \alpha + \delta_\alpha] \subset I_q^*$, there exist positive numbers $B_\alpha$ and $C_\alpha$, such that $|u_q(s)| \leq C_\alpha e^{R_\alpha|s|}$ for all $s \in S[\alpha - \delta_\alpha, \alpha + \delta_\alpha]$. Let $\alpha \in I_q^*$, $B_\alpha > B_\alpha$, $\beta \in (0, \frac{\pi}{2})$, $0 < \delta_\beta < \min\{\delta_\alpha, \frac{\pi}{2} - \beta\}$, $\theta \in \mathbb{R}$ and let $R$ be a sufficiently large number such that $\arg \log(z e^{i\theta}) \in (-\delta_\beta, \delta_\beta)$ for all $z \in S([-\alpha - \beta, -\alpha + \beta], R)$. It is easily seen that, for all $z \in S([-\alpha - \beta, -\alpha + \beta], R)$,

$$|\phi_\alpha^\theta(z) - \int_0^{re^{i\alpha}} u_q(s) e^{-sz} ds| \leq Ce^{-\delta|z|}$$

provided $R$ is sufficiently large, where $C$ and $\delta > 0$. Hence it follows that $\phi_\alpha^\theta \in (\mathcal{A}(\mathcal{A}(\mathcal{A}^{\leq -1^+}(I_q))^n$ and $\{\phi_\alpha^\theta : \alpha \in I_q^*, \theta \in \mathcal{I}\}$ is a representative of $L(u_q)|_{I_q} = f_{q-1}|_{I_q}$. From Lemma 3.10(ii) below we deduce that $\{\phi_\alpha^\theta : \alpha \in I_q^*, \theta \in \mathcal{I}\}$ also represents an element $f_q \in (\mathcal{A}/\mathcal{A}^{\leq -1^+}(I_q))^n$. Obviously, $f_q \bmod (\mathcal{A}^{\leq -1})^n = f_{q-1}|_{I_q}$. The fact that $f_{q-1}|_{I_q}$ is a solution of (2.7) in $(\mathcal{A}/\mathcal{A}^{\leq -1^+}(I_q))^n$ implies that $\Delta \phi_\alpha^\theta - \varphi_0 - E(z^{1/p}, \phi_\alpha^\theta) : \alpha \in I_q^*, \theta \in \mathcal{I}$ represents an element of $(\mathcal{A}^{\leq -1}/\mathcal{A}^{\leq -1^+}(I_q))^n$. As $|I_q| > \pi$, by Lemma 3.7, $f_q$ is a solution of (2.7) in $(\mathcal{A}/\mathcal{A}^{\leq -1^+}(I_q))^n$. 

\[\square\]
Note that Proposition 3.9 also holds if 0 is a singular direction of level 1, provided \( 0 \notin I_0^q \) (cf. Remark 4.13 below).

**Lemma 3.10.** — Let \( I \) be an open interval of \( \mathbb{R} \) and \( u : S(I) \to \mathbb{C} \) a holomorphic function, satisfying a growth condition of the form

\[
|u(s)| \leq Ce^{B|s|},
\]

where \( B \) and \( C \) are positive constants.

(i) Assume that \( 0 \in I \) and let \( I' \prec (\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2})) \) be an open interval with the property that \( \theta_-(I') < \theta_+(I') \). Let \( r > 0, B' > B \), and \( \theta \in I' \). There exist positive constants \( C' \) and \( \delta \) such that

\[
\left| \int_r^{\log(ze^{i\theta})/B'} u(s)e^{-sz}ds \right| \leq C'e^{-\delta|\log|z||}
\]

for all \( z \in \widehat{D}_{I'}(R) \), provided \( R \) is sufficiently large.

Let \( \theta_1, \theta_2 \in \tilde{I}' \). There exist positive constants \( C' \) and \( c \) such that

\[
\left| \int_{\log(ze^{i\theta_1})/B'}^{\log(ze^{i\theta_2})/B'} u(s)e^{-sz}ds \right| \leq C'e^{-c|z|}
\]

for all \( z \in \widehat{D}_{I'}(R) \), provided \( R \) is sufficiently large.

(ii) For \( j = 1, 2 \), let \( B_j > B, \theta_j \in \mathbb{R}, \alpha_j \in I \) and let \( I_j \prec (-\alpha_j - \frac{\pi}{2}, -\alpha_j + \frac{\pi}{2}) \). There exist positive constants \( C' \) and \( \delta \) such that

\[
\left| \int_0^{e^{i\alpha_1} \log(ze^{i\theta_1})/B_1} u(s)e^{-sz}ds - \int_0^{e^{i\alpha_2} \log(ze^{i\theta_2})/B_2} u(s)e^{-sz}ds \right| \leq C'e^{-\delta|z|}\log|z|}
\]

for all \( z \in S(I_1 \cap I_2, R) \), provided \( R \) is sufficiently large.

(iii) Assume that \( 0 \in I \) and let \( I' \prec (\phi_-(\frac{\pi}{2}), \phi_+(-\frac{\pi}{2})) \) be an open interval with the property that \( \theta_-(I') < \theta_+(I') \). Let \( B' > B, \theta \in \tilde{I}' \), \( \alpha \in I \) and \( I'' \prec (-\alpha - \frac{\pi}{2}, -\alpha + \frac{\pi}{2}) \). There exist positive constants \( C' \) and \( c \) such that

\[
\left| \int_{\log(ze^{i\theta})/B'}^{e^{i\alpha} \log(ze^{i\theta})/B'} u(s)e^{-sz}ds \right| \leq C'e^{-c|z|}
\]

for all \( z \in \widehat{D}_{I'}(R) \cap S(I''), \) provided \( R \) is sufficiently large.

**Proof.**

(i) As \( \widehat{D}_{I'}(R) \subset S((-\frac{\pi}{2}, \frac{\pi}{2}), R) \), we have, for all \( z \in \widehat{D}_{I'}(R) \),

\[
\left| \int_r^{\log(ze^{i\theta})/B'} u(s)e^{-sz}ds \right| \leq C(\Re z)^{-1}e^{B|\log(ze^{i\theta})/B' - r \Re z|}.
\]
With the aid of (2.4) it follows that, for all $z \in \hat{D}_I(R)$,
\[
\left| \int_R^{\log(ze^{i\theta})/B'} u(s)e^{-sz}ds \right| \leq C' e^{K_\theta |z| \frac{B}{B'} - \delta' - \frac{|s|}{\log |z|}}
\]
provided $R$ is sufficiently large, where $C'$, $K_\theta$ and $\delta' > 0$. As $\log(ze^{i\theta}) \to 0$ as $|z| \to \infty$, $\log(ze^{i\theta})/B' \in S(I)$ for all $z \in \hat{D}_I(R)$ if $R$ is sufficiently large. Let $\epsilon \in (0, \frac{\pi}{2})$ such that $(-\epsilon, \epsilon) \subset I$, and take $R$ so large that $|\arg \log(ze^{i\theta})| < \epsilon/2$ for all $z \in \hat{D}_I(R)$. Let $C_{12}(z)$ denote the arc of the circle $|s| = |\log ze^{i\theta}|/B'$ between $\arg s = 0$ and $\arg s = \arg \log(ze^{i\theta})$. For all $z \in \hat{D}_I(R)$ such that $|\arg z| \leq \frac{\pi}{2} - \epsilon$ we have
\[
\left| \int_{C_{12}(z)} u(s)e^{-sz}ds \right| = \left| \int_{C_{12}(z)} u(s)e^{-sz}ds \right| \\
\leq C e^{B|\log(ze^{i\theta})|/B' - |\log ze^{i\theta}| |s| \sin(\epsilon/2)/B'}
\]
whereas, for all $z \in \hat{D}_I(R)$ such that $|\arg z| \geq \frac{\pi}{2} - \epsilon$ and $s \in C_{12}(z)$, $|\arg z + \arg s| \leq |\arg(\log ze^{i\theta})|$, hence, with (2.5),
\[
\left| \int_{C_{12}(z)} u(s)e^{-sz}ds \right| \leq C e^{B|\log(ze^{i\theta})|/B' - \text{Re}(\psi(\theta))}/B' \\
\leq C' e^{K_\theta |z| \frac{B}{B'} - c'|z|}
\]
where $c' > 0$. From the above estimates the first statement of the lemma follows. Similarly, supposing that $\theta_1 < \theta_2$, we find
\[
\left| \int_{\log(ze^{i\theta_2})/B'}^{\log(ze^{i\theta_1})/B'} u(s)e^{-sz}ds \right| \leq C' e^{K|z| \frac{B}{B'} - \text{Re}(\psi(\theta_2))}/B' \int_{\theta_1}^{\theta_2} e^{-\text{Re}(\psi(\theta))}/B' d\theta \\
\leq C'' e^{K|z| \frac{B}{B'} - c'' |z|}
\]
where $C''$, $K$ and $c'' > 0$.

(ii) Without loss of generality we may take $\mathcal{I}_j = (-\alpha_j - \beta, -\alpha_j + \beta)$ for $j = 1, 2$, where $0 < \beta < \frac{\pi}{2}$. Let $\epsilon \in (0, \frac{\pi}{2} - \beta)$ such that, for $j = 1, 2$, $(\alpha_j - \epsilon, \alpha_j + \epsilon) \subset I$, and take $R$ so large that $|\arg \log(ze^{i\theta_j})| < \epsilon/2$ for all $z \in S(\mathcal{I}_1 \cap \mathcal{I}_2, R)$. Then, for $j = 1$ and 2, and all $z \in S(\mathcal{I}_1 \cap \mathcal{I}_2, R)$, 
$\cos(\arg z \log(ze^{i\theta_j}) + \alpha_j) \geq \sin(\epsilon/2)$. Let $s_j := e^{i\alpha_j \log(ze^{i\theta_j})}/B_1$, $j = 1, 2$
and let $C_{12}(z)$ denote the arc of the circle $|s| = |\log(z e^{i\theta})|/B_1$ between $s_1$ and $s_2$. For all $z \in S(I_1 \cap I_2, R)$, $C_{12}(z) \subset S(I)$ and $\cos(\arg z + \arg s) \geq \sin(\epsilon/2)$ for all $s \in C_{12}(z)$, hence

$$\left| \int_0^{s_1} u(s)e^{-sz}ds - \int_0^{s_2} u(s)e^{-sz}ds \right| \leq C \int_{C_{12}(z)} e^{e^{B|s|}|-s||z|} \cos(\arg z + \arg s)|ds|$$

$$\leq C|\alpha - \alpha_1||\log z|e^{B|\log(z e^{i\theta})|/B_1} - |\log(z e^{i\theta})|| \sin(\epsilon/2)/B_1$$

$$= e^{-\sin(\epsilon/2)/B_1} |\log z| \alpha/(1+o(1)) \quad \text{as} \quad z \to \infty \quad \text{in} \quad S(I_1 \cap I_2, R).$$

Furthermore, supposing that $B_1 \leq B_2$, we have for all $z \in S(I_1 \cap I_2, R)$,

$$\left| \int_0^{s_2} u(s)e^{-sz}ds - \int_0^{e^{i\alpha z} \log(z e^{i\theta})} u(s)e^{-sz}ds \right| \leq C \int_{e^{i\alpha z} \log(z e^{i\theta})} e^{e^{B|s|}|-\epsilon||z||s||ds|}$$

$$\leq C/(\epsilon||z||)e^{K|z| B_1 - \epsilon||z||1/B_2}$$

where $K > 0$ and $\epsilon' = \sin(\epsilon/2)$. The statement of the lemma now follows immediately.

(iii) Suppose $0 < \alpha < \pi$ and let $C_{12}(z)$ denote the arc of the circle $|z| = |\log(z e^{i\theta})|/B'$ between $\log(z e^{i\theta})/B'$ and $e^{i\alpha \log(z e^{i\theta})}/B'$. In view of (ii) it suffices to consider $z \in \hat{D}_R(R)$ with the property that $\arg z \leq -\alpha/2 - \delta$, where $0 < \delta < (\pi - \alpha)/2$. Then $\arg(z \log(z e^{i\theta})) \leq \arg z + \arg s \leq \alpha/2 - \delta + \arg(z e^{i\theta})$. As $-\pi/2 < \arg(z \log(z e^{i\theta})) \leq -\alpha/2 - \delta + \arg(z e^{i\theta}) < \alpha/2 + \delta - \arg(z e^{i\theta})$ if $R$ is sufficiently large, $\cos(\arg z + \arg s) \geq \cos \arg(z \log(z e^{i\theta}))$ for all $s \in C_{12}(z)$, provided $R$ is sufficiently large. Then we have, with (2.5),

$$\left| \int_{C_{12}(z)} u(s)e^{-sz}ds \right| \leq C \int_{C_{12}(z)} e^{e^{B|\log(z e^{i\theta})|/B' - |z \log(z e^{i\theta})| \cos(\arg z + \arg s)/B'|ds|}$$

$$\leq C' e^{e^{B|\log(z e^{i\theta})|/B' - \text{Re}(\psi(x))}/B'}$$

$$\leq C' e^{K \theta |z| B'/2 - \epsilon||z||}$$

where $C'$, $K_\theta$ and $\epsilon' > 0$. The proof for the case that $-\pi < \alpha < 0$ is similar. If $|\alpha| \geq \pi$, then $\hat{D}_R(R) \cap S(J') = \emptyset$ for all $R > 1$. ☐
In order to establish the accelero-summability of the formal solution of (2.7) in the case that 0 is not a singular direction of level 1, it remains to prove the existence of a solution $f_{q+1} \in (\widehat{A}(I_{q+1}))^n$ with the properties mentioned in definition 3.5. This will be done by suitably modifying some of the functions $\phi^\theta_0$ defined above.

**Proposition 3.11.** — Let $I_{q+1}$ be a large interval such that $\widehat{I}_{q+1} \subset [-\frac{\pi}{2}, \frac{\pi}{2}] \subset I_q$ and $\widehat{I}_{q+1} \cap \Theta(\Delta^c) = \emptyset$. Assume that $-I^c_q$ does not contain a singular direction of level 1. Then equation (2.7) has a solution $f_{q+1} \in (\widehat{A}(I_{q+1}))^n$ with the properties mentioned in definition 3.5.

**Proof.** — Let $\theta \in \widehat{I}_{q+1}$ and $y = w + \phi^\theta_0$. Then $(\phi_-(\theta), \phi_+(\theta)) \subset I_{q+1} \subset (\phi_-(\frac{\pi}{2}), \phi_+(\frac{\pi}{2}))$ and $y$ is a solution of the equation (2.7) if and only if $w$ satisfies the equation

\[
\Delta w(z) = G(z, w(z)) := E(z^{1/p}, w(z) + \phi^\theta_0(z)) + \varphi_0(z^{1/p}) - \Delta \phi^\theta_0(z).
\]

By Lemma 2.11, the function $\phi_r$ defined by $\phi_r(z) = \int_0^r u_q(s)e^{-sz}ds$, is an element of $(\widehat{A}(\phi_-(\frac{\pi}{2}), \phi_+(\frac{\pi}{2}))^n \subset (\widehat{A}(\phi_-(\theta), \phi_+(\theta)))^n$. From Lemma 3.10 (i) we infer that $\phi^\theta_0 - \phi_r \in (\widehat{A}^{\leq-1}(\phi_-(\theta), \phi_+(\theta)))^n$, hence $\phi^\theta_0 \in (\widehat{A}(\phi_-(\theta), \phi_+(\theta)))^n$. With proposition 3.9, (2.11) and Lemma 3.10 (iii) it follows that

\[
G(z, 0) = E(z^{1/p}, \phi^\theta_0(z)) + \varphi_0(z^{1/p}) - \Delta \phi^\theta_0(z) \in (\widehat{A}^{\leq-1}(\phi_-(\theta), \phi_+(\theta)))^n.
\]

According to [12, Theorem 1.2], with $I = \{\theta\}$, $k = 1^+$ and $l = 0$, the equation (3.8) has a solution $w_\theta \in (\widehat{A}^{\leq-1}(\phi_-(\theta), \phi_+(\theta)))^n$. Now, let $\theta_1, \theta_2 \in \widehat{I}_{q+1}$, $\theta_1 < \theta_2$, and let $y_i = w_{\theta_i} + \phi^\theta_0$, $i = 1, 2$. As $w_{\theta_i} \in (\widehat{A}^{\leq-1}(\phi_-(\theta_1), \phi_+(\theta_2)))^n$ for $i = 1, 2$, and, by Lemma 3.10(i), $\phi^\theta_0 - \phi^\theta_0 \in (\widehat{A}^{\leq-1}(\phi_-(\theta), \phi_+(\theta)))^n$, $y_1 - y_2 = (\widehat{A}^{\leq-1}(\phi_-(\theta_1), \phi_+(\theta_2)))^n$ as well. Both $y_1$ and $y_2$ are solutions of the nonlinear difference equation (2.7), so the difference $y_1 - y_2$ satisfies a homogeneous linear equation of the form (2.10), with $H \in \text{End}(n; (\widehat{A}(\phi_-(\theta_1), \phi_+(\theta_2)))$. By proposition 2.15, $\Delta$ and $H$ have a common canonical form $\Delta^c$ if $N$ is sufficiently large and so, by Corollary 2.14, $\text{Ker}(\Delta, \widehat{A}^{\leq-1}(\phi_-(\theta_1), \phi_+(\theta_2))) = \{0\}$. It follows that the solutions $w_{\theta} + \phi^\theta_0$, with $\theta \in \widehat{I}_{q+1}$, can be glued together, to define an analytic function $f_{q+1} \in \bigcap_{\theta \in \widehat{I}_{q+1}} (\widehat{A}(\phi_-(\theta), \phi_+(\theta)))^n = (\widehat{A}(I_{q+1}))^n$. □

**Remark 3.12.** — Let $I \prec I_{q+1}$ be a large interval, let $\theta \in \widehat{I}$ and let $f_{q+1} \in (\widehat{A}(I_{q+1}))^n$ be the unique solution of (1.1) with the properties mentioned in definition 3.5. Then the function $u_{q+1, \theta}$ defined by

\[
u_{q+1, \theta}(t) = \frac{1}{2\pi i} \int_{\delta D_1(R)} f_{q+1}(z)e^{t\psi_\theta(z)}d\psi_\theta(z), \quad \arg t = 0
\]
where $R$ is a sufficiently large positive number and $\delta \hat{D}_t(R)$ is described in the direction of increasing imaginary part, is quasi-analytic on the half line $\arg t = 0$. $u_{q+1,\theta}$ is a so-called weak accelerate of $u_q = B(f_{q-1})$. $u_{q+1,\theta}$ has exponential growth as $t \to \infty$ and $f_{q+1}$ can be represented by the Laplace integral

$$f_{q+1}(z) = y_0 + \int_0^{\infty} u_{q+1,\theta}(t)e^{-t\psi_\theta(z)}dt, \quad \text{Re} \psi_\theta(z) \geq c_\theta$$

where $c_\theta > 0$.

For a very general discussion of weak acceleration operators and their properties we refer the reader to [5, 7].

4. The general case

Let $I_{q-1}$ be an open interval of $\mathbb{R}$ such that $|I_{q-1}| > \pi/k_{q-1}$ and $[-\frac{\pi}{2}, \frac{\pi}{2}] \subset I_{q-1}$, and $f_{q-1} \in (\mathcal{A}/\mathcal{A}^{-1}(I_{q-1}))^n$. In the case that 0 is a singular direction of $\Delta^c$, of level 1, the matrix $((e^{-t} - 1)I_{q-1} - A_q)^{-1}$ in (3.3) has a singularity on the half line $\arg t = 0$ and, consequently, the Borel transform $u_q$ of $f_{q-1}$ cannot be continued analytically to this half line. In order to “bypass” in some sense, possible singularities of $u_q$ on $\arg t = 0$, we introduce a variable $r_\theta(z)$, equivalent to $z$ in the sense that

$$\lim_{z \to \infty} r_\theta(z)z^{-1} = 1.$$ 

**Definition 4.1.** — For all $z \in S((\pi, \pi), 1)$ and $\theta \in \mathbb{R}$ we define

$$r_\theta(z) = \frac{\psi_\theta(z)}{\log z} \quad \text{and} \quad \rho_\theta(z) = \text{Re} r_\theta(z).$$

We can illustrate the “bypassing” of a singularity on the half line $\arg t = 0$ with the following, very simple example.

**Example 4.2.** — For any $\theta \in \mathbb{R}$ and $R > 1$, the function

$$\phi_\theta(t) = \int_R^{\infty} e^{-z+tr_\theta(z)}dr_\theta(z)$$

is analytic in the half plane $\text{Re} t < 1$. For $\theta = 0$ we have

$$\phi_0(t) = \int_R^{\infty} e^{(t-1)z}dz = -\frac{e^{R(t-1)}}{t-1},$$

so $\phi_0$ has a simple pole at 1. For any $\theta \neq 0$, however, $\phi_\theta$ can be continued to a quasi-analytic function on the positive real axis. Let us consider the case that $\theta > 0$. By deformation of the path of integration we get, if $\text{Im} t > 0$,

$$\phi_\theta(t) = \int_R^{R+i\infty} e^{-z+tr_\theta(z)}dr_\theta(z).$$
Noting that, for all $z$ on the line $\text{Re } z = R$ and all $t \in \mathbb{R}$,
\[
\text{Re}(-z + t r_\theta(z)) = (t - 1 + \frac{\theta t \text{ arg } z}{|\log z|^2})R - \frac{\theta t \text{ Im } z \log |z|}{|\log z|^2}
\]
one easily verifies that the function defined by the right-hand side is continuous on the half plane $\text{Im } t \geq 0$ and $C^\infty$ on $\text{Im } t = 0$. Moreover, for any closed interval $[a, b] \subset (0, \infty)$, there exist positive constants $K$ and $A$ such that
\[
|\phi^{(m)}_\theta(t)| \leq K \int_R^{R+i\infty} |r_\theta(z)|^m e^{-\frac{\theta t \text{ Im } z \log |z|}{|\log z|^2}} |dr_\theta(z)| 
\leq K A^m (m \log m)^m \text{ for all } m \in \mathbb{N} \text{ and all } t \in [a, b].
\]
This implies that $\phi_\theta$ belongs to the Denjoy class $1D[a,b]$ and thus is quasi-analytic on the positive real axis.

For every $\theta \in \mathbb{R}$, the function $\phi_\theta$ has exponential growth of order 1 as $t \to \infty$ on the positive real axis. The function $f_\theta$, defined by the Laplace integral
\[
f_\theta(z) := \int_0^\infty \phi_\theta(t)e^{-t r_\theta(z)} dt
\]
is analytic in a domain of the form $\rho_\theta(z) > K' > 0$. In the case that $\theta = 0$ this obviously is a right half plane. In general, as we shall see, it contains a domain of the form $\tilde{D}_I(R)$ for every interval $I \prec (\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$. If $\theta \neq 0$ it does not contain a half plane, but is slightly ‘tilted’ and contains a part of either the positive (if $\theta < 0$) or negative (if $\theta > 0$) imaginary axis (cf. Figures 4.1 and 4.2).

**Remark 4.3.** — There is a certain amount of freedom in the choice of the variable $r_\theta(z)$. However, the “perturbation” $z - r_\theta(z)$ shouldn’t be too small. If, in example 4.2, $r_\theta(z)$ is defined as $r_\theta(z) = z + \frac{i\theta z}{(\log z)^2}$, we obtain the estimate
\[
|\phi^{(m)}_\theta(1)| = m^m (\log m)^{2m} (\theta e)^{-m+o(m)} \text{ as } m \to \infty
\]
which implies that $\phi_\theta$ is not quasi-analytic on any interval of the positive real axis containing 1. Moreover, in order to deal with the level $1^+$, the set $\text{Re } r_\theta(z) \geq 0$ (with $\theta \neq 0$) should contain a domain of the form $\tilde{D}_I(R)$, where $I$ is a large interval. This rules out larger perturbations of $z$ like $r_\theta(z) = z + \frac{i\theta z}{\log \log z}$. On the other hand, an alternative definition of the type $r_\theta(z) = z + (i\theta - 1)\frac{z}{\log z}$ would yield completely analogous results to those obtained with Definition 4.1, provided $\theta \neq 0$. The case $\theta = 0$ corresponds to a “pseudodeceleration” in the terminology used by Ecalle, serving to regularize the singularities of $u_q$ (cf. [6]).
Figure 4.1. The domain \( \hat{D}_{(\phi_-(\theta+\frac{\pi}{2}),\phi_+(\theta-\frac{\pi}{2}))} \) with \( \theta = -\frac{\pi}{4} \).

Figure 4.2. The domain \( \hat{D}_{(\phi_-(\theta+\frac{\pi}{2}),\phi_+(\theta-\frac{\pi}{2}))} \) with \( \theta = \frac{\pi}{4} \).

As the convolution equations obtained from (2.7) by applying a Borel transformation with respect to the variable \( r_\theta(z) \), in the case that \( \theta \neq 0 \),
appear quite unwieldy, we take a different approach here, more along the lines of the proof given by Ramis and Sibuya in [19]. However, instead of using an existence theorem for ordinary, analytic solutions of nonlinear difference equations, we use an existence theorem for quasi-function solutions, which considerably simplifies the argument. In this subsection we introduce 1-precise and $1^+$-precise quasi-functions ‘of the second kind’, which, instead of being defined on sectors, are defined on domains of the type $\hat{D}_I(R)$ and represent sections of the quotient sheaves $\hat{A}/\hat{A}^{\leq -1}$ and $\hat{A}/\hat{A}^{\leq -1^+}$. We show that $f_{q-1}(-\frac{\pi}{2}, \frac{\pi}{2})$ has a particular representative, which also represents an element of $(\hat{A}/\hat{A}^{\leq -1}(\mathbb{R}))^n$. Now, let $I_q$ be a large interval such that $|\hat{I}_q| > \pi$. On every large subinterval $I$ of $I_q$ such that $|I| \leq \pi$, we can modify this representative by means of exponentially small, $1^+$-precise quasi-function solutions of an associated difference equation, using a recent existence result for this type of solutions (Theorem 4.16), and obtain a solution of (2.7) in $(\hat{A}/\hat{A}^{\leq -1^+}(I))^n$. Moreover, this has the property that its restriction to any large subsector $I'$ of $I$ is represented by a solution of (2.7) in $(\hat{A}(I'))^n$, provided $\Theta(\Delta^c) \cap \hat{I}' = \emptyset$. Due to the fact that the difference of two solutions of (2.7) satisfies a homogeneous linear difference equation of the form (2.10) and by virtue of Theorem 2.13, these solutions can be glued together, resulting in a solution $f_q \in (\hat{A}/\hat{A}^{\leq -1^+}(I_q))^n$, with the property that $f_q|_{I_{q+1}}$ is represented by a solution $f_{q+1} \in (\hat{A}(I_{q+1}))^n$, provided $\Theta(\Delta^c) \cap \hat{I}_{q+1}^{-} = \emptyset$. $f_{q+1}$ is an accelero-sum of $f$ in a slightly weaker sense than that of Definition 3.5 (cf. Definition 4.11 below).

4.1. The quotient sheaves $\hat{A}/\hat{A}^{\leq -1}$ and $\hat{A}/\hat{A}^{\leq -1^+}$

Definition 4.4 (1-precise and $1^+$-precise quasi-functions). — Let $I$ be an interval of $\mathbb{R}$. A 1-precise quasi-function (of the second kind) on $I$ is a collection of functions $\{\phi_{\nu} : \nu \in \mathcal{N}\}$, where $\phi_{\nu} \in \hat{A}(I_{\nu})$, $\{I_{\nu} : \nu \in \mathcal{N}\}$ is an open covering of $I$, and $\phi_{\nu} - \phi_{\nu'} \in \hat{A}^{\leq -1}(I_{\nu} \cap I_{\nu'})$ for all $\nu$ and $\nu' \in \mathcal{N}$. Two 1-precise quasi-functions $\{\phi_{\nu} \in \hat{A}(I_{\nu}) : \nu \in \mathcal{N}\}$ and $\{\psi_{\mu} \in \hat{A}(I_{\mu}') : \mu \in \mathcal{M}\}$ on $I$ are considered equivalent if $\phi_{\nu} - \psi_{\mu} \in \hat{A}^{\leq -1}(I_{\nu} \cap I_{\mu}')$, for all $\nu \in \mathcal{N}$ and all $\mu \in \mathcal{M}$.

Similarly, a $1^+$-precise quasi-function (of the second kind) on $I$ is a collection of functions $\{\phi_{\nu} : \nu \in \mathcal{N}\}$, where $\phi_{\nu} \in \hat{A}(I_{\nu})$, $\{I_{\nu} : \nu \in \mathcal{N}\}$ is an open covering of $I$, and $\phi_{\nu} - \phi_{\nu'} \in \hat{A}^{\leq -1^+}(I_{\nu} \cap I_{\nu'})$ for all $\nu$ and $\nu' \in \mathcal{N}$. Two $1^+$-precise quasi-functions $\{\phi_{\nu} \in \hat{A}(I_{\nu}) : \nu \in \mathcal{N}\}$ and $\{\psi_{\mu} \in \hat{A}(I_{\mu}') : \mu \in \mathcal{M}\}$ on $I$ are considered equivalent if $\phi_{\nu} - \psi_{\mu} \in \hat{A}^{\leq -1^+}(I_{\nu} \cap I_{\mu}')$, for all $\nu \in \mathcal{N}$ and all $\mu \in \mathcal{M}$.
Obviously, 1-precise and $1^+$-precise quasi-functions of the second kind represent sections of the quotient sheaves $\widehat{A}/\widehat{A}^{\leq -1}$ and $\widehat{A}/\widehat{A}^{\leq -1^+}$, respectively. The following two lemmas provide us with the necessary link between $A/A^{\leq -1}$ and $A/A^{\leq -1^+}$ on one hand and $\widehat{A}/\widehat{A}^{\leq -1}$ and $\widehat{A}/\widehat{A}^{\leq -1^+}$ on the other.

**Lemma 4.5.** — For every $f \in A/A^{\leq -1}([−\frac{\pi}{2}, \frac{\pi}{2}])$ there exists a global section $\tilde{f}$ of $(\widehat{A}/\widehat{A}^{\leq -1})$, represented by a 1-precise quasi-function $\{\tilde{\phi}_\theta : \theta \in \mathbb{R}\}$ of the second kind, with the following properties:

(i) $\tilde{\phi}_\theta \in \widehat{A}(\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$,

(ii) If $\{\phi_\nu : \nu \in \mathcal{N}\}$, where $\phi_\nu \in A(I_\nu)$ and $\{I_\nu : \nu \in \mathcal{N}\}$ is an open covering of $[−\frac{\pi}{2}, \frac{\pi}{2}]$, is a representative of $f$, then, for any open interval $I'_\nu \prec I_\nu$ and any $I' < (\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$, there exist positive constants $R$, $c$ and $C$ such that

$$|\tilde{\phi}_\theta(z) - \phi_\nu(z)| \leq Ce^{-c|z|}$$

for all $z \in \widehat{D}_I(R) \cap S(I'_\nu)$.

(iii) $\tilde{\phi}_{\theta_1} - \tilde{\phi}_{\theta_2} \in \widehat{A}^{\leq -1}(\phi_-(\theta_1 + \frac{\pi}{2}), \phi_+(\theta_2 - \frac{\pi}{2}))$ if $\theta_1 < \theta_2$.

**Lemma 4.6.** — For every $f \in A/A^{\leq -1^+}([−\frac{\pi}{2}, \frac{\pi}{2}])$ there exists $\tilde{f}^+ \in (\widehat{A}/\widehat{A}^{\leq -1^+})(\mathbb{R})$, represented by a $1^+$-precise quasi-function $\{\tilde{\phi}_\theta^+ : \theta \in \mathbb{R}\}$ of the second kind, with the following properties:

(i) $\tilde{\phi}_\theta^+ \in \widehat{A}(\phi_-(\theta), \phi_+(\theta))$,

(ii) If $\{\phi_\nu : \nu \in \mathcal{N}\}$, where $\phi_\nu \in A(I_\nu)$ and $\{I_\nu : \nu \in \mathcal{N}\}$ is an open covering of $[−\frac{\pi}{2}, \frac{\pi}{2}]$, is a representative of $f$, then, for any $I'_\nu \prec I_\nu$ and any $I' < (\phi_-(\theta), \phi_+(\theta))$, there exist positive constants $R$, $c$ and $C$ such that

$$|\tilde{\phi}_\theta^+(z) - \phi_\nu(z)| \leq Ce^{-c|z|}$$

for all $z \in \widehat{D}_I(R) \cap S(I'_\nu)$.

(iii) $\tilde{\phi}_{\theta_1}^+ - \tilde{\phi}_{\theta_2}^+ \in \widehat{A}^{\leq -1}(\phi_-(\theta_1), \phi_+(\theta_2))$ if $\theta_1 < \theta_2$.

**Remark 4.7.** — Let $f \in A/A^{\leq -1}([−\frac{\pi}{2}, \frac{\pi}{2}])$ and $g \in A/A^{\leq -1^+}([−\frac{\pi}{2}, \frac{\pi}{2}])$ such that $f = g \mod A^{\leq -1}$. Let $\tilde{f}$ and $\tilde{g}^+$ denote the corresponding elements of $(\widehat{A}/\widehat{A}^{\leq -1})(\mathbb{R})$ and $(\widehat{A}/\widehat{A}^{\leq -1^+})(\mathbb{R})$, respectively. Then it is easily seen that $\tilde{f} = \tilde{g}^+$ mod $\widehat{A}^{\leq -1}$.

**Definition 4.8.** — Let $f \in A/A^{\leq -1}([−\frac{\pi}{2}, \frac{\pi}{2}])$ or $f \in A/A^{\leq -1^+}([−\frac{\pi}{2}, \frac{\pi}{2}])$, respectively, and let $\tilde{f}$ or $\tilde{f}^+$ denote the corresponding elements of $(\widehat{A}/\widehat{A}^{\leq -1})(\mathbb{R})$ and $(\widehat{A}/\widehat{A}^{\leq -1^+})(\mathbb{R})$, respectively. Let $I$ be an interval of $\mathbb{R}$. 

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Then by $f|_I$ we denote $\tilde{f}|_I \in \tilde{A}/\tilde{A}^{\leq -1}(I)$, or $f^+|_I \in \tilde{A}/\tilde{A}^{\leq -1+}(I)$, respectively.

In order to prove Lemma’s 4.5 and 4.6, we first derive some asymptotic properties of $\rho_\theta$. A straightforward computation shows that, for all $\theta$ and $\theta' \in \mathbb{R}$,

$$\rho_\theta(z) = \frac{(|\text{Re} \psi(z) + (\theta' - \theta + \arg z) \text{Im} z| \log |z| + \arg z(\arg z + \theta) \text{Re} z)^2}{|\log z|^2}.$$

Hence we deduce the estimates

$$\rho_\theta(z) = \frac{(\theta - \theta' + \frac{1}{2} \pi)|z|}{\log |z|} \left(1 + O\left(\frac{1}{\log |z|}\right)\right) \quad \text{as } z \to \infty$$

valid for any real $\theta' \neq \theta + \frac{\pi}{2}$ and all sufficiently large $R$, and

$$\rho_\theta(z) = \frac{(\theta' - \theta + \frac{1}{2} \pi)|z|}{\log |z|} \left(1 + O\left(\frac{1}{\log |z|}\right)\right) \quad \text{as } z \to \infty$$

valid for $\theta' \neq \theta - \frac{\pi}{2}$ and sufficiently large $R$ (cf. [12]). Furthermore, for any interval $I < (\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$, there exist positive numbers $R$ and $\delta$ such that

$$\rho_\theta(z) \gtrsim \delta \frac{|z|}{\log |z|} \quad \text{for all } z \in \hat{D}_I(R).$$

If, in addition, $0 \notin \tilde{T}$, there exist positive numbers $R$, $\delta_1$ and $\delta_2$ such that

$$\delta_1 \frac{|z|}{\log |z|} \leq \rho_\theta(z) \leq \delta_2 \frac{|z|}{\log |z|} \quad \text{for all } z \in \hat{D}_I(R).$$

From [12, Lemma 0.13]) and (4.6) we deduce the following result.

**Lemma 4.9.**

1. Let $\theta \in \mathbb{R}$, $I = (a, b)$ such that $a \neq 0 \neq b$, $I < (\phi_-(\theta + \frac{1}{2} \pi), \phi_+(\theta - \frac{1}{2} \pi))$ and let $R$ be a sufficiently large number. Let $f : \hat{D}_I(R) \to \mathbb{C}$ be a continuous function, holomorphic in int $\hat{D}_I(R)$. Then the following statements are equivalent.

   (i) There exist positive numbers $c$ and $C$, such that, for all $z \in \hat{D}_I(R)$,

   $$|f(z)| \leq Ce^{-c\rho_\theta(z)}.$$

   (ii) There exist positive numbers $\delta$ and $C$, such that, for all $z \in \hat{D}_I(R)$,

   $$|f(z)| \leq Ce^{-\delta \frac{|z|}{\log |z|}}.$$

2. Let $I$ be a large, open interval such that $|I| > \pi$. Then $\tilde{A}^{\leq -1}(I) = \{0\}$. 

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Remark 4.10. — From Lemma’s 4.9 and 2.12 we deduce that, for any open interval $I$ containing 0, $\hat{A}^{\leq -1}(I) \subset A^{\leq -1}(-(\frac{\pi}{2}, \frac{\pi}{2}))$ and $\hat{A}^{\leq -1}(I) < A^{\leq -1}(-(\frac{\pi}{2}, \frac{\pi}{2}))$. Obviously, $e^{-cr\theta(z)} \in \hat{A}^{\leq -1}(\phi_-(\theta + \frac{1}{2}\pi), \phi_+(\theta - \frac{1}{2}\pi))$ for all $c > 0$ and $\theta \in \mathbb{R}$. Statement 2 of Lemma 4.9 extends the well-known result that $A^{\leq -1}([-\frac{\pi}{2}, \frac{\pi}{2}]) = \{0\}$.

Proof of Lemma 4.5. — Let $\{\phi_\nu : \nu \in \{1, \ldots, N\}\}$ be a representative of $f$, with respect to a “good” open covering $\{I_\nu : \nu \in \{1, \ldots, N\}\}$ of $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (i.e. $I_\nu \cap I_\mu = \emptyset$ unless $|\nu - \mu| = 1$), such that $\inf I_\nu < \inf I_{\nu+1}$ for $\nu = 1, \ldots, N-1$. For all $\nu \in \{1, \ldots, N\}$ let $I'_\nu \prec I_\nu$, such that $\{I'_\nu : \nu \in \{1, \ldots, N\}\}$ is a good, open covering of $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Choose open subintervals $I''_\nu$ of $I_\nu$, such that $I'_\nu < I''_\nu \prec I_\nu$ for $\nu \in \{1, \ldots, N\}$ and let $\alpha_0 = \inf I'_1$, $\alpha_N = \sup I''_N$, $\alpha_\nu \in I''_\nu \cap I''_{\nu+1} \setminus (I'_\nu \cap I'_{\nu+1})$ for $\nu = 1, \ldots, N-1$, and let $R > 0$ such that $\phi_\nu$ is analytic and bounded on $S(I''_\nu, R)$ for all $\nu \in \{1, \ldots, N\}$. There exist $C'$ and $c' > 0$ such that, for all $z \in S(I''_\nu \cap I''_{\nu+1}, R)$ and $\nu = 1, \ldots, N-1$, $|\phi_\nu(z) - \phi_{\nu+1}(z)| \leq C' e^{-c'|z|}$.

Let $r < c'$. Then the function $\tilde{\varphi}_\theta$ defined, for all $z \in \cup_{\nu=1,\ldots,N} S((\alpha_{\nu-1}, \alpha_{\nu}), R)$, by

$$
\tilde{\varphi}_\theta(z) = \sum_{\nu=1}^{N} \int_{C_\nu} e^{r(r_\nu(\zeta) - r_\theta(z))} - \frac{1}{2\pi i(r_\nu(\zeta) - r_\theta(z))} \phi_\nu(\zeta) d\theta(\zeta)
$$

$$+
\sum_{\nu=0}^{N} \int_{\Re^\alpha_{\nu}}^\infty e^{i\alpha_{\nu}(\zeta) - i\alpha_{\nu}(z)} - \frac{1}{2\pi i(r_\nu(\zeta) - r_\theta(z))} (\phi_\nu - \phi_{\nu+1})(\zeta) d\theta(\zeta)$$


where $C_\nu$ denotes the arc of the circle $|z| = R$ from $\Re^{i\alpha_{\nu-1}}$ to $\Re^{i\alpha_{\nu}}$ and $\phi_0 \equiv \phi_N + 1 \equiv 0$, can be analytically continued to $S((\alpha_0, \alpha_N), R)$ by deformation of the paths of integration. Moreover, with Cauchy’s theorem it follows that, for all $z \in S(I'_\nu, R)$

$$
\tilde{\varphi}_\theta(z) - \phi_\nu(z) = \frac{e^{-rr_\theta(z)}}{2\pi i} \left( \sum_{\mu=1}^{N} \int_{C_\mu} \frac{e^{r\nu_\theta(\zeta)}}{r_\theta(\zeta) - r_\theta(z)} \phi_\mu(\zeta) d\theta(\zeta) \right)
$$

$$+
\sum_{\mu=0}^{N} \int_{\Re^\alpha_{\mu}}^\infty e^{i\alpha_{\mu}(\zeta)} - \frac{e^{r\nu_\theta(\zeta)}}{r_\theta(\zeta) - r_\theta(z)} (\phi_\mu - \phi_{\mu+1})(\zeta) d\theta(\zeta) .$$

Hence we deduce, with the aid of (4.5), that for any interval $I' \prec (\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$, there exist positive constants $c$ and $C$ such that

$$
|\tilde{\varphi}_\theta(z) - \phi_\nu(z)| \leq C e^{-c\frac{1}{2}|z|}$$

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for all $z \in \hat{D}_I'(R) \cap S(I'_\nu)$. It is easily seen that $\hat{\varphi}_\theta$ is independent of the choice of representative and the covering $\{I_\nu : \nu \in \{1, \ldots, N\}\}$. As $\hat{D}_I'(R) \subset \bigcup_{\nu \in \{1, \ldots, N\}} S(I'_\nu)$ for all sufficiently large $R$, it follows that $\hat{\varphi}_\theta \in \hat{\mathcal{A}}(\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$.

Now, let $\theta_1, \theta_2 \in \mathbb{R}$, $\theta_1 < \theta_2$. Using the estimates for $\hat{\varphi}_{\theta_\nu} - \varphi_\nu$ derived above and varying the $I'_\nu$, one easily shows that $\hat{\varphi}_{\theta_1} - \hat{\varphi}_{\theta_2} \in \mathcal{A}^{-1}(\phi_-(\theta_1 + \frac{\pi}{2}), \phi_+(\theta_2 - \frac{\pi}{2}))$. □

Note that, for any $\theta \in \mathbb{R}$, $\hat{\varphi}_\theta$ is a representative of $f\mid_{(-\frac{\pi}{2}, \frac{\pi}{2})}$. Lemma 4.6 can be proved similarly, with the aid of (2.5).

### 4.2. Accelero-summability of $\hat{f}$ in the general case

**Definition 4.11** (accelero-summability (generalization)). — Let $0 = k_0 < k_1 < \cdots < k_q = 1$, $I_0 = \mathbb{R}$ and let $I_h$, $h = 1, \ldots, q + 1$, be open intervals of $\mathbb{R}$ with the following properties:

- $[-\frac{\pi}{2}, \frac{\pi}{2}] \subset I_{q-1} \subset \cdots \subset I_1$ and $I_{q+1} \subset I_q$.
- $|I_h| > \frac{\pi}{k_h}$ for $h = 1, \ldots, q - 1$, $I_q$ and $I_{q+1}$ are large intervals and $|I_q| > \pi$.

$\hat{f} \in \mathbb{C}[\lceil z^{-1/p} \rceil]$ is called $(k_1, \ldots, k_q, 1^+)$-summable on $(I_1, \ldots, I_{q+1})$ with $(k_1, \ldots, k_q, 1^+)$-sum $f_{q+1}$, if there exist $f_h \in \mathcal{A}/\mathcal{A}^{\leq -k_h+1}(I_h)$, $h = 0, \ldots, q-1$, $f_q \in \mathcal{A}/\mathcal{A}^{\leq -1^+}(I_q)$ and $f_{q+1} \in \mathcal{A}(I_{q+1})$, with asymptotic expansion $\hat{f}$, such that

- $f_0(z e^{2p\pi i}) = f_0(z)$,
- $f_{h-1}|I_h = f_h$ mod $\mathcal{A}^{\leq -k_h}$, $h = 1, \ldots, q - 1$,
- $f_{q-1}|I_q = f_q$ mod $\mathcal{A}^{\leq -1}$,
- $f_{q+1}|I_{q+1} = f_{q+1}$ mod $\mathcal{A}^{\leq -1+}$.

Note that the role played by $I_q$ in this definition is quite different from that in definition 3.5. Here, we consider sections over $I_q$ of $\hat{\mathcal{A}}/\mathcal{A}^{\leq -1^+}$ and in definition 3.5 sections of $\mathcal{A}/\mathcal{A}^{\leq -1^+}$.

From Lemma 2.12, 2 we deduce that $f_{q+1}$ is determined uniquely by $f_q$. For, suppose $g_{q+1} \in \hat{\mathcal{A}}(I_{q+1})$ has the same properties as $f_{q+1}$. Then $f_{q+1} - g_{q+1} \in \mathcal{A}^{\leq -1^+}(I_{q+1})$, and thus, by Lemma 2.12, 2, $g_{q+1} \equiv f_{q+1}$. Similarly, it can be deduced from Lemma 4.14 below that $f_q$ is determined uniquely by $f_{q-1}$. Suppose $g_q \in \hat{\mathcal{A}}/\mathcal{A}^{\leq -1^+}(I_q)$ has the same properties as $f_q$. Then $f_q - g_q \in \mathcal{A}^{\leq -1}/\mathcal{A}^{\leq -1^+}(I_q)$ and from Lemma 4.14 it follows that $f_q = g_q$.

Now let $I_h$, $h = 1, \ldots, q + 1$, be open intervals of $\mathbb{R}$, satisfying the conditions of definition 3.5. Assume that $\hat{f}$ is $(k_1, \ldots, 1, 1^+)$-summable on
Let \((I_1, \ldots, I_q, I_{q+1})\) according to definition 3.5. With the aid of Lemma’s 4.5, 4.6 and Remark 4.7, replacing \(f_q\) by \(\tilde{f}_q^+\), it is easily seen that \(\tilde{f}\) is \((k_1, \ldots, 1, 1^+)\)-summable on \((I_1, \ldots, I_{q-1}, \mathbb{R}, I_{q+1})\) according to definition 4.11.

The main result of this paper is stated in the following theorem.

**Theorem 4.12.** Let \(F\) be a \(\mathbb{C}^n\)-valued function, analytic in a neighbourhood of \((\infty, y_0)\) for some \(y_0 \in \mathbb{C}^n\). Suppose that (1.1) has a formal solution \(\hat{f} \in \mathbb{C}^n[[z^{-1/p}]]\), with constant term \(y_0\), such that (1.2) holds, and that the corresponding difference operator \(\hat{\Delta} = \tau - \hat{\Delta}\) has positive levels \(k_1 < \cdots < k_q = 1\) and a level \(1^+\). Let \(I_h, h = 1, \ldots, q + 1\), be open intervals of \(\mathbb{R}\) with the following properties:

- \([-\pi, \pi/2] \subset I_{q-1} \subset \cdots \subset I_1\).
- \(|I_h| > \pi/k_h\) for \(h = 1, \ldots, q - 1\) and \(I_{q+1}\) is a large interval.
- \(I_h\) does not contain a Stokes pair of level \(k_h\) for \(h = 1, \ldots, q - 1\).
- \(I_{q+1} \cap \Theta(\hat{\Delta}) = \emptyset\).

I. Suppose that 0 is not a singular direction of level 1 and \([-\pi, \pi/2] \subset I_q \subset I_{q-1}\). Then \(\hat{f}\) is \((k_1, \ldots, k_{q-1}, 1, 1^+)\)-summable on \((I_1, \ldots, I_{q+1})\) in the sense of definition 3.5, and the sum is a solution of (1.1).

II. Suppose that 0 is a singular direction of level 1, \(I_q = (\phi_-(\pi/2), \infty)\) or \((\infty, \phi_+(-\pi/2))\) and \(I_{q+1} \subset I_q\). Then \(\hat{f}\) is \((k_1, \ldots, k_{q-1}, 1, 1^+)\)-summable on \((I_1, \ldots, I_{q+1})\) in the sense of definition 4.11, and the sum is a solution of (1.1).

**Remark 4.13.** Let \(I < I_{q+1}\) be a large interval, let \(\theta' \in \tilde{I}\) and let \(f_{q+1} \in (\hat{\Delta}(I_{q+1}))^n\) be the unique solution of (1.1) with the properties mentioned in definition 4.11. Then the function \(u_{q+1, \theta'}\) defined by

\[
u_{q+1, \theta'}(t) = \frac{1}{2\pi i} \int_{\delta \hat{D}_t(R)} f_{q+1}(z) e^{t\psi_\theta'(z)} d\psi_\theta'(z), \quad \arg t = 0
\]

where \(R\) is a sufficiently large positive number and \(\delta \hat{D}_t(R)\) is described in the direction of increasing imaginary part, is quasi-analytic on the half line \(\arg t = 0\) (like \(\phi_\theta\) in Example 4.2 it belongs to the Denjoy class \(\mathcal{D}[a, b]\) for any closed interval \([a, b] \subset (0, \infty)\). Let \(\theta \in -\tilde{I}_q^*\) (i.e. \(\theta < 0\) if \(I_q = (\phi_-(\pi/2), \infty)\), \(\theta > 0\) if \(I_q = (\infty, \phi_+(-\pi/2))\)) and let \(u_{q, \theta} := B_{1, \theta}(f_{q-1})\) denote the Borel transform of \(f_{q-1}\) with respect to the variable \(r_\theta(z)\), defined by
4.7 \quad u_{q, \theta}(t) = \frac{1}{2\pi i} \left( \sum_{\nu=1}^{N} \int_{C_{\nu}} \phi_{\nu}(z)e^{i\alpha_{\nu}z}dz \right) \\
+ \sum_{\nu=0}^{N} \int_{R_{e^{i\alpha_{\nu}}}} \left( \phi_{\nu} - \phi_{\nu+1} \right)(z)e^{i\alpha_{\nu}z}dz 
\]
where \( \{\phi_{\nu} : \nu \in \{1, \ldots, N\} \} \) is a representative of \( f_{q-1}|_{[-\frac{\pi}{2}, \frac{\pi}{2}]} \) with respect to a good, open covering \( \{\mathcal{I}_{\nu} : \nu \in \{1, \ldots, N\} \} \) of \( [-\frac{\pi}{2}, \frac{\pi}{2}] \) such that \( -\frac{\pi}{2} \in \mathcal{I}_{1} \) and \( \frac{\pi}{2} \in \mathcal{I}_{N} \), and \( \phi_{0} \equiv \phi_{N+1} \equiv 0 \). For \( \nu = 1, \ldots, N-1, \alpha_{\nu} \in \mathcal{I}_{\nu} \cap \mathcal{I}_{\nu+1}, \alpha_{0} \in \mathcal{I}_{1} \cap (\infty, -\frac{\pi}{2}) \) and \( \alpha_{N} \in \mathcal{I}_{N} \cap (\frac{\pi}{2}, \infty) \). \( C_{\nu} \) denotes the arc of the circle \(|z| = R \) from \( Re^{i\alpha_{\nu-1}} \) to \( Re^{i\alpha_{\nu}} \). It is easily seen that the function defined by the right-hand side is independent of the choice of representative \( \{\phi_{\nu} : \nu \in \{1, \ldots, N\} \} \). It is analytic in a sector of the form \( \{t \in S(\frac{\pi}{2} - \alpha_{N}, -\frac{\pi}{2} - \alpha_{0}) : |t| < r \} \), where \( r > 0 \). From the fact that \( f_{q-1}|_{I_{q}} = f_{q} \mod (A^{\leq -1})^{n} \) it can be derived that \( u_{q, \theta} \) can be continued quasi-analytically to the half line \( \arg t = 0 \). This quasi-analytic continuation can be represented by an expression similar to (4.7), obtained by deforming the paths of integration and replacing \( \{\phi_{\nu} : \nu \in \{1, \ldots, N\} \} \) with a representative of \( f_{q} \). \( u_{q+1, \theta'} \) is a weak accelerate of \( u_{q, \theta} \), provided \( |\theta' - \theta| < \frac{\pi}{2} \). (Cf. [14, Proposition 4 and Theorem 5] for the case that \( q = 1 \).)

On the other hand, in view of proposition 3.9, there exist \( \alpha < -\frac{\pi}{2}, \beta > \frac{\pi}{2} \) and \( f_{q}^{\pm} \in (A/\mathcal{A}^{\leq -1}(I_{q}^{\pm}))^{n} \), where \( I_{q}^{\pm} = (\alpha, \frac{\pi}{2}), I_{\hat{q}}^{\pm} = (-\frac{\pi}{2}, \beta), \) with the property that \( f_{q}^{\pm} \mod (A^{\leq -1})^{n} = f_{q-1}|_{I_{q}^{\pm}} \). This implies that, for every \( \theta \in \mathbb{R}, u_{q, \theta} \) can be analytically continued to the sectors \( S(\frac{\pi}{2} - \beta, 0) \) and \( S(0, -\alpha - \frac{\pi}{2}) \). One might expect that, for \( \arg t = 0, \lim_{\epsilon \downarrow 0} u_{q, \theta}(t + i\epsilon) = u_{q, \theta}(t) \) if \( \theta > 0 \) and \( \lim_{\epsilon \uparrow 0} u_{q, \theta}(t - i\epsilon) = u_{q, \theta}(t) \) if \( \theta < 0 \), but that is as yet an open question.

4.3. Another relative Watson Lemma

**Lemma 4.14 (“relative Watson Lemma” II).** — Let \( I \) be a large, open interval of \( \mathbb{R} \) such that \( |\tilde{I}| > \pi \). Then \( \hat{\mathcal{A}}^{\leq -1}/\hat{\mathcal{A}}^{\leq -1+}(I) = \{0\} \).

**Proof.** — We sketch a proof of this lemma, analogous to the one given in [15, 16]. Let \( \tilde{I} \in \hat{\mathcal{A}}^{\leq -1}/\hat{\mathcal{A}}^{\leq -1+}(I) \) and let \( I' < I \) be a large interval such that \( \tilde{I}' = (\theta_{1}, \theta_{2}) \) with \( \theta_{2} - \theta_{1} > \pi \). Similarly to the statement of Lemma 4.6, it is easily shown that \( f_{I'} \) admits a representative \( \{\phi_{1}, \phi_{2}\} \), where \( \phi_{j} \in \hat{\mathcal{A}}^{\leq -1}(\phi_{-(\theta_{j}), \phi_{+(\theta_{j})}}), \quad j = 1, 2, \) and \( \phi_{1} - \phi_{2} \in \hat{\mathcal{A}}^{\leq -1+}(\phi_{-(\theta_{j}), \phi_{+(\theta_{j})}}) \).
We shall show that $\phi_j \in \tilde{\mathcal{A}}^{< -1^+}((\phi_-(\theta_j), \phi_+(\theta_j)))$ for $j = 1, 2$. This implies $f|_{\Gamma'} = 0$ and, consequently, $f = 0$. By remark 4.10, $\phi_j \in \mathcal{A}^{< -1}(-\frac{\pi}{2}, \frac{\pi}{2})$, hence there exist positive numbers $c, C$ and $R$ such that, for $j = 1, 2$,
$$|\phi_j(z)| \leq Ce^{-cz}, \ z \in (R, \infty).$$
For all $\theta \in -\tilde{I}^* = (\theta_1 + \frac{\pi}{2}, \theta_2 - \frac{\pi}{2})$ and $j \in \{1, 2\}$ we define
\begin{equation}
\eta^\theta_j(s) = \int_R^\infty \phi_j(z)e^{-sr_\theta(z)}dr_\theta(z), \ \text{Re} \ s > -c.
\end{equation}
We begin by proving that, for $j = 1, 2$, $\eta^\theta_j$ is an entire function, satisfying a specific growth condition. $\eta^\theta_j$ is analytic in the half plane $\text{Re} \ s > -c$. By rotation of the path of integration, it can be analytically continued to the sector $|\arg(s + c)| < \pi$. It has at most exponential growth of order 1, uniformly on closed subsectors. Note that $\phi_-(\theta + \frac{\pi}{2}) > \phi_-(\theta_2)$ and $\phi_+(\theta - \frac{\pi}{2}) < \phi_+(\theta_1)$. Now choose $I_j^j = (a_j, b_j) \subset I_j := (\phi_-(\theta_j), \phi_+(\theta_j))$.
\begin{equation}
|a_2 < \phi_-(\theta + \frac{\pi}{2}) < a_1 < 0 < b_2 < \phi_+(\theta - \frac{\pi}{2}) < b_1. \ \text{Thus,} \ \vartheta(a_2) > \theta + \frac{\pi}{2} \ \text{and} \ \vartheta(b_1) < \theta - \frac{\pi}{2}. \ \text{By (4.3) and (4.4),} \ \rho_\theta(z) \ \text{is bounded above on} \ \tilde{C}_{a_2}(R) \ \text{and} \ \tilde{C}_{b_1}(R). \ \text{If} \ R \ \text{is sufficiently large, there exist} \ C' \ \text{and} \ c' > 0, \ \text{such that}
\end{equation}
$$|\phi_j(z)| \leq C'e^{-c'|z|/\log|z|}, \ z \in \tilde{D}_{I_j}(R).$$
Hence we can deform the path of integration in (4.8) into a path $\gamma_j$, consisting of the arc of the circle $|z| = R$ between $z = R$ and $z = z_{b_1}(R)$ and $\tilde{C}_{b_1}(R)$ if $j = 1$, or the arc of the circle $|z| = R$ between $z = R$ and $z = z_{a_2}(R)$ and $\tilde{C}_{a_2}(R)$ if $j = 2$. Noting that $\text{Im} r_\theta(z) = \text{Im} z + \theta \text{Re}(\frac{1}{\log z}) = \text{Im} z(1 + o(1))$ as $z \to \infty$ on $\gamma_j$, we conclude that $\eta^\theta_1$ is continuous on $-\pi \leq \arg(s + c) < \pi$, and has at most exponential growth of order 1, uniformly on sectors of the form $-\pi \leq \arg(s + c) \leq \pi - \delta$, for any $\delta > 0$, whereas $\eta^\theta_2$ is continuous on $-\pi < \arg(s + c) \leq \pi$, and has at most exponential growth of order 1, uniformly on sectors of the form $-\pi + \delta \leq \arg(s + c) \leq \pi$, for any $\delta > 0$. Moreover, it is easily seen that $\eta^\theta_1$ and $\eta^\theta_2$ are $C^\infty$ on $\arg s = -\pi$ and $\arg s = \pi$, respectively. For all $s$ such that $\arg s = -\pi$ and all $m \in \mathbb{N}_0$ we have
$$\left|\frac{d^m}{ds^m}\eta^\theta_j(s)\right| \leq \tilde{C}e^{c|s|} \int_{\gamma_1} |r_\theta(z)|^m e^{-c'|z|/\log|z|} |dr_\theta(z)|$$
where $\tilde{C}$ and $\tilde{c}$ are positive numbers, depending on $\theta$. Applying the method of Laplace to the integral, we obtain estimates of the form $|\frac{d^m}{ds^m}\eta^\theta_j(s)| \leq Ke^{c|s|}A^m(m \log m)^m$ for all $m \geq 2$, where $K$ and $A$ are positive constants (depending on $\theta$), proving that $\eta^\theta_j$ is quasi-analytic on the half line $\arg s = \pi$. 

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\(-\pi\) (cf. [9]). Similarly it is shown that, for all \(\theta \in (\theta_1 + \frac{\pi}{2}, \theta_2 - \frac{\pi}{2})\), \(\eta_2^\theta\) is quasi-analytic on \(\arg s = \pi\).

Obviously,

\[
\eta_1^\theta(s) - \eta_2^\theta(s) = \int_R^\infty (\phi_1(z) - \phi_2(z))e^{-s\theta_0(z)}d\theta_0(z).
\]

The fact that \(\phi_1 - \phi_2 \in \hat{A}^{\leq -1} (I_1 \cap I_2)\) implies the existence of positive constants \(C'', \delta'\) and \(R'\), such that, for all \(z \in (R', \infty)\),

\[
|\phi_1(z) - \phi_2(z)| \leq C''e^{-\delta'z\log z}.
\]

Consequently, if \(R > R'\), the integral can be estimated as follows

\[
\left|\int_R^\infty (\phi_1(z) - \phi_2(z)) e^{-s\theta_0(z)}d\theta_0(z)\right| \leq C'' \int_R^\infty e^{-\delta'z\log z + c''|s|z}dz, \ s \in \mathbb{C}
\]

where \(c''\) is a positive constant, depending on \(\theta\). The integrand on the right-hand side is maximal when \(\log z + 1 = c''|s|/\delta'\) and its maximum value is \(e^{\delta'c''|s|/\delta'-1}\). Hence we deduce that \(\eta_1^\theta - \eta_2^\theta\) is an entire function, satisfying a growth condition of the form

\[
|\eta_1^\theta(s) - \eta_2^\theta(s)| \leq C'''e^{B|s|}
\]

where \(C'''\) and \(B\) are positive constants, depending on \(\theta\). It now follows that, for \(j = 1, 2\) and \(\theta \in (\theta_1 + \frac{\pi}{2}, \theta_2 - \frac{\pi}{2})\), \(\eta_j^\theta\) is analytic in \(|\arg(s + c)| < \pi\), continuous on \(|\arg(s + c)| \leq \pi\) and quasi-analytic on \(\arg s = \pm \pi\), hence an entire function, satisfying a growth condition of the form

\[
\eta_j^\theta(s) \leq C'''e^{B|s|}
\]

where \(C'''\) is a positive constant, depending on \(\theta\).

Note that \(\phi_- (\theta_2) < \phi_- (\theta_1 + \pi) < \phi_- (\theta_1)\) and \(\phi_+ (\theta_2) < \phi_+ (\theta_2 - \pi) < \phi_+ (\theta_1)\). Now, let \(I_j = (a_j, b_j) \subset I_j\) such that \(a_2 < \phi_- (\theta_1 + \pi) < a_1 < 0 < b_2 < \phi_+ (\theta_2 - \pi) < b_1\). Choose \(\theta \in (\theta(a_2) - \frac{\pi}{2}, \theta_2 - \frac{\pi}{2})\). Then \(\theta \in (\theta_1 + \frac{\pi}{2}, \theta_2 - \frac{\pi}{2})\) and \((a_2, 0) < (\phi_-(\theta + \frac{\pi}{2}), \phi_+ (\theta - \frac{\pi}{2}))\). By (4.5), there exists \(\delta > 0\) such that

\[
\rho_0(z) \geq \delta \frac{|z|}{\log |z|} \text{ for all } z \in \widehat{D}_{(a_2, 0)}(R)
\]

\(\eta_2^\theta\) has at most exponential growth of order 1 as \(s \to \infty\), uniformly on \(S^+_{\alpha} := S[-\pi + \alpha, \pi]\), for any \(\alpha > 0\). Let \(\alpha\) and \(\beta \in (0, \frac{\pi}{2})\). Then, in view of (4.10),

\[-\frac{\pi}{2} < \arg r_\theta(z) < \frac{\pi}{2} - \alpha \text{ for all } z \in \widehat{D}_{I_2^\prime} (R') \cap S[-\pi, -\beta](\subset \widehat{D}_{(a_2, 0)}(R'))\]

provided \(R'\) is sufficiently large. Thus, according to the inversion formula.
we have, for all \( z \in \hat{D}_{I_2}'(R') \cap S[-\pi, -\beta]\),

\[
\phi(z) = \frac{1}{2\pi i} \int_{\partial S_+^\alpha} n^\alpha_2(s) e^{\sigma r_\theta(z)} ds.
\]

Replacing the path of integration by the path \( \Gamma \) (in \( C \)), consisting of the half line \( l_1 \) from \( \infty e^{-i(\pi-\alpha)} \) to \( \sigma/ \cos \alpha e^{-i(\pi-\alpha)} \), where \( \sigma := \frac{\cos \alpha}{B} \log \frac{\rho_\theta(z)}{B} \), the segment \( l_2 \) from \( \sigma/ \cos \alpha e^{-i(\pi-\alpha)} \) to \( -\sigma \) and the half line \( l_3 \) from \( -\sigma \) to \( -\infty \), we get, with the aid of (4.9),

\[
|\phi(z)| \leq C_2 \left[ \int_{l_1} e^{c_2|s|} e^{-\sigma \rho_\theta(z) - \text{Im } s \text{ Im } r_\theta(z)} |ds| + \int_{l_2} e^{c_2|s|} e^{-\sigma \rho_\theta(z) - \text{Im } s \text{ Im } r_\theta(z)} |ds| + \int_{-\infty}^{-\sigma} e^{(\rho_\theta(z) - c_2)s} ds \right]
\]

where \( C_2 \) and \( c_2 \) are positive constants. Using the (in)equalities

\[
\int_{l_1} e^{c_2|s|} e^{-\sigma \rho_\theta(z) - \text{Im } s \text{ Im } r_\theta(z)} |ds| \leq e^{-\sigma \rho_\theta(z)} \int_0^\infty e^{(c_2 + \text{Im } r_\theta(z) \sin \alpha)|s|} |ds| = \frac{e^{-\cos \alpha \log \frac{\rho_\theta(z)}{B_2}} \rho_\theta(z)}{-(\text{Im } r_\theta(z) \sin \alpha + c_2)} \text{ if } \text{Im } r_\theta(z) < 0
\]

\[
< -\frac{c_2}{\sin \alpha}, \quad \int_{l_2} e^{c_2|s|} e^{-\sigma \rho_\theta(z) - \text{Im } s \text{ Im } r_\theta(z)} |ds| \leq \sigma \tan \alpha e^{B_2 \sigma/\cos \alpha - \sigma \rho_\theta(z)} = \frac{\sin \alpha}{B_2} \log \frac{\rho_\theta(z)}{B_2} e^{\frac{c_2}{B_2}(1 - \cos \alpha \log \frac{\rho_\theta(z)}{B_2})}
\]

and

\[
\int_{-\infty}^{-\sigma} e^{(\rho_\theta(z) - c_2)s} ds = \frac{e^{-(\rho_\theta(z) - c_2) \cos \alpha \log \frac{\rho_\theta(z)}{B_2}}}{\rho_\theta(z) - c_2} \text{ if } \rho_\theta(z) > c_2
\]

and noting that both \( -\text{Im } r_\theta(z) < -\frac{c_2}{\sin \alpha} \) and \( \rho_\theta(z) > c_2 \) for all \( z \in \hat{D}_{I_2}'(R') \cap S[-\pi, -\beta] \) if \( R' \) is sufficiently large, we obtain an estimate of the form

\[
|\phi(z)| \leq C' e^{-c_2 \rho_\theta(z) \log \rho_\theta(z)}, \quad z \in \hat{D}_{I_2}'(R') \cap S[-\pi, -\beta],
\]

where \( C' \) and \( c_2 \) > 0, and, in view of (4.10), this implies \( |\phi(z)| \leq C' e^{-c' |z|} \) for \( z \in \hat{D}_{I_2}'(R') \cap S[-\pi, -\beta] \), where \( c' > 0 \), provided \( R' \) is sufficiently large. A similar estimate can be derived for \( \phi_1(z) \) for all \( z \in \hat{D}_{I_2}'(R') \cap S[\beta, \pi] \). Combining this with the fact that \( \phi_1 \in A^{\leq -1}(-\frac{\pi}{2}, \frac{\pi}{2}) \) and \( \phi_1 - \phi_2 \in \tilde{A}^{\leq -1+}(\mathcal{I}_1 \cap \mathcal{I}_2) \), we conclude that \( |\phi_1(z)| \leq C' e^{-c' |z|} \) for all \( z \in \hat{D}_{I_2}'(R') \cap S[-\pi, -\beta] \).
\[ \hat{D}_{I_1}(R'), \text{ where } C'_1 \text{ and } c'_1 > 0, \text{ provided } R' \text{ is sufficiently large. Hence } \]
\[ \phi_j \in \hat{A}^{\leq -1}(I_j) \text{ for } j = 1, 2. \]

4.4. The proof of Theorem 4.12

In the remaining part of this section we show that the condition \( \hat{I}_{q+1} \subset [-\frac{\pi}{2}, \frac{\pi}{2}] \) in Theorem 3.8 can be lifted and, in case II, we prove the existence, for appropriate intervals \( I_q \) and \( I_{q+1} \), of solutions \( f_q \in (\hat{A} / \hat{A}^{\leq -1}(I_q))^n \) and \( f_{q+1} \in (\hat{A}(I_{q+1}))^n \) of (1.1) satisfying the conditions of definition 4.11. With \( f_{q-1} \) we can associate an element \( f_{q-1} \) of \((\hat{A} / \hat{A}^{\leq -1}(\mathbb{R}))^n\), similarly to the scalar case (cf. Definition 4.8). By suitably modifying a representative of \( f_{q-1} \) we obtain a representative of \( f_q \). Without loss of generality we may assume that the equation is in the prepared form (2.7) and \( f_{q-1} \) in \( z^{-N/p}(\hat{A} / \hat{A}^{\leq -1}(\mathbb{R}))^n \) for some sufficiently large \( N \in \mathbb{N} \).

**Lemma 4.15.** — \( \hat{f}_{q-1} \) is a solution of (2.7) in \((\hat{A} / \hat{A}^{\leq -1}(\mathbb{R}))^n\). More precisely, it has a representative \( \{ \tilde{\varphi}_{q,\theta} : \theta \in \mathbb{R} \} \) with the following properties.

(i) For all \( \theta \in \mathbb{R} \), \( \tilde{\varphi}_{q,\theta} \in (\hat{A}(\varphi_-(\theta + \frac{\pi}{2}), \varphi_+(\theta - \frac{\pi}{2})))^n \).

(ii) \( \tilde{\varphi}_{q,\theta_1} - \tilde{\varphi}_{q,\theta_2} \in (\hat{A}^{\leq -1}(\varphi_-(\theta_1 + \frac{\pi}{2}), \varphi_+(\theta_2 - \frac{\pi}{2})))^n \) if \( \theta_1 < \theta_2 \).

(iii) For all \( \theta \in \mathbb{R} \),
\[ \Delta \tilde{\varphi}_{q,\theta}(z) - E(z^{1/p}, \tilde{\varphi}_{q,\theta}(z)) - \varphi_0(z^{1/p}) \in (\hat{A}^{\leq -1}(\varphi_-(\theta + \frac{\pi}{2}), \varphi_+(\theta - \frac{\pi}{2})))^n. \]

If 0 is not a singular direction of level 1, then \( f_q \) defines a section \( \hat{f}_q^{+} \in (\hat{A} / \hat{A}^{\leq -1}(\mathbb{R}))^n \), represented by a 1+-precise quasi-function \( \{ \tilde{\varphi}_{q,\theta}^{+} : \theta \in \mathbb{R} \} \) with the following properties.

(iv) For all \( \theta \in \mathbb{R} \), \( \tilde{\varphi}_{q,\theta}^{+} \in (\hat{A}(\varphi_-(\theta), \varphi_+(\theta)))^n \).

(v) \( \tilde{\varphi}_{q,\theta_1}^{+} - \tilde{\varphi}_{q,\theta_2}^{+} \in (\hat{A}^{\leq -1}(\varphi_-(\theta_1), \varphi_+(\theta_2)))^n \) if \( \theta_1 < \theta_2 \).

(vi) For all \( \theta \in \mathbb{R} \),
\[ \Delta \tilde{\varphi}_{q,\theta}^{+}(z) - E(z^{1/p}, \tilde{\varphi}_{q,\theta}^{+}(z)) - \varphi_0(z^{1/p}) \in (\hat{A}^{\leq -1}(\varphi_-(\theta), \varphi_+(\theta)))^n. \]

**Proof.** — For all \( \theta \in \mathbb{R} \), \( \tilde{\varphi}_{q,\theta} \) is defined similarly to \( \tilde{\varphi}_{\theta} \) in Lemma 4.5 and the first two statements of Lemma 4.15 follow immediately from that lemma.

Now, let \( \{ \phi_{\nu} : \nu \in \{1, \ldots, N'\} \} \), where \( \phi_{\nu} \in A(I_{\nu}) \) and \( \{ I_{\nu} : \nu \in \{1, \ldots, N'\} \} \) is a good, open covering of \([-\frac{\pi}{2}, \frac{\pi}{2}] \), be a representative of \( f_{q-1} \) \([-\frac{\pi}{2}, \frac{\pi}{2}] \). As \( f_{q-1} \) is a solution of (2.7) in \( z^{-N/p}(A / A^{\leq -1})(I_{q-1})^n \), we have, for all \( \nu \in \{1, \ldots, N'\} \),
\[ \Delta \phi_{\nu}(z) - E(z^{1/p}, \phi_{\nu}(z)) - \varphi_0(z^{1/p}) \in (A^{\leq -1}(I_{\nu}))^n. \]
Let $I' < (\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$ and $I'_\nu < I'_\nu$. From (2.11) and (4.1) we deduce the existence of positive numbers $C$ and $c$ such that

$$|\Delta(\phi_\nu(z) - \tilde{\phi}_{q,\theta}(z)) - (E(z^{1/p}, \phi_\nu(z)) - E(z^{1/p}, \tilde{\phi}_{q,\theta}(z)))| \leq Ce^{-c|z|}$$

for all $z \in \tilde{D}_I'(R) \cap S(I'_\nu)$, if $R$ is sufficiently large. Using (4.11) and varying $\nu$, we conclude that there exist $R'$, $C'$ and $c' > 0$ such that

$$|\Delta\tilde{\phi}_{q,\theta}(z) - E(z^{1/p}, \tilde{\phi}_{q,\theta}(z)) - \varphi_0(z^{1/p})| \leq C'e^{-c'|z|}$$

for all $z \in \tilde{D}_I'(R')$ and this implies $\Delta\tilde{\phi}_{q,\theta}(z) - E(z^{1/p}, \tilde{\phi}_{q,\theta}(z)) - \varphi_0(z^{1/p}) \in (\tilde{\mathcal{A}}^{\leq-1}(\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2})))^n$.

The statements concerning $\tilde{f}_q^+$ are proved analogously. □

First, consider the case that 0 is not a singular direction of level 1. Let $\{\tilde{\varphi}_{q,\theta}^+ : \theta \in \mathbb{R}\}$ be a representative of $\tilde{f}_q^+$ with the properties mentioned in Lemma 4.15. Analogously to the proof of proposition 3.11, with $f_0^q$ replaced by $\tilde{\varphi}_{q,\theta}^+$, $\theta \in \tilde{I}_{q+1}^-$ (so that $(\phi_-(\theta), \phi_+(\theta)) \subset I_{q+1}$), it can be proved that this proposition continues to hold when $I_{q+1}^\sim \not\subset [-\pi/2, \pi/2]$. Hence the first statement of the theorem follows.

Now, suppose that 0 is a singular direction of level 1 (case II). Let $\theta_1 < \theta_2$ and $I = (\phi_-(\theta_2 + \frac{\pi}{2}), \phi_+(\theta_1 - \frac{\pi}{2}))$. According to Lemma 4.15, $\{\tilde{\varphi}_{q,\theta_1}, \tilde{\varphi}_{q,\theta_2}\}$ represents a solution of (2.7) in $(\tilde{\mathcal{A}}/\tilde{\mathcal{A}}^{\leq-1}(I)^n$ (viz. $\tilde{f}_{q-1}^\sim|I|$). We shall show that $\tilde{\varphi}_{q,\theta_1}$ and $\tilde{\varphi}_{q,\theta_2}$ can be modified by exponentially small functions in such a manner that the resulting quasi-function is 1+-precise and represents a solution $f_q$ of (2.7) in $(\tilde{\mathcal{A}}/\tilde{\mathcal{A}}^{\leq-1^+}(I))^n$. For that purpose we need the following theorem.

**THEOREM 4.16.** — Let $I = (a, b)$ be a large, open interval such that $|\tilde{I}| \leq \pi$. Let $\Delta^c$ be a canonical difference operator (cf. (2.1) and (2.2)) and $\varphi : S(-\pi, \pi) \times \mathbb{C}^n \to \mathbb{C}^n$ a function of the form

$$\varphi(z, y) = \varphi_0(z) + A(z)y + \psi(z, y)$$

where $\varphi_0 \in \tilde{\mathcal{A}}^{\leq-1}(I)^n$, $A \in z^{-\nu_0-1/p} \text{End}(n; \tilde{\mathcal{A}}(I))$ with $\nu_0 = \max\{-d_j : j \in \{1, \ldots, n\}\}$ and $\psi$ has the following properties: for any open interval $I' \prec I$ there exists a positive number $R$, such that

(i) $\psi$ is holomorphic on $\tilde{D}_{I'}(R) \times U$, where $U \subset \mathbb{C}^n$ is a neighbourhood of 0,

(ii) $\psi$ admits an asymptotic expansion of the form $\sum_{m=m_0}^\infty \psi_m(z) z^{-m/p}$ as $z \to \infty$, uniformly on $\tilde{D}_{I'}(R) \times U$, where $m_0 \in \mathbb{Z}$ and the $\psi_m$ are holomorphic $\mathbb{C}^n$-valued functions,

(iii) $\psi_0(z, 0) = 0$ for all $z \in \tilde{D}_{I'}(R)$.
Furthermore, we assume: if \(-\frac{\pi}{2}\) is a Stokes direction of \(\Delta^c\) of level 1, then either \(a < \phi_-(\frac{\pi}{2})\) or \(b > \phi_+(-\frac{\pi}{2})\).

Let \(I_1 = (a, b_1)\) and \(I_2 = (a_2, b)\) be large, open subintervals of \(I\) such that \(\vartheta(a_2) \leq \vartheta(b_1)\) (i.e. \(I_1 \cap I_2\) is not a large interval), and \(\tilde{I}_i \cap \Theta(\Delta^c) = \emptyset\) for \(i = 1, 2\). Then the equation

\[
\Delta^c y(z) = \varphi(z, y(z))
\]

has a unique solution \(f \in \mathcal{A}^{\leq -1}/\mathcal{A}^{\leq -1 + 1}(I)^n\) with representative \(\{f_1, f_2\}\), where \(f_i\) is a solution of (4.12) in \(\mathcal{A}^{\leq -1}(I_1)^n\) and \(f|_{I_i} = f_i \mod (\mathcal{A}^{\leq -1 + 1})^n\). Moreover, if \(\tilde{I} \cap \Theta(\Delta^c) = \emptyset\), then (4.12) has a unique solution \(f \in \mathcal{A}^{\leq -1}(I)^n\).

In the case that \(\Delta^c\) has no levels \(\kappa_j \in (0, 1)\) and \(A(z) \sim 0\) as \(z \to \infty\), uniformly on \(\hat{D}_{1'}(R)\) for any open interval \(I' < I\) and some sufficiently large \(R\), this theorem follows easily from Theorems 4 and 2 in [14]. Using the same type of argument, it can be shown that these results continue to hold when \(\kappa_j \in (0, 1)\) for certain \(j \in \{1, \ldots, n\}\) and \(A \in z^{-\nu_0 - 1/p} \text{End}(n; \mathcal{A}(I))\), where \(\nu_0 = \max\{-d_j : j \in \{1, \ldots, n\}\}\) (cf. also [14, Remark 7]).

From Theorem 4.16 we derive the following result.

**Lemma 4.17.** — Let \(\Delta^c\) and \(\Delta\) be the difference operators in (2.7) and (2.9). Let \(I = (a, b)\) be a large, open interval such that either \(a < \phi_-(\frac{\pi}{2})\) or \(b > \phi_+(-\frac{\pi}{2})\) and \(|\tilde{I}| \leq \pi\). Let \(I_1 = (a, b_1)\) and \(I_2 = (a_2, b)\) be large subintervals of \(I\) such that \(\vartheta(a_2) \leq \vartheta(b_1)\) and \(\tilde{I}_i \cap \Theta(\Delta^c) = \emptyset\) for \(i = 1, 2\). Then (2.7) has a unique solution \(f\) in \((\mathcal{A}/\mathcal{A}^{\leq -1 + 1}(I))^n\) such that \(f\) mod \((\mathcal{A}^{\leq -1})^n = f_{q-1}1\) and \(f|_{I_i} = \phi_i \mod (\mathcal{A}^{\leq -1 + 1})^n\), where \(\phi_i\) is a solution of (2.7) in \((\mathcal{A}(I_i))^n\) for \(i = 1, 2\).

Moreover, \(\phi_1\) is unique if \(a < \phi_-(\frac{\pi}{2})\), \(\phi_2\) is unique if \(b > \phi_+(-\frac{\pi}{2})\).

**Proof.** — Let \(\tilde{I} = (\alpha, \beta), \theta \in [\beta - \frac{\pi}{2}, \alpha + \frac{\pi}{2}]\) and \(y = w + \nabla q, \theta\). Then \(I \subset (\phi_- (\theta + \frac{\pi}{2}), \phi_+ (\theta - \frac{\pi}{2}))\). \(y\) is a solution of the equation (2.7) if and only if \(w\) satisfies the equation

\[
\Delta w(z) = G(z, w(z)) := E(z^{1/p}, w(z) + \nabla q, \theta(z)) + \nabla_0(z^{1/p}) - \Delta \nabla q, \theta(z).
\]

According to (2.9), there exists \(L \in \text{Gl}(n; \mathbb{C}[z^{-1/p}][z^{1/p}])\) such that \(L(z)\Delta = \Delta^c + A(z)\), where \(A \in z^{-\nu_0 - 1/p} \text{End}(n; \mathbb{C}[z^{-1/p}])\) provided \(M\) is chosen sufficiently large (cf. §2.5). Thus, \(w\) is a solution of the equation (4.13) if and only if

\[
\Delta^c w(z) = \varphi(z, w(z)) := L(z)G(z, w(z)) - A(z)w(z).
\]

By Lemma 4.15, \(\varphi(z, 0) = L(z)(E(z^{1/p}, \nabla q, \theta(z)) + \nabla_0(z^{1/p}) - \Delta \nabla q, \theta(z)) \in (\mathcal{A}^{\leq -1}(\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2})))^n\). From the fact that \(\nabla q, \theta \in z^{-N/p}(\mathcal{A}(\phi_-(\theta + \frac{\pi}{2})  

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Let $2.7 \leq \phi(\theta - \frac{\pi}{2}))$, it follows that $\varphi$ admits an asymptotic expansion in $z^{-1/p}$ as $z \to \infty$, uniformly on $\tilde{D}_I(R) \times V$, for any interval $I' \subset (\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$, sufficiently large $R$ and some neighbourhood $V \subset \mathbb{C}^n$ of 0 (cf. [8, Lemma 14.3] and [10, Lemma 4.6]). Furthermore,

$$\varphi'_2(z, 0) = L(z)D_2E(z^{1/p}, \varphi_{q, \theta}(z)) - A(z).$$

As $D_2E(z^{1/p}, 0) = 0$, there exist positive constants $K$ and $K'$ such that $|D_2E(z^{1/p}, \varphi_{q, \theta}(z))| \leq K|\varphi_{q, \theta}(z)| \leq K'|z|^{-N/p}$ for all $z \in \tilde{D}_I(R)$, where $I' \subset (\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$ and $R$ sufficiently large ($K'$ and $R$ depend on $I'$). Consequently, $\varphi'_2(z, 0) \in z^{-v_0-1/p}$ $End(n; \hat{\mathcal{A}}(\phi_-(\theta + \frac{\pi}{2}), \phi_+(\theta - \frac{\pi}{2}))$ if $N$ is sufficiently large. According to Theorem 4.16, (4.14) has a unique solution $w \in \hat{\mathcal{A}}^{\leq-1}/\hat{\mathcal{A}}^{\leq-1}(I)^n$ with representative $\{w_1, w_2\}$, where $w_i$ is a solution of (4.14) in $\hat{\mathcal{A}}^{\leq-1}(I_i)^n$ with the property that $w_i \mod (\hat{\mathcal{A}}^{\leq-1})^n = w|_{I_i}$, for $i = 1, 2$. Now let $\phi_i = w_i + \hat{\varphi}_{q, \theta}$, $i = 1, 2$. Then $\phi_i \in \hat{\mathcal{A}}(I_i)^n$, $\phi_i$ is a solution of (2.7) for $i = 1, 2$ and $\phi_1 - \phi_2 = w_1 - w_2 \in \hat{\mathcal{A}}^{\leq-1}(I_1 \cap I_2)^n$. Moreover, $\phi_i - \hat{\varphi}_{q, \theta} = w_i \in (\hat{\mathcal{A}}^{\leq-1}(I_i))^n$ for $i = 1, 2$. The uniqueness of $\phi_1$ or $\phi_2$ follows immediately from the last statement of Theorem 4.16. $\{\phi_1, \phi_2\}$ represents an element $f \in (\hat{\mathcal{A}}/\hat{\mathcal{A}}^{\leq-1}(I))^n$ with the required properties.

From Lemma 4.17 we deduce the following proposition, which completes the proof of Theorem 4.12.

**Proposition 4.18.** — Let $I$ be a large, open interval such that $0 \notin \tilde{I}^*$ and $|\tilde{I}| > \pi$. Then (2.7) has a unique solution $f \in (\hat{\mathcal{A}}/\hat{\mathcal{A}}^{\leq-1}(I))^n$ with the property that $f|_{\tilde{I}} = f \mod (\hat{\mathcal{A}}^{\leq-1})^n$.

Moreover, if $J$ is a large, open subinterval of $I$ such that $\tilde{J} \cap \Theta(\Delta^c) = \emptyset$, there exists a solution $g \in (\hat{\mathcal{A}}(J))^n$ of (2.7) with the property that $f|_{J} = g \mod (\hat{\mathcal{A}}^{\leq-1})^n$.

**Proof.** — We shall prove the proposition for the case that $I$ (and hence $J$) is a bounded interval and leave the reader to extend the proof to the general case (i.e. $I \subset (-\infty, \phi_+(\frac{-\pi}{2}))$ or $(\phi_-(\frac{\pi}{2}), \infty)$). Let $-\tilde{I}^* = (\theta_1, \theta_2)$. Then $\tilde{I} = (\theta_1 - \frac{\pi}{2}, \theta_2 + \frac{\pi}{2})$ and $I = (\phi_-(\theta_2 + \frac{\pi}{2}), \phi_+(\theta_1 - \frac{\pi}{2})) := (a, b)$. Either $\theta_1 \geq 0$ or $\theta_2 \leq 0$. Suppose that $\theta_1 \geq 0$. The proof of the other case is similar. Let $I_1 = (a, b_1)$ and $I_2 = (a_2, b)$ be large subintervals of $I$ such that $\tilde{\vartheta}(a_2) \leq \tilde{\vartheta}(b)$, $|I_i| \leq \pi$ and $\tilde{I}_i \cap \Theta(\Delta^c) = \emptyset$ for $i = 1, 2$. Let $I' = (a', b') := (\phi_-(\theta_1 + \frac{\pi}{2}), \phi_+(\theta_1 - \frac{\pi}{2}))$ and let $I_1' = (a', b_1)'$ be a large subinterval of $I'$ such that $\tilde{I}'_1 \cap \Theta(\Delta^c) = \emptyset$. Note that $|\tilde{I}'_1| = \pi$ and thus $|\tilde{I}'_1| \leq \pi$ and $I_2 \subset I'$. Without loss of generality we may assume that $b_1 \leq b'$. Then $\tilde{I}_1 \cap I' = \tilde{I}_1 \cap I_1' = (a', b_1)$. $\theta_1 \geq 0$ implies that $a = \phi_-(\theta_2 + \frac{\pi}{2}) < a' = \phi_-(\theta_1 + \frac{\pi}{2}) \leq \phi_-(\theta_1 - \frac{\pi}{2})$. According to
Lemma 4.17, (2.7) has unique solutions \( \phi_1 \in (\hat{\mathcal{A}}(I_1))^{n} \), \( \phi_1' \in (\hat{\mathcal{A}}(I'_1))^{n} \) and \( f' \in \hat{\mathcal{A}}/\hat{\mathcal{A}}^{\leq -1'}(I')^{n} \), and a solution \( \phi_2 \in (\hat{\mathcal{A}}(I_2))^{n} \), such that \( \phi_1 \mod (\hat{\mathcal{A}}^{\leq -1})^{n} = f_{q-1}|_{I_1}, \) \( f' \mod (\hat{\mathcal{A}}^{\leq -1})^{n} = f_{q-1}|_{I'_1}, f'|_{I'_1} = \phi_1' \mod (\hat{\mathcal{A}}^{\leq -1'})^{n} \) and \( f'|_{I_2} = \phi_2 \mod (\hat{\mathcal{A}}^{\leq -1'})^{n} \). Obviously, \( \phi_1' \mod (\hat{\mathcal{A}}^{\leq -1})^{n} = f_{q-1}|_{I'_1} \), so \( \phi_1 - \phi_1' \in \hat{\mathcal{A}}^{\leq -1}(I_1 \cap I'_1)^{n} \). If \( I_1 \cap I'_1 \) is a large interval, then, by Lemma 4.17, \( \phi_1|_{I_1 \cap I'_1} \) is the unique solution of (2.7) in \( (\hat{\mathcal{A}}(I_1 \cap I'_1))^{n} \) with the property that \( \phi_1|_{I_1 \cap I'_1} \mod (\hat{\mathcal{A}}^{\leq -1})^{n} = f_{q-1}|_{I_1 \cap I'_1} \), so \( \phi_1' = \phi_1 \). Now assume \( I_1 \cap I'_1 \) is not a large interval (i.e. \( \hat{\mathcal{A}}(I_1 \cap I'_1)^{n} \)). As \( \phi_1' \) satisfies a homogeneous linear difference equation of the form (2.10) (with \( y_1 = \phi_1 \) and \( y_2 = \phi_1' \)). As \( \alpha' < \phi_1' \), \( I_1 \cap I'_1 \not\subset (\phi_1', \phi_1(-\frac{\pi}{2})) \) and thus, according to Theorem 2.13(iii), \( \ker(\hat{\Delta}, \hat{\mathcal{A}}^{\leq -1}(I_1 \cap I'_1)^{n}) = \ker(\hat{\Delta}, \hat{\mathcal{A}}^{\leq -1}(I_1 \cap I'_1)^{n}) = \ker(\hat{\Delta}, \hat{\mathcal{A}}^{\leq -1}(I_1 \cap I'_1)^{n}) \). Let \( f_1 := \phi_1 \mod (\hat{\mathcal{A}}^{\leq -1'})^{n} \). Then \( f_1|_{I_1 \cap I'_1} = f'|_{I_1 \cap I'_1} \), and, consequently, there exists a unique \( f \in \hat{\mathcal{A}}/\hat{\mathcal{A}}^{\leq -1'}(I_1 \cup I'_1)^{n} = \hat{\mathcal{A}}/\hat{\mathcal{A}}^{\leq -1'}(I')^{n} \) such that \( f|_{I_1} = f_1 \) and \( f|_{I'_1} = f' \). Obviously, \( f \mod (\hat{\mathcal{A}}^{\leq -1})^{n} = f_{q-1}|_{I'_1} \). Its uniqueness follows from Lemma 4.14. Moreover, \( f|_{I_2} = f'|_{I_2} = \phi_2 \mod (\hat{\mathcal{A}}^{\leq -1'})^{n} \).

Now suppose \( J = (c, d) \) is a large subinterval of \( I \) such that \( J \cap \Theta(\Delta c) = \emptyset \). To begin with, assume that \( |J| \leq \pi \). Let \( I_1 \) and \( I_2 \) as above. If \( J \subset I_1 \) or \( J \subset I_2 \), the last statement of the theorem immediately follows from the above argument. So suppose that \( a < c < a_2 \) and \( b_1 < d < b \). Without loss of generality we may assume that \( \vartheta(c) \leq \vartheta(b_1) \) and \( \vartheta(a_2) \leq \vartheta(d) \) (this can always be achieved by decreasing \( b_1 \) and increasing \( a_2 \), if necessary), taking care that \( \vartheta(b_1) \) remains less than \( \vartheta(a) \) and \( \vartheta(a_2) \geq \vartheta(b) \). Let \( I'' := (a, d), I'_1 := I_1 \) and \( I'_2 := J \). Then the above argument proves the existence of a unique solution \( f'' \in (\hat{\mathcal{A}}/\hat{\mathcal{A}}^{\leq -1'}(I''))^{n} \), of (2.7), with the following properties:

(i) \( f'' \mod (\hat{\mathcal{A}}^{\leq -1})^{n} = f_{q-1}|_{I''} \).

(ii) (2.7) has a solution \( \phi'_i \in (\hat{\mathcal{A}}(I_i''))^{n} \) such that \( f''|_{I_i''} = \phi'_i \mod (\hat{\mathcal{A}}^{\leq -1'})^{n} \) for \( i = 1, 2 \).

The uniqueness of \( \phi_1 \) implies that \( \phi_1' = \phi_1 \). Furthermore, \( \phi_2' - \phi_2 \in (\hat{\mathcal{A}}^{\leq -1}(I'_2 \cap I_2))^{n} \). Thus, \( \phi_2' - \phi_2 \) satisfies a homogeneous linear difference equation of the form (2.10), with \( y_1 \) and \( y_2 \) replaced by \( \phi_2' \) and \( \phi_2 \) and \( H \in \text{End}(n; \hat{\mathcal{A}}^{\leq -1}(I'_2 \cap I_2)) \). Note that \( \vartheta_-(I'_2 \cap I_2) = \vartheta(a_2) \leq \vartheta_+(I'_2 \cap I_2) = \vartheta(d) \). According to Theorem 2.13, \( \phi_2' \) can be written in the form

\[
\phi_2''(z) - \phi_2(z) = \sum_{j=1}^{n} z^{d_j} z^{e_{i_j}(z)} z^{\lambda_j} g_j(z)p_j(z)
\]
where \( g_j \in \hat{A}(I'') \cap I_2)^n [\log z] \) and the \( p_j \) are trigonometric polynomials, \( p_j \equiv 0 \) unless \( d_j < 0 \), or \( d_j = 0 \), \( k_j = 1 \) and \( \arg \mu_j = \pi \). As \( \phi_1 - \phi_2 \in (\hat{A}^{\leq -1}(I_1 \cap I_2))^n \), and \( \phi''_2 - \phi_2 \in (\hat{A}^{\leq -1}(I_1 \cap I_2))^n \subset (\hat{A}^{\leq -1+}(I_1 \cap I_2))^n \), we have \( \phi''_2 - \phi_2 \in (\hat{A}^{\leq -1}(I_2' \cap I_2))^n \cap (\hat{A}^{\leq -1}(I_1 \cap I_2))^n \). This implies that \( p_j \equiv 0 \) unless \( d_j < 0 \). Consequently, \( \phi''_2 - \phi_2 \in (\hat{A}^{\leq -1+}(I_2' \cap I_2))^n \) and thus \( f|_J = \phi''_2 \mod (\hat{A}^{\leq -1+})^n \).

Finally, suppose that \( |\tilde{J}| > \pi \). For any pair of large subintervals \( J_1 \) and \( J_2 \) of \( J \) such that \( |\tilde{J}_1| \leq \pi \), there exist solutions \( \phi'_1 \in (\hat{A}(J_1))^n \) and \( \phi'_2 \in (\hat{A}(J_2))^n \) of (2.7) such that \( f|_{J_1} = \phi'_i \mod (\hat{A}^{\leq -1+})^n \) for \( i = 1, 2 \). If, in addition, \( J_1 \cap J_2 \) is again a large interval, then, in view of Lemma 2.12 2, \( \phi'_1 - \phi'_2 \in (\hat{A}^{\leq -1+}(J_1 \cap J_2))^n = \{0\} \), so \( \phi'_1 = \phi'_2 \) and the result follows by glueing together all these solutions. \( \square \)

5. Appendix

In this section we compare the domains \( \hat{D}_I(R) \) to the domains \( D_I(R) \) and \( \hat{D}_I(R) \) used in previous papers ([11, 12, 14]).

**Definition 5.1 (“old domains”).** — Suppose that \( \Re \psi_\theta(z) > 1/e + |\theta| \). By \( D_\theta(z) \) we denote the domain

\[
D_\theta(z) := \{ z \in S_+: \Re \psi_\theta(\zeta) \geq \Re \psi_\theta(z) \}.
\]

Let \( I \) be a finite interval of \( \mathbb{R} \), such that \( \tilde{I} = [\theta_1, \theta_2] \). Let \( z \in S_+ \) such that \( \Re \psi_\theta(z) > 1/e + |\theta| \) for all \( \theta \in I \). By \( D_I(z) \) and \( \hat{D}_I(z) \) we denote the domains

\[
D_I(z) = \cap_{\theta \in I} D_\theta(z) = D_{\theta_1}(z) \cap D_{\theta_2}(z)
\]

and

\[
\hat{D}_I(z) = \cup_{\theta \in I} D_\theta(z).
\]

**Remark 5.2.** — Let \( I \) be an open interval, containing 0. If \( \theta_+(I) \leq \theta_+(I) \), \( \hat{A}(\tilde{I}) \), \( \hat{A}^{\leq -1}(I) \) and \( \hat{A}^{\leq -1+}(I) \) coincide with the sets \( \mathcal{A}(\tilde{I}) \), \( \mathcal{A}_{1,0}(\tilde{I}) \) and \( \mathcal{A}_{1+0}(\tilde{I}) \) defined in [14], respectively. If \( I \) is a large interval, \( \hat{A}(I) \) and \( \hat{A}^{\leq -1}(I) \) coincide with the sets \( \hat{A}(\tilde{I}) \) and \( \hat{A}_{1,0}(\tilde{I}) \) defined in [14], respectively. This is an immediate corollary to the following simple lemma.

**Lemma 5.3.** — Let \( I \) be an open interval, containing 0, and \( R > 1 \). If \( \theta_+(I) < \theta_+(I) \) there exists \( R' > R \) such that

\[
D_I(R') \subset \hat{D}_I(R).
\]
The picture on the left shows the “old” domains $D_{[-\pi,-\pi/4]}(6)$ and $\tilde{D}_{[-\pi,-\pi/4]}(6)$ (the large domain, bounded by $C_{-\pi}(6)$ and $C_{+\pi}(6)$). The corresponding domains in the picture on the right are $\hat{D}_{[\phi-(\pi),\phi+(\pi/4)]}(6)$ and $\tilde{D}_{[\phi-(\pi/2),\phi+(\pi)]}(6)$.

For every interval $I' \prec \tilde{I}$ there exists $R' > R$ such that
$$\hat{D}_{I'}(R') \subset D_{I'}(R).$$

If $I$ is a large interval, there exists $R' > R$ such that
$$\tilde{D}_{I}(R') \subset \hat{D}_{I}(R).$$

For every interval $I'$ such that $\tilde{I} \prec I'$, there exists $R' > R$ such that
$$\hat{D}_{I}(R') \subset \tilde{D}_{I'}(R).$$

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Geertrui Klara IMMINK
University of Groningen
Faculty of Economics
P.O. Box 800
9700 AV Groningen (The Netherlands)
g.k.immink@rug.nl