ANNALES

DE

L'INSTITUT

FOURIER

Erwan LANNEAU & Jean-Luc THIFFEAULT

On the minimum dilatation of pseudo-Anosov homeromorphisms on surfaces of small genus

Tome 61, nº 1 (2011), p. 105-144.

<http://aif.cedram.org/item?id=AIF_2011__61_1_105_0>

Ann. Inst. Fourier, Grenoble
61, 1 (2011) 105-144

ON THE MINIMUM DILATATION OF PSEUDO-ANOSOV HOMEROMORPHISMS ON SURFACES OF SMALL GENUS

by Erwan LANNEAU & Jean-Luc THIFFEAULT

Abstract. — We find the minimum dilatation of pseudo-Anosov homeomorphisms that stabilize an orientable foliation on surfaces of genus three, four, or five, and provide a lower bound for genus six to eight. Our technique also simplifies Cho and Ham's proof of the least dilatation of pseudo-Anosov homeomorphisms on a genus two surface. For genus $g = 2$ to 5, the minimum dilatation is the smallest Salem number for polynomials of degree $2g$.

Résumé. — Nous calculons la plus petite dilatation d'un homéomorphisme de type pseudo-Anosov laissant invariant un feuilletage mesuré orientable sur une surface de genre g pour $g = 3, 4, 5$. Nous donnons aussi une borne inférieure pour les genres $6, 7$ et 8. Nos techniques simplifient la preuve de Cho et Ham sur le calcul de la plus petite dilatation d'un homéomorphisme de type pseudo-Anosov sur une surface de genre 2. Pour $g = 2$ à 5, la plus petite dilatation est le plus petit nombre de Salem pour les polynomes à degré fixé $2g$.

1. Introduction

This paper concerns homeomorphisms of a compact oriented surface M to itself. There are natural equivalence classes of such homeomorphisms under isotopy, called isotopy classes or mapping classes. An irreducible mapping class is such that no power of its members preserves a nontrivial subsurface. By the Thurston–Nielsen classification [28], irreducible

Keywords: Pseudo-Anosov homeomorphism, small dilatation, flat surface.
Math. classification: 37D40, 37E30.

mapping classes are either finite-order or are of a type called pseudo-Anosov. The class of pseudo-Anosov homeomorphisms is by far the richest. One can think of such a homeomorphism ϕ as an Anosov (or hyperbolic) homeomorphism on $M\setminus\{\text{singularities}\}$. In particular, as for standard Anosov on the two dimensional torus, there exists a local Euclidean structure (with singularities) and two linear foliations ($\mathcal{F}^s$ and $\mathcal{F}^u$, called stable and unstable) such that ϕ expands the leaves of one foliation with a coefficient λ, and shrinks those of the other foliation with the same coefficient. The number λ is a topological invariant called the *dilatation* of ϕ; the number $\log(\lambda)$ is the *topological entropy* of ϕ.

Thurston proved that $\lambda + \lambda^{-1}$ is an algebraic integer (in fact, it is a Perron number) over $\mathbb{Q}$ of degree bounded by $4g-3$. In particular Newton's formulas imply that for each $g \geqslant 2$ the set of dilatations bounded from above by a constant is finite. Hence the minimum value δ_g of the dilatation of pseudo-Anosov homeomorphisms on M is well defined [2, 13]. It can be shown that the logarithm of δ_g is the length of the shortest geodesics on the moduli space of complex curves of genus g, $\mathcal{M}_g$ (for the Teichmüller metric).

Two natural questions arise. The first is how to compute δ_g explicitly for small $g \geqslant 2$. The second question asks if there is a unique (up to conjugacy) pseudo-Anosov homeomorphism with minimum dilatation in the modular group $\mathrm{Mod}(g)$. It is well known that $\delta_1 = \frac{1}{2}(3+\sqrt{5})$ and this dilatation is uniquely realized by the conjugacy class in $\mathrm{Mod}(1) = \mathrm{PSL}_2(\mathbb{Z})$ of the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. In principle these dilatations can be computed for any given g using train tracks. Of course actually carrying out this procedure, even for small values of g, seems impractical.

We know very little about the value of the constants δ_g. Using a computer and train tracks techniques for the punctured disc, Cho and Ham [7] proved that δ_2 is equal to the largest root of the polynomial $X^4 - X^3 - X^2 - X + 1$, $\delta_2 \simeq 1.72208$ [7]. One of the results of the present paper is an independent and elementary proof of this fact.

One can also ask about the uniqueness (up to conjugacy) of pseudo-Anosov homeomorphisms that realize δ_g. In genus 2, δ_2 is not unique due to the existence of the hyperelliptic involution and covering transformations (see Section 4 and Remark 4.1 for a precise definition). But, up to hyperelliptic involution and covering transformations, we prove the uniqueness of the conjugacy class of pseudo-Anosov homeomorphisms that realize δ_2, in the mapping class group of genus 2 surfaces, Mod(2) (see Theorem 1.1).

For $g > 1$ the estimate $2^{1/(12g-12)} \leqslant \delta_g \leqslant (2+\sqrt{3})^{1/g}$ holds [25, 12]. We will denote by δ_g^+ the minimum value of the dilatation of pseudo-Anosov homeomorphisms on a genus g surface with *orientable* invariant foliations. We shall prove:

Theorem 1.1. — *The minimum dilatation of a pseudo-Anosov homeomorphism on a genus two surface is equal to the largest root of the polynomial $X^4 - X^3 - X^2 - X + 1$,*

$$\delta_2 = \delta_2^+ = \frac{1}{4} + \frac{\sqrt{13}}{4} + \frac{1}{2}\sqrt{\frac{\sqrt{13}}{2} - \frac{1}{2}} \simeq 1.72208.$$

Moreover there exists a unique (up to conjugacy, hyperelliptic involution, and covering transformations) pseudo-Anosov homeomorphism on a genus two surface with dilatation δ_2.

Remark. — This answers Problem 7.3 and Question 7.4 of Farb [8] in genus two.

Theorem 1.2. — *The minimum value of the dilatation of pseudo-Anosov homeomorphisms on a genus g surface, $3 \leqslant g \leqslant 5$, with orientable invariant foliations is equal to the largest root of the polynomials in Table 1.1.*

g	polynomial	$\delta_g^+ \simeq$
3	$X^6 - X^4 - X^3 - X^2 + 1$	1.40127
4	$X^8 - X^5 - X^4 - X^3 + 1$	1.28064
5	$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$	1.17628

Table 1.1

All of the minimum dilatations for $2 \leqslant g \leqslant 5$ are Salem numbers [26]. In fact, their polynomials have the smallest Mahler measure over polynomials of their degree [4]. For $g = 5$, the dilatation is realized by the pseudo-Anosov homeomorphism described by Leininger [19] as a composition of Dehn twists about two multicurves. Its characteristic polynomial is the irreducible one having Lehmer's number as a root: this is the smallest known Salem number. The polynomial has the smallest known Mahler measure over all integral polynomials.

For $g = 3$ and 4, we have constructed explicit examples. We present two independent constructions in this paper: The first is given in term of Dehn twists on a surface; The second involves the *Rauzy–Veech* construction (see Appendix B).

THEOREM 1.3. — *The minimum value of the dilatation of pseudo-Anosov homeomorphisms on a genus g surface, $6 \leqslant g \leqslant 8$, with orientable invariant foliations is not less than the largest root of the polynomials in Table 1.2.*

In particular $\delta_6^+ \geqslant \delta_5^+$.

g	polynomial	$\delta_g^+ \gtrsim$
6	$X^{12} - X^7 - X^6 - X^5 + 1$	1.17628
7	$X^{14} + X^{13} - X^9 - X^8 - X^7 - X^6 - X^5 + X + 1$	1.11548
8	$X^{16} - X^9 - X^8 - X^7 + 1$	1.12876

Table 1.2

Remark 1.4. — Genus 6 is the first instance of a nondecreasing dilatation compared to the previous genus. This partially answers Question 7.2 of Farb [8] in the orientable case.

We have also found an example of a pseudo-Anosov homeomorphism on a genus 3 surface that stabilizes a non-orientable measured foliation, with dilatation δ_3^+. There is also evidence that $\delta_5 < \delta_5^+$ [1] (Section 6.1). In addition, Aaber & Dunfield [1] and Kin & Takasawa [15] have found a pseudo-Anosov homeomorphism realizing δ_7^+, and Hironaka [11] has done the same for δ_8^+. Hence, all the lower bounds in Table 1.2 except for genus 6 are known to be realized by a pseudo-Anosov homeomorphism.

Remark 1.5. — Our techniques also provide a way to investigate least dilatations of punctured discs. This will appear in the forthcoming paper [17]. Note that, for genus 3 to 8, none of the minimum dilatations realizing the bound can come from the lift of a pseudo-Anosov on a punctured disk (or any other lower-genus surface). Indeed, if the pseudo-Anosov comes from a lift, then composing this pseudo-Anosov with the hyperelliptic involution, one gets two pseudo-Anosov homeomorphisms, one with positive root when acting on homology, and one with negative root. Since the polynomials we find have only one sign of the dominant root when acting on homology, a lift is always ruled out. This is in contrast to the Hironaka & Kin [12] examples, which come from punctured disks.

Acknowledgments. The authors thank Christopher Leininger, Frédéric Le Roux, Jérôme Los, Sarah Matz, and Rupert Venzke for helpful conversations, and are grateful to Matthew D. Finn for help in finding pseudo-Anosov homeomorphisms in terms of Dehn twists. J-LT thanks the Centre de Physique Théorique de Marseille, where this work began, for its hospitality. J-LT was also supported by the Division of Mathematical Sciences of the US National Science Foundation, under grant DMS-0806821.

2. Background and tools

In this section we recall some general properties of dilatations and pseudo-Anosov homeomorphisms, namely algebraic and spectral radius properties. We also summarizes basic tools for proving our results (for example see [28, 9, 22, 23]).

To guide the reader, we will first outline the general method used to find the least dilatation δ_g^+:

Summary: to find the least dilatation δ_g^+ on a surface M of genus g.

1. *Start with a known pseudo-Anosov homeomorphism on M, with dilatation α, that stabilizes orientable foliations (we use the family in* [12]*).*
2. *Enumerate all reciprocal polynomials with Perron root less that α (see Section 2.2 for definitions, and Appendix A for an explicit algorithm). For genus $g > 2$, this requires a computer, but is a standard calculation.*
3. *Of these polynomials, eliminate the ones that are incompatible with the Lefschetz theorem (see Section 2.3). The remaining polynomial with the smallest root gives a lower bound on the least dilatation δ_g^+. For genus $g > 4$, this step requires a computer.*
4. *If possible, construct an explicit pseudo-Anosov homeomorphism on M having the lower bound in the previous step as a dilatation. We do this by either exhibiting a sequence of Dehn twists, or by*

the Rauzy–Veech construction (see Appendix B). This confirms that we have found δ_g^+.

2.1. Affine structures and affine homeomorphisms

To each pseudo-Anosov homeomorphism ϕ one can associate an affine structure on M for which ϕ is affine.

2.1.1. Affine structures

A surface of genus $g \geqslant 1$ is called a flat surface if it can be obtained by edge-to-edge gluing of polygons in the plane using translations or translations composed with $-\,\mathrm{id}$. We will call such a surface (M, q) where q is the form dz^2 defined locally. The metric on M has zero curvature except at the zeroes of q where the metric has *conical singularities* of angle $(k+2)\pi$ (with $k \geqslant -1$). The integer k is called the *degree* of the zero of q. A point that is not singular is *regular*. We will use the convention that a singular point of degree 0 is regular. A measured foliation M is a linear flow on this flat surface M for an affine structure.

The Gauss–Bonnet formula applied to the singularities reads $\sum_i k_i = 4g-4$. We will call the integer vector (or simply the stratum) $(k_1, \ldots, k_n)$ with $k_i \geqslant -1$ the *singularity data* of the measured foliation.

If one restricts gluing to translations only then the surface is called a translation surface; otherwise it is called a half-translation surface. For a translation surface the degree of all singularities is even; the converse is false in general.

There is a standard construction, *the orientating cover*, that produce a translation surface from a half-translation surface.

Construction 2.1. — *Let N be a half-translation surface with singularity data $(k_1, \ldots, k_n)$. Then there exists a translation surface M and a double branched cover $\pi : M \to N$, branched precisely over the singular points of odd degree. In addition π is the minimal double branched cover in this class.*

2.1.2. Affine homeomorphisms

A homeomorphism f is affine with respect to (M,q) if f permutes the singularities, f is a diffeomorphism on the complement of the singularities, and the derivative map Df of f is a constant matrix in $\mathrm{PSL}_2(\mathbb{R})$.

There is a standard classification of the elements of $\mathrm{PSL}_2(\mathbb{R})$ into three types: elliptic, parabolic and hyperbolic. This induces a classification of affine homeomorphisms. An affine homeomorphism is parabolic, elliptic, or pseudo-Anosov, respectively, if $|\operatorname{Tr}(Df)| = 2$, $\operatorname{Tr}(Df)| < 2$, or $|\operatorname{Tr}(Df)| > 2$, respectively (where Tr is the trace).

2.1.3. Pseudo-Anosov homeomorphisms

Since we are interested in pseudo-Anosov homeomorphisms we will assume that $|\operatorname{Tr}(Df)| > 2$. Then there exists an eigenvalue λ of Df such that $|\lambda| > 1$ and $\operatorname{Tr}(Df) = \lambda + \lambda^{-1}$. The two eigenvectors associated to λ and λ^{-1} determine two directions on the flat surface M, invariant by ϕ. Of course ϕ expends leaves of the stable foliation by the factor $|\lambda|$ and shrinks leaves of the unstable foliation by the same factor. We can assume that these directions are horizontal and vertical. In these coordinates (M,q), the pair of associated measured foliations (stable and unstable) of ϕ are given by the horizontal and vertical measured foliations $\operatorname{Im}(q)$ and $\operatorname{Re}(q)$ and the derivative of ϕ is the matrix $A = \begin{pmatrix} \pm\lambda^{-1} & 0 \\ 0 & \pm\lambda \end{pmatrix}$. By construction the dilatation $\lambda(\phi)$ of ϕ equals $|\lambda|$. The singularity data of a pseudo-Anosov ϕ is the singularity data of its invariant measured foliation.

The group $\mathrm{PSL}_2(\mathbb{R})$ naturally acts on the set of flat surfaces. With above notations the matrix A fixes the surface (M,q), that is, (M,q) can be obtained from $A \cdot (M,q)$ by "cutting" and "gluing" (*i.e.*, the two surfaces represent the same point in the moduli space). The converse is true: if A stabilizes a flat surface (M,q), then there exists an affine diffeomorphism $f : M \to M$ such that $Df = A$.

Masur and Smillie [21] proved the following result:

Theorem 2.1 (Masur, Smillie). — *For each integer partition $(k_1,\dots,$ $k_n)$ of $4g-4$ with $k_i \geqslant 0$ even, there is a pseudo-Anosov homeomorphism*

ϕ with singularity data $(k_1, \dots, k_n)$ that fixes an orientable measured foliation.

For each integer partition $(k_1, \dots, k_n)$ of $4g-4$ with $k_i \geqslant -1$, there is a pseudo-Anosov homeomorphism ϕ with singularity data $(k_1, \dots, k_n)$ that fixes a non-orientable measured foliation, with the following exceptions:

$$(1,-1),\ (1,3),\ \textit{and}\ (4).$$

Convention. — *For the remainder of this paper, unless explicitly stated (in particular in Section 4), we shall assume that pseudo-Anosov homeomorphisms preserve orientable measured foliations.*

For instance, if $g = 3$ and ϕ preserves an orientable measured foliation, then there are 5 possible strata for the singularity data of ϕ:

$$(8),\ (2,6),\ (4,4),\ (2,2,4),\ \text{and}\ (2,2,2,2).$$

2.2. Algebraic properties of dilatations

The next theorem follows from basic results in the theory of pseudo-Anosov homeomorphisms (see for example [28]).

Theorem 2.2 (Thurston). — *Let ϕ be a pseudo-Anosov homeomorphism on a genus g surface that leaves invariant an orientable measured foliation. Then*

(1) *The linear map ϕ_* defined on $H_1(M,\mathbb{R})$ has a simple eigenvalue $\rho(\phi_*) \in \mathbb{R}$ such that $|\rho(\phi_*)| > |x|$ for all other eigenvalues x;*
(2) *ϕ is affine, for the affine structure determined by the measured foliations, and the eigenvalues of the derivative $D\phi$ are $\rho(\phi_*)^{\pm 1}$;*
(3) *$|\rho(\phi_*)| > 1$ is the dilatation λ of ϕ.*

A *Perron root* is an algebraic integer $\lambda \geqslant 1$ all whose other conjugates satisfy $|\lambda'| < \lambda$. Observe that these are exactly the numbers that arise as the leading eigenvalues of Perron–Frobenius matrices. Since ϕ_* preserves a symplectic form, the characteristic polynomial χ_{ϕ_*} is a reciprocal degree $2g$ polynomial.

Remark 2.3. — The dilatation of a pseudo-Anosov homeomorphism ϕ is the Perron root of a reciprocal degree $2g$ polynomial, namely $\chi_{\phi_*}(X)$ if $\rho(\phi_*) > 0$ and $\chi_{\phi_*}(-X)$ otherwise.

There is a converse to Theorem 2.2, but the proof does not seem as well-known, so we include a proof here (see [3] Lemma 4.3).

Theorem 2.4. — *Let ϕ be a pseudo-Anosov homeomorphism on a surface M with dilatation λ. Then the following are equivalent:*

(1) *λ is an eigenvalue of the linear map ϕ_* defined on $H_1(M, \mathbb{R})$.*
(2) *The invariant measured foliations of ϕ are orientable.*

Proof. — Suppose the stable measured foliation on (M, q) is non-orientable. There exists a double branched cover $\pi : N \to M$ which orients the foliation (we denote by τ the involution of the covering). Let $[w]$ be an eigenvector of ϕ_* in $H^1(M, \mathbb{R})$ with eigenvalue λ. The vector $[w]$ pulls back to an eigenvector $[w']$ of the adjoint ϕ^* in $H^1(N, \mathbb{R})$ for the eigenvalue λ.

The stable foliation on N now also defines a cohomology class $[Re(\omega)]$ where $\omega^2 = \pi^* q$. By construction $[Re(\omega)]$ is an eigenvector for the eigenvalue λ. By Theorem 2.2 λ is simple so that the two classes $[Re(\omega)]$ and $[w']$ must be linearly dependent. But since $[w']$ is invariant by the deck transformation τ, while $[Re(\omega)]$ is sent to $-[Re(\omega)]$ by τ, we get a contradiction. □

Combining this theorem with two classical results of Casson–Bleiler [6] and Thurston [9] we get

Theorem 2.5. — *Let f be a homeomorphism on a surface M and let $P(X)$ be the characteristic polynomial of the linear map f_* defined on $H_1(M, \mathbb{R})$. Then one has*

(1) *If $P(X)$ is irreducible over $\mathbb{Z}$, has no roots of unity as zeroes, and is not a polynomial in X^k for $k > 1$, then f is isotopic to a pseudo-Anosov homeomorphism ϕ;*
(2) *In addition, if the maximal eigenvalue (in absolute value) of the action of f on the fundamental group is $\lambda > 1$, then the dilatation of ϕ is λ;*
(3) *In addition, if λ is the Perron root of $P(X)$, then ϕ leaves invariant orientable measured foliations.*

Proof. — The first point asserts that f is isotopic to a pseudo-Anosov homeomorphism ϕ [6, Lemma 5.1]. The second point asserts that ϕ has dilatation λ [9, Exposé 10]. Finally by the previous theorem, the last assumption implies that the invariant measured foliations of ϕ are orientable. □

We will need a more precise statement. The following has been remarked by Bestvina:

Proposition 2.6. — *The statement "P is irreducible over $\mathbb{Z}$" in part (1) of Theorem 2.5 can be replaced by "P is symplectically irreducible over $\mathbb{Z}$", meaning that P is not the product of two nontrivial reciprocal polynomials.*

2.3. Pseudo-Anosov homeomorphisms and the Lefschetz theorem

In this section, we recall the well-known Lefschetz theorem for *homeomorphisms* on compact surfaces (see for example [5]). If p is a fixed point of a homeomorphism f, we define the index of f at p to be the algebraic number $\mathrm{Ind}(f,p)$ of turns of the vector $(x, f(x))$ when x describes a small loop around p.

Theorem (Lefschetz theorem). — *Let f be a homeomorphism on a compact surface M. Denote by $\mathrm{Tr}(f_*)$ the trace of the linear map f_* defined on the first homology group $H_1(M,\mathbb{R})$. Then the Lefschetz number $L(f)$ is $2 - \mathrm{Tr}(f_*)$. Moreover the following equality holds:*

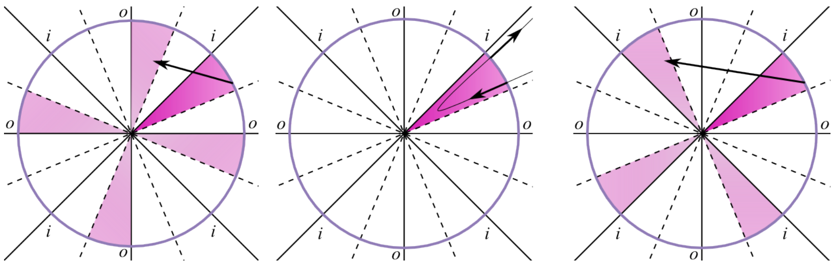
$$L(f) = \sum_{p=f(p)} \mathrm{Ind}(f,p).$$

For a pseudo-Anosov homeomorphism ϕ, if $\Sigma \in M$ is a singularity of the stable foliation of ϕ (of degree $2d$) then there are $2(d+1)$ emanating rays. The orientation of the foliation defines $d+1$ outgoing separatrices and $d+1$ ingoing separatrices.

Proposition 2.7. — *Let Σ be a fixed singularity of ϕ of degree $2d$ and let $\rho(\phi_*)$ be the leading eigenvalue of ϕ_*. Then*

- *If $\rho(\phi_*) < 0$ then ϕ exchanges the set of outgoing separatrices and the set of ingoing separatrices. Moreover* $\mathrm{Ind}(\phi,\Sigma) = 1$.
- *If $\rho(\phi_*) > 0$ then either*
  - *ϕ fixes each separatrix and* $\mathrm{Ind}(\phi,\Sigma) = 1 - 2(d+1) < 0$, *or*

– *ϕ permutes cyclically the outgoing separatrices (and ingoing separatrices) and* $\mathrm{Ind}(\phi, \Sigma) = 1$.



(a) (b)

Figure 2.1. Mapping of the $4(d+1)$ *hyperbolic sectors by* ϕ *near a degree* $2d = 6$ *singularity: (a)* $\rho(\phi_*) < 0$*: the sectors are permuted and the index is* 1*; (b)* $\rho(\phi_*) > 0$*: the sectors can either be fixed (left, index* $1 - 2(d+1) = -7$*) or permuted (right, index* 1*). The index is defined as the number of turns of a vector joining* x *to* $\phi(x)$ *as* x *travels counterclockwise around a small circle. The separatrices of the unstable foliation are alternately labeled ingoing (i) and outgoing (o). The grey areas indicate a hyperbolic sector and its possible images for each case.*

Proof of Proposition 2.7. — Obviously ϕ acts on the set of separatrices (namely the set of outgoing separatrices and ingoing separatrices). It is clear that $\rho(\phi_*) < 0$ if and only if ϕ exchanges these two sets. In that case, $\mathrm{Ind}(\phi, \Sigma) = 1$ for any fixed point Σ, since the tip of the vector $(x, f(x))$ never crosses the hyperbolic sector containing x and is thus constrained to make a single turn counterclockwise. (A hyperbolic sector is the region between adjacent ingoing and outgoing separatrices, see figure 2.1.)

If $\rho(\phi_*) > 0$ then ϕ fixes globally the set of outgoing separatrices. Let us assume that ϕ fixes an outgoing separatrix γ^u of the unstable foliation $\mathcal{F}^u$. Let γ_1^s and γ_2^s be two adjacent incoming separatrices for the stable foliation $\mathcal{F}^s$ that define a sector containing γ^u and another (ingoing) separatrix of $\mathcal{F}^u$. Since γ^u is fixed by ϕ, the sector determined by γ_1^s and γ_2^s is also fixed. ϕ preserves orientation so that γ_1^s (and so γ_2^s) is fixed.

Hence, the other separatrix of $\mathcal{F}^u$ in the sector is fixed. By induction, each separatrix of $\mathcal{F}^u$ is fixed.

There are $4(d+1)$ hyperbolic sectors. For each sector, the vector $(x, h(x)$ describes an angle of $-\pi$ plus the sector angle, $\pi/2(d+1)$. Thus the total angle is $4(d+1)(-\pi+\pi/2(d+1)) = 2\pi(1-2(d+1))$.

If ϕ has no fixed separatrices then clearly ϕ permutes the outgoing separatrices. In addition, ϕ is isotopic to a rotation, thus ϕ permutes cyclically the separatrices [18]. In that case $\operatorname{Ind}(\phi,\Sigma)=1$, for the same reason as the $\rho(\phi_*)<0$ case above. □

We will use the following corollaries:

Corollary 2.8 (Lefschetz theorem revisited for pseudo-Anosov homeomorphisms). — *Let* $\operatorname{Sing}(\phi)$ *be the set of fixed singularities of degree* $d>0$ *of the pseudo-Anosov homeomorphism* ϕ. *Let* $\operatorname{Fix}(\phi)$ *be the set of regular fixed points of* ϕ *(i.e., of degree* $d=0$*).*

Then if $\rho(\phi_*)>0$,

$$2-\operatorname{Tr}(\phi_*) = \sum_{\Sigma\in\operatorname{Sing}(\phi)} \operatorname{Ind}(\phi,\Sigma) - \#\operatorname{Fix}(\phi)$$

where $\operatorname{Ind}(\phi,\Sigma)=1$ *or* $1-2(d+1)$ *and* $2d$ *is the degree of* Σ.

If $\rho(\phi_*)<0$,

$$2-\operatorname{Tr}(\phi_*) = \#\operatorname{Sing}(\phi) + \#\operatorname{Fix}(\phi)\,.$$

Corollary 2.9. — *Let* Σ *be a fixed singularity of* ϕ *(of degree* $2d$*). Let us assume that* $\rho(\phi_*)>0$ *and* $\operatorname{Ind}(\phi,\Sigma)=1$. *Then*

$$\forall i=1,\dots d,\ \operatorname{Ind}(\phi^i,\Sigma)=1$$

and

$$\operatorname{Ind}(\phi^{d+1},\Sigma)=1-2(d+1).$$

We will use this corollary with $d=2$ and $d=4$ in the coming sections, so we prove it only for those cases.

Proof of Corollary 2.9. — If Σ is a singularity of degree 2 ($d=1$) then there are 2 outgoing separatrices. $\operatorname{Ind}(\phi,\Sigma)=1$ implies that ϕ permutes these two separatrices so that ϕ^2 fixes them. Hence $\operatorname{Ind}(\phi^2,\Sigma)=1-2(1+1)=-3$.

If Σ is a singularity of degree 4 ($d=2$) then there are three outgoing separatrices. $\operatorname{Ind}(\phi,\Sigma)=1$ implies that ϕ permutes cyclically these three separatrices. Hence $\operatorname{Ind}(\phi^2,\Sigma)=1$ and $\operatorname{Ind}(\phi^3,\Sigma)=1-2(2+1)=-5$. □

3. Genus three: A proof of Theorem 1.2 for $g = 3$

We write $\rho(P)$ for the largest root (in absolute value) of a polynomial P; for the polynomials we consider this is always real and with strictly larger absolute value than all the other roots, though it could have either sign. If $\rho(P) > 0$ then it is a Perron root; otherwise $\rho(P(-X))$ is a Perron root.

We find all reciprocal polynomials with a Perron root less than our candidate and then we test whether a polynomial is compatible with a given stratum. This is straightforward: we simply try all possible permutations of the singularities and separatrices, and calculate the contribution to the Lefschetz numbers for each iterate of ϕ. Then we see whether the deficit in the Lefschetz numbers can be exactly compensated by regular periodic orbits. If not, the polynomial cannot correspond to a pseudo-Anosov homeomorphism on that stratum.

We prove the theorems out of order since genus 3 is simplest. We know that $\delta_3^+ \leqslant \rho(X^3 - X^2 - 1) \simeq 1.46557$ (for instance see [12] or [17]). We will construct a pseudo-Anosov homeomorphism with a smaller dilatation than 1.46557 and prove that this dilatation is actually the least dilatation.

Recall that δ_3^+ is the Perron root of some reciprocal polynomial P of degree 6 (see Remark 2.3). As discussed in Appendix A, it is not difficult to find all reciprocal polynomials with a Perron root $\rho(P)$, $1 < \rho(P) < \rho(X^3 - X^2 - 1)$: there are only two, listed in Table 3.1 (see also Appendix A.2 for an alternate approach to this problem). Let us

polynomial	Perron root
$P_1 = (X^3 - X - 1)(X^3 + X^2 - 1)$	1.32472
$P_2 = X^6 - X^4 - X^3 - X^2 + 1$	1.40127

Table 3.1. List of all reciprocal monic degree 6 polynomials P with Perron root $1 < \rho(P) < \rho(X^3 - X^2 - 1) \simeq 1.46557$.

assume that $\delta_3^+ < \rho(X^3 - X^2 - 1)$ and see if we get a contradiction. We let ϕ be a pseudo-Anosov homeomorphism with $\lambda(\phi) = \delta_3^+$. By the above discussion there are only two possible candidates for a reciprocal annihilating polynomial P of the dilatation of ϕ, namely $\lambda(\phi) = \rho(P_i)$ for some $i \in \{1, 2\}$. In the next subsection we shall prove that there are no pseudo-Anosov homeomorphisms on a genus three surface (stabilizing orientable foliations) with a dilatation $\rho(P_1)$. We shall then show that a pseudo-Anosov homeomorphism with dilatation $\rho(P_2)$ exists on this surface.

3.1. First polynomial: $\lambda(\phi) = \rho(P_1)$

Let ϕ_* be the linear map defined on $H_1(X, \mathbb{R})$ and let χ_{ϕ_*} be its characteristic polynomial. By Theorem 2.2 the leading eigenvalue $\rho(\phi_*)$ of ϕ_* is $\pm\rho(P_1)$. The minimal polynomial of the dilatation of ϕ is $X^3 - X - 1$; thus if $\rho(\phi_*) > 0$ then $X^3 - X - 1$ divides χ_{ϕ_*}, otherwise $X^3 - X + 1$ divides χ_{ϕ_*}. Requiring the polynomial to be reciprocal leads to $\chi_{\phi_*} = P_1$ for the the first case and $\chi_{\phi_*} = P_1(-X) = (X^3 - X + 1)(X^3 - X^2 + 1)$ for the second.

The trace of ϕ_*^n (and so the Lefschetz number of ϕ^n) is easy to compute in terms of its characteristic polynomial. Let us analyze carefully the two cases depending on the sign of $\rho(\phi_*)$.

(1) If $\rho(\phi_*) < 0$ then $\chi_{\phi_*}(X) = P_1(-X) = (X^3-X+1)(X^3-X^2+1)$. Let $\psi = \phi^2$. Observe that ψ is a pseudo-Anosov homeomorphism and $\rho(\psi_*) > 0$ is a Perron root. >From Newton's formulas (see Appendix A), we have $\mathrm{Tr}(\phi_*) = -1$, $\mathrm{Tr}(\psi_*) = 3$, $\mathrm{Tr}(\psi_*^2) = -1$, and $\mathrm{Tr}(\psi_*^3) = 3$, so that $L(\phi) = 3$, $L(\psi) = -1$, $L(\psi^2) = 3$, and $L(\psi^3) = -1$.

As we have seen in Section 2, there are 5 possible strata for the singularity data of ϕ, and so for ψ, namely,

$$(8),\ (2,6),\ (4,4),\ (2,2,4),\ \text{and}\ (2,2,2,2).$$

Since $L(\psi^2) = 3$ there are at least 3 singularities (of index $+1$) fixed by ψ^2; thus we need only consider strata $(2,2,4)$ and $(2,2,2,2)$. (From Corollary 2.8 regular fixed points can only give negative index since $\rho(\psi_*^2) > 0$).

For stratum $(2,2,4)$, the single degree-4 singularity must be fixed, and its three outgoing separatrices must be fixed by ψ^3. The contribution to the index is then -5, which contradicts $L(\psi^3) = -1$ since there is no way to make up the deficit.

For stratum $(2,2,2,2)$, since ψ^2 fixes at least three singularities they account for $+3$ of the Lefschetz number $L(\psi^2) = 3$. But the fourth singularity must also be fixed by ψ^2, so it adds $+1$ or -3 to the Lefschetz number, depending on the permutation of its two separatrices. The only compatible scenario is that it adds $+1$, with the difference accounted by a single regular fixed point that contributes -1. Since all four singularities are thus fixed by $\psi^2 = \phi^4$, this means that their permutation $\sigma \in S_4$ must satisfy $\sigma^4 = \mathrm{id}$. There are three cases: either the singularities are all fixed by ϕ, they are permuted in groups of two, or they are cyclically permuted. For the first two cases, the singularities are also fixed by $\psi = \phi^2$, so by Corollary 2.9 they cannot contribute positively to ψ^2, which they must as we saw above. If the four singularities are all cyclically permuted, then they contribute nothing to $L(\phi) = 3$ and there is only one regular fixed point, so we get a contradiction here as well.

(2) If $\rho(\phi_*) > 0$ then $\chi_{\phi_*}(X) = P_1(X)$. We have $\mathrm{Tr}(\phi_*) = -1$ and $\mathrm{Tr}(\phi_*^2) = 3$, so that $L(\phi) = 3$ and $L(\phi^2) = -1$. Since $L(\phi) = 3$ there are at least 3 fixed singularities; thus we need only consider strata $(2,2,4)$ and $(2,2,2,2)$.

$L(\phi) = 3$ implies that all the singularities are necessarily fixed, with positive index. Let us denote by $\Sigma_1,\ \Sigma_2$ two degree-2 singularities. Since $\mathrm{Ind}(\phi, \Sigma_i) = 1$, by Corollary 2.9 one has $\mathrm{Ind}(\phi^2, \Sigma_i) = -3$, leading to $L(\phi^2) \leqslant -6 + 2 = -4$; but $L(\phi^2) = -1$, which is a contradiction.

3.2. Second polynomial: $\lambda(\phi) = \rho(P_2)$

As in the previous section, we can rule out most strata associated with P_2 both for positive ($P_2(X)$) or negative ($P_2(-X)$) dominant root. For $P_2(-X)$, however, there remain three strata that cannot be eliminated:

$$(8),\ (2,6),\ \text{and } (2,2,2,2).$$

We single out the last stratum, $(2,2,2,2)$, to illustrate that this is a candidate. Indeed, assume that three of the degree 2 singularities are cyclically

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$L(\phi^n)$	2	0	5	-4	7	-3	16	-12	23	-25	46	-55	80	-112	160
$L(2^3)$	0	0	3	0	0	3	0	0	3	0	0	-9	0	0	3
$L(2^1)$	1	1	1	-3	1	1	1	-3	1	1	1	-3	1	1	1
L_{ro}	1	-1	1	-1	6	-7	15	-9	19	-26	45	-43	79	-113	156

Table 3.2. For the first 15 iterates of ϕ, contribution to the Lefschetz numbers from the various orbits, for the polynomial $P_2(-X)$ from Table 3.1 on stratum $(2,2,2,2)$. The first row specifies the iterate of ϕ; the second the total Lefschetz number; the third the contribution from the three permuted degree-2 singularities; the fourth the contribution from the fixed degree-2 singularity; the fifth the contribution from the regular (degree 0) orbits. Note that $L(2^3)$, $L(2^1)$, and L_{ro} sum to L.

permuted, and the fourth one is fixed. For the triplet of singularities assume that the two ingoing (or outgoing) separatrices are permuted by ϕ^6, so they are fixed by ϕ^{12}. At iterates 3 and 9 the three singularities are fixed but their separatrices are permuted, and $\rho(\phi^3)$ and $\rho(\phi^9)$ are both negative, so by Proposition 2.7 the total contribution to the Lefschetz number from these three singularities is 3. At iterate 6 we have $\rho(\phi^6) > 0$ but the separatrices are permuted, so again from Proposition 2.7 the total contribution is 3. Finally, at iterate 12 the singularities and their separatrices are fixed, so the total contribution to $L(\phi^{12})$ is $3 \cdot (1-4) = -9$.

For the fixed singularity of degree 2, assume that the two separatrices are permuted by ϕ^2, so they are fixed by ϕ^4. Hence, the singularity contributes 1 to $L(\phi^n)$ except when n is a multiple of 4: we then have $\rho(\phi^n) > 0$ again by Proposition 2.7 the contribution is $1 - 4 = -3$. As can be seen in Table 3.2, the deficit in $L(\phi^n)$ can be exactly made up by introducing regular periodic orbits (it is easy to show that this can be done for arbitrary iterates). To complete the proof of 1.2 for $g = 3$, it remains to be shown that such a homeomorphism can be constructed.

3.3. Construction of a pseudo-Anosov homeomorphism by Dehn twists

We show how to realize in terms of Dehn twists a pseudo-Anosov homeomorphism whose dilatation is the Perron root of $P_2(X)$. The curves we

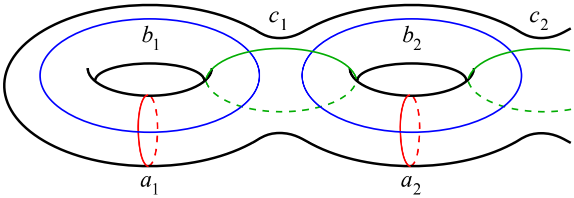


Figure 3.1. Curves used to define Dehn twists.

use for Dehn twists are shown in Figure 3.1. For example, a positive twist about c_1 is written T_{c_1}; a negative twist about b_2 is written $T_{b_2}^{-1}$.

Proposition 3.1. — *There exists a pseudo-Anosov homeomorphism on a genus* 3 *surface, stabilizing orientable foliations, and having for dilatation the Perron root of the polynomial* $P_2(X)$*.*

Proof. — Let us consider the sequence of Dehn twists

$$T_{a_1}T_{a_1}T_{b_1}T_{c_1}T_{a_2}T_{b_2}T_{c_2}T_{c_2}T_{a_3}^{-1}T_{b_3}^{-1}.$$

Its action on homology has $P_2(-X)$ as a characteristic polynomial. Since $P_2(X)$ is irreducible and has no roots that are also roots of unity [26], then by Theorem 2.5 the homeomorphism is isotopic to a pseudo-Anosov homeomorphism, say f (we also use Bestvina's remark, Proposition 2.6).

We can compute the dilatation of f by calculating the action on the fundamental group (or using the code described in the remark below). A straightforward calculation shows that the dilatation is the Perron root of the polynomial $P_2(X)$, so f must also stabilize a pair of orientable foliations. Hence, it realizes our systole δ_3^+. □

Remark 3.2. — To search for pseudo-Anosov homeomorphisms, we used a computer code written by Matthew D. Finn [10], which calculates the dilatation of words in terms of Dehn twists. The code uses the fast method of Moussafir [24] adapted to higher genus. Hence, we can examine a large number of words and find candidates with the required dilatation.

4. Genus two: A proof of Theorem 1.1

We prove theorem 1.1 in two parts: we first find the value of the systole δ_2, then demonstrate its uniqueness.

Recall that a surface M of genus g is called hyperelliptic if there exists an involution τ (called the hyperelliptic involution) with $2g+2$ fixed points. It is a classical fact that each genus two surface is hyperelliptic. The fixed points are also called the *Weierstrass points*. We now make more precise the qualification "up to hyperelliptic involution and covering transformation" of Theorem 1.1.

Remark 4.1. — If (M,q) is a hyperelliptic surface, then for each conjugacy class of a pseudo Anosov homeomorphism ϕ on M there exists another conjugacy class, namely $\tau\circ\phi$, having the same dilatation. For instance in genus 1 the two Anosov homeomorphisms $\phi = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\tau\circ\phi = \left(\begin{smallmatrix} -2 & -1 \\ -1 & -1 \end{smallmatrix}\right)$ have the same dilatation.

A second construction that produces another conjugacy class with the same dilatation is the following. Let ϕ be a pseudo-Anosov homeomorphism on a genus two surface M stabilizing a non-orientable foliation with singularity data $(1,1,2)$. Then there exists a branched double covering $\pi : M \to \mathbb{S}^2$ such that ϕ descends to a pseudo-Anosov $\tilde{\phi}$ on the sphere, fixing a non-orientable measured foliation and having singularity data $(-1,-1,-1,-1,-1,1,0)$ (see the proof of Theorem 1.1 below). Let the orientating double cover be $\pi' : N \to \mathbb{S}^2$. Now $\tilde{\phi}$ lifts to a new pseudo-Anosov homeomorphism φ on the genus-two surface N (stabilizing orientable foliations with singularity data (4)):

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d] & M \arrow[d] & N \arrow[r, "\varphi"] \arrow[dll] & N \arrow[dll] \\ \mathbb{S}^2 \arrow[r, "\tilde{\phi}"'] & \mathbb{S}^2 & & \end{tikzcd}$$

Now $\lambda(\phi) = \lambda(\tilde{\phi}) = \lambda(\varphi)$ (see also [16] for more details). But of course the conjugacy classes of ϕ and φ are not the same.

Finally we will use the following result.

Proposition 4.2. — *Let (M,q) be a genus two flat surface and let τ be the affine hyperelliptic involution. Let ϕ be an affine homeomorphism. Then ϕ commutes with τ.*

Proof of Proposition 4.2. — Let $\mathcal{P} = \{\mathcal{Q}_1,\ldots,\mathcal{Q}_6\}$ be the set of Weierstrass points, *i.e.*, the set of fixed points of τ.

Firstly let us show that ϕ preserves the set of Weierstrass points. Since $\phi^{-1}\circ\tau\circ\phi$ is a non-trivial involution, it is an automorphism of the complex surface, thus the fixed points of $\phi^{-1}\circ\tau\circ\phi$ are also Weierstrass points. Let p be a Weierstrass point. Then $\phi^{-1}\circ\tau\circ\phi(p) = p$ or $\tau\circ\phi(p) = \phi(p)$. Hence $\phi(p)$ is a fixed point of τ, and thus $\phi(p)$ is a Weierstrass point.

Now let $\psi = [\phi, \tau] = \phi\circ\tau\circ\phi^{-1}\circ\tau$ be the commutator of ϕ and τ. Since τ and ϕ are affine homeomorphisms, ψ is also an affine homeomorphism. The derivative of ψ is equal to the identity so that ψ is a translation. Since $\phi^{-1}\circ\tau(\mathcal{Q}_1) = \phi^{-1}(\mathcal{Q}_1) \in \mathcal{P}$ one has $\tau\circ\phi^{-1}\circ\tau(\mathcal{Q}_1) = \phi^{-1}(\mathcal{Q}_1)$ and $\psi(\mathcal{Q}_1) = \phi\circ\phi^{-1}(\mathcal{Q}_1) = \mathcal{Q}_1$. The translation ψ fixes a regular point. Thus it also fixes the separatrix issued from this point, and therefore $\psi = \mathrm{id}$ and ϕ commutes with τ. □

Proof of Theorem 1.1 (systole). — Let ϕ be a pseudo-Anosov homeomorphism with $\lambda(\phi) = \delta_2$. We know that δ_2^+ is the Perron root of $X^4 - X^3 - X^2 - X + 1$ (see Zhirov [30]; see also Appendix C for a different construction). Let us assume that $\delta_2 < \delta_2^+$. Thus ϕ preserves a pair of non-orientable measured foliations. The allowable singularity data for these foliations are $(2,2)$, $(1,1,2)$ or $(1,1,1,1)$. (Masur and Smillie [21] showed that (4) and $(1,3)$ cannot occur for non-orientable measured foliations.)

It is well known that each genus two surface is a branched double covering of the standard sphere. Let $\pi : M \to \mathbb{S}^2$ be the covering and τ the associated involution. It can be shown that τ is affine for the metric determined by ϕ (see [16]). Thus Proposition 4.2 applies and ϕ commutes with τ. Hence ϕ induces a pseudo-Anosov homeomorphism $\tilde{\phi}$ on the sphere $\mathbb{S}^2$ with the same dilatation. Of course $\tilde{\phi}$ leaves invariant a non-orientable pair of measured foliations. The singularity data for ϕ are $(2,2)$, $(1,1,2)$, or $(1,1,1,1)$; The singularity data for $\tilde{\phi}$ are respectively $(-1,-1,-1,-1,0,0)$, $(-1,-1,-1,-1,-1,1,0)$, or $(-1,-1,-1,-1,-1,-1,1,1)$. (For the first case, the singularity data cannot be $(-1,-1,-1,-1,-1,-1,2)$, otherwise the cover π would be the orientating cover — the branched points are precisely the singular points of odd degree, see Remark 2.1 — thus the foliations of ϕ would be orientable.)

There exists an (orientating) double covering $\pi' : N \to \mathbb{S}^2$ such that $\tilde{\phi}$ lifts to a pseudo-Anosov homeomorphism f on N that stabilizes an orientable measured foliation. Actually, since the deck group is $\mathbb{Z}/2\mathbb{Z}$, there are two lifts: f and $\tau\circ f$, where τ denote the hyperelliptic involution on N. Since $\mathrm{Tr}((\tau\circ f)_*) = -\mathrm{Tr}(f)$, there is one lift, say f, with $\rho(\chi_{f_*}) > 0$. By construction $\lambda(f) = \delta_2 = \rho(\chi_{f_*})$. Let us compute the genus of N using the singularity data of f as follows.

(1) If the singularities of ϕ are $(2,2)$ then the singularities of f are (0); thus N is a torus.
(2) If the singularities of ϕ are $(1,1,2)$ then the singularities of f are $(0,4)$; thus N is a genus two surface.
(3) If the singularities of ϕ are $(1,1,1,1)$ then the singularities of f are $(4,4)$; thus N is a genus three surface.

In the first case one has $\delta_2 \geqslant \delta_1$, but since $\delta_1 > \delta_2^+$ this contradicts the assumption $\delta_2 < \delta_2^+$. In the second case $\delta_2 \geqslant \delta_2^+$ which is also a contradiction. Let us analyze the third case. Since $\lambda(f) = \delta_2 < \delta_2^+$ and f preserves an orientable measured foliation on a genus three surface, Table 4.1 gives all possible minimal polynomials for δ_2 with $1 < \rho(P) < \rho(X^4 - X^3 - X^2 - X + 1)$ (see Appendix A). We will obtain a contradiction

polynomial	Perron root
$P_1 = (X^3 - X - 1)(X^3 + X^2 - 1)$	1.32472
$P_2 = X^6 - X^4 - X^3 - X^2 + 1$	1.40127
$P_3 = (X^3 + X - 1)(X^3 - X^2 - 1)$	1.46557
$P_4 = X^6 - X^5 - X^3 - X + 1$	1.50614
$P_5 = X^6 - X^5 - X^4 + X^3 - X^2 - X + 1$	1.55603
$P_6 = X^6 - 2X^5 + 3X^4 - 5X^3 + 3X^2 - 2X + 1$	1.56769
$P_7 = X^6 - X^4 - 2X^3 - X^2 + 1$	1.58235
$P_8 = X^6 - 2X^5 + 2X^4 - 3X^3 + 2X^2 - 2X + 1$	1.63557
$P_9 = X^6 - X^5 + X^4 - 4X^3 + X^2 - X + 1$	1.67114

Table 4.1. List of all reciprocal monic degree 6 *polynomials* P *such that the Perron root* $\lambda = \rho(P)$ *satisfies* $1 < \lambda < \rho(X^4 - X^3 - X^2 - X + 1) \simeq 1.72208$.

for each case. For each polynomial P_i, we calculate the Lefschetz number of iterates of f (see Table 4.2).

(1) Polynomial P_i for $i \in \{1,3,6,9\}$ cannot be a candidate since the number of singularities is 2 and $L(f)$ or $L(f^2)$ is greater than or equal to 3.

	$L(f)$	$L(f^2)$
P_1	3	
P_3		3
P_6		4
P_9		3

	$L(f)$	$L(f^3)$
P_2	2	-1
P_4	1	-2
P_5	1	1
P_7	2	-4

Table 4.2. Lefschetz number of iterates of the pseudo-Anosov homeomorphism f.

(2) Polynomial P_i for $i \in \{2, 4, 5, 7\}$ cannot be a candidate. Indeed the singularities are fixed with positive index, thus by Corollary 2.9 we should have $L(f^3) \leqslant -10$, but we know $L(f^3) \geqslant -4$ from Table 4.2.

Finally the last case we have to consider is P_8. In that case, the Lefschetz number of f is 0 and the Lefschetz number of f^3 is -3. Let Σ_1 and Σ_2 be the two singularities of f on N. Let us assume that the two singularities are fixed, so the index of f at Σ_i is necessarily positive. Then by Corollary 2.9 $\mathrm{Ind}(f^3, \Sigma_i) = -5$, so that $L(f^3) = -3 = -10 - \#\,\mathrm{Fix}(f^3)$ and $\#\,\mathrm{Fix}(f^3) = -7$, which is a contradiction. Hence Σ_1 and Σ_2 are exchanged by f, and therefore by f^3. The formula $L(f^3) = -3$ reads $\#\,\mathrm{Fix}(f^3) = 3$, so that f has a unique length 3 periodic orbit (and no fixed points). Recall also that f commutes with the hyperelliptic involution τ on N. This involution has exactly 8 fixed points on N: the two singularities and 6 regular points, which we will denote by $\{\Sigma_1, \Sigma_2, \mathcal{Q}_1, \dots, \mathcal{Q}_6\}$.

Let $\{\mathcal{S}, f(\mathcal{S}), f^2(\mathcal{S})\}$ be the length-3 orbit. Since $f \circ \tau = \tau \circ f$ the set $\{\tau(\mathcal{S}), \tau(f(\mathcal{S})), \tau(f^2(\mathcal{S}))\}$ is also a length-3 orbit and thus by uniqueness

$$\left\{\mathcal{S}, f(\mathcal{S}), f^2(\mathcal{S})\right\} = \left\{\tau(\mathcal{S}), \tau(f(\mathcal{S})), \tau(f^2(\mathcal{S}))\right\}.$$

If $\tau(\mathcal{S}) = \mathcal{S}$ then $\mathcal{S} = \mathcal{Q}_i$ for some i and $\{\mathcal{S}, f(\mathcal{S}), f^2(\mathcal{S})\}$ is a subset of $\{\mathcal{Q}_1, \dots, \mathcal{Q}_6\}$. Otherwise let us assume that $\tau(\mathcal{S}) = f(\mathcal{S})$. Applying f one gets $f^2(\mathcal{S}) = f(\tau(\mathcal{S})) = \tau(f(\mathcal{S})) = \tau^2(\mathcal{S}) = \mathcal{S}$ which is a contradiction. We get the same contradiction if $\tau(\mathcal{S}) = f^2(\mathcal{S})$. Therefore $\tau(\mathcal{S}) = \mathcal{S}$ and $\{\mathcal{S}, f(\mathcal{S}), f^2(\mathcal{S})\}$ is a subset of $\{\mathcal{Q}_1, \dots, \mathcal{Q}_6\}$.

Up to permutation one can assume that this set is $\{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3\}$. Since f preserves the set $\{\Sigma_1, \Sigma_2\}$ then f also preserves $\{\mathcal{Q}_4, \mathcal{Q}_5, \mathcal{Q}_6\}$. Hence

f has a fixed point or another length-3 periodic orbit, which is a contradiction. This ends the proof of the first part of Theorem 1.1. □

We now prove the uniqueness of the pseudo-Anosov homeomorphism realizing the systole in genus two, up to conjugacy, hyperelliptic involution, and covering transformations (see Remark 4.1).

Proof of Theorem 1.1 (uniqueness). — We will prove that there is no other construction that realizes the systole in genus two. The proof uses essentially McMullen's work [23]. Let ϕ and ϕ' be two pseudo-Anosov homeomorphisms on M with $\lambda(\phi) = \delta_2$ and let (M, q), (M', q') be the two associated flat surfaces.

The proof decomposes into 4 steps. We first show that one can assume that ϕ and ϕ' leave invariant an orientable measured foliation with singularity data (4). Then we show that we can assume, up to conjugacy, that the two surfaces (M, q) and (M', q') are isometric. Finally we show that the derivatives $D\phi$ and $D\phi'$ of the affine homeomorphism on M are conjugate. We then conclude that ϕ and ϕ are conjugated in the mapping class group Mod(2).

Step 4.3. — If the foliation is non-orientable then we have seen (proof of Theorem 1.1) that the singularity data of ϕ is $(1, 1, 2)$. By Remark 4.1 there exists a branched double covering $\pi : M \to \mathbb{P}^1$ such that ϕ descends to a pseudo-Anosov on the sphere $\mathbb{P}^1$ with singularity $(-1, -1, -1, -1, -1, 1, 0)$. Now the orientating cover $\tilde{\pi} : \tilde{M} \to \mathbb{P}^1$ gives a pseudo-Anosov homeomorphism $\tilde{\phi}$ on the genus 2 surface $\tilde{M}$, with orientable foliation and singularity data (4). In addition $\lambda(\phi) = \lambda(\tilde{\phi})$. Hence, from this discussion one can assume that ϕ stabilizes an orientable measured foliation. The singularity data of the measured foliation is either (4) or $(2, 2)$. Using the Lefschetz theorem, one shows that $(2, 2)$ is impossible.

Step 4.4. — Up to the hyperelliptic involution, we can assume that $\mathrm{Tr}(\phi) > 0$ and $\mathrm{Tr}(\phi') > 0$. There is natural invariant we can associate to a flat surface with a pseudo-Anosov homeomorphism φ: this is the trace field (see [14]), the number field generated by $\lambda(\varphi) + \frac{1}{\lambda(\varphi)}$. In our case of course the trace field of the surfaces (M, q) and (M', q') is the same since the dilatation of ϕ and ϕ' is the same. More precisely the trace field is $\mathbb{Q}[t]$, where $t = \delta_2 + \delta_2^{-1}$. A straightforward calculation gives that the minimal polynomial of t is $X^2 - X - 3$, so the trace field is $\mathbb{Q}(\sqrt{13})$.

Since the discriminant $\Delta = 13 \not\equiv 1 \bmod 8$, Theorem 1.1 of [23] implies that there exists a $A \in \mathrm{SL}_2(\mathbb{R})$ such that $A(M,q) = (M',q')$. (We can always assume that the area of the flat surfaces (M,q) and (M',q') is 1.) In particular there exists an affine homeomorphism $f : M \to M'$ such that $Df = A$. Hence $f^{-1}\phi' f$ is a pseudo-Anosov homeomorphism on the *same* affine surface (M,q).

Step 4.5. — Now the derivatives of the two affine maps ϕ and ϕ' (on the same flat surface (M,q)) belong to the Veech group of the surface (M,q). (This group has 3 cusps and genus zero — see [23], Theorem 9.8.) Using the Rauzy–Veech induction, we can check that $D\phi$ and $A^{-1}D\phi' A$ are conjugated in this group.

Step 4.6. — Thus there exists $B \in \mathrm{SL}_2(\mathbb{R})$ such that $D\phi = B^{-1}D\phi' B$. Now let $h : M \to M$ be such that $Dh = B$; hence one has $D\phi = Dh^{-1}D\phi' Dh$. Finally $h^{-1}\phi' h\phi^{-1}$ is an affine diffeomorphism with derivative map equal to the identity, and so it is a translation. Since the metric has a unique singularity (of type (4)), $h^{-1}\phi' h\phi^{-1} = \mathrm{id}$. We conclude that ϕ and ϕ' are conjugate in the mapping class group Mod(2), and the theorem is proved.

□

5. Genus four: A proof of Theorem 1.2 for $g = 4$

5.1. Polynomials

The techniques of the previous sections can also be applied to the genus 4 case. The only difference is that for genus four and higher we rely on a set of Mathematica scripts to test whether a polynomial is compatible with a given stratum. This is straightforward: we simply try all possible permutations of the singularities and separatrices, and calculate the contribution to the Lefschetz numbers for each iterate of ϕ. Then we see whether the deficit in the Lefschetz numbers can be exactly compensated by regular periodic orbits. If not, the polynomial cannot correspond to a pseudo-Anosov homeomorphism on that stratum.

polynomial	Perron root
$P_1 = X^8 - X^5 - X^4 - X^3 + 1$	1.28064
$P_2 = (X^3 - X - 1)(X^3 + X^2 - 1)(X - 1)^2$	1.32472
$P_3 = (X^3 - X - 1)(X^3 + X^2 - 1)(X + 1)^2$	1.32472
$P_4 = (X^3 - X - 1)(X^3 + X^2 - 1)(X^2 - X + 1)$	1.32472
$P_5 = (X^3 - X - 1)(X^3 + X^2 - 1)(X^2 + X + 1)$	1.32472
$P_6 = (X^3 - X - 1)(X^3 + X^2 - 1)(X^2 + 1)$	1.32472

Table 5.1. List of all reciprocal monic degree 8 *polynomials* P *with Perron root* $1 < \rho(P) < \rho(X^8 - X^7 + X^6 - X^5 - X^4 - X^3 + X^2 - X + 1) \simeq 1.34372$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$L(\phi^n)$	2	2	5	-2	7	-1	9	-2	14	-13	13	-17	28	-33	40
$L(10^1)$	1	1	1	1	1	1	1	1	1	1	1	-11	1	1	1
$L(2^1)$	1	1	1	-3	1	1	1	-3	1	1	1	-3	1	1	1
L_{ro}	0	0	3	0	5	-3	7	0	12	-15	11	-3	26	-35	38

Table 5.2. For the first 15 iterates of ϕ, *contribution to the Lefschetz numbers from the various orbits, for the polynomial* $P_1(-X)$ *from Table 5.1 on stratum* $(2, 10)$. *See the caption to Table 3.2 for details.*

Again, we start with $\delta_4^+ \leqslant \rho(X^8 - X^7 + X^6 - X^5 - X^4 - X^3 + X^2 - X + 1) \simeq 1.34372$ (for instance see [12] or [17]) and search for candidate polynomials with smaller dilatation (see Appendix A), shown in Table 5.1. Seeking a contradiction, we instead immediately find that $P_1(-X)$ is an allowable polynomial on strata

$$(2, 10), \ (2, 2, 2, 2, 4), \ \text{and} \ (2, 2, 2, 6).$$

As an example we show the contributions to the Lefschetz numbers in Table 5.2 on stratum $(2, 10)$. Each singularity is fixed (as they must be since there is only one of each type), and their separatrices are first fixed by ϕ^{12} (degree 10) and ϕ^4 (degree 2), respectively. We can easily show that the Lefschetz numbers are consistent for arbitrary iterate. It turns out that we can construct a pseudo-Anosov homeomorphism having this dilatation.

5.2. Construction of a pseudo-Anosov homeomorphism by Dehn twists

We use the same approach as in Section 3.3 to find the candidate word.

PROPOSITION 5.1. — *There exists a pseudo-Anosov homeomorphism on a genus* 4 *surface, stabilizing orientable foliations, and having for dilatation the Perron root of the polynomial* $P_1(X)$.

Proof. — Let us consider the sequence of Dehn twists

$$T_{a_1}T_{b_1}T_{c_1}T_{a_2}T_{b_2}T_{c_2}T_{b_3}T_{c_3}T_{b_4}.$$

Its action on homology has $P_1(-X)$ as a characteristic polynomial. Since $P_1(X)$ is irreducible and has no roots that are also roots of unity [26], then by Theorem 2.5 the homeomorphism is isotopic to a pseudo-Anosov homeomorphism, say f.

We compute the dilatation of f by calculating the action on the fundamental group, which shows that the dilatation is the Perron root of the polynomial $P_1(X)$. Hence, f must also stabilize a pair of orientable foliations, and it realizes our systole δ_4^+. $\square$

6. Higher genus

6.1. Genus five: A proof of Theorem 1.2 for $g = 5$

This time there is a known candidate with a lower dilatation than Hironaka & Kin's [12]: Leininger's pseudo-Anosov homeomorphism [19] having Lehmer's number $\simeq 1.17628$ as a dilatation. This pseudo-Anosov homeomorphism has invariant foliations corresponding to stratum (16). (The Lefschetz numbers are also compatible with stratum $(4, 4, 4, 4)$.) The polynomial associated with its action on homology has $\rho(P) < 0$. An exhaustive search (see Appendix A) leads us to conclude that there is no allowable polynomial with a lower dilatation, so there is nothing else to check.

As we finished this paper we learned that Aaber & Dunfield [1] have found a pseudo-Anosov homeomorphism with dilatation lower than δ_5^+ (stabilizing a non-orientable foliation), implying that $\delta_5 < \delta_5^+$.

6.2. Genus six: A proof of Theorem 1.3 for $g = 6$ (computer-assisted)

For genus 6, we have demonstrated that the Lefschetz numbers associated with $P(-X)$, with P the polynomial in Table 1.2, are compatible with stratum $(16, 4)$, with Lehmer's number as a root (Lehmer's polynomial is a factor). (There is another polynomial with the same dilatation that is compatible with the stratum (20).) We have not yet constructed an explicit pseudo-Anosov homeomorphism with this dilatation for genus 6, so Theorem 1.3 is a weaker form than 1.2: it only asserts that δ_6^+ is not less than this dilatation. Note, however, that whether or not this pseudo-Anosov homeomorphism exists this is the first instance where the minimum dilatation is not lower than for smaller genus.

6.3. Genus seven: A proof of Theorem 1.3 for $g = 7$ (computer-assisted)

Again, we have not constructed the pseudo-Anosov homeomorphism explicitly, but the Lefschetz numbers for the polynomial $P(-X)$, with P as in Table 1.2, are compatible with stratum $(2, 2, 2, 2, 2, 14)$.

As we finished this paper we learned that Aaber & Dunfield [1] and Kin & Takasawa [15] have found a pseudo-Anosov homeomorphism with dilatation equal to the systole δ_7^+.

6.4. Genus eight: A proof of Theorem 1.3 for $g = 8$ (computer-assisted)

Genus eight is roughly the limit of this brute-force approach: it takes our computer program about five days to ensure that we have the minimizing polynomial. The bound described in Appendix A yields 5×10^{12} cases for the traces, most of which do not correspond to integer-coefficient polynomials.

Yet again, we have not constructed the pseudo-Anosov homeomorphism explicitly, but the Lefschetz numbers for the polynomial $P(-X)$, with P as in Table 1.2, are compatible with stratum $(6, 22)$.

As we finished this paper we learned that Hironaka [11] has found a pseudo-Anosov homeomorphism with dilatation equal to the systole δ_8^+.

Examining the cases with even g leads to a natural question:

QUESTION 6.1. — *Is the minimum value of the dilatation of pseudo-Anosov homeomorphisms on a genus g surface, for g even, with orientable invariant foliations, equal to the largest root of the polynomial $X^{2g} - X^{g+1} - X^g - X^{g-1} + 1$?*

Appendix A. Searching for polynomials with small Perron root

A.1. Newton's formulas

The crucial task in our proofs is to find all reciprocal polynomials with a largest real root bounded by a given value α (typically the candidate minimum dilatation). Moreover, these must be allowable polynomials for a pseudo-Anosov homeomorphism: the largest root (in absolute value) must be real and strictly larger than all other roots, and it must be outside the unit circle in the complex plane.

The simplest way to find all such polynomials is to bound the coefficients directly. For example, in genus 3, If we denote an arbitrary reciprocal polynomial by $P(X) = X^6 + aX^5 + bX^4 + cX^3 + bX^2 + aX + 1$, we want to find all polynomials with Perron root smaller than $\alpha = \rho(X^3 - X^2 - 1) \simeq 1.46557$ (the candidate minimum dilatation at the beginning of Section 3). Let $t = \alpha + \alpha^{-1}$; a straightforward calculation assuming that half the roots of $P(X)$ are equal to α shows

$$|a| \leqslant 3t, \qquad |b| \leqslant 3(t^2 + 1), \qquad |c| \leqslant t(t^2 + 6).$$

Plugging in numbers, this means $|a| \leqslant 6$, $|b| \leqslant 18$, and $|c| \leqslant 26$. Allowing for $X \to -X$ since we only care about the absolute value of the largest root, we have a total of $12,765$ cases to examine. Out of these, only two polynomials actually have a root small enough and satisfy the other constraints (reality, uniqueness of largest root), as given in Section 3.

The problem with this straightforward approach (also employed by Cho and Ham for genus 2, see [7]) is that it scales very poorly with increasing genus. For genus 4, the number of cases is $9,889,930$; for genus 5, we have $63,523,102,800$ cases (we use for α the dilatation of Hironaka & Kin's pseudo-Anosov homeomorphism [12], currently the best general upper bound on δ_g). As g increases, the target dilatation α decreases, which should limit the number of cases, but the quantity $t = \alpha + \alpha^{-1}$ converges to unity, and the bound depends only weakly on $\alpha - 1$.

An improved approach is to start from Newton's formulas relating the traces to the coefficients: for a polynomial $P(X) = X^n + a_1 X^{n-1} + a_2 X^{n-2} + \ldots + a_{n-1} X + a_n$ which is the characteristic polynomial of a matrix M, we have

$$\operatorname{Tr}(M^k) = \begin{cases} -k a_k - \sum_{m=1}^{k-1} a_m \operatorname{Tr}(M^{k-m}), & 1 \leqslant k \leqslant n; \\ -\sum_{m=1}^{n} a_m \operatorname{Tr}(M^{k-m}), & k > n. \end{cases}$$

For a reciprocal polynomial, we have $a_{n-k} = a_k$. We can use these formulas to solve for the a_k given the first few traces $\operatorname{Tr}(M^k)$, $1 \leqslant k \leqslant g$ ($g = n/2$, n is even in this paper). We also have

Lemma A.1. — *If the characteristic polynomial $P(X)$ of a matrix M has a largest eigenvalue with absolute value r, then*

$$|\operatorname{Tr}(M^k)| \leqslant n\, r^k;$$

Furthermore, if $P(X)$ is reciprocal and of even degree, then

$$|\operatorname{Tr}(M^k)| \leqslant \tfrac{1}{2} n (r^k + r^{-k}).$$

Proof. — Obviously,

$$|\operatorname{Tr}(M^k)| = \left| \sum_{m=1}^{n} s_m^k \right| \leqslant \sum_{m=1}^{n} |s_m|^k \leqslant n\, r^k$$

where s_k are the eigenvalues of M. If the polynomial is reciprocal and n is even, then

$$|\operatorname{Tr}(M^k)| = \left| \sum_{m=1}^{n/2} (s_m^k + s_m^{-k}) \right| \leqslant \tfrac{1}{2} n (r^k + r^{-k}).$$

□

We now have the following prescription for enumerating allowable polynomials, given n and a largest root α:

(1) Use Lemma A.1 to bound the traces $\operatorname{Tr}(M^k) \in \mathbb{Z}$, $k = 1, \ldots, n/2$;

(2) For each possible set of $n/2$ traces, solve for the coefficients of the polynomial;
(3) If these coefficients are not all integers, move on to the next possible set of traces;
(4) If the coefficients are integers, check if the polynomial is allowable: largest eigenvalue real and with absolute value less than α, outside the unit circle, and nondegenerate;
(5) Repeat step 2 until we run out of possible values for the traces.

Let's compare with the earlier numbers for $g = 5$: assuming $\mathrm{Tr}(M) \geqslant 0$, we have $7,254,775$ cases to try, which is already a factor of 10^4 fewer than with the coefficient bound. Moreover, of these $7,194,541$ lead to fractional coefficients, and so are discarded in step 3 above. This only leaves $60,234$ cases, roughly a factor of 10^6 fewer than with the coefficient bound. Hence, with this simple approach we can tackle polynomials up to degree 16 ($g = 8$). More refined approaches will certainly allow higher degrees to be reached.

A final note on the numerical technique: we use Newton's iterative method to check the dominant root of candidate polynomials. A nice feature of polynomials with a dominant real root is that their graph is strictly convex upwards for x greater than the root (when that root is positive, otherwise for x less than the root). Hence, Newton's method is guaranteed to converge rapidly and uniquely for appropriate initial guess (typically, 5 iterates is enough for about 6 significant figures). If the method does *not* converge quickly, then the polynomial is ruled out.

A.2. Mahler measures

Another approach is to use the Mahler measure of a polynomial. If P is a degree $2g$ monic polynomial that admits a Perron root, say α, then the Mahler measure of P satisfies $M(P) \leqslant \alpha^g$. Thus to list all possible polynomials with a Perron root less than a constant α, we just have to list all possible polynomials with a Mahler measure less than α^g. Such lists already exist in the literature (for example in [4]).

Appendix B. Rauzy–Veech induction and pseudo-Anosov homeomorphisms

In this section we recall very briefly the basic construction of pseudo-Anosov homeomorphisms using the Rauzy–Veech induction (for details see [29], §8, and [27, 20]). We will use this to construct the minimizing pseudo-Anosov homeomorphisms in genus 3 and 4.

B.1. Interval exchange map

Let $I \subset \mathbb{R}$ be an open interval and let us choose a finite partition of I into $d \geqslant 2$ open subintervals $\{I_j,\ j = 1, \dots, d\}$. An interval exchange map is a one-to-one map T from I to itself that permutes, by translation, the subintervals I_j. It is easy to see that T is precisely determined by a permutation π that encodes how the intervals are exchanged, and a vector $\lambda = \{\lambda_j\}_{j=1,\dots,d}$ with positive entries that encodes the lengths of the intervals.

B.2. Suspension data

A suspension datum for T is a collection of vectors $\{\zeta_j\}_{j=1,\dots,d}$ such that

(1) $\forall j \in \{1, \dots, d\},\ Re(\zeta_j) = \lambda_j$;
(2) $\forall k,\ 1 \leqslant k \leqslant d-1,\ Im(\sum_{j=1}^{k} \zeta_j) > 0$;
(3) $\forall k,\ 1 \leqslant k \leqslant d-1,\ Im(\sum_{j=1}^{k} \zeta_{\pi^{-1}(j)}) < 0$.

To each suspension datum ζ, we can associate a translation surface $(M, q) = M(\pi, \zeta)$ in the following way. Consider the broken line L_0 on $\mathbb{C} = \mathbb{R}^2$ defined by concatenation of the vectors ζ_j (in this order) for $j = 1, \dots, d$ with starting point at the origin (see Figure B.1). Similarly, we consider the broken line L_1 defined by concatenation of the vectors $\zeta_{\pi^{-1}(j)}$ (in this order) for $j = 1, \dots, d$ with starting point at the origin. If the lines L_0 and L_1 have no intersections other than the endpoints, we can construct a translation surface S by identifying each side ζ_j on L_0 with the side ζ_j on L_1 by a translation. The resulting surface is a translation surface endowed with the form dz^2.

Let $I \subset M$ be the horizontal interval defined by $I = (0, \sum_{j=1}^{d} \lambda_j) \times \{0\}$. Then the interval exchange map T is precisely the one defined by the first return map to I of the vertical flow on M.

B.3. Rauzy–Veech induction

The Rauzy–Veech induction $\mathcal{R}(T)$ of T is defined as the first return map of T to a certain subinterval J of I (see [27, 20] for details).

We recall very briefly the construction. The *type* ε of T is defined by 0 if $\lambda_d > \lambda_{\pi^{-1}(d)}$ and 1 otherwise. We define a subinterval J of I by

$$J = \begin{cases} I \backslash T(I_{\pi^{-1}(d)}) & \text{if } T \text{ is of type } 0; \\ I \backslash I_d & \text{if } T \text{ is of type } 1. \end{cases}$$

The Rauzy–Veech induction $\mathcal{R}(T)$ of T is defined as the first return map of T to the subinterval J. This is again an interval exchange transformation, defined on d letters (see *e.g.*, [27]). Moreover, we can compute the data of the new map (permutation and length vector) by a combinatorial map and a matrix. We can also define the Rauzy–Veech induction on the space of suspensions. For a permutation π, we call the *Rauzy class* the graph of all permutations that we can obtain by the Rauzy–Veech induction. Each vertex of this graph corresponds to a permutation, and from each permutation there are two edges labelled 0 and 1 (the type). To each edge, one can associate a transition matrix that gives the corresponding vector of lengths.

B.4. Closed loops and pseudo-Anosov homeomorphisms

We now recall a theorem of Veech:

Theorem (Veech). — *Let γ be a closed loop, based at π, in a Rauzy class and $R = R(\gamma)$ be the product of the associated transition matrices. Let us assume that R is irreducible. Let λ be an eigenvector for the Perron eigenvalue α of R and τ be an eigenvector for the eigenvalue $\frac{1}{\alpha}$ of R. Then*

(1) *$\zeta = (\lambda, \tau)$ is a suspension data for $T = (\pi, \lambda)$;*

(2) *The matrix* $A = \left(\begin{smallmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{smallmatrix}\right)$ *is the derivative map of an affine pseudo-Anosov diffeomorphism* ϕ *on the suspension* $M(\pi,\zeta)$ *over* (π,λ)*;*
(3) *The dilatation of* ϕ *is* α*;*
(4) *All pseudo-Anosov homeomorphisms that fix a separatrix are constructed in this way.*

Since genus 4 is simpler to construct than genus 3, we present the genus 4 case first in detail, and briefly outline the construction of the other case.

B.5. Construction of an example for $g = 4$

We shall prove

Theorem B.1. — *There exists a pseudo-Anosov homeomorphism on a genus four surface, stabilizing orientable measured foliations, and having for dilatation the maximal real root of the polynomial* $X^8 - X^5 - X^4 - X^3 + 1$ *(namely* 1.28064...*).*

B.5.1. Construction of the translation surface for $g = 4$

Let $|\alpha| > 1$ be the maximal real root of the polynomial $P_1(X) = X^8 - X^5 - X^4 - X^3 + 1$ with $\alpha < -1$, so that $\alpha^8 + \alpha^5 - \alpha^4 + \alpha^3 + 1 = 0$. In the following, we will present elements of $\mathbb{Q}[\alpha]$ in the basis $\{\alpha^i\}_{i=0,\ldots,7}$. Thus the octuplet $(a_0, \ldots, a_7)$ stands for $\sum_{i=0}^{7} a_i \alpha^i$.

We start with the permutation $\pi = (5,3,9,8,6,2,7,1,4)$ and the closed Rauzy path

$$0 - 1 - 0 - 0 - 1 - 1 - 1 - 0 - 1 - 0 - 0 - 1 - 0 - 0.$$

The associated Rauzy–Veech matrix is

$$R = \begin{pmatrix} 1&1&0&0&0&0&0&0&0 \\ 0&0&1&1&1&1&1&1&0 \\ 0&0&0&0&0&1&0&1&1 \\ 1&0&0&1&0&0&1&0&0 \\ 0&0&1&0&1&0&0&0&0 \\ 0&0&0&1&1&0&0&0&0 \\ 0&0&0&0&0&1&1&0&0 \\ 1&0&0&0&0&1&1&1&0 \\ 1&1&0&1&0&0&1&0&0 \end{pmatrix}.$$

One checks that the characteristic polynomial of R is $Q(X)$ with the property that $Q(X)$ factors into $Q(X^4) = P_1(-X)S(X)$, where $S(X)$

is a polynomial. Let λ and τ be the corresponding eigenvectors for the Perron root α^4 of Q, expressed in the α-basis:

$$\begin{array}{ll}
\lambda_1 = (0,1,-2,1,-1,0,1,-1) & \tau_1 = (-1,0,0,0,0,-1,0,0)\\
\lambda_2 = (0,-1,1,0,1,0,-1,0) & \tau_2 = (0,0,-1,1,0,1,0,-1)\\
\lambda_3 = (-1,0,-1,0,0,-1,0,0) & \tau_3 = (0,0,-1,0,-1,0,0,-1)\\
\lambda_4 = (-1,2,-1,1,0,-1,1,0) & \tau_4 = (0,1,0,0,0,0,1,0)\\
\lambda_5 = (1,-1,1,0,0,1,0,0) & \tau_5 = (0,0,0,1,0,0,0,0)\\
\lambda_6 = (-1,1,-1,1,-1,-1,0,-1) & \tau_6 = (0,0,-1,0,0,0,1,0)\\
\lambda_7 = (1,-2,2,-2,1,1,-1,1) & \tau_7 = (0,0,0,0,0,0,-1,0)\\
\lambda_8 = (0,0,1,-1,1,0,0,1) & \tau_8 = (0,1,0,0,0,0,0,0)\\
\lambda_9 = (1,0,0,0,0,0,0,0) & \tau_9 = (-1,0,0,0,0,0,0,0).
\end{array}$$

For $i = 1,\dots,9$ we construct the vectors in $\mathbb{R}^2$ $\zeta_i = \left(\begin{smallmatrix}\lambda_i\\ \tau_i\end{smallmatrix}\right)$. The resulting surface $(M,q) = M(\pi,\zeta)$ is drawn in Figure B.1.

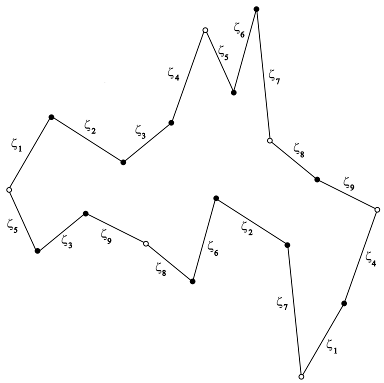


Figure B.1. Construction of (M,q). There are two singularities for the metric: one with conical angle 4π (hollow circles) and one with conical angle 12π (filled circles). The stratum is thus $(2,10)$.

B.5.2. Coordinates of the translation surface

By construction, the coordinates of (M,q) belong to $\mathbb{Q}[\alpha]$. We denote the vertices by p_i for $i = 1,\dots,18$ with $\mathrm{p}_1 = 0$ (see Figure B.2). Obviously for $i \leqslant 9$, $\mathrm{p}_i = \sum_{j=1}^{i} \zeta_j$, and for $i \geqslant 10$, $\mathrm{p}_i = \sum_{j=1}^{9} \zeta_j - \sum_{j=1}^{i-9} \zeta_{\pi^{-1}(j)}$. A direct calculation gives

$$
\begin{array}{ll}
\mathrm{p}_1 = ((0,0,0,0,0,0,0,0), & (0,0,0,0,0,0,0,0)) \\
\mathrm{p}_2 = ((0,1,-2,1,-1,0,1,-1), & (-1,0,0,0,0,-1,0,0)) \\
\mathrm{p}_3 = ((0,0,-1,1,0,0,0,-1), & (-1,0,-1,1,0,0,0,-1)) \\
\mathrm{p}_4 = ((-1,0,-2,1,0,-1,0,-1), & (-1,0,-2,1,-1,0,0,-2)) \\
\mathrm{p}_5 = ((-2,2,-3,2,0,-2,1,-1), & (-1,1,-2,1,-1,0,1,-2)) \\
\mathrm{p}_6 = ((-1,1,-2,2,0,-1,1,-1), & (-1,1,-2,2,-1,0,1,-2)) \\
\mathrm{p}_7 = ((-2,2,-3,3,-1,-2,1,-2), & (-1,1,-3,2,-1,0,2,-2)) \\
\mathrm{p}_8 = ((-1,0,-1,1,0,-1,0,-1), & (-1,1,-3,2,-1,0,1,-2)) \\
\mathrm{p}_9 = ((-1,0,0,0,1,-1,0,0), & (-1,2,-3,2,-1,0,1,-2)) \\
\mathrm{p}_{10} = ((0,0,0,0,1,-1,0,0), & (-2,2,-3,2,-1,0,1,-2)) \\
\mathrm{p}_{11} = ((1,-2,1,-1,1,0,-1,0), & (-2,1,-3,2,-1,0,0,-2)) \\
\mathrm{p}_{12} = ((1,-3,3,-2,2,0,-2,1), & (-1,1,-3,2,-1,1,0,-2)) \\
\mathrm{p}_{13} = ((0,-1,1,0,1,-1,-1,0), & (-1,1,-3,2,-1,1,1,-2)) \\
\mathrm{p}_{14} = ((0,0,0,0,0,-1,0,0), & (-1,1,-2,1,-1,0,1,-1)) \\
\mathrm{p}_{15} = ((1,-1,1,-1,1,0,0,1), & (-1,1,-1,1,-1,0,0,-1)) \\
\mathrm{p}_{16} = ((1,-1,0,0,0,0,0,0), & (-1,0,-1,1,-1,0,0,-1)) \\
\mathrm{p}_{17} = ((0,-1,0,0,0,0,0,0), & (0,0,-1,1,-1,0,0,-1)) \\
\mathrm{p}_{18} = ((1,-1,1,0,0,1,0,0), & (0,0,0,1,0,0,0,0))
\end{array}
$$

B.5.3. Construction of the pseudo-Anosov diffeomorphism

Let A be the hyperbolic matrix $\left(\begin{smallmatrix}\alpha^{-1} & 0\\ 0 & \alpha\end{smallmatrix}\right)$. Of course by construction A^4 stabilizes the translation surface (M,q) and hence there exists a pseudo-Anosov homeomorphism on M with dilatation α^4. We shall prove that this homeomorphism admits a root.

Let (M',q') be the image of (M,q) by the matrix A. We only need to prove that (M',q') and (M,q) defines the same translation surface, *i.e.*, one can cut and glue (M',q') in order to recover (M,q). This is

Theorem B.2. — *The surfaces (M',q') and (M,q) are isometric.*

Corollary B.3. — *There exists a pseudo-Anosov diffeomorphism $f : X \to X$ such that $Df = A$. In particular the dilatation of f is $|\alpha|$.*

Proof of Theorem B.2. — Using the two relations $\alpha^8 = -1 - \alpha^3 + \alpha^4 - \alpha^5$ and $\alpha^{-1} = \alpha^2 - \alpha^3 + \alpha^4 + \alpha^7$ and the relations that give the p_i, one gets by a straightforward calculation the coordinates $\mathrm{p'}_i = A\,\mathrm{p}_i$ of the surface (M', q'):

$$
\begin{array}{ll}
\mathrm{p'}_1 = ((0,0,0,0,0,0,0,0), & (0,0,0,0,0,0,0,0)) \\
\mathrm{p'}_2 = ((1,-2,1,-1,0,1,-1,0), & (0,-1,0,0,0,0,-1,0)) \\
\mathrm{p'}_3 = ((0,-1,1,0,0,0,-1,0), & (1,-1,0,0,0,1,0,0)) \\
\mathrm{p'}_4 = ((0,-2,2,-1,0,0,-1,1), & (2,-1,0,0,-1,1,0,0)) \\
\mathrm{p'}_5 = ((2,-3,4,-2,0,1,-1,2), & (2,-1,1,0,-1,1,0,1)) \\
\mathrm{p'}_6 = ((1,-2,3,-1,0,1,-1,1), & (2,-1,1,0,0,1,0,1)) \\
\mathrm{p'}_7 = ((2,-3,5,-3,0,1,-2,2), & (2,-1,1,-1,0,1,0,2)) \\
\mathrm{p'}_8 = ((0,-1,2,-1,0,0,-1,1), & (2,-1,1,-1,0,1,0,1)) \\
\mathrm{p'}_9 = ((0,0,1,0,0,0,0,1), & (2,-1,2,-1,0,1,0,1)) \\
\mathrm{p'}_{10} = ((0,0,0,1,-1,0,0,0), & (2,-2,2,-1,0,1,0,1)) \\
\mathrm{p'}_{11} = ((-2,1,-2,2,-1,-1,0,-1), & (2,-2,1,-1,0,1,0,0)) \\
\mathrm{p'}_{12} = ((-3,3,-3,3,-1,-2,1,-1), & (2,-1,1,-1,0,1,1,0)) \\
\mathrm{p'}_{13} = ((-1,1,0,1,-1,-1,0,0), & (2,-1,1,-1,0,1,1,1)) \\
\mathrm{p'}_{14} = ((0,0,0,0,-1,0,0,0), & (1,-1,1,-1,0,0,0,1)) \\
\mathrm{p'}_{15} = ((-1,1,-2,2,-1,0,1,-1), & (1,-1,1,0,0,0,0,0)) \\
\mathrm{p'}_{16} = ((-1,0,-1,1,-1,0,0,-1), & (1,-1,0,0,0,0,0,0)) \\
\mathrm{p'}_{17} = ((-1,0,0,0,0,0,0,0), & (1,0,0,0,0,0,0,0)) \\
\mathrm{p'}_{18} = ((-1,1,-1,1,0,0,0,-1), & (0,0,0,0,1,0,0,0))
\end{array}
$$

We will cut M into several pieces in order to recover M' such that the boundary gluings agree. Consider the decomposition in Figure B.2.

We enumerate the pieces on M from the left to the right. For instance, the first piece on M has coordinates $\mathrm{p}_1\,\mathrm{p}_2\,\mathrm{p}_3\,\mathrm{p}_{17}\,\mathrm{p}_{18}$. The corresponding piece on M' has coordinates $\mathrm{p'}_3\,\mathrm{p'}_4\,\mathrm{p'}_6\,\mathrm{p'}_8\,\mathrm{p'}_{14}$. The translation is $\overrightarrow{\mathrm{p'}_8\,\mathrm{p}_1} = \overrightarrow{\mathrm{p'}_{14}\,\mathrm{p}_2} = \overrightarrow{\mathrm{p'}_3\,\mathrm{p}_3} = \overrightarrow{\mathrm{p'}_4\,\mathrm{p}_{18}}$. □

piece #	coordinates on M	coordinates on M'	translation vectors
1	$\mathrm{p}_1\,\mathrm{p}_2\,\mathrm{p}_3\,\mathrm{p}_{17}\,\mathrm{p}_{18}$	$\mathrm{p'}_3\,\mathrm{p'}_4\,\mathrm{p'}_6\,\mathrm{p'}_8\,\mathrm{p'}_{14}$	$\overrightarrow{\mathrm{p'}_8\,\mathrm{p}_1} = \overrightarrow{\mathrm{p'}_{14}\,\mathrm{p}_2} = \overrightarrow{\mathrm{p'}_3\,\mathrm{p}_3} = \overrightarrow{\mathrm{p'}_4\,\mathrm{p}_{18}}$
2	$\mathrm{p}_3\,\mathrm{p}_{16}\,\mathrm{p}_{17}$	$\mathrm{p'}_{18}\,\mathrm{p'}_1\,\mathrm{p'}_{17}$	$\overrightarrow{\mathrm{p'}_{18}\,\mathrm{p}_3} = \overrightarrow{\mathrm{p'}_1\,\mathrm{p}_{16}} = \overrightarrow{\mathrm{p'}_{17}\,\mathrm{p}_2}$
3	$\mathrm{p}_3\,\mathrm{p}_4\,\mathrm{p}_{16}$	$\mathrm{p'}_6\,\mathrm{p'}_4\,\mathrm{p'}_5$	$\overrightarrow{\mathrm{p'}_6\,\mathrm{p}_3} = \overrightarrow{\mathrm{p'}_4\,\mathrm{p}_4} = \overrightarrow{\mathrm{p'}_5\,\mathrm{p}_{16}}$
4	$\mathrm{p}_4\,\mathrm{p}_5\,\mathrm{p}_{14}\,\mathrm{p}_{15}\,\mathrm{p}_{16}$	$\mathrm{p'}_{11}\,\mathrm{p'}_{12}\,\mathrm{p'}_{13}\,\mathrm{p'}_9\,\mathrm{p'}_{10}$	$\overrightarrow{\mathrm{p'}_{11}\,\mathrm{p}_4} = \overrightarrow{\mathrm{p'}_{12}\,\mathrm{p}_5} = \cdots = \overrightarrow{\mathrm{p'}_{10}\,\mathrm{p}_{16}}$
5	$\mathrm{p}_5\,\mathrm{p}_6\,\mathrm{p}_{14}$	$\mathrm{p'}_8\,\mathrm{p'}_6\,\mathrm{p'}_7$	$\overrightarrow{\mathrm{p'}_8\,\mathrm{p}_5} = \overrightarrow{\mathrm{p'}_6\,\mathrm{p}_6} = \overrightarrow{\mathrm{p'}_7\,\mathrm{p}_{14}}$
6	$\mathrm{p}_6\,\mathrm{p}_8\,\mathrm{p}_9\,\mathrm{p}_{10}\,\mathrm{p}_{11}\,\mathrm{p}_{13}\,\mathrm{p}_{14}$	$\mathrm{p'}_{15}\,\mathrm{p'}_{16}\,\mathrm{p'}_{17}\,\mathrm{p'}_1\,\mathrm{p'}_2\,\mathrm{p'}_3\,\mathrm{p'}_{14}$	$\overrightarrow{\mathrm{p'}_{15}\,\mathrm{p}_6} = \overrightarrow{\mathrm{p'}_{16}\,\mathrm{p}_8} = \cdots = \overrightarrow{\mathrm{p'}_{14}\,\mathrm{p}_{14}}$
7	$\mathrm{p}_6\,\mathrm{p}_7\,\mathrm{p}_8$	$\mathrm{p'}_8\,\mathrm{p'}_9\,\mathrm{p'}_{13}$	$\overrightarrow{\mathrm{p}_6\,\mathrm{p'}_9} = \overrightarrow{\mathrm{p}_7\,\mathrm{p'}_{13}} = \overrightarrow{\mathrm{p}_8\,\mathrm{p'}_8}$
8	$\mathrm{p}_{11}\,\mathrm{p}_{12}\,\mathrm{p}_{13}$	$\mathrm{p'}_{14}\,\mathrm{p'}_8\,\mathrm{p'}_{13}$	$\overrightarrow{\mathrm{p'}_{14}\,\mathrm{p}_{11}} = \overrightarrow{\mathrm{p'}_8\,\mathrm{p}_{12}} = \overrightarrow{\mathrm{p'}_{13}\,\mathrm{p}_{13}}$

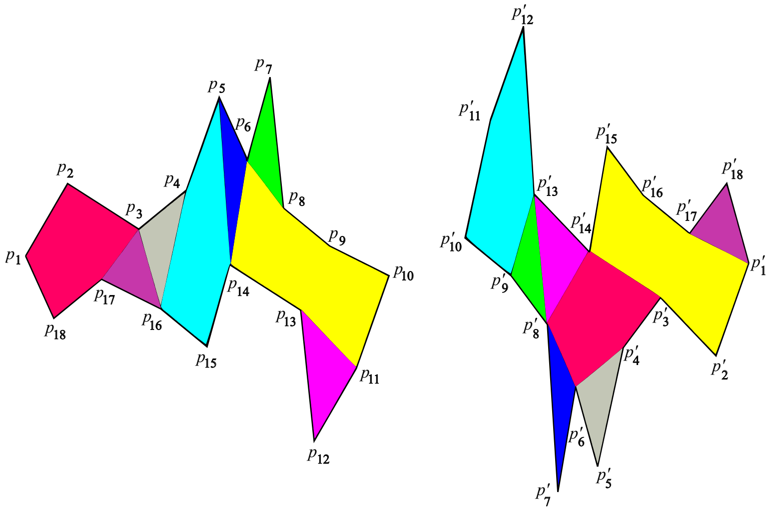


Figure B.2. Partition of (M,q) *and* $(M',q') = A(M,q)$.

B.6. Construction of an example for $g = 3$

We shall prove

THEOREM B.4. — *There exists a pseudo-Anosov homeomorphism on a genus three surface, stabilizing orientable measured foliations, and having for dilatation the maximal real root of the polynomial* $X^6 - X^4 - X^3 - X^2 + 1$ *(namely* 1.40127...*).*

Proof. — Let $|\alpha| > 1$ be the maximal real root of the polynomial $P_2(X) = X^6 - X^4 - X^3 - X^2 + 1$ with $\alpha < -1$, so that $\alpha^6 - \alpha^4 + \alpha^3 - \alpha^2 + 1 = 0$. We start with the permutation $\pi = (6,3,8,2,7,4,10,9,5,1)$ and the closed Rauzy path

$$1 - 1 - 1 - 0 - 0 - 1 - 0 - 1 - 0 - 0.$$

The associated Rauzy–Veech matrix is

$$R = \begin{pmatrix} 1&1&1&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&1&1\\ 0&0&1&0&0&1&0&0&0&0\\ 1&0&0&1&1&0&0&0&1&1\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0 \end{pmatrix}.$$

The associated translation surface and its image are presented in Figure B.3. □

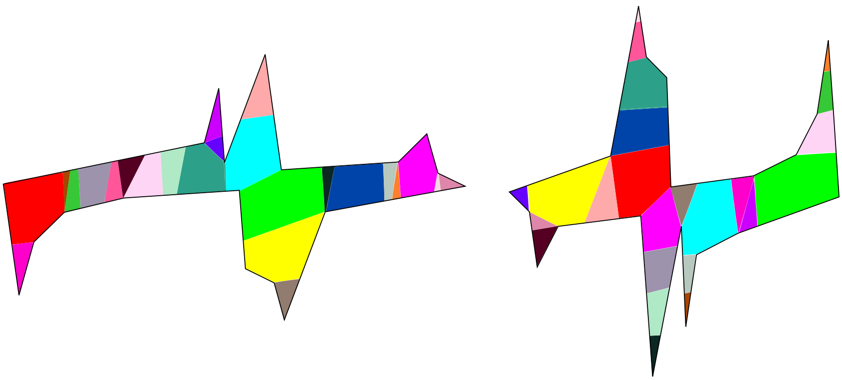


Figure B.3. Partition of (M,q) *and* $(M',q') = A(M,q)$.

Appendix C. Genus two

Let us consider the two sequences of Dehn twists on a genus two surface,

$$T_{a_1}^2 T_{c_1} T_{b_2} T_{a_2}^{-1} T_{b_1} \quad \text{and} \quad T_{a_1}^2 T_{b_2}^{-1} T_{c_1}^{-1} T_{a_2}^{-1} T_{b_1}.$$

Their actions on the first homology group are respectively $\begin{pmatrix} 1&-3&0&1\\ 1&-2&0&1\\ 0&2&2&-1\\ 0&1&1&0 \end{pmatrix}$ and $\begin{pmatrix} 1&-1&1&-1\\ 1&0&1&-1\\ 0&-1&-1&2\\ 0&0&-1&1 \end{pmatrix}$. The characteristic polynomials of these matrices are, respectively, $\mathrm{X}^4 - \mathrm{X}^3 - \mathrm{X}^2 - \mathrm{X} + 1$ and $\mathrm{X}^4 - \mathrm{X}^3 + 3\,\mathrm{X}^2 - \mathrm{X} + 1$; thus Theorem 2.5 implies that the isotopy classes of these homeomorphisms are pseudo-Anosov. Let ϕ_1 and ϕ_2 be the corresponding maps. One can calculate their dilatations from their action on the fundamental group [9]. We check that the dilatations, λ, are the same, namely the Perron root of the polynomial $X^4 - X^3 - X^2 - X + 1$ ($\lambda \simeq 1.72208$).

Theorem 2.4 thus implies that ϕ_1 fixes an orientable measured foliation, and hence $\delta_2^+ = \lambda(\phi_1)$ and ϕ_2 fixes a non-orientable measured foliation. We conclude that $\delta_2^- = \lambda(\phi_2)$.

These two homeomorphisms are related by covering transformations (see Remark 4.1).

BIBLIOGRAPHY

[1] J. W. Aaber & N. M. Dunfield, "Closed surface bundles of least volume", 2010, arXiv:1002.3423.

[2] P. Arnoux & J.-C. Yoccoz, "Construction de difféomorphismes pseudo-Anosov", *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), no. 1, p. 75-78.

[3] G. Band & P. Boyland, "The Burau estimate for the entropy of a braid", *Algebr. Geom. Topol.* **7** (2007), p. 1345-1378.

[4] D. W. Boyd, "Reciprocal polynomials having small measure", *Math. Comp.* **35** (1980), no. 152, p. 1361-1377.

[5] R. F. Brown, *The Lefschetz fixed point theorem*, Scott, Foresman and Co., Glenview, Ill.-London, 1971, vi+186 pages.

[6] A. J. Casson & S. A. Bleiler, *Automorphisms of surfaces after Nielsen and Thurston*, London Mathematical Society Student Texts, vol. 9, Cambridge University Press, Cambridge, 1988, iv+105 pages.

[7] J.-H. Cho & J.-Y. Ham, "The minimal dilatation of a genus-two surface", *Experiment. Math.* **17** (2008), no. 3, p. 257-267.

[8] B. Farb, "Some problems on mapping class groups and moduli space", in *Problems on mapping class groups and related topics*, Proc. Sympos. Pure Math., vol. 74, Amer. Math. Soc., Providence, RI, 2006, p. 11-55.

[9] A. Fathi, F. Laudenbach & V. Poénaru, "Travaux de Thurston sur les surfaces", in *Astérisque*, vol. 66–67, Société Mathématique de France, 1979.

[10] M. D. Finn, J.-L. Thiffeault & N. Jewell, "Topological entropy of braids on arbitrary surfaces", 2010, preprint.

[11] E. Hironaka, "Small dilatation pseudo-Anosov mapping classes coming from the simplest hyperbolic braid", 2009, arXiv:0909.4517.

[12] E. Hironaka & E. Kin, "A family of pseudo-Anosov braids with small dilatation", *Algebr. Geom. Topol.* **6** (2006), p. 699-738 (electronic).

[13] N. V. Ivanov, "Coefficients of expansion of pseudo-Anosov homeomorphisms", *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **167** (1988), no. Issled. Topol. 6, p. 111-116, 191, translation in J. Soviet Math., 52, (1990), pp. 2819–2822.

[14] R. Kenyon & J. Smillie, "Billiards in rational-angled triangles", *Comment. Math. Helv.* **75** (2000), p. 65-108.

[15] E. Kin & M. Takasawa, "Pseudo-Anosovs on closed surfaces having small entropy and the Whitehead sister link exterior", 2010, arXiv:1003.0545.

[16] E. Lanneau, "Hyperelliptic components of the moduli spaces of quadratic differentials with prescribed singularities", *Comment. Math. Helv.* **79** (2004), no. 3, p. 471-501.

[17] E. Lanneau & J.-L. Thiffeault, "Enumerating Pseudo-Anosov Homeomorphisms of the Punctured Disc", 2010, preprint, arXiv:1004.5344.

[18] F. Le Roux, "Homéomorphismes de surfaces: théorèmes de la fleur de Leau-Fatou et de la variété stable", *Astérisque* (2004), no. 292, p. vi+210.

[19] C. J. Leininger, "On groups generated by two positive multi-twists: Teichmüller curves and Lehmer's number", *Geom. Topol.* **8** (2004), p. 1301-1359 (electronic).

[20] S. Marmi, P. Moussa & J.-C. Yoccoz, "The cohomological equation for Roth-type interval exchange maps", *J. Amer. Math. Soc.* **18** (2005), no. 4, p. 823-872 (electronic).

[21] H. Masur & J. Smillie, "Quadratic differentials with prescribed singularities and pseudo-Anosov diffeomorphisms", *Comment. Math. Helv.* **68** (1993), no. 2, p. 289-307.

[22] H. Masur & S. Tabachnikov, "Rational billiards and flat structures", in *Handbook of dynamical systems, Vol. 1A*, North-Holland, Amsterdam, 2002, p. 1015-1089.

[23] C. T. McMullen, "Teichmüller curves in genus two: discriminant and spin", *Math. Ann.* **333** (2005), no. 1, p. 87-130.

[24] J.-O. Moussafir, "On the Entropy of Braids", *Func. Anal. and Other Math.* **1** (2006), p. 43-54.

[25] R. C. Penner, "Bounds on least dilatations", *Proc. Amer. Math. Soc.* **113** (1991), no. 2, p. 443-450.

[26] C. Pisot & R. Salem, "Distribution modulo 1 of the powers of real numbers larger than 1", *Compositio Math.* **16** (1964), p. 164-168 (1964).

[27] G. Rauzy, "Échanges d'intervalles et transformations induites", *Acta Arith.* **34** (1979), no. 4, p. 315-328.

[28] W. P. Thurston, "On the geometry and dynamics of diffeomorphisms of surfaces", *Bull. Amer. Math. Soc. (N.S.)* **19** (1988), no. 2, p. 417-431.

[29] W. A. Veech, "Gauss measures for transformations on the space of interval exchange maps", *Ann. of Math. (2)* **115** (1982), no. 1, p. 201-242.

[30] A. Y. Zhirov, "On the minimum dilation of pseudo-Anosov diffeomorphisms of a double torus", *Uspekhi Mat. Nauk* **50** (1995), no. 1(301), p. 197-198.

Manuscrit reçu le 21 mai 2009,
révisé le 21 janvier 2010,
accepté le 5 mars 2010.

Erwan LANNEAU
Université du Sud Toulon-Var
and Fédération de Recherches
des Unités de Mathématiques de Marseille
Centre de Physique Théorique (CPT)
UMR CNRS 6207,Luminy, Case 907
13288 Marseille Cedex 9 (France)
erwan.lanneau@cpt.univ-mrs.fr

Jean-Luc THIFFEAULT
University of Wisconsin
Department of Mathematics
Van Vleck Hall, 480 Lincoln Drive
Madison, WI 53706 (USA)
jeanluc@math.wisc.edu