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CONFORMALLY BENDING THREE-MANIFOLDS WITH BOUNDARY

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Abstract. — Given a three-dimensional manifold with boundary, the Cartan-Hadamard theorem implies that there are obstructions to filling the interior of the manifold with a complete metric of negative curvature. In this paper, we show that any three-dimensional manifold with boundary can be filled conformally with a complete metric satisfying a pinching condition: given any small constant, the ratio of the largest sectional curvature to (the absolute value of) the scalar curvature is less than this constant. This condition roughly means that the curvature is “almost negative”, in a scale-invariant sense.

Résumé. — Soit M une variété à bord de dimension trois, le théorème de Cartan-Hadamard implique qu'il existe des obstacles à remplir l'intérieur d'une variété avec une métrique complète de courbure négative. Dans cet article, nous montrons que toute variété à bord de dimension trois peut être remplie conformément avec une métrique complète satisfaisant une condition de pincement : on suppose que le rapport entre la plus grande courbure sectionnelle et la valeur absolue de la courbure scalaire est bornée par une constante (petite). Cette condition signifie que la courbure est “presque négative” dans un sens invariant d'échelle.

1. Introduction

In [8], Lohkamp proved the following:

Theorem 1.1. — *Let $M^n, n \geqslant 3$, be an open manifold, and g an arbitrary metric on M^n. Then there is a complete conformal metric $\hat{g} = e^{2u}g$ on M^n with negative Ricci curvature.*

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Subsequently, the authors of this article proved the following, which can be viewed as a refinement of Lohkamp's result (see [5]):

Theorem 1.2. — *Let $(M^n, \partial M^n, g)$ be a compact Riemannian manifold with boundary, and $n \geqslant 3$. Then there exists a unique, complete conformal metric $\hat{g} = e^{2u} g$ with $u \in C^\infty$ in the interior of M^n, such that*

(i) *The Ricci curvature $Ric(\hat{g}) < 0$,*

(ii) $\det[-Ric(\hat{g})]^{1/n} = (n-1)$,

(iii) $\lim_{x \to \partial M^n} e^{2u(x)} d(x)^2 = (n-1)^{-1}$, *where $d(x)$ is the distance to ∂M^n.*

Remark 1.3. — Prescribing the determinant of the Ricci curvature to be constant can be viewed as a way of "uniformizing" the open cone of conformal metrics with negative Ricci curvature. In addition, this condition implies that the conformal factor u in the statement of the theorem can be realized as the solution of a fully nonlinear PDE which determines the asymptotic behavior near the boundary.

Remark 1.4. — Theorem 1.2 can also be viewed as a generalization of the well known work of Loewner-Nirenberg [7] and Aviles-McOwen [1] on the singular Yamabe problem. In fact, this is a special case of a more general result: one can prescribe any of the elementary symmetric functions to have constant value, with the eigenvalues of the Ricci tensor in the appropriate cone of ellipticity. For the first symmetric function, i.e., the trace, this reduces to the scalar curvature.

Remark 1.5. — As in [7], the complete metric in the statement of the theorem is constructed by a limiting process; one begins by solving the Dirichlet problem with arbitrary conformal data on the boundary, then lets the boundary data go to infinity. In particular, we proved that a compact manifold with boundary can be conformally deformed to one with negative Ricci curvature, while leaving the boundary metric fixed.

Remark 1.6. — The properties (ii) and (iii) are identical to those satisfied by the hyperbolic metric when $(M^n, \partial M^n, g) = (B(0,1), \mathbb{S}^{n-1}, ds^2)$, where $B(0,1) \subset \mathbb{R}^n$ is the unit ball and ds^2 is the flat metric.

Remark 1.7. — As we noted in [5], these results can also be viewed as scalar versions of the problem of constructing Poincaré-Einstein metrics with prescribed conformal infinity; see Section 6 of [5].

In this article we are interested in the following question: To what extent can the condition of negative Ricci curvature be strengthened in Theorem 1.2? By the Cartan-Hadamard Theorem, there are topological obstructions to constructing a complete metric of negative *sectional* curvature: For example, let $(M^3, \partial M^3, g) = (\mathbb{S}^2 \times [0,1], \mathbb{S}^2 \cup \mathbb{S}^2, g_0 \oplus dt^2)$, where g_0 is the round metric on $\mathbb{S}^2$. If $\hat{g}$ were a complete conformal metric of negative curvature on $\mathbb{S}^2 \times (0,1)$, then its universal cover would be diffeomorphic to $\mathbb{R}^3$, an obvious contradiction.

Despite this obstruction, one can ask how *close* one can come to negative curvature. For closed manifolds this is quantified in Gromov's definition of *almost negative curvature*: M^n has almost negative curvature if for $\delta > 0$

$$(\operatorname{diam} M^n)^2 \cdot \max_{M^n} \kappa < \delta, \tag{1.1}$$

where $\max_{M^n} \kappa$ is the supremum of the sectional curavtures on M^n. Note that the diameter term is included to render the definition scale-invariant. In [3], Gromov proved that for any $\delta > 0$, $\mathbb{S}^3$ admits a metric of almost negative curvature. Later [2] Bavard extended this to any closed three-manifold.

Since we are considering metrics which are complete and conformally compact, we need to introduce a scale-invariant notion for having almost negative curvature in this context. The condition we impose amounts to a pointwise pinching condition: it says that for arbitrary $\delta > 0$, the ratio between the most positive sectional curvature and (absolute value of) the most negative sectional curvature is bounded by δ. Our results are also special to three dimensions, since (as we shall see in Section 2) a pinching condition on the sectional curvature can be reduced to a condition on the Ricci tensor. Our main result is

THEOREM 1.8. — *Let $(M^3, \partial M^3, g)$ be a compact three-dimensional Riemannian manifold with boundary, and let $\delta > 0$. Then there is a smooth, complete conformal metric $g_\delta = e^{2u} g$ defined in the interior of M^3 such that*

- (i) *The scalar curvature of g_δ is negative,*
- (ii) *For each $p \in M^3$, g_δ satisfies the pinching condition*

$$\frac{(\max \kappa)_p}{-(\min \kappa)_p} < \delta, \tag{1.2}$$

where $(\min \kappa)_p$ and $(\max \kappa)_p$ denote respectively the smallest and largest sectional curvatures at p.

Remark 1.9. — Since the scalar curvature of g_δ is negative, the smallest sectional curvature $(\min \kappa)_p$ at each point is necessarily negative.

Remark 1.10. — Although the inequality (1.2) allows for sectional curvatures to be arbitrarily large positive at a given point p, if one scales the metric to obtain $|Riem|_p = 1$, then the largest sectional curvature will be less than a constant times δ. A similar pinching condition, known as

Hamilton-Ivey pinching, arises in the study of Ricci flow on three-manifolds. Specifically, as one approaches a finite singular time and rescales solutions so that the curvature satisfies $|Rm| = 1$, one has that the smallest sectional curvature is at least $-\delta$, where δ is a constant going to zero as one blows up closer to the singular time. In particular this implies that ancient solutions to the Ricci flow on three manifolds have nonnegative sectional curvature. Similarly, if one were able to derive a convergent subsequence from the metrics $\{g_{\delta_i}\}$ of Theorem 1.8, then the limit would have nonpositive sectional curvatures. However, naive approaches to deriving this limit fail for PDE reasons as described below.

Remark 1.11. — It would be desirable to have a notion of almost pinching which implied an arbitrarily small upper bound on the curvature for a given asymptotical profile of the metric near infinity. However, as we are restricting to conformal metrics, there are regularity issues for the corresponding PDE; see the remark at the end of Section 6.

The paper is organized as follows: In Sections 2 and 3 we describe our pinching condition in more detail and write down the corresponding PDE. In Sections 4, 5, and 6 we prove global *a priori* bounds for solutions of the corresponding Dirichlet problem, and complete the proof of existence of solutions in Section 7. Finally, in Section 8 we use solutions of the Dirichlet problem to construct a complete metric by a standard limiting argument.

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2. The pinching condition

Let (M^3, g) be a three-dimensional Riemannian manifold, and let Ric and R denote the Ricci tensor and scalar curvature of g. If $\Pi \subset T_pM^3$ is a tangent plane, and $\nu \in T_pM^3$ is a unit normal to Π, then the sectional curvature of Π is given by the Einstein tensor acting on ν:

$$\kappa(\Pi) = \big(- Ric + \frac{1}{2} Rg\big)(\nu, \nu) \equiv S(\nu, \nu).$$

This follows from choosing an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of T_pM^3 with $\mathbf{e}_3 = \nu$, and using the standard decomposition of the curvature tensor to write

$$\begin{aligned} R_{1212} &= R_{11} + R_{22} - \frac{1}{2}R \\ &= (R - R_{33}) - \frac{1}{2}R \\ &= -R_{33} + \frac{1}{2}R \\ &= S(\nu,\nu). \end{aligned}$$

Since we are considering conformal deformations, we need the formulas giving the curvature of a conformal metric. Let $\hat{g} = e^{2u}g$, then the Einstein tensor $S(\hat{g})$ is given by

$$S(\hat{g}) = S + \nabla^2 u - \Delta u \cdot g - du \otimes du. \tag{2.1}$$

In particular, $\hat{g}$ will have negative sectional curvature provided

$$-S(\hat{g}) = (-S) - \nabla^2 u + \Delta u \cdot g + du \otimes du > 0. \tag{2.2}$$

As in [5], we will consider a Dirichlet problem: let $(M^3, \partial M^3, g)$ be a compact manifold with boundary; we want to solve

$$\det\big((-S) - \nabla^2 u + \Delta u \cdot g + du \otimes du\big)^{1/3} = e^{2u} \tag{2.3}$$

with

$$(-S) - \nabla^2 u + (\Delta u)g + du \otimes du > 0, \tag{2.4}$$

subject to various boundary conditions. As we will see, the condition (2.4) implies that (2.3) is elliptic. Also, we remark that the right-hand side of equation (2.3) is chosen to impose the appropriate behavior for complete solutions; see Section 8.

As noted in the introduction, there are obstructions to the existence of a solution of (2.3) defining a complete metric. On the PDE level this obstruction will manifest itself in the failure of C^2-estimates, as will be apparent in Section 6. Therefore, we will need to consider a regularized version of the equation:

$$\det\big((-S) - \nabla^2 u + (1+\epsilon)(\Delta u) \cdot g + du \otimes du\big)^{1/3} = e^{2u}. \tag{2.5}$$

The idea of regularizing by adding a trace term of this form goes back to work of Trudinger, and has been used in various geometric applications of fully nonlinear equations. For simplicity, we begin by considering solutions of (2.5) with zero Dirichlet data (that is, we are conformally fixing the boundary).

Theorem 2.1. — *For each $\epsilon > 0$, there is a unique solution of*

$$(*)_\epsilon \quad \det\big((-S) - \nabla^2 u + (1+\epsilon)(\Delta u)g + du \otimes du\big)^{1/3} = e^{2u} \quad \text{in } M^3,$$
$$u = 0 \quad \text{on } \partial M^3$$

which is smooth up to the boundary. Moreover, u satisfies the ellipticity condition

$$(2.6) \qquad (-S) - \nabla^2 u + (1+\epsilon)(\Delta u)g + du \otimes du > 0.$$

Before we describe the method of solving $(*)_\epsilon$, let us show how the existence of a solution implies the existence of a conformal metric satisfying the pinching condition of the main Theorem 1.8, but with the boundary fixed:

Proposition 2.2. — *Let $(M^3, \partial M^3, g)$ be a compact three-dimensional Riemannian manifold with boundary, and let $\delta > 0$. Then there is a conformal metric $g_\delta = e^{2u} g$ such that*

(*i*) *$g_\delta = g$ on ∂M^3,*

(*ii*) *The scalar curvature of g_δ is negative,*

(*iii*) *For each $p \in M^3$, g_δ satisfies the pinching condition*

$$(2.7) \qquad \frac{(\max \kappa)_p}{-(\min \kappa)_p} < \delta,$$

where $\min \kappa_p$ and $\max \kappa_p$ denote the smallest and largest sectional curvatures at p.

Proof. — By the work of [1] we may first conformally deform g such that the scalar curvature is negative. We still name this new conformally related metric g, and it suffices to show the proposition using this background metric.

For $\epsilon > 0$, let $u = u_\epsilon \in C^\infty(\bar{M}^3)$ be the unique solution of $(*)_\epsilon$with $\epsilon = 2\delta/3$, and denote $\hat{g} = e^{2u} g$. The Dirichlet condition in $(*)_\epsilon$obviously implies $\hat{g} = g$ on ∂M^3.

Fix $p \in M^3$. By our observations above, the smallest and largest sectional curvatures of $\hat{g}$ at p are given by the smallest and largest eigenvalues of the Einstein tensor $S(\hat{g})$. Choose a tangent plane $\Pi \subset T_p M^3$ with

$$\hat{\kappa}(\Pi) = (\max \hat{\kappa})_p,$$

where sectional curvatures of $\hat{g}$ will be designated with a hat. Let $\hat{n}$ denote a unit normal (w.r.t. $\hat{g}$) to Π; then by (2.1)

$$\begin{aligned}(\max \hat{\kappa})_p &= S(\hat{g})(\hat{n}, \hat{n}) \\ &= \{S + \nabla^2 u - \Delta u \cdot g - du \otimes du\}(\hat{n}, \hat{n}).\end{aligned}$$

Since u satisfies (2.6), we have

$$
\begin{aligned}
(\max \hat\kappa)_p &= \{S + \nabla^2 u - (1+\epsilon)(\Delta u)g - du \otimes du + \epsilon \Delta u \cdot g\}(\hat n, \hat n)\\
&< \epsilon(\Delta u) g(\hat n, \hat n)\\
&= \epsilon(\Delta u)e^{-2u}\hat g(\hat n, \hat n)\\
&= \epsilon(\Delta u)e^{-2u}.
\end{aligned}
\tag{2.8}
$$

Since each eigenvalue is greater than or equal to the smallest sectional curvature,

$$\mathrm{tr}_{\hat g} S(\hat g) \geqslant 3(\min \hat\kappa)_p. \tag{2.9}$$

Again using (2.1),

$$\mathrm{tr}_{\hat g} S(\hat g) = e^{-2u}(\mathrm{tr}\ S - 2\Delta u - |du|^2). \tag{2.10}$$

Combining these we get

$$(\Delta u)e^{-2u} \leqslant -\frac{3}{2}\epsilon(\min \hat\kappa)_p + \frac{1}{2}e^{-2u}(\mathrm{tr}\ S - |du|^2). \tag{2.11}$$

Since our background metric has negative scalar curvature, we have tr $S = R/2 < 0$, and hence

$$\epsilon(\Delta u)e^{-2u} < -\frac{3}{2}\epsilon(\min \hat\kappa)_p = -\delta(\min \hat\kappa)_p.$$

Comparing with (2.8), we get the pinching inequality (2.7). □

3. The equation

As in [5], we will use the continuity method to prove the existence of solutions to $(*)_\epsilon$. To this end, fix $\epsilon > 0$ and for $t \in [0,1]$ define

$$
\begin{aligned}
W_t = W_t[u] &= (1-t)g + t(-S) - \nabla^2 u + (1+\epsilon)(\Delta u)g + du \otimes du\\
&= S_t - \nabla^2 u + (1+\epsilon)(\Delta u)g + du \otimes du,
\end{aligned}
\tag{3.1}
$$

and

$$\Psi_t[u] = \det(W_t) - e^{6u}. \tag{3.2}$$

Consider the boundary value problem

$$
\begin{aligned}
\Psi_t[u] &= 0 \text{ in } M^3,\\
u &= 0 \text{ on } \partial M^3,
\end{aligned}
\tag{3.3}
$$

where once again we impose the condition

$$W_t[u] > 0. \tag{3.4}$$

Lemma 3.1. — *For $u \in C^2$ satisfying (3.4), equation (3.3) is elliptic for any $\epsilon \geqslant 0$.*

Proof. — Let $r = r_{ij}, N = N_{ij} \in \mathbb{R}^{n\times n}$, and assume

$$W_{ij} = N_{ij} - r_{ij} + (1+\epsilon)(r_{kk})\delta_{ij} > 0.$$

Then

$$\begin{aligned}\frac{\partial}{\partial r_{ij}} \det W &= (\det W) W^{\alpha\beta} \frac{\partial W_{\alpha\beta}}{\partial r_{ij}} \\ &= (\det W)\{-W^{ij} + (1+\epsilon)W^{kk}\delta_{ij}\} > 0,\end{aligned}$$

where $W^{\alpha\beta}$ are the components of W^{-1}. It follows that (3.3) is elliptic when u satisfies (3.4). □

Let

$$\Omega = \{t \in [0,1] : (3.3) \text{ admits a solution } u \in C^4 \text{ with } W_t[u] > 0\}.$$

Note $0 \in \Omega$, as $u \equiv 0$ is a solution of (3.3) with $t = 0$, hence $\Omega \neq \emptyset$.

Lemma 3.2. — *Ω is open.*

Proof. — Suppose $t_0 \in \Omega$, and let $u_0 \in C^4$ be the corresponding solution of (3.3). Define

$$u_s = u_0 + s\phi.$$

Then

$$\frac{\partial}{\partial s} W_{t_0}[u_s]\big|_{s=0} = -\nabla^2\phi + (1+\epsilon)(\Delta\phi)g + d\phi\otimes du_0 + du_0 \otimes d\phi.$$

Therefore, the linearization of Ψ_{t_0} is given by

$$\begin{aligned}&\mathcal{L}[u_0]\phi \\ &= \frac{\partial}{\partial s}\Psi_{t_0}[u_s]\big|_{s=0} \\ &= \det[W_{t_0}] W_{t_0}[u_0]^{\alpha\beta} \frac{\partial}{\partial s} W_{t_0}[u_s]_{\alpha\beta}\big|_{s=0} - 6\phi e^{6u_0} \\ &= e^{6u_0} W_{t_0}[u_0]^{\alpha\beta}\{-\nabla_\alpha\nabla_\beta\phi + (1+\epsilon)(\Delta\phi)g_{\alpha\beta} + \nabla_\alpha\phi\nabla_\beta u_0 + \nabla_\alpha u_0 \nabla_\beta \phi\} \\ &\quad - 6e^{6u_0}\phi \\ &= A^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi - 6e^{6u_0}\phi,\end{aligned} \tag{3.5}$$

where once again $W_{t_0}^{\alpha\beta}[u_0]$ denotes the inverse matrix, and

$$A^{\alpha\beta} = e^{6u_0}\{-W_{t_0}[u_0]^{\alpha\beta} + (1+\epsilon)W_{t_0}[u_0]^{\mu\mu} g_{\alpha\beta}\} > 0.$$

Because of the sign of the zeroth order term, it follows that the boundary value problem

$$\mathcal{L}[u_0]\phi = f \quad \text{in } M^3, \qquad \phi = 0 \quad \text{on } \partial M^3$$

has a unique solution. The result then follows from the implicit function theorem. □

Next, we show that (3.3) obeys a maximum principle:

Proposition 3.3. — *Suppose $u, v \in C^4(\bar{M}^3)$ satisfy*

$$\Psi_t[u] \geqslant \Psi_t[v] \quad \textit{in } M^3, \qquad u \leqslant v \quad \textit{on } \partial M^3.$$

Then $u \leqslant v$ in M^3. In particular, solutions of (3.3) are unique.

Proof. — The proof is standard; see Proposition 2.1 of [5]. □

Lemma 3.4. — *For each $0 \leqslant t \leqslant 1$, suppose u_t is a solution of (3.3). Then $u_t \leqslant 0$.*

Proof. — We may assume after scaling that $(-S) \leqslant g$ as bilinear forms. Then by convexity,

$$\begin{aligned} e^{6u_t} &= \Psi_t[u_t] + e^{6u_t} \\ &= \det\{S_t - \nabla^2 u_t + (1+\epsilon)(\Delta u_t)g + du_t \otimes du_t\} \\ &\leqslant \det\{g - \nabla^2 u_t + (1+\epsilon)(\Delta u_t)g + du_t \otimes du_t\} \\ &= \Psi_0[u_t] + e^{6u_t}. \end{aligned}$$

Therefore, $\Psi_0[u_t] \geqslant 0$; i.e., u_t is a subsolution of (3.3) with $t = 0$. It follows from Proposition 3.3 that $u_t \leqslant u_0 = 0$. □

Remark 3.5. — Suppose $u \in C^2$ is a solution of (3.3) in a ball $B(x, 2r) \subset M^3$. By the arithmetic-geometric mean inequality,

$$e^{2u} = \det(W_t[u])^{1/3} \leqslant \frac{1}{3}\mathrm{tr}(W_t[u]) = \frac{1}{3}\big(\mathrm{tr}\ S_t + (2+3\epsilon)\Delta u + |\nabla u|^2\big).$$

Let η be a smooth cut-off function supported in $B(x, 2r)$ with $\eta \equiv 1$ in $B(x, r)$. Applying the maximum principle to $V = \eta e^u$, we easily obtain

$$\max_{B(x,r)} e^u \leqslant C(g, r).$$

Hence, we always have interior sup-norm bounds on $u^+(x) = \max\{u(x), 0\}$.

In the following sections we prove various *a priori* estimates; these will establish that Ω is closed and thus $(*)_\epsilon$ has a solution.

4. C^0 estimates: construction of a subsolution

In this Section we adapt the construction of [5] to give a subsolution of $(*)_\epsilon$. By the maximum principle of Proposition 3.3, this will give an *a priori* lower bound for solutions. Since Lemma 3.4 gives an upper bound, this will establish the C^0-estimate.

As in [5], we begin by constructing a collar neighborhood of our manifold. Let $N = \partial M^3$ and consider the manifold $\bar{M}^3 = M^3 \cup (N \times [0,1]) / \sim$ where for $x \in N = \partial M^3$ we have $x \times 1 \sim x$. Using a standard partition of unity argument one may extend the metric g to a metric $\tilde{g}$ defined on $\tilde{M}^3$ such that $\tilde{g}_{|M^3} = g$. Consider a point $x_0 \in \partial M^3$. Fix a point $\bar{x} \in \tilde{M}^3 \setminus M^3$ in the connected component of N which contains x_0 chosen so that x_0 is the closest point to $\bar{x}$ which lies on the boundary. Let r denote geodesic distance from $\bar{x}$. We may arrange things so that $d(\bar{x}, \partial M^3) > \delta$ where δ only depends on the background metric.

Fix constants A and p whose exact size will be determined later, and let

$$\underline{\mathrm{u}} := A\left(\frac{1}{r^p} - \frac{1}{r(x_0)^p}\right)$$

Our goal is to show that $\underline{\mathrm{u}}$ is a subsolution of (3.3) for all $0 \leqslant t \leqslant 1$. First we recall the Hessian comparison theorem:

Lemma 4.1 (Hessian comparison theorem). — *Let (M^n, g) be a complete Riemannian manifold with $(\min \kappa)_x \geqslant K$ for each $x \in M^n$. For any point $p \in M^n$ the distance function $r(x) = d(x, P)$ satisfies*

$$\nabla^2 r \leqslant \frac{1}{n-1} H_K(r) g$$

where

$$H_K(r) = \begin{cases} (n-1)\sqrt{K}\cot\left(\sqrt{K}r\right) & K > 0 \\ \frac{n-1}{r} & K = 0 \\ (n-1)\sqrt{|K|}\coth\left(\sqrt{|K|}r\right) & K < 0 \end{cases}$$

Corollary 4.2. — *Let $(\tilde{M}^3, \tilde{g})$ be the metric constructed above, and let r denote the distance from a point $\bar{x} \in \tilde{M}^3 \setminus M^3$ chosen so that $d(\bar{x}, \partial M^3) > \delta > 0$ for some fixed small constant δ. Then there exists a constant $C = C(g)$ such that*

$$-\nabla^2 r(x) + (1+\epsilon)(\Delta r)(x) g \leqslant \frac{C}{r(x)} g$$

holds at any point x where r is smooth.

Proof. — At a point x where r is smooth, we can diagonalize $\nabla^2 r$:

$$\nabla^2 r(x) = \begin{pmatrix} \rho_1 & & \\ & \rho_2 & \\ & & \rho_3, \end{pmatrix}$$

hence

$$- \nabla^2 r(x) + (1+\epsilon)(\Delta r)(x) g =$$
$$\begin{pmatrix} \epsilon\rho_1 + (1+\epsilon)(\rho_2+\rho_3) & & \\ & \epsilon\rho_2 + (1+\epsilon)(\rho_1+\rho_3) & \\ & & \epsilon\rho_3 + (1+\epsilon)(\rho_1+\rho_2). \end{pmatrix}$$

By the Hessian Comparison Theorem and standard estimates for coth and cot,

$$\rho_i \leqslant \frac{C}{r(x)}, \quad 1 \leqslant i \leqslant 3.$$

Therefore,

$$-\nabla^2 r(x) + (1+\epsilon)(\Delta r)(x) g \leqslant \frac{C}{r(x)} g.$$

□

Lemma 4.3. — *For A and p chosen large enough with respect to constants depending only on g, at any point where r is smooth we have*

$$\Psi_t[\underline{u}] \geqslant 0.$$

Proof. —

By direct calculation,

$$\begin{aligned} \nabla \underline{u} &= - Apr^{-p-1} \nabla r, \\ \nabla^2 \underline{u} &= Ap(p+1) r^{-p-2} dr \otimes dr - Apr^{-p-1} \nabla^2 r, \\ \Delta \underline{u} &= Ap(p+1) r^{-p-2} - Apr^{-p-1} \Delta r. \end{aligned}$$

Thus

$$\begin{aligned} W_t[\underline{u}] &= S_t - \nabla^2 \underline{u} + (1+\epsilon)\Delta \underline{u} g + d\underline{u} \otimes d\underline{u} \\ &= S_t - Apr^{-p-1}\{-\nabla^2 r + (1+\epsilon)(\Delta r) g\} \\ &\quad + Ap^2 r^{-2p-2}[A - (1+\frac{1}{p}) r^p] dr \otimes dr + (1+\epsilon) Ap(p+1) r^{-p-2} g. \end{aligned}$$

Since $A, p > 0$, by Corollary 4.2 we have

$$-Apr^{-p-1}\{-\nabla^2 r + (1+\epsilon)(\Delta r) g\} \geqslant -C_1 Apr^{-p-2} g$$

for $C_1 = C_1(g)$. Also,

$$S_t \geqslant -C_2 g$$

for $C_2 = C_2(g)$. Therefore,

(4.1)

$$\begin{aligned} W_t[\underline{\mathrm{u}}] &\geqslant -C_2 g - C_1 A p r^{-p-2} g \\ &\qquad + Ap^2 r^{-p-2}[Ar^{-p} - (1+\frac{1}{p})] dr \otimes dr + (1+\epsilon) Ap(p+1) r^{-p-2} g \\ &= -C_2 g + Ap^2 r^{-2p-2}[A - (1+\frac{1}{p}) r^p] dr \otimes dr \\ &\qquad + Apr^{-p-2}[(1+\epsilon)(p+1) - C_1] g. \end{aligned}$$

Fix $p > C_1 + 1$. Since $r \leqslant C_3$ for some $C_3 = C_3(g)$, if

$$A > (1+\frac{1}{p}) C_3^p,$$

then the $dr \otimes dr$-term in (4.1) is positive. By choosing $A = A(C_2, C_3, p)$ larger still, we can arrange so that

$$Apr^{-p-2}[(1+\epsilon)(p+1) - C_1] \geqslant ApC_3^{-p-2} \geqslant C_2 + 1.$$

Therefore,

$$W_t[\underline{\mathrm{u}}] \geqslant g.$$

Since $\underline{\mathrm{u}} \leqslant 0$ by construction, it follows that

$$\Psi_t[\underline{\mathrm{u}}] \geqslant 0$$

as claimed. □

Remark 4.4. — In the construction of subsolutions for the Ricci curvature equation in [5], the gradient terms collectively have a sign which allows them to be essentially disregarded. However, in the preceding argument the structure of the gradient term in equation (3.3) plays a crucial role.

Remark 4.5. — Of course, we cannot say that $\underline{\mathrm{u}}$ is a classical subsolution, since it may fail to be differentiable on the cut locus. However, the comparison argument Lemma 4.1 of [5] shows that $\underline{\mathrm{u}}$ is majorized by any solution u_t:

Proposition 4.6. — *Given $\underline{\mathrm{u}}$ as in Lemma 4.3, for all $t \in [0,1]$, one has $\underline{\mathrm{u}} \leqslant u_t$.*

Proof. — This is Lemma 4.1 in [5]; we reproduce it here for the sake of completeness.

Fix a $t \in [0,1]$ and suppose that $\underline{\mathrm{u}} > u_t$ somewhere. We can fix a positive constant C and a point $x_1 \in M^3$ achieving the maximum of $\underline{\mathrm{u}} - u_t$, such that $\underline{u} - C \leqslant u_t$ and $(\underline{u} - C)(x_1) = u_t(x_1)$. It is clear by construction that this point must be inside of M^3. We also claim that $\underline{u}$, and equivalently, r, must be smooth at this point x_1. Indeed, if this were not the case, at x_1 there would be two geodesics γ_1, γ_2 which are each minimizing from $\bar{x}$ to x_1. Suppose $d(\bar{x}, x_1) = R$. Let γ_1 be given a unit speed parametrization in c. One concludes

$$\lim_{c \to R^-} \nabla r(\gamma_1(c)) \cdot \gamma_1' = 1. \tag{4.2}$$

We next claim that

$$\lim_{c \to R^+} \nabla r(\gamma_1(c)) \cdot \gamma_1' < 1. \tag{4.3}$$

Fix a constant $\epsilon > 0$ so small that $B_\epsilon(x_1)$ is geodesically convex. Consider the point $\tilde{x}_\epsilon = \gamma_1(R + \epsilon)$. Construct a new curve $\tilde{\gamma}$ from $\bar{x}$ to $\tilde{x}_\epsilon$ as follows: follow the geodesic γ_2 from $\bar{x}$ to $\gamma_2(R - \epsilon)$, then connect $\gamma_2(R - \epsilon)$ to $\tilde{x}_\epsilon$ by the unique geodesic in $B_\epsilon(x_1)$ between these two points. Recall that γ_1 and γ_2 are distinct geodesics. In particular, by uniqueness of solutions to ODE, it follows that $\gamma_1'(R) \neq \gamma_2'(R)$ since $\gamma_1(R) = \gamma_2(R)$. In particular, the triangle formed by the three points

$$\gamma_2(R - \epsilon), \gamma_1(R) = \gamma_2(R) = x_1, \quad \text{and} \quad \gamma_1(R + \epsilon) = \tilde{x}_\epsilon$$

is nondegenerate. It follows from the Toponogov comparison theorem that $d(\gamma_2(R - \epsilon), \tilde{x}_\epsilon)$ is strictly less than the sum of the lengths of the other two sides of the triangle, with the difference given in terms of a lower bound for the curvature of g. Specifically, there exists a $\delta > 0$ depending on this lower bound and the angles of the triangle so that

$$d(\gamma_2(R - \epsilon), \tilde{x}_\epsilon) \leqslant (2 - \delta)\, \epsilon$$

(In fact, since our triangle is very small, the curvature does not need to enter into the bound. One can forgo the Toponogov theorem and get a bound strictly in terms of the angles of the triangle). Using $\tilde{\gamma}$ as a test curve for the distance function, it follows that

$$d(\bar{x}, \tilde{x}_\epsilon) \leqslant R - \epsilon + (2 - \delta)\, \epsilon = R + \epsilon - \delta\epsilon.$$

Taking the limit as $\epsilon \to 0$, we immediately conclude that

$$\begin{aligned} \lim_{c \to R^+} \nabla r(\gamma_1(c)) \cdot \gamma_1' &= \lim_{\epsilon \to 0} \frac{r(\gamma_1(R + \epsilon)) - r(\gamma_1(R))}{\epsilon} \\ &\leqslant \lim_{\epsilon \to 0} \frac{R + \epsilon - \delta\epsilon - R}{\epsilon} \\ &< 1. \end{aligned}$$

We now finish the argument that $\underline{u}$ is smooth at x_1. Indeed, it follows from (4.2) and (4.3) by direct calculation that the derivative of the function $f(c) := \underline{u}(\gamma_1(c))$ jumps a certain positive amount at $c = R$. Considering next the smooth function $\psi(c) := u_t(\gamma_1(c))$, by assumption we have that $(\psi - f)(c)$ has a local minimum at $c = R$. Thus

$$\lim_{c\to R^-} f' - \psi' \geqslant \lim_{c\to R^+} f' - \psi'.$$

Since ψ is smooth, we therefore conclude

$$\lim_{c\to R^-} f' \geqslant \lim_{c\to R^+} f'.$$

This contradicts what we just showed about the left and right hand limits of f.

Given that $\underline{u}$ is smooth at x_1, using Lemma 4.3 the argument of Proposition 3.3 applies at this point to yield the required contradiction to the assumption that $\underline{u} > u_t$ somewhere. □

Corollary 4.7. — *There is a $C_0 = C_0(g) > 0$ such that for all $t\in[0,1]$,*

$$-C_0 \leqslant u_t \leqslant 0.$$

5. Gradient estimate

In this section we prove global C^1-estimates for solutions of (3.3).

5.1. Boundary gradient estimates

We begin by observing that our subsolution construction can be used to prove an *a priori* bound for the gradient on the boundary:

Lemma 5.1. — *There is a constant $C = C(g)$ such that for all $x_0 \in \partial M^3$ and for all $0 \leqslant t \leqslant 1$ we have*

$$\left|\frac{\partial}{\partial\nu} u_t\right| \leqslant C,$$

where ν denotes the interior normal to ∂M^3 at x_0.

Proof. — We can construct a subsolution $\underline{u}$ as in Lemma 4.3; from Proposition 4.6 and the fact that $u_t \leqslant 0$ it follows that for $x \in M^3$ near ∂M^3,

$$\frac{\underline{u}(x) - \underline{u}(x_0)}{d(x,x_0)} \leqslant \frac{u_t(x) - u_t(x_0)}{d(x,x_0)} \leqslant 0.$$

Letting $x \to x_0$, we conclude

$$-C \leqslant \frac{\partial}{\partial \nu} u_t \leqslant 0,$$

as claimed. □

Corollary 5.2. — *There is a constant $C = C(g)$ such that for all $0 \leqslant t \leqslant 1$,*

$$\sup_{\partial M^3} |\nabla u_t| \leqslant C.$$

Proof. — Since $u_t = 0$ on ∂M^3, all tangential derivatives vanish. Therefore, the Corollary is immediate from Lemma 5.1. □

5.2. Interior estimates for the gradient

Proposition 5.3. — *There is a constant $C = C(g)$ such that for all $0 \leqslant t \leqslant 1$,*

$$\sup_{M^3} |\nabla u_t| \leqslant C.$$

Proof. — Let

$$f(s) = -\log(2 - e^s). \tag{5.1}$$

Consider the function

$$H = |\nabla u|^2 e^{f(u)}, \tag{5.2}$$

where from now on we will suppress the subscript t on u. By Corollary 4.7,

$$\frac{1}{2 - e^{-C_0}} \leqslant e^f \leqslant 1.$$

If the maximum of H is attained at a point $\bar{p}$ on the boundary of M^3, then by Corollary 5.2

$$\max_{M^3} H = \max_{\partial M^3} H = H(\bar{p}) = e^{f(\bar{p})} |\nabla u(\bar{p})|^2 = |\nabla u(\bar{p})|^2 \leqslant C,$$

hence

$$\max_{M^3} |\nabla u|^2 = \max_{M^3} e^{-f} H \leqslant (2 - e^{-C_0}) \max_{M^3} H \leqslant C.$$

Now, assume the maximum of H occurs at an interior point $p \in M^3$. Choosing a local frame $\{e^k\}$ near p, we differentiate to get

$$\nabla_j H = 2e^f \nabla_j \nabla_k u \nabla_k u + f' e^f |\nabla u|^2 \nabla_j u, \tag{5.3}$$

and

$$\begin{aligned}(5.4)\quad \nabla_i\nabla_j H = 2e^f\nabla_i\nabla_j\nabla_k u\nabla_k u + 2e^f\nabla_i\nabla_k u\nabla_j\nabla_k u\\ + 2f'e^f\nabla_j\nabla_k u\nabla_i u\nabla_k u + 2f'e^f\nabla_i\nabla_k u\nabla_j u\nabla_k u\\ + f''e^f|\nabla u|^2\nabla_i u\nabla_j u + (f')^2e^f|\nabla u|^2\nabla_i u\nabla_j u\\ + f'e^f|\nabla u|^2\nabla_i\nabla_j u.\end{aligned}$$

Since p is a critical point of H, from (5.3) we see that

$$2\nabla_j\nabla_k u\nabla_k u e^f = -f'|\nabla u|^2 e^f\nabla_j u \tag{5.5}$$

holds at p. Substituting this into (5.4) gives (at p)

$$\begin{aligned}(5.6)\quad \nabla_i\nabla_j H = 2e^f\nabla_i\nabla_j\nabla_k u\nabla_k u + 2e^f\nabla_i\nabla_k u\nabla_j\nabla_k u\\ + [f'' - (f')^2]e^f|\nabla u|^2\nabla_i u\nabla_j u + f'e^f|\nabla u|^2\nabla_i\nabla_j u.\end{aligned}$$

If we commute derivatives in the leading term, we get

$$\begin{aligned}2e^f\nabla_i\nabla_j\nabla_k u\nabla_k u &= 2e^f\nabla_k\nabla_i\nabla_j u\nabla_k u + 2e^f R_{ikj\ell}\nabla_k u\nabla_\ell u\\ &= 2e^f\nabla_k\nabla_i\nabla_j u\nabla_k u + O(H).\end{aligned}$$

Substituting this into (5.6),

$$\begin{aligned}(5.7)\quad \nabla_i\nabla_j H = 2e^f\nabla_k\nabla_i\nabla_j u\nabla_k u + 2e^f\nabla_i\nabla_k u\nabla_j\nabla_k u\\ + [f'' - (f')^2]e^f|\nabla u|^2\nabla_i u\nabla_j u + f'e^f|\nabla u|^2\nabla_i\nabla_j u + O(H).\end{aligned}$$

Therefore, at p we have

$$\begin{aligned}(5.8)\quad & -\nabla_i\nabla_j H + (1+\epsilon)(\Delta H)g_{ij}\\ &= 2e^f\nabla_k\{-\nabla_i\nabla_j u + (1+\epsilon)(\Delta u)g_{ij}\}\nabla_k u\\ &+ 2e^f\{-\nabla_i\nabla_k u\nabla_j\nabla_k u + (1+\epsilon)|\nabla^2 u|^2 g_{ij}\}\\ &+ [f'' - (f')^2]e^f|\nabla u|^2\{-\nabla_i u\nabla_j u + (1+\epsilon)|\nabla u|^2 g_{ij}\}\\ & f'e^f|\nabla u|^2\{-\nabla_i\nabla_j u + (1+\epsilon)(\Delta u)g_{ij}\} + O(H).\end{aligned}$$

Recall the definition of (3.1):

$$(W_t)_{ij} \equiv W_{ij} = (S_t)_{ij} - \nabla_i\nabla_j u + (1+\epsilon)(\Delta u)g_{ij} + \nabla_i u\nabla_j u.$$

Then

$$
\begin{aligned}
-\nabla_i\nabla_j H &+ (1+\epsilon)(\Delta H)g_{ij}\\
&= 2e^f\nabla_k\{W_{ij} - \nabla_i u\nabla_j u - (S_t)_{ij}\}\nabla_k u\\
&\quad + 2e^f\{-\nabla_i\nabla_k u\nabla_j\nabla_k u + (1+\epsilon)|\nabla^2 u|^2 g_{ij}\}\\
&\quad + [f'' - (f')^2]e^f|\nabla u|^2\{-\nabla_i u\nabla_j u + (1+\epsilon)|\nabla u|^2 g_{ij}\}\\
&\quad + f'e^f|\nabla u|^2\{W_{ij} - \nabla_i u\nabla_j u - (S_t)_{ij}\} + O(H).
\end{aligned}
\tag{5.9}
$$

Using (5.5), the first term on the right-hand side can be written

$$
\begin{aligned}
&2e^f\nabla_k\{W_{ij} - \nabla_i u\nabla_j u - (S_t)_{ij}\}\nabla_k u\\
&= 2e^f\nabla_k W_{ij}\nabla_k u - 2e^f\nabla_k\nabla_i u\nabla_j u\nabla_k u - 2e^f\nabla_i u\nabla_j u\nabla_k u\nabla_k u\\
&\qquad - 2e^f\nabla_k(S_t)_{ij}\nabla_k u\\
&= 2e^f\nabla_k W_{ij}\nabla_k u + 2f'e^f|\nabla u|^2\nabla_i u\nabla_j u + O(H).
\end{aligned}
\tag{5.10}
$$

Also,

$$
2e^f\{-\nabla_i\nabla_k u\nabla_j\nabla_k u + (1+\epsilon)|\nabla^2 u|^2 g_{ij}\} \geqslant 0. \tag{5.11}
$$

Therefore, collecting the terms in (5.9) and (5.10) and using (5.11) we arrive at

$$
\begin{aligned}
&-\nabla_i\nabla_j H + (1+\epsilon)(\Delta H)g_{ij} \geqslant 2e^f\nabla_k W_{ij}\nabla_k u + f'e^f|\nabla u|^2 W_{ij}\\
&+ [-f'' + (f')^2 + f']e^f|\nabla u|^2\nabla_i u\nabla_j u + (1+\epsilon)[f'' - (f')^2]e^f|\nabla u|^4 g_{ij}\\
&\qquad + O(H).
\end{aligned}
\tag{5.12}
$$

From the definition of f in (5.1), one can check

$$
-f'' + (f')^2 + f' = 0.
$$

Also,

$$
[f''(u) - (f')^2(u)]e^{-f(u)} = e^u \geqslant e^{-C_0}.
$$

Therefore,

$$
\begin{aligned}
-\nabla_i\nabla_j H + (1+\epsilon)(\Delta H)g_{ij} &\geqslant 2e^f\nabla_k W_{ij}\nabla_k u + f'e^f|\nabla u|^2 W_{ij}\\
&\quad + \eta_0 H^2 g_{ij} + O(H),
\end{aligned}
\tag{5.13}
$$

where $\eta_0 = \eta_0(g) > 0$.

Since p is a maximum point of H,

$$
-\nabla_i\nabla_j H + (1+\epsilon)(\Delta H)g_{ij} \leqslant 0.
$$

As before, let W^{ij} denote the inverse of W; then at p

$$\begin{aligned}
0 &\geqslant W^{ij}\{-\nabla_i\nabla_j H + (1+\epsilon)(\Delta H)g_{ij}\} \\
&\geqslant 2e^f W^{ij}\nabla_k W_{ij}\nabla_k u + f' e^f |\nabla u|^2 W^{ij}W_{ij} + W^{ij}\{\eta_0 H^2 g_{ij} + O(H)\} \\
&= 2e^f\langle\nabla\log\det W, \nabla u\rangle + 3f'e^f|\nabla u|^2 + W^{ij}\{\eta_0 H^2 g_{ij} + O(H)\} \\
&= 12e^f|\nabla u|^2 + 3f'e^f|\nabla u|^2 + W^{ij}\{\eta_0 H^2 g_{ij} + O(H)\} \\
&\geqslant W^{ij}\{\eta_0 H^2 g_{ij} + O(H)\},
\end{aligned}$$

since $f' \geqslant 0$. It follows that $H(p) \leqslant C$, and therefore $|\nabla u| \leqslant C$ on M^3. □

6. C^2-estimates

6.1. Boundary estimates

Fix $x_0 \in \partial M^3$ and let $u = u_t$ be a solution of (3.3). We fix a small ball B_ρ centered at x_0 with $\rho > 0$ small, and introduce local coordinates $\{x^\ell\}$ so that $e_i = \partial/\partial x^i$ are tangent to ∂M^3 for $i = 1, 2$ and $e_n = \partial/\partial x^3$ is normal. As usual, we argue differently to estimate the various components of $\nabla^2 u$ at x_0.

Lemma 6.1. — *There is a constant $C = C(g)$ such that for all $0 \leqslant t \leqslant 1$,*

$$\sup_{\partial M^3} |\nabla_i\nabla_j u| \leqslant C.$$

Proof. — Since $\partial M^3 = \{x \mid u(x) = 0\}$,

$$\nabla_i\nabla_j u(x_0) = -\nabla_n u(x_0) A(e_i, e_j),$$

where A is the second fundamental form of ∂M^3. Since $|\nabla u| \leqslant C(g)$, the Lemma follows. □

Establishing a bound for $\nabla_i\nabla_n u(x_0)$ requires an auxiliary calculation. Let $\mathcal{L} = \mathcal{L}[u]$ denote the linearized operator defined in (3.5), and let $\phi = e_\alpha u$, where $\alpha = 1$ or 2. Note that $\phi\big|_{\partial M^3} = 0$. We will use a maximum principle argument to obtain a bound on the normal derivative of ϕ, thus giving the estimate for mixed second partials of u. We begin with a technical Lemma:

Lemma 6.2. — *There is a constant $\bar{C} = \bar{C}(g)$ such that*

$$|\mathcal{L}\phi| \leqslant \bar{C}\sum_j W^{jj}. \tag{6.1}$$

Proof. — Differentiating (3.3) with respect to e_α gives

$$\begin{aligned}0 &= e_\alpha \Psi_t[u]\\ &= (\det W)W^{ij}\big\{(\nabla_\alpha S_t)_{ij} - \nabla_\alpha\nabla_i\nabla_j u + (1+\epsilon)\nabla_\alpha(\Delta u)g_{ij}\\ &\quad + \nabla_\alpha\nabla_i u\nabla_j u + \nabla_i u\nabla_\alpha\nabla_j u\big\} - 6e^{6u}\nabla_\alpha u.\end{aligned}$$

Commuting derivatives,

$$\begin{aligned}-\nabla_\alpha\nabla_i\nabla_j u &= -\nabla_i\nabla_j\phi + Rm_g * \nabla u,\\ (1+\epsilon)\nabla_\alpha(\Delta u) &= (1+\epsilon)\Delta\phi + Rm_g * \nabla u.\end{aligned}$$

Therefore,

$$\begin{aligned}0 &= (\det W)W^{ij}\big\{-\nabla_i\nabla_j\phi + (1+\epsilon)(\Delta\phi)g_{ij}\\ &\quad + \nabla_i\phi\nabla_j u + \nabla_i u\nabla_j\phi + (\nabla_\alpha S_t)_{ij} + Rm * \nabla u\big\} - 6e^{6u}\nabla_\alpha u.\\ &= \mathcal{L}\phi + e^{6u}W^{ij}\big\{(\nabla_\alpha S_t)_{ij} + Rm * \nabla u\big\},\end{aligned}$$

hence

$$\begin{aligned}|\mathcal{L}\phi| &= \big|e^{6u}W^{ij}\big\{-(\nabla_\alpha S_t)_{ij} + Rm * \nabla u\big\}\big|\\ &\leqslant \bar{C}\mathrm{tr}\ W^{-1},\end{aligned}$$

the last line following from the C^0- and C^1-estimates. The Lemma follows. □

LEMMA 6.3. — *There is a constant $C = C(g,\epsilon)$ such that for all $0 \leqslant t \leqslant 1$,*

$$\sup_{\partial M^3} |\nabla_\alpha\nabla_n u| \leqslant C.$$

Proof. — As in the construction of a subsolution, construct a collar neighborhood of ∂M^3 and choose a point $\bar{x} \in \tilde{M}^3 \setminus M^3$ with $d(x_0,\bar{x}) = d(\partial M^3,\bar{x}) = \delta/3 << 1$, where $\delta > 0$ is chosen small enough so that $B = B_\delta(x_0)$ is geodesically convex. Let $r(x) = r = dist(x,\bar{x})$, and define

$$v(x) = \frac{1}{r(x)^p} - \frac{1}{r(x_0)^p}.$$

Note that v is smooth in $U = M^3 \cap B$. Also, using the C^1-estimates and the calculations from Section 4, it follows that

$$\nabla^2 v + (1+\epsilon)(\Delta v)g + dv\otimes du + du\otimes dv \geqslant \epsilon p(p+1)r^{-p-2}g - Cpr^{-p-2}g,$$

where $C = C(\delta,g)$. Since $v \leqslant 0$ on U, for $p >> 1$ large (depending on ϵ^{-1}, C, and $\bar{C}$, where $\bar{C}$ is the constant in (6.1)) we conclude

$$\begin{aligned}\mathcal{L}v &\geqslant p[\epsilon p - C]r^{-p-2}\mathrm{tr}\ W^{-1}\\ &\geqslant 2\bar{C}\mathrm{tr}\ W^{-1}.\end{aligned}$$

Therefore,

$$\mathcal{L}(\phi - v) \leqslant 0.$$

By the maximum principle, the minimum of $\phi - v$ is attained on the boundary of U. Note that $\phi - v \geqslant 0$ on $\partial U \cap \partial M^3$, since $\phi \equiv 0$ and $v \leqslant 0$ there. Next, consider the component $U \cap \partial B$. Due to the C^1-estimate, on this set we have $\phi \geqslant -C$ for some $C = C(g)$. Also, for $x \in \partial B$, by the triangle inequality

$$\begin{aligned}\delta = d(x, x_0) &\leqslant d(x, \bar{x}) + d(\bar{x}, x_0) \\ &= d(x, \bar{x}) + \delta/3,\end{aligned}$$

hence

$$r(x) \geqslant \frac{2}{3}\delta.$$

It follows that

$$\begin{aligned}v(x) &= \frac{1}{r(x)^p} - \frac{1}{r(x_0)^p} \\ &\leqslant \frac{1}{(\frac{2}{3}\delta)^p} - \frac{1}{(\delta/3)^p} \\ &= \left(\frac{1}{\delta}\right)^p \left[\left(\frac{3}{2}\right)^p - 3^p\right].\end{aligned}$$

Therefore, by choosing p larger still if necessary, we have $v << 0$ on $U \cap \partial B$, hence $\phi - v > 0$ there. It follows that the minimum of $\phi - v$ occurs at x_0, and the (interior) normal derivative is non-negative; this implies

$$\nabla_n \nabla_\alpha u \geqslant -C.$$

However, using Lemma 6.2 it is clear we can apply a similar argument using $-\phi$ instead of ϕ to obtain an upper bound, and the Lemma follows. □

LEMMA 6.4. — *There is a constant $C = C(g, \epsilon)$ such that for all $0 \leqslant t \leqslant 1$,*

$$|\nabla_n \nabla_n u| \leqslant C. \tag{6.2}$$

Proof. — We may assume that our local coordinates are normal at x_0. Then the matrix of W at x_0 is given by

$$W = \begin{pmatrix} \epsilon u_{nn} & 0 & 0 \\ 0 & (1+\epsilon)u_{nn} & 0 \\ 0 & 0 & (1+\epsilon)u_{nn} \end{pmatrix} + O(1),$$

where we are using the fact that

$$|u_{ij}| + |u_{in}| + |u_i| + |u_n| \leqslant C(\epsilon, g).$$

Since the trace of W is positive,

$$0 < \mathrm{tr}\ W = (2+3\epsilon)u_{nn} + C,$$

hence

$$u_{nn} \geqslant -C.$$

On the other hand, if $u_{nn} \geqslant N >> 1$ where N is large, then

$$C \geqslant e^{6u} = \det W \geqslant C_\epsilon N^3,$$

which implies an upper bound on u_{nn}. □

Summarizing the preceding Lemmas gives

Proposition 6.5. — *There is a constant $C = C(\epsilon, g)$ such that for all $0 \leqslant t \leqslant 1$,*

$$\sup_{\partial M^3} |\nabla^2 u_t| \leqslant C.$$

6.2. Interior estimates

Proposition 6.6. — *There is a constant $C = C(\epsilon, g)$ such that for all $0 \leqslant t \leqslant 1$,*

$$\sup_{M^3} |\nabla^2 u_t| \leqslant C.$$

Proof. — As in the proof of the gradient estimate we suppress the subscript t.

We begin by observing that it suffices to prove a bound for $\epsilon|\Delta u|$. This follows from a standard argument: since W is positive definite, $\sigma_2(W) > 0$, where $\sigma_2(\cdot)$ denotes the second elementary symmetric polynomial. Thus

$$\begin{aligned} (6.3)\quad 0 < 2\sigma_2(W) &= -|W|^2 + (\mathrm{tr}\ W)^2 \\ &= -|\nabla^2 u|^2 + (3 + 8\epsilon + 6\epsilon^2)(\Delta u)^2 + \nabla^2 u * \nabla u * \nabla u + \cdots, \end{aligned}$$

where "$\cdots$" denotes terms which involve ∇u or S_t, and for tensors A, B the notation $A * B$ means contractions of the tensor product of A and B. Since $|\nabla u|$ is bounded, (6.3) implies

$$\epsilon|\nabla^2 u| \leqslant C\big(\epsilon|\Delta u| + 1\big).$$

Moreover, since the trace of W is positive,

$$0 < \mathrm{tr}\ W = \mathrm{tr}\ S_t + (2+3\epsilon)\Delta u + |\nabla u|^2, \tag{6.4}$$

hence by the gradient estimate

$$\Delta u \geqslant -C.$$

Therefore, we only need to establish an upper bound on $\epsilon\Delta u$.

To this end, let

$$Q = \epsilon\Delta u + 2|\nabla u|^2. \tag{6.5}$$

If the maximum of Q is attained at a boundary point, then the conclusion follows from Proposition 6.5. Therefore, assume $\max Q$ is attained at an interior point $p \in M^3$. At p,

$$\begin{aligned} \nabla Q &= 0, \\ -\nabla^2 Q + (1+\epsilon)(\Delta Q)g &\leqslant 0. \end{aligned}$$

Introduce a local frame field near p. Then

$$\begin{aligned} \nabla_j Q &= \epsilon\nabla_j(\Delta u) + 2\nabla_j|\nabla u|^2 \\ &= \epsilon\nabla_j(\Delta u) + 4\nabla_j\nabla_k u\nabla_k u, \end{aligned} \tag{6.6}$$

and

$$\begin{aligned} \nabla_i\nabla_j Q &= \epsilon\nabla_i\nabla_j(\Delta u) + 4\nabla_i\big(\nabla_j\nabla_k u\nabla_k u\big) \\ &= \epsilon\nabla_i\nabla_j(\Delta u) + 4\nabla_i\nabla_j\nabla_k u\nabla_k u + 4\nabla_i\nabla_k u\nabla_j\nabla_k u. \end{aligned} \tag{6.7}$$

Commuting derivatives in the leading terms above gives

$$\begin{aligned} &\epsilon\nabla_i\nabla_j(\Delta u) + 4\nabla_i\nabla_j\nabla_k u\nabla_k u \\ &\qquad = \epsilon\Delta\big(\nabla_i\nabla_j u\big) + 4\nabla_k\big(\nabla_i\nabla_j u\big)\nabla_k u + O(\epsilon|\nabla^2 u| + |\nabla u|^2), \end{aligned}$$

hence

$$\begin{aligned} \nabla_i\nabla_j Q = {}& \epsilon\Delta\big(\nabla_i\nabla_j u\big) + 4\nabla_k\big(\nabla_i\nabla_j u\big)\nabla_k u \\ &+ 4\nabla_i\nabla_k u\nabla_j\nabla_k u + O(\epsilon|\nabla^2 u| + |\nabla u|^2). \end{aligned} \tag{6.8}$$

Therefore,

$$\begin{aligned} & -\nabla_i\nabla_j Q + (1+\epsilon)(\Delta Q)g_{ij} \\ &= \epsilon\Delta\big\{-\nabla_i\nabla_j u + (1+\epsilon)(\Delta u)g_{ij}\big\} + 4\nabla_k\big\{-\nabla_i\nabla_j u + (1+\epsilon)(\Delta u)g_{ij}\big\}\nabla_k u \\ &\qquad + 4\big\{-\nabla_i\nabla_k u\nabla_j\nabla_k u + (1+\epsilon)|\nabla^2 u|^2 g_{ij}\big\} + O(\epsilon|\nabla^2 u| + |\nabla u|^2). \end{aligned} \tag{6.9}$$

Recall

$$(W_t)_{ij} = W_{ij} = (S_t)_{ij} - \nabla_i\nabla_j u + (1+\epsilon)(\Delta u)g_{ij} + \nabla_i u\nabla_j u,$$

so that

$$\begin{aligned} & -\nabla_i\nabla_j Q + (1+\epsilon)(\Delta Q)g_{ij} \\ &= \epsilon\Delta\big\{W_{ij} - \nabla_i u\nabla_j u - (S_t)_{ij}\big\} + 4\nabla_k\big\{W_{ij} - \nabla_i u\nabla_j u - (S_t)_{ij}\big\}\nabla_k u \\ &\qquad + 4\big\{-\nabla_i\nabla_k u\nabla_j\nabla_k u + (1+\epsilon)|\nabla^2 u|^2 g_{ij}\big\} + O(\epsilon|\nabla^2 u| + |\nabla u|^2). \end{aligned} \tag{6.10}$$

For the first two terms on the right-hand side of (6.10), at p we have

$$\begin{aligned}
&\epsilon\Delta\big\{W_{ij}-\nabla_i u\nabla_j u-(S_t)_{ij}\big\}+4\nabla_k\big\{W_{ij}-\nabla_i u\nabla_j u-(S_t)_{ij}\big\}\nabla_k u\\
&\quad=\epsilon\Delta W_{ij}-\epsilon\nabla_i(\Delta u)\nabla_j u-\epsilon\nabla_i u\nabla_j(\Delta u)-2\epsilon\nabla_i\nabla_k u\nabla_j\nabla_k u\\
&+4\nabla_k W_{ij}\nabla_k u-4\nabla_i\nabla_k u\nabla_j u\nabla_k u-4\nabla_i u\nabla_j\nabla_k u\nabla_k u+O(1+|\nabla u|^2)\\
&\quad=\epsilon\Delta W_{ij}+4\nabla_k W_{ij}\nabla_k u-\nabla_i Q\nabla_j u-\nabla_i u\nabla_j Q\\
&\qquad-2\epsilon\nabla_i\nabla_k u\nabla_j\nabla_k u+O(1+|\nabla u|^2)\\
&\quad=\epsilon\Delta W_{ij}+4\nabla_k W_{ij}\nabla_k u-2\epsilon\nabla_i\nabla_k u\nabla_j\nabla_k u+O(1+|\nabla u|^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&0\geqslant-\nabla_i\nabla_j Q+(1+\epsilon)(\Delta Q)g_{ij}=\epsilon\Delta W_{ij}+4\nabla_k W_{ij}\nabla_k u\\
&+4\Big\{-(1+\frac{\epsilon}{2})\nabla_i\nabla_k u\nabla_j\nabla_k u+(1+\epsilon)|\nabla^2 u|^2 g_{ij}\Big\}+O(\epsilon|\nabla^2 u|+|\nabla u|^2)\\
&\geqslant\epsilon\Delta W_{ij}+4\nabla_k W_{ij}\nabla_k u+2\epsilon|\nabla^2 u|^2 g_{ij}+O(\epsilon|\nabla^2 u|+|\nabla u|^2).
\end{aligned}\tag{6.11}$$

Pairing the both sides with W^{ij} and summing, we have

$$\begin{aligned}
&0\geqslant W^{ij}\big\{-\nabla_i\nabla_j Q+(1+\epsilon)(\Delta Q)g_{ij}\big\}\\
&\geqslant\epsilon W^{ij}\Delta W_{ij}+4W^{ij}\nabla_k W_{ij}\nabla_k u+W^{ij}\big\{2\epsilon|\nabla^2 u|^2 g_{ij}+O(\epsilon|\nabla^2 u|+|\nabla u|^2)\big\}.
\end{aligned}\tag{6.12}$$

Since $\det W$ is a concave function when $W>0$,

$$\begin{aligned}
W^{ij}\Delta W_{ij}&\geqslant\Delta\log\det W\\
&=\Delta\log(e^{6u})=6\Delta u.
\end{aligned}$$

Also,

$$\begin{aligned}
W^{ij}\nabla_k W_{ij}\nabla_k u&=\langle\nabla\log\det W,\nabla u\rangle\\
&=\langle\nabla\log(e^{6u}),\nabla u\rangle\\
&=6|\nabla u|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&0\geqslant W^{ij}\big\{-\nabla_i\nabla_j Q+(1+\epsilon)(\Delta Q)g_{ij}\big\}\\
&\geqslant 6\epsilon\Delta u+24|\nabla u|^2+W^{ij}\big\{2\epsilon|\nabla^2 u|^2 g_{ij}+O(\epsilon|\nabla^2 u|+|\nabla u|^2)\big\},
\end{aligned}\tag{6.13}$$

from which it easily follows that

$$\epsilon|\nabla^2 u|\leqslant C\tag{6.14}$$

at p. Hence, $Q\leqslant C$ at p, and the Proposition follows. □

Remark 6.7. — It is clear that the preceding interior estimates can be localized: for a solution $u \in C^4$ defined in a ball $B(p, 2r) \subset M^3$, we have

$$\max_{B(x,r)} |\nabla^2 u| \leqslant C(g, r, \max_{B(x,2r)} |\nabla u|). \tag{6.15}$$

Along the same lines, one can localize the interior gradient estimate of Section 5:

$$\max_{B(x,r)} |\nabla u| \leqslant C(g, r, \max_{B(x,2r)} |u|). \tag{6.16}$$

Putting these together, one arrives at the *a priori* local estimate

$$\max_{B(x,r)} \Big[|\nabla u| + |\nabla^2 u| \Big] \leqslant C(g, r, \max_{B(x,2r)} |u|).$$

Remark 6.8. — If the Hessian of u has eigenvalues $\{\epsilon^{-1}, (\epsilon - 1)/2, (\epsilon - 1)/2\}$, then $|\nabla^2 u| \sim \epsilon^{-1}$ while $\det[-\nabla^2 u + (1 + \epsilon)(\Delta u) g] \sim 1$. Therefore, the estimate in (6.14) appears to be optimal.

7. The proof of Theorem 2.1

We saw in Section 3 that the set

$$\Omega = \{t \in [0, 1] : (3.3) \text{ admits a solution } u \in C^4 \text{ with } W_t[u] > 0\}$$

is non-empty and open. By the estimates of Sections 4 - 6, we have the *a priori* bounds

$$\|u\|_{L^\infty} + \|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^\infty} \leqslant C(\epsilon, g),$$

for any solution of (3.3). It follows from the result of Evans [4] and Krylov [6] that

$$\|u\|_{C^{2,\alpha}} \leqslant C,$$

for some $\alpha > 0$. Then, by the Schauder estimates we have estimates on the C^k-norm of solutions, for all $k \geqslant 1$. Therefore, Ω is closed, hence $\Omega = [0, 1]$. It follows that $(*)_\epsilon$admits a solution u which is smooth up to the boundary, and uniqueness follows from the maximum principle.

To simplify the exposition we only considered zero boundary values; however, it is straightforward to extend this result to arbitrary smooth boundary data:

Corollary 7.1. — *Let $\varphi \in C^\infty(\partial M^3)$. For each $\epsilon > 0$ there is a unique solution of*

$$(*)_\epsilon \quad \det\big((-S) - \nabla^2 u + (1+\epsilon)(\Delta u) g + du \otimes du\big)^{1/n} = e^{2u} \quad \textit{in } M^3,$$
$$u = \varphi \quad \textit{on } \partial M^3$$

which is smooth up to the boundary. Moreover, u satisfies the ellipticity condition with the ellipticity condition

$$(-S) - \nabla^2 u + (1+\epsilon)(\Delta u) g + du \otimes du > 0. \tag{7.1}$$

8. Building a complete metric

In this Section we follow the arguments of [7] and [5] to show that one can construct complete metrics in the interior of M^3 satisfying the pinching condition of Theorem 1.8. For $j = 1, 2, \ldots$ we let u_j denote the unique solution given by Corollary 7.1 with $u_j = j$ on ∂M^3. By the maximum principle, the sequence $\{u_j\}$ is monotone increasing. Also, by the Remark at the end of Section 3, for each compact set K in the interior of M^3 we have $\max u_j \leqslant C(K)$. By the interior estimates above, we can take the limit

$$u = \lim_{j \to \infty} u_j$$

to obtain a smooth solution of

$$\det\big((-S) - \nabla^2 u + (1+\epsilon)(\Delta u) g + du \otimes du\big)^{1/3} = e^{2u}$$

in the interior of M^3. All that remains to demonstrate is the $\hat{g} = e^{2u} g$ is complete.

Although it is possible to adapt the arguments of [5], Section 5, to obtain more precise control of the asymptotic behavior of u near ∂M^3, we will use a simpler argument to prove a lower bound of the growth rate. This will suffice to prove completeness, which is sufficient for our purposes. Let $\rho = \rho(x)$ denote the distance to the boundary, and let $\theta > 0$ be small. Choose $\rho_0 > 0$ small enough so that ρ is smooth on the collar neighborhood

$$\mathcal{U}_0 = \{x \in M^3 \ : \ \rho(x) < \rho_0 \}.$$

Define

$$\underline{\mathrm{w}} = -\log(\rho + \theta) + \log(\rho_0 + \theta).$$

Then $\underline{\mathrm{w}}$ is smooth in $\mathcal{U}_0$, and

$$\underline{\mathrm{w}} = \log(1 + \rho_0/\theta) \quad \text{on } \partial M^3,$$
$$\underline{\mathrm{w}} = 0 \quad \text{on } M^3 \cap \{\rho = \rho_0\}.$$

One can compute

$$\begin{aligned}W[\underline{\mathrm{w}}] &= (-S) - \nabla^2\underline{\mathrm{w}} + (1+\epsilon)(\Delta\underline{\mathrm{w}})g + d\underline{\mathrm{w}}\otimes d\underline{\mathrm{w}}\\ &= (-S) + \frac{1}{\rho+\theta}\big[\nabla^2\rho - (1+\epsilon)(\Delta\rho)g\big] + \frac{(1+\epsilon)}{(\rho+\theta)^2}g\\ &= \frac{1}{(\rho+\theta)^2}\Big\{(1+\epsilon)g + (\rho+\theta)\big[\nabla^2\rho - (1+\epsilon)(\Delta\rho)g\big] + (\rho+\theta)^2(-S)\Big\}.\end{aligned}$$

For fixed $\epsilon > 0$, if $\rho_0, \theta > 0$ are chosen small enough, then

$$W[\underline{\mathrm{w}}] \geqslant \frac{1}{(\rho+\theta)^2}g,$$

hence

$$\begin{aligned}\Psi_t[\underline{\mathrm{w}}] &= (\det W[\underline{\mathrm{w}}])^{1/3} - e^{2\underline{\mathrm{w}}}\\ &\geqslant \frac{1}{(\rho+\theta)^2} - \frac{(\rho_0+\theta)^2}{(\rho+\theta)^2}\\ &\geqslant 0,\end{aligned}$$

for $\rho_0, \theta > 0$ small enough. Therefore, for $j >> 1$ large, $u_j \geqslant \underline{\mathrm{w}}$ on $\partial\mathcal{U}_0$, so by the maximum principle $u_j \geqslant \underline{\mathrm{w}}$ on $\mathcal{U}_0$. Letting $j \to \infty$, we conclude

$$e^{2u(x)} \geqslant \frac{(\rho_0+\theta)^2}{(\rho(x)+\theta)^2}$$

near ∂M^3. Since $\theta > 0$ was arbitrary, we get

$$e^{2u(x)} \geqslant \frac{C}{\rho(x)^2},$$

and it follows that $e^{2u}g$ is complete.

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