ANNALES

DE

L'INSTITUT FOURIER

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Tome 60, nº 6 (2010), p. 2115-2135.

<http://aif.cedram.org/item?id=AIF_2010__60_6_2115_0>

Ann. Inst. Fourier, Grenoble
60, 6 (2010) 2115-2135

THE C^1 INVARIANCE OF THE ALGEBRAIC MULTIPLICITY OF A HOLOMORPHIC VECTOR FIELD

by Rudy ROSAS

Abstract. — We prove that the algebraic multiplicity of a holomorphic vector field at an isolated singularity is invariant by C^1 equivalences.

Résumé. — On démontre que la multiplicité algébrique d'une singularité d'un champ de vecteurs holomorphe est invariante par C^1-equivalences.

1. Introduction

Given a curve $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, singular at $0 \in \mathbb{C}^2$, we define its *algebraic multiplicity* as the degree of the first nonzero jet of f, that is, $\nu(f) = \nu$ where

$$f = f_\nu + f_{\nu+1} + \cdots$$

is the Taylor development of f and $f_\nu \neq 0$. A well known result by Burau [2] and Zariski [15] states that ν is a *topological invariant*, that is, given $\widetilde{f} : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and a homeomorphism $\mathrm{h} : U \to \widetilde{U}$ between neighborhoods of $0 \in \mathbb{C}^2$ such that $\mathrm{h}(f^{-1}(0) \cap U) = \widetilde{f}^{-1}(0) \cap V$ then $\nu(f) = \nu(\widetilde{f})$. Consider now a holomorphic vector field Z in $\mathbb{C}^2$ with a singularity at $0 \in \mathbb{C}^2$. If

$$Z = Z_\nu + Z_{\nu+1} + \cdots, Z_\nu \neq 0$$

we define $\nu = \nu(Z)$ as the *algebraic multiplicity* of Z. The vector field Z defines a holomorphic foliation by curves $\mathcal{F}$ with isolated singularity in a neighborhood of $0 \in \mathbb{C}^2$ and the algebraic multiplicity $\nu(Z)$ depends only on the foliation $\mathcal{F}$. A natural question, posed by J.F.Mattei is: is $\nu(\mathcal{F})$ a

Keywords: Algebraic multiplicity, holomorphic vector field, holomorphic foliation.
Math. classification: 37F75.

topological invariant of $\mathcal{F}$?. In [4], the authors give a positive answer if $\mathcal{F}$ is a *generalized curve*, that is, if the desingularization of $\mathcal{F}$ does not contain complex saddle-nodes. In this work, we consider the problem in dimension $n \geqslant 2$ and impose conditions on the topological equivalence. Let $\mathcal{F}$ be a holomorphic foliation by curves of a neighborhood U of $0 \in \mathbb{C}^n$ with a unique singularity at $0 \in \mathbb{C}^n (n \geqslant 2)$. We assume that $\mathcal{F}$ is generated by the holomorphic vector field

$$V = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i},\ a_i \in \mathcal{O}_U,\ g.c.d.(a_1, a_2, \ldots, a_n) = 1.$$

The algebraic multiplicity of $\mathcal{F}$ (at $0 \in \mathbb{C}^n$) is the minimum vanishing order at $0 \in \mathbb{C}^n$ of the functions a_i. Let $\widetilde{\mathcal{F}}$ be another holomorphic foliation by curves of a neighborhood $\widetilde{U}$ of $0 \in \mathbb{C}^n$ and let $\mathrm{h} : U \to \widetilde{U}$ be a topological equivalence between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$, that is, a homeomorphism taking leaves of $\mathcal{F}$ to leaves of $\widetilde{\mathcal{F}}$. Let $\pi : \widehat{\mathbb{C}^n} \to \mathbb{C}^n$ be the quadratic blow up with center at $0 \in \mathbb{C}^n$. Clearly the map $h := \pi^{-1}\,\mathrm{h}\,\pi$ is a homeomorphism between $\pi^{-1}(U\backslash\{0\})$ and $\pi^{-1}(\widetilde{U}\backslash\{0\})$. Then we prove the following:

Theorem 1.1. — *Suppose that h extends to the divisor $\pi^{-1}(0)$ as a homeomorphism between $\pi^{-1}(U)$ and $\pi^{-1}(\widetilde{U})$. Then the algebraic multiplicities of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are the same.*

If h is a C^1 diffeomorphism, we prove that h extends to the divisor. Thus, we obtain that the algebraic multiplicity is invariant by C^1 equivalences:

Theorem 1.2. — *Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be two foliations by curves of neighborhoods U and $\widetilde{U}$ of $0 \in \mathbb{C}^n$, $n \geqslant 2$. Let $\mathrm{h} : U \to \widetilde{U}$ be a C^1 equivalence between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$, that is, a C^1 diffeomorphism taking leaves of $\mathcal{F}$ to leaves of $\widetilde{\mathcal{F}}$. Then the algebraic multiplicities of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are equal.*

It is known that there exists a unique way of extending the pull back foliation $\pi^*(\mathcal{F}|_{U\backslash\{0\}})$ to a singular analytic foliation $\mathcal{F}_0$ on $\pi^{-1}(U)$ with singular set of codimension $\geqslant 2$. We say that $\mathcal{F}_0$ is the strict transform of $\mathcal{F}$ by π. Let $\widetilde{\mathcal{F}}_0$ be the strict transform of $\widetilde{\mathcal{F}}$ by π. In order to prove Theorem 1.1 we show that the algebraic multiplicity of $\mathcal{F}$ depends on the Chern class of the tangent bundle of $\mathcal{F}_0$. To relate the Chern classes of the tangent bundles of $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$ we use the following theorem (see [7]).

Theorem 1.3. — *Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be foliations by curves on the complex manifolds M and $\widetilde{M}$ respectively. Let $c(T\mathcal{F})$ denote the Chern class of the tangent bundle $T\mathcal{F}$ of $\mathcal{F}$. Let $h : M \to \widetilde{M}$ be a topological equivalence between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ and consider the map $h^* : H^2(M, \mathbb{Z}) \to H^2(\widetilde{M}, \mathbb{Z})$ induced in the cohomology. Then $h^*(c(T\mathcal{F})) = c(T\widetilde{\mathcal{F}})$.*

Clearly the homeomorphism $h : \pi^{-1}(U\backslash\{0\}) \to \pi^{-1}(\widetilde{U}\backslash\{0\})$ is a topological equivalence between $\mathcal{F}_0|_{\pi^{-1}(U\backslash\{0\})}$ and $\widetilde{\mathcal{F}}_0|_{\pi^{-1}(\widetilde{U}\backslash\{0\})}$. To be able to apply Theorem 1.3 we show that h extends as a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$. This is the non trivial part of the proof. Thus, we prove the following.

Theorem 1.4. — *Let V and $\widetilde{V}$ be complex manifolds, let $Y \subset V$ and $\widetilde{Y} \subset \widetilde{V}$ be analytic subvarieties of codimension $\geqslant 1$ and, let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be holomorphic foliations by curves on V and $\widetilde{V}$ respectively. Suppose there is a homeomorphism h between V and $\widetilde{V}$ with $h(Y) = \widetilde{Y}$ and such that $h|_{V\backslash Y}$ is a topological equivalence between $\mathcal{F}|_{V\backslash Y}$ and $\widetilde{\mathcal{F}}|_{\widetilde{V}\backslash\widetilde{Y}}$. Then h is a topological equivalence between $\mathcal{F}$ and $\widetilde{\mathcal{F}}$.*

This paper is organized as follows. In section 2 we prove Theorem 1.4. In section 3 we relate the algebraic multiplicity of the foliation and the Chern class of its strict transform, and prove Theorem 1.1. Finally, section 4 discusses the C^1 case.

The contents of this paper originally comprised a Ph.D. dissertation at Instituto de Matematica Pura e Aplicada, Rio de Janeiro. The author would like to thank his advisor, César Camacho, for guidance and support. I also thank Alcides Lins Neto, Paulo Sad, Luis Gustavo Mendes and specially Jorge Vitório Pereira for the remarks that helps in the redaction of the present paper.

2. An extension theorem

This section is devoted to prove Theorem 1.4. We start with some definitions. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{B} = \{z \in \mathbb{C}^{n-1} : ||z|| < 1\}$ where $n \geqslant 2$. Let M be a complex manifold of complex dimension n and let D be a subset of M homeomorphic to a disc. We say that D is a *singular disc* if for all $x \in D$ there exist a neighborhood $\mathcal{D}$ of x in D, and an injective holomorphic function $f : \mathbb{D} \to M$ such that $f(\mathbb{D}) = \mathcal{D}$, $f(0) = x$. If $f'(0) = 0$ we say that x is a *singularity* of D, otherwise x is a *regular point* of D (this does not depend on f). The set S of singularities of D is discrete and closed in D and we have that $D\backslash S$ is a complex submanifold of M. Thus, if x is a regular point of D, there is a neighborhood U of x in M and holomorphic coordinates (w, z), $w \in \mathbb{B}$, $z \in \mathbb{D}$ on U such that $D \cap U$ is represented by $(w = 0)$. If D does not have singularities we say that it is a *regular disc*. In this case, by uniformization, there is a holomorphic map

$f : E \to M$, where $E = \mathbb{D}$ or $\mathbb{C}$, such that f is a biholomorphism between E and D.

Example. — Let $\mathcal{F}$ be a holomorphic foliation by curves on the complex manifold M and let $D \subset M$ be a topological disc contained in a leaf of $\mathcal{F}$. Then D is a regular disc.

The following Lemma will be fundamental in the proof of Theorem 1.4.

Lemma 2.1. — *Let $F : \mathbb{D} \times [0,1] \to \mathbb{C}^n$ be a continuous map such that for all $t \in [0,1]$, the map $F(*,t) : \mathbb{D} \to \mathbb{C}^n$ is a homeomorphism onto its image. Thus, we have a continuous family of discs $D_t := F(\mathbb{D} \times \{t\})$. Suppose D_t is a regular disc for each $t > 0$. Then D_0 is a singular disc.*

Proof. — We give a sketch of the proof. Let $p = F(x_0, 0)$ be any point in D_0. Let $\mathrm{U} \subset \mathbb{D}$ be a disc centered at x_0 and such that $\overline{\mathrm{U}} \subset \mathbb{D}$. Let $t_k > 0$ be such that $t_k \to 0$ as $k \to \infty$ an define $\mathcal{D}_k = F(\mathrm{U} \times \{t_k\})$. By uniformization there is a holomorphic map $f_k : \mathbb{D} \to \mathbb{C}^n$ which is a biholomorphism between $\mathbb{D}$ and $\mathcal{D}_k$. We may assume that $f_k(0) = F(x_0, t_k)$ for all k and it is well known that f_k extends as a homeomorphism $f_k : \overline{\mathbb{D}} \to \overline{\mathcal{D}}_k$. By Montel's theorem we can assume that f_k converges uniformly on compact sets to a holomorphic function $f : \mathbb{D} \to \mathbb{C}^n$, $f(0) = p$. Clearly it is sufficient to show that f is not a constant function ($f \not\equiv p$). Let $\mathbb{S}^1 := \partial \mathbb{D}$ and consider for each k the homeomorphism

$$\varphi_k := f_k|_{\mathbb{S}^1} : \mathbb{S}^1 \to \partial \mathcal{D}_k.$$

By taking a subsequence if necessary, it is not difficult to see that we may assume that φ_k converges a.e. to a function

$$\varphi : \mathbb{S}^1 \to \partial \mathcal{D}_0.$$

Fix $x \in \mathbb{D}$. Since $\{\varphi_k\}$ is uniformly bounded, by the dominated convergence theorem we have that

$$\frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{\varphi_k(w)}{w - x} dw \to \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{\varphi(w)}{w - x} dw \tag{2.1}$$

as $k \to \infty$. By Cauchy's Integral Formula the left part of (2.1) is equal to $f_k(x)$ and, since $f_k(x) \to f(x)$, we conclude that

$$f(x) = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{\varphi(w)}{w - x} dw. \tag{2.2}$$

Finally, it is not difficult to prove from this equation that $f \equiv p$ implies $\varphi = p$ a.e., which is a contradiction because $\varphi(\mathbb{S}^1) \subset \partial \mathcal{D}_0$ and $p \notin \partial \mathcal{D}_0$. $\square$

We now show that Theorem 1.4 is a consequence of the following theorem.

THEOREM 2.2. — *Let $\mathcal{F}$ be a foliation by curves on the complex manifold M. Let $X \subset M$ be an analytic subvariety of codimension $\geqslant 1$. Suppose that:*

- (i) *$\mathcal{F}$ is generated by a holomorphic vector field.*
- (ii) *There exists a homeomorphism $h : \Sigma \times D \to M$, where Σ is a ball in $\mathbb{C}^{n-1}$ and D is a disc in $\mathbb{C}$.*
- (iii) *If $D_z := h(\{z\} \times D)$ then for all z: either D_z is contained in X, or $D_z \cap X$ is discrete and $D_z \backslash X$ is contained in a leaf of $\mathcal{F}$.*

Then $\mathcal{F}$ is regular and the sets D_z are the leaves of $\mathcal{F}$.

Proof of Theorem 1.4. — Let p be a point in Y which is regular for $\mathcal{F}$. Let Σ denote a ball in $\mathbb{C}^{n-1}$ and D a disc in $\mathbb{C}$. Consider a neighborhood W of p on which $\mathcal{F}$ is a product foliation, that is, $W \simeq \Sigma \times D$ and the sets $\{z\} \times D$ are the leaves of $\mathcal{F}|_W$. We take W small enough such that $\widetilde{\mathcal{F}}$ restricted to $M := h(W)$ is generated by a holomorphic vector field. Let X be the intersection between M and $\widetilde{Y}$. We will show that the hypothesis of Theorem 2.2 hold for $\widetilde{\mathcal{F}}$ restricted to M. Hypothesis (i) and (ii) of 2.2 evidently hold. Let $D_z = h(\{z\} \times D)$. Then it is easy to see that

ASSERTION 1. — *For all $z \in \Sigma$, either $\{z\} \times D$ is contained in Y, or $S'_z := (\{z\} \times D) \cap Y$ is discrete and closed in $\{z\} \times D$.*

Suppose that D_z is not contained in X. Let $S_z = h(S'_z)$, where S'_z is given by Assertion 1. Then S_z is discrete in D_z. Observe that $(\{z\} \times D)\backslash S'_z$ is contained in a leaf of $\mathcal{F}|_{M\backslash Y}$. Then, since $h|_{M\backslash Y}$ is a topological equivalence between $\mathcal{F}|_{V\backslash Y}$ and $\widetilde{\mathcal{F}}|_{\widetilde{V}\backslash\widetilde{Y}}$, it follows that

$$D_z\backslash S_z = h((\{z\} \times D)\backslash S'_z)$$

is contained in a leaf of $\widetilde{\mathcal{F}}$. Thus, hypothesis (iii) of 2.2 holds. Then $\widetilde{\mathcal{F}}$ is regular on $M = h(W)$ and every D_z is contained in a leaf of $\widetilde{\mathcal{F}}$. Therefore we conclude:

ASSERTION 2. — *If p is a point in Y which is regular for $\mathcal{F}$, then p is mapped by h to a regular point of $\widetilde{\mathcal{F}}$. Moreover, there exists a neighborhood Ω of p in its leaf which is mapped by h onto a neighborhood of $h(p)$ in its leaf.*

Now, by using Assertion 2 for h and h^{-1}, we deduce that p is regular for $\mathcal{F}$ if and only if $h(p)$ is regular for $\widetilde{\mathcal{F}}$. Hence

$$h(\mathrm{Sing}(\mathcal{F})) = \mathrm{Sing}(\widetilde{\mathcal{F}}).$$

It remains to prove that h maps any leaf of $\mathcal{F}$ onto a leaf of $\widetilde{\mathcal{F}}$. Let p be a regular point of $\mathcal{F}$. Let L be the leaf of $\mathcal{F}$ passing through p and let

$\widetilde{L}$ be the leaf of $\widetilde{\mathcal{F}}$ passing through $h(p)$. Let A be the set of points in L which are mapped by h into $\widetilde{L}$. By Assertion 2, if $x \in A$ there exists a neighborhood of x in L_p contained in A. Therefore A is open. Now, let $x \notin A$. Then $h(x) \notin \widetilde{L}$. Thus, if $L' \neq L$ is the leaf of $\widetilde{\mathcal{F}}$ passing through $h(x)$ it follows by Assertion 2 that there exists a neighborhood Ω of x in L which is mapped by h into $L' \neq \widetilde{L}$, hence Ω is contained in $L \backslash A$. Then A is also closed and it follows by conectedness that $A = L$, that is, $h(L) \subset \widetilde{L}$. Analogously, we prove that $h^{-1}(\widetilde{L}) \subset L$. Therefore $h(L) = \widetilde{L}$. □

We proceed now to prove Theorem 2.2.

Proposition 2.3. — *Let $\mathcal{F}$ be a foliation by curves on the complex manifold M. Let $X \subset M$ be an analytic subvariety of codimension $\geqslant 1$. Suppose that:*

(*i*) *There exists a homeomorphism $h : \Sigma \times D \to M$, where Σ is a ball in $\mathbb{C}^{n-1}$ and D is a disc in $\mathbb{C}$.*

(*ii*) *If $D_z := h(\{z\} \times D)$ then for all z: either D_z is contained in X, or D_z is contained in a leaf of $\mathcal{F}$.*

Consider $z' \in \Sigma$ and suppose that $D_{z'}$ is a singular disc. Let $S_{z'}$ the set of singularities of $D_{z'}$. Then $D_{z'} \backslash S_{z'}$ is contained in a leaf of $\mathcal{F}$.

Proof. — It is sufficient to prove the following.

Assertion. — *If $p \in D_{z'} \backslash S_{z'}$ then p has a neighborhood in $D_{z'} \backslash S_{z'}$ contained in a leaf of $\mathcal{F}$.*

Suppose Assertion holds. Let L be a leaf of $\mathcal{F}$ and let $x \in (D_{z'} \backslash S_{z'}) \cap L$. By Assertion, there is a neighborhood Δ of x in $D_{z'} \backslash S_{z'}$ such that $\Delta \subset L$. Then $\Delta \subset (D_{z'} \backslash S_{z'}) \cap L$ and it follows that the intersection of $D_{z'} \backslash S_{z'}$ with any leaf is open in $D_{z'} \backslash S_{z'}$. Then, since $D_{z'} \backslash S_{z'}$ is connected, we have that it is contained in a unique leaf.

Proof of Assertion. — Let p in $D_{z'} \backslash S_{z'}$. Since p is a regular point of the singular disc $D_{z'}$, on a neighborhood $U \subset M$ of p we may consider coordinates (w, y), $w \in \mathbb{B}$, $y \in \mathbb{D}$ with $p = (0, 0)$ and such that $D_{z'} \cap U$ is represented by $(w = 0)$. Suppose that $p = h(z', t')$. Let Σ' be a ball in Σ containing z' and let D' be a disc in D containing t'. Then $W = \Sigma' \times D'$ is a neighborhood of (z', t') and, by taking W small enough, we assume $h(\overline{W}) \subset U$. Let $D'_z = h(\{z\} \times D')$. Note that $D'_{z'} \subset D_{z'} \cap U$, hence $D'_{z'}$ is contained in $(w = 0)$. Let $g : U \to \mathbb{D}$ be the projection $g(w, y) = y$. Consider $z \in \Sigma'$ and suppose $D_z \backslash X \neq \emptyset$. By hypothesis (ii), D_z is contained in a leaf of $\mathcal{F}$. Therefore D'_z is contained in leaf of $\mathcal{F}$ and we have that $g|_{D'_z} : D'_z \to \mathbb{D}$ is a holomorphic map. Remember that $D'_{z'} \subset (w = 0)$.

Then $g|_{D'_{z'}} : D'_{z'} \to \mathbb{D}$ is given by $(0,y) \to y$ and is therefore a one to one map. Then $g(D'_{z'})$ is a disc in $\mathbb{D}$ with $g(\partial D'_{z'})$ as boundary. Note that $p = (0,0) \in D'_{z'}$, hence 0 is contained in the disc $g(D'_{z'})$. Therefore the curve $g(\partial D'_{z'})$ winds once around 0. By the continuity of h we assume Σ' small enough such that $g(\partial D'_z)$ is homotopic to $g(\partial D'_{z'})$ in $\mathbb{D}\backslash\{0\}$ for all $z \in \Sigma'$. Then $g(\partial D'_z)$ winds once around 0 and $g|_{D'_z}$ has therefore a unique zero. In other words, the plaque D'_z intersects $Y = \mathbb{B} \times \{0\} \subset U$ at a unique point. Thus, we can define the map $f : h(W)\backslash X \to Y$ by $f(D'_z\backslash X) = D'_z \cap Y$ whenever $D'_z\backslash X \neq \emptyset$. We have that f is holomorphic because it is constant along the leaves and, restricted to any transversal, is a holonomy map. Since f is bounded and X has codimension $\geqslant 1$, by the generalized Riemann's extension theorem, f extends to a holomorphic function on $h(W)$. Observe that f restricted to Y is the identity map, then f is a submersion in a neighborhood V of Y. Hence f defines a regular foliation $\mathcal{N}$ on V. It is easy to see that $\mathcal{N}$ coincides with $\mathcal{F}$ on $V\backslash X$, thus $\mathcal{N} = \mathcal{F}$. Therefore $p \in Y$ is a regular point of $\mathcal{F}$.

Now, by reducing the neighborhood $W = \Sigma' \times D'$ of (z',t'), we may assume that $h(W)$ is contained in a neighborhood of p where $\mathcal{F}$ is given by a submersion f. Obviously $D'_{z'}$ is a neighborhood of p in D_z. We shall prove that $D'_{z'}$ is contained in a leaf of $\mathcal{F}$ (the leaf passing through p). If $D'_{z'}$ is not contained in X, so is $D_{z'}$ and, by hypothesis (ii), we have that $D'_{z'}$ is contained in a leaf of $\mathcal{F}$. On the other hand, suppose that $D'_{z'}$ is contained in X. Then there exists a sequence of points $z_k \to z'$ such that $h(\{z_k\} \times D)$ is not contained in X, otherwise $h(\Sigma'' \times D) \subset X$ for some neighborhood $\Sigma'' \subset \Sigma$ of z', which is a contradiction because X has codimension $\geqslant 1$. Thus, by (ii), we have that D'_{z_k} is contained in a leaf of $\mathcal{F}$ for all k. Recall $D'_{z_k} \subset h(W)$ is contained in a domain where $\mathcal{F}$ is given by the submersion f. Then f is constant over $D'_{z_k} = h(\{z_k\} \times D')$ and in particular, for all $t \in D'$ we have $f(h(z_k,t)) = f(h(z_k,t'))$. Then:

$$\begin{aligned} f(h(z',t)) &= f(h(\lim_{k\to\infty} z_k, t)) = \lim_{k\to\infty} f(h(z_k,t)) \\ &= \lim_{k\to\infty} f(h(z_k,t')) = f(h(\lim_{k\to\infty} z_k, t')) \\ &= f(h(z',t')). \end{aligned}$$

Therefore, for all $t \in D'$ we have that $h(z',t)$ and $h(z',t')$ are contained in the same leaf. It follows that $D'_{z'}$ is contained in the leaf passing trough $h(z',t')$. Thus, Assertion is proved. □

Proposition 2.4. — *Let $\mathcal{F}$ be a foliation by curves on the complex manifold M such that:*

(*i*) *$\mathcal{F}$ is generated by a holomorphic vector field.*

(*ii*) *There exists a homeomorphism $h : \Sigma \times D \to M$, where Σ is a ball in $\mathbb{C}^{n-1}$ and D is a disc in $\mathbb{C}$.*

(*iii*) *For all z, there is a discrete closed set $S_z \subset D_z := h(\{z\} \times D)$ such that $D_z \backslash S_z$ is contained in a leaf of $\mathcal{F}$.*

Then $\mathcal{F}$ is regular and the sets D_z are the leaves of $\mathcal{F}$.

The following lemmas are easy consequences of well known facts and we left the proofs to the reader.

Lemma 2.5. — *Let $f : \overline{\mathbb{D}} \to \mathbb{C}$ be smooth, and holomorphic on $\mathbb{D}$. Suppose that f is regular on $\mathbb{S}^1 := \overline{\mathbb{D}}$. Then f is a regular map if and only if the curve $f|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{C}$ has degree 1*[(1)]*.*

Lemma 2.6. — *Let M be a complex manifold and $D \subset M$ a singular disc. Then there exists a holomorphic injective map $g : E \to M$, where $E = \mathbb{D}$ or $\mathbb{C}$, such that $g(E) = D$.*

Lemma 2.7. — *Let $\mathcal{D} \subset \mathbb{C}^n$ be a set homeomorphic to a disc such that for some point $p \in \mathcal{D}$ the annulus $\mathcal{D}\backslash\{p\}$ is a complex submanifold. Then $\mathcal{D}$ is a singular disc.*

Proof of Proposition 2.4. —

Assertion 1. — *For all z, we have that D_z is a singular disc and the sets $D_z \backslash \operatorname{Sing}(\mathcal{F})$ are the nonsingular leaves of $\mathcal{F}$.*

Proof. — Let $x \in D_z$. Since S_z is a discrete closed subset of D_z, there is a disc $\mathcal{D} \subset D_z$ with $x \in \mathcal{D}$ such that $\mathcal{D}\backslash\{x\} \subset D_z \backslash S_z$. Then, from hypothesis (*iii*), $\mathcal{D}\backslash\{x\}$ is contained in a leaf of $\mathcal{F}$. If $\mathcal{D}$ is small enough, we may think that $\mathcal{D}$ is contained in $\mathbb{C}^n$. Hence, by applying Lemma 2.7, there exists a holomorphic injective map $g : \mathbb{D} \to M$ with $g(\mathbb{D}) = \mathcal{D}$. Since that $x \in D_z$ was arbitrary, it follows that D_z is a singular disc.

Let L be a leaf of $\mathcal{F}$ and suppose that $x \in L \cap (D_z \backslash \operatorname{Sing}(\mathcal{F}))$ for some z. Take $\mathcal{D} \subset D_z$ as above. We assume $\mathcal{D}$ small enough such that it is contained in a neighborhood U of x where $\mathcal{F}$ is trivial and given by the submersion f. Then $\mathcal{D}\backslash\{x\}$ is contained in a leaf of $\mathcal{F}|_U$ and f is therefore constant over $\mathcal{D}\backslash\{x\}$. Hence, by continuity, f is constant over $\mathcal{D}$. Then $\mathcal{D}$ is contained in a leaf of $\mathcal{F}|_U$ and we have therefore $\mathcal{D} \subset L$. Thus we have $\mathcal{D} \subset L \cap (D_z \backslash \operatorname{Sing}(\mathcal{F}))$. It follows that $L \cap (D_z \backslash \operatorname{Sing}(\mathcal{F}))$ is an open subset of both L and $D_z \backslash \operatorname{Sing}(\mathcal{F})$ for all L and z. Now, fix a leaf L.

[(1)] The degree of a parameterized regular curve in the plane is defined as the winding number around 0 of its velocity vector.

Since the intersection of L with any $D_z \setminus \mathrm{Sing}(\mathcal{F})$ is open in L, it follows by connectedness that L is contained in a unique $D_z \setminus \mathrm{Sing}(\mathcal{F})$. For this $D_z \setminus \mathrm{Sing}(\mathcal{F})$, we also have that its intersection with any leaf is open in $D_z \setminus \mathrm{Sing}(\mathcal{F})$. Again by connectedness $D_z \setminus \mathrm{Sing}(\mathcal{F})$ is contained in a unique leaf, thus we necessarily have that $D_z \setminus \mathrm{Sing}(\mathcal{F}) = L$. Therefore Assertion 1 is proved.

Fix $p \in M$. We have $p \in D_{z'}$ for some $z' \in \Sigma$. Take p' in $D_{z'} \setminus S_{z'}$. From hypothesis (iii), p' is a regular point of $\mathcal{F}$. We have $p' = h(z', t')$ with $t' \in D$. If $B \subset \Sigma$ is a ball containing z', then $\Sigma_0 := B \times \{t'\}$ is a $(n-1)$ ball passing through (z', t'). We assume B small enough such that $\overline{\Sigma}_0$ is mapped by h into a neighborhood W of p' where $\mathcal{F}$ is equivalent to a product foliation. Let $\widetilde{\Sigma}$ (submanifold of W) be a global transversal to $\mathcal{F}|_W$. If w is a point contained in $\overline{h(\Sigma_0)}$, the leaf of $\mathcal{F}|_W$ passing through it intersects $\widetilde{\Sigma}$ in a unique point $\psi(w)$. We claim that ψ is a homeomorphism of $h(\Sigma_0)$ onto its image. Since $\overline{h(\Sigma_0)}$ is compact, it suffices to prove that ψ is injective on $\overline{h(\Sigma_0)}$. Suppose that w_1 and w_2 are two points in $\overline{h(\Sigma_0)}$ contained in the same leaf L of $\mathcal{F}|_W$. From Assertion 1, we have that $L \subset D_z$ for some z. Then $h^{-1}(L) \subset \{z\} \times D$, hence $h^{-1}(w_1)$ and $h^{-1}(w_2)$ are two different points in the intersection of $(z \times D)$ with $\overline{\Sigma}_0$, which is a contradiction because $\overline{\Sigma}_0 \subset \Sigma \times \{t'\}$ intersects $(z \times D)$ only at (z, t').

If we redefine $\widetilde{\Sigma}$ as $\widetilde{\Sigma} = \psi(h(\Sigma_0))$, it follows from above that for all $z \in B$, D_z intersects $\widetilde{\Sigma}$ at the unique point $\psi(h(z, t_0))$. Thus we may define the map

$$g : V = h(B \times \mathbb{D}) \to \widetilde{\Sigma},$$

$$g(D_z) = D_z \cap \widetilde{\Sigma}.$$

By Assertion 1, each leaf of $\mathcal{F}$ is contained in some D_z. Then g is constant along the leaves. Therefore, since the restriction of g to any transversal is a holonomy map, we have that g is holomorphic on $V \setminus \mathrm{Sing}(\mathcal{F})$. Actually, since $\mathrm{Sing}(\mathcal{F})$ has codimension $\geqslant 2$, g is holomorphic on V.

Consider $x \in \widetilde{\Sigma} \setminus g(\mathrm{Sing}(\mathcal{F}))$. Then $D = g^{-1}(x)$ does not intersect $\mathrm{Sing}(\mathcal{F})$. Clearly D is equal to some D_z. Then, by Assertion 1, $D \setminus \mathrm{Sing}(\mathcal{F}) = D$ is a leaf of $\mathcal{F}$. Thus, we conclude that for all $x \in \widetilde{\Sigma} \setminus g(\mathrm{Sing}(\mathcal{F}))$, the leaf passing through x is simply connected. Moreover, since $\mathrm{Sing}(\mathcal{F})$ has codimension $\geqslant 2$, we have that $g(\mathrm{Sing}(\mathcal{F}))$ has codimension $\geqslant 1$ in $\widetilde{\Sigma}$ and we have therefore that:

Assertion 2. — *For all x in a dense subset of $\widetilde{\Sigma}$, the leaf passing through x is simply connected.*

Let Z be a holomorphic vector field which generates $\mathcal{F}$ on V and φ the local complex flow of Z. Let L be a leaf of $\mathcal{F}|_V$ and let x_L be its intersection with $\widetilde{\Sigma}$ ($g(L) = \{x_L\}$). There exists $\varepsilon_L > 0$ such that $\varphi(x_L, *)$ maps the disc $|t| < \varepsilon_L$ biholomorphically onto a neighborhood D_L of x_L in L. Thus, given any x in D_L there exists a unique $\tau_L(x)$ with $|\tau_L(x)| < \varepsilon_L$ such that $\varphi(x_L, \tau_L(x)) = x$. The function $\tau_L : D_L \to \mathbb{C}$ is the complex time between x_L and x. Clearly τ_L is holomorphic on D_L.

Assertion 3. — *The function τ_L can be analytically continued on L along any path $\gamma : [0,1] \to L$ with $\gamma(0) = x_L$.*

Proof . — Since γ does not intersect $\mathrm{Sing}(\mathcal{F})$ there exists $\delta > 0$ such that for all x in $\gamma([0,1])$, the map $\varphi(x, *)$ is a biholomorphism between $\mathbb{D}_{2\delta}$ and its image. Denote x_L by x_0 and let $0 = s_0 < s_1 < \cdots < s_r = 1$ and $x_1 = \gamma(s_1), \ldots, x_r = \gamma(s_r)$ be such that:

(*i*) The open sets $\varphi(x_i, \mathbb{D}_\delta)$ for $i = 0, \ldots, r$ cover $\gamma([0,1])$.

(*ii*) x_i is contained in $\varphi(x_{i-1}, \mathbb{D}_\delta)$ for $i = 1, \ldots, r$.

For each $i = 0, \ldots, r$ let $\tau_i' : \varphi(x_i, \mathbb{D}_{2\delta}) \to \mathbb{D}_{2\delta}$ be defined by $\varphi(x_i, \tau_i'(x)) = x$. Let $x \in \varphi(x_{i-1}, \mathbb{D}_\delta) \cap \varphi(x_i, \mathbb{D}_\delta)$. Let $t_i = \tau_{i-1}'(x_i)$ for $i = 1, \ldots, r$ and define $t_0 = 0$. Clearly, $|t_i|$ and $|\tau_i'(x)|$ are less than δ, hence $|t_i + \tau_i'(x)| < 2\delta$ and we have that

$$\begin{aligned} \varphi(x_{i-1}, t_i + \tau_i'(x)) &= \varphi(\varphi(x_{i-1}, t_i), \tau_i'(x)) \\ &= \varphi(\varphi(x_{i-1}, \tau_{i-1}'(x_i)), \tau_i'(x)) \\ &= \varphi(x_i, \tau_i'(x)) \\ &= x. \end{aligned}$$

Then, by definition of τ_{i-1}' we obtain:

$$t_i + \tau_i'(x) = \tau_{i-1}'(x). \tag{2.3}$$

For each $i = 1, \ldots, r$ let τ_i be the holomorphic function on $\varphi(x_i, \mathbb{D}_\delta)$ defined by

$$\tau_i = \tau_i' + t_0 + \cdots + t_i.$$

By using (2.3) we deduce that $\tau_{i-1} = \tau_i$ on $\varphi(x_{i-1}, \mathbb{D}_\delta) \cap \varphi(x_i, \mathbb{D}_\delta)$. Moreover, it follows from the definition that τ_0 is equal to τ_L in a neighborhood of $x_0 = x_L$. Therefore, $\tau_0, \ldots, \tau_r$ give an analytic continuation of τ_L along γ.

Assertion 4. — *Let L be any leaf of $\mathcal{F}|_V$ and let $\gamma', \gamma'' : [0,1] \to L$ be paths such that $\gamma'(0) = \gamma''(0) = x_L$ and $\gamma'(1) = \gamma''(1) = x \in L$. Let τ_L' be the analytic continuation of τ_L along γ' and let τ_L'' be the analytic continuation of τ_L along γ''. Then $\tau_L'(x) = \tau_L''(x)$. Thus, τ_L extends as a* *holomorphic function on L. Therefore we may define $\tau : V\setminus \mathrm{Sing}(\mathcal{F}) \to \mathbb{C}$ by $\tau = \tau_L$ on L. Then τ is holomorphic on $U\setminus \mathrm{Sing}(\mathcal{F})$ and extends to U because $\mathrm{Sing}(\mathcal{F})$ has codimension $\geqslant 2$. Moreover, if restricted to a leaf, τ is a regular map. In particular, τ is a submersion on $U\setminus \mathrm{Sing}(\mathcal{F})$.*

Proof . — Fix L and denote x_L by x_0. Let $0 = s_0 < \cdots < s_r = 1$, let $\Sigma_0, \ldots, \Sigma_r$ be transversals to the foliation at the points $x_0, x_1 = \gamma(s_1), \ldots, x_r = \gamma(s_r)$ respectively, and let $\delta > 0$ with the following properties:

- (*i*) $\Sigma_0 \subset \widetilde{\Sigma}$.
- (*ii*) The flow φ maps $\Sigma_i \times \mathbb{D}_{2\delta}$ biholomorphically onto its image, for all $i = 0, \ldots, r$.
- (*iii*) The transversal Σ_i is contained in $\varphi(\Sigma_{i-1} \times \mathbb{D}_\delta)$, for all $i = 1, \ldots, r$.
- (*iv*) For all $i = 1, \ldots, r$ we have that $\Sigma_i = h_i(\Sigma_0)$, where h_i is the holonomy map along γ.

Denote by V' the union of the sets $\varphi(\Sigma_i \times \mathbb{D}_\delta)$ for $i = 0, \ldots, r$. Consider $x \in V'$ and let L_x be the leaf passing through x. Let $k \in \{0, \ldots, r\}$ be such that $x \in \varphi(\Sigma_k \times \mathbb{D}_\delta)$. Then L_x intersects Σ_k and it follows from hypothesis (iv) that L_x intersects each Σ_i. Since $\Sigma_0 \subset \widetilde{\Sigma}$ we have that L_x intersects Σ_0 in a unique point and, by (iv), the same holds for each Σ_i. Then we may define $\rho_i : V' \to \Sigma_i$ such that $\rho_i(x)$ is the point of intersection between L_x and Σ_i. Let $\tau'_i(x) \in \mathbb{D}_\delta$ be defined by $\varphi(\rho_i(x), \tau'_i(x)) = x$. Since $\rho_i(x) \in \Sigma_i$, by hypothesis (*iii*) we have that $\rho_i(x) \in \varphi(\Sigma_{i-1} \times \mathbb{D}_\delta)$ for $i = 1, \ldots, r$. Then for $i = 1, \ldots, r$ we may define $t_i : V' \to \mathbb{D}_\delta$ as $t_i = \tau'_{i-1} \circ \rho_i$. Define $t_0 : V' \to \mathbb{D}_\delta$ as the zero function. Clearly, ρ_i, τ_i and t_i are holomorphic functions. We proceed as in the proof of Assertion 3. Let $x \in \varphi(\Sigma_i \times \mathbb{D}_\delta) \cap \varphi(\Sigma_{i-1} \times \mathbb{D}_\delta)$. Since $|t_i(x)|$ and $|\tau'_i(x)|$ are less than δ, then $|t_i(x) + \tau'_i(x)| < 2\delta$. Thus, by hypothesis (*ii*), $\varphi(\rho_{i-1}(x), t_i(x) + \tau'_i(x))$ is well defined and:

$$\begin{aligned}
\varphi(\rho_{i-1}(x), t_i(x) + \tau'_i(x)) &= \varphi(\varphi(\rho_{i-1}(x), t_i(x)), \tau'_i(x)) \\
&= \varphi(\varphi(\rho_{i-1}(x), \tau'_{i-1} \circ \rho_i(x)), \tau'_i(x)) \\
&= \varphi(\rho_i(x), \tau'_i(x)) \\
&= x.
\end{aligned}$$

Then by definition of τ'_{i-1} we deduce that

$$t_i(x) + \tau'_i(x) = \tau'_{i-1}(x).$$

Thus, the holomorphic functions on $\varphi(\Sigma_i \times \mathbb{D}_\delta)$ defined as

$$\tau_i(x) = \tau'_i(x) + t_0(x) + \cdots + t_i(x) \tag{2.4}$$

for each $i = 0, \ldots, r$ are such that

$$\tau_i = \tau_{i-1}$$

on $\varphi(\Sigma_i \times \mathbb{D}_\delta) \cap \varphi(\Sigma_{i-1} \times \mathbb{D}_\delta)$. Observe that for any leaf L', the restriction $\tau_0|_{L'}$ coincides with $\tau_{L'}$ on a neighborhood of $x_{L'}$. Then $\tau_0|_{L'}, \cdots, \tau_r|_{L'}$ give an analytic continuation of $\tau_{L'}$. Thus, $\tau_r|_L$ is the analytic continuation of τ_L along γ', hence $\tau_r(x) = \tau_L'(x)$. We denote τ_r by τ'. Analogously we construct τ'' for γ''. Then we have that $\tau''|_{L'}$ is an analytic continuation of $\tau_{L'}$ and, $\tau''|_L$ is the analytic continuation of τ_L along γ'', hence $\tau''(x) = \tau_L(x)$. By Assertion 2, we may take a sequence $\{x_k\}$ of points in Σ_0 with $x_k \to x$ as $k \to \infty$ and such that the leaf L_k passing through x_k is simply connected for all k. From above $\tau'|_{L_k}$ and $\tau''|_{L_k}$ are analytic continuations of τ_{L_k}. Since L_k is simply connected and, by Assertion 2, τ_{L_k} has an analytic continuation along any path, then $\tau'|_{L_k}$ and $\tau''|_{L_k}$ coincide on a neighborhood of x_k. In particular, $\tau'(x_k) = \tau''(x_k)$. Making $k \to \infty$ it follows by continuity that $\tau'(x) = \tau''(x)$, that is, $\tau_L'(x) = \tau_L''(x)$. Therefore, τ_L extends to L.

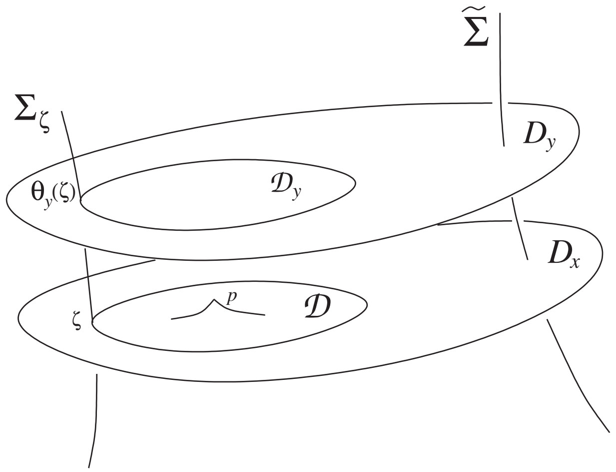
We define $\tau : V \backslash \operatorname{Sing}(\mathcal{F}) \to \mathbb{C}$ by $\tau|_L = \tau_L$. From above, τ coincides with the holomorphic function τ' on a neighborhood of the point x (arbitrary point). Then τ is holomorphic. Finally, remember (equation 2.4) that on a neighborhood of any non singular point, τ is expressed as

$$\tau_r(x) = \tau_r'(x) + t_0(x) + \cdots + t_r(x).$$

If we restrict x to a leaf, the first term of the sum above is a regular map and the other terms are constants. Hence τ is a regular map of any leaf. This finishes the proof of Assertion 4.

Given $x \in \widetilde{\Sigma}$, we know that $g^{-1}(x)$ is equal to D_z for some z. We denote $g^{-1}(x)$ by D_x. Thus, we have $p \in D_x$ for $x = g(p)$. It follows from hypothesis (iii) that there is a disc $\mathcal{D}' \subset D_x$ containing p such that $\mathcal{D}' \backslash \{p\}$ is contained in a leaf. Lemma 2.7 implies that there is a holomorphic bijective map $f : \Omega \to \mathcal{D}'$, $f(0) = p$, where $\Omega \subset \mathbb{C}$ is a disc containing $\mathbb{D}$. Thus if $\mathcal{D} = f(\mathbb{D})$, we have that $f : \overline{\mathbb{D}} \to \overline{\mathcal{D}}$ is holomorphic and regular on $\overline{\mathbb{D}} \backslash \{0\}$. Since $\overline{\mathcal{D}} \backslash \{p\}$ is contained in a leaf and by Assertion 3 we have that τ is a submersion on $U \backslash \operatorname{Sing}(\mathcal{F})$, then there exists a neighborhood V of $\partial\Delta$ on which τ defines a foliation by transversal balls along $\partial\Delta$. If we denote by Σ_ζ the transversal passing trough $\zeta \in \partial\Delta$ we have that τ is constant along Σ_ζ. Recall that $y \in \widetilde{\Sigma}$ is the unique point in the intersection of D_y and $\widetilde{\Sigma}$. It follows from the transversal uniformity of the foliation that if $y \in \widetilde{\Sigma}$ is close to x then D_y intersects only one time each transversal Σ_ζ. Let $\theta_y(\zeta)$ be the intersection of D_y with Σ_ζ. Since $\theta_y(\zeta)$ and ζ are both contained in

Σ_ζ, we have that $\tau(\theta_y(\zeta)) = \tau(\zeta)$ for all $\zeta \in \partial\Delta$. Note that $\theta_y := \theta_y(\partial\Delta)$ is a smooth Jordan curve in D_y. By Assertion 2, we may choose y such that D_y is a leaf. We consider $\mathcal{D}_y \subset D_y$, the regular disc bounded by θ_y.



Let $f_y : \mathbb{D} \to \mathcal{D}_y$ be a uniformization map. Since θ_y is a smooth Jordan curve, f_y extends as a diffeomorphism $f_y : \overline{\mathbb{D}} \to \overline{\mathcal{D}}_y$ (see [14], p.323). By Assertion 3, we have that τ is regular on $\overline{\mathcal{D}}_y$. It follows that $\tau \circ f_y : \overline{\mathbb{D}} \to \mathbb{C}$ is a regular map. Therefore, by Lemma 2.5, the curve $\tau \circ f_y : \mathbb{S}^1 \to \mathbb{C}$ has degree 1. Remember that $\tau(\theta_y(\zeta)) = \tau(\zeta)$ for all $\zeta \in \partial\Delta$, thus $\tau(\partial\mathcal{D}_y) = \tau(\partial\mathcal{D})$. Then

$$\tau \circ f_y(\mathbb{S}^1) = \tau(\partial\mathcal{D}_y) = \tau(\partial\mathcal{D}) = \tau \circ f(\mathbb{S}^1).$$

Therefore $\tau \circ f : \mathbb{S}^1 \to \mathbb{C}$ is only a reparametrization of $\tau \circ f_y : \mathbb{S}^1 \to \mathbb{C}$, hence $\tau \circ f : \mathbb{S}^1 \to \mathbb{C}$ is regular and has degree 1. Again by Lemma 2.5, $\tau \circ f : \mathbb{D} \to \mathbb{C}$ is also a regular map and in particular, $\tau \circ f$ is locally injective. Therefore there exists a disc $\mathrm{U} \subset \mathbb{D}$, centered at 0, such that $\tau \circ f$ is injective on $\overline{\mathrm{U}}$. Then

$$\tau \circ f(\partial\,\mathrm{U})$$

is a Jordan curve in $\mathbb{C}$. We also denote $f(\mathrm{U})$ by $\mathcal{D}$. Again, let Σ_ζ be the transversal ball through $\zeta \in \partial\mathcal{D}$. Proceeding as above, if Σ' is a small enough ball in $\widetilde{\Sigma}$ containing $x = g(p)$, we obtain that for all $y \in \Sigma'$ the set D_y intersects each Σ_ζ at the unique point $\theta_y(\zeta)$. Thus we have the Jordan curve θ_y in D_y such that $\tau(\theta_y) = \tau(\partial D)$. Remember that $\tau(\partial D) = \tau \circ f(\partial\,\mathrm{U})$ is a Jordan curve in $\mathbb{C}$. It follows that $\tau(\theta_y)$ is Jordan curve in $\mathbb{C}$ for all y. Let $\mathcal{D}_y \subset D_y$ be the disc bounded by θ_y. Since D_y is a singular disc, by Lemma 2.6, there is an injective holomorphic map $f_y : E \to M$, where

$E = \mathbb{D}$ or $\mathbb{C}$, such that $f_y(E) = D_y$. Let $\Omega_y \subset E$ be such that $f_y(\Omega_y) = \mathcal{D}_y$. Clearly Ω_y is a disc and $f_y(\partial\Omega_y) = \partial\mathcal{D}_y$. Then

$$\tau \circ f_y(\partial\Omega_y) = \tau(\partial\mathcal{D}_y)$$

is, from above, a Jordan curve in $\mathbb{C}$. Hence we deduce that the holomorphic function $\tau \circ f_y : \Omega_y \to \mathbb{C}$ is injective on $\overline{\Omega}_y$. Thus, since f_y is injective, we conclude that

$$\tau : \overline{\mathcal{D}}_y \to \mathbb{C}$$

is injective for all $y \in \Sigma'$.

Denote by W the union of the discs $\mathcal{D}_y$ for all $y \in \Sigma'$. It is easy to see that W is a neighborhood of p. Define

$$F : \overline{W} \to \widetilde{\Sigma} \times \mathbb{C}$$

$$F(w) = (g(w), \tau(w))$$

Assertion 5. — *F is a biholomorphism between W and its image.*

Proof. — Clearly F is holomorphic on W. We shall prove that F injective on $\overline{W}$. Suppose $F(w) = F(w')$. Then $g(w) = g(w') = y$, hence $w, w' \in D_y$ and, since $\overline{W} \cap D_y = \overline{\mathcal{D}}_y$, we have $w, w' \in \overline{\mathcal{D}}_y$. On the other hand $\tau(w) = \tau(w')$ and since τ is injective on $\overline{\mathcal{D}}_y$ we conclude that $w = w'$. Now, since $\overline{W}$ is compact, F is a homeomorphism onto its image and it follows that F is a biholomorphism.

Now, we will prove that $p \in W$ is regular for $\mathcal{F}$. Let $\mathcal{N}$ be the regular foliation on $\widetilde{\Sigma} \times \mathbb{C}$ whose leaves are the sets $\{*\} \times \mathbb{C}$. Let $\mathcal{F}'$ be the pull-back foliation of $\mathcal{N}$ by the biholomorphism F. Then $\mathcal{F}'$ is regular and it is easy to see that $\mathcal{F}'$ coincides with $\mathcal{F}$ out on a open set of W (out of $\operatorname{Sing}(\mathcal{F})$). Then $\mathcal{F}' = \mathcal{F}$ on W and $\mathcal{F}$ is therefore regular at p. Since $p \in U$ was arbitrary, we have proved that $\operatorname{Sing}(\mathcal{F})$ is empty. Then, from Assertion 1, the sets D_z are the leaves of $\mathcal{F}$. The proof of Proposition 2.4 is complete. □

Proof of Theorem 2.2. —

Assertion 1. — *Let $z \in \Sigma$ such that D_z is not contained in X. Then D_z is contained in a leaf of $\mathcal{F}$.*

Proof. — Take $t_0 \in D_z$ such that $h(z, t_0) \notin X$. Since X is closed in M, if Σ' is a small enough neighborhood (ball) of z in Σ, we have that $h(z', t_0) \notin X$ for all $z' \in \Sigma'$. Hence, for all $z' \in \Sigma'$ we have that $D_{z'}$ is not contained in X. Then, by hypothesis (ii), $S_{z'} := D_{z'} \cap X$ is discrete and $D_{z'} \backslash S_{z'}$ is contained in a nonsingular leaf of $\mathcal{F}$. Therefore, $\mathcal{F}$ restricted to $M' := h(\Sigma' \times D)$ satisfies the hypothesis of Proposition 2.4 and we have therefore that D_z is contained in a leaf of $\mathcal{F}$.

ASSERTION 2. — *Let $z \in \Sigma$ such that D_z is contained in X. Then D_z is a singular disc.*

Proof . — Let $x \in D_z$, $x = h(z,t)$. Let $\Sigma' \subset \Sigma$ be a neighborhood (a ball) of z and $D' \subset D$ be a neighborhood (a disc) of t. If Σ' and D' are small enough, we may assume that $M' := h(\Sigma' \times D')$ is a domain in $\mathbb{C}^n$. Since X has codimension $\geqslant 1$, there is a path $x_s = h(z_t, t_s)$ in M' such that $x_0 = x$ and $x_s \notin X$ for all $s > 0$. Then $D_s := D_{z_s}$ is not contained in X for all $s > 0$ and it follows by Assertion 1 that D_s is contained in a leaf. Hence D_s is a regular disc for all $s > 0$. Then, we may apply Lemma 2.1 to the family of discs D_s and conclude that $D_z = D_0$ is a singular disc.

ASSERTION 3. — *Let z be such that $D_z \subset X$. Let S_z be the set of singularities of the singular disc D_z. Then $D_z \backslash S_z$ is contained in a leaf of $\mathcal{F}$.*

Proof . — By Assertion 2, if D_z is not contained in X we have that D_z is contained in a leaf of $\mathcal{F}$. Therefore, the hypothesis of Proposition 2.3 holds for $\mathcal{F}$ and Assertion 3 follows.

Let z be such that D_z is not contained in X. By hypothesis (iii) of 2.2, we have that $S_z := D_z \cap X$ is discrete and $D_z \backslash S_z$ is contained in a leaf of $\mathcal{F}$. From this and Assertion 3 we conclude: for all z there is a discrete set S_z such that $D_z \backslash S_z$ is contained in a leaf of $\mathcal{F}$. Therefore the hypothesis of Proposition 2.4 holds and Theorem 2.2 follows. □

3. The algebraic multiplicity and the Chern class of the tangent bundle of the strict transform

Let $\mathcal{F}_0$, $\widetilde{\mathcal{F}}_0$ and h as in §1.

PROPOSITION 3.1. — *If h extends to the divisor as a homeomorphism between $\pi^{-1}(U)$ and $\pi^{-1}(\widetilde{U})$, then the extension also denoted by h is a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$.*

Proof. — Is a direct application of Theorem 1.4. □

Proof of Theorem 1.1. — Suppose that $\mathcal{F}$ is generated on U by the holomorphic vector field

$$V = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i},\ a_i \in \mathcal{O}_U,\ g.c.d.(a_1, a_2, \ldots, a_n) = 1.$$

For each $j = 1, 2, \ldots, n$, let $U_j = (x_j \neq 0)$ and $U'_j = \pi^{-1}(U_j)$. Let $V_j = \pi^*(V|_{U_j})$. If $(x^j_1, \ldots, x^j_n)$ are coordinates on U'_j such that

$$\pi(x^j_1, \ldots, x^j_n) = (x^j_j x^j_1, \ldots, x^j_j, \ldots, x^j_j x^j_n),$$

then

$$V_j = a_j \frac{\partial}{\partial x^j_j} + \sum_{i=1, i\neq j}^{n} \frac{a_i - x^j_i a_j}{x^j_j} \frac{\partial}{\partial x^j_i},$$

where $a_i = a_i \circ \pi$ for $i = 1, \ldots, n$. On U'_j, $\mathcal{F}_0$ is defined by the vector field

$$W_j = \frac{1}{(x^j_j)^{r-\xi}} V_j,$$

where r is the algebraic multiplicity of V at $0 \in \mathbb{C}^n$ and $\xi = 1$ or 0 depending on the divisor being invariant or not by $\mathcal{F}_0$. Evidently $V_i = V_j$ on $U'_i \cap U'_j$. Then

$$W_i = \left(x^j_j / x^i_i\right)^{r-\xi} W_j \quad \text{on} \quad U'_i \cap U'_j.$$

It follows from this equation that the tangent bundle $T\mathcal{F}_0$ of $\mathcal{F}_0$ is isomorphic to $L^{\xi - r}$, where L is the line bundle associated to the divisor $E = \pi^{-1}(0)$. Then the Chern class $c(T\mathcal{F}_0)$ of $T\mathcal{F}_0$ is equal to $(\xi - r)c(L)$. It is natural consider E as an element in $H_{n-2}(U', \mathbb{Z})$, where $U' = \pi^{-1}(U)$. We know that $c(L)$ is equal to $d(E) \in H^2(U', \mathbb{Z})$, the dual of E. Therefore

$$c(T\mathcal{F}_0) = (\xi - r)d(E).$$

On the other hand, make $\widetilde{U}' = \pi^{-1}(\widetilde{U})$ and observe that the divisor E is invariant by $\mathcal{F}_0$ if and only if it is by $\widetilde{\mathcal{F}}_0$. Then analogously we have

$$c(T\widetilde{\mathcal{F}}_0) = (\xi - \widetilde{r})\widetilde{d}(E),$$

where $\widetilde{r}$ is the algebraic multiplicity of $\widetilde{\mathcal{F}}$ and $\widetilde{d}(E) \in H^2(\widetilde{U}', \mathbb{Z})$ is the dual of E. By Proposition 3.1 we have that $h : U' \to \widetilde{U}'$ is a topological equivalence between $\mathcal{F}_0$ and $\widetilde{\mathcal{F}}_0$. Then Theorem 1.3 implies that

$$(\xi - r)h^*(d(E)) = (\xi - \widetilde{r})\widetilde{d}(E). \tag{3.1}$$

We may assume that U is a ball in $\mathbb{C}^n$. Thus, we have that U' is a tubular neighborhood of E and therefore $H^2(U', \mathbb{Z}) \simeq \mathbb{Z}$. Since the cohomology is invariant by homeomorphisms, we also have $H^2(\widetilde{U}', \mathbb{Z}) \simeq \mathbb{Z}$. Can be proved that $d(E)$ and $\widetilde{d}(E)$ are generators of $H^2(U', \mathbb{Z})$ and $H^2(\widetilde{U}', \mathbb{Z})$ respectively. Then we have that $h^*(d(E)) = \widetilde{d}(E)$ or $h^*(d(E)) = -\widetilde{d}(E)$. By using this in (3.1) we obtain $r = \widetilde{r}$ or $r + \widetilde{r} = 2\xi$. The second possibility implies $r = \widetilde{r} = \xi$, since $r \geqslant 1$, $\widetilde{r} \geqslant 1$ and $\xi = 1$ or 0. Therefore we conclude that $r = \widetilde{r}$. □

Remark. — Under the hypothesis of Theorem 1.1, we have another invariants. The restriction of $\mathcal{F}_0$ to the divisor is a foliation with $\mathrm{Sing}(\mathcal{F}_0)$ as singular set. It is well known that this foliation coincides out of the singular set with a unique foliation $\mathcal{N}$ of codimension $\geqslant 2$ in the divisor (the saturated foliation). We will say that $\mathcal{N}$ is the foliation induced by $\mathcal{F}_0$ in the divisor. Let $\widetilde{\mathcal{N}}$ be the foliation induced by $\widetilde{\mathcal{F}}_0$ in the divisor. It follows from Theorem 1.4 that $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are topologically equivalent. Thus, since the divisor is isomorphic to $\mathbb{P}^{n-1}$, Theorem 1.3 implies that $d(\mathcal{N}) = d(\widetilde{\mathcal{N}})$. In other words, the degree of the foliation induced in the divisor is invariant.

From above, $\mathcal{F}_0$ is generated by the holomorphic vector fields W_i and

$$W_i = \left(x_j^j/x_i^i\right)^{r-\xi} W_j \quad \text{on} \quad U_i' \cap U_j',$$

where $\xi = 1$ or 0. Let $x \in U_i' \cap U_j'$. Let $x^i = (x_1^i, \ldots, x_n^i)$ be the coordinates of x in U_i' and let $x^j = (x_1^j, \ldots, x_n^j)$ be the coordinates of x in U_j'. Since $\pi(x^i) = \pi(x^j)$, we have that

$$(x_i^i x_1^i, \ldots, x_i^i, \ldots, x_i^i x_n^i) = (x_j^j x_1^j, \ldots, x_j^j, \ldots, x_j^j x_n^j),$$

hence $x_j^j/x_i^i = x_j^i$. Replacing in last equation we obtain:

$$W_i = \left(x_j^i\right)^{r-\xi} W_j \quad \text{on} \quad U_i' \cap U_j'. \tag{3.2}$$

Observe that $\pi^{-1}(0) \cap U_i'$ is represented by $(x_i^i = 0)$. Recall that $\pi^{-1}(0)$ is canonically isomorphic to $\mathbb{P}^{n-1}$. A point p in $\pi^{-1}(0) \cap U_i'$ given by

$$(x_1^i(p), \ldots, 0_i, \ldots, x_n^i(p))$$

is represented in homogeneous coordinates by

$$[z_1 : \cdots : z_n](p) = [x_1^i(p) : \cdots : 1_i : \cdots : x_n^i(p)],$$

hence $x_j^i(p) = (z_j/z_i)(p)$. Thus, if $\mathcal{U}_i = U_i' \cap \pi^{-1}(0)$ and $J_i = W_i|_{\mathcal{U}_i}$, it follows from (3.2) that

$$J_i = (z_j/z_i)^{r-\xi} J_j \quad \text{on} \quad \mathcal{U}_i \cap \mathcal{U}_j. \tag{3.3}$$

Let S be the union of the components of codimension 2 of $\mathrm{Sing}\,(\mathcal{F}_0)$. Then S is the codimension 1 part (respect to the divisor) of the zero set of $\{J_i\}$. Each J_i may be expressed as $J_i = f_i Z_i$, where f_i is a holomorphic function on $\mathcal{U}_i$ and the vector field Z_i has singular set of codimension $\geqslant 2$. It follows from (3.3) that

$$Z_i = (f_j/f_i)\,(z_j/z_i)^{r-\xi} Z_j \ \text{ on } \ \mathcal{U}_i \cap \mathcal{U}_j.$$

From this equation, it is not difficult to conclude that

$$r = d(\mathcal{N}) - \deg(S) - 1 + \xi,$$

where $\deg(S)$ is the degree of S as a divisor of $\pi^{-1}(0)$. Then, since the algebraic multiplicity and the degree of the foliation induced in the divisor are invariants, we deduce that the degree of the codimension 1 part of the singular set of the strict transform is also an invariant. Moreover it is not difficult to see that $h(S) = \widetilde{S}$, where $\widetilde{S}$ is the union of the components of codimension 2 of $\mathrm{Sing}(\widetilde{\mathcal{F}}_0)$.

4. The case C^1

In this section we prove Theorem 1.2. In view of Theorem 1.1, it is sufficient to show the following.

Proposition 4.1. — *Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be two foliations by curves of neighborhoods U and $\widetilde{U}$ of $0 \in \mathbb{C}^n$. Let $\mathrm{h} : U \to \widetilde{U}$ be a C^1 equivalence. Let $h : \pi^{-1}(U \backslash \{0\}) \to \pi^{-1}(\widetilde{U} \backslash \{0\})$ be as before. Then h can be extended to the divisor as a homeomorphism between $\pi^{-1}(U)$ and $\pi^{-1}(\widetilde{U})$.*

We start the proof.

Proposition 4.2. — *Under the conditions of Proposition 4.1, we have that $d\,\mathrm{h}(0) : \mathbb{C}^n \to \mathbb{C}^n$ maps complex lines onto complex lines. Furthermore, if $J : \mathbb{C}^n \to \mathbb{C}^n$ is the conjugation $J(z) = \bar{z}$, then either $d\,\mathrm{h}(0) : \mathbb{C}^n \to \mathbb{C}^n$ is a $\mathbb{C}$-linear isomorphism, or $d\,\mathrm{h}(0) = Q \circ J$, where $Q : \mathbb{C}^n \to \mathbb{C}^n$ is a $\mathbb{C}$-linear isomorphism. Thus, $d\,\mathrm{h}(0)$ induces a diffeomorphism of $\mathbb{P}^{n-1}$ onto itself.*

Proof. — let L be a complex line, $0 \in L \subset \mathbb{C}^n$. There exists $\mathbb{C}$-linear functions $A_i : \mathbb{C}^n \to \mathbb{C}$ for $i = 1, \ldots, (n-1)$, such that

$$L = \{z \in \mathbb{C}^n : \ A_i(z) = 0 \ \text{ for all } \ i = 1, \ldots, (n-1)\}.$$

Let $V : U \to \mathbb{C}^n$ be a holomorphic vector field which generates $\mathcal{F}$. The set:

$$B = \{z \in \mathbb{C}^n : \ A_i \circ V(z) = 0 \text{ for all } \quad i = 1, \ldots, (n-1)\}$$

is an analytic variety and it is easy to see that $0 \in B$. Then, there exists a complex curve contained in B and passing through 0. In particular there exists a sequence of points $z_k \in \mathbb{C}^n \backslash \{0\}$, $z_k \to 0$, such that $A_i \circ V(z_k) = 0$ for all $k \in \mathbb{N}$ and all $i = 1, 2, \ldots, (n-1)$. In other words, $T_{z_k}\mathcal{F} = L$ for all $k \in \mathbb{N}$. Now, since h is a C^1 equivalence, $d\,\mathrm{h}_{z_k}(T_{z_k}\mathcal{F}) = T_{\mathrm{h}(z_k)}\widetilde{\mathcal{F}}$, that is, $d\,\mathrm{h}_{z_k}(L) = T_{\mathrm{h}(z_k)}\widetilde{\mathcal{F}}$ is a complex line for all $k \in \mathbb{N}$. Making $k \to \infty$, since $\mathrm{h} \in C^1$ and the space of complex lines of $\mathbb{C}^n$ is compact, we obtain that $d\ h_0(L)$ is also a complex line. The second part of the proposition is an immediate consequence of the following lemma. □

Lemma 4.3. — *Let $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ $(n \geqslant 2)$ be a $\mathbb{R}$-linear isomorphism. Identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and assume that A maps complex lines onto complex lines. Then, either A is a $\mathbb{C}$-linear isomorphism, or $A = Q \circ J$ with $Q : \mathbb{C}^n \to \mathbb{C}^n$ a $\mathbb{C}$-linear isomorphism.*

Proof. — Since A maps any complex line onto a complex line, for all $v \in \mathbb{C}^n \backslash \{0\}$ there exists $\theta(v) \in \mathbb{C}\backslash\{0\}$ such that $A(iv) = \theta(v)A(v)$. Let v_1 and v_2 be two $\mathbb{C}$-linearly independent vectors. Then

$$A(iv_1 + iv_2) = A(iv_1) + A(iv_2) = \theta(v_1)A(v_1) + \theta(v_2)A(v_2).$$

Moreover:

$$\begin{aligned} A(iv_1 + iv_2) &= A(i(v_1 + v_2)) = \theta(v_1 + v_2)A(v_1 + v_2) \\ &= \theta(v_1 + v_2)A(v_1) + \theta(v_1 + v_2)A(v_2). \end{aligned}$$

From the equations above, we obtain:

$$(\theta(v_1) - \theta(v_1 + v_2))A(v_1) + ((\theta(v_2) - \theta(v_1 + v_2))A(v_2) = 0. \tag{4.1}$$

Let L_1 and L_2 be the complex lines generated by v_1 and v_2 respectively. Since v_1 and v_2 are $\mathbb{C}$-linearly independent, we have that L_1 and L_2 are different. This implies, since A is an isomorphism, that $A(L_1)$ and $A(L_2)$ are different complex lines. Then, since $A(L_1)$ and $A(L_2)$ are generated by $A(v_1)$ and $A(v_2)$ respectively, we have that $A(v_1)$ and $A(v_2)$ are $\mathbb{C}$-linearly independent. Thus, it follows from equation (4.1) that

$$\theta(v_1) = \theta(v_1 + v_2) = \theta(v_2).$$

It is now easy to see that $\theta(v) = \theta_0, \ \forall v \in \mathbb{C}^n\backslash\{0\}$. We know that there exists two $\mathbb{C}$-linear transformations $P : \mathbb{C}^n \to \mathbb{C}^n$ and $Q : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$A(z) = P(z) + Q(\bar{z}), \ \text{ for all } \ z \in \mathbb{C}^n.$$

Then

$$A(iz) = iP(z) - iQ(\bar{z}).$$

On the other hand

$$A(iz) = \theta_0 A(z) = \theta_0 P(z) + \theta_0 Q(\bar{z}), \ \text{ for all } \ z \in \mathbb{C}^n.$$

consequently

$$(\theta_0 - i)P(z) + (\theta_0 + i)Q(\bar{z}) = 0.$$

Since, as functions of z, $(\theta_0 - i)P$ and $(\theta_0 + i)Q \circ J$ are holomorphic and anti-holomorphic respectively, we have that

$$(\theta_0 - i)P \equiv 0, \ \ (\theta_0 + i)Q \circ J \equiv 0.$$

From this it is easy to see that either $P = 0$, or $Q = 0$. This proves the lemma. $\square$

Definition 4.4. — *Let $\{z_k\}$ be a sequence of points in $\mathbb{C}^n \backslash \{0\}$. Let L be a complex line in $\mathbb{C}^n$. We say that $\{z_k\}$ is tangent to L at 0 if $z_k \to 0$ and every accumulation point of $\{z_k/||z_k||\}$ is contained in L.*

Let $\pi : \widehat{\mathbb{C}^n} \to \mathbb{C}^n$ be the blow up at $0 \in \mathbb{C}^n$. We know that $\pi^{-1}(0)$ is naturally isomorphic to $\mathbb{P}^{n-1}$. Thus, for each $p \in \pi^{-1}(0)$ we denote by L_p the respective complex line in $\mathbb{C}^n$. The following fact is well known and we left the proof to the reader:

Proposition 4.5. — *Let $\{p_k\}$ be a sequence of points in $\widehat{\mathbb{C}^n} \backslash \pi^{-1}(0)$. Then $p_k \to p \in \pi^{-1}(0)$ if and only if $\{\pi(p_k)\}$ is tangent to L_p at 0.*

Proof of Proposition 4.1. — Let $p \in \pi^{-1}(0)$ and $\{p_k\}$ any sequence of points in $\pi^{-1}(U) \backslash \pi^{-1}(0)$ such that $p_k \to p$.

Since $h \in C^1$, we have

$$\mathrm{h}(\pi(p_k)) = d\,\mathrm{h}_0(\pi(p_k)) + r(\pi(p_k)), \text{ where } \frac{r(\pi(p_k))}{||\pi(p_k)||} \to 0 \text{ as } k \to 0.$$

Then

$$\frac{\mathrm{h}(\pi(p_k))}{||\pi(p_k)||} = d\,\mathrm{h}_0\left(\frac{(\pi(p_k))}{||\pi(p_k)||}\right) + \frac{r(\pi(p_k))}{||\pi(p_k)||}. \tag{4.2}$$

By proposition 4.5, $\pi(p_k)$ is tangent to L_p at 0, hence any point of accumulation of the sequence $\{(\pi(p_k))/\|\pi(p_k)||\}$ is contained in L_p. Thus, it is easy to see from equation (4.2) that any point of accumulation of the sequence $\{\mathrm{h}(\pi(p_k))/||\pi(p_k)||\}$ is contained in $d\,\mathrm{h}_0(L_p)$ and the same holds for the sequence

$$\frac{\mathrm{h}(\pi(p_k))}{||\,\mathrm{h}(\pi(p_k))||} = \frac{\mathrm{h}(\pi(p_k))}{||\pi(p_k)||}\frac{||\pi(p_k)||}{||\,\mathrm{h}(\pi(p_k))||}.$$

From proposition 4.2 we have that $d\,\mathrm{h}_0(L_p)$ is a complex line. Then $\{\mathrm{h}(\pi(p_k)\})$ is tangent to $d\,\mathrm{h}_0(L_p)$ at 0. It follows by proposition 4.5 that $\pi^{-1} \circ \mathrm{h} \circ \pi(p_k) = h(p_k) \to q$, where $q \in \pi^{-1}(0)$ is such that $L_q = d\,\mathrm{h}_0(L_p)$. We extend h by making $h(p) = d\,\mathrm{h}_0(L_p)$ for all p in $\pi^{-1}(0)$. Finally, it is easy to prove that $h : \pi^{-1}(U) \to \pi^{-1}(\widetilde{U})$ is a homeomorphism. $\square$

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Manuscrit reçu le 18 juin 2009,
accepté le 21 septembre 2009.

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