Rudy ROSAS

The $C^1$ invariance of the algebraic multiplicity of a holomorphic vector field

<http://aif.cedram.org/item?id=AIF_2010__60_6_2115_0>
THE $C^1$ INVARIANCE OF THE ALGEBRAIC MULTIPLICITY OF A HOLOMORPHIC VECTOR FIELD

by Rudy ROSAS

ABSTRACT. — We prove that the algebraic multiplicity of a holomorphic vector field at an isolated singularity is invariant by $C^1$ equivalences.

RéSUMÉ. — On démontre que la multiplicité algébrique d’une singularité d’un champ de vecteurs holomorphe est invariante par $C^1$-équivalences.

1. Introduction

Given a curve $f : (\mathbb{C}^2,0) \to (\mathbb{C},0)$, singular at $0 \in \mathbb{C}^2$, we define its algebraic multiplicity as the degree of the first nonzero jet of $f$, that is, $\nu(f) = \nu$ where

$$f = f_\nu + f_{\nu+1} + \cdots$$

is the Taylor development of $f$ and $f_\nu \neq 0$. A well known result by Burau [2] and Zariski [15] states that $\nu$ is a topological invariant, that is, given $\tilde{f} : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ and a homeomorphism $h : U \to \tilde{U}$ between neighborhoods of $0 \in \mathbb{C}^2$ such that $h(f^{-1}(0) \cap U) = \tilde{f}^{-1}(0) \cap V$ then $\nu(f) = \nu(\tilde{f})$. Consider now a holomorphic vector field $Z$ in $\mathbb{C}^2$ with a singularity at $0 \in \mathbb{C}^2$. If

$$Z = Z_\nu + Z_{\nu+1} + \cdots, Z_\nu \neq 0$$

we define $\nu = \nu(Z)$ as the algebraic multiplicity of $Z$. The vector field $Z$ defines a holomorphic foliation by curves $\mathcal{F}$ with isolated singularity in a neighborhood of $0 \in \mathbb{C}^2$ and the algebraic multiplicity $\nu(Z)$ depends only on the foliation $\mathcal{F}$. A natural question, posed by J.F.Mattei is: is $\nu(\mathcal{F})$ a

Keywords: Algebraic multiplicity, holomorphic vector field, holomorphic foliation.
Math. classification: 37F75.
topological invariant of $\mathcal{F}$?. In [4], the authors give a positive answer if $\mathcal{F}$ is a generalized curve, that is, if the desingularization of $\mathcal{F}$ does not contain complex saddle-nodes. In this work, we consider the problem in dimension $n \geq 2$ and impose conditions on the topological equivalence. Let $\mathcal{F}$ be a holomorphic foliation by curves of a neighborhood $U$ of $0 \in \mathbb{C}^n$ with a unique singularity at $0 \in \mathbb{C}^n(n \geq 2)$. We assume that $\mathcal{F}$ is generated by the holomorphic vector field

$$V = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i}, \quad a_i \in \mathcal{O}_U, \quad \text{g.c.d.}(a_1, a_2, \ldots, a_n) = 1.$$ 

The algebraic multiplicity of $\mathcal{F}$ (at $0 \in \mathbb{C}^n$) is the minimum vanishing order at $0 \in \mathbb{C}^n$ of the functions $a_i$. Let $\tilde{\mathcal{F}}$ be another holomorphic foliation by curves of a neighborhood $\tilde{U}$ of $0 \in \mathbb{C}^n$ and let $h : U \rightarrow \tilde{U}$ be a topological equivalence between $\mathcal{F}$ and $\tilde{\mathcal{F}}$, that is, a homeomorphism taking leaves of $\mathcal{F}$ to leaves of $\tilde{\mathcal{F}}$. Let $\pi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be the quadratic blow up with center at $0 \in \mathbb{C}^n$. Clearly the map $h := \pi^{-1} h \pi$ is a homeomorphism between $\pi^{-1}(U \setminus \{0\})$ and $\pi^{-1}(\tilde{U} \setminus \{0\})$. Then we prove the following:

**Theorem 1.1.** Suppose that $h$ extends to the divisor $\pi^{-1}(0)$ as a homeomorphism between $\pi^{-1}(U)$ and $\pi^{-1}(\tilde{U})$. Then the algebraic multiplicities of $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are the same.

If $h$ is a $C^1$ diffeomorphism, we prove that $h$ extends to the divisor. Thus, we obtain that the algebraic multiplicity is invariant by $C^1$ equivalences:

**Theorem 1.2.** Let $\mathcal{F}$ and $\tilde{\mathcal{F}}$ be two foliations by curves of neighborhoods $U$ and $\tilde{U}$ of $0 \in \mathbb{C}^n$, $n \geq 2$. Let $h : U \rightarrow \tilde{U}$ be a $C^1$ equivalence between $\mathcal{F}$ and $\tilde{\mathcal{F}}$, that is, a $C^1$ diffeomorphism taking leaves of $\mathcal{F}$ to leaves of $\tilde{\mathcal{F}}$. Then the algebraic multiplicities of $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are equal.

It is known that there exists a unique way of extending the pull back foliation $\pi^*(\mathcal{F}|_{U \setminus \{0\}})$ to a singular analytic foliation $\mathcal{F}_0$ on $\pi^{-1}(U)$ with singular set of codimension $\geq 2$. We say that $\mathcal{F}_0$ is the strict transform of $\mathcal{F}$ by $\pi$. Let $\tilde{\mathcal{F}}_0$ be the strict transform of $\tilde{\mathcal{F}}$ by $\pi$. In order to prove Theorem 1.1 we show that the algebraic multiplicity of $\mathcal{F}$ depends on the Chern class of the tangent bundle of $\mathcal{F}_0$. To relate the Chern classes of the tangent bundles of $\mathcal{F}_0$ and $\tilde{\mathcal{F}}_0$ we use the following theorem (see [7]).

**Theorem 1.3.** Let $\mathcal{F}$ and $\tilde{\mathcal{F}}$ be foliations by curves on the complex manifolds $M$ and $\tilde{M}$ respectively. Let $c(T\mathcal{F})$ denote the Chern class of the tangent bundle $T\mathcal{F}$ of $\mathcal{F}$. Let $h : M \rightarrow \tilde{M}$ be a topological equivalence between $\mathcal{F}$ and $\tilde{\mathcal{F}}$ and consider the map $h^* : H^2(M, \mathbb{Z}) \rightarrow H^2(\tilde{M}, \mathbb{Z})$ induced in the cohomology. Then $h^*(c(T\mathcal{F})) = c(T\tilde{\mathcal{F}})$.
Clearly the homeomorphism $h : \pi^{-1}(U\setminus\{0\}) \to \pi^{-1}(\tilde{U}\setminus\{0\})$ is a topological equivalence between $F_0|_{\pi^{-1}(U\setminus\{0\})}$ and $\tilde{F}_0|_{\pi^{-1}(\tilde{U}\setminus\{0\})}$. To be able to apply Theorem 1.3 we show that $h$ extends as a topological equivalence between $F_0$ and $\tilde{F}_0$. This is the non trivial part of the proof. Thus, we prove the following.

**Theorem 1.4.** — Let $V$ and $\tilde{V}$ be complex manifolds, let $Y \subset V$ and $\tilde{Y} \subset \tilde{V}$ be analytic subvarieties of codimension $\geq 1$ and, let $F$ and $\tilde{F}$ be holomorphic foliations by curves on $V$ and $\tilde{V}$ respectively. Suppose there is a homeomorphism $h$ between $V$ and $\tilde{V}$ with $h(Y) = \tilde{Y}$ and such that $h|_{V\setminus Y}$ is a topological equivalence between $F|_{V\setminus Y}$ and $\tilde{F}|_{\tilde{V}\setminus \tilde{Y}}$. Then $h$ is a topological equivalence between $F$ and $\tilde{F}$.

This paper is organized as follows. In section 2 we prove Theorem 1.4. In section 3 we relate the algebraic multiplicity of the foliation and the Chern class of its strict transform, and prove Theorem 1.1. Finally, section 4 discusses the $C^1$ case.

The contents of this paper originally comprised a Ph.D. dissertation at Instituto de Matematica Pura e Aplicada, Rio de Janeiro. The author would like to thank his advisor, César Camacho, for guidance and support. I also thank Alcides Lins Neto, Paulo Sad, Luis Gustavo Mendes and specially Jorge Vitório Pereira for the remarks that helps in the redaction of the present paper.

## 2. An extension theorem

This section is devoted to prove Theorem 1.4. We start with some definitions. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{B} = \{z \in \mathbb{C}^{n-1} : ||z|| < 1\}$ where $n \geq 2$. Let $M$ be a complex manifold of complex dimension $n$ and let $D$ be a subset of $M$ homeomorphic to a disc. We say that $D$ is a singular disc if for all $x \in D$ there exist a neighborhood $\mathcal{D}$ of $x$ in $D$, and an injective holomorphic function $f : \mathbb{D} \to M$ such that $f(\mathbb{D}) = D$, $f(0) = x$. If $f'(0) = 0$ we say that $x$ is a singularity of $D$, otherwise $x$ is a regular point of $D$ (this does not depend on $f$). The set $S$ of singularities of $D$ is discrete and closed in $D$ and we have that $D\setminus S$ is a complex submanifold of $M$. Thus, if $x$ is a regular point of $D$, there is a neighborhood $U$ of $x$ in $M$ and holomorphic coordinates $(w, z)$, $w \in \mathbb{B}$, $z \in \mathbb{D}$ on $U$ such that $D\cap U$ is represented by $(w = 0)$. If $D$ does not have singularities we say that it is a regular disc. In this case, by uniformization, there is a holomorphic map.
$f: E \to M$, where $E = \mathbb{D}$ or $\mathbb{C}$, such that $f$ is a biholomorphism between $E$ and $D$.

**Example.** — Let $\mathcal{F}$ be a holomorphic foliation by curves on the complex manifold $M$ and let $D \subset M$ be a topological disc contained in a leaf of $\mathcal{F}$. Then $D$ is a regular disc.

The following Lemma will be fundamental in the proof of Theorem 1.4.

**Lemma 2.1.** — Let $F: \mathbb{D} \times [0,1] \to \mathbb{C}^n$ be a continuous map such that for all $t \in [0,1]$, the map $F(*,t): \mathbb{D} \to \mathbb{C}^n$ is a homeomorphism onto its image. Thus, we have a continuous family of discs $D_t := F(\mathbb{D} \times \{t\})$. Suppose $D_t$ is a regular disc for each $t > 0$. Then $D_0$ is a singular disc.

**Proof.** — We give a sketch of the proof. Let $p = F(x_0,0)$ be any point in $D_0$. Let $U \subset \mathbb{D}$ be a disc centered at $x_0$ and such that $U \subset \mathbb{D}$. Let $t_k > 0$ be such that $t_k \to 0$ as $k \to \infty$ and define $D_k = F(U \times \{t_k\})$. By uniformization there is a holomorphic map $f_k: \mathbb{D} \to \mathbb{C}^n$ which is a biholomorphism between $\mathbb{D}$ and $D_k$. We may assume that $f_k(0) = F(x_0,t_k)$ for all $k$ and it is well known that $f_k$ extends as a homeomorphism $f_k: \overline{\mathbb{D}} \to \overline{D_k}$. By Montel’s theorem we can assume that $f_k$ converges uniformly on compact sets to a holomorphic function $f: \mathbb{D} \to \mathbb{C}^n$, $f(0) = p$. Clearly it is sufficient to show that $f$ is not a constant function ($f \not\equiv p$). Let $S^1 := \partial \mathbb{D}$ and consider for each $k$ the homeomorphism

$$\varphi_k := f_k|_{S^1} : S^1 \to \partial D_k.$$ 

By taking a subsequence if necessary, it is not difficult to see that we may assume that $\varphi_k$ converges a.e. to a function $\varphi: S^1 \to \partial D_0$.

Fix $x \in \mathbb{D}$. Since $\{\varphi_k\}$ is uniformly bounded, by the dominated convergence theorem we have that

$$\frac{1}{2\pi i} \int_{S^1} \frac{\varphi_k(w)}{w-x} dw \to \frac{1}{2\pi i} \int_{S^1} \frac{\varphi(w)}{w-x} dw$$

as $k \to \infty$. By Cauchy’s Integral Formula the left part of (2.1) is equal to $f_k(x)$ and, since $f_k(x) \to f(x)$, we conclude that

$$f(x) = \frac{1}{2\pi i} \int_{S^1} \frac{\varphi(w)}{w-x} dw.$$ 

Finally, it is not difficult to prove from this equation that $f \equiv p$ implies $\varphi = p$ a.e., which is a contradiction because $\varphi(S^1) \subset \partial D_0$ and $p \notin \partial D_0$. □

We now show that Theorem 1.4 is a consequence of the following theorem.
Theorem 2.2. — Let \( \mathcal{F} \) be a foliation by curves on the complex manifold \( M \). Let \( X \subset M \) be an analytic subvariety of codimension \( \geq 1 \). Suppose that:

(i) \( \mathcal{F} \) is generated by a holomorphic vector field.

(ii) There exists a homeomorphism \( h : \Sigma \times D \to M \), where \( \Sigma \) is a ball in \( \mathbb{C}^{n-1} \) and \( D \) is a disc in \( \mathbb{C} \).

(iii) If \( D_z := h(\{z\} \times D) \) then for all \( z \): either \( D_z \) is contained in \( X \), or \( D_z \cap X \) is discrete and \( D_z \setminus X \) is contained in a leaf of \( \mathcal{F} \).

Then \( \mathcal{F} \) is regular and the sets \( D_z \) are the leaves of \( \mathcal{F} \).

Proof of Theorem 1.4. — Let \( p \) be a point in \( Y \) which is regular for \( \mathcal{F} \). Let \( \Sigma \) denote a ball in \( \mathbb{C}^{n-1} \) and \( D \) a disc in \( \mathbb{C} \). Consider a neighborhood \( W \) of \( p \) on which \( \mathcal{F} \) is a product foliation, that is, \( W \simeq \Sigma \times D \) and the sets \( \{z\} \times D \) are the leaves of \( \mathcal{F}|_W \). We take \( W \) small enough such that \( \tilde{\mathcal{F}} \) restricted to \( M := h(W) \) is generated by a holomorphic vector field. Let \( X \) be the intersection between \( M \) and \( \tilde{Y} \). We will show that the hypothesis of Theorem 2.2 hold for \( \tilde{\mathcal{F}} \) restricted to \( M \). Hypothesis (i) and (ii) of 2.2 evidently hold. Let \( D_z = h(\{z\} \times D) \). Then it is easy to see that

Assertion 1. — For all \( z \in \Sigma \), either \( \{z\} \times D \) is contained in \( Y \), or \( S'_z := (\{z\} \times D) \cap Y \) is discrete and closed in \( \{z\} \times D \).

Suppose that \( D_z \) is not contained in \( X \). Let \( S_z = h(S'_z) \), where \( S'_z \) is given by Assertion 1. Then \( S_z \) is discrete in \( D_z \). Observe that \( (\{z\} \times D) \setminus S'_z \) is contained in a leaf of \( \mathcal{F}|_W \). Then, since \( h|_{M \setminus Y} \) is a topological equivalence between \( \mathcal{F}|_{V \setminus Y} \) and \( \tilde{\mathcal{F}}|_{\tilde{V} \setminus \tilde{Y}} \), it follows that

\[
D_z \setminus S_z = h((\{z\} \times D) \setminus S'_z)
\]

is contained in a leaf of \( \tilde{\mathcal{F}} \). Thus, hypothesis (iii) of 2.2 holds. Then \( \tilde{\mathcal{F}} \) is regular on \( M = h(W) \) and every \( D_z \) is contained in a leaf of \( \tilde{\mathcal{F}} \). Therefore we conclude:

Assertion 2. — If \( p \) is a point in \( Y \) which is regular for \( \mathcal{F} \), then \( p \) is mapped by \( h \) to a regular point of \( \tilde{\mathcal{F}} \). Moreover, there exists a neighborhood \( \Omega \) of \( p \) in its leaf which is mapped by \( h \) onto a neighborhood of \( h(p) \) in its leaf.

Now, by using Assertion 2 for \( h \) and \( h^{-1} \), we deduce that \( p \) is regular for \( \mathcal{F} \) if and only if \( h(p) \) is regular for \( \tilde{\mathcal{F}} \). Hence

\[
h(\text{Sing}(\mathcal{F})) = \text{Sing}(\tilde{\mathcal{F}}).
\]

It remains to prove that \( h \) maps any leaf of \( \mathcal{F} \) onto a leaf of \( \tilde{\mathcal{F}} \). Let \( p \) be a regular point of \( \mathcal{F} \). Let \( L \) be the leaf of \( \mathcal{F} \) passing through \( p \) and let
be the leaf of $\tilde{F}$ passing through $h(p)$. Let $A$ be the set of points in $L$
which are mapped by $h$ into $\tilde{L}$. By Assertion 2, if $x \in A$ there exists a
neighborhood of $x$ in $L_p$ contained in $A$. Therefore $A$ is open. Now, let
$x \notin A$. Then $h(x) \notin \tilde{L}$. Thus, if $L' \neq L$ is the leaf of $\tilde{F}$ passing through
$h(x)$ it follows by Assertion 2 that there exists a neighborhood $\Omega$ of $x$ in $L$
which is mapped by $h$ into $L' \neq \tilde{L}$, hence $\Omega$ is contained in $L \setminus A$. Then $A$
is also closed and it follows by connectedness that $A = L$, that is, $h(L) \subset \tilde{L}$.
Analogously, we prove that $h^{-1}(\tilde{L}) \subset L$. Therefore $h(L) = \tilde{L}$. □

We proceed now to prove Theorem 2.2.

**Proposition 2.3.** — Let $F$ be a foliation by curves on the complex
manifold $M$. Let $X \subset M$ be an analytic subvariety of codimension $\geq 1$.
Suppose that:

(i) There exists a homeomorphism $h : \Sigma \times D \rightarrow M$, where $\Sigma$ is a ball
in $\mathbb{C}^{n-1}$ and $D$ is a disc in $\mathbb{C}$.

(ii) If $D_z := h(\{z\} \times D)$ then for all $z$: either $D_z$ is contained in $X$, or
$D_z$ is contained in a leaf of $F$.

Consider $z' \in \Sigma$ and suppose that $D_{z'}$ is a singular disc. Let $S_{z'}$ the set of
singularities of $D_{z'}$. Then $D_{z'} \setminus S_{z'}$ is contained in a leaf of $F$.

**Proof.** — It is sufficient to prove the following.

**Assertion.** — If $p \in D_{z'} \setminus S_{z'}$ then $p$ has a neighborhood in $D_{z'} \setminus S_{z'}$
contained in a leaf of $F$.

Suppose Assertion holds. Let $L$ be a leaf of $F$ and let $x \in (D_{z'} \setminus S_{z'}) \cap L$.
By Assertion, there is a neighborhood $\Delta$ of $x$ in $D_{z'} \setminus S_{z'}$ such that $\Delta \subset L$.
Then $\Delta \subset (D_{z'} \setminus S_{z'}) \cap L$ and it follows that the intersection of $D_{z'} \setminus S_{z'}$
with any leaf is open in $D_{z'} \setminus S_{z'}$. Then, since $D_{z'} \setminus S_{z'}$ is connected, we have
that it is contained in a unique leaf.

**Proof of Assertion.** — Let $p$ in $D_{z'} \setminus S_{z'}$. Since $p$ is a regular point of
the singular disc $D_{z'}$, on a neighborhood $U \subset M$ of $p$ we may consider
coordinates $(w, y)$, $w \in \mathbb{B}$, $y \in \mathbb{D}$ with $p = (0, 0)$ and such that $D_{z'} \cap U$ is
represented by $(w = 0)$. Suppose that $p = h(z', t')$. Let $\Sigma'$ be a ball in $\Sigma$
containing $z'$ and let $D'$ be a disc in $D$ containing $t'$. Then $W = \Sigma' \times D'$
is a neighborhood of $(z', t')$ and, by taking $W$ small enough, we assume
$h(\overline{W}) \subset U$. Let $D'_z = h(\{z\} \times D')$. Note that $D'_z \subset D_{z'} \cap U$, hence
$D'_z$ is contained in $(w = 0)$. Let $g : U \rightarrow \mathbb{D}$ be the projection $g(w, y) = y$.
Consider $z \in \Sigma'$ and suppose $D_z \setminus X \neq \emptyset$. By hypothesis (ii), $D_z$ is contained
in a leaf of $F$. Therefore $D'_z$ is contained in leaf of $F$ and we have that
$g|_{D'_z} : D'_z \rightarrow \mathbb{D}$ is a holomorphic map. Remember that $D'_{z'} \subset (w = 0)$.
Then \( g|_{D'_z} : D'_z \to \mathbb{D} \) is given by \((0, y) \to y\) and is therefore a one to one map. Then \( g(D'_z) \) is a disc in \( \mathbb{D} \) with \( g(\partial D'_z) \) as boundary. Note that \( p = (0, 0) \in D'_z \), hence 0 is contained in the disc \( g(D'_z) \). Therefore the curve \( g(\partial D'_z) \) winds once around 0. By the continuity of \( h \) we assume \( \Sigma' \) small enough such that \( g(\partial D'_z) \) is homotopic to \( g(\partial D'_z) \) in \( \mathbb{D} \setminus \{0\} \) for all \( z \in \Sigma' \). Then \( g(\partial D'_z) \) winds once around 0 and \( g|_{D'_z} \) has therefore a unique zero. In other words, the plaque \( D'_z \) intersects \( Y = \mathbb{B} \times \{0\} \subset U \) at a unique point. Thus, we can define the map \( f : h(W)\setminus X \to Y \) by \( f(D'_z \setminus X) = D'_z \cap Y \) whenever \( D'_z \setminus X \neq \emptyset \). We have that \( f \) is holomorphic because it is constant along the leaves and, restricted to any transversal, is a holonomy map. Since \( f \) is bounded and \( X \) has codimension \( \geq 1 \), by the generalized Riemann’s extension theorem, \( f \) extends to a holomorphic function on \( h(W) \). Observe that \( f \) restricted to \( Y \) is the identity map, then \( f \) is a submersion in a neighborhood \( V \) of \( Y \). Hence \( f \) defines a regular foliation \( \mathcal{N} \) on \( V \). It is easy to see that \( \mathcal{N} \) coincides with \( \mathcal{F} \) on \( V \setminus X \), thus \( \mathcal{N} = \mathcal{F} \). Therefore \( p \in Y \) is a regular point of \( \mathcal{F} \).

Now, by reducing the neighborhood \( W = \Sigma' \times D' \) of \((z', t')\), we may assume that \( h(W) \) is contained in a neighborhood of \( p \) where \( \mathcal{F} \) is given by a submersion \( f \). Obviously \( D'_z \) is a neighborhood of \( p \) in \( D_z \). We shall prove that \( D'_z \) is contained in a leaf of \( \mathcal{F} \) (the leaf passing through \( p \)). If \( D'_z \) is not contained in \( X \), so is \( D_z \) and, by hypothesis (ii), we have that \( D'_z \) is contained in a leaf of \( \mathcal{F} \). On the other hand, suppose that \( D'_z \) is contained in \( X \). Then there exists a sequence of points \( z_k \to z' \) such that \( h(\{z_k\} \times D) \) is not contained in \( X \), otherwise \( h(\Sigma'' \times D) \subset X \) for some neighborhood \( \Sigma'' \subset \Sigma \) of \( z' \), which is a contradiction because \( X \) has codimension \( \geq 1 \). Thus, by (ii), we have that \( D'_z \) is contained in a leaf of \( \mathcal{F} \) for all \( k \). Recall \( D'_z \) is contained in a domain where \( \mathcal{F} \) is given by the submersion \( f \). Then \( f \) is constant over \( D'_z \) and \( f(h(z_k, t)) = f(h(z_k, t')) \) and in particular, for all \( t \in D' \) we have \( f(h(z_k, t)) = f(h(z_k, t')) \). Then:

\[
\begin{align*}
  f(h(z', t)) &= f(h(\lim_{k \to \infty} z_k, t)) = \lim_{k \to \infty} f(h(z_k, t)) \\
  &= \lim_{k \to \infty} f(h(z_k, t')) = f(h(\lim_{k \to \infty} z_k, t')) \\
  &= f(h(z', t')).
\end{align*}
\]

Therefore, for all \( t \in D' \) we have that \( h(z', t) \) and \( h(z', t') \) are contained in the same leaf. It follows that \( D'_z \) is contained in the leaf passing through \( h(z', t') \). Thus, Assertion is proved.

**Proposition 2.4.** — Let \( \mathcal{F} \) be a foliation by curves on the complex manifold \( M \) such that:
(i) $\mathcal{F}$ is generated by a holomorphic vector field.

(ii) There exists a homeomorphism $h : \Sigma \times D \to M$, where $\Sigma$ is a ball in $\mathbb{C}^{n-1}$ and $D$ is a disc in $\mathbb{C}$.

(iii) For all $z$, there is a discrete closed set $S_z \subset D_z := h(\{z\} \times D)$ such that $D_z \setminus S_z$ is contained in a leaf of $\mathcal{F}$.

Then $\mathcal{F}$ is regular and the sets $D_z$ are the leaves of $\mathcal{F}$.

The following lemmas are easy consequences of well known facts and we left the proofs to the reader.

**Lemma 2.5.** — Let $f : \mathbb{D} \to \mathbb{C}$ be smooth, and holomorphic on $\mathbb{D}$. Suppose that $f$ is regular on $S^1 := \mathbb{S}^1$. Then $f$ is a regular map if and only if the curve $f|_{S^1} : S^1 \to \mathbb{C}$ has degree 1\(^{(1)}\).

**Lemma 2.6.** — Let $M$ be a complex manifold and $D \subset M$ a singular disc. Then there exists a holomorphic injective map $g : E \to M$, where $E = \mathbb{D}$ or $\mathbb{C}$, such that $g(E) = D$.

**Lemma 2.7.** — Let $D \subset \mathbb{C}^n$ be a set homeomorphic to a disc such that for some point $p \in D$ the annulus $D \setminus \{p\}$ is a complex submanifold. Then $D$ is a singular disc.

**Proof of Proposition 2.4.** —

**Assertion 1.** — For all $z$, we have that $D_z$ is a singular disc and the sets $D_z \setminus \text{Sing}(\mathcal{F})$ are the nonsingular leaves of $\mathcal{F}$.

**Proof.** — Let $x \in D_z$. Since $S_z$ is a discrete closed subset of $D_z$, there is a disc $\mathcal{D} \subset D_z$ with $x \in \mathcal{D}$ such that $\mathcal{D} \setminus \{x\} \subset D_z \setminus S_z$. Then, from hypothesis (iii), $\mathcal{D} \setminus \{x\}$ is contained in a leaf of $\mathcal{F}$. If $\mathcal{D}$ is small enough, we may think that $\mathcal{D}$ is contained in $\mathbb{C}^n$. Hence, by applying Lemma 2.7, there exists a holomorphic injective map $g : \mathbb{D} \to M$ with $g(\mathbb{D}) = D$. Since that $x \in D_z$ was arbitrary, it follows that $D_z$ is a singular disc.

Let $L$ be a leaf of $\mathcal{F}$ and suppose that $x \in L \cap (D_z \setminus \text{Sing}(\mathcal{F}))$ for some $z$. Take $\mathcal{D} \subset D_z$ as above. We assume $\mathcal{D}$ small enough such that it is contained in a neighborhood $U$ of $x$ where $\mathcal{F}$ is trivial and given by the submersion $f$. Then $\mathcal{D} \setminus \{x\}$ is contained in a leaf of $\mathcal{F}|_U$ and $f$ is therefore constant over $\mathcal{D} \setminus \{x\}$. Hence, by continuity, $f$ is constant over $\mathcal{D}$. Then $\mathcal{D}$ is contained in a leaf of $\mathcal{F}|_U$ and we have therefore $\mathcal{D} \subset L$. Thus we have $\mathcal{D} \subset L \cap (D_z \setminus \text{Sing}(\mathcal{F}))$. It follows that $L \cap (D_z \setminus \text{Sing}(\mathcal{F}))$ is an open subset of both $L$ and $D_z \setminus \text{Sing}(\mathcal{F})$ for all $L$ and $z$. Now, fix a leaf $L$.

\(^{(1)}\) The degree of a parameterized regular curve in the plane is defined as the winding number around 0 of its velocity vector.
Since the intersection of $L$ with any $D_z \setminus \text{Sing}(F)$ is open in $L$, it follows by connectedness that $L$ is contained in a unique $D_z \setminus \text{Sing}(F)$. For this $D_z \setminus \text{Sing}(F)$, we also have that its intersection with any leaf is open in $D_z \setminus \text{Sing}(F)$. Again by connectedness $D_z \setminus \text{Sing}(F)$ is contained in a unique leaf, thus we necessarily have that $D_z \setminus \text{Sing}(F) = L$. Therefore Assertion 1 is proved.

Fix $p \in M$. We have $p \in D_{z'}$ for some $z' \in \Sigma$. Take $p'$ in $D_{z'} \setminus S_{z'}$. From hypothesis (iii), $p'$ is a regular point of $F$. We have $p' = h(z', t')$ with $t' \in D$. If $B \subset \Sigma$ is a ball containing $z'$, then $\Sigma_0 := B \times \{t'\}$ is a $(n - 1)$ ball passing through $(z', t')$. We assume $B$ small enough such that $\overline{\Sigma_0}$ is mapped by $h$ into a neighborhood $W$ of $p'$ where $F$ is equivalent to a product foliation. Let $\tilde{\Sigma}$ (submanifold of $W$) be a global transversal to $F|_W$. If $w$ is a point contained in $h(\Sigma_0)$, the leaf of $F|_W$ passing through it intersects $\tilde{\Sigma}$ in a unique point $\psi(w)$. We claim that $\psi$ is a homeomorphism of $h(\Sigma_0)$ onto its image. Since $h(\Sigma_0)$ is compact, it suffices to prove that $\psi$ is injective on $\overline{h(\Sigma_0)}$. Suppose that $w_1$ and $w_2$ are two points in $\overline{h(\Sigma_0)}$ contained in the same leaf $L$ of $F|_W$. From Assertion 1, we have that $L \subset D_z$ for some $z$. Then $h^{-1}(L) \subset \{z\} \times D$, hence $h^{-1}(w_1)$ and $h^{-1}(w_2)$ are two different points in the intersection of $(z \times D)$ with $\overline{\Sigma_0}$, which is a contradiction because $\Sigma_0 \subset \Sigma \times \{t'\}$ intersects $(z \times D)$ only at $(z, t')$.

If we redefine $\tilde{\Sigma}$ as $\tilde{\Sigma} = \psi(h(\Sigma_0))$, it follows from above that for all $z \in B$, $D_z$ intersects $\tilde{\Sigma}$ at the unique point $\psi(h(z, t_0))$. Thus we may define the map

$$g : V = h(B \times \mathbb{D}) \to \tilde{\Sigma},$$

$$g(D_z) = D_z \cap \tilde{\Sigma}.$$  

By Assertion 1, each leaf of $F$ is contained in some $D_z$. Then $g$ is constant along the leaves. Therefore, since the restriction of $g$ to any transversal is a holonomy map, we have that $g$ is holomorphic on $V \setminus \text{Sing}(F)$. Actually, since $\text{Sing}(F)$ has codimension $\geq 2$, $g$ is holomorphic on $V$.

Consider $x \in \tilde{\Sigma} \setminus g(\text{Sing}(F))$. Then $D = g^{-1}(x)$ does not intersect $\text{Sing}(F)$. Clearly $D$ is equal to some $D_z$. Then, by Assertion 1, $D \setminus \text{Sing}(F) = D$ is a leaf of $F$. Thus, we conclude that for all $x \in \tilde{\Sigma} \setminus g(\text{Sing}(F))$, the leaf passing through $x$ is simply connected. Moreover, since $\text{Sing}(F)$ has codimension $\geq 2$, we have that $g(\text{Sing}(F))$ has codimension $\geq 1$ in $\tilde{\Sigma}$ and we have therefore that:

**Assertion 2.** — For all $x$ in a dense subset of $\tilde{\Sigma}$, the leaf passing through $x$ is simply connected.
Let $Z$ be a holomorphic vector field which generates $\mathcal{F}$ on $V$ and $\varphi$ the local complex flow of $Z$. Let $L$ be a leaf of $\mathcal{F}|_V$ and let $x_L$ be its intersection with $\tilde{\Sigma} \ (g(L) = \{x_L\})$. There exists $\varepsilon_L > 0$ such that $\varphi(x_L, \ast)$ maps the disc $|t| < \varepsilon_L$ biholomorphically onto a neighborhood $D_L$ of $x_L$ in $L$. Thus, given any $x$ in $D_L$ there exists a unique $\tau_L(x)$ with $|\tau_L(x)| < \varepsilon_L$ such that $\varphi(x_L, \tau_L(x)) = x$. The function $\tau_L : D_L \to \mathbb{C}$ is the complex time between $x_L$ and $x$. Clearly $\tau_L$ is holomorphic on $D_L$.

**Assertion 3.** — The function $\tau_L$ can be analytically continued on $L$ along any path $\gamma : [0, 1] \to L$ with $\gamma(0) = x_L$.

**Proof.** — Since $\gamma$ does not intersect $\text{Sing}(\mathcal{F})$ there exists $\delta > 0$ such that for all $x$ in $\gamma([0, 1])$, the map $\varphi(x, \ast)$ is a biholomorphism between $D_{2\delta}$ and its image. Denote $x_L$ by $x_0$ and let $0 = s_0 < s_1 < \cdots < s_r = 1$ and $x_1 = \gamma(s_1), \ldots, x_r = \gamma(s_r)$ be such that:

(i) The open sets $\varphi(x_i, D_{\delta})$ for $i = 0, \ldots, r$ cover $\gamma([0, 1])$.

(ii) $x_i$ is contained in $\varphi(x_{i-1}, D_{\delta})$ for $i = 1, \ldots, r$.

For each $i = 0, \ldots, r$ let $\tau'_i : \varphi(x_i, D_{2\delta}) \to D_{2\delta}$ be defined by $\varphi(x_i, \tau'_i(x)) = x$. Let $x \in \varphi(x_{i-1}, D_{\delta}) \cap \varphi(x_i, D_{\delta})$. Let $t_i = \tau'_{i-1}(x_i)$ for $i = 1, \ldots, r$ and define $t_0 = 0$. Clearly, $|t_i|$ and $|\tau'_i(x)|$ are less than $\delta$, hence $|t_i + \tau'_i(x)| < 2\delta$ and we have that

$$
\varphi(x_{i-1}, t_i + \tau'_i(x)) = \varphi(\varphi(x_{i-1}, t_i), \tau'_i(x))
= \varphi(\varphi(x_{i-1}, \tau'_{i-1}(x_i)), \tau'_i(x))
= \varphi(x_i, \tau'_i(x))
= x.
$$

Then, by definition of $\tau'_{i-1}$ we obtain:

$$
(2.3) \quad t_i + \tau'_i(x) = \tau'_{i-1}(x).
$$

For each $i = 1, \ldots, r$ let $\tau_i$ be the holomorphic function on $\varphi(x_i, D_{\delta})$ defined by

$$
\tau_i = \tau'_i + t_0 + \cdots + t_i.
$$

By using (2.3) we deduce that $\tau_{i-1} = \tau_i$ on $\varphi(x_{i-1}, D_{\delta}) \cap \varphi(x_i, D_{\delta})$. Moreover, it follows from the definition that $\tau_0$ is equal to $\tau_L$ in a neighborhood of $x_0 = x_L$. Therefore, $\tau_0, \ldots, \tau_r$ give an analytic continuation of $\tau_L$ along $\gamma$.

**Assertion 4.** — Let $L$ be any leaf of $\mathcal{F}|_V$ and let $\gamma', \gamma'' : [0, 1] \to L$ be paths such that $\gamma'(0) = \gamma''(0) = x_L$ and $\gamma'(1) = \gamma''(1) = x \in L$. Let $\tau'_L$ be the analytic continuation of $\tau_L$ along $\gamma'$ and let $\tau''_L$ be the analytic continuation of $\tau_L$ along $\gamma''$. Then $\tau'_L(x) = \tau''_L(x)$. Thus, $\tau_L$ extends as a
holomorphic function on \( L \). Therefore we may define \( \tau : V \setminus \text{Sing}(F) \to \mathbb{C} \) by \( \tau = \tau_L \) on \( L \). Then \( \tau \) is holomorphic on \( U \setminus \text{Sing}(F) \) and extends to \( U \) because \( \text{Sing}(F) \) has codimension \( \geq 2 \). Moreover, if restricted to a leaf, \( \tau \) is a regular map. In particular, \( \tau \) is a submersion on \( U \setminus \text{Sing}(F) \).

**Proof.** — Fix \( L \) and denote \( x_L \) by \( x_0 \). Let \( 0 = s_0 < \cdots < s_r = 1 \), let \( \Sigma_0, \ldots, \Sigma_r \) be transversals to the foliation at the points \( x_0, x_1 = \gamma(s_1), \ldots, x_r = \gamma(s_r) \) respectively, and let \( \delta > 0 \) with the following properties:

(i) \( \Sigma_0 \subset \tilde{\Sigma} \).

(ii) The flow \( \varphi \) maps \( \Sigma_i \times D_\delta \) biholomorphically onto its image, for all \( i = 0, \ldots, r \).

(iii) The transversal \( \Sigma_i \) is contained in \( \varphi(\Sigma_{i-1} \times D_\delta) \), for all \( i = 1, \ldots, r \).

(iv) For all \( i = 1, \ldots, r \) we have that \( \Sigma_i = h_i(\Sigma_0) \), where \( h_i \) is the holonomy map along \( \gamma \).

Denote by \( V' \) the union of the sets \( \varphi(\Sigma_i \times D_\delta) \) for \( i = 0, \ldots, r \). Consider \( x \in V' \) and let \( L_x \) be the leaf passing through \( x \). Let \( k \in \{0, \ldots, r\} \) be such that \( x \in \varphi(\Sigma_k \times D_\delta) \). Then \( L_x \) intersects \( \Sigma_k \) and it follows from hypothesis (iv) that \( L_x \) intersects each \( \Sigma_i \). Since \( \Sigma_0 \subset \tilde{\Sigma} \) we have that \( L_x \) intersects \( \Sigma_0 \) in a unique point and, by (iv), the same holds for each \( \Sigma_i \). Then we may define \( \rho_i : V' \to \Sigma_i \) such that \( \rho_i(x) \) is the point of intersection between \( L_x \) and \( \Sigma_i \). Let \( \tau_i'(x) \in D_\delta \) be defined by \( \varphi(\rho_i(x), \tau_i'(x)) = x \). Since \( \rho_i(x) \in \Sigma_i \), by hypothesis (iii) we have that \( \rho_i(x) \in \varphi(\Sigma_{i-1} \times D_\delta) \) for \( i = 1, \ldots, r \). Then for \( i = 1, \ldots, r \) we may define \( t_i : V' \to D_\delta \) as \( t_i = \tau'_{i-1} \circ \rho_i \). Define \( t_0 : V' \to D_\delta \) as the zero function. Clearly, \( \rho_i \), \( \tau_i \) and \( t_i \) are holomorphic functions. We proceed as in the proof of Assertion 3. Let \( x \in \varphi(\Sigma_i \times D_\delta) \). Since \( |t_i(x)| \) and \( |\tau_i'(x)| \) are less than \( \delta \), then \( |t_i(x) + \tau_i'(x)| < 2\delta \). Thus, by hypothesis (ii), \( \varphi(\rho_{i-1}(x), t_i(x) + \tau_i'(x)) \) is well defined and:

\[
\varphi(\rho_{i-1}(x), t_i(x) + \tau_i'(x)) = \varphi(\rho_{i-1}(x), t_i(x)), \tau_i'(x)) = \varphi(\rho_{i-1}(x), \tau_i' \circ \rho_i(x)), \tau_i'(x)) = \varphi(\rho_i(x), \tau_i'(x)) = x.
\]

Then by definition of \( \tau_{i-1}' \) we deduce that

\[
t_i(x) + \tau_i'(x) = \tau_{i-1}'(x).
\]

Thus, the holomorphic functions on \( \varphi(\Sigma_i \times D_\delta) \) defined as

\[
(2.4) \quad \tau_i(x) = \tau_i'(x) + t_0(x) + \cdots + t_i(x)
\]
for each \(i = 0, \ldots, r\) are such that
\[
\tau_i = \tau_{i-1}
\]
on \(\varphi(\Sigma_i \times \mathbb{D}_\delta) \cap \varphi(\Sigma_{i-1} \times \mathbb{D}_\delta)\). Observe that for any leaf \(L'\), the restriction \(\tau_0|_{L'}\) coincides with \(\tau_{L'}\) on a neighborhood of \(x_{L'}\). Then \(\tau_0|_{L'}, \ldots, \tau_r|_{L'}\) give an analytic continuation of \(\tau_{L'}\). Thus, \(\tau_r|_L\) is the analytic continuation of \(\tau_L\) along \(\gamma'\), hence \(\tau_r(x) = \tau_r'(x)\). We denote \(\tau_r\) by \(\tau'\). Analogously we construct \(\tau''\) for \(\gamma''\). Then we have that \(\tau''|_{L'}\) is an analytic continuation of \(\tau_{L'}\), and, \(\tau''|_L\) is the analytic continuation of \(\tau_L\) along \(\gamma''\), hence \(\tau''(x) = \tau_L(x)\). By Assertion 2, we may take a sequence \(\{x_k\}\) of points in \(\Sigma_0\) with \(x_k \to x\) as \(k \to \infty\) and such that the leaf \(L_k\) passing through \(x_k\) is simply connected for all \(k\). From above \(\tau'|_{L_k}\) and \(\tau''|_{L_k}\) are analytic continuations of \(\tau_{L_k}\). Since \(L_k\) is simply connected and, by Assertion 2, \(\tau_{L_k}\) has an analytic continuation along any path, then \(\tau'|_{L_k}\) and \(\tau''|_{L_k}\) coincide on a neighborhood of \(x_k\). In particular, \(\tau'(x_k) = \tau''(x_k)\). Making \(k \to \infty\) it follows by continuity that \(\tau'(x) = \tau''(x)\), that is, \(\tau_L'(x) = \tau_L''(x)\). Therefore, \(\tau_L\) extends to \(L\).

We define \(\tau : V \setminus \text{Sing}(\mathcal{F}) \to \mathbb{C}\) by \(\tau|_L = \tau_L\). From above, \(\tau\) coincides with the holomorphic function \(\tau'\) on a neighborhood of the point \(x\) (arbitrary point). Then \(\tau\) is holomorphic. Finally, remember (equation 2.4) that on a neighborhood of any non singular point, \(\tau\) is expressed as
\[
\tau_r(x) = \tau_r'(x) + t_0(x) + \cdots + t_r(x).
\]
If we restrict \(x\) to a leaf, the first term of the sum above is a regular map and the other terms are constants. Hence \(\tau\) is a regular map of any leaf. This finishes the proof of Assertion 4.

Given \(x \in \tilde{\Sigma}\), we know that \(g^{-1}(x)\) is equal to \(D_z\) for some \(z\). We denote \(g^{-1}(x)\) by \(D_z\). Thus, we have \(p \in D_z\) for \(x = g(p)\). It follows from hypothesis \((iii)\) that there is a disc \(\mathcal{D}' \subset D_x\) containing \(p\) such that \(\mathcal{D}' \setminus \{p\}\) is contained in a leaf. Lemma 2.7 implies that there is a holomorphic bijective map \(f : \Omega \to \mathcal{D}',\ f(0) = p\), where \(\Omega \subset \mathbb{C}\) is a disc containing \(\mathbb{D}\). Thus if \(\mathcal{D} = f(\mathbb{D})\), we have that \(f : \overline{\mathbb{D}} \to \mathcal{D}\) is holomorphic and regular on \(\mathbb{D} \setminus \{0\}\). Since \(\overline{\mathcal{D}} \setminus \{p\}\) is contained in a leaf and by Assertion 3 we have that \(\tau\) is a submersion on \(U \setminus \text{Sing}(\mathcal{F})\), then there exists a neighborhood \(V\) of \(\partial \Delta\) on which \(\tau\) defines a foliation by transversal balls along \(\partial \Delta\). If we denote by \(\Sigma_\zeta\) the transversal passing trough \(\zeta \in \partial \Delta\) we have that \(\tau\) is constant along \(\Sigma_\zeta\). Recall that \(y \in \tilde{\Sigma}\) is the unique point in the intersection of \(D_y\) and \(\tilde{\Sigma}\). It follows from the transversal uniformity of the foliation that if \(y \in \tilde{\Sigma}\) is close to \(x\) then \(D_y\) intersects only one time each transversal \(\Sigma_\zeta\). Let \(\theta_y(\zeta)\) be the intersection of \(D_y\) with \(\Sigma_\zeta\). Since \(\theta_y(\zeta)\) and \(\zeta\) are both contained in
the invariance of the algebraic multiplicity

\[ \Sigma_\zeta, \text{ we have that } \tau(\theta_y(\zeta)) = \tau(\zeta) \text{ for all } \zeta \in \partial \Delta. \]

Note that \( \theta_y := \theta_y(\partial \Delta) \) is a smooth Jordan curve in \( D_y \). By Assertion 2, we may choose \( y \) such that \( D_y \) is a leaf. We consider \( D_y \subset D_y \), the regular disc bounded by \( \theta_y \).

Let \( f_y : \mathbb{D} \to D_y \) be a uniformization map. Since \( \theta_y \) is a smooth Jordan curve, \( f_y \) extends as a diffeomorphism \( f_y : \overline{\mathbb{D}} \to \overline{D_y} \) (see [14], p.323). By Assertion 3, we have that \( \tau \) is regular on \( D_y \). It follows that \( \tau(\partial D_y) = \tau(\partial \Delta) \). By Lemma 2.5, the curve \( \tau \circ f_y : S^1 \to \mathbb{C} \) has degree 1. Remember that \( \tau(\partial D_y) = \tau(\partial \Delta) \) is a Jordan curve in \( \mathbb{C} \). It follows that \( \tau(\theta_y(\zeta)) = \tau(\zeta) \) for all \( \zeta \in \partial \Delta \), thus \( \tau(\partial D_y) = \tau(\partial \Delta) \).

\[ \tau(\theta_y(\zeta)) = \tau(\zeta) \]

Therefore \( \tau \circ f : S^1 \to \mathbb{C} \) is only a reparametrization of \( \tau \circ f_y : S^1 \to \mathbb{C} \), hence \( \tau \circ f : S^1 \to \mathbb{C} \) is regular and has degree 1. Again by Lemma 2.5, \( \tau \circ f : \mathbb{D} \to \mathbb{C} \) is also a regular map and in particular, \( \tau \circ f \) is locally injective. Therefore there exists a disc \( U \subset \mathbb{D} \), centered at 0, such that \( \tau \circ f \) is injective on \( U \). Then

\[ \tau \circ f(\partial U) \]

is a Jordan curve in \( \mathbb{C} \). We also denote \( f(U) \) by \( D \). Again, let \( \Sigma_\zeta \) be the transversal ball through \( \zeta \in \partial D \). Proceeding as above, if \( \Sigma' \) is a small enough ball in \( \Sigma \) containing \( x = g(p) \), we obtain that for all \( y \in \Sigma' \) the set \( D_y \) intersects each \( \Sigma_\zeta \) at the unique point \( \theta_y(\zeta) \). Thus we have the Jordan curve \( \theta_y \) in \( D_y \) such that \( \tau(\theta_y) = \tau(\partial D) \). Remember that \( \tau(\partial D) = \tau \circ f(\partial U) \) is a Jordan curve in \( \mathbb{C} \). It follows that \( \tau(\theta_y) \) is Jordan curve in \( \mathbb{C} \) for all \( y \). Let \( D_y \subset D_y \) be the disc bounded by \( \theta_y \). Since \( D_y \) is a singular disc, by Lemma 2.6, there is an injective holomorphic map \( f_y : E \to M \), where
\[ E = \mathbb{D} \text{ or } \mathbb{C}, \text{ such that } f_y(E) = D_y. \] Let \( \Omega_y \subset E \) be such that \( f_y(\Omega_y) = D_y. \) Clearly \( \Omega_y \) is a disc and \( f_y(\partial \Omega_y) = \partial D_y. \) Then
\[
\tau \circ f_y(\partial \Omega_y) = \tau(\partial D_y)
\]
is, from above, a Jordan curve in \( \mathbb{C}. \) Hence we deduce that the holomorphic function \( \tau \circ f_y : \Omega_y \to \mathbb{C} \) is injective on \( \Omega_y. \) Thus, since \( f_y \) is injective, we conclude that
\[
\tau : \overline{D}_y \to \mathbb{C}
\]
is injective for all \( y \in \Sigma'. \)

Denote by \( W \) the union of the discs \( D_y \) for all \( y \in \Sigma'. \) It is easy to see that \( W \) is a neighborhood of \( p. \) Define
\[
F : \overline{W} \to \tilde{\Sigma} \times \mathbb{C}
\]
\[
F(w) = (g(w), \tau(w))
\]

**Assertion 5.** — \( F \) is a biholomorphism between \( W \) and its image.

**Proof.** — Clearly \( F \) is holomorphic on \( W. \) We shall prove that \( F \) injective on \( \overline{W}. \) Suppose \( F(w) = F(w'). \) Then \( g(w) = g(w') = y, \) hence \( w, w' \in D_y \) and, since \( \overline{W} \cap D_y = \overline{D}_y, \) we have \( w, w' \in \overline{D}_y. \) On the other hand \( \tau(w) = \tau(w') \) and since \( \tau \) is injective on \( \overline{D}_y \) we conclude that \( w = w'. \) Now, since \( \overline{W} \) is compact, \( F \) is a homeomorphism onto its image and it follows that \( F \) is a biholomorphism.

Now, we will prove that \( p \in W \) is regular for \( F. \) Let \( \mathcal{N} \) be the regular foliation on \( \tilde{\Sigma} \times \mathbb{C} \) whose leaves are the sets \( \{\ast\} \times \mathbb{C}. \) Let \( \mathcal{F}' \) be the pull-back foliation of \( \mathcal{N} \) by the biholomorphism \( F. \) Then \( \mathcal{F}' \) is regular and it is easy to see that \( \mathcal{F}' \) coincides with \( \mathcal{F} \) out on a open set of \( W \) (out of \( \text{Sing}(\mathcal{F}) \)). Then \( \mathcal{F}' = \mathcal{F} \) on \( W \) and \( F \) is therefore regular at \( p. \) Since \( p \in U \) was arbitrary, we have proved that \( \text{Sing}(\mathcal{F}) \) is empty. Then, from Assertion 1, the sets \( D_z \) are the leaves of \( \mathcal{F}. \) The proof of Proposition 2.4 is complete. \( \square \)

**Proof of Theorem 2.2.** —

**Assertion 1.** — Let \( z \in \Sigma \) such that \( D_z \) is not contained in \( X. \) Then \( D_z \) is contained in a leaf of \( \mathcal{F}. \)

**Proof.** — Take \( t_0 \in D_z \) such that \( h(z, t_0) \notin X. \) Since \( X \) is closed in \( M, \) if \( \Sigma' \) is a small enough neighborhood (ball) of \( z \) in \( \Sigma, \) we have that \( h(z', t_0) \notin X \) for all \( z' \in \Sigma'. \) Hence, for all \( z' \in \Sigma' \) we have that \( D_{z'} \) is not contained in \( X. \) Then, by hypothesis (ii), \( S_{z'} := D_{z'} \cap X \) is discrete and \( D_{z'} \setminus S_{z'} \) is contained in a nonsingular leaf of \( \mathcal{F}. \) Therefore, \( \mathcal{F} \) restricted to \( M' := h(\Sigma' \times D) \) satisfies the hypothesis of Proposition 2.4 and we have therefore that \( D_z \) is contained in a leaf of \( \mathcal{F}. \)
Assertion 2. — Let $z \in \Sigma$ such that $D_z$ is contained in $X$. Then $D_z$ is a singular disc.

Proof. — Let $x \in D_z$, $x = h(z, t)$. Let $\Sigma' \subset \Sigma$ be a neighborhood (a ball) of $z$ and $D' \subset D$ be a neighborhood (a disc) of $t$. If $\Sigma'$ and $D'$ are small enough, we may assume that $M' := h(\Sigma' \times D')$ is a domain in $\mathbb{C}^n$. Since $X$ has codimension $\geq 1$, there is a path $x_s = h(z_t, t_s)$ in $M'$ such that $x_0 = x$ and $x_s \notin X$ for all $s > 0$. Then $D_s := D_{z_s}$ is not contained in $X$ for all $s > 0$ and it follows by Assertion 1 that $D_s$ is contained in a leaf. Hence $D_s$ is a regular disc for all $s > 0$. Then, we may apply Lemma 2.1 to the family of discs $D_s$ and conclude that $D_z = D_0$ is a singular disc.

Assertion 3. — Let $z$ be such that $D_z \subset X$. Let $S_z$ be the set of singularities of the singular disc $D_z$. Then $D_z \setminus S_z$ is contained in a leaf of $\mathcal{F}$.

Proof. — By Assertion 2, if $D_z$ is not contained in $X$ we have that $D_z$ is contained in a leaf of $\mathcal{F}$. Therefore, the hypothesis of Proposition 2.3 holds for $\mathcal{F}$ and Assertion 3 follows.

Let $z$ be such that $D_z$ is not contained in $X$. By hypothesis $(iii)$ of 2.2, we have that $S_z := D_z \cap X$ is discrete and $D_z \setminus S_z$ is contained in a leaf of $\mathcal{F}$. From this and Assertion 3 we conclude: for all $z$ there is a discrete set $S_z$ such that $D_z \setminus S_z$ is contained in a leaf of $\mathcal{F}$. Therefore the hypothesis of Proposition 2.4 holds and Theorem 2.2 follows. □

3. The algebraic multiplicity and the Chern class of the tangent bundle of the strict transform

Let $\mathcal{F}_0$, $\tilde{\mathcal{F}}_0$ and $h$ as in §1.

Proposition 3.1. — If $h$ extends to the divisor as a homeomorphism between $\pi^{-1}(U)$ and $\pi^{-1}(\tilde{U})$, then the extension also denoted by $h$ is a topological equivalence between $\mathcal{F}_0$ and $\tilde{\mathcal{F}}_0$.

Proof. — Is a direct application of Theorem 1.4. □

Proof of Theorem 1.1. — Suppose that $\mathcal{F}$ is generated on $U$ by the holomorphic vector field

$$V = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i}, \quad a_i \in \mathcal{O}_U, \quad \text{g.c.d.}(a_1, a_2, \ldots, a_n) = 1.$$
For each \( j = 1, 2, \ldots, n \), let \( U_j = (x_j \neq 0) \) and \( U_j' = \pi^{-1}(U_j) \). Let \( V_j = \pi^*(V|_{U_j}) \). If \((x_{1j}', \ldots, x_{nj}')\) are coordinates on \( U_j' \) such that
\[
\pi(x_{1j}', \ldots, x_{nj}') = (x_{1j}x_{1j}', \ldots, x_{jj}x_{jj}', \ldots, x_{nj}x_{nj}'),
\]
then
\[
V_j = a_j \frac{\partial}{\partial x_{jj}'} + \sum_{i=1, i \neq j}^n \frac{a_i - x_{ij}a_j}{x_{jj}'} \frac{\partial}{\partial x_{ij}'}
\]
where \( a_i = a_i \circ \pi \) for \( i = 1, \ldots, n \). On \( U_j' \), \( \mathcal{F}_0 \) is defined by the vector field
\[
W_j = \frac{1}{(x_{jj}')^{r-\xi}} V_j,
\]
where \( r \) is the algebraic multiplicity of \( V \) at \( 0 \in \mathbb{C}^n \) and \( \xi = 1 \) or \( 0 \) depending on the divisor being invariant or not by \( \mathcal{F}_0 \). Evidently \( V_i = V_j \) on \( U_i' \cap U_j' \). Then
\[
W_i = \left(\frac{x_{jj}'}{x_{ii}'}\right)^{r-\xi} W_j \quad \text{on} \quad U_i' \cap U_j'.
\]
It follows from this equation that the tangent bundle \( T\mathcal{F}_0 \) of \( \mathcal{F}_0 \) is isomorphic to \( L^{\xi-r} \), where \( L \) is the line bundle associated to the divisor \( E = \pi^{-1}(0) \). Then the Chern class \( c(T\mathcal{F}_0) \) of \( T\mathcal{F}_0 \) is equal to \((\xi-r)c(L)\). It is natural consider \( E \) as an element in \( H_{n-2}(U', \mathbb{Z}) \), where \( U' = \pi^{-1}(U) \). We know that \( c(L) \) is equal to \( d(E) \in H^2(U', \mathbb{Z}) \), the dual of \( E \). Therefore
\[
c(T\tilde{\mathcal{F}}_0) = (\xi-r)d(E).
\]
On the other hand, make \( \tilde{U}' = \pi^{-1}(\tilde{U}) \) and observe that the divisor \( E \) is invariant by \( \mathcal{F}_0 \) if and only if it is by \( \tilde{\mathcal{F}}_0 \). Then analogously we have
\[
c(T\tilde{\mathcal{F}}_0) = (\xi-\tilde{r})d(E),
\]
where \( \tilde{r} \) is the algebraic multiplicity of \( \tilde{\mathcal{F}} \) and \( \tilde{d}(E) \in H^2(\tilde{U}', \mathbb{Z}) \) is the dual of \( E \). By Proposition 3.1 we have that \( h : U' \to \tilde{U}' \) is a topological equivalence between \( \mathcal{F}_0 \) and \( \tilde{\mathcal{F}}_0 \). Then Theorem 1.3 implies that
\[
(\xi-r)h^*(d(E)) = (\xi-\tilde{r})\tilde{d}(E).
\]
We may assume that \( U \) is a ball in \( \mathbb{C}^n \). Thus, we have that \( U' \) is a tubular neighborhood of \( E \) and therefore \( H^2(U', \mathbb{Z}) \simeq \mathbb{Z} \). Since the cohomology is invariant by homeomorphisms, we also have \( H^2(\tilde{U}', \mathbb{Z}) \simeq \mathbb{Z} \). Can be proved that \( d(E) \) and \( \tilde{d}(E) \) are generators of \( H^2(U', \mathbb{Z}) \) and \( H^2(\tilde{U}', \mathbb{Z}) \) respectively. Then we have that \( h^*(d(E)) = \tilde{d}(E) \) or \( h^*(d(E)) = -\tilde{d}(E) \). By using this in (3.1) we obtain \( r = \tilde{r} \) or \( r + \tilde{r} = 2\xi \). The second possibility implies \( r = \tilde{r} = \xi \), since \( r \geq 1, r \geq 1 \) and \( \xi = 1 \) or \( 0 \). Therefore we conclude that \( r = \tilde{r} \).
\[\square\]
Remark. — Under the hypothesis of Theorem 1.1, we have another invariants. The restriction of \( F_0 \) to the divisor is a foliation with \( \text{Sing}(F_0) \) as singular set. It is well known that this foliation coincides out of the singular set with a unique foliation \( N \) of codimension \( \geq 2 \) in the divisor (the saturated foliation). We will say that \( N \) is the foliation induced by \( F_0 \) in the divisor. Let \( \tilde{N} \) be the foliation induced by \( \tilde{F}_0 \) in the divisor. It follows from Theorem 1.4 that \( N \) and \( \tilde{N} \) are topologically equivalent. Thus, since the divisor is isomorphic to \( \mathbb{P}^{n-1} \), Theorem 1.3 implies that \( d(N) = d(\tilde{N}) \). In other words, the degree of the foliation induced in the divisor is invariant.

From above, \( F_0 \) is generated by the holomorphic vector fields \( W_i \) and

\[
W_i = \left( \frac{x_j}{x_i} \right)^{r-\xi} W_j \quad \text{on} \quad U_i' \cap U_j',
\]

where \( \xi = 1 \) or \( 0 \). Let \( x \in U_i' \cap U_j' \). Let \( x^i = (x_1^i, \ldots, x_n^i) \) be the coordinates of \( x \) in \( U_i' \) and let \( x^j = (x_1^j, \ldots, x_n^j) \) be the coordinates of \( x \) in \( U_j' \). Since \( \pi(x^i) = \pi(x^j) \), we have that

\[
(x_i^j x_1^i, \ldots, x_i^j, \ldots, x_i^j x_i^i) = (x_j^j x_1^j, \ldots, x_j^j, \ldots, x_j^j x_i^i),
\]

hence \( x_j^i / x_i^i = x_j^i \). Replacing in last equation we obtain:

\[
(3.2) \quad W_i = (x_j^i)^{r-\xi} W_j \quad \text{on} \quad U_i' \cap U_j'.
\]

Observe that \( \pi^{-1}(0) \cap U_i' \) is represented by \( (x_i^j = 0) \). Recall that \( \pi^{-1}(0) \) is canonically isomorphic to \( \mathbb{P}^{n-1} \). A point \( p \) in \( \pi^{-1}(0) \cap U_i' \) given by

\[
(x_1^i(p), \ldots, 0_i, \ldots, x_n^i(p))
\]

is represented in homogeneous coordinates by

\[
[z_1 : \cdots : z_n](p) = [x_1^i(p) : \cdots : 1_i : \cdots : x_n^i(p)],
\]

hence \( x_j^i(p) = (z_j / z_i)(p) \). Thus, if \( U_i = U_i' \cap \pi^{-1}(0) \) and \( J_i = W_i|U_i \), it follows from (3.2) that

\[
(3.3) \quad J_i = (z_j / z_i)^{r-\xi} J_j \quad \text{on} \quad U_i \cap U_j.
\]

Let \( S \) be the union of the components of codimension 2 of \( \text{Sing}(F_0) \). Then \( S \) is the codimension 1 part (respect to the divisor) of the zero sets of \( \{ J_i \} \). Each \( J_i \) may be expressed as \( J_i = f_i Z_i \), where \( f_i \) is a holomorphic function on \( U_i \) and the vector field \( Z_i \) has singular set of codimension \( \geq 2 \). It follows from (3.3) that

\[
Z_i = (f_j / f_i) (z_j / z_i)^{r-\xi} Z_j \quad \text{on} \quad U_i \cap U_j.
\]

From this equation, it is not difficult to conclude that

\[
r = d(N) - \deg(S) - 1 + \xi,
\]
where deg($S$) is the degree of $S$ as a divisor of $\pi^{-1}(0)$. Then, since the algebraic multiplicity and the degree of the foliation induced in the divisor are invariants, we deduce that the degree of the codimension 1 part of the singular set of the strict transform is also an invariant. Moreover it is not difficult to see that $h(S) = \tilde{S}$, where $\tilde{S}$ is the union of the components of codimension 2 of Sing($\tilde{F}_0$).

4. The case $C^1$

In this section we prove Theorem 1.2. In view of Theorem 1.1, it is sufficient to show the following.

**Proposition 4.1.** — Let $F$ and $\tilde{F}$ be two foliations by curves of neighborhoods $U$ and $\tilde{U}$ of $0 \in \mathbb{C}^n$. Let $h : U \rightarrow \tilde{U}$ be a $C^1$ equivalence. Let $h : \pi^{-1}(U \backslash \{0\}) \rightarrow \pi^{-1}(\tilde{U} \backslash \{0\})$ be as before. Then $h$ can be extended to the divisor as a homeomorphism between $\pi^{-1}(U)$ and $\pi^{-1}(\tilde{U})$.

We start the proof.

**Proposition 4.2.** — Under the conditions of Proposition 4.1, we have that $d h(0) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ maps complex lines onto complex lines. Furthermore, if $J : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is the conjugation $J(z) = \bar{z}$, then either $d h(0) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a $c$-linear isomorphism, or $d h(0) = Q \circ J$, where $Q : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a $c$-linear isomorphism. Thus, $d h(0)$ induces a diffeomorphism of $\mathbb{P}^{n-1}$ onto itself.

**Proof.** — let $L$ be a complex line, $0 \in L \subset \mathbb{C}^n$. There exists $c$-linear functions $A_i : \mathbb{C}^n \rightarrow \mathbb{C}$ for $i = 1, \ldots, (n - 1)$, such that

$$L = \{ z \in \mathbb{C}^n : A_i(z) = 0 \text{ for all } i = 1, \ldots, (n - 1) \}.$$  

Let $V : U \rightarrow \mathbb{C}^n$ be a holomorphic vector field which generates $F$. The set:

$$B = \{ z \in \mathbb{C}^n : A_i \circ V(z) = 0 \text{ for all } i = 1, \ldots, (n - 1) \}$$

is an analytic variety and it is easy to see that $0 \in B$. Then, there exists a complex curve contained in $B$ and passing through 0. In particular there exists a sequence of points $z_k \in \mathbb{C}^n \backslash \{0\}$, $z_k \rightarrow 0$, such that $A_i \circ V(z_k) = 0$ for all $k \in \mathbb{N}$ and all $i = 1, 2, \ldots, (n - 1)$. In other words, $T_{z_k}F = L$ for all $k \in \mathbb{N}$. Now, since $h$ is a $C^1$ equivalence, $d h_{z_k}(T_{z_k}F) = T_{h(z_k)}\tilde{F}$, that is, $d h_{z_k}(L) = T_{h(z_k)}\tilde{F}$ is a complex line for all $k \in \mathbb{N}$. Making $k \rightarrow \infty$, since $h \in C^1$ and the space of complex lines of $\mathbb{C}^n$ is compact, we obtain that $d h_0(L)$ is also a complex line. The second part of the proposition is an immediate consequence of the following lemma. □
LEMMA 4.3. — Let $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ ($n \geq 2$) be a $\mathbb{R}$-linear isomorphism. Identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and assume that $A$ maps complex lines onto complex lines. Then, either $A$ is a $\mathbb{C}$-linear isomorphism, or $A = Q \circ J$ with $Q : \mathbb{C}^n \to \mathbb{C}^n$ a $\mathbb{C}$-linear isomorphism.

Proof. — Since $A$ maps any complex line onto a complex line, for all $v \in \mathbb{C}^n \setminus \{0\}$ there exists $\theta(v) \in \mathbb{C} \setminus \{0\}$ such that $A(iv) = \theta(v)A(v)$. Let $v_1$ and $v_2$ be two $\mathbb{C}$-linearly independent vectors. Then

\[
A(iv_1 + iv_2) = A(iv_1) + A(iv_2) = \theta(v_1)A(v_1) + \theta(v_2)A(v_2).
\]

Moreover:

\[
A(iv_1 + iv_2) = \theta(v_1 + v_2)A(v_1 + v_2) = \theta(v_1 + v_2)A(v_1) + \theta(v_1 + v_2)A(v_2).
\]

From the equations above, we obtain:

\[
(\theta(v_1) - \theta(v_1 + v_2))A(v_1) + ((\theta(v_2) - \theta(v_1 + v_2))A(v_2) = 0.
\]

Let $L_1$ and $L_2$ be the complex lines generated by $v_1$ and $v_2$ respectively. Since $v_1$ and $v_2$ are $\mathbb{C}$-linearly independent, we have that $L_1$ and $L_2$ are different. This implies, since $A$ is an isomorphism, that $A(L_1)$ and $A(L_2)$ are different complex lines. Then, since $A(L_1)$ and $A(L_2)$ are generated by $A(v_1)$ and $A(v_2)$ respectively, we have that $A(v_1)$ and $A(v_2)$ are $\mathbb{C}$-linearly independent. Thus, it follows from equation (4.1) that

\[
\theta(v_1) = \theta(v_1 + v_2) = \theta(v_2).
\]

It is now easy to see that $\theta(v) = \theta_0$, $\forall v \in \mathbb{C}^n \setminus \{0\}$. We know that there exists two $\mathbb{C}$-linear transformations $P : \mathbb{C}^n \to \mathbb{C}^n$ and $Q : \mathbb{C}^n \to \mathbb{C}^n$ such that

\[
A(z) = P(z) + Q(\bar{z}), \text{ for all } z \in \mathbb{C}^n.
\]

Then

\[
A(iz) = iP(z) - iQ(\bar{z}).
\]

On the other hand

\[
A(iz) = \theta_0A(z) = \theta_0P(z) + \theta_0Q(\bar{z}), \text{ for all } z \in \mathbb{C}^n.
\]

consequently

\[
(\theta_0 - i)P(z) + (\theta_0 + i)Q(\bar{z}) = 0.
\]

Since, as functions of $z$, $(\theta_0 - i)P$ and $(\theta_0 + i)Q \circ J$ are holomorphic and anti-holomorphic respectively, we have that

\[
(\theta_0 - i)P \equiv 0, \ (\theta_0 + i)Q \circ J \equiv 0.
\]
From this it is easy to see that either $P = 0$, or $Q = 0$. This proves the lemma.

**Definition 4.4.** Let $\{z_k\}$ be a sequence of points in $\mathbb{C}^n \setminus \{0\}$. Let $L$ be a complex line in $\mathbb{C}^n$. We say that $\{z_k\}$ is tangent to $L$ at $0$ if $z_k \to 0$ and every accumulation point of $\{z_k/||z_k||\}$ is contained in $L$.

Let $\pi : \mathbb{C}^n \to \mathbb{C}^n$ be the blow up at $0 \in \mathbb{C}^n$. We know that $\pi^{-1}(0)$ is naturally isomorphic to $\mathbb{P}^{n-1}$. Thus, for each $p \in \pi^{-1}(0)$ we denote by $L_p$ the respective complex line in $\mathbb{C}^n$. The following fact is well known and we left the proof to the reader:

**Proposition 4.5.** Let $\{p_k\}$ be a sequence of points in $\mathbb{C}^n \setminus \pi^{-1}(0)$. Then $p_k \to p \in \pi^{-1}(0)$ if and only if $\{\pi(p_k)\}$ is tangent to $L_p$ at $0$.

**Proof of Proposition 4.1.** Let $p \in \pi^{-1}(0)$ and $\{p_k\}$ any sequence of points in $\pi^{-1}(U) \setminus \pi^{-1}(0)$ such that $p_k \to p$.

Since $h \in C^1$, we have

$$h(\pi(p_k)) = dh_0(\pi(p_k)) + r(\pi(p_k)), \quad \text{where} \quad \frac{r(\pi(p_k))}{||\pi(p_k)||} \to 0 \text{ as } k \to 0.$$

Then

$$\frac{h(\pi(p_k))}{||\pi(p_k)||} = dh_0 \left( \frac{\pi(p_k)}{||\pi(p_k)||} \right) + \frac{r(\pi(p_k))}{||\pi(p_k)||}.$$

By proposition 4.5, $\pi(p_k)$ is tangent to $L_p$ at $0$, hence any point of accumulation of the sequence $\{(\pi(p_k))/||\pi(p_k)||\}$ is contained in $L_p$. Thus, it is easy to see from equation (4.2) that any point of accumulation of the sequence $\{h(\pi(p_k))/||\pi(p_k)||\}$ is contained in $dh_0(L_p)$ and the same holds for the sequence

$$\frac{h(\pi(p_k))}{||h(\pi(p_k))||} = \frac{h(\pi(p_k))}{||\pi(p_k)||} \cdot \frac{||\pi(p_k)||}{||h(\pi(p_k))||}.$$

From proposition 4.2 we have that $dh_0(L_p)$ is a complex line. Then $\{h(\pi(p_k))\}$ is tangent to $dh_0(L_p)$ at $0$. It follows by proposition 4.5 that $\pi^{-1} \circ h \circ \pi(p_k) = h(p_k) \to q$, where $q \in \pi^{-1}(0)$ is such that $L_q = dh_0(L_p)$. We extend $h$ by making $h(p) = dh_0(L_p)$ for all $p \in \pi^{-1}(0)$. Finally, it is easy to prove that $h : \pi^{-1}(U) \to \pi^{-1}(U)$ is a homeomorphism.

**BIBLIOGRAPHY**

THE $C^1$ INVARIANCE OF THE ALGEBRAIC MULTIPURITY

2135


Manuscrit reçu le 18 juin 2009,
accepté le 21 septembre 2009.

Rudy ROSAS
Pontificia Universidad Católica del Perú
Av Universitaria 1801
San Miguel, Lima 32 (Perú)
Instituto de Matemática y Ciencias Afines (IMCA)
Jr. los Biólogos 245
La Molina, Lima (Perú)
rudy.rosas@pucp.edu.pe
rudy@imca.edu.pe