Pham Hoang HIEP

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HÖLDER CONTINUITY OF SOLUTIONS TO THE MONGE-AMPÈRE EQUATIONS ON COMPACT KÄHLER MANIFOLDS

by Pham Hoang HIEP

Abstract. — We study Hölder continuity of solutions to the Monge-Ampère equations on compact Kähler manifolds. T. C. Dinh, V.A. Nguyen and N. Sibony have shown that the measure $\omega^n_u$ is moderate if $u$ is Hölder continuous. We prove a theorem which is a partial converse to this result.

Résumé. — Nous étudions la continuité de Hölder des solutions des équations de Monge-Ampère sur des variétés Kählériennes compactes. T. C. Dinh, V.A. Nguyen et N. Sibony ont prouvé que $\omega^n_u$ est modéré si $u$ est Hölder-continue. Nous démontrons dans quelques cas la réciproque de ce résultat.

1. Introduction

Let $X$ be a compact $n$-dimensional Kähler manifold equipped with a fundamental form $\omega$ satisfying $\int_X \omega^n = 1$. An upper semicontinuous function $\varphi: X \to [-\infty, +\infty)$ is called $\omega$-plurisubharmonic ($\omega$-psh) if $\varphi \in L^1(X)$ and $\omega_\varphi := \omega + dd^c \varphi \geq 0$. By $\text{PSH}(X, \omega)$ (resp. $\text{PSH}^{-}(X, \omega)$) we denote the set of $\omega$-psh (resp. negative $\omega$-psh) functions on $X$. The complex Monge-Ampère equation $\omega^n_u = f \omega^n$ was solved for smooth positive $f$ in the fundamental work of S. T. Yau (see [31]). Later S. Kołodziej showed that there exists a continuous solution if $f \in L^p(\omega^n)$, $f \geq 0$, $p > 1$ (see [24]). Recently in [27] he proved that this solution is Hölder continuous in this case (see also [18] for the case $X = \mathbb{C}P^n$). In Corollary 1.2 in [16] the authors have shown that the measure $\omega^n_u$ is moderate if $u$ is Hölder continuous. The main result is the following theorem which give a partial answer to the converse problem:

Keywords: Hölder continuity, complex Monge-Ampère operator, $\omega$-plurisubharmonic functions, compact Kähler manifolds.

Theorem A. — Let \( \mu \) be a non-negative Radon measure on \( X \) such that
\[
\mu(B(z,r)) \leq Ar^{2n-2+\alpha},
\]
for all \( B(z,r) \subset X \) (\( A, \alpha > 0 \) are constants). Then for every \( f \in L^p(d\mu) \) with \( p > 1 \), \( \int_X f d\mu = 1 \), there exists a Hölder continuous \( \omega \)-psh function \( u \) such that \( \omega_n^u = fd\mu \).

The following results are simple applications of Theorem A:

Corollary B. — Let \( \varphi \in \text{PSH}(X,\omega) \) be a Hölder continuous function. Then for every \( f \in L^p(\omega_\varphi \wedge \omega^{n-1}) \) with \( p > 1 \), \( \int_X f \omega_\varphi \wedge \omega^{n-1} = 1 \), there exists a Hölder continuous \( \omega \)-psh function \( u \) such that \( \omega_n^u = f \omega_\varphi \wedge \omega^{n-1} \).

Corollary C. — Let \( S \) be a \( C^1 \) smooth real hypersurface in \( X \) and \( V_S \) be the volume measure on \( S \). Then for every \( f \in L^p(dV_S) \) with \( p > 1 \), \( \int_X f dV_S = 1 \), there exists a Hölder continuous \( \omega \)-psh function \( u \) such that \( \omega_n^u = f dV_S \).

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2. Preliminaries

First we recall some elements of pluripotential theory that will be used throughout the paper. Details can be found in [2]–[3], [5]–[6], [4], [7], [9]–[8], [13]–[15], [19]–[20], [21], [23]–[27], [28], [29]–[30], [32]–[33].

2.1. In [24] Kołodziej introduced the capacity \( C_X \) on \( X \) by
\[
C_X(E) := \sup \left\{ \int_E \omega^n_\varphi : \varphi \in \text{PSH}(X,\omega), -1 \leq \varphi \leq 0 \right\}
\]
for all Borel sets \( E \subset X \).

2.2. In [19] Guedj and Zeriahi introduced the Alexander capacity \( T_X \) on \( X \) by
\[
T_X(E) = e^{-\sup_x V^*_E, X}
\]
for all Borel sets \( E \subset X \). Here \( V^*_E, X \) is the global extremal \( \omega \)-psh function for \( E \) defined as the smallest upper semicontinuous majorant of \( V^*_{E, X} \) i.e,
\[
V_{E, X}(z) = \sup \left\{ \varphi(z) : \varphi \in \text{PSH}(X,\omega), \varphi \leq 0 \text{ on } E \right\}.
\]
2.3. The following definition was introduced in [18]: A probability measure $\mu$ on $X$ is said to satisfy the condition $\mathcal{H}(\alpha, A)$ ($\alpha, A > 0$) if

$$\mu(K) \leq AC_X(K)^{1+\alpha},$$

for any Borel subset $K$ of $X$.

A probability measure $\mu$ on $X$ is said to satisfy the condition $\mathcal{H}(\infty)$ if for any $\alpha > 0$ there exist $A(\alpha) > 0$ dependent on $\alpha$ such that

$$\mu(K) \leq A(\alpha)C_X(K)^{1+\alpha},$$

for any Borel subset $K$ of $X$.

2.4. The following definition was introduced in [17]: A measure $\mu$ is said to be moderate if for any open set $U \subset X$, any compact set $K \subset U$ and any compact family $\mathcal{F}$ of plurisubharmonic functions on $U$, there are constants $\alpha > 0$ such that

$$\sup \left\{ \int_K e^{-\alpha \varphi} d\mu : \varphi \in \mathcal{F} \right\} < +\infty.$$

2.5. The following class of $\omega$-psh functions was investigated by Guedj and Zeriahi in [20]:

$$\mathcal{E}(X, \omega) = \left\{ \varphi \in \text{PSH}(X, \omega) : \lim_{j \to \infty} \int_{\{\varphi > -j\}} \omega^n_{\text{max}(\varphi, -j)} = \int_X \omega^n = 1 \right\}.$$

Let us also define

$$\mathcal{E}^{-}(X, \omega) = \mathcal{E}(X, \omega) \cap \text{PSH}^{-}(X, \omega).$$

We refer to [20] for the properties of the class $\mathcal{E}(X, \omega)$.

2.6. $S$ is called a $C^1$ smooth real hypersurface in $X$ if for all $z \in X$ there exists a neighborhood $U$ of $z$ and $\chi \in C^1(U)$ such that $S \cap U = \{ z \in U : \chi(z) = 0 \}$ and $D\chi(z) \neq 0$ for all $z \in S \cap U$.

Next we state a well-known result needed for our work.

2.7. **Proposition.** — Let $\mu$ be a non-negative Radon measure on $X$ such that $\mu(B(z, r)) \leq Ar^{2n-2+\alpha}$ for all $B(z, r) \subset X$ ($A, \alpha > 0$ are constants). Then $\mu \in \mathcal{H}(\infty)$.

**Proof.** — By Theorem 7.2 in [33] and Proposition 7.1 in [19] we can find $\epsilon, C > 0$ such that

$$\mu(K) \leq Ah^{2n-2+\alpha}(K) \leq \frac{AC}{\alpha} T_X(K)^{\epsilon \alpha} \leq \frac{ACe}{\alpha} e^{-\frac{\epsilon \alpha}{C_X(K)^{\frac{1}{\alpha}}}}.$$
for all Borel subsets $K$ of $X$, where $h^{2n-2+\alpha}$ is the Hausdorff content of dimension $2n-2+\alpha$. This implies that $\mu \in H(\infty)$. □

3. Stability of the solutions

The stability estimate of solutions to the Monge-Ampère equations on compact Kähler manifolds was obtained by Kołodziej ([24]). Recently, in [12] S. Dinew and Z. Zhang proved a stronger version of this estimate. We will show a generalization of the stability theorem by S. Kołodziej. As a first step we have the following proposition. This proof follows ideas of the proof of Theorem 2.5 in [11]. We include a proof for the reader’s convenience.

3.1. Proposition. — Let $\varphi, \psi \in E^-(X, \omega)$ be such that $\omega^n_\varphi \in H(\alpha, A)$. Then there exist constants $t \in \mathbb{R}$ and $C(\alpha, A) \geq 0$ such that

$$\int \{\{\varphi - \psi - t\} \geq a\} (\omega^n_\varphi + \omega^n_\psi) \leq C(\alpha, A)a^{n+1},$$

here $a = \left[\int_X \|\omega^n_\varphi - \omega^n_\psi\|\right]^{\frac{2n+3}{n+1}}$.

Proof. — Since $\int \{\varphi - \psi - t\} \geq a\} (\omega^n_\varphi + \omega^n_\psi) \leq 2$, it suffices to consider the case when $a$ is small. Set $\epsilon = \frac{1}{2} \inf \left\{\int \{\varphi - \psi - t\} \omega^n_\varphi : t \in \mathbb{R}\right\}$. Hence

$$\int \{\varphi - \psi - t\} \leq 1 - 2\epsilon$$

for all $t \in \mathbb{R}$. Set

$$t_0 = \sup\left\{t \in \mathbb{R} : \int \{\varphi, \psi - t\} \omega^n_\varphi \leq 1 - \epsilon\right\}$$

Replacing $\psi$ by $\psi + t_0$ we can assume that $t_0 = 0$. Then $\int \{\varphi \leq \psi + a\} \omega^n_\varphi \leq 1 - \epsilon$ and $\int \{\varphi - a \leq \psi\} \omega^n_\varphi \geq 1 - \epsilon$. Hence

$$\int \{\varphi - a \leq \psi\} \omega^n_\varphi \leq 1 - \int \{\varphi + a \leq \psi\} \omega^n_\varphi = 1 - \int \{\varphi \leq \psi\} \omega^n_\varphi + \int \{a < \varphi \leq \psi\} \omega^n_\varphi \leq 1 - \epsilon.$$

Since $\int \{\varphi \leq a\} \omega^n_\varphi \leq 1$ we can choose $s \in [-a + a^{n+2}, a - a^{n+2}]$ satisfying

$$\int \{\varphi - s \leq a^{n+2}\} \omega^n_\varphi \leq 2a^{n+1}.$$
Replacing $\psi$ by $\psi + s$ we can assume that $s = 0$. One easily obtains the following inequalities

\[(1)\quad \int_{\{\phi < \psi + a^{n+2}\}} \omega^n_{\phi} \leq 1 - \epsilon, \quad \int_{\{\psi < \phi + a^{n+2}\}} \omega^n_{\phi} \leq 1 - \epsilon, \quad \int_{\{|\phi - \psi| < a^{n+2}\}} \omega^n_{\phi} \leq 2a^{n+1}.
\]

By \cite{20} we can find $\rho \in \mathcal{E}(X, \omega)$, such that

\[(2)\quad \omega^n_{\phi} = \frac{1}{1 - \epsilon} \omega^n_{\phi} + c 1_{\{\phi < \psi\}} \omega^n_{\phi} + \sup_X \rho = 0,
\]

($c \geq 0$ is chosen so that the measure has total mass 1). For simplicity of notation we set $\beta = \frac{n+1}{1+\alpha}$. Set

\[U = \{(1 - a^{n+2+\beta})\phi < (1 - a^{n+2+\beta})\psi + a^{n+2+\beta}\rho \} \subset \{\phi < \psi\}.
\]

From Theorem 2.1 in \cite{15} and (2) we get

\[(3)\quad \omega^{n-1}_\phi \wedge \omega_{(1-a^{n+2+\beta})\psi+a^{n+2+\beta}\rho} \geq (1 - a^{n+2+\beta})\omega^{n-1}_\phi \wedge \omega_{\psi} + \frac{a^{n+2+\beta}}{(1-\epsilon)\frac{1}{\pi}} \omega^n_{\phi},
\]

on $U$. From Theorem 2.3 in \cite{15}, Lemma 2.6 in \cite{11} and (3) we obtain

\[
(1 - a^{n+2+\beta}) \int_U \omega^{n-1}_\phi \wedge \omega_{\psi} + \frac{a^{n+2+\beta}}{(1-\epsilon)\frac{1}{\pi}} \int_U \omega^n_{\phi} \\
\leq \int_U \omega_{(1-a^{n+2+\beta})\psi+a^{n+2+\beta}\rho} \wedge \omega^{n-1}_{\phi} \\
\leq \int_U \omega_{(1-a^{n+2+\beta})\phi} \wedge \omega^{n-1}_{\phi} \\
= (1 - a^{n+2+\beta}) \int_U \omega^n_{\phi} + a^{n+2+\beta} \int_U \omega \wedge \omega^{n-1}_{\phi} \\
\leq (1 - a^{n+2+\beta}) \left( \int_U \omega^{n-1}_\phi \wedge \omega_{\psi} + 2a^{2n+3+\beta} \right) + a^{n+2+\beta} \int_U \omega \wedge \omega^{n-1}_{\phi}.
\]

Hence

\[(4)\quad \frac{1}{(1-\epsilon)\frac{1}{\pi}} \int_U \omega^n_{\phi} \leq 2a^{n+1} + \int_U \omega \wedge \omega^{n-1}_{\phi}.
\]

From Proposition 3.6 in \cite{19} and (4) we get
\[ (5) \quad \frac{1}{(1-\epsilon)\pi} \left[ \int_{\{ \phi \leq -a^{n+2} \}} \omega_{\phi}^{a} - C_1(\alpha, A)a^{n+1} \right] \leq \frac{1}{(1-\epsilon)\pi} \left[ \int_{\{ \phi \leq -a^{n+2} \}} \omega_{\phi}^{a} - A[C_{X}(\{ \rho \leq -\frac{1}{2a}\})]^{1+\alpha} \right] \]
\[ \leq \frac{1}{(1-\epsilon)\pi} \int_{U} \omega_{\phi}^{a} \]
\[ \leq 2a^{n+1} + \int_{U} \omega \wedge \omega_{\phi}^{a-1} \]
\[ \leq 2a^{n+1} + \int_{\{ \phi < \psi \}} \omega \wedge \omega_{\phi}^{a-1}, \]

Similarly to \( \rho \) we define \( \vartheta \in E(X, \omega) \), such that
\[ \omega_{\vartheta}^{a} = \frac{1}{1-\epsilon}1_{\{ \phi < \psi \}} \omega_{\phi}^{a} + l1_{\{ \psi \leq \phi \}} \omega_{\psi}^{a} \text{ and } \sup_{X} \vartheta = 0, \]

(\( l \) plays the same role as \( c \) above). Set
\[ V = \{ (1-a^{n+2+\beta})\psi < (1-a^{n+2+\beta})\phi + a^{n+2+\beta} \phi \} \subset \{ \psi < \phi \}. \]

We get
\[ (6) \quad \frac{1}{(1-\epsilon)\pi} \left[ \int_{\{ \psi \leq -a^{n+2} \}} \omega_{\phi}^{a} - C_1(\alpha, A)a^{n+1} \right] \leq 2a^{n+1} + \int_{\{ \psi < \phi \}} \omega \wedge \omega_{\phi}^{a-1}. \]

From (1), (5) and (6) we obtain
\[ \frac{1}{(1-\epsilon)\pi} \left[ 1 - 2a^{n+1} - 2C_1(\alpha, A)a^{n+1} \right] \]
\[ \leq \frac{1}{(1-\epsilon)\pi} \left[ \int_{\{ |\phi - \psi| > a^{n+1} \}} \omega_{\phi}^{a} - 2C_1(\alpha, A)a^{1+\alpha} \right] \]
\[ \leq 4a^{n+1} + 1. \]

Hence
\[ \epsilon \leq 1 - \left[ \frac{1 - 2(C_1(\alpha, A) + 1)a^{n+1}}{4a^{n+1} + 1} \right]^{n} \leq C_2(\alpha, A)a^{n+1}. \]

This implies that there exists \( t \in \mathbb{R} \) satisfying
\[ \int_{\{ |\phi - \psi - t| > a \}} \omega_{\phi}^{a} \leq 2C_2(\alpha, A)a^{n+1}. \]
Finally we have
\[
\int_{\{\|\varphi - \psi - t\| > a\}} (\omega^n_\varphi + \omega^n_\psi) = 2 \int_{\{\|\varphi - \psi - t\| > a\}} \omega^n_\varphi + \int_{\{\|\varphi - \psi - t\| > a\}} (\omega^n_\psi - \omega^n_\varphi) \leq 2C_2(\alpha, A)a^{n+1} + a^{2n+3+\beta} \leq C(\alpha, A)a^{n+1}.
\]

The second step in proving our stability theorem is the following

**3.2. Proposition.** — Let \(\varphi, \psi \in \mathcal{E}^-(X, \omega)\) be such that \(\omega^n_\varphi, \omega^n_\psi \in \mathcal{H}(\alpha, A)\). Then there exist constants \(t \in \mathbb{R}\) and \(C(\alpha, A) \geq 0\) such that
\[
C_X(\{\|\varphi - \psi - t\| > a\}) \leq C(\alpha, A)a,
\]
here \(a = [\int_X \|\omega^n_\varphi - \omega^n_\psi\|]^{\frac{1}{2n+3+\beta}}\).

**Proof.** — Since \(C_X(\{\|\varphi - \psi - t\| > a\}) \leq C_X(X) = 1\), it suffices to consider the case when \(a\) is small. Without loss of generality we can assume that \(\sup_X \varphi = \sup_X \psi = 0\). By Remark 2.5 in [18] there exists \(M(\alpha, A) > 0\) such that \(\|\varphi\|_{L^\infty(X)} < M(\alpha, A), \|\psi\|_{L^\infty(X)} < M(\alpha, A)\). By Proposition 3.1 we can find \(t > 0\) such that
\[
\int_{\{\|\varphi - \psi - t\| > a\}} (\omega^n_\varphi + \omega^n_\psi) \leq C_1(\alpha, A)a^{n+1}.
\]

We consider the case \(a < \min(1, \frac{1}{C_1(\alpha, A)})\). Since \(\int_{\{\|\varphi - \psi - t\| > a\}} (\omega^n_\varphi + \omega^n_\psi) < 1\) we get \(\{\|\varphi - \psi - t\| > a\} \neq X\). This implies that \(|t| \leq \sup_X |\varphi - \psi| + 1 \leq M(\alpha, A) + 1\). Replacing \(\psi\) by \(\psi + t\) we can assume that \(t = 0\) and \(\|\psi\|_{L^\infty(X)} < 2M(\alpha, A) + 1\). Using Lemma 2.3 in [18] for \(s = \frac{a}{2}, t = \frac{a}{2(2M(\alpha, A) + 1)}\) we get
\[
C_X(\{\varphi - \psi < -a\}) \leq C_X\left(\left\{\varphi - \psi < -\frac{a}{2} - \frac{a}{2(2M(\alpha, A) + 1)}\right\}\right) \leq \frac{2^n(2M(\alpha, A) + 1)^n}{a^n} \int_{\{\|\varphi - \psi < -a\}} \omega^n_\varphi \leq 2^n(2M(\alpha, A) + 1)^nC_1(\alpha, A)a.
\]

Similarly we get
\[
C_X(\{\psi - \varphi < -a\}) \leq 2^n(2M(\alpha, A) + 1)^nC_1(\alpha, A)a.
\]

Combination of these inequalities yields
\[
C_X(\{\|\varphi - \psi| > a\}) \leq C(\alpha, A)a.
\]

Now we prove the promised generalization of Kołodziej stability theorem (Theorem 1.1 in [27]).
3.3. Theorem. — Let \( \varphi, \psi \in \mathcal{E}^-(X, \omega) \) be such that \( \sup_X \varphi = \sup_X \psi = 0 \) and \( \omega^n_\varphi, \omega^n_\psi \in \mathcal{H}(\alpha, A) \). Then there exists \( C(\alpha, A) > 0 \) such that

\[
\sup_X |\varphi - \psi| \leq C(\alpha, A) \left[ \int_X \|\omega^n_\varphi - \omega^n_\psi\| \right]^{\frac{\min(1, \frac{\alpha}{n})}{2n+3+\frac{4n+3+1}{\alpha+1}}}.
\]

Proof. — Set

\[
a = \left[ \int_X \|\omega^n_\varphi - \omega^n_\psi\| \right]^{\frac{1}{2n+3+\frac{4n+3+1}{\alpha+1}}}.
\]

By Proposition 3.2 there exists \( C_1(\alpha, A) > 0 \) and \( t \in \mathbb{R} \) such that \( |t| \leq M(\alpha, A) + 1 \) and

\[
C_X(\{|\varphi - \psi - t| > a\}) \leq C_1(\alpha, A)a.
\]

Moreover, by Proposition 2.6 in [18] there exists \( C_2(\alpha, A) > 0 \) such that

\[
\sup_X |\varphi - \psi - t| \leq 2a + C_2(\alpha, A)[C_X(\{|\varphi - \psi - t| > a\})]^{\frac{2}{3}}
\leq 2a + C_2(\alpha, A)[C_1(\alpha, A)a]^{\frac{2}{3}}
\leq C_3(\alpha, A)a^{\min(1, \frac{\alpha}{n})}.
\]

Moreover, since \( \sup_X \varphi = \sup_X \psi = 0 \) we obtain \( |t| \leq C_3(\alpha, A)a^{\min(1, \frac{\alpha}{n})} \).

Combination of these inequalities yields

\[
\sup_X |\varphi - \psi| \leq \sup_X |\varphi - \psi - t| + |t| \leq 2C_3(\alpha, A)a^{\min(1, \frac{\alpha}{n})}
= C(\alpha, A) \left[ \int_X \|\omega^n_\varphi - \omega^n_\psi\| \right]^{\frac{\min(1, \frac{\alpha}{n})}{2n+3+\frac{4n+3+1}{\alpha+1}}}.
\]

\[ \square \]

3.4. Corollary. — Let \( \mu \) be a non-negative Radon measure on \( X \) such that \( \mu(B(z, r)) \leq Ar^{2n-2+\alpha} \) for all \( B(z, r) \subset X \) (\( A, \alpha > 0 \) are constants). Given \( p > 1, M > 0, \varepsilon > 0 \) and \( f, g \in L^p(d\mu) \) with \( \|f\|_{L^p(d\mu)}, \|g\|_{L^p(d\mu)} \leq M \) and \( \int_X f d\mu = \int_X g d\mu = 1 \). Assume that \( \varphi, \psi \in \mathcal{E}^-(X, \omega) \) satisfy \( \omega^n_\varphi = f d\mu, \omega^n_\psi = g d\mu \) and \( \sup_X \varphi = \sup_X \psi = 0 \). Then there exists \( C(\alpha, A, M, \varepsilon) > 0 \) such that

\[
\sup_X |\varphi - \psi| \leq C(\alpha, A, M, \varepsilon) \left[ \int_X |f - g| d\mu \right]^{\frac{1}{2n+3+\alpha}}.
\]

Proof. — By Hölder inequality we have

\[
\int_K f d\mu \leq \|f\|_{L^p(d\mu)} \mu(K)^{1 - \frac{1}{p}} \leq M[\mu(K)]^{1 - \frac{1}{p}},
\]

\[
\int_K g d\mu \leq \|g\|_{L^p(d\mu)} \mu(K)^{1 - \frac{1}{p}} \leq M[\mu(K)]^{1 - \frac{1}{p}},
\]

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for any Borel subset $K$ of $X$. By Proposition 2.7 we get $f \, d\mu, g \, d\mu \in \mathcal{H}(\infty)$. Using Theorem 3.3 we can find $C(\alpha, A, M, \varepsilon) > 0$ such that

$$
\|\phi - \psi\|_X \leq C(\alpha, A, M, \varepsilon) \left[ \int_X |f - g| \, d\mu \right]^{\frac{1}{2n+3+\varepsilon}}.
$$

□

4. Local estimates in Potential theory

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$). By $\text{SH}(\Omega)$ (resp. $\text{SH}^-(\Omega)$) we denote the set of subharmonic (resp. negative subharmonic) functions on $\Omega$. For each $u \in \text{SH}(\Omega)$ and $\delta > 0$ we denote

$$
\tilde{u}_\delta(x) = \frac{1}{c_n \delta^n} \int_{B_\delta} u(x + y) \, dV_n(y),
$$

$$
u_\delta(x) = \sup_{y \in B_\delta} u(x + y),
$$

for $x \in \Omega_\delta = \{ x \in \Omega : d(x, \partial \Omega) > \delta \}$. Here $B_\delta = \{ x \in \mathbb{R}^n : |x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}} < \delta \}$ and $c_n$ is the volume of the unit ball $B_1$. We state some results which will be used in our main theorems.

**4.1. Theorem.** — Let $\mu$ be a non-negative Radon measure on $\Omega$ such that $\mu(B(z, r)) \leq Ar^{n-2+\alpha}$ for all $B(z, r) \subset D \subset \subset \Omega$ ($A, \alpha > 0$ are constants). Then for $K \subset \subset D$ and $\varepsilon > 0$ there exists $C(\alpha, A, K, \epsilon)$ such that

$$
\int_K [\tilde{u}_\delta - u] \, d\mu \leq C(\alpha, A, K, \epsilon) \int_D \Delta u \, \delta^{\frac{n+\varepsilon}{n+2+\varepsilon}},
$$

for all $u \in \text{SH}(\Omega)$, where $\Delta$ is the Laplace operator.

**Proof.** — Since the change of radii of the balls does not affect the statement we can assume that $\Omega = B_4$, $D = B_3$, $K = B_1$ and $u$ is smooth on $B_4$. By [22] we have

$$
u(x) = \int_{B_2} G(x, z) \Delta u(z) + h(x),$$

where $G(x, y)$ is the fundamental solution of Laplace equation and $h$ is harmonic in $B_2$. By Fubini theorem we have

$$
\int_{B_1} [\tilde{u}_\delta(x) - u(x)] \, d\mu(x)
$$

$$
= \int_{B_1} \frac{1}{c_n \delta^n} \int_{B_\delta} [u(x + y) - u(x)] \, dV_n(y) \, d\mu(x)
$$

$$
\cdot \frac{1}{c_n \delta^n} \int_{B_1} \int_{B_3} \int_{B_2} [G(x + y, z) - G(x, z)] \Delta u(z) \, dV_n(y) \, d\mu(x)
$$
\[
\begin{align*}
F(y, z) &= \int_{B_1} [G(x + y, z) - G(x, z)]d\mu(x).
\end{align*}
\]

It is enough to prove that \( F(y, z) \leq C(\alpha, A, s)\delta^{\frac{\alpha-s}{1+s}} \) for all \( y \in B_\delta, z \in B_2 \).

We consider two cases:

**Case 1: \( n = 2 \).** For \( y \in B_\delta, z \in B_2, \delta < \frac{1}{2} \), we have

\[
F(y, z) = \int_{B_1} [\ln|x + y - z| - \ln|x - z|]d\mu(x)
\]

\[
= \int_{B_1 \cap \{|x-z|\geq |y|^{\frac{1}{1+\alpha}}\}} \ln|1 + \frac{y}{x - z}|d\mu(x) + \int_{B_1 \cap \{|x-z|<|y|^{\frac{1}{1+\alpha}}\}} \ln|1 + \frac{y}{x - z}|d\mu(x)
\]

\[
\leq \int_{B_1 \cap \{|x-z|\geq |y|^{\frac{1}{1+\alpha}}\}} \ln(1 + |y|^{\frac{\alpha}{1+\alpha}})d\mu(x) + \ln 4 \int_{B_1 \cap \{|x-z|<|y|^{\frac{1}{1+\alpha}}\}} d\mu + \int_{B_1 \cap \{|x-z|<|y|^{\frac{1}{1+\alpha}}\}} \ln \frac{1}{|x - z|}d\mu(x)
\]

\[
\leq |y|^{\frac{\alpha}{1+\alpha}} \mu(B_1) + A|y|^{\frac{\alpha}{1+\alpha}} \ln 4
\]

\[
+ |y|^{\frac{\alpha}{1+\alpha}} \int_{B_1 \cap \{|x-z|<|y|^{\frac{1}{1+\alpha}}\}} \frac{1}{|x - z|^{\alpha - \epsilon}} \ln \frac{1}{|x - z|}d\mu(x)
\]

\[
\leq A(1 + \ln 4)|y|^{\frac{\alpha}{1+\alpha}} + |y|^{\frac{\alpha}{1+\alpha}} C_1(\alpha, \epsilon) \int_{\{|x-z|<1\}} \frac{d\mu(x)}{|x - z|^{\alpha - \frac{\epsilon}{2}}}
\]

\[
\leq A(1 + \ln 4)|y|^{\frac{\alpha}{1+\alpha}} + C_1(\alpha, \epsilon)|y|^{\frac{\alpha}{1+\alpha}} \sum_{j=0}^{\infty} \int_{\{2^{-j-1} \leq |x-z| < 2^{-j}\}} \frac{d\mu(x)}{|x - z|^{\alpha - \frac{\epsilon}{2}}}
\]

\[
\leq A(1 + \ln 4)|y|^{\frac{\alpha}{1+\alpha}} + C_1(\alpha, \epsilon)|y|^{\frac{\alpha}{1+\alpha}} A \sum_{j=0}^{\infty} 2^{-(j+1)(\alpha - \frac{\epsilon}{2}) - j \alpha}
\]

\[
\leq C(\alpha, A, \epsilon)|y|^{\frac{\alpha}{1+\alpha}} \leq C(\alpha, A, \epsilon)\delta^{\frac{\alpha-s}{1+s}}.
\]

**Case 2: \( n \geq 3 \).** Similarly for \( y \in B_\delta, z \in B_2, \delta < \frac{1}{2} \), we have

\[
F(y, z) = \int_{B_1} \left[-\frac{1}{|x + y - z|^{n-2}} + \frac{1}{|x - z|^{n-2}}\right]d\mu(x)
\]
\[ \int_{B_1 \cap \{|x-z| \geq |y|^{\frac{1}{n-\alpha}}\}} \frac{|x+y-z|^{n-2} - |x-z|^{n-2}}{|x+y-z|^{n-2}} d\mu(x) + \int_{\{|x-z| < |y|^{\frac{1}{n-\alpha}}\}} \frac{d\mu(x)}{|x-z|^{n-2}} \]

\[ \leq C_2(\alpha) |y|^{\frac{n-\alpha}{1+n}} \int_{B_1 \cap \{|x-z| \geq |y|^{\frac{1}{n-\alpha}}\}} d\mu(x) + |y|^{\frac{n-\alpha}{1+n}} \int_{\{|x-z| < |y|^{\frac{1}{n-\alpha}}\}} \frac{d\mu(x)}{|x-z|^{n-2+\alpha-\epsilon}} \]

\[ \leq AC_2(\alpha) |y|^{\frac{n-\alpha}{1+n}} + |y|^{\frac{n-\alpha}{1+n}} \int_{\{|x-z| < 1\}} \frac{d\mu(x)}{|x-z|^{n-2+\alpha-\epsilon}} \]

\[ \leq C(\alpha, A, \epsilon) |y|^{\frac{n-\alpha}{1+n}} \]

\[ \leq C(\alpha, A, \epsilon) \delta^{\frac{\alpha-\epsilon}{2(n-\alpha)}}. \]

\[ \square \]

**4.2. Theorem.** Let \( \mu \) be a non-negative Radon measure on \( \Omega \) such that \( \mu(B(z,r)) \leq Ar^{n-2+\alpha} \) for all \( B(z,r) \subset D \subset \subset \Omega \) \( A, \alpha > 0 \) are constants. Then for \( K \subset \subset D \) and \( \epsilon > 0 \) there exists \( C(\alpha, A, K, \epsilon) \) such that

\[ \int_K [u_\delta - u] d\mu \leq C(\alpha, A, K, \epsilon) \|u\|_{L^\infty(\Omega)} \delta^{\frac{\alpha-\epsilon}{2(n-\alpha)}}, \]

for all \( u \in SH \cap L^\infty(\Omega) \).

We need a well-known lemma:

**4.3. Lemma.** Let \( u \in SH \cap L^\infty(\Omega) \). Then

\[ |\tilde{u}_\delta(x) - \tilde{u}_\delta(y)| \leq \frac{\|u\|_{L^\infty(\Omega)} |x-y|}{\delta}, \]

for all \( x, y \in \Omega_\delta \).

**Proof.** Proof of Theorem 4.2 By Lemma 4.3 we have

\[ u_\delta(x) = \sup_{y \in B_\delta} u(x+y) \leq \sup_{y \in B_\delta} \tilde{u}_\delta^{\frac{1}{2}}(x+y) \leq \tilde{u}_\delta^{\frac{1}{2}}(x) + \delta^{\frac{1}{2}} \|u\|_{L^\infty(\Omega)}. \]

By Theorem 4.1 we get

\[ \int_K [u_\delta - u] d\mu \leq \int_K [\tilde{u}_\delta^{\frac{1}{2}} - u] d\mu + \|u\|_{L^\infty(\Omega)} \mu(K) \delta^{\frac{1}{2}} \]

\[ \leq C(\alpha, A, K, \epsilon) \|u\|_{L^\infty(\Omega)} \delta^{\frac{\alpha-\epsilon}{2(n-\alpha)}}. \]

Next we state a well-known result is a direct consequence of the Jensen formula (see [1]) \( \square \)
4.4. Proposition. — Let $u \in SH(B_2)$ be such that $|u(x) - u(y)| \leq A|x - y|^{\alpha}$ for all $x, y \in B_2$. Then there exists $C(\alpha, A) > 0$ such that

$$\int_{B(x, r)} \Delta u \leq C(\alpha, A)r^{n-2+\alpha},$$

for all $B(x, r) \subset B_1$.

5. Main results

Proof of Theorem A. — We use the same scheme as the proof of Theorem 2.1 in [27]. From Corollary 3.4 and from Theorem 4.2 we can replace $\omega^n$ by $d\mu$. This implies that $u$ is Hölder continuous with the Hölder exponent dependent on $\alpha, A, p, X$ and $\|f\|_{L^p(d\mu)}$.

Proof of Corollary B. — It follows from Proposition 4.4 and Theorem A.

Proof of Corollary C. — Direct application of Theorem A.

BIBLIOGRAPHY


