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# THE GEOMETRIC COMPLEX FOR ALGEBRAIC CURVES WITH CONE-LIKE SINGULARITIES AND ADMISSIBLE MORSE FUNCTIONS

**by Ursula LUDWIG**

Abstract. — In a previous note the author gave a generalisation of Witten's proof of the Morse inequalities to the model of a complex singular curve $X$ and a stratified Morse function $f$. In this note a geometric interpretation of the complex of eigenforms of the Witten Laplacian corresponding to small eigenvalues is provided in terms of an appropriate subcomplex of the complex of unstable cells of critical points of $f$.

Résumé. — Dans une note précédente, l'auteur a donné une généralisation de la preuve de Witten des inégalités de Morse pour le cas modèle d'une courbe algébrique complexe singulière et d'une fonction de Morse stratifiée. Le but de cette note est de donner une interprétation géométrique du complexe des formes propres du Laplacien de Witten pour des petites valeurs propres à l'aide d'un sous-complexe approprié du complexe des cellules instables.

## 1. Introduction

Let $M$ be a smooth compact manifold of dimension $\dim(M) = n$. Let $f : M \to \mathbb{R}$ be a Morse function on $M$, i.e. a function such that for each critical point $p$ ($df(p) = 0$) the Hessian $\mathrm{Hess}_p(f)$ of $f$ in $p$ is non degenerate (as a symmetric bilinear form on $T_pM$). The number of negative eigenvalues of the Hessian $\mathrm{Hess}_p(f)$ is called the index of $f$ in $p$. We denote by $\mathrm{Crit}_i(f)$ the set of critical points of $f$ of index $i$ and by $c_i(f) := \#\mathrm{Crit}_i(f)$. The celebrated Morse inequalities state that there is a relation between the number of critical points of $f$ and the Betti numbers of $M$.

A way to prove the Morse inequalities is to show the existence of a complex $(C_*, \partial_*)$ of vector spaces such that $\dim C_i = c_i(f)$ and such that

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the homology of the complex is isomorphic to the singular homology of $M$. The Morse inequalities follow from the existence of such a complex by a simple algebraic argument. The existence of a complex with the above properties has been shown by geometrical methods by Thom and Smale: The chain groups of the Thom-Smale complex are generated by the critical points of $f$, the boundary operator is defined by "counting trajectories" of the negative gradient flow (for a generic metric) between critical points of index difference 1.

In [16] Witten proposed a different, purely analytical proof of the Morse inequalities. A rigorous account of the analytic proof of the Morse inequalities using semi-classical analysis has been done in [8]. The main idea of Witten's method consists in deforming the de Rham complex $(\Omega^*(M), d)$ by means of the Morse function $f$ into a complex $(\Omega^*(M), d_t)$, where $d_t = e^{-tf} d e^{tf}$ and $t \in (0, \infty)$ denotes the deformation parameter. The map $\omega \mapsto e^{tf}\omega$ induces an isomorphism of the two complexes and therefore

$$H^*\big((\Omega^*(M), d_t)\big) \simeq H^*\big((\Omega^*(M), d)\big) \simeq H^*(M). \tag{1.1}$$

The last isomorphism in (1.1) is just the well-known de Rham isomorphism. Let us denote by $\delta_t := e^{tf}\delta e^{-tf}$ the adjoint of $d_t$ and by

$$\Delta_t = d_t\delta_t + \delta_t d_t$$

the Witten Laplacian. The Hodge theorem for the deformed complex $(\Omega^*(M), d_t)$ states that

$$\ker(\Delta_t) \simeq H^*\big((\Omega^*(M), d_t)\big).$$

The advantage of the deformed complex compared to the initial de Rham complex is that the spectral properties of the Witten Laplacian are "nice". In particular one can show that for large deformation parameter $t$ there is a "gap" in the spectrum of the Witten Laplacian, i.e. $\mathrm{spec}(\Delta_t) \cap (e^{-ct}, Ct) = \emptyset$ for some $c, C > 0$. Moreover, for $0 \leqslant i \leqslant \dim(M)$, the number of eigenvalues (counted with multiplicities) of $\Delta_{t|\Omega^i(M)}$ contained in the interval $[0,1]$ is equal to $c_i(f)$. We denote by $\mathbb{F}_t^i \subset \Omega^i(M)$ the $c_i(f)$-dimensional vector space generated by the eigenspaces of $\Delta_{t|\Omega^i(M)}$ corresponding to eigenvalues in $[0,1]$. One thus gets a finite dimensional subcomplex $(\mathbb{F}_t^*, d_t)$ of $(\Omega^*, d_t)$, with

$$H^*\big((\mathbb{F}_t^*, d_t)\big) \simeq \ker(\Delta_t) \simeq H^*(M)$$

and as indicated above the Morse inequalities (for cohomology) follow.

Witten further suggested in [16] that under some genericity conditions from the complex $(\mathbb{F}_t^*, d_t)$ one can recover the Thom-Smale complex associated to the Morse function $f$ by letting $t \to \infty$. Again a rigorous proof based on semi-classical analysis can be found in [8]. In [1] Bismut and Zhang gave another proof of this "comparison theorem" using a result of Laudenbach in [9] describing the geometry of the boundary of the unstable cells of the singular points of $f$. (The result is used in the sequel to give an extension of a theorem of Cheeger and Müller on the relation between the Ray-Singer analytic torsion and the Reidemeister torsion.)

In [10] (see also [12]) a generalisation of Witten's proof of the Morse inequalities to the model of a singular complex algebraic curve and stratified Morse functions (in the sense of the theory developed by Goresky and MacPherson in [7]) is given. The model functions considered in [12] were called admissible Morse functions. One can assume that all singularities $p \in \Sigma := \mathrm{Sing}(X)$ are unibranched. For $p \in \Sigma$ we denote by $m(p)$ the multiplicity of $X$ at $p$. In this situation the Witten method consists in deforming the complex $(\mathcal{C}, d)$ of $\mathrm{L}^2$-integrable forms (instead of the de Rham complex) by means of an admissible Morse functions $f$. One can then show that also in this situation the "spectral gap" theorem for the Witten Laplacian holds and the vector space $\mathbb{F}_t^i$ of eigenforms of the Witten Laplacian to small eigenvalues has dimension

$$\dim \mathbb{F}_t^i = c_i(f) := \begin{cases} c_i(f_{|X\setminus\Sigma}) & i = 0, 2, \\ c_1(f_{|X\setminus\Sigma}) + \sum_{p\in\Sigma}(m(p)-1) & i = 1. \end{cases}$$

The below Morse inequalities for the $\mathrm{L}^2$-Betti numbers $b_i^{(2)}(X)$ of $X$ now follow by a simple algebraic argument:

$$\begin{aligned} &\sum_{i=0}^{k}(-1)^{k-i}c_i(f) \geqslant \sum_{i=0}^{k}(-1)^{k-i}b_i^{(2)}(X) \text{ for } k = 0, 1, \\ &\sum_{i=0}^{2}(-1)^{i}c_i(f) = \sum_{i=0}^{2}(-1)^{i}b_i^{(2)}(X). \end{aligned} \tag{1.2}$$

Since the situation treated in [10] is a model for a singular algebraic curve and certain stratified Morse functions on it as explained in [10], from (1.2) one gets back the Morse inequalities for intersection homology of middle perversity which were already known by [7].

The goal of this note is to generalise the second part of Witten's program to the singular situation described above, i.e. to provide a geometric interpretation of the complex $(\mathbb{F}_t^*, d_t)$.

First one has to investigate the structure of the unstable set of points in $\mathrm{Crit}(f) := \mathrm{Crit}(f_{|X\setminus\Sigma}) \cup \Sigma$. Using the structure of the boundary of the unstable sets one can then construct a subcomplex $(C_*^{u'}, \partial_*)$ of the complex of unstable cells as well as an appropriate basis

$$\{e_1^p, p \in \mathrm{Crit}(f) \setminus \Sigma\} \cup \{e_i^p, p \in \Sigma, i = 1, \dots, m(p) - 1\}.$$

The main result of this article is a comparison theorem between the combinatorial complex $(C_*^{u'}, \partial_*)$ and the complex $(\mathbb{F}_t^*, d_t)$. Let us denote by $\{\Psi_1^p(t), p \in \mathrm{Crit}(f) \setminus \Sigma\} \cup \{\Psi_i^p(t), p \in \Sigma, i = 1, \dots, m-1\}$ the basis of $\mathbb{F}_t$ constructed in [10] (and recalled in section 2.2). The map

$$R(t) : \mathrm{Hom}\big((C_*^{u'}, \partial_*), \mathbb{R}\big) \to (\widetilde{\mathbb{F}_t}^*, d), \quad [e_i^p]^* \mapsto e^{tf}\Psi_i^p(t), \tag{1.3}$$

is an isomorphism into a subcomplex $(\widetilde{\mathbb{F}}_t, d)$ of the complex of $\mathrm{L}^2$-integrable forms. One can show that integration yields a well-defined morphism of complexes

$$P_{\infty,t} : \quad (\widetilde{\mathbb{F}}_t^*, d) \quad \longrightarrow \quad \mathrm{Hom}\big((C_*^{u'}, \partial_*), \mathbb{R}\big). \tag{1.4}$$

We are now ready to state the two main results of the article. The two theorems below generalise Theorem 6.11 and Theorem 6.12 in [3] respectively to the singular situation. Denote by $\mathcal{F} \in \mathrm{End}(\mathrm{Hom}(C_i^{u'}, \mathbb{R}))$ the homomorphism which acts on $[e_j^p]^*$ by multiplication with $f(p)$. With $\mathcal{I} \in \mathrm{End}(\mathrm{Hom}(C_i^{u'}, \mathbb{R}))$ we denote multiplication by $i$.

Theorem 1.1. — *The asymptotic behaviour of $P_{\infty,t} \circ R(t)$ as $t \to \infty$ is*

$$P_{\infty,t} \circ R(t) = e^{t\mathcal{F}} \left(\frac{\pi}{t}\right)^{(\mathcal{I}-1)/2} (1 + O(e^{-ct})). \tag{1.5}$$

*In particular for large $t$ the linear map of vector spaces $P_{\infty,t}$ is an isomorphism.*

Theorem 1.2. — *There exists $c > 0$ such that for $t \to \infty$,*

$$R(t)^{-1} \circ d \circ R(t) = \sqrt{\frac{t}{\pi}}(1 + O(e^{-ct}))^{-1} e^{-t\mathcal{F}} \partial^* e^{t\mathcal{F}} (1 + O(e^{-ct})).$$

This note is organised as follows: In Section 2 basic facts on the $\mathrm{L}^2$-cohomology of a singular space having cone-like singularities are recalled. Also, for convenience of the reader the results in [10] are summarised. In particular the construction of the basis

$$\{\Psi_1^p(t), p \in \mathrm{Crit}(f) \setminus \Sigma\} \cup \{\Psi_i^p(t), p \in \Sigma, i = 1, \dots, m-1\}$$

of $\mathbb{F}_t$ is explained.

Section 3.1 describes the structure of the boundary of the unstable set for the critical points of the admissible Morse function, thus extending the result in [9] to the singular situation. In Section 3.2 we define the subcomplex $(C_*^{u'}, \partial_*)$ of the complex of unstable cells. One can moreover show that integration of $\mathrm{L}^2$-integrable forms on the cells in $(C_*^{u'}, \partial_*)$ is well-defined and Stokes formula holds.

In Section 4.1 the two comparison theorems Theorem 1.1 and Theorem 1.2 are proved by extending the proofs of the corresponding statements for smooth manifolds in [3]. Section 4.2 shortly explains the duality pairing for the geometric complexes of the stable and unstable cells and its compatibility with the Poincaré duality.

Some explicit computations are postponed to the appendix. The results contained in this paper were announced in [11].

This article together with [10] is a first attempt to make the Witten method accessible to singular spaces. Note however that the generalisation of both parts to higher dimensional spaces promises to be more involved, one of the problems thereby being that the natural metrics on algebraic varieties, i.e. those induced from a metric on projective space are in general not of cone-type.

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## 2. Witten deformation on a singular curve by means of an admissible Morse function

### 2.1. $\mathrm{L}^2$-cohomology and the Witten deformation

In this note we deal with the following situation: Let $(X, g)$ be the model of an algebraic curve with unibranched singularities, i.e. $(X, g)$ is a Riemannian space of $\dim X = 2$ with cone-like singularities $\Sigma := \{p_1, \ldots, p_N\}$ of multiplicities $m_i = m(p_i) \in \mathbb{N}$, $m_i \geqslant 2$. More precisely

- $X$ is a topological space, such that $X - \Sigma$ is a smooth manifold. $g$ is a Riemannian metric on $X - \Sigma$.
- There exist open neighbourhoods $U(p_i)$ of $p_i$ in $X$, $i \in \{1, \ldots, N\}$, such that $X - \bigcup_{i=1}^N U(p_i)$ is a smooth compact manifold with boundary.
- The open set $(U(p_i) - p_i, g_{|U_i - p_i})$ is isometric to
$$(\mathrm{cone}_\epsilon(S^1_{m_i}), dr^2 + r^2 d\varphi^2)$$
for some $\epsilon > 0$. Hereby for $m \in \mathbb{N}$ we denote by $S^1_m$ the circle of length $2\pi m$ and by $\mathrm{cone}_\epsilon(S^1_m) := \big\{(r, \varphi) \mid r \in (0, \epsilon), \varphi \in S^1_m\big\}$.

Definition 2.1. — *Let $f : X \to \mathbb{R}$ be a continuous function, which is smooth outside the singularities of $X$. The function $f$ is called an admissible Morse function if the following conditions are satisfied*

(1) *Each critical point $p \in X - \Sigma$ of $f$ is a non-degenerate critical point.*
(2) *Let $p \in \Sigma$ be a singular point of $X$. Then there exist $a_p, b_p \in \mathbb{R}, (a_p, b_p) \neq (0,0)$, such that the function $f$ has the following form in local coordinates $(r, \varphi)$ near $p$:*

$$f(r,\varphi) = f(p) + r(a_p \cos(\varphi) + b_p \sin(\varphi)).$$

*Remark 2.1.* — Note that after change of coordinates on the link $S^1_m$ and rescaling $t \rightsquigarrow t' = \sqrt{a_p^2 + b_p^2} \cdot t$ one can always assume that $(a_p, b_p) = (1,0)$ in the above definition.

We denote by $\mathrm{Crit}(f) := \{p \in X \setminus \Sigma \mid df(p) = 0\} \cup \Sigma$ the set of critical points of $f$. For $p \in X - \Sigma$ a (smooth) critical point of $f$ the index $\mathrm{ind}(p)$ of $f$ in $p$ is defined as the number of negative eigenvalues of the Hessian of $f$ in $p$. Each singular point $p \in \Sigma$ of $X$ is considered to be a critical point of $f$ of index $\mathrm{ind}(p) = 1$. For $i = 0,1,2$ we denote by $\mathrm{Crit}_i(f)$ the set of critical points of $f$ of index $i$.

A Riemannian singular space $X$ of $\dim X = 2$ as above is a metric model for a singular complex projective algebraic curve. An admissible Morse function on $X$ is a model for a stratified Morse function on a complex curve in the sense of the theory developed by Goresky/MacPherson in [7]. Let us explain this in more detail: Let $C \subset \mathbb{P}^n(\mathbb{C})$ be a complex projective algebraic curve. Let $p \in C$ be a singular point of $C$ and denote by $C_j$, $j = 1, \dots, s$, the analytic branches of $C$ at $p$. Then for each branch $C_j$ there exist open neighbourhoods $V_j \subset \mathbb{C}$ of 0 resp. $U(p) \subset \mathbb{P}^n(\mathbb{C})$ of $p$, as well as affine coordinates $z_1, \dots, z_n$ on $U(p)$ and a normalisation map defined by

$$\begin{aligned} \pi : V_j \subset \mathbb{C} &\to U(p) \cap C_j \\ t &\mapsto (z_1(t), \dots, z_n(t)) = (t^{m_j}, t^{q_{j2}} f_{j2}(t), \dots, t^{q_{jn}} f_{jn}(t)), \end{aligned}$$

such that $\pi_{|V_j - \{0\}}$ is a biholomorphic map. Hereby $m_j < q_{j2} < q_{j3} < \dots < q_{jn}$ and $f_{jk}(0) \neq 0$, for $k = 2, \dots, n$. The multiplicity $m_j$ of $C_j$ at $p$ is an analytic invariant, i.e. it does not depend on the choice of local coordinates $z_1, \dots, z_n$.

We denote by $\widetilde{g}$ the Riemannian metric on $C_j$ induced by the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$. Then the metric $\pi^* \widetilde{g}$ on $V_j - \{0\} \subset \mathbb{C}$ is isometric to the metric $(m_j^2 \mid t \mid^{2(m_j - 1)} + O(\mid t \mid^{2m_j - 1})) dt \otimes d\bar{t}$. Moreover the map

$$\Pi : (\mid t \mid, \arg(t)) \to (\mid t \mid^{m_j}, m_j \cdot \arg(t))$$

induces an isometry from $(V_j - \{0\}, \pi^*\widetilde{g})$ to

$$(\mathrm{cone}(S^1_{m_j}), (1 + O(r^{1/m_j}))(dr^2 + r^2 d\varphi^2))$$

(see e.g. [13]). Thus in particular $(C_j, \widetilde{g})$ is quasi-isometric to a cone-like singularity of multiplicity $m_j$.

The affine line $l := \{z_2 = \ldots = z_n = 0\}$ is the tangent line to the irreducible branch $C_j$. Let $F : \mathbb{P}^n(\mathbb{C}) \cap U(p) \to \mathbb{C}$ be a holomorphic function such that $f := \mathrm{Re}(F)_{|C} : C \cap U(p) \to \mathbb{R}$ is a stratified Morse function in the sense of [7] (Part II). The non-degeneracy condition in [7] (for the branch $C_j$) implies that locally near $p$ the function $F$ has the form

$$F = F(p) + \sum a_i z_i + O(z^2),$$

where $a_1 \neq 0$. One checks easily that

$$f \circ \pi \circ \Pi^{-1} :$$
$$\mathrm{cone}(S^1_m) \longrightarrow \mathbb{R},\ (r, \varphi) \mapsto r(\mathrm{Re}(a)\cos(\varphi) - \mathrm{Im}(a)\sin(\varphi)) + O(r^{1+\delta})$$

for some $\delta > 0$. The leading term is thus of the form given in Definition 2.1 (2).

Let us now recall the main features of the L$^2$-cohomology of $(X, g)$. Let $(\Omega^*_0(X - \Sigma), d)$ be the de Rham complex of differential forms acting on smooth forms with compact support. An *ideal boundary condition* for the elliptic complex $(\Omega^*_0(X - \Sigma), d)$ is a choice of closed extensions $D_k$ of $d_k$ in the Hilbert space of square integrable $k$-forms, such that

$$D_k(\mathrm{dom}(D_k)) \subset \mathrm{dom}(D_{k+1}).$$

We then get a Hilbert complex

$$0 \to \mathrm{dom}(D_0) \xrightarrow{D_0} \ldots\ldots \xrightarrow{D_{n-1}} \mathrm{dom}(D_n) \to 0.$$

(See [4] for the general theory for Hilbert and Fredholm complexes). The minimal and maximal extension of $d$

$$d_{\min} := \overline{d} = \text{ closure of } d,$$
$$d_{\max} := \delta^* = \text{ adjoint of the formal adjoint } \delta \text{ of } d$$

are examples of *ideal boundary conditions*. As shown in [5] in the case of cone-like singularities we have *uniqueness* of *ideal boundary conditions*, i.e.

$$d_{k,\min} = d_{k,\max} \text{ for all } k. \tag{2.1}$$

The equation (2.1) is also called the L$^2$-Stokes theorem. We denote by $(\mathcal{C}, d, <\ ,\ >)$ the unique extension of the de Rham complex

$$(\Omega^*_0(X - \Sigma), d, <, >)$$

to a Hilbert complex. The cohomology of this complex is the so-called L$^2$-cohomology of $X$

$$H^i_{(2)}(X) := \ker d_{i,\max}/\operatorname{im} d_{i-1,\max} = \ker d_{i,\min}/\operatorname{im} d_{i-1,\min}.$$

Note that the validity of (2.1) does not imply the essential self-adjointness of the Beltrami-Laplace operator $\Delta_{|\Omega_0^*(X\setminus\Sigma)} = d\delta + \delta d$ (acting on smooth compactly supported forms). Instead it is equivalent to the self-adjointness of the particular extension $\Delta = d_{\min}\delta_{\min} + \delta_{\min}d_{\min}$.

Moreover the L$^2$-Hodge theorem for Riemannian spaces with cone-like singularities (see [5], Section 1) states that the complex $(\mathcal{C}, d, < \ , \ >)$ is Fredholm and that the canonical maps

$$\ker(\Delta_i) \to H^i_{(2)}(X), i = 0, \ldots, n, \tag{2.2}$$

are isomorphisms. (In particular range$(\overline{d_i})$ is closed for all $i$ and therefore reduced and unreduced L$^2$-cohomology coincide in this case.)

By generalising Witten's idea to the singular situation described above one can deform the complex of L$^2$-forms by means of an admissible Morse function. I.e. one starts with the differential complex

$$(\Omega_0^*(X - \Sigma), d_t, < \ , \ >),$$

where $d_t = e^{-tf} de^{tf}$ and $t \in (0, \infty)$. As shown in [10] the complex

$$(\Omega_0^*(X - \Sigma), d_t, < \ , \ >)$$

also has *unique ibc*. We denote the unique extension of

$$(\Omega_0^*(X - \Sigma), d_t, < \ , \ >)$$

into a Hilbert complex by $(\mathcal{C}_t, d_t, < \ , \ >)$. It is not difficult to see that there is an isomorphism of Hilbert complexes:

$$e^{-tf} : (C, d, <, >_t) \to (C_t, d_t, <, >),$$

where by $< \alpha, \beta >_t = \int \alpha \wedge * \beta e^{-2tf}$ we denote the twisted metric. Therefore the complex $(\mathcal{C}_t, d_t, <, >)$ is also a Fredholm complex whose cohomology is isomorphic to the L$^2$-cohomology of $X$, i.e.

$$H^*\big((\mathcal{C}_t, d_t, <, >)\big) \simeq H^*_{(2)}(X). \tag{2.3}$$

Let us denote by $\delta_t$ the formal adjoint of the operator $d_t$ with respect to the L$^2$-metric $< \ , \ >$ and by $\Delta_{t|\Omega_0^*(X-\Sigma)} = (d_t + \delta_t)^2$ the corresponding Laplacian (acting on smooth compactly supported forms). Then we have the following identities (c.f. e.g. [3], Proposition 5.5.) on smooth forms with compact support outside the singularity:

$$\begin{aligned} d_t &= d + tdf\wedge\ , \\ \delta_t &= e^{tf}\delta e^{-tf} = \delta + t\nabla f \lrcorner\ , \\ \Delta_t &= \Delta + t^2\|\nabla f\|^2 + t(\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}), \end{aligned} \tag{2.4}$$

where we denote by $\mathcal{L}_{\nabla f} = d(\nabla f \lrcorner\ ) + \nabla f \lrcorner\, d$ the Lie derivative in the direction of the gradient vector field $\nabla f$ and by $\mathcal{L}^*_{\nabla f}$ its adjoint. Note that the operator $M_f := \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}$ is a zeroth order operator.

The Hodge theorem holds for the complex $(\mathcal{C}_t, d_t, <,>)$, i.e.

$$H^i((\mathcal{C}_t, d_t, <,>)) \simeq \ker d_{t,i} \cap \ker \delta_{t,i-1} \simeq \ker \Delta_{t,i}, \tag{2.5}$$

where $\Delta_t$ denotes the closed extension of $\Delta_{t|\Omega_0^*(X\setminus\Sigma)}$ with domain:

$$\operatorname{dom}(\Delta_t) = \big\{\Phi \bigm| \Phi, d_t\Phi, \delta_t\Phi, d_t\delta_t\Phi, \delta_t d_t\Phi \in \mathrm{L}^2\big(\Lambda^*(X-\Sigma)\big)\big\}. \tag{2.6}$$

Note however that $\operatorname{dom}(\Delta_t) \neq \operatorname{dom}(\Delta)$ in the presence of singularities and therefor we have to indicate carefully the domain of definition of each operator. The operator $\Delta_t$ is called the Witten Laplacian. It is a self-adjoint, nonnegative, discrete operator. The main result in [10] states

Theorem 2.2. — (a) **(Spectral gap theorem)** *Let $(X, g)$ be a Riemannian space as above and let $f : X \to \mathbb{R}$ be an admissible Morse function. Then there exist constants $C_1, C_2, C_3 > 0$ and $t_0 > 0$ depending on $X$ and $f$ such that for any $t > t_0$*

$$\operatorname{spec}(\Delta_t) \cap (C_1 e^{-C_2 t}, C_3 t) = \emptyset.$$

(b) *For large parameter $t$, the subcomplex $(\mathbb{F}_t, d_t, <,>)$ of the complex $(\mathcal{C}_t, d_t, <,>)$ generated by the eigenforms of $\Delta_{t,i}$ to eigenvalues in $[0,1]$ satisfies*

$$\operatorname{rk}(\mathbb{F}_t^i) = c_i(f) := \begin{cases} \#\mathrm{Crit}_i(f_{|X-\Sigma}) & \text{for } i = 0, 2, \\ \#\mathrm{Crit}_1(f_{|X-\Sigma}) + \sum_{p\in\Sigma}(m(p)-1) & \text{for } i = 1. \end{cases} \tag{2.7}$$

By (2.3) and (2.5) we know that the cohomology of the complex

$$(\mathbb{F}_t^*, d_t, <,>)$$

is isomorphic to the $\mathrm{L}^2$-cohomology of $X$. By a standard algebraic argument one can therefore deduce the Morse inequalities (1.2) from Theorem 2.2.

## 2.2. Construction of a basis of $\mathbb{F}_t$

In the remaining of this section we recall the construction of a basis of $(\mathbb{F}_t, d_t, <,>)$ (see [10], [12]). This basis will be needed in Section 4 to construct the comparison morphism. Let us first construct a local model for the Witten Laplacian near a singular point $p$ of multiplicity $m(p)$ of $X$. Let us denote by $\mathrm{cone}(S^1_m)$ the infinite cone $(0,\infty)\times S^1_m$ equipped with the metric $dr^2 + r^2 d\varphi^2$. We consider $f : \mathrm{cone}(S^1_m) \to \mathbb{R}$, $f(r,\varphi) = r\cos\varphi$. By $\mathbf{\Delta}_{|\Omega_0^*(\mathrm{cone}(S^1_m))}$ we denote the Laplace operator acting on compactly supported smooth forms on $\mathrm{cone}(S^1_m)$. Let us consider the operator

$$\mathbf{\Delta}_{t,i|\Omega_0^*(\mathrm{cone}(S^1_m))} := d_{t,i-1}\delta_{t,i-1} + \delta_{t,i} d_{t,i}. \tag{2.8}$$

In view of the formulas in (2.4) we get

$$\mathbf{\Delta}_{t,i|\Omega_0^*(\mathrm{cone}(S^1_m))} = \mathbf{\Delta}_{i|\Omega_0^*(\mathrm{cone}(S^1_m))} + t^2 \text{ for } i = 0,1,2. \tag{2.9}$$

The model Witten Laplacian $\mathbf{\Delta}_t$ is defined to be the self-adjoint extension of $\mathbf{\Delta}_{t|\Omega_0^*(\mathrm{cone}(S^1_m))}$ with domain

$$\mathrm{dom}(\mathbf{\Delta}_t) = \big\{\Phi \;\big|\; \Phi, d_t\Phi, \delta_t\Phi, d_t\delta_t\Phi, \delta_t d_t\Phi \in \mathrm{L}^2\big(\Lambda^*(\mathrm{cone}(S^1_m))\big)\big\}.$$

Theorem 2.3. — (a) $\mathrm{spec}(\mathbf{\Delta}_{t,i}) = \big[t^2,\infty\big)$ *in case* $i = 0,2$.

(b) $\mathrm{spec}(\mathbf{\Delta}_{t,1}) = \{0\} \cup \big[t^2,\infty\big)$ *and* $\dim\ker(\mathbf{\Delta}_{t,1}) = m-1$. *Moreover* $\{\omega_j^p(t) \mid j = 1,\dots,m-1\}$ *(see Appendix) form an ONB of* $\ker(\mathbf{\Delta}_{t,1})$.

*Proof.* — See [10], [12]. □

We make the additional assumption that in a neighbourhood of $p \in \mathrm{Crit}_k(f) - \Sigma$, $k = 0,1,2$ the metric $g$ is the Euclidean flat metric. The local model operator $\mathbf{\Delta}^p_{t,i}$ is closely related to a harmonic oscillator and has been studied in [16]. It is well-known that $\mathbf{\Delta}^p_t$ is a nonnegative, essentially self-adjoint, elliptic operator with $\ker\mathbf{\Delta}^p_t = \ker\mathbf{\Delta}^p_{t,k} = \mathrm{span}\{\omega_1^p(t)\}$, where $\omega_1^p(t) := \sqrt{t/\pi}\, e^{-t\|x\|^2/2} dx_1 \wedge \ldots \wedge dx_k$. (Hereby $x_1,\dots,x_k$ denote the coordinates in the Morse Lemma, i.e.

$$f = f(p) - 1/2(x_1^2 + \ldots + x_k^2) + 1/2(x_{k+1}^2 + \ldots + x_2^2)$$

near $p$.)

Now we choose $\epsilon > 0$ such that the open neighbourhoods $B_{2\epsilon}(p)$, $p \in \mathrm{Crit}(f)$, are pairwise disjoint. We denote by $\nu_\epsilon : [0,\infty) \to [0,1]$ a smooth cut-off function, which equals 1 in the interval $[0,\epsilon/2]$ and is equal to 0 in the interval $(\epsilon,\infty)$. The index set $I_p$ is defined by

$$I_p := \begin{cases} \{1,\dots,m-1\} & \text{for } p \in \Sigma \text{ of multiplicity } m, \\ \{1\} & \text{for } p \in \mathrm{Crit}(f) \setminus \Sigma. \end{cases}$$

For $p \in \mathrm{Crit}(f)$, $j \in I_p$ we define

$$\Phi_j^p(t) := \beta_j^p(t)^{-1}\nu_\epsilon(r)\omega_j^p(t)$$

where $\beta_j^p(t) = \|\nu_\epsilon(r)\omega_j^p(t)\| = \|\omega_j^p(t)\| + O(e^{-ct})$. The forms $\Phi_j^p(t)$ can be identified with forms in $(\mathcal{C}_t, d_t, <,>)$.

We denote by $P(t,[0,1])$ the orthogonal projection operator from $\mathcal{C}_t$ on $\mathbb{F}_t$ (with respect to the metric $<,>$). Then

$$\{\Psi_j^p(t) := P(t,[0,1])(\Phi_j^p(t)), p \in \mathrm{Crit}(f), j \in I_p\}$$

is a basis for $\mathbb{F}_t$. In section 4 we will need the following proposition:

Proposition 2.4. — *The set*

$$\{\Phi_j^p(t) \mid p \in \mathrm{Crit}(f), j \in I_p\}$$

*forms an ONB of* $\mathrm{span}\{\Phi_j^p(t) \mid p \in \mathrm{Crit}(f), j \in I_p\}$ *with respect to the* $\mathrm{L}^2$*-norm.*

*Proof.* — The proposition follows from Lemma 5.1 in the Appendix, the definition of the $\beta^p(t)$'s and the fact that, by construction, for $p \neq q$ the supports of $\Phi_j^p$ and $\Phi_k^q$ are disjoint, for all $j \in I_p, k \in I_q$. □

## 3. The geometric complex

### 3.1. Unstable/stable sets of critical points

In this section we will show that the singular space $X$ has a decomposition

$$X = \bigsqcup_{p \in \mathrm{Crit}(f)} W^u(p), \tag{3.1}$$

where for a critical point $p \in \mathrm{Crit}(f)$ we denote by $W^u(p)$ its unstable set (see Definition below). The main result of this section is Proposition 3.2, which describes the boundary of each $W^u(p)$, or in other terms "the way the cells are attached to each other".

Let us denote by $-\nabla_g f$ the negative gradient vector field of $f$. Note that $-\nabla_g f$ is defined only on $X \setminus \Sigma$. We denote by $\Phi$ the induced flow. The flow $\Phi$ is not defined for all time $t \in \mathbb{R}$. However we can define the stable/unstable set for all critical points of $f$ (including points $p \in \Sigma$):

Let $p \in \mathrm{Crit}(f)$. Then the stable (resp. unstable) set of $p$ is defined as follows:

$$\begin{aligned} W^s(p) = \Big\{x \in X \mid \exists\ t^+(x) > 0, \\ \text{such that } \lim_{t \to t^+(x)} \Phi(x,t) = p\Big\} \cup \{p\}, \\ \Big(\text{resp. } W^u(p) = \Big\{x \in X \mid \exists\ t^-(x) < 0, \\ \text{such that } \lim_{t \to t^-(x)} \Phi(x,t) = p\Big\} \cup \{p\}\Big). \end{aligned}$$

Thus for a critical point $p \in \mathrm{Crit}(f) \setminus \Sigma$ the definition above coincides with the usual definition of the stable/unstable set. For $p \in \Sigma$ by the above definition we included $p \in W^{u/s}(p)$.

If $p \in X - \Sigma$ is a critical point of $f$ of index $\mathrm{ind}(p)$ it is well-known that the stable (resp. unstable manifold) is a (non closed) manifold of dimension $\dim W^s(p) = 2 - \mathrm{ind}(p)$ (resp. $\dim W^u(p) = \mathrm{ind}(p)$), see e.g. [14].

Proposition 3.1. — *Let $p \in \Sigma$ be a singular point of $X$ of multiplicity $m$, then $W^u(p) - \{p\}$ as well as $W^s(p) - \{p\}$ are manifolds of dimension 1 having $m$ connected components $W_j^{u/s}(p)$, $j \in \mathbb{Z}/m(p)\mathbb{Z}$.*

*Proof.* — The picture below describes the gradient flow in the neighbourhood of a singular point (the picture below represents the case $m = 3$; note that $\varphi \in [0, 6\pi]$). By definition of an admissible Morse function there exists a neighbourhood $U(p)$ of $p$ in $X$ such that $f$ has the following form in local coordinates $(r, \varphi)$ near $p$:

$$f(r,\varphi) = f(p) + r\big(a_p \cos(\varphi) + b_p \sin(\varphi)\big).$$

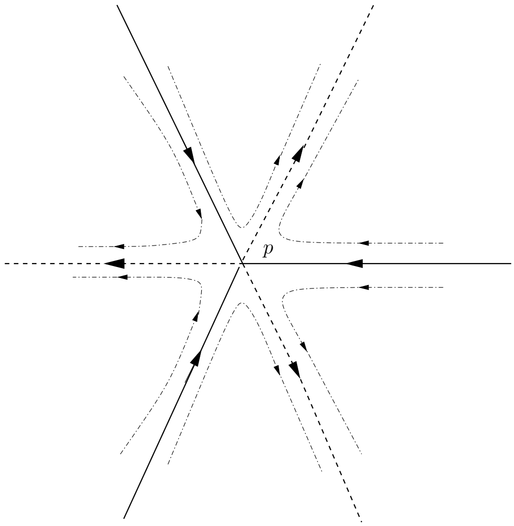
As explained in Remark 2.1 we can assume that $a_p = 1$ and $b_p = 0$. Then $-\nabla f(r,\varphi) = -(\cos(\varphi), -r^{-1}\sin(\varphi))$. In particular $-\nabla f(r,(2k+1)\pi) = (1,0)$ and $-\nabla f(r, 2k\pi) = (-1,0)$. Therefore it is easy to see that the local stable (resp. unstable) set of $p$ are given by

$$W^s(p) \cap U(p) = \bigsqcup_{i \in \widetilde{I}_p} W_i^{s,\mathrm{loc}}(p) \bigsqcup \{p\} \tag{3.2}$$

$$\Big(\text{resp. } W^u(p) \cap U(p) = \bigsqcup_{i \in \widetilde{I}_p} W_i^{u,\mathrm{loc}}(p) \bigsqcup \{p\},\Big) \tag{3.3}$$

where for $i \in \widetilde{I}_p := \mathbb{Z}/m(p)\mathbb{Z}$ we denote by

$$\begin{aligned} W_i^{s,\mathrm{loc}}(p) &:= \big\{(r,\varphi) \in U(p) \mid r \in \mathbb{R}^+, \varphi = 2i\pi\big\}, \\ W_i^{u,\mathrm{loc}}(p) &:= \big\{(r,\varphi) \in U(p) \mid r \in \mathbb{R}^+, \varphi = (2i+1)\pi\big\}. \end{aligned} \tag{3.4}$$




Thus the assumption holds locally near $p$. The global statement is shown as usual by "moving the charts" by means of the flow. □

An orientation is chosen for all $W^u(p)$, $p \in \mathrm{Crit}(f) - \Sigma$. For $p \in \Sigma$, $j \in \widetilde{I}_p$ the cells $W_j^u(p)$ are oriented by the negative gradient flow. We denote by $-W^u(p)$ the cell $W^u(p)$ with its opposite orientation.

It is easy to see that by perturbing the metric $g$ outside of a neighbourhood of $\Sigma$ we can assume that the gradient vector field $\nabla_g f$ is Morse-Smale, i.e. all intersections of stable and unstable manifolds are transversal.

The following proposition is a generalisation of Proposition 2 in [9] to the present situation:

Proposition 3.2. — *Let $f$ be an admissible Morse function such that $\nabla_g f$ satisfies the Morse-Smale condition. Then for each critical point $p \in \mathrm{Crit}(f) - \Sigma$ the closure $\overline{W^u(p)}$ is a stratified space. Let $p \in \mathrm{Crit}(f) \setminus \Sigma$. Then the strata of $\overline{W^u(p)} \setminus W^u(p)$ can be of the following form:*

(a) $W^u(q)$, *for* $q \in X - \Sigma$, $\mathrm{ind}(q) < \mathrm{ind}(p)$,
(b) $W_j^u(q)$, *for* $q \in \Sigma$, $j \in \widetilde{I}_q$ *and* $1 = \mathrm{ind}(q) < \mathrm{ind}(p)$,
(c) $\{q\}$, *for* $q \in \Sigma$, $\mathrm{ind}(q) < \mathrm{ind}(p)$.

*Moreover the strata of type (b) "come in pairs", i.e. if there exists $j \in \widetilde{I}_q$ such that $W_j^u(q) \subset \partial W^u(p)$ then $W_{j-1}^u(q) \subset \partial W^u(p)$ or $W_{j+1}^u(q) \subset \partial W^u(p)$. Moreover if $W^u(p)$ has 2 connected components near $W_j^u(q)$ then*

*$W_j^u(q)$ is the boundary of one of these, while $-W_j^u(q)$ is the boundary of the other one.*

*Moreover for $p \in \Sigma$, $i \in \widetilde{I}_p$ we have $W_i^u(p) \simeq (0,1)$ and $\overline{W_i^u(p)} \simeq [0,1]$ where one end of the compactification corresponds to $p$ and the other end corresponds to some $q \in \mathrm{Crit}_0(f)$.*

*Remark 3.3.* — The analogous result holds for the closures of the stable cells.

*Proof.* — Note first that by the Morse-Smale condition we can always assume that the critical values are pairwise distinct. Note moreover that the statement of the proposition is obvious if $p$ is a critical point of index 0 or 1.

Let $p$ be a critical point of index 2. For $a \in \mathbb{R}$ we denote by $X^a := W^u(p) \cap f^{-1}(a)$. If $a < f(p)$ is such that $[a, f(p)]$ contains no critical value, then

$$X^a \simeq S^1. \tag{3.5}$$

As $a$ decreases this remains true as long as we don't pass a critical value.

Let now $a_1$ be the first critical value of $f$ with $a_1 < f(p)$. By our assumption on the critical values of the Morse function there is a unique critical point $q_1 \in \mathrm{Crit}(f)$ with $a_1 = f(q_1)$.

*Case 1.* — If $W^u(p) \cap W^s(q_1) = \emptyset$ then $X^{a_1-\epsilon} \simeq S^1$.

*Case 2.* — Assume that $W^u(p) \cap W^s(q_1) \neq \emptyset$. Then $\mathrm{ind}(q_1) < 2$. If $\mathrm{ind}(q_1) = 0$. Then obviously $X^{a_1} \simeq \{*\}$ and $X^{a_1-\epsilon} = \emptyset$. (Thus $\overline{W^u(p)} = W^u(p) \cup \{q_1\}$ in this case.)

Therefore assume $\mathrm{ind}(q_1) = 1$. Then $X^{a_1-\epsilon}$ is no longer a closed manifold. If $q_1 \notin \Sigma$ this is a consequence of Lemma 4 in [9] and the situation is already well-understood. If $q_1 \in \Sigma$, let us denote by $(Y_1, Y_1')$ the pair of sets:

$$(Y_1, Y_1') := (X^{a_1+\epsilon}, X^{a_1+\epsilon} \cap W^s(q_1)) \simeq \left( S^1, \bigsqcup_{i \in \widetilde{I}'_{q_1} \subset \widetilde{I}_{q_1}} \{*_i\} \right). \tag{3.6}$$

We have the following easy lemma:

Lemma 3.4. — *Each connected component of $Y_1 - Y_1'$ is mapped diffeomorphically (by means of the flow) to a submanifold of $f^{-1}(a_1 - \epsilon)$, which is diffeomorphic to an open interval and the closure of which is either homeomorphic to $S^1$ (case (i)) or to $[0,1]$ (case (ii)).*

We continue the proof of Proposition 3.2. In case (i) we deduce that there exists a $j \in \widetilde{I}_{q_1}$ such that $W^u(p)$ has 2 connected components near $W^u_j(q_1)$, $W^u_j(q_1)$ is the oriented boundary of one of them while $-W^u_j(q_1)$ is the oriented boundary of the other one. In case (ii) we deduce that there is a $j$ such that $W^u_j(q_1) \cup -W^u_{j-1}(q_1) \subset \partial W^u(p)$.

We continue the process by studying the set $Y_2 := \overline{\Phi(Y_1 - Y_1', \mathbb{R}) \cap X^{a_1-\epsilon}}$ when passing the next critical point $q_2 \in \mathrm{Crit}(f)$. Again $Y_2$ stays unchanged if $W^u(p) \cap W^s(q_2) = \emptyset$. If $W^u(p) \cap W^s(q_2) \neq \emptyset$ and $\mathrm{ind}(q_2) = 0$ at least one of the connected components of $Y_2$ will be mapped to $q_2$ under the flow. Let us now assume that $W^u(p) \cap W^s(q_2) \neq \emptyset$ and $\mathrm{ind}(q_2) = 1$. Denote by

$$(Y_2, Y_2') := (Y_2, Y_2 \cap W^s(q_2)). \tag{3.7}$$

Then by [9] and a generalised version of Lemma 3.4 we deduce that each connected component of $Y_2 - Y_2'$ is mapped under the flow into a submanifold of $f^{-1}(a_2 - \epsilon)$ which is diffeomorphic to one of the following intervals:

$$(0,1) \text{ or } [0,1) \text{ or } (0,1] \text{ or } [0,1]. \tag{3.8}$$

Each open end corresponds to a boundary component $W^u_j(q_2), j \in \widetilde{I}_{q_2}$ of $W^u(p)$. Again this components come in pairs.

Since $\mathrm{Crit}(f)$ is finite the process described above finishes after a finite number of steps and we therefore get the result.

□

### 3.2. The complex of unstable cells $(C^u_*, \partial_*)$ and its subcomplex $(C^{u'}_*, \partial_*)$

In this section we define the chain complex $(C^u_*, \partial_*)$ "generated by the unstable manifolds" of the critical points of $f$. The chain groups of the complex $(C^u_*, \partial_*)$ are defined as follows

$$\begin{aligned} C^u_2 &:= \bigoplus_{p \in \mathrm{Crit}_2(f)} \mathbb{R} \cdot [W^u(p)], \\ C^u_1 &:= \bigoplus_{\substack{p \in \mathrm{Crit}_1(f) \\ p \notin \Sigma}} \mathbb{R} \cdot [W^u(p)] \oplus \bigoplus_{\substack{p \in \Sigma \\ j \in \widetilde{I}_p}} \mathbb{R} \cdot [W^u_j(p)], \\ C^u_0 &:= \bigoplus_{p \in \mathrm{Crit}_0(f)} \mathbb{R} \cdot [W^u(p)] \oplus \bigoplus_{p \in \Sigma} \mathbb{R} \cdot [\{p\}]. \end{aligned} \tag{3.9}$$

Note that since $\mathrm{Crit}(f)$ is finite the above chain groups are well-defined. The boundary of a generator $\sigma \in C_i^u$ is defined by

$$\partial\sigma = \sum n(\sigma,\theta)\cdot\theta, \tag{3.10}$$

where the sum is taken over all generators of $C_{i-1}$ and where $n(\sigma,\theta) = 0$ if $\theta$ is not in the closure of $\sigma$. Moreover if $\theta$ is in the closure of $\sigma$ we define $n(\sigma,\theta)$ as follows: Near $\theta$ the cell $\sigma$ has $n = n_+ + n_-$ connected components such that $\theta$ is the oriented boundary of $n_+$ of these and $-\theta$ is the oriented boundary of the other $n_-$. Then

$$n(\sigma,\theta) = n_+ - n_-. \tag{3.11}$$

It is not difficult to verify that $\partial^2 = 0$.

In the case of a Morse function on a smooth manifold one can give an interpretation of the coefficients $n(\sigma,\theta)$ by counting trajectories of the gradient flow between critical points of index difference 1. In our situation we can not do so.

For $p \in \Sigma$ and $j \in I_p$ denote by $\sigma_j^u := W_j^u(p) \cup -W_{j-1}^u(p) \cup \{p\}$ and by $[\sigma_j^u] := [W_j^u(p)] - [W_{j-1}^u(p)]$. We denote by

$$\begin{aligned}
C_2^{u'} &:= C_2^u,\\
C_1^{u'} &:= \bigoplus_{\substack{p\in\mathrm{Crit}_1(f)\\ p\notin\Sigma}} \mathbb{R}\cdot[W^u(p)] \bigoplus \mathrm{span}\big\{[\sigma_j^u(p)] \mid p\in\Sigma, j\in I_p\big\},\\
C_0^{u'} &:= \bigoplus_{p\in\mathrm{Crit}_0(f)} \mathbb{R}\cdot[W^u(p)].
\end{aligned} \tag{3.12}$$

Proposition 3.5. — *$(C_*^{u'},\partial_*)$ is a subcomplex of $(C_*^u,\partial_*)$.*

*Proof.* — From Proposition 3.2 we deduce that for all $\sigma \in C_2^{u'}$ we have $\partial\sigma \in C_1^{u'}$, and for all $\sigma \in C_1^{u'}$ we have $\partial\sigma \in C_0^{u'}$. □

*Remark 3.6.* — The decomposition of $X$ into unstable cells is a CW-decomposition of $X$ and therefor we have $H_*((C_*^u,\partial_*)) \simeq H_{\mathrm{sing}}(X)$. Since $X$ has dimension 2 and is (topologically) normal we have moreover that $H_{\mathrm{sing}}(X) \simeq IH_*(X)$ (see [6], section 4.2 and 4.3). It is not difficult to show that the inclusion of complexes $(C_*^{u'},\partial_*) \hookrightarrow (C_*^u,\partial_*)$ is a quasi-isomorphism. Thus the complex $(C_*^{u'},\partial_*)$ computes the intersection homology of $X$. Note however that the cells $\sigma_j^u$ are not allowed in the sense of intersection homology.

### 3.3. Stokes theorem

Denote by $(\Omega^*(X\setminus\Sigma),d)$ the differential complex of smooth forms on $X\setminus\Sigma$. Let

$$0\longrightarrow\mathcal{D}^0\xrightarrow{d_{\mathcal{D},0}}\mathcal{D}^1\xrightarrow{d_{\mathcal{D},1}}\mathcal{D}^2\longrightarrow 0 \tag{3.13}$$

be the differential complex with

$$\mathcal{D}^i:=\{\alpha\in \mathrm{L}^2(\Lambda^i(X))\cap\Omega^*(X\setminus\Sigma),\ d\alpha\in\mathrm{L}^2(\Lambda^{i+1}(X))\}. \tag{3.14}$$

For $i=0,1,2$ define by

$$H^i_{(2),\mathcal{D}}(X)=\ker d_{\mathcal{D},i}/\operatorname{im} d_{\mathcal{D},i-1} \tag{3.15}$$

the $i$-th cohomology of the complex $(\mathcal{D},d)$. There is a natural morphism

$$i_{(2)}:H^*_{(2),\mathcal{D}}(X)\to H^*_{(2)}(X), \tag{3.16}$$

which by [5] is an isomorphism.

We can now use the results in Section 3.1 to prove the following proposition

Proposition 3.7. — *Let $i=1,2$ and $\omega\in\mathcal{D}^{i-1}$. Denote by $\sigma$*

(1) *$\sigma:=\overline{W^u(p)}$, where $p\in\mathrm{Crit}_i(f)\setminus\Sigma$ or*
(2) *$\sigma:=\overline{\sigma^u_j(p)}$, where $p\in\Sigma$, $j\in I_p$ (in case $i=1$).*

*Then the Stokes formula holds, i.e. one has*

$$\int_\sigma d\omega=\int_{\partial\sigma}\omega. \tag{3.17}$$

*In particular both sides of* (3.17) *are well-defined.*

*Proof.* — As shown in Proposition 6 in [9] the claim holds if $\sigma\cap\Sigma=\emptyset$. To prove the proposition it is therefore enough to treat the following 2 cases.

*Case 1.* — $\sigma=\overline{W^u(p)}$ where $p\in\mathrm{Crit}_2(f)$ and $\sigma\cap\Sigma=q$. Let $B_\epsilon(q)$ be an $\epsilon$-neighbourhood of $q$ in $X$. Then the usual Stokes formula gives

$$\int_{\sigma\setminus B_\epsilon(q)}d\omega=\int_{\partial(\sigma\setminus B_\epsilon(q))}\omega. \tag{3.18}$$

We get the claim by letting $\epsilon\to 0$: Since $\omega\in\mathrm{L}^2(\Lambda^1(X))$ and $d\omega\in\mathrm{L}^2(\Lambda^2(X))$ we have $\omega=O(r^\beta)dr+O(r^\gamma)d\varphi$ for some $\beta>-1$ and $\gamma>0$ and $d\omega=O(r^\alpha)drd\varphi$ for some $\alpha>0$. Therefore we get for the right hand side of (3.18):

$$\int_{\sigma\cap B_\epsilon(q)}d\omega=\int_{\sigma\cap B_\epsilon(q)}O(r^\alpha)drd\varphi\leqslant C\epsilon^\alpha\to 0. \tag{3.19}$$

Moreover for the left hand side of (3.18) we use that

$$\int_{\partial(\sigma\cap B_\epsilon(q))}\omega=\int_{\partial(\sigma\cap B_\epsilon(q))}O(r^\beta)dr+O(r^\gamma)d\varphi\to 0. \tag{3.20}$$

*Case 2.* — $\sigma=\overline{\sigma_j^u(p)}$ for $p\in\Sigma$, $j\in I_p$ can be treated similarly.

□

*Remark 3.8.* — Note that as a corollary of the above proposition we get a second proof of the fact that $(C^{u'},\partial_*)$ is a complex, i.e. that $\partial^2=0$.

## 4. Relation to the geometric complex

### 4.1. Proof of the main theorems

For $p\in\mathrm{Crit}(f)\setminus\Sigma$ define $e_1^p:=[W^u(p)]$, $W^u(e_1^p):=W^u(p)$. For $p\in\Sigma$ and $j\in I_p$ define

$$e_j^p:=\sum_{l\in I_p}a_{lj}[\sigma_l^u(p)],\quad W^u(e_j^p):=\sum_{l\in I_p}a_{lj}\sigma_l^u(p) \tag{4.1}$$

where $A=\left(a_{lj}\right)_{l,j}\in\mathrm{GL}(m-1,\mathbb{R})$ is defined in Lemma 5.3. Let us equip $C_i^{u'}$ with the unique metric such that $\{e_j^p\mid p\in\mathrm{Crit}(f),j\in I_p\}$ is an orthonormal base.

We denote by $J_i(t):\mathrm{Hom}(C_i^{u'},\mathbb{R})\longrightarrow\mathcal{C}_t^i$ the linear map defined by $J_i(t)([e_j^p]^*)=\Phi_j^p(t)$. From Proposition 2.4 we deduce that $J_i(t)$ is an isometry from $\mathrm{Hom}(C_i^{u'},\mathbb{R})$ into the image of $J_i(t)$.

Denote by $(\mathbb{F}_t,d_t,<,>)$ the subcomplex of $(\mathcal{C}_t,d_t,<,>)$ generated by the eigenforms of $\Delta_t$ to eigenvalues $\lambda\in[0,1]$. We denote by $P(t,[0,1])$ the orthogonal projection operator from $\mathcal{C}_t$ on $\mathbb{F}_t$ (with respect to the metric $<,>$).

Proposition 4.1. — *There exist a constant $c>0$ and a* $\mathrm{L}^2$*-integrable function $\rho:X\to\mathbb{R}$ such that for all $v\in\mathrm{Hom}(C_i^{u'},\mathbb{R})$ and all $x\in X$:*

$$\left|[(P_i(t,[0,1])\circ J_i(t)-J_i(t))v](x)\right|=\rho(x)O(e^{-ct})\|v\|. \tag{4.2}$$

*Proof.* — See [10] Proposition 7.1 (see also [12]). □

We denote by $\widetilde{\Delta_t}$ the Laplacian associated to the complex $(\mathcal{C},d,<,>_t)$. Denote by $(\widetilde{\mathbb{F}}_t,d,<,>_t)$ the subcomplex of $(\mathcal{C},d,<,>_t)$ generated by the eigenforms of $\widetilde{\Delta_t}$ to eigenvalues $\lambda\in[0,1]$. Denote by $\widetilde{P}(t,[0,1])$ the orthogonal projection from $\mathcal{C}$ to $\widetilde{\mathbb{F}}_t$ with respect to the metric $<,>_t$. Then

Proposition 4.2. — *The map $\omega \mapsto e^{ft}\omega$ induces an isomorphism of complexes*

$$(\mathbb{F}_t, d_t, <,>) \longrightarrow (\widetilde{\mathbb{F}}_t, d, <,>_t). \tag{4.3}$$

*Moreover* $\widetilde{P}(t,[0,1]) = e^{tf}P(t,[0,1])e^{-tf}$.

*Proof.* — Obvious. □

Denote by $R_i(t)$ the linear map

$$R_i(t): \begin{array}{ccc} \mathrm{Hom}(C_i^{u'},\mathbb{R}) & \longrightarrow & \widetilde{\mathbb{F}}_t^i \\ {[e_j^p]^*} & \mapsto & e^{ft}P(t,[0,1]) \circ J(t)[e_j^p]^*. \end{array} \tag{4.4}$$

Denote by $R_i(t)^*$ the adjoint of $R_i(t)$ with respect to $<,>_t$.

Proposition 4.3. — *There exists $c > 0$ such that for $t \to \infty$*

$$R_i(t)^* R_i(t) = 1 + O(e^{-ct}). \tag{4.5}$$

*In particular for $t$ large enough $R_i(t)$ is an isomorphism of vectorspaces.*

*Proof.* — See [10], Corollary 7.2. □

Note that by elliptic regularity the complex $(\widetilde{\mathbb{F}}_t, d)$ may be considered as a subcomplex of the complex $(\mathcal{D}, d)$ and therefore, by Proposition 3.7, the integration morphism

$$\begin{aligned} P_{\infty,t} : \widetilde{\mathbb{F}}_t^i &\longrightarrow \mathrm{Hom}\big(C_i^{u'},\mathbb{R}\big) \\ \omega &\longmapsto \sum_{\substack{p\in \mathrm{Crit}_i(f)\\ j\in I_p}} \left(\int_{\overline{W^u(e_j^p)}} \omega\right) [e_j^p]^* \end{aligned} \tag{4.6}$$

is well-defined. By Stokes theorem $P_{\infty,t}$ is a morphism of complexes. By Hodge theory

$$H^*((\widetilde{\mathbb{F}}_t^{\,*}, d)) \simeq H^*_{(2)}(X). \tag{4.7}$$

*Proof of Theorem 1.1.* — The proof of Theorem 1.1 is analogous to that of Theorem 6.11 in [1], which makes use of the result of Laudenbach (see Proposition 2 in [9]) on the structure of the boundary of the unstable manifolds of critical points. In the present situation the structure of the boundary of the unstable manifolds is described in Proposition 3.2.

Let us fix $q \in \mathrm{Crit}_i(f)$. Using Proposition 4.1 we get

$$\begin{aligned}
(4.8)\quad & P_{\infty,t} \circ R_i(t)[e_k^q]^* \\
&= \sum_{\substack{p \in \mathrm{Crit}_i(f) \\ j \in I_p}} \left( \int_{\overline{W^u(e_j^p)}} R_i(t)[e_k^q]^* \right) [e_j^p]^* \\
&= \sum_{\substack{p \in \mathrm{Crit}_i(f) \\ j \in I_p}} \left( \int_{\overline{W^u(e_j^p)}} e^{ft} (J(t)[e_k^q]^* + r_k^q O(e^{-ct})) \right) [e_j^p]^* \\
&= \sum_{\substack{p \in \mathrm{Crit}_i(f) \\ j \in I_p}} e^{f(p)t} \left( \int_{\overline{W^u(e_j^p)}} e^{(f-f(p))t} (\Phi_k^q + r_k^q O(e^{-ct})) \right) [e_j^p]^*,
\end{aligned}$$

where $r_k^q$ is an $i$-form the pointwise norm of which is majorated by an $\mathrm{L}^2$-integrable function $\rho$, i.e. near the singularities of $X$:

$$\rho = \begin{cases} O(r^\alpha) & \text{for } i = 0, \\ O(r^\alpha) dr + O(r^\beta) d\varphi & \text{for } i = 1, \\ O(r^\beta) dr \wedge d\varphi & \text{for } i = 2, \end{cases} \tag{4.9}$$

where $\alpha > -1, \beta > 0$.

To prove the proposition it is now enough to evaluate the integrals

$$\int_{\overline{W^u(e_p^j)}} \left( e^{(f-f(p))t} \Phi_q^k + e^{(f-f(p))t} r_q^k O(e^{-ct}) \right) \tag{4.10}$$

where $p, q \in \mathrm{Crit}_i(f)$, $j \in I_p, k \in I_q$.

The function $f$ is decreasing along flow lines of the negative gradient flow and therefore $e^{(f-f(p))t} < 1$ on $\overline{W^u(e_j^p)}$. Note moreover that $\overline{W^u(e_j^p)} \cap U(\Sigma) \neq \emptyset$ only for $p \in \mathrm{Crit}_2(f)$ or $p \in \Sigma$. Therefore using (4.9) we get for the second part of the integral (4.10):

$$\int_{\overline{W^u(e_j^p)}} e^{(f-f(p))t} r_k^q O(e^{-ct}) = O(e^{-ct}). \tag{4.11}$$

If $p \neq q$ then according to Proposition 3.2 $q \notin \overline{W^u(e_j^p)}$ and therefore, by definition of the $\Phi_k^q$'s, $\Phi_k^q = O(e^{-ct})$ on $\overline{W^u(e_j^p)}$. Thus

$$\int_{\overline{W^u(e_j^p)}} e^{(f-f(p))t} \Phi_k^q = O(e^{-ct}). \tag{4.12}$$

If $p = q \notin \Sigma$ we get as in [3] (Theorem 6.11) that

$$\int_{\overline{W^u(q)}} e^{(f-f(q))t}\Phi_1^q = \left(\frac{\pi}{t}\right)^{(i-1)/2}(1+O(e^{-ct})). \tag{4.13}$$

Moreover if $p = q \in \Sigma$ we get using Lemma 5.3 and the definition of $\Phi_k^q$ that

$$\int_{\overline{W^u(e_j^q)}} e^{(f-f(q))t}\Phi_k^q = \delta_{jk} + O(e^{-ct}). \tag{4.14}$$

□

*Proof of Theorem 1.2.* — According to Theorem 1.1 and Proposition 4.3 for $t$ large enough $P_{\infty,t}\circ R(t)$ as well as $R(t)$ are isomorphisms and therefore

$$R(t)^{-1}\circ d\circ R(t) = \big(P_{\infty,t}\circ R(t)\big)^{-1}\circ P_{\infty,t}\circ d\circ P_{\infty,t}^{-1}\circ\big(P_{\infty,t}\circ R(t)\big). \tag{4.15}$$

Moreover $P_{\infty,t}$ is a chain homomorphism, i.e. $P_{\infty,t}\circ d\circ P_{\infty,t}^{-1} = \partial^*$.

Using Theorem 1.1 and equation (4.15) we get

$$R(t)^{-1}\circ d\circ R(t) = \big(1+O(e^{-ct})\big)^{-1}\left(\frac{\pi}{t}\right)^{N/2} e^{-t\mathcal{F}}\partial^* e^{t\mathcal{F}}\left(\frac{\pi}{t}\right)^{N/2}\big(1+O(e^{-ct})\big)$$

and since $\partial^*$ increases the degree by 1 we get the claim. □

## 4.2. Poincaré Duality

Let us choose an orientation of all unstable cells such that the orientation of all maximal cells is compatible with a chosen orientation on $X$. The stable cells $W^s(p)$, $p\in\mathrm{Crit}(f)\setminus\Sigma$ are naturally co-oriented. For $p\in\Sigma$ we orient the $m(p)$ connected components $W_j^s(p)$ of the stable set by the flow. For $j\in I_p$ we define $\sigma_j^s(p) := -[W_j^s] + [W_{j-1}^s]$. We denote by $(C_*^{s'},\partial_*)$ the subcomplex of the complex of stable cells generated by $\{W^s(p)\mid p\in\mathrm{Crit}(f)\setminus\Sigma\}\cup\{\sigma_j^s(p)\mid p\in\Sigma, j\in I_p\}$. Note that this is just the combinatorial complex associated to the admissible Morse function $-f$. We define a bilinear form

$$p_f : C_{2-k}^{s'}\times C_k^{u'}\to\mathbb{R} \tag{4.16}$$

by

$$p_f(W^s(p),W^u(p)) = 1 \text{ for } p\in\mathrm{Crit}(f)\setminus\Sigma. \tag{4.17}$$

Moreover if $p \in \Sigma$ and $i, j \in I_p$ set

$$p_f(\sigma_i^s(p), \sigma_j^u(p)) = \begin{cases} 1 & \text{if } j = i-1, \\ -1 & \text{if } j = i, \\ 0 & \text{else.} \end{cases} \tag{4.18}$$

(See the picture below for $m = 3$.)

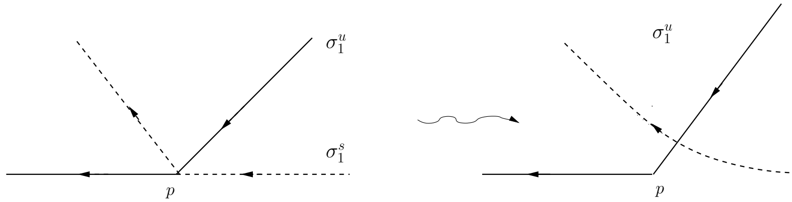



*Figure 4.1. We define the intersection of $\sigma_i^u$ and $\sigma_j^s$ by "moving" $\sigma_j^s$ away from $p$.*

All other intersections are $= 0$. Note that

$$p_f(\partial W^u(p), W^s(q)) = \pm p_f(W^u(p), \partial W^s(q)). \tag{4.19}$$

The bilinear form $p_f$ yields an identification $C_{2-*}^{s'} \simeq \mathrm{Hom}(C_*^{u'}, \mathbb{R})$ and induces an isomorphism

$$H_*\big((C_{2-*}^{s'}, \partial_*)\big) \simeq \mathrm{Hom}\big(H_*\big((C_*^{u'}, \partial_*)\big), \mathbb{R}\big). \tag{4.20}$$

Let $(\widetilde{\mathbb{F}_t}^-, d)$ be the complex of small eigenvalues for the admissible Morse function $-f$. Then there is a natural pairing:

$$P : \widetilde{\mathbb{F}_t}^k \times \widetilde{\mathbb{F}_t}^{-,2-k} \to \mathbb{R}, \qquad (\alpha, \omega) \to \int \alpha \wedge \omega, \tag{4.21}$$

which induces the Poincaré duality pairing $H_{(2)}^k(X) \simeq \mathrm{Hom}(H_{(2)}^{2-k}(X), \mathbb{R})$ for $\mathrm{L}^2$-cohomology.

Let us denote by $I_{\infty,t} : \widetilde{\mathbb{F}_t}^k \to C_{2-k}^{s'}$ the composition of $P_{\infty,t}$ with the isomorphism $\mathrm{Hom}(C_*^{u'}, \mathbb{R}) \simeq C_{2-*}^{s'}$. Similarly we define

$$I_{\infty,t}^- : \widetilde{\mathbb{F}_t}^{2-k,-} \to C_k^{u'}. \tag{4.22}$$

Proposition 4.4. — *For $t >> t_0$ the diagram*

(4.23)
$$\begin{array}{ccc} \widetilde{\mathbb{F}}_t^{\,k} \times \widetilde{\mathbb{F}}_t^{\,-,2-k} & \xrightarrow{P} & \mathbb{R} \\ {\scriptstyle I_{\infty,t}\times I^-_{\infty,t}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ C^{s'}_{2-k} \times C^{u'}_k & \xrightarrow[p_f^*]{} & \mathbb{R} \end{array}$$

*commutes up to a term of order* $O(e^{-ct})$.

*Proof.* — The proposition follows using Lemma 5.4. □

## 5. Appendix

Let $p \in \Sigma$ be a singular point of $X$ of multiplicity $m$. We do all computations for the case $m$ odd only, the case $m$ even can be treated similarly. Recall that locally near $p$ the Morse function has the form $f = f(p) + r\cos(\varphi)$. By $K_\nu$ we denote the modified Bessel function (of the second kind) of order $\nu$ (see [15]).

A basis for the kernel of the model Witten Laplacian $\mathbf{\Delta}_t^f$ is given by $\{\gamma_\nu^1, \gamma_\nu^2 \mid \nu = \frac{1}{m}, \dots \frac{m-1}{2m}\}$, where

$$\begin{aligned} \gamma_\nu^1 &= \frac{1}{\pi}\big[tr\big(K_{\nu-1}\cos(\nu\varphi) + K_\nu\cos((\nu-1)\varphi)\big)d\varphi \\ &\qquad + t\big(K_{\nu-1}\sin(\nu\varphi) - K_\nu\sin((\nu-1)\varphi)\big)dr\big] \\ \gamma_\nu^2 &= \frac{1}{\pi}\big[tr\big(K_{\nu-1}\sin(\nu\varphi) + K_\nu\sin((\nu-1)\varphi)\big)d\varphi \\ &\qquad + t\big(-K_{\nu-1}\cos(\nu\varphi) + K_\nu\cos((\nu-1)\varphi)\big)dr\big]. \end{aligned} \tag{5.1}$$

Similarly a basis for the model Witten Laplacian $\mathbf{\Delta}_t^{-f}$ (where $f$ has been replaced by $-f$) is given by $\{*\gamma_\nu^1, *\gamma_\nu^2 \mid \nu = \frac{1}{m}, \dots \frac{m-1}{2m}\}$. We denote by

$$\nu_j := \begin{cases} \frac{j}{m} & \text{for } 1 \leqslant j \leqslant \frac{m-1}{2}, j \in \mathbb{N}, \\ \frac{j}{m} - \frac{m-1}{2m} & \text{for } \frac{m-1}{2} < j \leqslant m-1, j \in \mathbb{N}, \end{cases} \tag{5.2}$$

and by $\beta_j := \sqrt{\frac{\sin(\nu_j\pi)}{m}}$.

Define for $j = 1, \dots, m-1$ the 1-forms

$$\omega_j^u := \begin{cases} \beta_j \gamma_{\nu_j}^1 & \text{for } 1 \leqslant j \leqslant \frac{m-1}{2}, \\ \beta_j \gamma_{\nu_j}^2 & \text{for } \frac{m-1}{2} < j \leqslant m-1, \end{cases} \tag{5.3}$$

and

$$\omega_j^s := \begin{cases} *\beta_j \gamma^1_{\nu_j} & \text{for } 1 \leqslant j \leqslant \frac{m-1}{2}, \\ *\beta_j \gamma^2_{\nu_j} & \text{for } \frac{m-1}{2} < j \leqslant m-1. \end{cases} \tag{5.4}$$

Note that the forms $e^{tf}\omega_j^u$ as well as $e^{-tf}\omega_j^s$ are exact.

Lemma 5.1. — *The set* $\{\omega_j^u \mid j = 1, \dots, m-1\}$ *is an ONB for* $\ker \mathbf{\Delta}_t^f$. *The set* $\{\omega_j^s \mid j = 1, \dots, m-1\}$ *is an ONB for* $\ker \mathbf{\Delta}_t^{-f}$.

*Proof.* — We only give the proof for the identity

$$\int_{\text{cone}} \gamma_\nu^i \wedge *\gamma_\nu^i = \frac{m}{\sin(\nu\pi)},$$

the other identities are shown in a similar way.

$$\begin{aligned} &\int_{\text{cone}} \gamma_\nu^i \wedge *\gamma_\nu^i \\ &\quad = \int_{\text{cone}} \frac{t^2}{\pi^2} r\big(K_{1-\nu}(tr)^2 + K_\nu(tr)^2\big)\big(\cos^2(\nu\varphi) + \sin^2(\nu\varphi)\big) dr d\varphi \\ &\quad = 2\pi m \int_0^\infty \frac{t^2}{\pi^2} r\big(K_{1-\nu}(tr)^2 + K_\nu(tr)^2\big) dr \\ &\quad = 2\pi m \int_0^\infty \frac{1}{\pi^2} y\big(K_{1-\nu}(y)^2 + K_\nu(y)^2\big) dy \\ &\quad = \frac{m}{\pi}\left(\frac{\nu\pi}{\sin(\nu\pi)} + \frac{(1-\nu)\pi}{\sin((1-\nu)\pi)}\right) = \frac{m}{\sin(\nu\pi)}. \end{aligned}$$

□

For $i \in I_p = \{1, \dots, m-1\}$ set $\widetilde{\sigma}_i^u := \mathbb{R}_+ \cdot [L_i^u] - \mathbb{R}_+ \cdot [L_{i-1}^u]$, where

$$L_i^u := \{(r, \varphi) \in \operatorname{cone}(S_m^1) \mid \varphi = (2i+1)\pi\}$$

and similarly

$$\widetilde{\sigma}_i^s := \mathbb{R}_+ \cdot [L_i^s] - \mathbb{R}_+ \cdot [L_{i-1}^s],$$

where $L_i^s := \{(r, \varphi) \in \operatorname{cone}(S_m^1) \mid \varphi = 2i\pi\}$.

Lemma 5.2. — *We have for* $i, j \in I_p$*:*

$$\int_{\widetilde{\sigma}_i^u} e^{tr\cos\varphi} \omega_j^u = \alpha(j, i), \qquad \int_{\widetilde{\sigma}_i^s} e^{-tr\cos\varphi} \omega_j^s = \beta(j, i), \tag{5.5}$$

*where*

$$\alpha(j, i) := \begin{cases} 2\beta_j \cos(2i\nu_j\pi) & \textit{for } j \leqslant \frac{m-1}{2}, \\ 2\beta_j \sin(2i\nu_j\pi) & \textit{for } j > \frac{m-1}{2}, \end{cases} \tag{5.6}$$

*and*

$$\beta(j,i) := \begin{cases} 2\beta_j \sin(\nu_j(2i-1)\pi) & \text{for } j \leqslant \frac{m-1}{2}, \\ -2\beta_j \cos(\nu_j(2i-1)\pi) & \text{for } j > \frac{m-1}{2}. \end{cases}$$

*Proof.* — The computation for $\int_{\widetilde{\sigma}_i^s} e^{-tr\cos\varphi}\omega_j^s$ being similar, we only compute $\int_{\widetilde{\sigma}_i^u} e^{tr\cos\varphi}\omega_j^u$,

$$= \begin{cases} \frac{\beta_{\nu_j}}{\pi} \cdot \int te^{-tr}(K_{1-\nu_j} + K_{\nu_j})dr \cdot \big(\sin((2i+1)\nu_j\pi) \\ \qquad - \sin((2i-1)\nu_j\pi)\big), j \leqslant \frac{m-1}{2}, \\ -\frac{\beta_{\nu_j}}{\pi} \cdot \int te^{-tr}(K_{1-\nu_j} + K_{\nu_j})dr\big(\cos((2i+1)\nu_j\pi) \\ \qquad - \cos((2i-1)\nu_j\pi)\big), j > \frac{m-1}{2}, \end{cases}$$

$$= \begin{cases} \frac{\beta_{\nu_j}}{\pi} \frac{\pi}{\sin(\nu_j\pi)} \cdot 2\sin(\pi\nu_j)\cos(2i\pi\nu_j) \text{ for } j \leqslant \frac{m-1}{2}, \\ \frac{\beta_{\nu_j}}{\pi} \frac{\pi}{\sin(\nu_j\pi)} \cdot 2\sin(\pi\nu_j)\sin(2i\pi\nu_j) \text{ for } j > \frac{m-1}{2}, \end{cases}$$

$$= \begin{cases} \frac{1}{2\sin(\nu_j\pi)}(\beta(j,i+1) - \beta(j,i)) = 2 \cdot \beta_{\nu_j} \cdot \cos(2i\pi\nu_j) \text{ for } j \leqslant \frac{m-1}{2}, \\ \frac{1}{2\sin(\nu_j\pi)}(\beta(j,i+1) - \beta(j,i)) = 2 \cdot \beta_{\nu_j} \cdot \sin(2i\pi\nu_j) \text{ for } j > \frac{m-1}{2}. \end{cases}$$

□

LEMMA 5.3. — *The matrix $\alpha = (\alpha(i,j))_{i,j} \in M(m-1, m-1, \mathbb{R})$ (resp. $\beta = (\beta(i,j))_{i,j} \in M(m-1, m-1, \mathbb{R})$) is invertible with inverse $A := \alpha^{-1}$ (resp. $B := \beta^{-1}$). Moreover*

$$\int_{\widetilde{e_i^u}} \omega_j^u = \delta_{ij}, \text{ where } \widetilde{e_i^u} := \sum_{l=1}^{m-1} a_{li}\widetilde{\sigma}_l^u, \tag{5.7}$$
$$\int_{\widetilde{e_i^s}} \omega_j^s = \delta_{ij}, \text{ where } \widetilde{e_i^s} := \sum_{l=1}^{m-1} b_{li}\widetilde{\sigma}_l^s.$$

*Proof.* — We only show the claims for $\alpha$. For $i = 1, \ldots, m$ let us denote by $\xi_i$ the $m$-th roots of unity $\xi_i := e^{2i\pi/m}$.

The invertibility of the matrix $\alpha$ is equivalent to the invertibility of the matrix

$$C := \begin{pmatrix} \operatorname{Re}(\xi_1) & \ldots & \operatorname{Re}\left(\xi_1^{m-1}\right) \\ \vdots & & \vdots \\ \operatorname{Re}\left(\xi_{\frac{m-1}{2}}\right) & \ldots & \operatorname{Re}\left(\xi_{\frac{m-1}{2}}^{m-1}\right) \\ \operatorname{Im}(\xi_1) & \ldots & \operatorname{Im}\left(\xi_1^{m-1}\right) \\ \vdots & & \vdots \\ \operatorname{Im}\left(\xi_{\frac{m-1}{2}}\right) & \ldots & \operatorname{Im}\left(\xi_{\frac{m-1}{2}}^{m-1}\right) \end{pmatrix} \in M(m-1, m-1, \mathbb{R}). \tag{5.8}$$

Applying appropriate elementary transformations of rows to the matrix $C$ we get (up to sign) the matrix

$$\widetilde{C} := \begin{pmatrix} \xi_1 & \dots & \xi_1^{m-1} \\ \vdots & & \vdots \\ \xi_{m-1} & \dots & \xi_{m-1}^{m-1} \end{pmatrix}. \tag{5.9}$$

We can complete $\widetilde{C}$ to the Vandermonde matrix

$$C' := \begin{pmatrix} 1 & \xi_1 & \dots & \xi^{m-1} \\ \vdots & & & \vdots \\ 1 & \xi_m & \dots & \xi_m^{m-1} \end{pmatrix} \in M(m, m, \mathbb{R}). \tag{5.10}$$

Since $\xi_i \neq \xi_j$ for $i \neq j$ the matrix $C'$ is invertible. Then the matrix $\alpha$ is also invertible.

Using the result of Lemma 5.2 and the definition of $\widetilde{e_i^u}$ and $A$ we get

$$\int_{\widetilde{e_i^u}} \omega_p^j = \sum_l a_{li}\alpha(j,l) = \delta_{ji}. \tag{5.11}$$

□

Lemma 5.4. — *Denote by $F := \big((\widetilde{\sigma_i^s}, \widetilde{\sigma_k^u})\big)_{i,k} \in \mathrm{GL}_{m-1}(\mathbb{R})$ the intersection matrix with*

$$f_{ik} := \begin{cases} -1 & \text{if } k = i, \\ 1 & \text{if } k = i-1, \\ 0 & \text{else.} \end{cases} \tag{5.12}$$

*Then*

$$\big((\widetilde{e_i^s}, \widetilde{e_k^u})\big)_{i,k} = E. \tag{5.13}$$

*Proof.* — By definition of the $\widetilde{e_i^{u/s}}$ we have $\big((\widetilde{e_i^s}, \widetilde{e_l^u})\big)_{i,l} = B^t F A$. We prove the equivalent statement $F = \beta^t \alpha$. We get for $\sum_{j=1}^{m-1} \beta(j,i)\alpha(j,k)$

$$
\begin{aligned}
&= \sum_{j=1}^{m-1} \frac{1}{2\sin(\nu_j\pi)} \big(\beta(j,k+1) - \beta(j,k)\big)\beta(j,i) \\
&= \sum_{j<m/2} \frac{4\beta_j^2}{2\sin(\nu_j\pi)} \Big[\big(\sin(\nu_j(2k+1)\pi) - \sin(\nu_j(2k-1)\pi)\big) \\
&\qquad \sin(\nu_j(2i-1)\pi) + \big(\cos(\nu_j(2k+1)\pi) - \cos(\nu_j(2k-1)\pi)\big) \\
&\qquad\quad \cos(\nu_j(2i-1)\pi)\Big] \\
&= \sum_{j<m/2} \frac{2}{m}\Big[\cos\big((2k-2i+2)\nu_j\pi\big) - \cos\big((2k-2i)\nu_j\pi\big)\Big].
\end{aligned} \tag{5.14}
$$

Since $\sum_{j=1}^{m} \cos(2l\nu_j\pi) = 1 + 2\sum_{1\leqslant j<m/2} \cos(2l\nu_j\pi) = 0$ for $l \neq 0$ (which is just a consequence of the egality $1+\xi+\xi^2+\ldots+\xi^{m-1} = \frac{1-\xi^m}{1-\xi} = 0$ for every $m$th root of unity $\xi \neq 1$) we get

$$
\sum_{j=1}^{m-1} \alpha(j,i)\beta(j,k) = \begin{cases} \frac{2}{m}(-1/2 - (m-1)/2) = -1 & \text{if } k = i, \\ \frac{2}{m}((m-1)/2 + 1/2) = 1 & \text{if } k = i-1, \\ \frac{2}{m}(-1/2+1/2) = 0 & \text{else .} \end{cases} \tag{5.15}
$$

□

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