ANNALES

DE

L'INSTITUT FOURIER

Sergey IVASHKOVICH & Alexandre SUKHOV

Schwarz Reflection Principle, Boundary Regularity and Compactness for J-Complex Curves

Tome 60, nº 4 (2010), p. 1489-1513.

<http://aif.cedram.org/item?id=AIF_2010__60_4_1489_0>

Ann. Inst. Fourier, Grenoble
60, 4 (2010) 1489-1513

SCHWARZ REFLECTION PRINCIPLE, BOUNDARY REGULARITY AND COMPACTNESS FOR J-COMPLEX CURVES

by Sergey IVASHKOVICH & Alexandre SUKHOV

Abstract. — We establish the Schwarz Reflection Principle for J-complex discs attached to a real analytic J-totally real submanifold of an almost complex manifold with real analytic J. We also prove the precise boundary regularity and derive the precise convergence in Gromov compactness theorem in $\mathcal{C}^{k,\alpha}$-classes.

Résumé. — Nous établissons le Principe de Symétrie de Schwarz pour les disques complexes attachés à une sous-variété analytique réelle et totalement réelle d'une variété presque complexe munie d'une structure presque complexe analytique réelle. Nous prouvons également la régularité au bord précise de ces disques et nous en déduisons la convergence exacte dans le théorème de compacité de Gromov dans les classes $\mathcal{C}^{k,\alpha}$.

1. Introduction

1.1. Reflection Principle

Denote by Δ the unit disc in $\mathbb{C}$, by $\mathbf{S}$ - the unit circle. Let $\beta \subset \mathbf{S}$ be a non-empty open subarc of $\mathbf{S}$.

Theorem 1.1 (Reflection Principle). — *Let (X, J) be a real analytic almost complex manifold and W a real analytic J-totally real submanifold of X. Let $u : \Delta \to X$ be a J-holomorphic map continuous up to β and such that $u(\beta) \subset W$. Then u extends to a neighborhood of β as a (real analytic) J-holomorphic map.*

Keywords: Almost complex structure, totally real manifold, holomorphic disc, reflection principle.
Math. classification: 32Q65, 32H40.

The case of integrable J is due to H. A. Schwarz [15]. Indeed, one can find local holomorphic coordinates in a neighborhood of $u(p)$ for a taken $p \in \beta$ such that $W = \mathbb{R}^n$ in these coordinates and now the Schwarz Reflection Principle applies. In our case there is no such reflection, since a general almost complex structure doesn't admits any local (anti)-holomorphic maps. But the extension result still holds.

One can put Theorem 1.1 into a more general form of Carathéodory, [3]. For this recall that the cluster set $\mathsf{cl}(\mathsf{u}, \beta)$ of u at β consists of all limits $\mathsf{lim}_{k\to\infty} u(\zeta_k)$ for all sequences $\{\zeta_k\} \subset \Delta$ converging to β. In [5] it was proved that if the cluster set $\mathsf{cl}(\mathsf{u}, \beta)$ of a J-holomorphic map $u : \Delta \to X$ is compactly contained in a totally real submanifold W then u smoothly extends to β. Therefore we derive the following

Corollary 1.2. — *In the conditions of the Theorem 1.1 the assumption of continuity of u up to β and $u(\beta) \subset W$ one can replace by the assumption that $\overline{u(\Delta)}$ is compact and the cluster set $\mathsf{cl}(\mathsf{u}, \beta)$ is contained in W.*

1.2. Boundary Regularity

For the proof of our Reflection Principle we need to study not only real analytic boundary values but also the smooth ones (with finite smoothness). For our method to work we need the precise regularity and a certain kind of uniqueness of smooth J-complex discs attached to a J-totally real submanifold. The result obtained is the following

Theorem 1.3 (Boundary Regularity). — *Let $u : (\Delta, \beta) \to (X, W)$ be a J-holomorphic map of class $L^{1,2} \cap \mathcal{C}^0(\Delta \cup \beta)$, where W is J-totally real. Then:*

(i) *for any integer $k \geqslant 0$ and real $0 < \alpha < 1$ if $J \in \mathcal{C}^{k,\alpha}$ and $W \in \mathcal{C}^{k+1,\alpha}$ then u is of class $\mathcal{C}^{k+1,\alpha}$ on $\Delta \cup \beta$;*

(ii) *for $k \geqslant 1$ the condition $u \in L^{1,2} \cap \mathcal{C}^0(\Delta \cup \beta)$ and $u(\beta) \subset W$ can be replaced by the assumption that $\overline{u(\Delta)}$ is compact and the cluster set $\mathsf{cl}(\mathsf{u}, \beta)$ is contained in W.*

Remark 1.4. — If J is of class $\mathcal{C}^0$ and W of $\mathcal{C}^1$ then $u \in \mathcal{C}^\alpha$ up to β for all $0 < \alpha < 1$. This was proved in [10], Lemma 3.1.

For integrable J the result of Theorem 1.3 is due to E. Chirka [4]. For non-integrable J weaker versions of this Theorem were obtained in [5, 7, 12].

Namely, the $\mathcal{C}^{k,\alpha}$-regularity of u up to β was achieved there under the same assumptions. The precise inner regularity was obtained by J.-C. Sikorav in [16].

1.3. Compactness Theorem

The precise regularity of J-complex discs attached to a J-totally real submanifolds of Theorem 1.3 allows also to get the precise convergence in Gromov compactness theorem. We refer to the Subsection 6.1 and to [9, 10] for the relevant notions and definitions.

Theorem 1.5 (Compactness Theorem). — *Let a sequence $\{J_n\}$ of almost complex structures of class $\mathcal{C}^{k,\alpha}$, $k \geqslant 0, 0 < \alpha < 1$, on a Riemannian manifold (X, h) converge on a compact subset $K \subset X$ in $\mathcal{C}^{k,\alpha}$-topology to an almost complex structure J. Let a sequence $\boldsymbol{W}_n = \{(W_i, f_{n,i})\}_{i=1}^m$ of J_n-totally real immersed submanifolds of X of class $\mathcal{C}^{k+1,\alpha}$ converge in $\mathcal{C}^{k+1,\alpha}$-topology to a J-totally real immersion $\boldsymbol{W} = \{(W_i, f_i)\}_{i=1}^m$. Suppose that all $\boldsymbol{W}_n$ and $\boldsymbol{W}$ have only weak transverse self-intersections. Let furthermore, $\{(\bar{C}_n, u_n)\}$ be a sequence of stable J_n-complex curves over X, parametrized by a fixed oriented compact real surface with boundary $\bar{\Sigma} = (\Sigma, \partial\Sigma)$, such that:*

(i) *$u_n(C_n) \subset K$ and that there exists a constant M such that $\mathsf{area}[\mathsf{u_n}(\mathsf{C_n})] \leqslant \mathsf{M}$ for all n;*

(ii) *$(\bar{C}_n, u_n)$ satisfy the totally real boundary conditions $(\boldsymbol{W}_n, \boldsymbol{\beta}, \boldsymbol{u}_n^{(b)})$ with $u_{n,i}^{(b)}(\beta_i) \subset W_i$ for all n and i.*

Then there exits a subsequence $\{(\overline{C}_{n_k}, u_{n_k})\}$ of $\{(\overline{C}_n, u_n)\}$ and parametrizations $\sigma_{n_k} : \overline{\Sigma} \to \overline{C}_{n_k}$, such that $(C_{n_k}, u_{n_k}, \sigma_{n_k})$ converges in $\mathcal{C}^{k+1,\alpha}$-topology up to boundary to a stable J-complex curve $(\overline{C}, u, \sigma)$ over X and this $(\overline{C}, u)$ satisfies the totally real boundary conditions $(\boldsymbol{W}, \boldsymbol{\beta}, \boldsymbol{u}^{(b)})$ with some $\mathcal{C}^{k+1,\alpha}$-continuous maps $u_k^b : \beta_k \to W_i$.

The novelty here with respect [12, 6] is that there is no loss in both of regularity and convergence of curves up to the boundary. In [10] an analog of Theorem 1.5 was proved for the special case $k = \alpha = 0$.

1.4. Proofs

The interior analyticity of J-holomorphic discs in analytic almost complex manifolds follows from classical results on elliptic regularity in the real analytic category, see, for instance, [2]. However the real analyticity up to the boundary is not a consequence of the known results since we do not deal with a boundary problem of the Dirichlet type. The direct application of the reflection principle (in the form of Vekua, for example) also leads to technical complications because of the non-linearity of the Cauchy-Riemann operator on an almost complex manifold. So our approach is different and is based on the reduction of the boundary regularity to a non-linear Riemann-Hilbert type boundary-value problem.

This paper is organized in the following way.

1. In §3, using [10] and [7] we prove Theorem 1.3. First we establish the $\mathcal{C}^{1,\alpha}$-regularity of u if $J \in \mathcal{C}^{\alpha}$ and then, using a sort of "geometric bootstrap", we obtain the $\mathcal{C}^{k+1,\alpha}$-regularity of u if $J \in \mathcal{C}^{k,\alpha}$.

2. In §4 we prove the solvability and uniqueness of a Riemann-Hilbert type boundary-value problem in Sobolev classes - the principal new tool of this paper. In §5 we adapt this method to the real analytic case. Then the uniqueness, both in Sobolev and in real analytic categories gives the proof of the Reflection Principle of Theorem 1.1. The novelty in §4 and 5, is a non-standard choice of the smoothness classes and an application of the Riemann-Hilbert boundary-value problem in the resolving of boundary regularity questions.

3. In §6 we recall necessary notions and prove the Compactness Theorem.

4. We end up with the formulation of open questions in §7.

We would like to express our gratitude to J.-F. Barraud who turned our attention to the question of extendability of J-holomorphic maps through totally real submanifolds in real analytic category.

2. Preliminaries

In what follows (X, J) will denote a pair which consists of a real analytic manifold X and an almost complex structure J on it. Note that by the well-known theorem of Whitney any smooth manifold can be endowed with a compatible atlas with real analytic transition mappings, *i.e.,* the condition of real analyticity of X doesn't lead to any loss of generality. The regularity of J will be specified in each statement. By $J_{st} = i\mathsf{Id}$ denote the standard complex structure of $\mathbb{C}^n$ (as well as of $\mathbb{C}$ and of Δ). Everywhere throughout the paper we use the notation $L^{k,p}$ for Sobolev space of functions with generalized partial derivatives of class L^p up to the order k. A $\mathcal{C}^1$-map (or

$\mathcal{C}^0 \cap L^{1,1}_{\mathrm{loc}}$-map) $u : \Delta \to X$ is called *J-holomorphic* if for every (or almost every) $\zeta \in \Delta$

$$du(\zeta) \circ J_{st} = J(u(\zeta)) \circ du(\zeta) \tag{2.1}$$

as mappings of tangent spaces $T_\zeta \Delta \to T_{u(\zeta)} X$. The image $u(\Delta)$ is called then a *J-complex* disc. Every almost complex manifold (X, J) of complex dimension n can be locally viewed as the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ equipped with an almost complex structure which is a small deformation of J_{st}. To see this fix a point $p \in X$, choose a coordinate system such that $p = 0$, make an $\mathbb{R}$-linear change of coordinates in order to have $J(0) = J_{st}$ and rescale, i. e., consider $J(tz)$ for $t > 0$ small enough. Then the equation (2.1) of J-holomorphicity of a map $u : \Delta \to B$ can be written in local coordinates ζ on Δ and z on $\mathbb{C}^n$ as the following first order quasilinear system of partial differential equations

$$u_{\overline{\zeta}} - A_J(u)\overline{u}_{\overline{\zeta}} = 0, \tag{2.2}$$

where $A_J(z)$ is the complex $n \times n$ matrix of the operator whose composite with complex conjugation is equal to the endomorphism $(J_{st} + J(z))^{-1}(J_{st} - J(z))$ (which is an anti-linear operator with respect to the standard structure J_{st}). Since $J(0) = J_{st}$, we have $A_J(0) = 0$. So in a sufficiently small neighborhood of the origin the norm $\| A_J \|_{L^\infty}$ is also small which implies the ellipticity of the system (2.2).

We recall some classical integral transformations. Let Ω be a relatively compact domain in $\mathbb{C}$ bounded by a finite number of transversely intersecting smooth curves. Denote by T^{CG}_Ω the Cauchy-Green transform in Ω:

$$\left(T^{CG}_\Omega h\right)(\zeta) = \frac{1}{2\pi i} \int\int_\Omega \frac{h(\tau) d\tau \wedge d\overline{\tau}}{\tau - \zeta}. \tag{2.3}$$

Denote also by

$$\left(K_\Omega h\right)(\zeta) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{h(\tau) d\tau}{\tau - \zeta} \tag{2.4}$$

the Cauchy transform.

Proposition 2.1. — *For every integer $k \geqslant 0$, real $0 < \alpha < 1$ and real $p > 1$ the following holds:*

(i) $T^{CG}_\Omega : \mathcal{C}^{k,\alpha}(\bar{\Omega}) \longrightarrow \mathcal{C}^{k+1,\alpha}(\bar{\Omega})$ *(resp.* $T^{CG}_\Omega : L^{k,p}(\Omega) \longrightarrow L^{k+1,p}(\Omega)$*) is a bounded linear operator and* $(T^{CG}_\Omega h)_{\overline{\zeta}} = h$ *for any* $h \in \mathcal{C}^{k,\alpha}(\Omega)$ *(resp. for any* $h \in L^{k,p}(\Omega)$*).*

(ii) *Let h be a bounded real analytic function on Ω, then $T^{CG}_\Omega h$ is real analytic on Ω. If, furthermore, $\Omega = \Delta$, h is a sum of a converging* *series $h(\zeta,\overline{\zeta}) = \Sigma h_{k,l}\zeta^k\bar{\zeta}^l$ and is continuous up to the boundary $\partial\Delta$, then for any $\zeta\in\Delta$ we have*

$$T_{\Delta}^{CG}h(\zeta) = H(\zeta,\overline{\zeta}) - \big(K_{\Delta}H\big)(\zeta), \tag{2.5}$$

where H is a primitive of h with respect to $\overline{\zeta}$.

The proof of (i) is contained, for instance, in [18], Theorems 1.32 and 1.37. For the statement (ii) about real analyticity see [18], p. 26. Relation (2.5) is in fact nothing but the Cauchy-Green formula. We shall also need the Schwarz integral transform on Δ:

$$\left(T^{SW}h\right)(\zeta) = \frac{1}{2\pi i}\int_{\partial\Delta}\frac{\tau+\zeta}{\tau-\zeta}\cdot\frac{h(\tau)}{\tau}d\tau. \tag{2.6}$$

Denote by $\mathcal{O}^{k,\alpha}(\Omega)$ (resp. $\mathcal{O}^{k,p}(\Omega)$) the Banach space of holomorphic maps $g:\Omega\longrightarrow\mathbb{C}^n$ of class $\mathcal{C}^{k,\alpha}(\bar{\Omega})$ (resp. $L^{k,p}(\bar{\Omega})$). This space is equipped with the norm $\| g \|_{\mathcal{C}^{k,\alpha}(\bar{\Omega})}$ (resp. $\| g \|_{L^{k,p}(\bar{\Omega})}$).

Proposition 2.2. — *$T^{SW}{:}\mathcal{C}^{k,\alpha}(\mathbf{S})\longrightarrow\mathcal{O}^{k,\alpha}(\Delta)$ (resp. $T^{SW}{:}L^{k,p}(\mathbf{S})\longrightarrow\mathcal{O}^{k,p}(\Delta)$) and $K_\Omega : \mathcal{C}^{k,\alpha}(\partial\Omega)\longrightarrow\mathcal{O}^{k,\alpha}(\Omega)$ (resp. $K_\Omega : L^{k,p}(\partial\Omega)\longrightarrow\mathcal{O}^{k,p}(\Omega)$) are bounded linear operators. For any real-valued function $\psi\in\mathcal{C}^{k,\alpha}(\mathbf{S})$ one has*

$$\mathsf{Re}\,(\mathsf{T}^{\mathsf{SW}}\psi)|_{\mathbf{S}} = \psi \tag{2.7}$$

and

$$\mathsf{Im}\,(\mathsf{T}^{\mathsf{SW}}\psi)(0) = 0.$$

For the proof see, for instance, [18], Theorem 1.10.

We shall need to consider the space of traces of functions from $L^{1,p}(\Delta)$ on the unit circle $\mathbf{S}$, see [13]. We say that a function ϕ defined on $\mathbf{S}$ is in the space $T^{1,p}(\mathbf{S})$, $p>2$, if there exists a function $u\in L^{1,p}(\Delta)$ such that $u|_{\mathbf{S}} = \phi$. Let us point out that by the Sobolev imbedding we have $T^{1,p}(\mathbf{S})\subset\mathcal{C}^{\alpha}(\mathbf{S})$ with $\alpha=(p-2)/p$. On the other hand if $1/2<\alpha<1$ then $\mathcal{C}^{\alpha}(\mathbf{S})\subset T^{1,p}(\mathbf{S})$ for every $p<(1-\alpha)^{-1}$. Indeed, given $\phi\in\mathcal{C}^{\alpha}(\mathbf{S})$ its Schwarz integral $u=T^{SW}\phi$ is of class $\mathcal{O}^{\alpha}(\overline{\Delta})$ and so by the classical theorem of Hardy-Littlewood, see [13], one has

$$\left|\frac{du(\zeta)}{d\zeta}\right| \leqslant \frac{C}{(1-|\zeta|)^{1-\alpha}}.$$

This implies that $\frac{du(\zeta)}{d\zeta}\in L^p(\Delta)$ for $p(1-\alpha)<1$. Now the Poisson integral

$$P\phi := \mathsf{Re}\,\mathsf{T}^{\mathsf{SW}}(\mathsf{Re}\,\phi) + \mathsf{i}\mathsf{Re}\,\mathsf{T}^{\mathsf{SW}}(\mathsf{Im}\,\phi) \tag{2.8}$$

gives an extension of ϕ of class $L^{1,p}(\Delta)$.

PROPOSITION 2.3. — *If $\phi \in T^{1,p}(\mathbf{S})$ then $T^{SW}\phi \in L^{1,p}(\Delta)$.*

Proof. — Let $u \in L^{1,p}(\Delta)$ be an extension of ϕ. By the Cauchy-Green formula

$$u = K_\Delta \phi + T^{CG}_\Delta u_{\overline{\zeta}}. \tag{2.9}$$

Hence the Cauchy type integral in the right hand (and therefore the Schwarz integral) is of class $L^{1,p}(\Delta)$, which proves the proposition. □

Finally, we introduce the norm on the space $T^{1,p}(\mathbf{S})$ by setting

$$\|\phi\|_{T^{1,p}} := \|P\phi\|_{L^{1,p}(\Delta)}$$

where $P\phi$ denotes the Poisson integral (2.8). Obviously, $T^{1,p}(\mathbf{S})$ is a Banach space. It follows from (2.8) that the convergence in $\mathcal{C}^\alpha(\mathbf{S})$, $\alpha > 1/2$ implies the convergence in $T^{1,p}(\mathbf{S})$, *i.e.*, $\mathcal{C}^\alpha(\mathbf{S})$ is a closed subspace in $T^{1,p}(\mathbf{S})$. Furthermore, if a sequence of functions $\{u_n\}$ converges in $L^{1,p}$, then their traces $\phi_n := u_n|_\mathbf{S}$ converge in $T^{1,p}(\mathbf{S})$. This follows from the Cauchy-Green representation (2.9) since the convergence of the Cauchy type integral implies the convergence of the Schwarz and Poisson integrals.

Remark 2.4. — In conclusion of this section we point out that the notion of the trace space $T^{1,p}(\partial\Omega)$ on the boundary of a domain Ω can be extended to a much larger class of simply connected domains by means of the Riemann mapping theorem and the classical theory of boundary properties of conformal mappings. For instance, all above definitions and properties admits an immediate generalization to the case where Δ is replaced by a simply connected domain bounded by a finite number of real analytic arcs with transversal intersections. The special case of the upper semi-disc Δ^+ will be important for our considerations. In what follows we simply write $T^{1,p}$ in the case of the unit circle.

3. Boundary Regularity in Hölder Classes

In this section we shall first use a version of a Reflection Principle proposed in [10] to prove the case $k = 0$ of Theorem 1.3. Then using the "geometric bootstrap" from [7] we obtain $\mathcal{C}^{k+1,\alpha}$-regularity of complex discs in $\mathcal{C}^{k,\alpha}$-regular structures for all $k \geqslant 1$, thus proving the Theorem 1.3.

3.1. Reflection Principle-I

In this subsection the structure J is supposed to be of class $\mathcal{C}^\alpha$ only. A J-totally real submanifold W of X will be supposed to have $\mathcal{C}^{1,\alpha}$-regularity. Similarly to the integrable case, a real submanifold W of an almost complex manifold (X,J) is called *J-totally real* if $T_pW \cap J(T_pW) = \{0\}$ at every point p of W. If n is the complex dimension of X, then any totally real submanifold of X is locally contained in a totally real submanifold of real dimension n. So in what follows we assume that W is n-dimensional.

First we make a suitable change of coordinates.

Lemma 3.1. — *One can find coordinates in a neighborhood V of $p \in W$ such that in these coordinates $V = \mathbb{R}^{2n}$, $W = \mathbb{R}^n$, $J|_{\mathbb{R}^n} = J_{\mathsf{st}}$ and $J(x,y) - J_{\mathsf{st}} = O(||y||^\alpha)$.*

Proof. — After a change of coordinates of class $\mathcal{C}^{1,\alpha}$ we can suppose that in some neighborhood of $p = 0$ our manifold W coincides with $\mathbb{R}^n$. Next we are looking for a $\mathcal{C}^{1,\alpha}$-diffeomorphism $\varphi = (\varphi_1, ...\varphi_{2n})$ in a neighborhood of the origin such that

(1) $\varphi_j(x,0) = x_j$ for $j = 1,...,n$;
(2) $\varphi_j(x,0) = 0$ for $j = n+1,...,2n$;
(3) $\frac{\partial\varphi}{\partial y_j}(x,0) = J(x,0)\left(\frac{\partial}{\partial x_j}\right)$ for $j = 1,...,n$.

Such $\mathcal{C}^{1,\alpha}$-diffeomorphism exists due to the Trace theorem, see [17]. In new coordinates given by φ we shall clearly have $W = \mathbb{R}^n$, $J|_{\mathbb{R}^n} = J_{\mathsf{st}}$ and $J(x,y) - J_{\mathsf{st}} = O(||y||^\alpha)$ due to $\mathcal{C}^\alpha$-regularity of J. □

In what follows the disc Δ with an arc β on its boundary $\mathbf{S}$ will be suitable for us to change by the upper half-disc $\Delta^+ = \{\zeta : \mathsf{Re}\,\zeta > 0\}$ and the segment $(-1,1)$. Let $u : (\Delta^+, \beta) \to (X, W)$ be a J-holomorphic map of class $L^{1,p}$ up to $\beta = (-1,1)$ for some $p > 2$.

The following lemma will prove the case $k = 0$ of Theorem 1.3 and will be used in the proof of the same case of Theorem 1.5 in the last section.

Lemma 3.2. — *Let J be of class $\mathcal{C}^\alpha$ and W be J-totally real of class $\mathcal{C}^{1,\alpha}$. Let $u : (\Delta^+, \beta) \to (X, W)$ be J-holomorphic of class $\mathcal{C}^0 \cap L^{1,2}$ up to β. Then u is of class $\mathcal{C}^{1,\alpha}$ up to β.*

Proof. — We can assume that $W = \mathbb{R}^n$ and $J(x,y) - J_{\mathsf{st}} = O(||y||^\alpha)$. On the trivial bundle $\Delta^+ \times \mathbb{R}^{2n} \to \Delta^+$ we consider the following linear complex structure: $J_u(z)[\xi] = J(u(z))[\xi]$ for $\xi \in \mathbb{R}^{2n}$ and $z \in \Delta^+$. At this point we stress that J_u is defined only on $\Delta^+ \times \mathbb{R}^{2n}$. Denote by τ the standard conjugation in $\Delta \subset \mathbb{C}$ as well as the standard conjugation in $\mathbb{R}^{2n} = \mathbb{C}^n$. Now we extend J_u to $\Delta \times \mathbb{R}^{2n}$ by setting

$$\tilde{J}_u(z)[\xi] = -\tau J_{u(\tau z)}[\tau \xi] \text{ for} z \in \Delta \text{ and } \xi \in \mathbb{R}^{2n}. \tag{3.1}$$

We consider now u as a section (over Δ^+) of the trivial bundle $E = \mathbb{R}^{2n} \times \Delta \to \Delta$ and endow E with the complex structure $\tilde{J}_u$. Complex structure $\tilde{J}_u$ defines a $\bar{\partial}$-operator $\bar{\partial}_{\tilde{J}_u} w = \partial_x w + \tilde{J}_u \partial_y w$ on $L^{1,p}$-sections of E for all $1 \leqslant p < \infty$ (for this only continuity of $\tilde{J}_u$ is needed). Remark that u is $\tilde{J}_u$-holomorphic on Δ^+. By F we denote the totally real subbundle $\Delta \times \mathbb{R}^n \to \Delta$ of E.

Definition 3.3. — *Define the "extension by reflection" operator* $\mathsf{ext} : \mathsf{L}^1(\Delta^+, \mathsf{E}) \to \mathsf{L}^1(\Delta, \mathsf{E})$ *by setting*

$$\mathsf{ext}(\mathsf{w})(\mathsf{z}) = \tau \mathsf{w}(\tau \mathsf{z}) \tag{3.2}$$

for $z \in \Delta^-$ *and* $w \in L^1(\Delta^+, E)$. *We shall also write* $\tilde{w}$ *for* $\mathsf{ext}(\mathsf{w})$.

Note that if w is continuous up to β and takes on β values in the subbundle F then $\mathsf{ext}(\mathsf{w})$ stays continuous. By the reflection principle of Theorem 1.1 from [10] we know that $\mathsf{ext} : \mathsf{L}^{1,p}(\Delta^+, \mathsf{E}, \mathsf{F}) \to \mathsf{L}^{1,p}(\Delta, \mathsf{E})$ is a continuous operator for all $1 \leqslant p < \infty$ and that $\bar{\partial}_{\tilde{J}_u} \tilde{w} = 0$ if $\bar{\partial}_{J_u} w = 0$. Let $\tilde{u}$ be the extension of u, in particular, $\tilde{u}$ is $\tilde{J}_u$-holomorphic on Δ of class $L^{1,2}(\Delta)$. First a priori estimate (1.1) from [9] insures that $\tilde{u} \in L^{1,p}_{loc}$ for all $p < \infty$. In particular $\tilde{u} \in \mathcal{C}^\gamma$ for every $\gamma < 1$. Therefore $\tilde{J}_u$ is of class $\mathcal{C}^\delta$, where $\delta = \alpha\gamma$.

Set $v = \rho \tilde{u}$, where ρ is a cut-off function equal to 1 in $\Delta_{\frac{1}{2}}$. Then from (2.2) we get

$$v_{\bar{\zeta}} - A_J(\tilde{u}) \overline{v_{\zeta}} = g, \tag{3.3}$$

where the function $g = \left[\rho_{\bar{\zeta}} - \rho_\zeta A_J(\tilde{u})\right] \tilde{u}$ is of class $\mathcal{C}^\delta(\Delta)$. Observe that $v_{\bar{\zeta}} - A_J(\tilde{u}) \overline{v_{\zeta}} = \left(v - T^{CG}_\Delta A_J(\tilde{u}) \overline{v_{\zeta}}\right)_{\bar{\zeta}}$. Elliptic regularity implies that $v - T^{CG}_\Delta A_J(\tilde{u}) \overline{v_{\zeta}}$ is of class $\mathcal{C}^{1,\delta}$ and invertibility of the operator $\mathsf{Id} - \mathsf{T}^{\mathsf{CG}}_\Delta \mathsf{A}_\mathsf{J}(\tilde{\mathsf{u}}) \overline{\frac{\partial}{\partial \mathsf{z}}}$ in $\mathcal{C}^{1,\delta}$ gives that v is in $\mathcal{C}^{1,\delta}$.

One can repeat this step once more to get $\mathcal{C}^{1,\alpha}$-regularity of v on Δ and therefore of u up to β. □

3.2. Cluster Sets on Totally Real Submanifolds

Since the case $k = 0$ of the Theorem 1.3 is already proved, we restrict ourselves in the future with $k \geqslant 1$. Fix an almost complex manifold (X, J)

with J of class $\mathcal{C}^{1,\alpha}$ and a J-totally real submanifold W of class $\mathcal{C}^{2,\alpha}$. Let $u : \Delta \to X$ be a bounded J-holomorphic map of the unit disc into X. Suppose that $\mathsf{cl}(\mathsf{u},\beta) \Subset \mathsf{W}$, where β is some non-empty open subarc of the boundary.

We use the Proposition 4.1 from [5] and observe that u is in Sobolev class $L^{1,p}$ up to β for all $p < 4$. In particular u is $\mathcal{C}^{\beta}$-regular up to β with $\beta = 1 - \frac{2}{p}$ (this means for all $\beta < \frac{1}{2}$). Lemma 3.2 implies now the following

Corollary 3.4. — *Let $J \in \mathcal{C}^{1,\alpha}$ and $W \in \mathcal{C}^{2,\alpha}$. If $u : (\Delta^+,\beta) \to (X,W)$ is a bounded J-holomorphic map with $\mathsf{cl}(\mathsf{u},\beta) \Subset \mathsf{W}$ then $u \in \mathcal{C}^{1,\alpha}(\Delta^+ \cup \beta)$.*

Let's stress here that $\mathcal{C}^{1,\alpha}$ is not the optimal regularity of u, it should be $\mathcal{C}^{2,\alpha}$. This will be achieved in the next subsection.

3.3. Geometric bootstrap and boundary $\mathcal{C}^{k+1,\alpha}$-Regularity

Having proved the $\mathcal{C}^{1,\alpha}$-regularity of complex discs in $\mathcal{C}^{\alpha}$-regular structures we are going to use the "geometric bootstrap" to obtain $\mathcal{C}^{k+1,\alpha}$-regularity of complex discs in $\mathcal{C}^{k,\alpha}$-regular structures.

We need to lift an almost complex structure from the manifold X to its tangent bundle TX. In local coordinates the lift J^c is defined by

$$J^c = \begin{pmatrix} J_i^h & 0 \\ t^a \partial_a J_i^h & J_i^h \end{pmatrix},$$

where t^a are coordinates in the tangent space. This lift is invariantly defined, see [7] for more details. After that we can use the induction on k. Really, if $J \in \mathcal{C}^{1,\alpha}$ then $J_n^c \in \mathcal{C}^{\alpha}$. Further we lift J-holomorphic map $u : \Delta \to X$ to J^c-holomorphic map $u^c : \Delta \to TX$. This lift is defined as

$$u^c(\zeta) = (u(\zeta), du(\zeta)(e_1)), \tag{3.4}$$

where $e_1 = (1,0)$.

Now remark that if W is a J-totally real submanifold in X then TW is a J^c-totally real submanifold in TX. Really, let $v \in T(TW) \cap J^c(T(TW))$. If $v = (v_1, v_2)$ in the trivialisation $T(TX) = TX \oplus TX$ then $v_1 \in TW \cap J(TW)$, implying that $v_1 = 0$. Hence $v_2 \in TW \cap J(TW)$, implying that $v_2 = 0$. Therefore $v = 0$.

Further the lift $u^c : \Delta^+ \to TX$ of a J-holomorphic map $u : \Delta^+ \to X$ with boundary values in W has its boundary values in TW (this is clear from the formula (3.4)). Applying Lemma 3.2 to u^c and TW we prove that the first derivative of u with respect to ξ is of class $\mathcal{C}^{1,\alpha}$ on $\Delta^+ \cup \beta$. Here $\zeta = \xi + i\eta$. The J^c-holomorphicity equation

$$\frac{\partial u^c}{\partial \eta} = J^c(u^c)\frac{\partial u^c}{\partial \xi} \tag{3.5}$$

implies that $\frac{\partial u^c}{\partial \eta}$ is also of class $\mathcal{C}^{1,\alpha}$ on $\Delta^+ \cup \beta$. Therefore $u \in \mathcal{C}^{2,\alpha}$ up to β. By induction we conclude that u is of class $\mathcal{C}^{k+1,\alpha}$ up to β if $J \in \mathcal{C}^{k,\alpha}$ and W is of class $\mathcal{C}^{k+1,\alpha}$.

Theorem 1.3 is proved.

4. Riemann-Hilbert Boundary-Value Problem

In this Section we develop one of the main tools of this paper - a sort of a Riemann-Hilbert problem. This will be used in the proof of Theorem 1.1. Denote by $\mathbf{S}^+ = \{e^{i\theta} : \theta \in]0, \pi[\}$ the upper semi-circle.

For the local considerations of the present Section we suppose that $X = \mathbb{C}^n$ and $W = i\mathbb{R}^n = \{z = x + iy : x = 0\}$ and that $J \in \mathcal{C}^{1,\alpha}$ is a small deformation of J_{st}. Fix also a J_{st}-holomorphic map $u^0 : \Delta \longrightarrow \mathbb{C}^n$ of class $L^{1,p}(\Delta)$, $p > 2$, such that $u^0(\mathbf{S}^+) \subset i\mathbb{R}^n$ (so that u^0 extends holomorphically to a neighborhood of $\mathbf{S}^+$ by the classical Schwarz Reflection Principle). For J close enough to J_{st} we will establish the existence and uniqueness of a J-holomorphic disc u close enough to u^0 satisfying the boundary condition $u(\mathbf{S}^+) \subset i\mathbb{R}^n$.

Therefore for J close enough to J_{st} we study the solutions of (2.2) satisfying the boundary condition

$$\mathsf{Re}\,\mathsf{u}|_{\mathbf{S}^+} = 0. \tag{4.1}$$

Denote by $T^{1,p}_0$ the Banach space of ($\mathbb{R}^n$ -valued) functions $\varphi \in T^{1,p}$ vanishing on $\mathbf{S}^+$. This space is equipped with the standard norm $\| \varphi \|_{T^{1,p}}$. Set now $\varphi^0 := \mathsf{Re}\,\mathsf{u}^0|\mathbf{S}$ and remark that $\varphi^0 \in T^{1,p}_0$ because $\mathsf{Re}\,\mathsf{u}^0|_{\mathbf{S}^+} = 0$. We replace the condition (4.1) for the solutions of the partial differential equation (2.2) by the condition

$$\mathsf{Re}\,\mathsf{u}|_{\mathbf{S}} = \varphi, \tag{4.2}$$

where $\varphi \in T_0^{1,p}$. Therefore we consider the following boundary-value problem

$$
\begin{cases} u_{\bar\zeta} - A_J(u)\overline{u}_{\bar\zeta} = 0, \\ \mathsf{Re\,u}|_{\mathbf{S}} = \varphi, \\ \mathsf{Im\,u(0) = a}, \end{cases} \tag{4.3}
$$

for the given initial data $\varphi \in T_0^{1,p}$, $a \in \mathbb{R}^n$.

Lemma 4.1. — *If J is close enough to J_{st} in $\mathcal{C}^{1,\alpha}$-norm then there exists a neighborhood U of φ^0 in $T_0^{1,p}$, a neighborhood U' of $a^0 := \mathsf{Im\,u^0(0)}$ in $\mathbb{R}^n$ and a neighborhood V of u^0 in $L^{1,p}(\Delta)$ such that for each $\varphi \in U$ and $a \in U'$ the boundary problem (4.3) admits a unique solution $u \in V$.*

Proof. — Consider the operator

$$L_J : L^{1,p}(\Delta) \longrightarrow L^p(\Delta) \times T^{1,p} \times \mathbb{R}^n$$

defined by

$$L_J : u \mapsto \begin{pmatrix} u_{\bar\zeta} - A_J(u)\overline{u}_{\bar\zeta} \\ \mathsf{Re\,u}|_{\mathbf{S}} \\ \mathsf{Im\,u(0)} \end{pmatrix}.$$

L_J smoothly depends on the parameter J. Denote by $\dot{L}_J(u)$ the Fréchet derivative of L_J at u. $\dot{L}_J$ is continuous on the couple (J, u) and at J_{st}-holomorphic u^0 the derivative $\dot{L}_{J_{\mathsf{st}}}(u^0)$ is particularly simple:

$$\dot{L}_{J_{\mathsf{st}}}(u^0) : L^{1,p}(\Delta) \longrightarrow L^p(\Delta) \times T^{1,p} \times \mathbb{R}^n$$

$$\dot{L}_{J_{\mathsf{st}}}(u^0) : \dot{u} \mapsto \begin{pmatrix} \dot{u}_{\bar\zeta} \\ \mathsf{Re\,\dot{u}}|_{\mathbf{S}} \\ \mathsf{Im\,\dot{u}(0)} \end{pmatrix}.$$

Let's see that $\dot{L}_{J_{\mathsf{st}}}(u^0)$ is an isomorphism. Indeed, given $h \in L^p(\Delta)$, $\psi \in T^{1,p}$ and $a \in \mathbb{R}^n$ then the function

$$\dot{u} = T_{\Delta}^{CG}h - i\mathsf{Im}\,\left(\mathsf{T}_{\Delta}^{\mathsf{CG}}\mathsf{h(0)}\right) + \mathsf{ia} + \mathsf{T}^{\mathsf{SW}}(\psi - \mathsf{Re}\,(\mathsf{T}_{\Delta}^{\mathsf{CG}}\mathsf{h})|\mathbf{S})$$

is of class $L^{1,p}(\Delta)$ and satisfies the equation

$$\dot{L}_{J_{\mathsf{st}}}(u^0)(\dot{u}) = \begin{pmatrix} h \\ \psi \\ a \end{pmatrix}.$$

Uniqueness of $\dot{u}$ is obvious. Therefore by the Implicit Function Theorem every L_J is a $\mathcal{C}^1$-diffeomorphism of neighborhoods of u^0 in $L^{1,p}(\Delta)$ and of $(0, \varphi_0, a_0)$ in $L^p(\Delta) \times T^{1,p} \times \mathbb{R}^n$. Since $T_0^{1,p}$ is a closed subspace of $T^{1,p}$ the Lemma 4.1 follows. □

Let us formulate a corresponding statement in Hölder classes. Let $k \geqslant 1$. Fix a J_{st}-holomorphic map $u^0 : \Delta \longrightarrow \mathbb{C}^n$ of class $\mathcal{C}^{k+1,\alpha}(\overline{\Delta})$ such that $u^0(\mathbf{S}^+) \subset i\mathbb{R}^n$. For every positive integer k denote by $\mathcal{C}_0^{k,\alpha}(\mathbf{S})$ the Banach space of ($\mathbb{R}^n$ -valued) functions $\varphi \in \mathcal{C}^{k,\alpha}(\mathbf{S})$ vanishing on $\mathbf{S}^+$. This space is equipped with the standard norm $\| \varphi \|_{\mathcal{C}^{k,\alpha}(\mathbf{S})}$. Set now $\varphi^0 := \mathsf{Re}\,\mathsf{u}^0|\mathbf{S}$. $\varphi^0 \in \mathcal{C}_0^{k+1,\alpha}(\mathbf{S})$ because $\mathsf{Re}\,\mathsf{u}^0|_{\mathbf{S}^+} = 0$. Therefore we consider the boundary value problem (4.3) for the given initial data $\varphi \in \mathcal{C}_0^{k+1,\alpha}(\mathbf{S})$, $a \in \mathbb{R}^n$.

Lemma 4.2. — *Suppose $k \geqslant 1$. If J is close enough to J_{st} in $\mathcal{C}^{k,\alpha}$-norm then for every $1 \leqslant l \leqslant k$:*

- (i) *there exists a neighborhood U of φ^0 in $\mathcal{C}_0^{l+1,\alpha}(\mathbf{S})$, a neighborhood U' of $a^0 := \mathsf{Im}\,\varphi^0(0)$ in $\mathbb{R}^n$ and a neighborhood V of u^0 in $\mathcal{C}^{l+1,\alpha}(\bar{\Delta})$ such that for each $\varphi \in U$ and $a \in U'$ the boundary problem (4.3) admits a unique solution $u \in V$;*
- (ii) *the unit disc Δ can be replaced in part (i) of the present Lemma by any bounded simply connected domain Ω with $\mathcal{C}^\infty$ boundary and $\mathbf{S}^+$ can be replaced by any non-empty open arc.*

Proof. — The part (ii) follows from (i) by the Riemann mapping theorem and the classical theorems on the boundary regularity of conformal maps. For the proof of part consider the operator

$$L_J : \mathcal{C}^{l+1,\alpha}(\bar{\Delta}) \longrightarrow \mathcal{C}^{l,\alpha}(\bar{\Delta}) \times \mathcal{C}^{l+1,\alpha}(\mathbf{S}) \times \mathbb{R}^n,$$

defined as in the proof of Lemma 4.1, but this time in other smoothness classes. One can literally repeat the arguments used there also in this case. □

5. Reflection Principle-II: Real Analytic Case

Let us turn now to the proof of Theorem 1.1. We shall proceed in two steps.

5.1. Small deformations of the standard structure

Here we consider the case when J is a small real analytic deformation of J_{st} and $W = i\mathbb{R}^n$. First we introduce suitable Banach spaces of real analytic functions using the complexification.

Denote by $\Delta^2 = \Delta \times \Delta$ the standard bidisc in $\mathbb{C}^2$. We define the space $\mathcal{C}_\omega^{1,\alpha}(\Delta)$ consisting of functions u (or $\mathbb{C}^n$-valued maps) of class $\mathcal{C}^{1,\alpha}(\bar{\Delta})$ with the following properties:

- (i) u is a sum of a power series $u(\zeta) = \sum_{kl} u_{kl}\zeta^k\overline{\zeta}^l$ for $\zeta \in \Delta$.

(ii) The "polarization" $\hat{u}$ of u defined by $\hat{u}(\zeta,\xi) = \sum_{kl} u_{kl}\zeta^k\xi^l$ is a function holomorphic on Δ^2 and of class $\mathcal{C}^{1,\alpha}(\overline{\Delta}^2)$.

(iii) The mixed derivative $\frac{\partial^2\hat{u}}{\partial\zeta\partial\xi}$ is of class $\mathcal{C}^{\alpha}(\overline{\Delta}^2)$.

We define the norm of u as following:

$$\|u\|_{\mathcal{C}^{1,\alpha}_{\omega}(\Delta)} = \|\hat{u}\|_{\mathcal{C}^{1,\alpha}(\overline{\Delta}^2)} + \left\|\frac{\partial^2\hat{u}}{\partial\zeta\partial\xi}\right\|_{\mathcal{C}^{\alpha}(\overline{\Delta}^2)}.$$

Since u is the restriction of $\hat{u}$ onto the totally real diagonal $\{\xi = \bar{\zeta}\}$, the polarization $\hat{u}$ is uniquely determined by u and therefore $\mathcal{C}^{1,\alpha}_{\omega}(\Delta)$ equipped with this norm is a Banach space.

Remark 5.1. — One has the following continuous inclusion $\mathcal{O}^{1,\alpha}(\Delta) \subset \mathcal{C}^{1,\alpha}_{\omega}(\Delta)$: for $u \in \mathcal{O}^{1,\alpha}(\Delta)$ the corresponding $\hat{u}$ is simply $\hat{u}(\zeta,\xi) = u(\zeta)$. Really, for such $\hat{u}$ one has $\frac{\partial^2\hat{u}}{\partial\zeta\partial\xi} = 0$.

We denote by $\mathcal{C}^{1,\alpha}_{\omega}(\partial\Delta^+)$ the space of real functions φ on $\partial\Delta^+$ such that there exists a function $v \in \mathcal{O}^{1,\alpha}(\Delta)$ satisfying the condition $\mathsf{Re}\,\mathsf{v}|_{\partial\Delta^+} = \varphi$. In particular such function φ is real analytic on the interval $(-1,1)$. The holomorphic function v is unique up to an imaginary constant so we always assume that $\mathsf{Im}\,\mathsf{v}(0) = 0$. We define the norm of φ as a $\mathcal{C}^{1,\alpha}$ norm of the corresponding function v on $\bar{\Delta}$. Then $\mathcal{C}^{1,\alpha}_{\omega}(\partial\Delta^+)$ equipped with this norm, is a Banach space.

Furthermore, denote by $\mathcal{C}^{1,\alpha}(\partial\Delta^+)$ the space of real continuous functions on $\partial\Delta^+$ which are of class $\mathcal{C}^{1,\alpha}$ on the closed upper semi-circle and on the interval $[-1,1]$. Finally we denote by $\mathcal{C}^{1,\alpha}_0(\partial\Delta^+)$ the space of real functions of class $\mathcal{C}^{1,\alpha}(\partial\Delta^+)$ vanishing on the interval $[-1,1]$. The following statement is a consequence of the reflection principle.

Lemma 5.2. — *For every function $\varphi \in \mathcal{C}^{1,\alpha}_0(\partial\Delta^+)$ there exists $u \in \mathcal{O}^{1,\alpha}(\Delta)$ such that $\mathsf{Re}\,\mathsf{u}|_{\partial\Delta^+} = \varphi$. In particular, the space $\mathcal{C}^{1,\alpha}_0(\partial\Delta^+)$ is a subspace of $\mathcal{C}^{1,\alpha}_{\omega}(\partial\Delta^+)$.*

Proof. — Let φ be a function of class $\mathcal{C}^{1,\alpha}_0(\partial\Delta^+)$. Solving the Dirichlet problem for φ in the upper semi-disc, we obtain a harmonic function h in Δ^+ continuous on $\overline{\Delta}^+$ such that $h|_{\partial\Delta^+} = \varphi$. Since h vanishes on $[-1,1]$ it extends harmonically on Δ by the classical reflection principle for harmonic functions. Namely, its extension h^* is defined by $h^*(\zeta) = -h(\overline{\zeta})$ for ζ in the lower semi-disc Δ^-. Thus we obtain a function $\tilde{h}$ harmonic on Δ and continuous on $\overline{\Delta}$. Since the restriction φ of h on the closed upper semi-circle is a function of class $\mathcal{C}^{1,\alpha}$, it follows easily by the definition of the reflection h^* that the restriction $\tilde{\varphi} := \tilde{h}|_{\partial\Delta}$ of $\tilde{h}$ on $\partial\Delta$ is a function of class $\mathcal{C}^{1,\alpha}(\partial\Delta)$. Then the Schwarz integral $T^{SW}\tilde{\varphi}$ gives by Proposition 2.2 a function of class $\mathcal{O}^{1,\alpha}(\Delta)$ whose real part coincides with $\tilde{h}$. ☐

LEMMA 5.3. — *If $u \in \mathcal{C}^{1,\alpha}_\omega(\Delta)$ then $\mathsf{Re}\, \mathsf{u}|_{\partial\Delta^+} \in \mathcal{C}^{1,\alpha}_\omega(\partial\Delta^+)$.*

Proof. — Let $\hat{u}(\zeta,\xi) = \sum u_{kl}\zeta^k\xi^l$ be the polarization of u holomorphic in the bidisc Δ^2 (that is $u(\zeta) = \hat{u}(\zeta,\overline{\zeta})$). Then the function $h(\zeta) = \hat{u}(\zeta,\zeta)$ is of class $\mathcal{O}^{1,\alpha}(\Delta)$ and $h|_{[-1,1]} = u|_{[-1,1]}$. Denote b φ the restriction of $\mathsf{Re}\,(\mathsf{u}-\mathsf{h})$ to $\partial\Delta^+$. Then $\varphi \in \mathcal{C}^{1,\alpha}_0(\partial\Delta^+)$ and by Lemma 5.2 there exists a function $v \in \mathcal{O}^{1,\alpha}(\Delta)$ such that $\mathsf{Re}\,\mathsf{v}|\partial\Delta^+ = \varphi$. Since the function $h+v$ is of class $\mathcal{O}^{1,\alpha}(\Delta)$, its real part gives the desired extension of the function $\mathsf{Re}\,\mathsf{u}|\partial\Delta^+$. ☐

We suppose everywhere below that our almost complex structure J (and therefore A_J in the equation for J holomorphic curves) is a real analytic matrix-valued function given by a convergent power series $\sum a_{kl}z^k\overline{z}^l$ with the radius of convergence big enough. The equation (2.2) on Δ can be rewritten in the form

$$(5.1)\qquad (u - T^{CG}_\Delta A_J(u)\overline{u}_{\overline{\zeta}})_{\overline{\zeta}} = 0,$$

where T^{CG}_Δ denotes the Cauchy - Green transform on Δ. Define the map

$$\Phi_J : \mathcal{C}^{1,\alpha}(\bar{\Delta}) \to \mathcal{C}^{1,\alpha}(\bar{\Delta}),$$

as

$$(5.2)\qquad \Phi_J : u \mapsto u - T^{CG}_\Delta A_J(u)\overline{u}_{\overline{\zeta}}.$$

Equation (5.1) means that u is J-holomorphic if and only if $\Phi_J u$ is holomorphic with respect to J_{st}. The following lemma explains the choice of smoothness classes in this Section and is the principal step in the proof of Theorem 1.1.

LEMMA 5.4. — *For J close to J_{st} the operator Φ_J establishes a diffeomorphism of neighborhoods of zero in the space $\mathcal{C}^{1,\alpha}_\omega(\Delta)$.*

Proof. — First we prove that Φ_J maps the space $\mathcal{C}^{1,\alpha}_\omega(\Delta)$ to itself. Given function $u \in \mathcal{C}^{1,\alpha}_\omega(\Delta)$ denote the function $A_J(u)\overline{u}_{\overline{\zeta}}$ by h. We need to prove that $T^{CG}_\Delta h$ belongs to $\mathcal{C}^{1,\alpha}_\omega(\Delta)$. Consider the polarization $\hat{h}(\zeta,\xi) = \hat{h}(\zeta,\xi)$ of h. By Proposition 2.1 we have the representation

$$(5.3)\qquad T^{CG}_\Delta h(\zeta) = \hat{H}(\zeta,\overline{\zeta}) - \frac{1}{2\pi i}\int_{\partial\Delta}\frac{\hat{H}(\tau,\overline{\tau})d\tau}{\tau-\zeta}.$$

where

$$(5.4)\qquad \hat{H}(\zeta,\xi) = \int_{[0,\xi]}\hat{h}(\zeta,\omega)d\omega$$

is a primitive of $\hat{h}$ with respect to ξ. Let's study the primitive (5.4) of $\hat{h}$ first. We point out that the function $\hat{h}$ is of class $\mathcal{C}^{0,\alpha}(\overline{\Delta}^2)$. Furthermore, the condition (iii) of the definition of the space $\mathcal{C}^{1,\alpha}_{\omega}(\Delta)$ implies that $\partial\hat{h}/\partial\zeta$ is of class $\mathcal{C}^{0,\alpha}(\overline{\Delta}^2)$. Now the derivation of the integral (5.4) with respect to ζ and ξ gives that $\hat{H}$ satisfies conditions (i), (ii), (iii) of the definition of the space $\mathcal{C}^{1,\alpha}_{\omega}(\Delta)$.

By Proposition 2.2 the Cauchy integral in the right hand side of (6.3) represents a function of class $\mathcal{O}^{1,\alpha}(\Delta)$ and so also belongs to the space $\mathcal{C}^{1,\alpha}_{\omega}(\Delta)$.

Thus we obtain that $\Phi_J(u)$ belongs to $\mathcal{C}^{1,\alpha}_{\omega}(\Delta)$. Since the Fréchet derivative of Φ_J with respect to u at $u=0$ and $J=J_{st}$ is the identity map, the lemma follows from the inverse mapping theorem. $\square$

Hence Φ_J is a diffeomorphism between neighborhoods of zero in the manifolds of J-holomorphic and J_{st}-holomorphic maps of class $\mathcal{C}^{1,\alpha}_{\omega}(\Delta)$. In particular, J-holomorphic maps form a Banach submanifold in $\mathcal{C}^{1,\alpha}_{\omega}(\Delta)$ in a neighborhood of zero. We denote this manifold as $\mathcal{O}^{1,\alpha}_{\omega,J}(\Delta)$.

Remark 5.5. — Note that $\mathcal{O}^{1,\alpha}_{\omega,J_{\mathsf{st}}}(\Delta) = \mathcal{O}^{1,\alpha}(\Delta)$ and that $\mathcal{O}^{1,\alpha}_{\omega,J}(\Delta) = \Phi_J(\mathcal{O}^{1,\alpha}(\Delta))$.

We use the notation $\mathcal{O}^{1,\alpha}_{\omega,J,0}(\Delta)$ for the submanifold of such $u\in\mathcal{O}^{1,\alpha}_{\omega,J}(\Delta)$ that $\mathsf{Re}\,\mathsf{u}|_{[-1.1]}\equiv 0$. Its diffeomorphic image under Φ_J we denote as $\mathcal{M}_{J,0} := \Phi_J\left(\mathcal{O}^{1,\alpha}_{\omega,J,0}(\Delta)\right)$. By $R_{\partial\Delta^+}$ denote "taking real part and restriction to $\partial\Delta^+$" operator. One has the following commutative diagram:

$$
\begin{array}{ccccccc}
\mathcal{C}^{1,\alpha}_{\omega}(\Delta) & \supset & \mathcal{O}^{1,\alpha}(\Delta) & \xrightarrow{\Phi_J^{-1}} & \mathcal{O}^{1,\alpha}_{\omega,J}(\Delta) & & \\
 & & \uparrow i & & \uparrow i & & \\
\mathcal{C}^{1,\alpha}_{\omega}(\Delta) & \supset & \mathcal{M}_{J,0} & \xrightarrow{\Phi_J^{-1}} & \mathcal{O}^{1,\alpha}_{\omega,J,0}(\Delta) & \xrightarrow{R_{\partial\Delta^+}} & \mathcal{C}^{1,\alpha}_{0}(\partial\Delta^+)
\end{array} \quad , \tag{5.5}
$$

where both i-s are natural imbeddings. For an unknown map v from $\mathcal{O}^{1,\alpha}_{\omega,J,0}(\Delta)$ and given $\varphi\in\mathcal{C}^{1,\alpha}_{0}(\partial\Delta^+)$ consider the system

$$
\begin{cases} R_{\partial\Delta^+}v = \varphi, \\ \mathsf{Im}\,\mathsf{v}(0) = \mathsf{a}. \end{cases} \tag{5.6}
$$

Fix a J_{st}-holomorphic map $v^0\in\mathcal{O}^{1,\alpha}(\Delta)$ such that $\mathsf{Re}\,\mathsf{v}^0|_{[-1,1]}\equiv 0$ and set $\varphi^0 = \mathsf{Re}\,\mathsf{v}^0|_{\partial\Delta^+}$.

Lemma 5.6. — *For real analytic J close enough to J_{st} in $\mathcal{C}^{1,\alpha}$-norm there exists a neighborhood U of φ^0 in $\mathcal{C}^{1,\alpha}_{0}(\partial\Delta^+)$, a neighborhood U' of $a^0 := \mathsf{Im}\,\mathsf{v}^0(0)$ in $\mathbb{R}^n$ and a neighborhood V of v^0 in $\mathcal{C}^{1,\alpha}_{\omega}(\Delta)$ such that for $\varphi\in U$ and $a\in U'$ the system (5.6) admits a unique solution $v\in V\cap\mathcal{O}^{1,\alpha}_{\omega,J,0}(\Delta)$.*

Proof. — Setting $v = \Phi_J^{-1} w$ we replace (5.6) by the boundary value problem for an unknown map $w \in \mathcal{M}_{J,0}$, *i.e.*, we shall write it as

$$\begin{cases} R_{\partial\Delta^+}\Phi_J^{-1} w = \varphi, \\ \mathsf{Im}\,\Phi_J^{-1}\mathsf{w}(0) = \mathsf{a}. \end{cases} \tag{5.7}$$

At $J = J_{\mathsf{st}}$ that $\Phi_J = \mathsf{Id}$ and $\mathcal{M}_{J_{\mathsf{st}},0} = \mathcal{O}_0^{1,\alpha}(\Delta) := \{w \in \mathcal{O}^{1,\alpha}(\Delta) : \mathsf{Re}\,\mathsf{w}|[-1,1] \equiv 0\}$. The surjectivity condition for the operator obtained by the linearization of (5.7) at $w^0 = v^0$ and $J = J_{\mathsf{st}}$ and of the operator Φ_J^{-1} is reduced to the resolution of the following system

$$\begin{cases} R_{\partial\Delta^+}\dot{w} = \psi, \\ \mathsf{Im}\,\dot{\mathsf{w}}(0) = \mathsf{a}, \end{cases} \tag{5.8}$$

for an arbitrary given function $\psi \in \mathcal{C}_0^{1,\alpha}(\partial\Delta^+)$, arbitrary $a \in \mathbb{R}^n$ and an unknown map $\dot{w} \in \mathcal{O}_0^{1,\alpha}(\Delta)$. By Lemma 5.2 we obtain a solution for any given right hand side of (5.8). The uniqueness of $\dot{w}$ is obvious. Therefore the linearization of (5.7) is a bijective operator at $J = J_{\mathsf{st}}$. By continuity it will be bijective from $T_0\mathcal{M}_{J,0}$ to $\mathcal{C}_0^{1,\alpha}(\partial\Delta^+) \oplus \mathbb{R}^n$ also for J close to J_{st}. Now the Implicit Function Theorem implies the desired statement. □

5.2. General case

Now we prove Theorem 1.1. According to Theorem 1.3 (which we will prove in the next section) the map u is $\mathcal{C}^\infty$ smooth up to the arc β. We replace the unit disc by the upper semi-disc Δ^+ and β by the interval $(-1,1)$. By the classical results on the interior regularity of pseudo-holomorphic maps we can assume that the map u is real analytic in a neighborhood of $\bar{\Delta}^+ \backslash [-1,1]$. Furthermore, the statement is local so shrinking Δ^+ if necessary we assume that u is of class $\mathcal{C}^\infty(\overline{\Delta^+})$. We can also assume that X is the unit ball of $\mathbb{C}^n$ equipped with a real analytic almost complex structure J with $J(0) = J_{st}$ and that $W = i\mathbb{R}^n$ and $u(0) = 0$.

Since our considerations are local, we can reduce them to the case of a small deformation of the standard complex structure. Indeed, our map u admits the expansion $u(\zeta) = b\zeta + o(|\zeta|)$ near the origin. For $t > 0$ consider the real analytic structures $J_t(z) = J(tz)$. They tend to J_{st} as t tends to 0. Maps $u^t(\zeta) = t^{-1}u(t\zeta)$ are J_t-holomorphic and tend to the map $u^0 : \zeta \longrightarrow b\zeta$ as $t \longrightarrow 0$ which is viewed as a J_{st}-holomorphic map.

For t small enough let v^t be the solution of the boundary-value problem (5.6) with $\varphi^t := \mathsf{Re}\,\mathsf{u}^{\mathsf{t}}|\partial\Delta^+$ given by Lemma 5.6. This solution is unique in the class of solutions real analytically extendable past $(-1,1)$. However, this still does not give the desired analyticity of u^t since this boundary-value problem could admit solutions in other smoothness classes of maps. So we need the uniqueness statement of Lemma 4.1. Let $H:\Delta\to\Delta^+$ be a biholomorphic map fixing the points -1 and 1. Then H is of class $\mathcal{C}^{1/2}(\overline{\Delta})$ and extends analytically through the open upper and lower semi-circles. In particular, $H\in L^{1,p}(\Delta)$ for any $p<4$. Since $v^t\in\mathcal{C}^{1,\alpha}(\overline{\Delta}^+)$ the composition $v^t\circ H$ is in $L^{1,p}(\Delta)$. At the same time $\varphi^t\circ H$ belongs to $T^{1,p}$ for every $p<4$. Applying the uniqueness statement of Lemma 4.1 we conclude that $v^t\circ H = u^t\circ H$ which implies that the maps u^t are real analytic up to $(-1,1)$ for t small enough. This proves finally that u extends as a real analytic map past $(-1,1)$. Since it satisfies the real analytic condition (2.1) on an open set, the extension is a J-holomorphic map. This proves the Theorem 1.1.

6. Compactness

6.1. Compactness theorem: definitions

We start with recalling some notions and definitions relevant to the formulation of Gromov compactness theorem as it is stated in [9, 10]. For a more detailed exposition we refer to these papers.

1. Structures. We fix a Riemannian manifold (X,h), a compact subset $K\Subset X$ and a sequence of almost-complex structures J_n of class $\mathcal{C}^{k,\alpha}$ on X, which converge on K in $\mathcal{C}^{k,\alpha}$-topology to an almost-complex structure J, $k\geqslant 0, 0<\alpha<1$. The latter is supposed to be defined on the whole X. Areas of J_n-complex curves will be measured with respect to the "hermitizations" $h_{J_n}(\cdot,\cdot):=\frac{1}{2}(h(\cdot,\cdot)+h(J_n\cdot,J_n\cdot))$ of h (which converge to h_J).

2. Curves. We are given a sequence $\{C_n\}$ of *nodal curves with boundary*, parametrized by the same compact, oriented real surface $(\Sigma,\partial\Sigma)$ with boundary. We suppose that some parametrization $\delta_n:\bar\Sigma\to\bar C_n$ are given as well as some J_n-holomorphic maps $u_n:C_n\to X$ such that (C_n,u_n) becomes a *stable curve over* (X,J_n), see Definitions 2.1 - 2.3 from [10].

3. Boundedness of areas. We suppose that h_{J_n}-areas of $u_n(C_n)$ are uniformly bounded and that $u_n(C_n)\subset K$ for all n.

4. Weak transversality. Let $f : W \to X$ be an immersion of class $\mathcal{C}^{k+1,\alpha}$ of a real n-dimensional manifold W into a real $2n$-dimensional manifold X and $x \in f(W)$ a point of self-intersection, so that $f^{-1}(x) = \{w_1, ..., w_d\} \subset W$ with $d \geqslant 2$. We say that $f(W)$ has *weakly transverse self-intersection in* x if there exist neighborhoods $U_i \subset W$ of w_i such that for any pair $w_i \neq w_j$ the intersection $f(U_i) \cap f(U_j)$ is a $\mathcal{C}^{k+1,\alpha}$-submanifold in X of dimension equal to $\mathsf{dim}\left(\mathsf{df}(\mathsf{T}_{\mathsf{w_i}}\mathsf{W}) \cap \mathsf{df}(\mathsf{T}_{\mathsf{w_j}}\mathsf{W})\right)$.

5. Totally real submanifolds. We fix a collection $\{W_i\}_{i=1}^m$ of real manifolds of real dimension $n = \mathsf{dim}_{\mathbb{C}}\mathsf{X}$ and J-totally real immersions $f_i : W_i \to X$, i=1,...,m, of class $\mathcal{C}^{k+1,\alpha}$. We call $\boldsymbol{W} = \{(W_i, f_i)\}_{i=1}^m$ the J-totally real immersed submanifold of X. We suppose that the immersion $\boldsymbol{f} = \{f_i\}$ has only *weakly transverse* self-intersections. More precisely that means that each f_i has only weakly transverse self-intersections and every pair f_i, f_j intersect in the same manner.

Furthermore, we fix a sequence $\boldsymbol{W}_n = \{(W_i, f_{n,i})\}_{i=1}^m$ of J_n-totally real immersed submanifolds (with weakly transverse self-intersections), which converge in $\mathcal{C}^{k+1,\alpha}$-sense to a J-totally real submanifold $\boldsymbol{W} = \{(W_i, f_i)_{i=1}^m\}$ (again, with weakly transverse self-intersections!).

Remark 6.1. — The condition of a weak transverse self-intersection is crucial in applications. The reason is that if, for example, a totally real manifold W is immersed into $\mathbb{C}^n$ with transversal self-intersections then its product with, say a circle $W \times \mathbb{S}^1$ will be immersed into $\mathbb{C}^{n+1}$, but its self-intersections will be only weakly transverse! This construction repeatedly occurs in applications. See more about this in [10].

6. Boundary conditions. We fix a collection of arcs with disjoint interiors $\boldsymbol{\beta} = \{\beta_k\}_{k=1}^M$, which defines a decomposition of the boundary $\partial\Sigma = \cup_k \beta_k$. We assume that every boundary point b of Σ, which is mapped by the parametrization $\delta_n : \bar{\Sigma} \to C_n$ into a boundary node a of C_n, is the endpoint of two arcs and that a itself is an endpoint for four arcs $\delta_n(\beta_k) =: \beta_{n,k}$. Our basic assumption is that the same collection $\boldsymbol{\beta}$ serves for *all* curves C_n. Totally real boundary conditions $(\boldsymbol{W}, \boldsymbol{\beta}, \boldsymbol{u}^{(b)})$ is the data, which includes $\boldsymbol{W} = \{(W_i, f_i)\}$, $\boldsymbol{\beta} = \{\beta_k\}$ and continuous maps $\boldsymbol{u}^{(b)} = \{u_k^{(b)} : \beta_k \to W_i\}, i = 1, ..., m, k = 1, ..., M$. Here several different β_k-s could be mapped into the same W_i.

We shall suppose that the sequence of curves $(\overline{C}_n, u_n)$ satisfy the *totally real boundary conditions* $(\boldsymbol{W}_n, \boldsymbol{\beta}, \boldsymbol{u}_n^{(b)})$ in the sense that there are given continuous maps $u_{n,k}^{(b)} : \beta_k \to W_i$ with $f_{n,i} \circ u_{n,k}^{(b)} = u_n|_{\beta_{n,k}} = u_n \circ \delta_n \mid_{\beta_k}$. Here $\boldsymbol{u}_n^{(b)} = \{u_{n,k}^{(b)}\}$.

7. Description of convergence. Compactness theorem states that under the assumptions described above there exist a subsequence (C_{n_k}, u_{n_k}) which converge in the following sense.

Definition 6.2. — *We say that the sequence $(\overline{C}_n, u_n)$ of stable J_n-complex curves over X, which satisfies the totally real boundary conditions $(\boldsymbol{W}_n, \beta, \boldsymbol{u}_n^{(b)})$ converges up to the boundary to a stable J-complex curve $(\overline{C}, u)$ over X if the parametrization $\sigma_n : \overline{\Sigma} \to \overline{C}_n$ and $\sigma : \overline{\Sigma} \to \overline{C}$ can be chosen in such a way that the following holds:*

(i) *$u_n \circ \sigma_n$ converges to $u \circ \sigma$ in $C^0(\overline{\Sigma}, X)$-topology; moreover, $u_n^{(b)}$ uniformly on $\partial\Sigma$ converge to some $u^{(b)}$ such that $(\bar{C}, u)$ satisfies the totally real boundary condition $(\boldsymbol{W}, \boldsymbol{\beta}, \boldsymbol{u}^{(b)})$;*

(ii) *if $\{a_k\}$ is the set of the nodes of C_∞ and $\gamma_k := \sigma^{-1}(a_k)$ are the corresponding circles and arcs in $\overline{\Sigma}$, then for any compact subset $R \Subset \overline{\Sigma} \backslash \cup_k \gamma_k$ there exists $n_0 = n_0(K)$, such that $\sigma_n^{-1}(\{a_k\}) \cap K = \varnothing$ for all $n \geqslant n_0$ and complex structures $\sigma_n^* j_{C_n}$ smoothly converge to $\sigma^* j_C$ on R, $n \geqslant n_0$;*

(iii) *on any compact subset $R \Subset \overline{\Sigma} \backslash \cup_k \gamma_k$ the convergence $u_n \circ \sigma_n \to u \circ \sigma$ is in $\mathcal{C}^{k+1,\alpha}$-topology.*

Note that in this definition it is *ad hoc* supposed that all $\bar{C}_n$-s and also $\bar{C}$ can be parametrized by the same real surface $\bar{\Sigma}$ and that one is allowed to change parametrization from initial δ_n to an appropriate σ_n. Note also that no convergence of $\boldsymbol{u}_n^{(b)})$ is a priori supposed. It comes as the statement of the Theorem.

6.2. Generalized Giraud-Calderon-Zygmund Inequality

Recall the following Giraud Inequality (or estimate): *for all* $1 < p < \infty$ *there exists a constant* G_p *such that for all* $u \in L^p(\Delta, \mathbb{C}^n)$ *one has*

$$\left\|(\partial \circ T_\Delta^{CG})(u)\right\|_{L^p(\Delta)} \leqslant G_p \cdot \|u\|_{L^p(\Delta)} . \tag{6.1}$$

In Hölder norms an analogous statement is due to Calderon and Zygmund. Namely: *for every* $0 < \alpha < 1$ *there exists* C_α *such that for all* $u \in \mathcal{C}^\alpha(\Delta, \mathbb{C}^n)$ *one has*

$$\left\|(\partial \circ T_\Delta^{CG})(u)\right\|_{C^\alpha(\Delta)} \leqslant G_\alpha \cdot \|u\|_{C^\alpha(\Delta)} . \tag{6.2}$$

We shall need the following generalization of (6.1) and (6.2) to $\bar{\partial}$-type operators. Let $J = J(\zeta)$ be a bounded (resp. $\mathcal{C}^\alpha$-continuous) operator in the trivial bundle $\Delta \times \mathbb{R}^{2n}$ which satisfies $J^2(\zeta) \equiv -\mathsf{Id}$ for all $\zeta \in \Delta$. It defines a natural $\bar{\partial}$-type operator $\bar{\partial}_J : L^{1,p}(\Delta, \mathbb{R}^{2n}) \to L^p(\Delta, \mathbb{R}^{2n})$ (resp. $\bar{\partial}_J : \mathcal{C}^{1,\alpha}(\Delta, \mathbb{R}^{2n}) \to \mathcal{C}^{\alpha}(\Delta, \mathbb{R}^{2n})$) as follows

$$\bar{\partial}_J u = \frac{\partial u}{\partial \xi} + J(\zeta)\frac{\partial u}{\partial \eta}. \tag{6.3}$$

Lemma 6.3. — *For any $p > 2$ there exist $\varepsilon_p > 0$ and $C(p, \|J - J_{\mathsf{st}}\|_{L^\infty}) < \infty$ $\big($resp. for any $0 < \alpha < 1$ there exist $\varepsilon_\alpha > 0$ and $C(\alpha, \|J - J_{\mathsf{st}}\|_\alpha)\big)$ such that for any $J \in L^\infty(\Delta, End(\mathbb{R}^{2n}))$, $J^2 \equiv -\mathsf{Id}$ with $\|J - J_{\mathsf{st}}\|_{L^\infty(\Delta)} < \varepsilon_p$ $\big($resp. any $J \in \mathcal{C}^\alpha$ with $\|J - J_{\mathsf{st}}\|_{\mathcal{C}^\alpha} < \varepsilon_\alpha\big)$ any $u \in L^p(\Delta, \mathbb{R}^{2n})$ $\big($resp. any $u \in \mathcal{C}^\alpha(\Delta, \mathbb{R}^{2n})\big)$ with compact support in Δ one has*

$$\|du\|_{L^p(\Delta,\mathbb{R}^{2n})} \leqslant C(p, \|J - J_{\mathsf{st}}\|_{L^\infty}) \left\|\bar{\partial}_J u\right\|_{L^p(\Delta,\mathbb{R}^{2n})}, \tag{6.4}$$

and respectively

$$\|du\|_{\mathcal{C}^\alpha(\Delta,\mathbb{R}^{2n})} \leqslant C(\alpha, \|J - J_{\mathsf{st}}\|_{\mathcal{C}^\alpha}) \left\|\bar{\partial}_J u\right\|_{\mathcal{C}^\alpha(\Delta,\mathbb{R}^{2n})}. \tag{6.5}$$

Proof. — For the proof of (6.4) see Lemma 1.2 in [9]. The proof of (6.5) follows the same lines. For $u \in \mathcal{C}^\alpha(\mathbb{C}, \mathbb{R}^{2n})$ it holds that

$$\begin{aligned} \left\|(\bar{\partial}_J \circ T_\Delta^{CG} - \bar{\partial}_{J_{\mathsf{st}}} \circ T_\Delta^{CG})u\right\|_{\mathcal{C}^\alpha(\Delta)} &\leqslant \|J - J_{\mathsf{st}}\|_{\mathcal{C}^\alpha(\Delta)} \cdot \left\|d(T_\Delta^{CG} u)\right\|_{\mathcal{C}^\alpha(\Delta)} \\ &\leqslant \|J - J_{\mathsf{st}}\|_{\mathcal{C}^\alpha(\Delta)} (1 + G_\alpha) \|u\|_{\mathcal{C}^\alpha(\Delta)}, \end{aligned} \tag{6.6}$$

where G_α is the constant from (6.2). For the standard structure in $\mathbb{C}^n$ the operator $\bar{\partial}_{J_{\mathsf{st}}} \circ T_\Delta^{CG} : \mathcal{C}^\alpha(\Delta, \mathbb{C}^n) \to \mathcal{C}^\alpha(\Delta, \mathbb{C}^n)$ is the identity. So from (6.6) we see that there exists $\varepsilon_\alpha = \frac{1}{1+G_\alpha}$ such that if $\|J - J_{\mathsf{st}}\|_{\mathcal{C}^\alpha(\Delta)} < \varepsilon_\alpha$, then $\bar{\partial}_J \circ T_\Delta^{CG} : \mathcal{C}^\alpha(\Delta, \mathbb{C}^n) \to \mathcal{C}^\alpha(\Delta, \mathbb{C}^n)$ is an isomorphism. Moreover, since $\bar{\partial}_J \circ T_\Delta^{CG} = \bar{\partial}_{J_{\mathsf{st}}} \circ T_\Delta^{CG} + (\bar{\partial}_J - \bar{\partial}_{J_{\mathsf{st}}}) \circ T_\Delta^{CG}$, we have

$$(\bar{\partial}_J \circ T_\Delta^{CG})^{-1} = \left[\mathsf{Id} + (\bar{\partial}_{\mathsf{J}} - \bar{\partial}_{\mathsf{J_{st}}}) \circ \mathsf{T}_\Delta^{\mathsf{CG}}\right]^{-1} = \sum_{n=0}^{\infty} (-1)^n [(\bar{\partial}_J - \bar{\partial}_{J_{\mathsf{st}}}) \circ T_\Delta^{CG}]^n. \tag{6.7}$$

This shows, in particular, that $(\bar{\partial}_J \circ T_{\mathbb{C}}^{CG})^{-1}$ does not depend on $0 < \alpha < 1$ provided that $\|J - J_{\mathsf{st}}\|_{\mathcal{C}^\alpha(\Delta)} < \varepsilon_\alpha$.

Put $h = u - T^{CG}_{\Delta} \circ \bar\partial_{J_{\mathsf{st}}} u$. Then $\bar\partial_{J_{\mathsf{st}}} h = 0$. So h is holomorphic and descends at infinity. Thus $h \equiv 0$, which implies $u = (T_{CG} \circ \bar\partial_{J_{\mathsf{st}}})u$. Write

$$
\begin{aligned}
\|du\|_{\mathcal{C}^{\alpha}(\Delta)} &\leqslant (1 + C_{\alpha}) \left\|\bar\partial_{J_{\mathsf{st}}} u\right\|_{\mathcal{C}^{\alpha}(\Delta)} \\
&= (1 + C_{\alpha}) \left\|(\bar\partial_{J} \circ T^{CG}_{\Delta})^{-1} (\bar\partial_{J} \circ T^{CG}_{\Delta}) \bar\partial_{J_{\mathsf{st}}} u\right\|_{\mathcal{C}^{\alpha}(\Delta)} \\
&= (1 + C_{\alpha}) \left\|(\bar\partial_{J} \circ T^{CG}_{\Delta})^{-1} (\bar\partial_{J} u\right\|_{\mathcal{C}^{\alpha}(\Delta)} \\
&\leqslant (1 + C_{\alpha}) \sum_{n=0}^{\infty} \left\|(\bar\partial_{J} - \bar\partial_{J_{\mathsf{st}}}) \circ T^{CG}_{\Delta}\right\|^{n}_{\mathcal{C}^{\alpha}(\Delta)} \cdot \left\|\bar\partial_{J} u\right\|_{\mathcal{C}^{\alpha}(\Delta)} \\
&\leqslant C(\alpha, \|J - J_{\mathsf{st}}\|_{\mathcal{C}^{\alpha}}) \cdot \left\|\bar\partial_{J} u\right\|_{\mathcal{C}^{\alpha}(\Delta)},
\end{aligned} \tag{6.8}
$$

provided that $\|J - J_{\mathsf{st}}\|_{\mathcal{C}^{\alpha}} < \varepsilon_{\alpha}$. □

Corollary 6.4. — *If $J_n \to J$ in $\mathcal{C}^{k,\alpha}$-norm on compact $K \Subset X$, $k \geqslant 0, 0 < \alpha < 1$, and J_n-holomorphic maps $u_n : \Delta \to K \Subset X$ uniformly converge to $u : \Delta \to X$ then u_n converge to u in $\mathcal{C}^{k+1,\alpha}$-topology on compacts in Δ.*

Proof. — This will be done in three steps.

Step 1. $\mathcal{C}^{\alpha}$-convergence. First we prove the $\mathcal{C}^{\alpha}$-convergence (which is implicitly contained in [9]). For this we need only *uniform* convergence of *continuous* structures. Consider all u_n as a sections of the trivial bundle $\Delta \times \mathbb{R}^{2n}$ which are holomorphic with respect to the pulled back structures $J_n \circ u_n$, *i.e.*, $\bar\partial_{J_n \circ u_n} u_n = 0$. Theorem 6.2.5 from [14] implies that for every $2 \leqslant p < \infty$

$$
\|u_n\|_{L^{1,p}(\Delta(1/2))} \leqslant C_p\big(\mu(J_n \circ u_n)\big) \, \|u_n\|_{L^2(\Delta)}, \tag{6.9}
$$

where the constant $C_p\big(\mu(J_n \circ u_n)\big)$ crucially depends not only on p but also on the modulus of continuity of $J_n \circ u_n$. This gives us the boundedness of u_n in $L^{1,p}(\Delta(1/2))$ for all p and therefore in $\mathcal{C}^{\gamma}$ for all $\gamma < 1$ by Sobolev imbedding theorem. And the last in its turn by the Ascoli theorem implies the $\mathcal{C}^{\gamma}$-convergence for all γ, in particular, for our α in question.

Step 2. $\mathcal{C}^{1,\alpha}$-convergence. Take a cut-off function φ and write using (6.5) (all constants below are denoted by the same letter C, but are different):

$$
\begin{aligned}
(6.10)\quad \left\|d\left[\varphi(u_n-u_m)\right]\right\|_{\mathcal{C}^\alpha} &\leqslant C\left\|\bar\partial_{J_n\circ u_n}\left[\varphi(u_n-u_m)\right]\right\|_{\mathcal{C}^\alpha}\\
&\leqslant C\left\|u_n-u_m\right\|_{\mathcal{C}^\alpha}+C\left\|\varphi\bar\partial_{J_n}u_m\right\|_{\mathcal{C}^\alpha}\\
&\leqslant C\left\|u_n-u_m\right\|_{\mathcal{C}^\alpha}+\left\|J_n\circ u_n-J_m\circ u_m\right\|_{\mathcal{C}^\alpha}\left\|\varphi du_m\right\|_{\mathcal{C}^\alpha}\\
&\leqslant C\left\|u_n-u_m\right\|_{\mathcal{C}^\alpha}+C\left\|J_n\circ u_n-J_m\circ u_m\right\|_{\mathcal{C}^\alpha}\left[\left\|d(\varphi u_m)\right\|_{\mathcal{C}^\alpha}+\left\|u_m\right\|_{\mathcal{C}^\alpha}\right].
\end{aligned}
$$

In the same manner from (6.5) we get that

$$
\left\|d(\varphi u_m)\right\|_{\mathcal{C}^\alpha}\leqslant C\left\|\bar\partial_{J_m\circ u_m}(\varphi u_m)\right\|_{\mathcal{C}^\alpha}\leqslant C\left\|u_m\right\|_{\mathcal{C}^\alpha},
$$

and the latter is bounded. (6.10) implies now $\mathcal{C}^{1,\alpha}$-convergence of u_n to u.

Step 3. $\mathcal{C}^{k+1.\alpha}$-convergence. We already lifted an almost complex structure from the manifold X to its tangent bundle TX, see (3.4). Since u_n already converge in $\mathcal{C}^{1,\alpha}$-topology by Step 2, lifts u_n^c converge in $\mathcal{C}^\alpha$-topology. And therefore again by Step 2 they converge in $\mathcal{C}^{1,\alpha}$-topology. The rest is obvious. □

6.3. Proof of the Compactness theorem

In [10] it was proved that there exists a subsequence u_{n_k} which converges as in Definition 6.2 with the only difference that in (iii) the convergence was in $L^{1,p}$-topology for all $p<\infty$. That implies $\mathcal{C}^\alpha$-convergence for all $0<\alpha<1$. We recall that structures J_n in [10] where supposed to be only continuous and uniformly, *i.e.*, $\mathcal{C}^0$-convergent to J. All we need to do here is to improve convergence to $\mathcal{C}^{k+1,\alpha}$ in the case when J_n converge in $\mathcal{C}^{k,\alpha}$ and W_n in $\mathcal{C}^{k+1,\alpha}$.

The statement we need is purely local and therefore the totally real manifolds can be supposed to be imbedded. β, as before, stays for the interval $(-1,1)$.

LEMMA 6.5. — *Let $k\geqslant 0$ and let a sequence $\{J_n\}$ of almost complex structures of class $\mathcal{C}^{k,\alpha}$ converge in $\mathcal{C}^{k,\alpha}$-topology to J, a sequence $\{W_n\}$ of imbedded J_n-totally real submanifolds of class $\mathcal{C}^{k+1,\alpha}$ converge in $\mathcal{C}^{k+1,\alpha}$-topology to a J-totally real W, and a sequence $u_n:(\Delta^+,\beta)\to(X,W_n)$ of J_n-holomorphic maps of class $\mathcal{C}^0\cap L^{1,2}$ converge in $\mathcal{C}^0\cap L^{1,2}$-topology to $u:(\Delta^+,\beta)\to(X,W)$. Then u_n converge to u in $\mathcal{C}^{k+1,\alpha}$-sense up to β.*

Proof. — We start with the case $k=0$. An obvious modification of Lemma 3.1 provides us a sequence $\{\varphi^n\}$ of $\mathcal{C}^{1,\alpha}$- diffeomorphisms converging in $\mathcal{C}^{1,\alpha}$-topology to φ and such that they all satisfy the properties 1)-3) stated there. Repeating the considerations of the proof of Lemma 3.2 to each J_n and u_n we get a uniform bound on the norms of "extensions by reflection" $\|\tilde{u}_n\|_{\mathcal{C}^{1,\alpha}}$ and this implies the statement of the Lemma for $k = 0$ by Corollary 6.4.

Having convergence in $\mathcal{C}^{1,\alpha}$ topology we obtain the convergence in higher regularity classes via geometric bootstrap of subsection 3.3. Really, if $J_n \to J$ in $\mathcal{C}^{1,\alpha}$-topology then $J_n^c \to J^c$ in $\mathcal{C}^{\alpha}$-topology. Further we lift J_n-holomorphic maps $u_n : \Delta \to X$ to J^c-holomorphic maps $u_n^c : \Delta \to TX$ and apply the case $k = 0$. □

7. Open Questions

At the end we would like to turn the attention of a reader to some open questions.

1. Let (X, J) be a real analytic almost complex manifold and W a real analytic J-totally real submanifold of X. Let C^+ be J-complex curve in $X \setminus W$. Does there exists a neighborhood V of W and a J-complex curve C^- in $V \setminus W$ (reflection of C^+) such that $\overline{(C^+ \cup C^-)} \cap V$ is a J-complex curve in V?

For integrable J the answer is yes and is due to H. Alexander, see [1].

2. The following question is a particular case of the previous one. Let C be a J-complex curve in the complement of a point. Will its closure $\bar{C}$ be a J-complex curve?

3. This question was communicated to us by J.-C. Sikorav. Define a J-holomorphic map as a differentiable map $u : \Delta \to X$ such that (2.1) is satisfied at every point. Prove that $u \in L^{1,2}_{loc}$ (and therefore u is a J-holomorphic map in the usual sense).

BIBLIOGRAPHY

[1] H. Alexander, "Continuing 1-dimensional Analytic Sets.", *Math. Ann.* **191** (1971), p. 143-144.

[2] L. Bers, F. John & M. Schechter, *Partial differential equations*, J. Wiley and Sons, 1964.

[3] C. Carathéodory, "Zum Schwarzschen Spiegelungsprinzip", *Comm. Math. Helv.* **19** (1946), no. 1, p. 263-278.

[4] E. Chirka, "Regularity of boundaries of analytic sets", *Math. USSR Sbornik* **43** (1983), p. 291-335.

[5] B. Coupet, H. Gaussier & A. Sukhov, "Fefferman's mapping theorem on almost complex manifold in complex dimension two", *Math. Z.* **250** (2005), p. 59-90.

[6] V. Fromm, "Gromov Compactness in Hölder Spaces and Minimal Connections on Jet Bundles", math. SG/0808.0415.

[7] H. Gaussier & A. Sukhov, "On the geometry of model almost complex manifolds with boundary", *Math. Z.* **254** (2006), p. 567-589.

[8] S. Ivashkovich & J.-P. Rosay, "Schwarz-type lemmas for solutions of $\overline{\partial}$-inequalities and complete hyperbolicity of almost complex manifolds", *Annales Inst. Fourier* **54** (2004), p. 2387-2435.

[9] S. Ivashkovich & V. Shevchishin, "Gromov Compactness Theorem for J-Complex Curves with Boundary", *Int. Math. Res. Notices* **22** (2000), p. 1167-1206.

[10] ———, "Reflection Principle and J-Complex Curves with Boundary on Totally Real Immersions", *Communications in Contemporary Mathematics* **4** (2002), p. 65-106.

[11] L. Lempert & R. Szöke, "The tangent bundle of an almost complex manifold", *Canad. Math. Bull.* **44** (2001), p. 70-79.

[12] D. McDuff & D. Salamon, *J-holomorphic curves and symplectic topology*, AMS Colloquium Publ., vol. 52, AMS, Providence, RI, 2004.

[13] V. Monakhov, *Boundary-value problems with free boundary for elliptic systems of equations*, Translations of Mathematical Monographs, vol. 57, AMS, Providence, RI, 1983, 522 pp. (Originally published by Nauka, Novosibirsk, 1977).

[14] C. Morrey, *Multiple integrals in the calculus of variations*, Springer Verlag, 1966.

[15] H. A. Schwarz, "Über einige Abbildungsaufgaben", *Journal für reine und angewandte Mathematik* **70** (1869), p. 105-120 (see pages 106-107), See also Gesammelte mathematische Abhandlungen, Springer (1892), 66-67. Or the Second Edition, Bronx, N.Y., Chelsea Pub. Co. (1972).

[16] J.-C. Sikorav, "Some properties of holomorphic curves in almost complex manifolds", in *Holomorphic curves in symplectic geometry* (M. Audin, J. Lafontaine, ed.), Progress in Mathematics, vol. 117, Birkhäuser, 1994, Ch. V, p. 165-189.

[17] H. Triebel, *Theory of Function Spaces*, Birkhäuser, 1983.

[18] I. N. Vekua, *Generalized analytic functions*, Fizmatgiz, Moscow, 1959, English translation - Pergamon Press, London, and Addison-Welsey, Reading, Massachusetts (1962).

[19] K. Yano & S. Ishihara, *Tangent and cotangent bundles*, Marcel Dekker, NY, 1973.

Manuscrit reçu le 30 juin 2009,
accepté le 23 novembre 2009.

Sergey IVASHKOVICH
U.F.R. de Mathématiques
Université de Lille-1
59655 Villeneuve d'Ascq (France)
and
IAPMM Acad. Sci. Ukraine
Lviv, Naukova 3b, 79601 Ukraine (Ukraine)
ivachkov@math.univ-lille1.fr

Alexandre SUKHOV
U.F.R. de Mathématiques
Université de Lille-1
59655 Villeneuve d'Ascq (France)
sukhov@math.univ-lille1.fr