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GEOMETRY OF THE GENUS 9 FANO 4-FOLDS

by Frédéric HAN

Abstract. — We study the geometry of a general Fano variety of dimension four, genus nine, and Picard number one. We compute its Chow ring and give an explicit description of its variety of lines. We apply these results to study the geometry of non quadratically normal varieties of dimension three in a five dimensional projective space.

Résumé. — On étudie la géométrie d'une variété générale de Fano de dimension quatre, de genre neuf, et de nombre de Picard un. On calcule son anneau de Chow, et l'on donne une description simple et explicite de sa variété des droites. On utilise alors ces résultats pour étudier des propriétés géométriques de variétés de dimension 3 non quadratiquement normales dans un espace projectif de dimension cinq.

Introduction:

Let W be a 6 dimensional vector space over the complex numbers endowed with a non degenerate symplectic form ω. Let G_ω be the Grassmannian of ω-isotropic 3-dimensional vector subspaces of W. We will also denote by P_ω the 13 dimensional projective space spanned by G_ω in the Plücker embedding. Considering this embedding, the intersection of G_ω with a generic codimension 2 linear subspace is the Mukai model of a smooth Fano manifold of dimension 4, genus 9, index 2 and Picard number 1.

On a genus 9 Fano variety with Picard number 1, Mukai's construction gives a natural rank 3 vector bundle, but in dimension 4, another phenomena appears. In the first part of this article, we will explain how to construct on a general Fano 4-fold B of genus 9, a canonical set of four stable vector bundles of rank 2, and prove that they are rigid. In next parts, we study the consequences on the geometry of the 4-fold.

Keywords: Fano manifold, variety of lines, secant variety, quadratic normality, vector bundles, virtual section, symplectic grassmannian.
Math. classification: 14J45, 14J35, 14J60, 14J30, 14M15, 14M07.

Indeed, this “four-ality” (cf [12]) is also present in the geometry of the lines included in this Fano 4-fold, and also in its Chow ring. In section 2, we show explicitly that the variety of lines in B is a hyperplane section of $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$. Then in section 3, we compute the Chow ring of B which appears to have a rich structure in codimension 2.

The four bundles can embed B in a Grassmannian $G(2,6)$, and the link with the order one congruence of lines discovered by E. Mezzetti and P. de Poi in [14] is explained in section 4. In particular we prove that the generic Fano variety of genus 9 and dimension 4 can be obtained by their construction, and explain the choices involved. We also describe in this part the normalization of the non quadratically normal variety they constructed, and also its variety of plane cubics.

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Notations. — In all the paper but section 4.1, B will be a general double hyperplane section of G_w. For any u in G_w, the corresponding plane of $\mathbb{P}(W)$ will be noted π_u. Furthermore, the restriction of a hyperplane H in P_w to G_w will be noted $\bar{H}$.

1. Construction of rank 2 vector bundles on B

This part is devoted to the construction of a canonical set of four stable and rigid rank two vector bundles on B using a classical technique of modification. Those bundles were already known to A. Iliev and K. Ranestad in [8] where they are constructed by projection. They described the link between some moduli spaces of vector bundles in terms of linear sections of the dual variety of G_w, with main application to the genus 9 Fano threefold case. Note also that many of the results of this section are obtained in a universal way in derived categories by A. Kuznetsov in [11], but we detail this short description to use it in the next sections.

Let's first recall some classical geometric properties of G_w (cf [6]). The union of the tangent spaces to G_w is a quartic hypersurface of P_w, so a general line of P_w has naturally 4 marked points. Dually, as the variety B is given by a pencil L of hyperplane sections of G_w, there are in this pencil, four hyperplanes $H_1, \ldots, H_4$ tangent to G_w. Denoting by u_i the unique contact point of H_i with G_w, we will first construct a rank two sheaf on $H_i \cap G_w$ with singular locus u_i, and its restriction to B will be a vector bundle.

1.1. Data associated to a tangent hyperplane section

Let $u \in G_w$, and H be a general hyperplane tangent to G_w at u. We consider the following hyperplane section of G_w:

$$\bar{H}_u = \{v \in G_w, \pi_v \cap \pi_u \neq \emptyset\}.$$

The following lemma is proved in [6]:

Lemma 1.1. — *There exists a conic C in π_u such that $v \in H \cap \bar{H}_u \iff \pi_v \cap C \neq \emptyset$. For H general containing the tangent space of G_w at u, C is smooth. Furthermore, $H \cap \bar{H}_u$ contains the tangent cone $T_u G_w \cap G_w = \{v \in G_w | \dim(\pi_v \cap \pi_u) \geqslant 1\}$ which is embedded in P_w as a cone over a Veronese surface.*

Consequently, we will assume that H is such that C is smooth. Now, we consider the following incidence variety:

$$Z_H = \{(p, v) \in C \times \bar{H} | p \in \pi_v\}$$

and denote by q_1 and q_2 the projections from $C \times G_w$ to C and to G_w.

Corollary 1.2. — *The incidence variety Z_H is smooth.*

The previous lemma implies that for any point p of C the fiber $q_1^{-1}(p)$ consist of all the isotropic planes containing p, so $q_1^{-1}(p)$ is a smooth 3-dimensional quadric (cf [6] proposition 3.2). So Z_H is a fibration in smooth quadratic threefolds over $\mathbb{P}_1$.

Notations. — Let σ be the class of a point of C, and denote by L be the vector space $H^0\mathcal{O}_C(\sigma)$ viewed as SL_2-representation. In all the paper, we will identify L with its dual, and denote by $S_i L$ the symmetric power of order i of L.

For any integers a and b, the sheaf $q_1^*\mathcal{O}_C(a.\sigma) \otimes q_2^*(\mathcal{O}_{G_w}(b))$ on Z_H will be denoted $\mathcal{O}_{Z_H}(a, b)$.

Let K and Q be the tautological[(1)] bundles of rank three on G_w, such that the following sequence is exact:

$$0 \to K \to W \otimes \mathcal{O}_{G_w} \to Q \to 0.$$

Proposition 1.3. — *For $i > 0$ we have $R^i q_{2*}\mathcal{O}_{Z_H}(1,0) = 0$, and the resolution of $q_{2*}\mathcal{O}_{Z_H}(1,0)$ as a $\mathcal{O}_{G_w}$-module is given by the following exact sequence:*

$$(1.1)\quad 0 \to S_3 L \otimes \mathcal{O}_{G_w}(-1) \to L \otimes \bigwedge^2 Q^{\vee} \to L \otimes \mathcal{O}_{G_w} \to q_{2*}\mathcal{O}_{Z_H}(1,0) \to 0$$

(1) Remark that on G_w the bundles Q and $K^{\vee}$ are isomorphic

Proof. — We consider the injection from $q_1^*(\mathcal{O}_C(-2\sigma))$ to $W \otimes \mathcal{O}_{C\times G_w}$ given by the conic C. The incidence Z_H is the locus where the map from $q_1^*(\mathcal{O}_C(-2\sigma)) \oplus q_2^* K$ to $W \otimes \mathcal{O}_{C\times G_w}$ is not injective, hence Z_H is obtained in $C \times G_w$ as the zero locus of a section of the bundle $\mathcal{O}_C(2\sigma) \boxtimes Q$.

Let $\mathcal{K}^{\cdot}$ be the Koszul complex $\bigwedge^{i}(\mathcal{O}_C(-2\sigma) \boxtimes Q^\vee)$ of this section. We obtain proposition 1.3 from the Leray spectral sequence applied to $\mathcal{K}^{\cdot}$ twisted by $\mathcal{O}_{C\times G_w}(\sigma)$. □

Restricting the above surjection $L \otimes \mathcal{O}_{G_w} \to q_{2*}\mathcal{O}_{Z_H}(1,0)$ to the hyperplane section $\bar{H}$, we obtain:

Proposition 1.4. — *The sheaf $\mathcal{E}$ on $\bar{H}$ defined by the following exact sequence:*

$$0 \to \mathcal{E} \to L \otimes \mathcal{O}_{\bar{H}} \to q_{2*}\mathcal{O}_{Z_H}(1,0) \to 0$$

is reflexive of rank 2, has $c_1(\mathcal{E}) = -1$ and is locally free outside u.

Proof. — From lemma 1.1 we have $q_2(Z_H) = H \cap \bar{H}_u$. So set theoretically the support of $q_{2*}\mathcal{O}_{Z_H}(1,0)$ is $H \cap \bar{H}_u$, and $\mathcal{E}$ has rank 2 on $\bar{H}$. The exact sequence (1.1) in proposition 1.3 proves that the $\mathcal{O}_{G_w}$-module $q_{2*}\mathcal{O}_{Z_H}(1,0)$ has local projective dimension 2. So we obtain from the Auslander-Buchsbaum formula that its local depth is 4. Therefore, this module is locally Cohen-Macaulay and $\mathcal{E}$ is reflexive from [5] Corollary 1.5. We also deduce from the sequence (1.1) that the Chern polynomial of $q_{2*}\mathcal{O}_{Z_H}(1,0)$ at order 2 is: $1 - 2c_2(Q)$. But on G_w we have the relation $2c_2(Q) = (c_1(Q))^2$, so $c_1(\mathcal{E}) = -1$ and the support of $q_{2*}\mathcal{O}_{Z_H}(1,0)$ is the reduced scheme $H \cap \bar{H}_u$. As the equation of H annihilates $q_{2*}\mathcal{O}_{Z_H}(1,0)$, we can also consider this sheaf as a $\mathcal{O}_{\bar{H}}$-module with the same local depth. Remarking that $\bar{H} - \{u\}$ is smooth of dimension 5, we obtain from the Auslander-Buchsbaum formula, that the $\mathcal{O}_{\bar{H},x}$-module $(q_{2*}\mathcal{O}_{Z_H}(1,0))_x$ has projective dimension 1 for every closed point x of $\bar{H} - \{u\}$, so $\mathcal{E}$ is locally free outside u. □

Furthermore, we deduce from Bott's theorem on G_w the following:

Corollary 1.5. — *We have the following equality $L = H^0(\mathcal{O}_{Z_H}(1,0)) = H^0(q_{2*}\mathcal{O}_{Z_H}(1,0))$, and for $i > 0$, all the groups $H^i(\mathcal{O}_{Z_H}(1,0))$ and $H^i(q_{2*}\mathcal{O}_{Z_H}(1,0))$ are zero. For $i \geqslant 0$ all the groups $H^i(q_{2*}\mathcal{O}_{Z_H}(1,-1))$ and $H^i(q_{2*}\mathcal{O}_{Z_H}(1,-1))$ are zero.*

Proof. — We will prove that, on the isotropic Grassmannian G_w, all the cohomology groups of the bundles $\bigwedge^{i} Q^\vee$ and $(\bigwedge^{i} Q^\vee)(-1)$ vanish for $i \in \{1,2,3\}$. Indeed, with the notations of [18] 4.3.3 and 4.3.4, they correspond to the partitions $(0,0,-1)$, $(0,-1,-1)$, $(-1,-1,-1)$, $(-1,-1,-2)$,

$(-1,-2,-2)$, $(-2,-2,-2)$. Now recall that the half sum of positive roots is $\rho = (3,2,1)$, so $\alpha+\rho$ either contains a 0 or is $(2,1,-1)$. So in all cases $\alpha+\rho$ is invariant by a signed permutation, and the sheaves have zero cohomology. The corollary is now a direct consequence of these vanishing and of proposition 1.3 and its proof. □

Corollary 1.6. — *We have: $\forall i \geqslant 0, h^i(\mathcal{E}) = h^i(\mathcal{E}(-1)) = 0$. The vector space $V = H^0(\mathcal{E}(1))$ has dimension 6 and $h^i(\mathcal{E}(1)) = 0$ for $i > 0$. Furthermore, $\mathcal{E}(1)$ is generated by its global sections.*

Proof. — The vanishing of the cohomology of $\mathcal{E}$ and $\mathcal{E}(-1)$ is a direct consequence of the definition of $\mathcal{E}$ and of the previous corollary.

To prove the second assertion, we restrict the sequence (1.1) to the hyperplane section $\bar{H}$, and we obtain the following monad[2]:

$$0 \to S_3L \otimes \mathcal{O}_{\bar{H}}(-1) \to L \otimes \bigwedge^2 Q^\vee_{\bar{H}} \to \mathcal{E} \to 0$$

whose cohomology is $\mathcal{T}or^1(q_{2*}(\mathcal{O}_{Z_H}(1,0)), \mathcal{O}_{\bar{H}})$ which is equal to $q_{2*}(\mathcal{O}_{Z_H}(1,-1))$ because we proved in proposition 1.4 that its support is a subscheme of H, so the multiplication by the equation of H from $q_{2*}(\mathcal{O}_{Z_H}(1,0)) \otimes \mathcal{O}_{G_w}(-1)$ to $q_{2*}(\mathcal{O}_{Z_H}(1,0)) \otimes \mathcal{O}_{G_w}$ is the zero map. Twisting this monad by $\mathcal{O}_{\bar{H}}(1)$ and using corollary 1.5, we obtain that $H^0(\mathcal{E}(1))$ is the quotient of $L \otimes W$ by $S_3L \oplus L$ because $W = H^0(Q_{\bar{H}})$ and $Q = (\bigwedge^2 Q^\vee)(1)$. Furthermore, the right part of the monad gives a sujection from $L \otimes Q_{\bar{H}}$ to $\mathcal{E}(1)$. Since $L \otimes Q_{\bar{H}}$ is generated by its global sections, so is $\mathcal{E}(1)$.

The vanishing of $h^i(\mathcal{E}(1))$ for $i > 0$ is a corollary of the vanishing of $h^i(q_{2*}(\mathcal{O}_{Z_H}(1,0))$, $h^i(Q_{\bar{H}})$ and $h^i(\mathcal{O}_{\bar{H}})$ for $i > 0$. □

Remark 1.7. — The two vector spaces V and W of dimension 6 have different roles. More precisely, the conic C gives a marked subspace of W so that we have the following SL_2-equivariant sequences:

$$0 \to S_2L \to W \to S_2L \to 0 \quad \text{and} \quad 0 \to L \to V \to S_3L \to 0$$

1.2. The 4 rank 2 vector bundles on B

The pencil of hyperplanes defining B contains the 4 tangent hyperplanes H_i, so we can apply the previous construction to construct a rank 2 sheaf $\mathcal{E}_i$ on each of the $\bar{H}_i$, and define by E_i the restriction of $\mathcal{E}_i$ to B. Because B is smooth, it doesn't contain the contact points u_i, so E_i is locally free on B.

[2] A monad is a complex of vector bundles given by an injection followed by a surjection (cf [15])

Corollary 1.8. — *All the cohomology groups of the vector bundles E_i vanish. In particular, the rank 2 vector bundles E_i are stable. The vector space $H^0(E_i(1))$ has dimension 6, and we have $H^j(E_i(1)) = 0$ for $j > 0$. The bundles $E_i(1)$ are generated by their global sections.*

Proof. — It is a direct consequence of corollary 1.6, because B is a hyperplane section of $\bar{H}_i$. (Note that the stability condition for E_i is equivalent to $h^0(E_i) = 0$, because $c_1(E_i) = -1$ and E_i has rank 2). □

1.3. The restricted incidences

Now, for each of the 4 hyperplanes H_i containing B and tangent to G_w at the point u_i, let C_i be the conic of the projective plane π_{u_i} constructed in 1.1. Consider the restriction of the incidence variety Z_{H_i} to B. In other words, let Z_i, Z_i' be:

$$Z_i = \{(p, v) \in C_i \times B | p \in \pi_v\}, \quad Z_i' = \{(p, v) \in Z_i | \dim(\pi_v \cap \pi_{u_i}) > 0\}$$

and still denote by q_1 and q_2 the projections from $C_i \times G_w$ to C_i and G_w.

Lemma 1.9. — *Let p be a fixed point of C_i. The scheme $Z_{i,p} = q_2(q_1^{-1}(p) \cap Z_i)$ is a 2 dimensional irreducible quadric in P_w. For a general choice of p on C_i, the quadric $Z_{i,p}$ is smooth. The restriction of q_2 to Z_i' is a double cover of a Veronese surface $V_i = q_2(Z_i')$.*

Proof. — In fact, we already mentioned that $\{v \in G_w | p \in \pi_v\}$ is a smooth quadric of dimension 3 (cf [6] prop 3.2), so it doesn't contain planes. This scheme is included in H_i, so $Z_{i,p}$ is just a hyperplane section of this smooth quadric, so it is always irreducible, and for a general p it is smooth because B is also general. Consider the cone $\{v \in G_w | \dim(\pi_{u_i} \cap \pi_v) > 0\}$ described in lemma 1.1. As $u_i \notin B$, the surface V_i is the intersection of this cone with a hyperplane which doesn't contain the vertex u_i, so it's a Veronese surface. □

Notations. — We denote by σ_i the class of a point on C_i, and by h_3 the class of a hyperplane in P_w (the Plücker embedding of G_w).

Proposition 1.10. — *Let Π be the following projective bundle, and h be the class of $\mathcal{O}_\Pi(1)$. The incidence Z_i is a divisor of class $2h$ in*

$$\Pi = Proj(\mathcal{O}_{C_i}(2\sigma_i) \oplus S_2L \otimes \mathcal{O}_{C_i}).$$

Furthermore we have $h_3 \sim h + 2\sigma_i$ and σ_i is also the class of the fiber of a point on the base of the fibration Π . The divisor Z_i' of Z_i is equivalent to the restriction to Z_i of $h - 2\sigma_i$.

Proof. — Denote by e_i the vector bundle image of the map from $\mathcal{O}_{C_i}(-2\sigma_i)$ to $W \otimes \mathcal{O}_{C_i}$ associated to the embedding of C_i in π_{u_i}, and by $e_i^\perp$ its orthogonal with respect to ω. Choose an element ϕ' of $\bigwedge^3 W^\vee$ such that $\ker \phi'$ gives a hyperplane section of G_w containing B and different from the $\bar{H}_i$ (i.e $\phi'(u_i) \neq 0$). We can remark that the incidence Z_i is given over C_i by the isotropic 2-dimensional subspaces l of $\frac{e_i^\perp}{e_i}$ such that $\phi'(e_i \wedge (\wedge^2 l)) = 0$. Indeed, the condition $\phi_i(e_i \wedge (\wedge^2 l)) = 0$ is already satisfied by the definition of C_i and lemma 1.1 (here ϕ_i denotes a trilinear form of kernel H_i).

The bundle $e_i^\perp$ is isomorphic to $S_2 L \otimes \mathcal{O}_{C_i} \oplus L \otimes \mathcal{O}_{C_i}(-\sigma_i)$, and the trivial factor $S_2 L$ corresponds to the plane π_{u_i}. So the bundle $\frac{e_i^\perp}{e_i}$ is isomorphic to $L \otimes \mathcal{O}_{C_i}(\sigma_i) \oplus L \otimes \mathcal{O}_{C_i}(-\sigma_i)$ where those factors are isotropic for the symplectic form induced by ω. We can take local basis s_0, s_1 and s_2, s_3 of each factors such that the form induced by ω is $p_{0,2} + p_{1,3}$ where $p_{j,k}$ denotes the Plücker coordinates associated to the s_j.

So the relative isotropic Grassmannian $G_w(2, \frac{e_i^\perp}{e_i})$ is the intersection of $G(2, \frac{e_i^\perp}{e_i})$ with $\mathbb{P}(\mathcal{O}_{C_i}(2\sigma_i) \oplus \mathcal{O}_{C_i}(-2\sigma_i) \oplus S_2 L \otimes \mathcal{O}_{C_i})$ in $\mathbb{P}(\bigwedge^2 \frac{e_i^\perp}{e_i})$, and the factor $\mathcal{O}_{C_i}(2\sigma_i)$ still corresponds to $s_0 \wedge s_1$.

Now we need to compute the kernel of the map $e_i \otimes \bigwedge^2 (\frac{e_i^\perp}{e_i}) \xrightarrow{\phi'} \mathcal{O}_{C_i}$. But the assumption $\phi'(u_i) \neq 0$ proves that it is $\mathcal{O}_{C_i}(-4\sigma_i) \oplus L \otimes L \otimes \mathcal{O}_{C_i}(-2\sigma_i)$.

So we have an exact sequence:

$$0 \to \mathcal{O}_{C_i}(-2\sigma_i) \oplus S_2 L \otimes \mathcal{O}_{C_i} \to \bigwedge^2 (\frac{e_i^\perp}{e_i}) \xrightarrow{\binom{\omega}{\phi'}} \mathcal{O}_{C_i} \oplus e_i^\vee \to 0$$

and Z_i is a divisor of class $2h$ in $Proj(\mathcal{O}_{C_i}(2\sigma_i) \oplus S_2 L \otimes \mathcal{O}_{C_i})$. The relation $h_3 \sim h + 2\sigma_i$ is given by the map $e_i \otimes \bigwedge^2 (\frac{e_i^\perp}{e_i}) \to \bigwedge^3 W$.

The divisor Z_i' of Z_i is locally given by the vanishing of the exterior product with $s_0 \wedge s_1$ so it is equivalent to $(h - 2\sigma_i)_{|Z_i}$. □

Corollary 1.11. — *The two dimensional quadrics $(Z_{i,p})_{p \in C_i}$ are smooth except in four cases where they are irreducible cones.*

Proof. — From the definition of $Z_{i,p}$ in lemma 1.9, for any point p of C_i, the quadric $Z_{i,p}$ is the image of $q_1^{-1}(p) \cap Z_i$ by the restriction of the linear system $|h_3|$. As the restriction of h_3 and h to the fibers of $q_1 : Z_i \to C_i$ are equivalent from the proposition 1.10, we just have to study the section of $\mathcal{O}_\Pi(2h)$ found in the proposition 1.10. This section corresponds to a section of $S_2(\mathcal{O}_{C_i}(2\sigma_i) \oplus S_2 L \otimes \mathcal{O}_{C_i})$, and the determinant of this quadratic form is a section of $\mathcal{O}_{C_i}(4\sigma_i)$. Lemma 1.9 now implies that there are only four irreducible cones. □

1.4. Rigidity of E_i

We will now study the relation between the conormal bundle of Z_i in $C_i \times B$ and the bundle E_i.

Lemma 1.12. — *We have the following exact sequence:*

$$0 \to \mathcal{O}_{C_i \times B}(-\sigma_i - h_3) \to q_2^* E_i \to \mathcal{I}_{Z_i}(\sigma_i) \to q_2^*(R^1 q_{2*}\mathcal{I}_{Z_i})(-\sigma_i) \to 0$$

Proof. — Consider q_2 as a fibration in $\mathbb{P}_1$. The resolution of the diagonal of $\mathbb{P}_1 \times \mathbb{P}_1$ gives the relative Beilinson's spectral sequence (cf [15] chap 2 §3):

$$E^1_{a,b} = (\bigwedge^{-a} \omega_{q_2}(\sigma_i)) \otimes R^b q_{2*}(\mathcal{I}_{Z_i}((1+a)\sigma_i)) \Longrightarrow \mathcal{I}_{Z_i}(\sigma_i).$$

By the definition of E_i (cf prop 1.4) we have $E_i = q_{2*}(\mathcal{I}_{Z_i}(\sigma_i))$. Furthermore, the projection $q_2(Z_i)$ is a hyperplane section of B, so $q_{2*}\mathcal{I}_{Z_i} = \mathcal{O}_B(-h_3)$. Now remark that $R^1 q_{2*}(\mathcal{I}_{Z_i}(\sigma_i)) = 0$, because the restriction of q_2: $Z_i \xrightarrow{q_{2|Z_i}} q_2(Z_i)$ has all its fibers of length at most 2. The non zero terms in E^1 are:

$$\begin{array}{|c|c|l}
\multicolumn{2}{r}{\uparrow E^1_{0,\cdot}} & \\
\hline
q_2^*(R^1 q_{2*}\mathcal{I}_{Z_i})(-\sigma_i) & & \\
\hline
\mathcal{O}_B(-\sigma_i - h_3) & E_i & \longrightarrow E^1_{\cdot,0} \\
\hline
\end{array}$$

and the spectral sequence ends at E^2. The filtration of $I_{Z_i}(\sigma_i)$ is given by the sequence:

$$0 \to E^2_{0,0} \to I_{Z_i}(\sigma_i) \to E^2_{-1,-1} \to 0.$$

So we obtain the lemma by the definition of E^2 and the above values of E^1. □

NB: The support of $R^1 q_{2*}\mathcal{I}_{Z_i}$ is the natural scheme structure (cf [3]) on the scheme of fibers of q_2 intersecting Z_i in length 2 or more. It is the Veronese surface $V_i = q_2(Z_i')$. So the previous lemma can now be translated in the following:

Corollary 1.13. — *The scheme $q_2^{-1}(V_i) \cup Z_i$ is in $C_i \times B$ the zero locus of a section of the bundle $q_2^* E_i(\sigma_i + h_3)$.*

This gives also a geometric description of the marked pencil of sections of $E_i(h_3)$ given by the natural inclusion $L \subset V$ found in remark 1.7. Indeed, if we fix a point p on C_i, the restriction to $q_1^{-1}(p)$ of the section obtained in corollary 1.13 gives with the notations of lemma 1.9 the following:

Corollary 1.14. — *For any point p on the conic C_i, the vector bundle $E_i(h_3)$ has a section vanishing on $Z_{i,p} \cup V_i$.*

We can now study the restriction of $q_2^* E_i$ to Z_i.

PROPOSITION 1.15. — *The restriction $q_2^* E_{i|Z_i}$ of the vector bundle $q_2^* E_i$ to Z_i fits into the following exact sequence:*

$$0 \to \mathcal{O}_{Z_i}(h_3 - 3\sigma_i) \to q_2^* E_{i|Z_i}(h_3) \to \mathcal{O}_{Z_i}(3\sigma_i) \to 0$$

Proof. — Fix a point p on C_i, and consider the corresponding section of $E_i(h_3)$ constructed in corollary 1.14. Its pull back gives a section of $q_2^* E_i(h_3)$ vanishing on $q_2^{-1}(Z_{i,p} \cup V_i)$, so its restriction to Z_i gives a section of $q_2^* E_{i|Z_i}(h_3 - \sigma_i - Z_i')$. Now, using the computation of the class of Z_i' in Z_i made in proposition 1.10, namely that $\mathcal{O}_{Z_i}(Z_i')$ is $\mathcal{O}_{Z_i}(h_3 - 4\sigma_i)$, it gives a section of $(q_2^* E_i)_{|Z_i}(3\sigma_i)$. We have to prove that it is a non vanishing section. To obtain this, we compute the second Chern class of $(q_2^* E_i)_{|Z_i}(3\sigma_i)$. We will show that its image in the Chow ring of $C_i \times B$ is zero. Denote by a_i the second Chern class of E_i. From lemma 1.13, we obtain the class of Z_i in $C_i \times B$: $[Z_i] = a_i + h_3.\sigma_i - [V_i]$. So we can compute $[Z_i].c_2(q_2^* E_i(3\sigma_i))$. It is $(a_i + h_3.\sigma_i - [V_i]).(a_i - 3h_3\sigma_i)$, but we will compute in proposition 3.7 the Chow ring of B, and this class vanishes. □

COROLLARY 1.16. — *The vector bundles E_i are rigid, in other words:* $\mathrm{Ext}^1(E_i, E_i) = 0$.

Proof. — From corollary 1.13 we deduce an exact sequence on $C_i \times B$:

$$0 \to q_2^* E_i(-\sigma_i) \to q_2^*(E_i) \otimes q_2^*(E_i)(h_3) \to q_2^* E_i(h_3 + \sigma_i) \\ \to (q_2^* E_i(h_3 + \sigma_i))_{|Z_i \cup q_2^{-1}(V_i)} \to 0$$

All the cohomology groups of the bundle $q_2^* E_i(-\sigma_i)$ are zero, and the corollary 1.8 gives $H^0(q_2^* E_i(h_3 + \sigma_i)) = L \otimes V$ and $H^1(q_2^* E_i(h_3 + \sigma_i)) = 0$. The liaison exact sequence:

$$0 \to \mathcal{O}_{q_2^{-1}(V_i)}(-Z_i') \to \mathcal{O}_{Z_i \cup q_2^{-1}(V_i)} \to \mathcal{O}_{Z_i} \to 0$$

twisted by $q_2^*(E_i(h_3 + \sigma_i))$ is:

$$0 \to q_2^* E_i(h_3) \otimes \mathcal{O}_{q_2^{-1}(V_i)}(\sigma_i - Z_i') \to q_2^* E_i(h_3 + \sigma_i))_{|Z_i \cup q_2^{-1}(V_i)} \\ \to q_2^* E_{i|Z_i}(h_3 + \sigma_i) \to 0$$

As $\sigma_i - Z_i'$ has degree -1 along the fibers of $q_2 : q_2^{-1}(V_i) \to V_i$, all the cohomology groups of the bundle $q_2^* E_i(h_3) \otimes \mathcal{O}_{q_2^{-1}(V_i)}(\sigma_i - Z_i')$ vanish, so the cohomology of $q_2^* E_i(h_3 + \sigma_i))_{|Z_i \cup q_2^{-1}(V_i)}$ can be computed from its restriction to Z_i. Propositions 1.15 and 1.10 show that $H^0(q_2^* E_{i|Z_i}(h_3 + \sigma_i)) = S_2 L \oplus S_2 L \oplus S_4 L$. In conclusion, we have the exact sequence:

$$0 \to \mathrm{Hom}(E_i, E_i) \to L \otimes V \to S_2 L \oplus S_2 L \oplus S_4 L \to \mathrm{Ext}^1(E_i, E_i) \to 0$$

By corollary 1.8, the bundle E_i is stable, so it is simple, in other words we have $\mathrm{Hom}(E_i, E_i) = \mathbb{C}$, and the above exact sequence gives $\mathrm{Ext}^1(E_i, E_i) = 0$. □

2. The variety of lines in B

Remark 2.1. — Let δ be an isotropic line in $\mathbb{P}(W)$. The set of isotropic planes of $\delta^{\perp}$ containing δ form a line in $G_w(3, W)$, and all the lines in $G_w(3, W)$ are of this type for a unique element of $G_w(2, W)$. In other words, the variety of lines in $G_w(3, W)$ is naturally isomorphic with $G_w(2, W)$.

Notations. — A point of $G_w(2, W)$ will be denoted by a lower case letter, and its corresponding line in $G_w(3, W)$ by the associated upper case letter. The Plücker hyperplane section of $G_w(j, W)$ will be noted h_j, the tautological subbundle by K_j, and the variety of lines included in B will be denoted F_B. Let I be the following incidence variety:

$$I = Proj\left(\left(\frac{K_2^{\perp}}{K_2}\right)^{\vee}(h_2)\right) = \{(\delta, p) \in F_B \times B | p \in \Delta\} \subset G_w(2, W) \times G_w(3, W)$$

The projections from I to F_B and B will be denoted by f_1 and f_2.

$$\begin{tikzcd} & I \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & Z_i \arrow[dl, "q_2"'] \arrow[dr, "q_1"] & \\ G_w(2,W) \supset F_B & & B \subset G_w(3,W) & & C_i \simeq \mathbb{P}_1 \end{tikzcd}$$

LEMMA 2.2. — *The variety F_B is obtained in $G_w(2, W)$ as the zero locus of a section of the bundle $\left(\frac{K_2^{\perp}}{K_2}\right)^{\vee}(h_2) \oplus \left(\frac{K_2^{\perp}}{K_2}\right)^{\vee}(h_2)$. For a general choice of B, F_B is smooth and irreducible with $\omega_{F_B} = \mathcal{O}_{F_B}(-h_2)$.*

Proof. — As B is a double hyperplane section of $G_w(3, W)$, we have from the previous definition of I the equality: $f_{1*}(f_2^* \mathcal{O}_{G_w(3,W)}(h_3)) = \left(\frac{K_2^{\perp}}{K_2}\right)^{\vee}(h_2)$. So F_B is the vanishing locus of a section of $\left(\frac{K_2^{\perp}}{K_2}\right)^{\vee}(h_2) \oplus \left(\frac{K_2^{\perp}}{K_2}\right)^{\vee}(h_2)$ and its dualising sheaf is $\mathcal{O}_{F_B}(-h_2)$. The choice of a generic 2-dimensional subspace of $H^0(\mathcal{O}_{G_w(3,W)}(h_3))$ corresponds to the choice of a generic section of $\left(\frac{K_2^{\perp}}{K_2}\right)^{\vee}(h_2) \oplus \left(\frac{K_2^{\perp}}{K_2}\right)^{\vee}(h_2)$. Hence, for a general choice of B, the variety F_B will be smooth because $K_2^{\perp}(h_2)$ is globally generated, and so is $\frac{K_2^{\perp}}{K_2}(h_2) \simeq \left(\frac{K_2^{\perp}}{K_2}\right)^{\vee}(h_2)$.

We obtain the vanishing of the first cohomology group of the ideal $\mathcal{I}_{F_B}$ of F_B in $G_w(2, W)$ from the exact sequence:

$$0 \to \mathcal{O}_{G_w(2,W)}(-h_2) \to \left(\frac{K_2^{\perp}}{K_2}\right)^{\vee} \oplus \left(\frac{K_2^{\perp}}{K_2}\right)^{\vee} \to \mathcal{I}_{F_B} \to 0.$$

So we have $h^0(\mathcal{O}_{F_B}) = 1$ and F_B is connected and smooth, so it is irreducible. □

2.1. A morphism from F_B to $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$

Each of the four conics C_i will enable us to construct a morphism from F_B to $\mathbb{P}_1$. We have the following geometric hint to expect at least a rational map: a general element δ of F_B gives an isotropic 2-dimensional subspace L_δ of W. In general, the projectivisation of $L_\delta^\perp$ intersects the plane containg C_i in a point p. There is at least an element m of Δ such that $p \in \pi_m$, so m is in $H_i \cap \bar{H}_{u_i}$ because it is in $\Delta \subset B \subset H_i$. Now, the definition of H_i and lemma 1.1 prove that p must be on C_i.

But to show that this rational map is everywhere defined, we will use the vector bundle E_i. We start by constructing line bundles on F_B.

Lemma 2.3. — *A line Δ included in B intersects the Veronese surface V_i if and only if it is included in the hyperplane section $\bar{H}_{u_i} = q_2(Z_i)$ of B. Moreover, any such line is contained in a quadric Z_{i,p_i} for a unique point p_i of C_i. The set $v_i = \{\delta \in F_B | \Delta \subset \bar{H}_{u_i}\}$ is equal to $f_1(f_2^{-1}(V_i))$ and it is a divisor in F_B.*

Proof. — Let δ be an element of F_B such that there is a point v in the intersection $\Delta \cap V_i$. By lemma 1.9, the plane π_v intersects π_{u_i} in a line and it contains the line $\mathbb{P}(L_\delta)$. So the intersection $\mathbb{P}(L_\delta) \cap \pi_{u_i}$ is not empty and is included in $\pi_{v'}$ for any point v' of Δ. So Δ is included in $\bar{H}_{u_i}$.

Now if the line Δ is included in $\bar{H}_{u_i}$, we have from lemma 1.1 that for any $b \in \Delta$, the plane π_b intersects the conic C_i. Let's first prove that the line $\mathbb{P}(L_\delta)$ must intersect C_i in some point p_i. Indeed, if it was not the case, the intersection of π_b with C_i would cover C_i as b vary in Δ, so $\mathbb{P}(L_\delta)$ would be orthogonal to C_i and it would be in the plane π_{u_i}, but any line in this plane intersects C_i.

So the line Δ is in the quadric Z_{i,p_i}. Note that $\mathbb{P}(L_\delta) \cap C_i$ can't contain another point because the line Δ can't be included in the Veronese surface V_i. Furthermore, proposition 1.10 implies that Z'_{i,p_i} is a plane section of the quadric Z_{i,p_i}. As this section is irreducible because V_i doesn't contain lines, the intersection of Δ and V_i is a single point. In conclusion we have $v_i = f_1(f_2^{-1}(V_i))$ and the varieties v_i and V_i have the same dimension, so v_i is a divisor in F_B. □

Lemma 2.4. — *For any point p_i of C_i, the scheme $f_2^{-1}(Z_{i,p_i})$ has several irreducible components of dimension 2, but some of these components are contracted by f_1 to a curve.*

Proof. — The components of $f_2^{-1}(Z_{i,p_i})$ corresponding to the lines included in Z_{i,p_i} are contracted to curves. $\square$

Notations. — Denote by A_{i,p_i} the 2 dimensional part of $f_1(f_2^{-1}(Z_{i,p_i}))$.

PROPOSITION 2.5. — *The sheaf $f_{1*}f_2^*E_i$ is a line bundle on F_B. There is a natural map μ_i from $L\otimes\mathcal{O}_{F_B}$ to the dual bundle of $f_{1*}f_2^*E_i$. The image of μ_i is also a line bundle on F_B, we will denote it by $\mathcal{O}_{F_B}(\alpha_i)$. By construction, for any $p_i\in C_i$, the divisor A_{i,p_i} will be in the linear system $|\mathcal{O}_{F_B}(\alpha_i)|$, and we have $f_{1*}f_2^*E_i=\mathcal{O}_{F_B}(-\alpha_i-v_i)$.*

Proof. — By corollary 1.8, the bundle E_i is a quotient of $6\mathcal{O}_B(-1)$ and by proposition 1.4, it is a subsheaf of $2\mathcal{O}_B$ with $c_1(E_i)=-1$. So its restriction to any line Δ included in B must be $\mathcal{O}_\Delta\oplus\mathcal{O}_\Delta(-1)$. As a consequence, we have $R^1f_{1*}f_2^*E_i=0$ and $f_{1*}f_2^*E_i$ is a line bundle. Denote this line bundle by $\mathcal{O}_{F_B}(-\alpha_i')$. Dualising and twisting the exact sequence defining E_i in proposition 1.4, we obtain the following one:

$$0\to L\otimes\mathcal{O}_B(-2h_3)\to E_i(-h_3)\to\mathcal{L}_i\to 0 \tag{2.1}$$

where $\mathcal{L}_i$ is supported on the hyperplane section $\bar{H}_{u_i}$, and is singular along the Veronese surface V_i. As the incidence I is $Proj\left(\left(\frac{K_2^\perp}{K_2}\right)^{\vee}(h_2)\right)$ (where K_2 is the tautological subbundle of $W\otimes\mathcal{O}_{G_w(2,W)}$), the relative dualising sheaf ω_{f_1} is $\mathcal{O}_I(2h_2-2h_3)$. So we have $R^1f_{1*}(f_2^*E_i(-h_3))=\mathcal{O}_{F_B}(\alpha_i'-2h_2)$. Therefore the base locus of the map:

$$L\otimes\mathcal{O}_{F_B}(-2h_2)\to\mathcal{O}_{F_B}(\alpha_i'-2h_2)$$

is the support of $R^1f_{1*}f_2^*(\mathcal{L}_i)$. We will now prove that this sheaf is a line bundle on the divisor v_i defined in lemma 2.3.

The morphism f_1 defined at the beginning of section 2, is projective of relative dimension 1, and the sheaf $f_2^*E_i(-h_3)$ is flat over F_B. So by base change (cf [4] 11.2 and 12.11), for any point δ of F_B, the fiber $(R^1f_{1*}f_2^*E_i(-h_3))_\delta$ is $H^1(E_i(-h_3)\otimes\mathcal{O}_\Delta)$, where Δ is the line in B corresponding to δ. The restriction of the sequence (2.1) to Δ gives the surjection:

$$2\mathcal{O}_\Delta(-2)\to\mathcal{O}_\Delta(-1)\oplus\mathcal{O}_\Delta(-2)\to\mathcal{L}_i\otimes\mathcal{O}_\Delta\to 0 \tag{2.2}$$

When the line Δ is not in the hyperplane section $\bar{H}_{u_i}$, the sheaf $\mathcal{L}_i\otimes\mathcal{O}_\Delta$ is supported by the point $\bar{H}_{u_i}\cap\Delta$, so in this case we have $h^1(\mathcal{L}_i\otimes\mathcal{O}_\Delta)=0$. Now, when the line Δ is in $\bar{H}_{u_i}$, the sheaf $\mathcal{L}_i\otimes\mathcal{O}_\Delta$ has generic rank 1 because the Veronese surface V_i can't contain the line Δ. We have proved in lemma 2.3 that Δ intersects V_i, hence for any element of L, the corresponding section of $E_i(h_3)$ vanishes on Δ. So the map $2\mathcal{O}_\Delta(-2)\to\mathcal{O}_\Delta(-2)$ induced by the sequence (2.2) is zero, and for any δ in v_i, we have $h^1(\mathcal{L}_i \otimes \mathcal{O}_\Delta) = 1$, therefore $R^1 f_{1*} f_2^*(\mathcal{L}_i)$ is a line bundle on v_i.

So our pencil of sections of $(f_{1*} f_2^* E_i)^{\mathrm{v}}$ can now be interpreted as a base point free pencil of sections of $(f_{1*} f_2^* E_i)^{\mathrm{v}}(-v_i)$. In other words, the image of μ_i is the line bundle $\mathcal{O}_{F_B}(\alpha_i) = (f_{1*} f_2^* E_i)^{\mathrm{v}}(-v_i)$. By definition A_{i,p_i} is the closure of $\{\delta \in F_B | \text{length}(\Delta \cap Z_{i,p_i}) = 1\}$ which was identified set theoretically with an element of the linear system $|\alpha_i|$, so we conclude the proof with the following lemma 2.6 □

Lemma 2.6. — *For a generic choice of a point p_i on C_i, the support of the sheaf $R^1 f_{1*}(f_2^* I_{Z_{i,p_i} \cup V_i}(-h_3))$ represents the class α'_i, and all its irreducible components are reduced.*

Proof. — First notice that the point p_i on C_i gives from the corollary 1.13 a section of $E_i(h_3)$, so the exact sequence:

$$0 \to \mathcal{O}_B(-2h_3) \to E_i(-h_3) \to I_{Z_{i,p_i} \cup V_i}(-h_3) \to 0.$$

Applying f_2^* and f_{1*} to the previous sequence, we obtain a section of $\mathcal{O}_{F_B}(\alpha'_i)$ vanishing on the support of the sheaf $R^1 f_{1*}(f_2^* I_{Z_{i,p_i} \cup V_i}(-h_3))$. But this is the definition in [3] of the scheme structure on the set of lines included in B and intersecting $Z_{i,p_i} \cup V_i$. So to show that this scheme structure is reduced on each component, we have to prove that Z_{i,p_i} and V_i are not contained in the ramification locus of the morphism: $f_2 : I \to B$. We will do those two cases simultaneously by taking a general point m on $Z_{i,p_i} \cap V_i$. Such a point is the intersection of Z_{i,p_i} and another quadric Z_{i,p'_i}, so there pass four distinct lines through m, and from lemma 2.3 there are no other lines in B through m because m is only on two element of $(Z_{i,p})_{p \in C_i}$. So the point m is not in the ramification of the morphism $f_2 : I \to B$ because it is of degree four (cf lemma 3.1, or remind that the tangent cone of $G_w(3, W)$ is a cone over a Veronese surface). □

2.2. Description of the morphism

Denote by $\tilde{\mu_i}$ the morphisms from F_B to C_i given by the surjections $\mu_i : H^0(\mathcal{O}_{C_i}(\sigma_i))^{\mathrm{v}} \otimes \mathcal{O}_{F_B} \twoheadrightarrow \mathcal{O}_{F_B}(\alpha_i)$ constructed in the previous section. In the following, we prove that the morphism from F_B to $C_1 \times C_2 \times C_3 \times C_4$ is an embedding for a generic B, and that its image is a hyperplane[(3)] section.

As we need to allow cases where a point and a line do not generate a plane, we will sometimes work in affine spaces.

(3) with respect to $|\mathcal{O}_{C_1 \times C_2 \times C_3 \times C_4}(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)|$

Notations. — Let $\mathcal{C}_B$ be the affine cone over B. A vector corresponding to a point of a projective space will be denoted by the same letter with a tilde. For a point p_i of the conic C_i, we consider:

$$F_{p_i} = \{\delta \in G_w(2,W) | \tilde{\delta} \wedge \tilde{p_i} \in \mathcal{C}_B\}$$

Unfortunately, we have to remark that $\tilde{\mu_i}^{-1}(p_i)$ is not exactly $F_{p_i} \cap F_B$:

LEMMA 2.7. — *The intersection $F_{p_i} \cap F_B$ is equal to $G_w(2, p_i^{\perp}) \cap F_B$. It contains $\tilde{\mu_i}^{-1}(p_i)$ and residual curves corresponding to the lines included in the quadric Z_{i,p_i}.*

Proof. — If δ be an element of F_{p_i}, then we have the inclusion $< L_\delta, p_i > \subset < L_\delta, p_i >^{\perp}$. So δ is in $G_w(2, p_i^{\perp})$ and $F_{p_i} \subset G_w(2, p_i^{\perp})$.

Now, let δ be an element of $G_w(2, p_i^{\perp}) \cap F_B$. We have from remark 2.1:

$$f_1(f_2^{-1}(Z_{i,p_i})) = \{\delta \in F_B | \exists v \in B,\ \mathbb{P}(L_\delta) \subset \pi_v \subset \mathbb{P}(L_\delta^{\perp}) \text{ and } p_i \in \pi_v\}.$$

Furthermore, this remark also implies that any isotropic plane of $\mathbb{P}(W)$ containing $\mathbb{P}(L_\delta)$ is an element of the line Δ, so the vector $\tilde{\delta} \wedge \tilde{p_i}$ is either 0 or corresponds to an element of B. So we have $\tilde{\delta} \wedge \tilde{p_i} \in \mathcal{C}_B$. So $G_w(2, p_i^{\perp}) \cap F_B \subset F_{p_i} \cap F_B$.

Now remark that in $G_w(2,W)$, the scheme $G_w(2, p_i^{\perp})$ is the zero locus of a section of $K_2^{\vee}$. The restriction of this section to F_B vanishes on the divisor A_{i,p_i} defined in proposition 2.5, so $F_B \cap F_{p_i}$ contains a divisor of class α_i and the zero locus of a section of $(K_2^{\vee})_{|F_B}(-\alpha_i)$ corresponding to the lines included in Z_{i,p_i}. □

Nevertheless, we have the following:

LEMMA 2.8. — *For the generic double hyperplane section B of $G_w(3,W)$, we can find points p_i (resp. p_j) in the conics C_i (resp. C_j) such that $F_{p_i} \cap F_{p_j}$ is a smooth conic in $G_w(2,W)$. Furthermore, this conic is in F_B and represents the class $\alpha_i . \alpha_j$.*

Proof. — We can choose p_i and p_j respectively in C_i and C_j such that, p_i is not in $p_j^{\perp}$. So, for any element δ of $F_{p_i} \cap F_{p_j}$, the vectors $\tilde{\delta} \wedge \tilde{p_i}$ and $\tilde{\delta} \wedge \tilde{p_j}$ are different from 0 and give two distinct points of $\Delta \cap B$. This implies that Δ is in B, and that the intersection $F_{p_i} \cap F_{p_j}$ is automatically included in F_B. We also deduce from the assumption $p_i \notin p_j^{\perp}$ that the isotropic Grassmannian $G_w(2, p_i^{\perp} \cap p_j^{\perp})$ is a smooth 3-dimensional quadric.

Denote by ϕ_k the equation of the hyperplane H_k. Among the four equations $(\phi_k(\tilde{\delta} \wedge \tilde{p}_l) = 0)_{k,l \in \{i,j\}}$ defining the intersection $F_{p_i} \cap F_{p_j}$ in $G_w(2, p_i^{\perp} \cap p_j^{\perp})$, the following two are automatically satisfied: $(\phi_k(\tilde{\delta} \wedge \tilde{p}_k) = 0)_{k \in \{i,j\}}$. Indeed, recall that the vectors $\tilde{\delta} \wedge \tilde{p}_i$ and $\tilde{\delta} \wedge \tilde{p}_j$ are in the affine cone over

G_w because $d \in G_w(2, p_i^\perp \cap p_j^\perp)$. Then conclude with lemma 1.1 because p_i is in C_i and p_j is in C_j. Consequently, the intersection $F_{p_i} \cap F_{p_j}$ is defined in $G_w(2, p_i^\perp \cap p_j^\perp)$ by the two equations $\phi_i(\tilde{\delta} \wedge \tilde{p}_j) = 0$ and $\phi_j(\tilde{\delta} \wedge \tilde{p}_i) = 0$.

Let L_δ be the intersection of $p_j^\perp$ and the 3 dimensional vector space U_i represented by the contact point u_i of H_i. The corresponding point δ is an element of $G_w(2, p_i^\perp \cap p_j^\perp)$ such that $\phi_j(\tilde{p}_i \wedge \tilde{\delta}) \neq 0$ (because $\tilde{p}_i \wedge \tilde{\delta}$ corresponds to u_i, and B is smooth), but recall from lemma 1.1 that $\phi_i(\tilde{p}_j \wedge \tilde{\delta}) = 0$. So $F_{p_i} \cap F_{p_j}$ is defined by two independent hyperplane sections of a smooth 3 dimensional quadric. So it is a (may be singular) conic.

Note that from the genericity of B we could also assume that $\mathbb{P}(p_i^\perp) \cap C_j = \{a_1, a_2\}$ and $\mathbb{P}(p_j^\perp) \cap C_i = \{b_1, b_2\}$ are 4 distinct points. According to lemma 1.1, the lines (p_i, a_1), (p_i, a_2), (p_j, b_1), (p_j, b_2) define points of $F_{p_i} \cap F_{p_j}$, but no three of those lines are in the same plane. So the conic $F_{p_i} \cap F_{p_j}$ contains four points with no trisecant line, hence it must be smooth. To conclude that this conic represent the class $\alpha_i.\alpha_j$, we just have to prove that the residual curves of $F_{p_i} \cap F_B$ found in lemma 2.7 have empty intersection with F_{p_j}. But if δ is such that $\Delta \subset Z_{i,p_i}$, the line $\mathbb{P}(L_\delta)$ contains the point p_i which is not orthogonal to p_j, so $\delta \notin F_{p_j}$. $\square$

Lemma 2.9. — *When i, j, k are distinct, the morphism from F_B to $C_i \times C_j \times C_k$ is dominant.*

Proof. — According to lemma 2.8, for a generic choice of $p_i \in C_i$ and $p_j \in C_j$, the subvariety $F_{p_i} \cap F_{p_j}$ of F_B is a smooth conic, and we can also assume that the line $(p_i p_j)$ doesn't intersect C_k and that $p_i \notin p_j^\perp$.

If the induced map from $F_{p_i} \cap F_{p_j}$ to C_k is not dominant, then there is a point $p_k \in C_k$ such that $F_{p_i} \cap F_{p_j} \subset F_{p_k}$. In that case, for any element δ of $F_{p_i} \cap F_{p_j}$ the corresponding line $\mathbb{P}(L_\delta)$ is in $\mathbb{P}(< p_i, p_j, p_k >^\perp)$. As $p_k \notin (p_i p_j)$, the vector space $< p_i, p_j, p_k >^\perp$ has dimension 3, and it is not isotropic because $p_i \notin p_j^\perp$. So the intersection $F_{p_i} \cap F_{p_j}$ is in the line $G_w(2, < p_i, p_j, p_k >^\perp)$ and it is not a smooth conic.

So the projection from $F_{p_i} \cap F_{p_j}$ to C_k is dominant and therefore, the map $F_B \to C_i \times C_j \times C_k$ is dominant. $\square$

At this point, we need more details about the embedding of F_B in $G_w(2, W)$. As $G_w(2, W)$ is a hyperplane section of $G(2, W)$, we will do the computations in $G(2, W)$.

Remark 2.10. — Let c_2 be the second Chern class of the rank two tautological subsheaf K_2. The Chow ring of $G(2, W)$ is

$$\mathbb{Q}[h_2, c_2]/(h_2^5 + 3h_2c_2^2 - 4h_2^3c_2, -h_2^4c_2 + 3h_2^2c_2^2 - c_2^3).$$

We obtain from this description of the Chow ring the following computations:

Lemma 2.11. — *The class of F_B in the Chow ring of $G_w(2,W)$ is $4.(h_2^2-c_2)^2$, and F_B has degree 24 in $G_w(2,W)$. In the Chow ring of F_B, we have the extra relation: $h_2^2 = 3c_2$.*

Proof. — We just have to remark that $K_2^{\perp}$ is isomorphic to the dual of the tautological quotient and obtain the class of F_B from its description in the lemma 2.2. □

Lemma 2.12. — *In the Chow ring of F_B we have $\alpha_i^2 = 0$, $\alpha_i.\alpha_j.h_2 = 2$ and $\alpha_i.\alpha_j.\alpha_k = 1$ when i,j,k are distinct.*

Proof. — In proposition 2.5 we have proved that the map $\mu_i : 2\mathcal{O}_{F_B} \to \mathcal{O}_{F_B}(\alpha_i)$ is onto. So $\alpha_i^2 = 0$ because it is the class of the support of the cokernel of μ_i. The equality $\alpha_i.\alpha_j.h_2 = 2$ is a direct consquence of lemma 2.8.

From lemma 2.9, for a generic choice of p_i, p_j, p_k, the intersection $F_{p_i} \cap F_{p_j} \cap F_{p_k}$ is not empty. Furthermore, this intersection is included in the smooth conic $F_{p_i} \cap F_{p_j}$ and the line $G_w(2, p_i^{\perp} \cap p_j^{\perp} \cap p_k^{\perp})$. So it must be a point because the intersection of a smooth conic and a line in the 3-dimensional smooth quadric $G_w(2, p_i^{\perp} \cap p_j^{\perp})$ must be empty or a point. So $\alpha_i.\alpha_j.\alpha_k = 1$. □

At this point we need more informations on the position of the conics. We start with the following:

Lemma 2.13. — *For any integer n, the union of four general $\mathbb{P}_n$ in $\mathbb{P}_{2n+1}$ has $n+1$ quadrisecant lines.*

Proof. — Denote by $\bar{Q}$ the tautological quotient of rank $2n$ on the Grassmannian $G(2, 2n+2)$. The class of the lines intersecting a fixed $\mathbb{P}_n$ is the special Schubert cycle $c_n(\bar{Q})$. Remark that $(c_n(\bar{Q}))^2$ is a sum of $n+1$ distinct self-dual Schubert cycles. So its square is $(n+1)c_{2n}(\bar{Q})$ and the lemma is proved. □

Lemma 2.14. — *The rational map κ from $G(2, H^0(\mathcal{O}_{G_w(3,W)}(1)))$ to the symmetric product of order 4 of $G_w(3,W)$ defined by $\kappa(B) = \{u_1, u_2, u_3, u_4\}$ is dominant. In particular for a generic B, the intersection $\pi_{u_i} \cap \pi_{u_j}$ is empty for $i \neq j$, and there is no line intersecting every C_i for $i \in \{1, \dots, 4\}$.*

Proof. — The Grassmannian $G_w(3,W)$ is naturally embedded in P_w and in its dual space because $\bigwedge^3 W$ is self dual. Let $(u_i)_{i\in\{1\dots4\}}$ be a generic element of $G_w(3,6)^4$. The projectified tangent space to $G_w(3,6)$ at u_i is six dimensional, but in $P^{\vee}_w$ a set of four projective spaces of dimension six has a quadrisecant line (cf lemma 2.13). Taking one of these lines for the equations of B proves that κ is dominant.

So for a generic B, the four planes $(\pi_{u_i})_{i\in\{1,\dots,4\}}$ have empty intersection two by two, and there are three lines intersecting all of them. We will prove that none of those lines are isotropic by producing the following example:

Take an orthogonal decomposition of W into a sum of non isotropic spaces of dimension 2: $W=\bigoplus_{i=0}^{2} L_i$ and chose a basis (e_i,e'_i) of L_i. Consider the projectivisation of $<(e_i)_{i\in\{0\dots2\}}>$, $<(e'_i)_{i\in\{0\dots2\}}>$, $<(e_i+e'_i)_{i\in\{0\dots2\}}>$, $<((i+1)e_i+e'_i)_{i\in\{0\dots2\}}>$. The lines $\mathbb{P}(L_i)$ are not isotropic and intersect the four spaces, so we have to show that the union of these four spaces has a finite number of 4-secant lines. Let A and B be the points $\sum_{i=0}^{2} a_i.(e_i+e'_i)$ and $\sum_{i=0}^{2} b_i.((i+1)e_i+e'_i)$. The line (AB) intersects the first space if and only if $(a_i)_{i\in\{0\dots2\}}$ and $(b_i)_{i\in\{0\dots2\}}$ are proportional, and it intersects the second space if and only if $(a_i)_{i\in\{0\dots2\}}$ and $((i+1)b_i)_{i\in\{0\dots2\}}$ are proportional, so it intersects the four spaces if and only if it is one of the $\mathbb{P}(L_i)$. From this example, the open set of $G_w(3,W)^4$ corresponding to union of four planes with exactly three non isotropic 4-secant lines is not empty.

Now we claim that for a generic B, none of the 4-secant lines to $\bigcup_{i=1}^{4}\pi_{u_i}$ intersects every C_i. Indeed, if l is such a line, suppose that it meets C_i at p_i. As l is not isotropic, the intersection $l^{\perp}\cap\pi_{u_1}$ is a line D such that $p_1\notin D$. By lemma 1.1 the planes $<D,p_2>$ and $<D,p_3>$ are two distinct elements of B. This implies that $<D,p_1>$ is also in B because it is linearly dependent with $<D,p_2>$ and $<D,p_3>$. So B is not smooth because this plane is u_1. □

Notations. — Let $\bar{F}$ be the image of F_B in $C_1\times C_2\times C_3\times C_4$ and denote by ψ the morphism:

$$\psi: F_B\to\bar{F}.$$

LEMMA 2.15. — *The variety $\bar{F}$ is a smooth divisor in $C_1\times C_2\times C_3\times C_4$ of class $\sum_{i=1}^{4}\sigma_i$ and ψ is a birational morphism.*

Proof. — We deduce from lemma 2.12 that the composition of ψ with the projection on $C_1\times C_2\times C_3$ is birational. So ψ is also birational, and $\bar{F}$ is a divisor in $C_1\times C_2\times C_3\times C_4$. Let i,j,k,l be four distinct elements of $\{1,\dots,4\}$. We obtain the coefficient of σ_l in $[\bar{F}]$ from the relation $\psi^*([\bar{F}].\sigma_i.\sigma_j.\sigma_k)=\alpha_i.\alpha_j.\alpha_k$ and lemma 2.12.

First, we prove that $\bar{F}$ is normal. As a divisor in a smooth variety satisfies Serre's property S2, we just have to prove the regularity in codimension 1. So consider a general double hyperplane[(4)] section $\bar{\Gamma}$ of $\bar{F}$. Let Γ be $\psi^{-1}(\bar{\Gamma})$. Apply Bertini's theorem to the smooth variety F_B to obtain that Γ is a smooth curve. So Γ is the normalisation of $\bar{\Gamma}$. Now compute their arithmetic genus. The curve $\bar{\Gamma}$ is a triple hyperplane section of $C_1 \times C_2 \times C_3 \times C_4$, so its dualising sheaf is $\omega_{\bar{\Gamma}} = \mathcal{O}_{\bar{\Gamma}}(\sum_{i=1}^4 \sigma_i)$, and it has degree $(\sum_{i=1}^4 \sigma_i)^4 = 24$. From the adjunction formula and the lemma 2.2, we have $\omega_\Gamma = \mathcal{O}_\Gamma(-h_2 + 2.\sum_{i=1}^4 \alpha_i)$. So we obtain the degree of ω_Γ by computing $(-h_2 + 2.\sum_{i=1}^4 \alpha_i).(\sum_{i=1}^4 \alpha_i)^2$ on F_B. So the degree of ω_Γ is also 24 by lemma 2.12. Therefore the normalisation Γ of $\bar{\Gamma}$ has the same arithmetic genus as $\bar{\Gamma}$, so they are isomorphic (cf [4] IV.1 ex 1.8). In conclusion $\bar{F}$ is regular in codimension 1, so it is normal.

We are now able to prove that $\bar{F}$ is smooth. Assume that the element $(p_1, \dots, p_4)$ of $C_1 \times \cdots \times C_4$ is a singular point of $\bar{F}$. The normality of $\bar{F}$ and the smoothness of F_B imply that the fiber $\psi^{-1}(p_1, \dots, p_4)$ is not finite. Recall that it is a subset of $G_w(2, p_1^\perp \cap \cdots \cap p_4^\perp)$, so $p_1, \dots, p_4$ are in a single projective plane π (cf lemma 2.14). As $\bar{F}$ is a hyperplane section singular at $(p_1, \dots, p_4)$, the 4 lines of $C_1 \times C_2 \times C_3 \times C_4$ containing $(p_1, \dots, p_4)$ are in $\bar{F}$. We can assume that one of the 4 points (say p_1), is such that the following linear spans $< p_1, p_2, p_3 >$, $< p_1, p_2, p_4 >$, $< p_1, p_3, p_4 >$ are equal to π. Remark that the pull back of the 3 distinct curves $\{p_1\} \times \{p_2\} \times \{p_3\} \times C_4$, $\{p_1\} \times \{p_2\} \times C_3 \times \{p_4\}$, $\{p_1\} \times C_2 \times \{p_3\} \times \{p_4\}$ by ψ are 3 distinct varieties of dimension strictly positive in $G_w(2, \pi^\perp)$. So π is an isotropic plane, since otherwise $G_w(2, \pi^\perp)$ would be a line. So from lemma 1.1 π is an element of B. In that situation, any of these three varieties gives a two-dimensional cone included in B with vertex π. So B is not smooth at π because the tangent cone at a smooth point of a double hyperplane section of $G_w(3, W)$ is described in the following lemma 2.16.

Lemma 2.16. — *The tangent cone $C_{B,m}$ of B at a smooth point m is either the cone over a scheme of length 4 or the union of a line with a cone over a smooth conic.*

Proof. — Recall that the tangent cone to $G_w(3, W)$ is a cone over a Veronese surface. As the variety B is smooth at m, the basis of $C_{B,m}$ is the intersection of this Veronese surface by two independent hyperplanes. Their restriction on $\mathbb{P}_2$ gives two independent conics. So their intersection is either a scheme of length 4 or the union of a point with a line. This proves

(4) with respect to the embedding of $C_1 \times C_2 \times C_3 \times C_4$ by $\mathcal{O}(\sum_{i=1}^4 \sigma_i)$

the lemma because the basis of $C_{B,m}$ is the image of this intersection by the Veronese embedding of $\mathbb{P}_2$ in $\mathbb{P}_5$. $\square$

This concludes the proof of lemma 2.15. $\square$

LEMMA 2.17. — *The map ψ is an isomorphism, and the hyperplane section h_2 of F_B is linearly equivalent to $\alpha_1+\alpha_2+\alpha_3+\alpha_4$.*

Proof. — The smoothness of $\bar{F}$ gives a non zero map $\psi^*(\omega_{\bar{F}}) \to \omega_{F_B}$. From lemma 2.11, this section of $\omega_{F_B} \otimes (\psi^*(\omega_{\bar{F}}))^{\vee}$ vanishes on an effective divisor of class $\alpha_1+\alpha_2+\alpha_3+\alpha_4-h_2$ in the Picard group of F_B. To obtain the result, we show that its intersection with h_2^2 is zero. Computations on F_B with lemma 2.11 give $h_2^3 = 24$. So the proof will be finished after the following lemma: $\square$

LEMMA 2.18. — *The image of the class α_i in the Chow ring of $G_w(2,W)$ is $h_2.c_2.2(h_2^2-c_2)$, and in the Chow ring of F_B we have $\alpha_i.h_2^2 = 6$.*

Proof. — First, choose a hyperplane H' such that $B = G_w(3,W) \cap H' \cap H_i$. Let $F_{H'}$ and F_{H_i} be the varieties of lines included respectively in $\bar{H}'$ and $\bar{H}_i$, and denote by Y the Grassmannian $G_w(2,p_i^{\perp})$. We proved in lemma 2.7 that α_i is represented by the 2-dimensional part of $Y \cap F_B$ for some point p_i on the conic C_i. From the definition of B, we have $F_B = F_{H'} \cap F_{H_i}$. The varieties $F_{H'}$ and F_{H_i} represent the class $c_2\left(\left(\frac{K_2^{\perp}}{K_2}\right)^{\vee}(h_2)\right)$ in the Chow ring of $G_w(2,W)$, but the intersection $Y \cap F_{H_i}$ has codimension one in Y. Indeed, for any l in Y the point $< p_i, l >$ of $G_w(3,W)$ is already in $\bar{H}_i$. The restriction of the sheaf $(K_2)^{\perp}$ to Y has a section given by p_i, and the intersection $Y \cap F_{H_i}$ is the vanishing locus of a section of $\frac{K_{2|Y}^{\perp}}{\mathcal{O}_Y \oplus K_{2|Y}}(h_2)$. Remark that Y represents the class c_2 in $G_w(2,W)$ to conclude that the class α_i is equivalent to $h_2.c_2.c_2\left(\left(\frac{K_2^{\perp}}{K_2}\right)^{\vee}(h_2)\right)$ in the Chow ring of $G_w(2,W)$.

With remark 2.10, we obtain the equality $h_2^2.\alpha_i = 6$. This ends the proof of lemma 2.18, and also of lemma 2.17. $\square$

In conclusion, lemmas 2.15 and 2.17 give the following:

THEOREM 2.19. — *The variety of lines included in a generic double hyperplane section of $G_w(3,W)$ is a smooth hyperplane section of $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$.*

3. The Chow ring of B

We keep the notations of section 2. We study here the Chow ring of B. The codimension 2 part of the Chow ring of B will appear to be surprisingly bigger than the codimension 2 part of the Chow ring of $G_w(3,W)$. To understand this ring, we will compare it to the Chow ring of I via the projection $f_2 : I \to B$.

Lemma 3.1. — *The variety B contains 16 quadric cones of dimension 2. For any point m different from the 16 vertices of these cones, the fiber $f_2^{-1}(\{m\})$ has length 4.*

Proof. — We proved in lemma 2.16 that the tangent cone of B is either four lines (with multiplicity) or the union of a line with a two-dimensional quadric cone with smooth basis. So if m is not the vertex of a two-dimensional quadric cone, the fiber $f_2^{-1}(\{m\})$ has length 4.

Now, let Γ be such a quadric cone included in B, lemma 3.5 of [6] says that there is a point e of $\mathbb{P}(W)$ included in all the planes π_u for u in Γ. So Γ is the intersection of the 3-dimensional quadric $Q_e = \{v \in G_w(3,W) | e \in \pi_v\}$ with the pencil of hyperplanes containing B. This proves that one of them contains Q_e. We deduce from proposition 3.3 of [6] that this hyperplane must be one of the H_i defined in section 1 and that e is on one of the four conics C_i. But we proved in the lemma 1.11 that there are only four cones for each C_i. $\square$

Notations. — In the sequel, we will denote the Chow ring over $\mathbb{Q}$ of a variety X by A_X. Let $c'_1 = h_3$, c'_2 and c'_3 be the Chern classes of the tautological quotient Q_3 of $W \otimes \mathcal{O}_{G_w(3,W)}$. Denote by h'_3 the class $f_2^* h_3$ in A_I, and let a_i be the second Chern class of the bundle E_i.

First, we deduce from the section 2 and lemma 2.2, that the Chow ring of the incidence variety I is:

Lemma 3.2. — *The Chow ring of I is*

$$A_I = \frac{A_{F_B}[h'_3]}{(h'^2_3 - 2.h_2.h'_3 + \frac{4}{3}.h_2^2)}$$

At this point, we need to get more informations about F_B.

Lemma 3.3. — *Let i, j, k, l be four distinct elements of $\{1, \ldots, 4\}$. The variety F_B can be naturally identified with the blow up of $C_j \times C_k \times C_l$ in an elliptic sextic curve, and v_i is the exceptional divisor $h_2 - 2.\alpha_i$.*

Proof. — In the previous section, we proved that F_B is the vanishing locus of a section of $\mathcal{O}_{C_i \times C_j \times C_k \times C_l}(\sum_{a=1}^{4} \sigma_a)$. This section gives an inclusion of $(H^0(\mathcal{O}_{C_i}(\sigma_i)))^{\vee}$ into $H^0(\mathcal{O}_{C_j \times C_k \times C_l}(\sigma_j + \sigma_k + \sigma_l))$. So we can consider C_i as a marked pencil of hyperplane sections of $C_j \times C_k \times C_l$ such that F_B

is the incidence variety:

$$\{(x,h)|x \in C_j \times C_k \times C_l, h \in C_i, h(x) = 0\}$$

In other words, the variety F_B is identified with the blow up of $C_j \times C_k \times C_l$ in the intersection of this Segre threefold with the marked pencil of hyperplanes. The contraction is given by the linear system $|h_2 - \alpha_i|$ and the class of the exceptional divisor is $h_2 - 2\alpha_i$. By the adjunction formula, this intersection is an elliptic sextic curve.

We will now prove that v_i is the exceptional divisor. Let a, b be distinct elements of $\{j,k,l\}$. Choose a point p_a on C_a and another one p_b on C_b such that $\mathbb{P}(p_a^\perp \cap p_b^\perp) \cap C_i = \emptyset$. The divisor v_i was constructed in lemma 2.3 as the set of lines included in the hyperplane section $\bar{H}_{u_i}$ of B. So any element δ of v_i has its corresponding line Δ in $\bar{H}_{u_i}$. Consequently the line $\mathbb{P}(L_\delta)$ intersects C_i. So $\mathbb{P}(L_\delta)$ can't be in $\mathbb{P}(p_a^\perp \cap p_b^\perp)$, and we have the relation $\alpha_a.\alpha_b.v_i = 0$. Now, lemma 2.17 implies $(h_2-\alpha_i)^2.v_i = 0$, so set theoretically, v_i is the exceptional divisor. But we proved in lemma 2.6 that v_i is reduced. □

Lemma 3.4. — *The projection $f_{1*}(f_2^* a_i)$ of the second Chern class of $f_2^*E_i$ to A_{F_B} is $h_2 - \alpha_i$. Furthermore, we have in A_I the equality: $f_2^* a_i = h_3'.(h_2-\alpha_i) + h_2(2\alpha_i - h_2)$*

Proof. — From lemma 3.3 and proposition 2.5, the first Chern class of $f_{1*}(f_2^*E_i)$ is $-\alpha_i - v_i = \alpha_i - h_2$. We will obtain the first assertion from Riemann-Roch-Grothendieck's theorem:

We have from the description of I at the beginning of section 2 the equality $\omega_{f_1} = \mathcal{O}_I(2h_2 - 2h_3')$. So the Todd class of the relative tangent bundle and the Chern character of $f_2^*(E_i)$ are at order two:

$$\mathrm{td}(T_{f_1}) = 1 + (h_3' - h_2) + \frac{(h_3'-h_2)^2}{3}, \ \mathrm{ch}(f_2^*(E_i)) = 2 - h_3' + \frac{h_3'^2 - 2f_2^*(a_i)}{2}.$$

From lemma 3.2, their product modulo terms of order two is: $2 + h_3 - f_2^*(a_i) + P(h_2)$ (where $P(h_2)$ is a polynomial in h_2). But we obtained in the proof of proposition 2.5 the vanishing of $R^1 f_{1*} f_2^* E_i$, so the first Chern class of $f_{1*}f_2^*E_i$ is from Riemann-Roch-Grothendieck's theorem $-f_{1*}f_2^*(a_i)$, and we have the equality $f_{1*}f_2^*(a_i) = h_2 - \alpha_i$.

To obtain the second assertion, we will study the cokernel of the evaluation map $f_1^* f_{1*} f_2^* E_i \to f_2^* E_i$. Again remark the vanishing of $R^1 f_{1*}(f_2^*E_i)$. So the relative Beilinson's spectral sequence (cf proof of lemma 1.12) ends at the first level and we have the following exact sequence:

$$0 \to f_1^* f_{1*} f_2^* E_i \to f_2^* E_i \to \omega_{f_1}(h_3') \otimes f_1^* R^1 f_{1*}(f_2^*(E_i(-h_3))) \to 0.$$

From the propositions 2.5 and 3.3, the sheaf $f_{1*}f_2^*E_i$ is $\mathcal{O}_{F_B}(\alpha_i - h_2)$. But we have already obtained in the proof of proposition 2.5 that $R^1 f_{1*}(f_2^*(E_i(-h_3))$ is a line bundle on F_B, so the previous sequence is the following extension:

$$(3.1) \qquad 0 \to \mathcal{O}_I(f_1^*\alpha_i - f_1^*h_2) \to f_2^*E_i \to \mathcal{O}_I(f_1^*h_2 - f_1^*\alpha_i - h_3') \to 0,$$

and the computation of $f_2^*a_i$ follows. □

Lemma 3.5. — *Let $\gamma \in A_B^2$. The class $f_2^*\gamma$ can be written in A_I by $h_3'.\gamma_0 + \gamma_1$ for some elements γ_i in the vector space generated in $A_{F_B}^{i+1}$ by $h_2^i\alpha_1, \ldots, h_2^i\alpha_4$. Moreover, we have: $2h_2.\gamma_0 + \gamma_1 \in \mathbb{Q}.h_2^2$.*

Proof. — We can first find classes γ_i in $A_{F_B}^{i+1}$ such that $f_2^*\gamma = h_3'.\gamma_0 + \gamma_1$. Then remark that A_B^3 is one dimensional. Indeed, the Picard group of B is $\mathbf{Z}$, the family of rational curves I is unsplit, and from lemma 2.16 B is covered by these curves because the tangent cone at any point of B is never empty. From the proposition 1.1 of [1], B is rationally chain connected for I, and from the proposition 3.13.3 of [10] $A_1(B)$ is one dimensional.

So $h_3.\gamma$ is proportional to h_3^3. But from the relation $h_3'^2 = 2.h_2.h_3' - \frac{4}{3}.h_2^2$, the class $f_{1*}h_3'^3$ is proportional to h_2^2. So the class $f_{1*}f_2^*(\gamma.h_3)$ is also in $\mathbb{Q}.h_2^2$, and we have $2h_2.\gamma_0 + \gamma_1 \in \mathbb{Q}.h_2^2$. We can now conclude that γ_1 is in the vector space generated by $h_2\alpha_1, \ldots, h_2\alpha_4$ because the description of F_B in lemma 3.3 says that γ_0 and h_2 are in the vector space generated by $\alpha_1, \ldots, \alpha_4$. □

Lemma 3.6. — *For all i in $\{1, \ldots, 4\}$, the class a_i is in the affine space $\frac{1}{2}.f_{2*}(h_2\alpha_i) + \mathbb{Q}.h_3^2$. Moreover, we have in A_B the relation $2(a_1 + a_2 + a_3 + a_4) = 3h_3^2$, and the classes (a_1, a_2, a_3, a_4) form a basis of the vector space A_B^2.*

Proof. — Denote by $\mathcal{V}$ the union of the vertices of the cones described in the lemma 3.1. The codimensions of $\mathcal{V}$ in B and of $f_2^{-1}(\mathcal{V})$ in I are respectively 4 and 3. So we have the isomorphism: $A_B^2 \simeq A_{B-\mathcal{V}}^2$, and $A_I^2 \simeq A_{I-\{f_2^{-1}(\mathcal{V})\}}^2$. From lemma 3.1, the map $f_2 : (I - \{f_2^{-1}(\mathcal{V})\}) \to (B - \mathcal{V})$ is flat and finite of degree 4. Therefore the restriction of $f_{2*}f_2^*$ to A_B^2 is by the projection formula $4.id_{A_B^2}$.

So, by lemma 3.5, A_B^2 is generated by the set of classes $\{f_{2*}(h_3'.\alpha_i), f_{2*}(h_2\alpha_i)\}_{i=1\ldots4}$. As the Picard group of B is generated by h_3, we obtain from the projection[5] formula that all the classes $f_{2*}(h_3'.\gamma_0)$ are proportional to h_3^2. Now, we use the relation $\frac{4}{3}.h_2^2 = 2.h_2.h_3' - h_3'^2$ to

[5] recall that h_3' is by definition $f_2^*(h_3)$

eliminate h_2^2 in the expression of $f_2^* a_i$ found in lemma 3.4. Consequently a_i is in the affine space $\frac{1}{2}.f_{2*}(h_2\alpha_i) + \mathbb{Q}.h_3^2$.

So we have proved that A_B^2 is generated by $h_3^2, a_1, a_2, a_3, a_4$. Moreover, we have $f_{1*}f_2^* a_i = h_2 - \alpha_i$ and $(h_2 - \alpha_i)_{i=\{1,\dots,4\}}$ is a free family in $A^1_{F_B}$, so the family $(a_1, \dots, a_4)$ is free in A_B^2.

To obtain the relation with h_3^2, we substitute the expression of $f_2^* a_i$ of lemma 3.4 in the relation found in lemma 2.17. We eliminate $h'_3.h_2$ with the relation $\frac{4}{3}.h_2^2 + h_3^{'2} = 2.h_2.h'_3$, and we obtain: $\sum\limits_{i=1}^{4} f_2^* a_i = \frac{3}{2} h_3^{'2}$. □

So we are now ready to describe the Chow ring of B.

Proposition 3.7. — *The Chow ring of B is $\mathbb{Q}[h_3, a_1, a_2, a_3, a_4]/\mathcal{I}$ where $\mathcal{I}$ is generated by $3h_3^2 - 2.\sum_{i=1}^4 a_i$, $(8h_3.a_i - 3h_3^3)_{i\in\{1,\dots,4\}}$, $(8.a_i.a_j - h_3^4)_{i\neq j,(i,j)\in\{1,\dots,4\}^2}$. (The class of a point is $\frac{a_1.a_2}{2}$, the class of the Veronese surface V_i is $2a_i - \frac{1}{2}h_3^2$, and $[V_i]^2$ is 4 points).*

Proof. — As A^i_B is known to be one dimensional for $i \in \{0, 1, 3, 4\}$, we just need to find the relations. This is done by computing degrees because we have found the structure of A_B^2 in lemma 3.6. The relations $(8h_3.a_i - 3h_3^3)$ are consequences of the facts that the degree of $G_w(3, 6)$ is 16 and that a_i can be represented by the union of a quadric Z_{i,p_i} and the Veronese surface V_i.

We have from lemma 3.5 the equality $f_2^*[V_i] = h'_3.v_i + \gamma_1$ for some element γ_1 of $h_2^2.\mathbb{Q} + h_2\alpha_i.\mathbb{Q}$. Moreover, by lemma 3.3 v_i is $h_2 - 2\alpha_i$, so we obtain from lemma 3.6 that $[V_i]$ is in the vector space generated by h_3^2 and a_i and $[Z_{i,p_i}]$ also. From the degrees of these varieties, we have: $[V_i] = 2a_i - \frac{1}{2}h_3^2$ and $[Z_{i,p_i}] = \frac{1}{2}h_3^2 - a_i$. The last relations are deduced from the equalities $[V_i].[V_j] = 0$ for $i \neq j$. □

4. Application to quadratic normality

In this part, we explain the link of our construction with the congruence of lines found in [14]. They start with the intersection of $G(2, 6)$ by a very particular $\mathbb{P}_{11}$. Denote by Γ this five dimensional intersection. Recall that the choice of this $\mathbb{P}_{11}$ is proved in [13] to be unique up to the action of GL_6. After that, they choose a general quadric containing a fixed subscheme(6) of Γ to obtain a reducible intersection with Γ. Then they check with Macaulay2 that one of these irreducible components is smooth of degree 16 and sectional genus 9.

(6) This particular subscheme contains the singular locus of Γ.

Here we prove that the generic Fano 4-fold of genus 9 with Picard number 1 can be obtained by their construction, and that the choice we need to do is generically finite. More precisely, the choice of Γ corresponds to the choice of a tangent hyperplane H to $G_w(3,6)$, and the choice of the quadric corresponds to the choice of a generic element in the twelve dimensional linear system $|\mathcal{O}_{\bar{H}}(h_3)|$.

Proposition 4.1. — *For any i in $\{1,2,3,4\}$, the bundle $E_i(h_3)$ gives an embedding of the Fano manifold B in the Grassmannian $G(2,6)$ as the congruence of lines constructed in theorems 8,9,10 of* [14].

Proof. — Their description of the congruence is in terms of equations, so we need to make an adapted choice of coordinates to get the link. Choose a basis $w_0,\dots,w_5$ of W such that $\omega^\vee = w_0\wedge w_3 + w_1\wedge w_4 + w_2\wedge w_5$. Let A and B be the vector spaces generated respectively by w_0,w_1,w_2 and w_3,w_4,w_5. Note that the form ω gives an identification between $A^\vee$ and B. The decomposition $W = A\oplus B$ gives a decomposition of $\bigwedge^3 W$. We can represent an element of $\bigwedge^3 W$ as in [6] by (a,X,Y,b), with $a\in\wedge^3 A$, $b\in\wedge^3 B$, $X\in Hom(A,B)$, $Y\in Hom(B,A)$. The equations of $G(3,6)$ are $\wedge^2 X = aY$, $\wedge^2 Y = bX$, $YX = ab.I_3$, and to obtain $G_w(3,6)$ we add the linear relations $X = {}^tX$ and $Y = {}^tY$. So take $X = \begin{pmatrix} x_0 & x_1 & x_2\\ x_1 & x_5 & x_3\\ x_2 & x_3 & x_4\end{pmatrix}$ and $Y = \begin{pmatrix} y_0 & y_1 & y_2\\ y_1 & y_5 & y_3\\ y_2 & y_3 & y_4\end{pmatrix}$.

Now we need to choose a hyperplane H as in section 1.1. Consider the hyperplane $y_5 = y_2$. It is tangent to $G_w(3,6)$ in $[w_0\wedge w_1\wedge w_2]$ and contains $[w_3\wedge w_4\wedge w_5]$. Moreover, the conic described in lemma 1.1 is parameterized by $\lambda^2 w_0 + \lambda\mu w_1 + \mu^2 w_2$, so the incidence Z_H is given in $\mathbb{P}_1\times\bar{H}$ by the equations:

$$a = 0, X.\begin{pmatrix}\lambda^2\\ \lambda\mu\\ \mu^2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix} -\mu & \lambda & 0\\ 0 & -\mu & \lambda\end{pmatrix}.Y = \begin{pmatrix}0&0&0\\0&0&0\end{pmatrix}$$

The 6 dimensional vector space $H^0(\mathcal{I}_{Z_H}(1,1))$ is generated by

$$(-a.\lambda, -a.\mu, \lambda y_1 - \mu y_0, \lambda y_2 - \mu y_1, \lambda y_3 - \mu y_2, \lambda y_4 - \mu y_3).$$

The sheaf $\mathcal{E}$ constructed in proposition 1.4 is the image of the map:

$$6\mathcal{O}_{\bar{H}}(-1) \xrightarrow{\begin{pmatrix} a & 0 & y_0 & y_1 & y_2 & y_3\\ 0 & a & -y_1 & -y_2 & -y_3 & -y_4\end{pmatrix}} 2\mathcal{O}_{\bar{H}}.$$

Now remark that it is exactly the map of [14], Theorem 9. A generic hyperplane section of $G_w(3,6)$ gives a linear relation between $a,(x_i),(y_i),b$. As it is exactly the relation 29 of [14], we have obtained the identification with their congruence of lines. □

4.1. Geometry of the focal locus of Mezzetti-de Poi's congruence

Here we choose one of the 4 bundles, say E_1.

Notations. — Denote by Ξ the projective bundle $Proj(E_1(h_3))$, by h its relative hyperplane class, and by p_B the projection from Ξ to its basis B. We proved in corollary 1.8 that the linear system $|h|$ gives a morphism from Ξ to $\mathbb{P}_5$. Denote it by r.

$$B \xleftarrow{p_B} \Xi \xrightarrow{r} \mathbb{P}_5$$

LEMMA 4.2. — *The morphism $r : \Xi \to \mathbb{P}_5$ is birational. The class of the exceptional divisor $\mathcal{R}$ in Ξ is $4h - h_3$.*

Proof. — The degree of r is given by the length of the degeneracy locus of a generic map $5\mathcal{O}_B \to E_1(h_3)$, so it is the fourth Segre class of E_1. This class is $h_3^4 - 3h_3^2 a_1 + a_1^2$, and from proposition 3.7 we obtain the relations $h_3^4 = 16$, $a_1 h_3^2 = 6$, $a_1^2 = 3$, so r is birational. As the canonical sheaf of B is $\mathcal{O}_B(-2h_3)$, the canonical sheaf of Ξ is $\mathcal{O}_\Xi(-h_3 - 2h)$. So the class of the exceptional divisor of r is $4h - h_3$. □

Notations. — Denote by X the locus $r(\mathcal{R})$, often called the focal locus of the congruence of lines B. Recall from [14] that X has dimension 3, degree 6, and is singular along a rational smooth cubic curve C.

The manifold B gives the family $LB = \{r(\mathbb{P}(E_{1,u}))|u \in B\}$ of lines in $\mathbb{P}_5$. From lemma 4.2, any element of LB not included in X intersects X in length 4. We can now describe easily the normalisation $\tilde{X}$ of X. We will study its geometry to point out the analogies with the Palatini threefold. To get a better comparison with the Palatini threefold, some proofs will be valid in a more general setting (cf propositions 4.7, 4.10, 4.11). Take the following:

DEFINITION 4.3. — *We will say that a variety B is "a congruence satisfying (4.3)" if it is a smooth subvariety of $G(2,6)$ of dimension 4 such that:*

- *The projection to $r : \Xi \to \mathbb{P}_5$ is birational, with an irreducible exceptional divisor $\mathcal{R}$ contracted to a codimension 2 subvariety X of $\mathbb{P}_5$.*

- *The canonical sheaf of B is the restriction of $\mathcal{O}_{G(2,6)}(-2)$.*
- *The variety F_B of lines included in B has pure dimension 3, and the projection f_2 is onto.*

The restriction to B of the tautological rank two bundle will be denoted by E_1 and its determinant by $\mathcal{O}_B(-h_3)$. The incidence variety $Proj(E_1(h_3))$ will again be noted Ξ. The projections from Ξ to $\mathbb{P}_5$ and B will be r and p_B. Still denote by I the incidence variety (point of B)/(line of B). The projections from I to F_B and B will be f_1 and f_2.

The congruences studied in the proposition 4.1 will be called "congruences of type MdP".

Example 4.4. — A generic codimension four linear section of $G(2,6)$ is a congruence satisfying (4.3). It is the Mukai model for a genus 8 Fano 4-fold. Its focal locus X is smooth and irreducible, it will be called a Palatini threefold. In this example, the morphism f_2 is also of degree four.

Proof. — Such a linear section of $G(2,6)$ is a smooth congruence of order 1. By push-forward from Ξ, we have the exact sequence:

$$0 \to 4\mathcal{O}_{\mathbb{P}_5} \xrightarrow{w} \Omega^1_{\mathbb{P}_5}(2) \to I_X(4) \to 0.$$

So, by the genericity of w, X is smooth and irreducible. The degree of f_2 is the number of lines included in B through a general point v of B. But the tangent cone of $G(2,6)$ at v is a cone over $\mathbb{P}_1 \times \mathbb{P}_3$. So its intersection with B is four lines. Therefore, the projection f_2 is surjective and of degree 4 over B. The variety F_B is embedded in the incidence variety $F_{1,3}$ (point of $\mathbb{P}_5$)/(plane of $\mathbb{P}_5$). Denote by j and j' the projections from the incidence variety point/line/plane to $F_{1,3}$ and $G(2,6)$. The variety F_B is defined in $F_{1,3}$ as the vanishing locus of a generic section of the globally generated vector bundle $\overset{4}{\underset{i=1}{\oplus}} j_*(j'^*\mathcal{O}_{G(2,6)}(1))$, so it has pure dimension 3. □

Remark 4.5. — The lemma 4.2 is true for any congruence satisfying (4.3).

Proof. — By assumption r is birational, and the computation of the class of $\mathcal{R}$ is the same because the canonical divisor of B is not changed. □

We should remark that a congruence of type MdP will satisfy (4.3) after the proof of the proposition 4.7 and its corollary 4.8. Before that, we consider again the situation of a congruence of type MdP to obtain the following:

Lemma 4.6. — *The restriction of f_2 to $f_1^{-1}(v_i)$ is a morphism of degree two from $f_1^{-1}(v_i)$ to $\bar{H}_{u_i}$. We have the equalities $f_{2*}f_1^*(\alpha_i) = h_3$ and $4h_3 = f_{2*}f_1^*(h_2)$.*

Proof. — The equality $f_2(f_1^{-1}(v_i)) = \bar{H}_{u_i}$ is just a corollary of lemma 2.3. To compute the degree of this morphism, we chose a generic point b of $\bar{H}_{u_i}$. From the definition of v_i in lemma 2.3, this degree is the number of lines included in $\bar{H}_{u_i}$ containing b. The lines of $\bar{H}_{u_i}$ are described in lemma 2.3 as the lines included in the quadrics $(Z_{i,p_i})_{p_i \in C_i}$. As b is generic in $\bar{H}_{u_i}$, it is not in the Veronese surface V_i, so there is only one point p_i of C_i such that b is in the quadric Z_{i,p_i}. So the only lines of $\bar{H}_{u_i}$ through b are the two rulings of Z_{i,p_i} containing b. Therefore the restriction of f_2 to $f_1^{-1}(v_i)$ has degree two over $\bar{H}_{u_i}$ and we have $f_{2*}f_1^*(v_i) = 2h_3$.

The description of F_B in lemma 3.3 as a blowup gives the relation $v_i = h_2 - 2.\alpha_i$. So we have from lemma 2.17 the equality $8h_3 = \sum_{i=1}^4 f_{2*}f_1^*(v_i) = 2f_{2*}f_1^*(h_2)$ and we obtain $f_{2*}f_1^*(\alpha_i) = h_3$. □

Proposition 4.7. — *Let B be a congruence satisfying (4.3). The projective bundle $Proj(f_2^*(E_1(h_3)))$ over I has a section ρ such that the restriction of $f_2^*(\mathcal{O}_\Xi(h))$ to $\rho(I)$ is the line bundle $(f_1^*(f_{1*}f_2^*(E_1)))^{\mathrm{v}}$. Moreover, we have $f_2(\rho(I)) = \mathcal{R}$, and this equality remains true for any congruence of type MdP.*

Proof. — In the proof of lemma 3.4 we proved that the cokernel of the evaluation map $f_1^*f_{1*}f_2^*E_1 \to f_2^*E_1$ is a line bundle over I (cf the extension (3.1)). We should point out here that it was obtained from the fact that the restriction of E_1 to any line Δ of B is $\mathcal{O}_\Delta \oplus \mathcal{O}_\Delta(-1)$. In particular it is true for any congruence of lines satisfying (4.3) because E_1 is the tautological bundle of rank two.

As the cokernel of $f_1^*f_{1*}f_2^*E_1 \to f_2^*E_1$ is invertible, it must be the line bundle $(f_1^*f_{1*}f_2^*E_1)^{\mathrm{v}}(-h_3)$. So we have a surjection from $f_2^*(E_1(h_3))$ to the line bundle $(f_1^*f_{1*}f_2^*E_1)^{\mathrm{v}}$. This defines a section ρ of the fibration $Proj(f_2^*(E_1(h_3)))$ such that the restriction of $f_2^*(\mathcal{O}_\Xi(h))$ to $\rho(I)$ is $(f_1^*(f_{1*}f_2^*(E_1)))^{\mathrm{v}}$. As this bundle is the pull back of a line bundle of F_B, the composition $r \circ f_2 \circ \rho$ contracts the fibers of $f_1 : I \to F_B$. So we have the inclusion: $f_2(\rho(I)) \subset \mathcal{R}$. Therefore in the case of a congruence satisfying (4.3), the equality follows from the irreducibility of $\mathcal{R}$ and the dimension of F_B.

Now consider a congruence of type MdP. From the proposition 2.5, the class of $\rho(I)$ in $Proj(f_2^*(E_1(h_3)))$ is $f_2^*(h - h_3) + f_1^*(h_2) - f_1^*(\alpha_1)$. The projection formula gives $f_{2*}([\rho(I)]) = 4(h - h_3) + f_{2*}f_1^*(h_2) - f_{2*}f_1^*(\alpha_1)$

and by the lemmas 4.6 and 4.2 we have $f_{2*}([\rho(I)]) = 4h - h_3 = [\mathcal{R}]$. This achieve the proof of the equality $f_2(\rho(I)) = \mathcal{R}$ for that case. $\square$

Corollary 4.8. — *The focal locus X of Mezzetti-de Poi's congruence is a projection of $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ from a line*[7]*. The variety of pencils of lines belonging to LB is $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ blown up in the elliptic curve of degree 6 defined by the double cover of the cubic curve C via this projection.*

Proof. — From the propositions 2.5 and 4.7, the restriction of $f_2^*(h)$ to $\rho(I)$ is $f_1^*(h_2 - \alpha_1)$, and by lemma 2.17, it is also $f_1^*(\alpha_2+\alpha_3+\alpha_4)$. Moreover, we have by proposition 4.7 the equalities $r(f_2(\rho(I))) = r(\mathcal{R}) = X$. So the composition $r \circ f_2 \circ \rho$ is given by a linear subsystem of dimension five of $|f_1^*(\alpha_2+\alpha_3+\alpha_4)|$. So the morphism $r \circ f_2 \circ \rho$ from I to X can be decomposed like this:

$$r \circ f_2 \circ \rho : \quad I \xrightarrow{f_1} F_B \xrightarrow{r_2} C_2 \times C_3 \times C_4 \xrightarrow{r_1} X, \tag{4.1}$$

where r_2 is the contraction of the exceptional divisor v_1, and r_1 is the projection of $C_2 \times C_3 \times C_4$ from a line of $\mathbb{P}(H^0(\mathcal{O}_{C_2\times C_3\times C_4}(\sigma_2+\sigma_3+\sigma_4))^{\vee})$.

We will prove that the elliptic curves $r_2(v_1)$ and $r_1^{-1}(C)$ are the same. Indeed, let p_1 be generic in C_1, and b be generic in Z_{1,p_1}. Denote by $\delta_{p_1,b}$ and $\delta'_{p_1,b}$ the two points of F_B corresponding to the rulings of Z_{1,p_1} containing b. From lemma 2.3 and the genericity of p_1 and b, we obtain that $\delta_{p_1,b}$ and $\delta'_{p_1,b}$ are generic in v_1.

From the decomposition (4.1) of $r \circ f_2 \circ \rho$, the point $r_1(r_2(\delta_{p_1,b}))$ and $r \circ f_2 \circ \rho(\delta_{p_1,b}, b)$ are equal. From the equality $f_2 = p_B \circ f_2 \circ \rho$, this point is on the line $r(\mathbb{P}(p_B^{-1}(b)))$ because the element $(\delta_{p_1,b}, b)$ of I is mapped by f_2 to b. For the same reason, the point $r_1(r_2(\delta'_{p_1,b}))$ is also on the line $r(\mathbb{P}(p_B^{-1}(b)))$. But from the genericity of b and p_1, this line intersects the curve $r_1(r_2(v_1))$ in at most one point. So the two points $r_1(r_2(\delta_{p_1,b}))$ and $r_1(r_2(\delta'_{p_1,b}))$ are equal. As b is generic in Z_{1,p_1} we obtain that all the rulings of Z_{1,p_1} have the same image m_{p_1} by $r_1 \circ r_2$. So the curve $(r_1 \circ r_2)^{-1}(\{m_{p_1}\})$ has two irreducible components, and m_{p_1} is in the singular locus C of X. Moreover this construction gives an identification between the cubic curve C and C_1.

We conclude the proof of the corollary with the remark that the pencils of lines belonging to LB are (by definition) the lines included in B, so they are parameterized by F_B. $\square$

Remark 4.9. — From the proposition 4.7, the corollary 4.8 and the lemma 2.2, a congruence of type MdP satisfies (4.3).

[7] Note that we don't prove that this line is generic.

We can also obtain a description of the plane curves in X related to this family of lines:

Proposition 4.10. — *Let B be a congruence satisfying (4.3). Any point of F_B gives a plane in $\mathbb{P}_5$ intersecting X in a point and a plane cubic.*

Moreover, if B is of type MdP, those plane cubics are singular in a point of C. In that case, only 12 lines included in X are in LB.

Proof. — Let B be a congruence satisfying (4.3). Any point δ of F_B corresponds to a line Δ included in B. The restriction of the tautological bundle E_1 to Δ is $\mathcal{O}_\Delta \oplus \mathcal{O}_\Delta(-1)$. Lemma 4.2 implies that the intersection of $\mathcal{R}$ with $p_B^{-1}(\Delta)$ is given by a section of $(S_4(E_1))(3h_3)$. This intersection contains the exceptional divisor of $Proj(\mathcal{O}_\Delta \oplus \mathcal{O}_\Delta(1))$ which is contracted to a point by r. So the plane $r(p_B^{-1}(\Delta))$ intersects X in a cubic curve and a unique residual point included in all the lines $(r(\mathbb{P}(E_{1,p})))_{p\in\Delta}$.

Now we assume that B is of type MdP. From the lemma 3.1, and the equality $f_2(\rho(I)) = \mathcal{R}$ of lemma 4.7, the projection $\mathcal{R} \to B$ is finite of degree four except over 16 points of B. To understand why the plane cubic described above is singular, we first notice that it is the image of the following curve T_Δ of F_B: The closure in F_B of the lines included in B intersecting Δ and different from Δ. A general Δ intersects the hyperplane section $\bar{H}_{u_1}$ in a single point b not lying on the Veronese V_1. From corollaries 1.14 and 1.15, there is a single quadric Z_{1,p_1} containing b. The two lines containing b and included in this quadric are included in $\bar{H}_{u_1}$. So, from lemma 2.3, they correspond in F_B to two involutive points on the exceptional divisor v_1 of F_B with respect to the degree two morphism described in lemma 4.6. In conclusion, for a general Δ, the curve T_Δ intersects the exceptional divisor v_1 in 2 points that have the same image by $r_1 \circ r_2$. So the plane cubic is singular at this point of C.

For the same reason, if b is one of the four vertices of the cones of $\bar{H}_{u_1} \cap B$, the line $\mathbb{P}(p_B^{-1}(b))$ is contracted by r to a point on the curve C. This is why only twelve lines of X belong to the family LB. □

When a subvariety of $\mathbb{P}_n$ is not k-normal, there are elements of $|\mathcal{O}_X(k)|$ that can't be extended to a hypersurface of $\mathbb{P}_n$ of degree k. Such a divisor will be called a virtual section. The first example of non linearly normal variety, is a Veronese surface in $\mathbb{P}_4$. We can remark that this surface has a natural duality, and also a marked virtual section of $\mathcal{O}(1)$. In the case of a non 2-normal threefold of $\mathbb{P}_5$ we have shown above that we can expect some correspondence *between points and plane cubics instead of a duality. We point out here that there is also a naturally marked virtual section of $\mathcal{O}_X(2)$ associated to the construction and study its geometry.*

PROPOSITION 4.11. — *Let B be a congruence satisfying (4.3). The ramification divisor $\mathcal{D}_{p_B}$ of the restriction of p_B to $\mathcal{R}$ is the pullback of a virtual section of $\mathcal{O}_X(2)$.*

Proof. — Locally $\mathcal{D}_{p_B}$ is defined in the projective bundle Ξ by the vanishing of the two partial derivatives of the equation of $\mathcal{R}$ relatively to p_B. So we define $\mathcal{D}_{p_B}$ in Ξ as the support of the cokernel of the polarization map:

$$p_B^*(E_1(h_3))^\vee \to \mathcal{O}_\Xi(\mathcal{R} - h).$$

Therefore, it is the vanishing locus of a section ζ of $p_B^*(E_1)(h_3 + \mathcal{R} - h)$. From the isomorphism $E_1 \simeq E_1^\vee(-h_3)$, the Koszul complex of this section is:

$$0 \to \omega_{p_B}^\vee(-2\mathcal{R}) \to p_B^*(E_1)(h - \mathcal{R}) \xrightarrow{{}^t\zeta} \mathcal{O}_\Xi \to \mathcal{O}_{\mathcal{D}_{p_B}} \to 0$$

where ω_{p_B} is the relative dualising sheaf of $p_B : \Xi \to B$. By the Euler formula, the tautological map from $\mathcal{O}_\Xi(-\mathcal{R})$ to $p_B^*(E_1)(h - \mathcal{R})$ composed with ${}^t\zeta$ is the multiplication by the equation of $\mathcal{R}$ from $\mathcal{O}_\Xi(-\mathcal{R})$ to $\mathcal{O}_\Xi$. So we have the commutative diagram:

(4.2)

$$\begin{array}{ccccccccccc}
 & & & & \mathcal{O}_\Xi(-\mathcal{R}) & \to & \mathcal{O}_\Xi & \to & \mathcal{O}_\mathcal{R} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \omega_{p_B}^\vee(-2\mathcal{R}) & \to & p_B^*(E_1)(h-\mathcal{R}) & \xrightarrow{{}^t\zeta} & \mathcal{O}_\Xi & \to & \mathcal{O}_{\mathcal{D}_{p_B}} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & & & \\
0 & \to & \omega_{p_B}^\vee(-2\mathcal{R}) & \to & \omega_{p_B}^\vee(-\mathcal{R}) & & & & & &
\end{array} \cdot$$

From the snake lemma, the kernel of the surjection from $\mathcal{O}_\mathcal{R}$ to $\mathcal{O}_{\mathcal{D}_{p_B}}$ is $\omega_{p_B}^\vee(-\mathcal{R}) \otimes \mathcal{O}_\mathcal{R}$. So the ramification $\mathcal{D}_{p_B}$ is given on $\mathcal{R}$ by the vanishing of a section of the line bundle $\omega_{p_B}(\mathcal{R}) \otimes \mathcal{O}_\mathcal{R}$. By lemma 4.2, this line bundle is $r^*(\mathcal{O}_X(2))$. Let $I_{\mathcal{D}_{p_B}}$ be the ideal of $\mathcal{D}_{p_B}$ in Ξ. We have the exact sequence:

$$0 \to \mathcal{O}_\Xi(-\mathcal{R}) \to p_B^*(E_1) \otimes \omega_{p_B}(h) \to I_{\mathcal{D}_{p_B}} \otimes \omega_{p_B}(\mathcal{R}) \to 0.$$

By the projection formula, all the sheaves $R^i p_{B*}(p_B^*(E_1) \otimes \omega_{p_B}(h))$ are zero for i in $\{0,1\}$. From lemma 4.2, the sheaf $p_{B*}(\mathcal{O}_\Xi(-\mathcal{R}))$ vanishes also. So, by the Leray spectral sequence, the groups $H^0(p_B^*(E_1) \otimes \omega_{p_B}(h))$ and $H^1(p_B^*(E_1) \otimes \omega_{p_B}(h))$ are zero, and we have the equalities:

$$h^0(I_{\mathcal{D}_{p_B}} \otimes \omega_{p_B}(\mathcal{R})) = h^1(\mathcal{O}_\Xi(-\mathcal{R})) = h^0(R^1 p_{B*}(\mathcal{O}_\Xi(-\mathcal{R}))).$$

By relative duality, the sheaf $R^1 p_{B*}(\mathcal{O}_\Xi(-\mathcal{R}))$ is the dual of $p_{B*}(\omega_{p_B}(\mathcal{R}))$. So we have the equalities:

$$h^0(I_{\mathcal{D}_{p_B}}(2h)) = h^0(S_2(E_1)) = h^0(\mathcal{O}_\Xi(2h - 2h_3)).$$

Assume that the linear system $|\mathcal{O}_\Xi(2h-2h_3)|$ contains some element $\mathcal{R}'$. The restriction of $\mathcal{R}'$ to any line Δ included in B is twice the exceptional divisor of $Proj(\mathcal{O}_\Delta \oplus \mathcal{O}_\Delta(1))$. But this divisor is contracted to a point by r, so $\mathcal{R}'$ is in the exceptionnal divisor of r. But from lemma 4.2 and $\mathcal{R}$ would be reducible. So the existence of $\mathcal{R}'$ contradicts one of the properties of definition 4.3. In conclusion we have $h^0(I_{\mathcal{D}_{p_B}}(2h)) = 0$. So $r(\mathcal{D}_{p_B})$ is a virtual section of $\mathcal{O}_X(2)$. □

Now using properties that are strongly related to the geometry of congruences of type MdP, we will obtain a more detailled description of this virtual section. The corollary 4.8 gives a natural geometric relation between the marking of a line in the following two projective spaces: The span of the first and second variety of the third row of the extended Freudenthal magic square (cf [7] *p101). We should also point out that the invariant algebra for the θ-group $SL_2 \otimes \bigwedge^3 Sp_6$ (cf* [9] *table III) is freely generated by polynomials of degree* 2, 6, 8, 12. *Moreover, the same result is true for the invariant algebra of the semi direct product of $(SL(2,\mathbb{C}))^{\otimes 4}$ with the permutation group σ_4 (cf* [16]*). Is the following configuration related to another case of the table of* [9] *with invariants of the same degrees?*

Proposition 4.12. — *The ramification locus $\mathcal{R}_{f_2}$ of the morphism $f_2 : I \to B$ is $f_1^{-1}(\Sigma)$ for some surface Σ in $|\mathcal{O}_{F_B}(h_2)|$. The canonical sheaf of Σ is trivial, and Σ contains the* 16 *rational curves parameterizing the* 16 *cones of B. The image of Σ in X is the virtual section $r(\mathcal{D}_{p_B})$.*

Proof. — The canonical divisor of I can be computed from §2, and we get that $\mathcal{R}_{f_2} \sim f_1^* h_2$. The 16 contracted curves must be in $\mathcal{R}_{f_2}$. We proved in proposition 4.7 the equality $f_2(\rho(I)) = \mathcal{R}$. So we obtain the inclusion $(f_2 \circ \rho)^{-1}(\mathcal{D}_{p_B}) \subset \mathcal{R}_{f_2}$ from the equality $f_2 = p_B \circ f_2 \circ \rho$. So the virtual section $r(\mathcal{D}_{p_B})$ is included in $r_2 \circ r_1(\Sigma)$. We obtain the last assertion from the fact that $r_2 \circ r_1(\Sigma)$ is also in $|\mathcal{O}_X(2)|$ by the lemma 3.3. $\square$

4.2. A rank 2 reflexive sheaf on $\mathbb{P}_5$

A general double hyperplane section of X is by corollary 4.8 a smooth elliptic curve of degree 6. From Serre's construction, it is the vanishing locus of a section of a rank 2 vector bundle on $\mathbb{P}_3$, unique up to isomorphism. Curiously, there is a way to globalize this construction over $\mathbb{P}_5$. In this part we will construct an SL_2-equivariant rank 2 reflexive sheaf on $\mathbb{P}_5$ using classical techniques developed in mathematical instanton studies (cf [2], [17]). Consider again vector spaces L and V of respective dimensions 2 and 6. This construction will be essentially unique up to the SL_6 action. Indeed, it could be constructed from a tangent hyperplane to $G_w(3,6)$ or as follows:

Recall from [13] that $S_2L \otimes \bigwedge^2 V$ has an $SL_2 \times SL_6$-orbit whose points correspond to nets of alternating forms of constant rank 4. Let $\beta \in S_2L \otimes \bigwedge^2 V$ be an element of this orbit. This element was considered by E. Mezzetti and P. de Poi in [14] to construct the special $\mathbb{P}_{11}$ containing their congruence of lines. For instance, take the following β:

$$\begin{pmatrix} 0 & u^2 & 2uv & v^2 & 0 & 0 \\ -u^2 & 0 & 0 & 0 & 0 & 0 \\ -2uv & 0 & 0 & 0 & 0 & u^2 \\ -v^2 & 0 & 0 & 0 & 0 & 2uv \\ 0 & 0 & 0 & 0 & 0 & v^2 \\ 0 & 0 & -u^2 & -2uv & -v^2 & 0 \end{pmatrix}$$

We can remark that β viewed as an element of $\bigwedge^2(L \otimes V)$ has rank 6 because it can be represented by:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

Denote by W the six dimensional image of this map, we will identify W with its dual via the induced alternating form. The inclusion $W \subset L \otimes V$ gives a map β' from W to $L \otimes \mathcal{O}_{\mathbb{P}_5}(1)$, and we can construct a complex:

$$L \otimes \mathcal{O}_{\mathbb{P}_5}(-1) \xrightarrow{{}^t\beta'} W \otimes \mathcal{O}_{\mathbb{P}_5} \xrightarrow{\beta'} L \otimes \mathcal{O}_{\mathbb{P}_5}(1) \tag{4.3}$$

with ${}^t\beta'$ injective, whose middle cohomology is a rank 2 reflexive sheaf K. The right cohomology is a sheaf $\mathcal{L}$, supported on a smooth rational cubic curve C, isomorphic to $\mathcal{O}_{\mathbb{P}_1}(4)$. So we can compute from the complex (4.3) some of the invariants of K:

Corollary 4.13. — *The sheaf K has rank 2 and Chern numbers $c_1(K) = 0$, $c_2(K) = 2$, $c_3(K) = 0$, $c_4(K) = -15$. Its singular locus is the cubic curve C. We have $H^0(K(1)) = 0$ and $H^0(K(2)) = 13$, and $H^1(K) = 1$.*

In particular, a section of $K(2)$ vanishes on a 3-fold X' of degree 6 singular along C. From the exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}_5}(-2) \to K \to I_{X'}(2) \to 0,$$

we have the equalities $h^1(I_{X'}(2)) = h^1(K) = 1$. Therefore X' is not quadratically normal.

So we have constructed from a fixed Γ a linear family of projective dimension twelve of non quadratically normal varieties of degree 6 and singular along the cubic curve C.

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