ANNALES

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Jacques-Élie FURTER & Angela Maria SITTA

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Tome 60, nº 4 (2010), p. 1363-1400.

<http://aif.cedram.org/item?id=AIF_2010__60_4_1363_0>

Ann. Inst. Fourier, Grenoble
60, 4 (2010) 1363-1400

PATH FORMULATION FOR MULTIPARAMETER $\mathbb{D}_3$-EQUIVARIANT BIFURCATION PROBLEMS

by Jacques-Élie FURTER & Angela Maria SITTA (*)

Abstract. — We implement a singularity theory approach, the path formulation, to classify $\mathbb{D}_3$-equivariant bifurcation problems of corank 2, with one or two distinguished parameters, and their perturbations. The bifurcation diagrams are identified with sections over paths in the parameter space of a $\mathbb{D}_3$-miniversal unfolding F_0 of their cores. Equivalence between paths is given by diffeomorphisms liftable over the projection from the zero-set of F_0 onto its unfolding parameter space. We apply our results to degenerate bifurcation of period-3 subharmonics in reversible systems, in particular in the 1:1-resonance.

Résumé. — Nous utilisons une approche de la théorie des singularités pour classifier des problèmes de bifurcation $\mathbb{D}_3$-équivariants de corang 2, avec un ou deux paramètres de bifurcation distingués, et leurs perturbations. Les diagrammes de bifurcation sont identifiés avec des sections sur des chemins dans l'espace des paramètres d'un déployement miniversel $\mathbb{D}_3$-équivariant F_0 de leur noyau. Les équivalences entre les chemins sont données par des difféomorphismes qui se relèvent le long de la projection de l'ensemble des zéros de F_0 dans l'espace de ses paramètres. Nos résultats sont appliqués aux bifurcations dégénérées de solutions sous-harmoniques de période 3 dans des systèmes dynamiques réversibles, en particulier dans la résonance 1 :1.

1. Introduction

This work is about the implementation of a singularity theory approach, the *path formulation*, to classify multiparameter bifurcation problems and their perturbations. We chose problems in corank 2 equivariant with respect to the dihedral group $\mathbb{D}_3$ because their classification is only complete up to codimension 1 in the literature and such problems occur regularly because $\mathbb{D}_3$ is isomorphic to the permutation group $\mathbb{S}_3$, for instance in the

Keywords: Equivariant bifurcation, degenerate bifurcation, path formulation, singularity theory, 1:1-resonance, reversible systems, subharmonic bifurcation.
Math. classification: 37G40,58K70,58K40,34F10,34F15.
(*) Research supported by FAPESP and CAPES.

bifurcation of period-3 orbits in reversible systems. Other problems include five dimensional actions of $\mathbb{O}(3)$ ([23, 21]), permutations of 3 objects ([2]) or as the Weyl group of the isotropy subgroup $(\mathbb{S}_k)^3$ in $\mathbb{S}_{3k}$-equivariant problems ([13]). We show how to use the path formulation approach to study bifurcations of higher codimension and how versatile it is in dealing with parameter structures.

Singularity theory is a powerful tool to systematically classify and qualitatively analyze bifurcation problems and their perturbations. For problems corresponding to bifurcation equations of the type $f(x,\lambda)=0$, where $x\in\mathbb{R}^n$ are the state variables, $\lambda\in\mathbb{R}$ is the bifurcation parameter and $f:\mathbb{R}^{n+1}\to\mathbb{R}^n$ is a smooth bifurcation function, Golubitsky-Schaeffer introduced in [20] a parametrized version of contact equivalence theory. Two bifurcation functions f,g are **bifurcation equivalent** around $(0,0)$ if there exist smooth changes of co-ordinates (T,X,L) with

$$g(x,\lambda)=T(x,\lambda)f(X(x,\lambda),L(\lambda)) \tag{1.1}$$

where $T:\mathbb{R}^{n+1}\to \mathrm{GL}(n,\mathbb{R})$ is a map of invertible matrices and $(X,L):\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ is a local diffeomorphism around the origin. Note that (1.1) preserves the λ-slice structure of the **bifurcation diagrams**, the zero-sets of the bifurcation functions. Around a bifurcation point of f, where the Jacobian matrix $(\frac{\partial f}{\partial x})$ is singular, the equivalence (1.1) is arguably the best approach to classify systematically the possible normal form for f, solve its recognition problem and study its deformations.

In such analysis it is sensible to consider f as a germ of function at the origin to be able to state results that will persist on any neighbourhood of the origin. A **germ of a function** around a point is its equivalence class under the filtration by the neighbourhoods of that point. In general, we use the notation $f:(\mathbb{R}^n,x_0)\to\mathbb{R}^p$ to denote the germ of f around x_0. To make sense of equations like $f=0$, when $f:(\mathbb{R}^{n+1},0)\to\mathbb{R}^n$, we define **germs of sets**, or **germs of varieties**, at a point using filtrations by neighbourhoods of that point. Unless otherwise stated, we only consider here smooth germs. We denote by $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ a germ such that $f(0)=0$. We also know from singularity theory that the sets of germs of functions have nice algebraic properties (of local rings, noetherian in the complex case) that we can exploit in our algebraic calculation.

The equivalence (1.1) is extended to cover equivariant bifurcation germs, satisfying $f(\gamma x,\lambda)=\gamma f(x,\lambda)$ where $\gamma\in\Gamma$, a compact group acting on $\mathbb{R}^n$, by requiring additional equivariant structure on (T,X,L), see (2.2) here and [24] for a comprehensive account. In [25], the abstract theory for multiparameter equivariant bifurcation problems has been shown to obey the general abstract framework for singularity theory of Damon [10].

We implement an alternative singularity theory approach, the *path formulation*, to classify $\mathbb{D}_3$-equivariant one and two-parameter bifurcation germs of corank 2 and their perturbations. In the path formulation, a bifurcation germ f is viewed as an l-parameter deformation (unfolding) of its **core** f_0, $f_0(x) = f(x,0)$, with parameter λ. Hence, if the core has an appropriate miniversal a-parameter unfolding $F_0(x,\alpha)$, $\alpha \in (\mathbb{R}^a, 0)$, bifurcation diagrams are diffeomorphic to sections $\{(x,\lambda) \ : \ F_0(x,\bar{\alpha}(\lambda)) = 0\}$ over **paths** $\lambda \mapsto \bar{\alpha}(\lambda)$ in the parameter space of F_0 (see Section 2.2.1). Equivalence (1.1) is then replaced by equivalence (2.5) between paths given by diffeomorphisms liftable over the projection $\pi_{F_0} : (F_0^{-1}(0), 0) \to (\mathbb{R}^a, 0)$. We believe that the path formulation offers conceptual, even computational, advantages when we have multidimensional parameter problems. Once the initial set-up for the core has been obtained, it is simpler to take account in the algebraic calculations of the different path structures. But, if required, explicit changes of co-ordinates are easier to get from (1.1) because liftable diffeomorphisms are difficult to handle explicitly.

The path formulation has already been used in [3] in the context of $\mathbb{D}_4$-equivariant gradient problems, reproducing the classification of [22] obtained by classical means. For $k \geqslant 5$, some $\mathbb{D}_k$-equivariant normal forms have already been identified in the literature (see [6, 24]), but a systematic classification for all k is rather lengthy and the path formulation involves some other technical issues that are better discussed elsewhere.

1.1. Main results

Due to the technical nature of the main parts of this paper, we state here a self-contained version of our main results. Our bifurcation germs $f : (\mathbb{R}^{2+l}, 0) \to (\mathbb{R}^2, 0)$ are equivariant with respect to the standard action of $\mathbb{D}_3$ on $\mathbb{R}^2$, $f(\gamma z, \lambda) = \gamma f(z,\lambda)$, $\forall \gamma \in \mathbb{D}_3$, where $z = (x,y) \in (\mathbb{R}^2, 0)$. We establish two types of results. We classify one or two-parameter bifurcation germs of low codimension (a measure of their degeneracy) modulo changes of co-ordinates of type (1.1) respecting the $\mathbb{D}_3$-equivariance and give for each class a polynomial model, their **normal form**. To study perturbations of f, we introduce the notion of b**-parameter unfoldings** of f, $b \in \mathbb{N}$, as germs $F : (\mathbb{R}^{2+l+b}, 0) \to (\mathbb{R}^2, 0)$ such that $F(z,\lambda,0) = f(z,\lambda)$ and $F(\gamma z, \lambda, \beta) = \gamma F(z,\lambda,\beta)$, $\forall \gamma \in \mathbb{D}_3$. For each normal form, we give a distinguished polynomial unfolding of the normal form, a **miniversal unfolding**. From it, we can recover any information on the behaviour of the unfoldings of f modulo changes of co-ordinates, using the minimal number of parameters necessary. We use the path formulation to establish the classifications. The cores $f_0 : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ are $\mathbb{D}_3$-equivariant maps. Each core of finite codimension, say N, has a polynomial normal form and a polynomial miniversal unfolding $F_0 : (\mathbb{R}^{2+N}, 0) \to (\mathbb{R}^2, 0)$. Given an l-parameter path $\bar{\alpha} : (\mathbb{R}^l, 0) \to (\mathbb{R}^N, 0)$, the **pull-back** $\bar{\alpha}^* F_0$ defines a bifurcation germ $(\bar{\alpha}^* F_0)(z, \lambda) = F_0(z, \bar{\alpha}(\lambda))$. An unfolding A of $\bar{\alpha}$ determines an unfolding $A^* F_0$ of $\bar{\alpha}^* F_0$. We define in (2.5) an equivalence for the paths. Our fundamental result states that the theories of bifurcation and path equivalences coincide for problems of finite codimension.

Theorem (Theorem 5.1). — *Let f be a $\mathbb{D}_3$-equivariant bifurcation germ with a finite codimension core f_0, there is a path $\bar{\alpha}$ such that f is bifurcation equivalent to $\bar{\alpha}^* F_0$. The codimensions of f, with respect to bifurcation equivalence, and of $\bar{\alpha}$, with respect to path equivalence, are equal. A map A is a miniversal unfolding of $\bar{\alpha}$, with respect to path equivalence, if and only if $A^* F_0$ is a miniversal unfolding for $\bar{\alpha}^* F_0$, with respect to bifurcation equivalence. Finally, for finite codimension problems, two paths $\bar{\alpha}, \bar{\beta}$ are path equivalent if and only if $\bar{\alpha}^* F_0$ and $\bar{\beta}^* F_0$ are bifurcation equivalent.*

An essential ingredient for the theory, and calculations, of path equivalence is the module $\mathrm{Derlog}^*(F_0)$ of smooth vector fields liftable over the projection $\pi_{F_0} : (F_0^{-1}(0), 0) \to (\mathbb{R}^N, 0)$, as defined in (2.7). Like in the non equivariant case ([26]), we show that its sub-module of analytic liftable vector fields, obtained through the complexification procedure described in Section 2.2.3, has a useful algebraic structure.

Theorem (Theorem 4.3). — *For every $\mathbb{D}_3$-equivariant core f_0 of finite codimension N, the module of analytic vector fields liftable over the projection π_{F_0} is a free module of rank N.*

In the smooth case, the N analytic generators produce enough vector fields one can integrate to achieve the path equivalence between paths of finite codimension and calculate their miniversal unfoldings. Liftable vector fields are tangent to the discriminant of π_{F_0}. Like in the non-equivariant case (see [26]), the converse is true in our context for analytic vector fields.

Corollary (Corollary 4.4). — *An analytic vector field is liftable over π_{F_0} if and only if it is tangent to the discriminant of π_{F_0}.*

The second sets of results are explicit classifications. The ring of $\mathbb{D}_3$-invariant polynomials in two variables is generated by $u = x^2 + y^2$ and

$v = x^3 - 3xy^2$. The $\mathbb{D}_3$-equivariant polynomial maps on $\mathbb{R}^2$ form a free module over the ring of $\mathbb{D}_3$-invariants, generated by $Z_1 = (x, y)$ and $Z_2 = (x^2 - y^2, -2xy)$. We show that there are four normal forms of the cores up to topological codimension 3.

Proposition (Proposition 3.3). — *There are four classes of cores of topological codimension $i \leqslant 3$, denoted by C_i^3. In cartesian co-ordinates, miniversal unfoldings are the following. The normal forms are obtained by setting the unfoldings parameters α_i, $i = 1, 2, 3$, to 0.*

(1) *C_1^3, see (3.12): $\alpha_1 Z_1 + \epsilon Z_2$,*
(2) *C_2^3, see (3.13): $(\epsilon_1 u + \alpha_1) Z_1 + (\epsilon_2 v + \alpha_2) Z_2$,*
(3) *C_3^3, see (3.14): $(\epsilon_1 u + \alpha_1) Z_1 + (\epsilon_3 v^2 + \alpha_3 v + \alpha_2) Z_2$,*
(4) *$C_{3^*}^3$, see (3.15): $\big((\mu + \alpha_0) v + \alpha_3 u + \alpha_1\big) Z_1 + (\epsilon_4 u + \alpha_2) Z_2$,*

where $\epsilon^2 = \epsilon_i^2 = 1$, $i = 1 \ldots 4$. The asterisk * indicates that the parameter μ is **modal**, *invariant with respect to smooth equivalences.*

Finally, we classify the one and two-parameter paths of low codimension. Each of the normal form for the paths is given with a miniversal unfolding.

Theorem. — *There are 8 one-parameter paths of topological codimension $k \leqslant 2$, given in Theorem 5.2, and there are 5 two-parameter paths of topological codimension $k \leqslant 1$, given in Theorem 5.3.*

In Section 2, we define our main technical terms and concepts. In (2.5) we define the equivalence on paths reproducing the classification and analysis of the bifurcation equivalence (1.1) in the context of $\mathbb{D}_3$-equivariant germs. In Section 3, we discuss $\mathbb{D}_3$-equivariant cores, with their classification up to (topological) codimension 3 in Proposition 3.3. For each core C_i^3, $i = 1, 2, 3, 3^*$, we determine a miniversal unfolding and calculate in Proposition 4.2 the generators of its sub-module of $\mathrm{Derlog}^*(C_i^3)$ of analytic liftable vector fields. Those generators are central to the theory because they are used to perform the algebraic calculations we need, like the tangent spaces of the paths we classify. With those results, we prove our main results in Section 5. Theorem 5.1 states that, for finite codimension germs, path and bifurcation equivalences lead to the same classification. Some one-parameter bifurcation germs of topological codimension 1 or 2 have been obtained in [6, 24]. In Theorem 5.2, resp. Theorem 5.3, we complete the classification of one-parameter, resp. two-parameter, paths, up to topological codimension 2, resp. 1. In Section 6, we describe the bifurcation diagrams of the normal forms, and their miniversal unfoldings, needed for our application to the degenerate bifurcations of period-3 points in reversible maps at the 1:1-resonance. In each of our diagrams we can find a bifurcation of period-3 points for some reversible problem, but the 1:1-resonance has particular interest because it corresponds to a degenerate behaviour of the bifurcation parameters. Details of the low codimension problems for that resonance are presented in Section 7. We finish, in Section 8, commenting on variational problems and the $\mathbb{D}_k$-classification, $k \geqslant 4$.

2. Path versus bifurcation equivalences

2.1. Notations and preliminary structures

Via the notation $z = x + iy$ we identify $\mathbb{C}$ and $\mathbb{R}^2$. Let $\theta = \exp(\frac{2i\pi}{3})$ be a third root of unity. The standard action of $\mathbb{D}_3$ on $\mathbb{C}$ is generated by the rotation $z \mapsto \theta z$ and the reflection $z \mapsto \bar{z}$. We deal with smooth $\mathbb{D}_3$-equivariant l-parameter bifurcation germs $f : (\mathbb{C} \times \mathbb{R}^l, 0) \mapsto \mathbb{C}$ with $\lambda \in (\mathbb{R}^l, 0)$ and $f(\gamma z, \lambda) = \gamma f(z, \lambda)$, $\forall \gamma \in \mathbb{D}_3$. We denote their set by $\vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\lambda)}$. The superscript ^{o} denotes the evaluation of the germ at the origin and subscripts denote derivatives, for instance $f^o_{\lambda\lambda} = \frac{\partial^2 f}{\partial \lambda^2}(0,0)$. Because $\mathbb{D}_3$ acts absolutely irreducibly on $\mathbb{C}$, $f^o = 0$ if $f \in \vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\lambda)}$.

For a local ring R, we denote by $\mathcal{M}$ its maximal ideal and the R-module generated by the elements $\{m_i\}_{i=1}^k$ is denoted by $< m_1 \dots m_k >_R$. The following description of rings and modules of functions is well-known.

Lemma 2.1 ([24], p.178). — *Let $\alpha \in (\mathbb{R}^a, 0)$ be any parameter,*

1. *The ring $\mathcal{E}^{\mathbb{D}_3}_{(z,\alpha)}$ of smooth $\mathbb{D}_3$-invariant germs from $(\mathbb{C} \times \mathbb{R}^a, 0) \to \mathbb{R}$ is generated by the invariant polynomials $u = z\bar{z}$, $v = \frac{1}{2}(z^3 + \bar{z}^3)$ and α, that is, any $f \in \mathcal{E}^{\mathbb{D}_3}_{(z,\alpha)}$ can be written as $p(u, v, \alpha)$ for some smooth $p : (\mathbb{R}^{2+a}, 0) \to \mathbb{R}$.*
2. *The free $\mathcal{E}^{\mathbb{D}_3}_{(z,\alpha)}$-module $\vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\alpha)}$ of smooth $\mathbb{D}_3$-equivariant map germs from $(\mathbb{C} \times \mathbb{R}^a, 0) \to \mathbb{C}$ is generated by $Z_1(z) = z$ and $Z_2(z) = \bar{z}^2$. Every $f \in \vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\alpha)}$ can be uniquely written as*

$$f(z, \alpha) = pZ_1 + qZ_2 = p(u, v, \alpha)\, z + q(u, v, \alpha)\, \bar{z}^2 \tag{2.1}$$

   for some $p, q \in \mathcal{E}^{\mathbb{D}_3}_{(z,\alpha)}$. In that case, we denote f by $[p, q]$.
3. *The free $\mathcal{E}^{\mathbb{D}_3}_{(z,\alpha)}$-module $\mathrm{M}^{\mathbb{D}_3}_{(z,\alpha)}$ of smooth $\mathbb{D}_3$-equivariant map germs of 2×2-matrices satisfying $M(\gamma z, \alpha)\gamma = \gamma M(z, \alpha)$, $\forall \gamma \in \mathbb{D}_3$, is generated by the maps $M_1(z)\omega = \omega$, $M_2(z)\omega = z\,\mathrm{re}(z\bar{\omega})$, $M_3(z)\omega = \bar{z}\bar{\omega}$ and $M_4(z)\omega = z\,\mathrm{re}(z^2\omega)$, linear in $\omega \in \mathbb{C}$.*

2.1.1. Bifurcation equivalence (see [24])

Let $f, g \in \vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\lambda)}$, we say that f is $\mathbb{D}_3$**-bifurcation equivalent** to g if

$$f(z,\lambda) = T(z,\lambda)\, g(Z(z,\lambda), L(\lambda)) \tag{2.2}$$

where

$$T : (\mathbb{R}^{2+l}, 0) \to \mathrm{GL}(2,\mathbb{R}), Z : (\mathbb{R}^{2+l}, 0) \to (\mathbb{R}^2, 0) \text{ and } L : (\mathbb{R}^l, 0) \to (\mathbb{R}^l, 0)$$

are smooth germs such that T^o, Z^o_z have positive entries and $\det(L^o_\lambda) > 0$. Moreover, we ask for the $\mathbb{D}_3$-equivariance of T and Z, that is,

$$T(\gamma z,\lambda)\gamma = \gamma T(z,\lambda), \quad Z(\gamma z,\lambda) = \gamma Z(z,\lambda), \quad \forall \gamma \in \mathbb{D}_3, \forall (z,\lambda) \in (\mathbb{R}^{2+l}, 0).$$

The relation (2.2) means that the zero-sets $f^{-1}(0)$ and $g^{-1}(0)$ are diffeomorphic under (Z, L) which preserves the orientation, the symmetry, as well as the λ-slice structure of the zero-set.

The $\mathbb{D}_3$-bifurcation equivalences (T, Z, L) form a group by composition, denoted by $\mathcal{K}^{\mathbb{D}_3}_\lambda$, acting on $\vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\lambda)}$ via (2.2). It is a semi-direct product of the subgroups of matrices $T \in \mathrm{M}^{\mathbb{D}_3}_{(z,\lambda)}$, with T^o of positive entries, and the subgroup of orientation-preserving diffeomorphisms (Z, L) where $Z \in \vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\lambda)}$ and $L : (\mathbb{R}^l, 0) \to (\mathbb{R}^l, 0)$. The d-parameter deformations of f, called **unfoldings** of f, are $\mathbb{D}_3$-equivariant germs $F : (\mathbb{R}^{2+l+d}, 0) \to (\mathbb{R}^2, 0)$ such that $F(z,\lambda,0) = f(z,\lambda)$. To compare them, we say that a d_1-parameter unfolding F_1 of f **maps into** a d_2-parameter unfolding F_2 of f if

$$F_2(z,\alpha_2) = T(z,\alpha_2) F_1(Z(z,\alpha_2), A(\alpha_2)) \tag{2.3}$$

where T, Z are unfoldings of the identity in their respective category and $A : (\mathbb{R}^{d_2}, 0) \to (\mathbb{R}^{d_1}, 0)$ is in general not invertible. We say that an unfolding F of f is **versal** if any other unfolding of f maps into F, meaning that we get all possible perturbations of $f^{-1}(0)$ via $F^{-1}(0)$. The versal unfoldings of f with the minimum number of parameters are called **miniversal**.

Following [24], the calculation of the **extended tangent space** $\mathcal{T}_e\mathcal{K}^{\mathbb{D}_3}_\lambda(f)$ of f is fundamental in the theory. Its exact form here is given in (2.8). The real dimension as a vector space of the **normal space**

$$\mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}_\lambda(f) = \vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\lambda)} / \mathcal{T}_e\mathcal{K}^{\mathbb{D}_3}_\lambda(f)$$

is called the **codimension** of f, denoted by $\mathrm{cod}(f)$. When $\mathrm{cod}(f) = k$ is finite, a miniversal unfolding F of f needs k parameters and F is miniversal if and only if $< F^o_{\alpha_1} \dots F^o_{\alpha_k} >_{\mathbb{R}}$ projects onto a basis of $\mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}_\lambda(f)$. Finite codimension is also a necessary and sufficient condition for f to be **finitely determined**, that is, equivalent to a finite segment of its Taylor series expansion. In particular, if f is finitely determined, a normal form for f and a basis of $\mathcal{N}_e\mathcal{K}_\lambda^{\mathbb{D}_3}(f)$ can be chosen to be polynomials.

In this paper, more groups of equivalence will be considered beside $\mathcal{K}_\lambda^{\mathbb{D}_3}$. When needed, we use the terminology $\mathcal{G}$-codimension, $\mathcal{G}$-miniversal unfolding, and so on, to emphasize the underlying group of equivalences.

2.2. Paths and path equivalence

2.2.1. Paths

We can view each bifurcation germ $f \in \vec{\mathcal{E}}_{(z,\lambda)}^{\mathbb{D}_3}$ as an l-parameter unfolding of its core f_0. When $\lambda = 0$, (2.2) reduces to the classical $\mathbb{D}_3$-equivariant contact equivalence without parameters between $f_0, g_0 \in \vec{\mathcal{E}}_z^{\mathbb{D}_3}$:

$$g_0(z) = T(z) f_0(Z(z))$$

where $T \in \mathrm{M}_z^{\mathbb{D}_3}$ and $Z \in \vec{\mathcal{E}}_z^{\mathbb{D}_3}$ are locally invertible with positive entries for T^o, Z_z^o. Such equivalences (T, Z) form a group $\mathcal{K}^{\mathbb{D}_3}$ acting on $\vec{\mathcal{E}}_z^{\mathbb{D}_3}$. We first classify the cores and their miniversal unfoldings under $\mathcal{K}^{\mathbb{D}_3}$. The germ f is of **finite core** if the $\mathcal{K}^{\mathbb{D}_3}$-codimension of f_0 is finite, say N. In that case, f_0 and a $\mathcal{K}^{\mathbb{D}_3}$-miniversal unfolding of f_0, $F_0(z, \alpha)$, $\alpha \in \mathbb{R}^N$, can be cast to be polynomials. Because f is an l-parameter unfolding of f_0, from the $\mathcal{K}^{\mathbb{D}_3}$-theory of unfoldings and (2.3), there is a mapping of unfoldings $(T, Z, \bar{\alpha})$ where $(T, Z, I) \in \mathcal{K}_\lambda^{\mathbb{D}_3}$, I is the identity map, such that

$$f(z, \lambda) = T(z, \lambda)\, F_0(Z(z, \lambda), \bar{\alpha}(\lambda)) \tag{2.4}$$

where $\bar{\alpha} : (\mathbb{R}^l, 0) \to (\mathbb{R}^N, 0)$ is a **path** associated with f. Note that $\bar{\alpha}$ has no reason to be invertible. But (2.4) means that f and the pull-back $\bar{\alpha}^* F_0$ are $\mathbb{D}_3$-bifurcation equivalent with **strong equivalence** (T, Z, I). We denote by $\vec{\mathcal{P}}_{0,\lambda}^N$ the **space of smooth paths** $\bar{\alpha} : (\mathbb{R}^l, 0) \to (\mathbb{R}^N, 0)$ associated with F_0, contained into $\vec{\mathcal{P}}_\lambda^N$, the space of smooth maps $(\mathbb{R}^l, 0) \to \mathbb{R}^N$.

2.2.2. Path equivalence

Fix a core f_0 and a $\mathcal{K}^{\mathbb{D}_3}$-miniversal unfolding F_0. To compare paths such that the sections of $\bar{\alpha}^* F_0$ over them are equivalent bifurcation diagrams, we say that the paths $\bar{\alpha}, \bar{\beta} : (\mathbb{R}^l, 0) \to (\mathbb{R}^N, 0)$ are **path equivalent** if

$$\bar{\alpha}(\lambda) = H(\lambda, \bar{\beta}(L(\lambda))) \tag{2.5}$$

where $L : (\mathbb{R}^l, 0) \to (\mathbb{R}^l, 0)$ and the smooth λ-parametrized family $H(\cdot, \lambda) : (\mathbb{R}^N, 0) \to (\mathbb{R}^N, 0)$ are diffeomorphisms whose linear parts are path connected to the identity. We also require that H lifts to a λ-family of equivariant diffeomorphism preserving $F_0^{-1}(0)$. That is, there exists a smooth λ-family $\Phi(\cdot, \lambda) : (\mathbb{R}^{2+N}, 0) \to (\mathbb{R}^{2+N}, 0)$ of $\mathbb{D}_3$-equivariant diffeomorphisms preserving $F_0^{-1}(0)$ such that $H \circ \pi_{F_0} = \pi_{F_0} \circ \Phi$ on $F_0^{-1}(0)$ where $\pi_{F_0} : F_0^{-1}(0) \to (\mathbb{R}^N, 0)$ is the restriction to $F_0^{-1}(0)$ of the natural projection $(\mathbb{R}^{2+N}, 0) \to (\mathbb{R}^N, 0)$. For fixed F_0, the path equivalences (H, L) form a group, denoted by $\mathcal{K}^*$, acting on $\vec{\mathcal{P}}^N_{0,\lambda}$ via (2.5). The explicit determination of the diffeomorphisms H in (2.5) is nearly impossible: we cannot in general simplify H to a multiplication by a matrix. But we can characterize the tangent spaces needed for the algebraic calculations pertinent to the singularity theory for $\mathcal{K}^*$. Let $\bar{\alpha} \in \vec{\mathcal{P}}^N_{0,\lambda}$ be a path. Consider all one-parameter families $t \mapsto (H(t), L(t))$, $t \in (\mathbb{R}, 0)$, of unfoldings of the identity in $\mathcal{K}^*$, that is, $H(0)(\lambda, \alpha) = \alpha$ and $L(0)(\lambda) = \lambda$. Differentiating $t \mapsto H(t)(\lambda, \bar{\alpha}(L(t)(\lambda)))$ at $t = 0$, we get the **extended tangent space** of $\bar{\alpha}$

$$\mathcal{T}_e\mathcal{K}^*(\bar{\alpha}) = < \bar{\alpha}_\lambda >_{\mathcal{E}_\lambda} + \bar{\alpha}^*(\mathrm{Derlog}^*(F_0))_{\mathcal{E}_\lambda}, \tag{2.6}$$

where $\mathrm{Derlog}^*(F_0)$ is the module of liftable vector fields, the **algebraic Derlog** of F_0. A vector field $\xi : (\mathbb{R}^N, 0) \to (\mathbb{R}^N, 0)$ is **liftable** if

$$(F_0)_z(z, \alpha)\, Z(z, \alpha) + (F_0)_\alpha(z, \alpha)\, \xi(\alpha) = T(z, \alpha)\, F_0(z, \alpha)\,, \tag{2.7}$$

where Z and T are $\mathbb{D}_3$-equivariant in z. The **$\mathcal{K}^*$-codimension** of a path $\bar{\alpha}$ is $\mathrm{cod}(\bar{\alpha}) = \dim_{\mathbb{R}} \vec{\mathcal{P}}^N_\lambda / \mathcal{T}_e\mathcal{K}^*(\bar{\alpha})$.

2.2.3. Liftable vector fields, discriminant and complexification

Let $F_0 \in \vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\alpha)}$, the **discriminant** Δ^{F_0} of π_{F_0} is the set

$$\Delta^{F_0} = \{\alpha \in (\mathbb{R}^N, 0) \ : \ \exists z \in (\mathbb{C}, 0), F_0(z, \alpha) = 0 = \det(F_0)_z(z, \alpha))\}$$

of values of α where $F_0(z, \alpha) = 0$ has a local bifurcation. By projecting down along π_{F_0}, any liftable diffeomorphism H in (2.5) preserves the discriminant Δ^{F_0}. Because of that, the group of contact equivalence preserving Δ^{F_0} in the target, $\mathcal{K}_{\Delta^{F_0}}$ in the notation of [10], was used in [31]. Its extended tangent space is formed using the module $\mathrm{Derlog}(\Delta^{F_0})$ of the vector fields tangent to the discriminant Δ^{F_0}. Our interest in smooth germs introduces a difficulty. From [1], the module of smooth vector fields tangent to a discriminant is not necessarily finitely generated, even when F_0 is a polynomial, and Δ^{F_0} can be too small to characterize the liftable vector fields. We are only interested in finitely determined germs, therefore germs equivalent to polynomials. Because the essential calculations can be done in the analytic category, we are going to be able to work with smooth real germs. In [11], it is shown that the sets of smooth and analytic vector fields tangent to Δ^{F_0} differ by a submodule of infinitely flat vector fields. Therefore, in the smooth case, when germs are finitely determined, the subset of analytic vector fields will be sufficient to perform the algebraic calculation we need.

Often in singularity theory, the underlying algebra has a full geometrical interpretation in the holomorphic situation. To use techniques of algebraic geometry, we need to be able to complexify our situation, use geometrical ideas and come back taking real slices of our results. Explicit details are as follows. The important point is that the real and complex algebras will coincide and we need to keep track of the signs that exist in the real case. The injection $(x, y) \hookrightarrow (z, \bar{z})$ induces on $\mathbb{C}^2$ the complexification of our action of $\mathbb{D}_3$ given by $\kappa(z_1, z_2) = (z_2, z_1)$ and $\theta(z_1, z_2) = (\theta z_1, \theta^2 z_2)$, where $\theta^3 = 1$. We denote by $\mathcal{O}_z^{\mathbb{D}_3}$, resp. $\vec{\mathcal{O}}_z^{\mathbb{D}_3}$, the ring, resp. the $\mathcal{O}_z^{\mathbb{D}_3}$-module, of $\mathbb{D}_3$-invariant, resp. $\mathbb{D}_3$-equivariant, holomorphic germs $(\mathbb{C}^2, 0) \to \mathbb{C}$, resp. $(\mathbb{C}^2, 0) \to \mathbb{C}^2$, generated by the invariants $u^{\mathbb{C}} = z_1 z_2$ and $v^{\mathbb{C}} = \frac{1}{2}(z_1^3 + z_2^3)$, resp. generated by $Z_1^{\mathbb{C}} = (z_1, z_2)$ and $Z_2^{\mathbb{C}} = (z_2^2, z_1^2)$. We shall use the notation $\mathcal{O}$ instead of $\mathcal{E}$ to denote sets of analytic, instead of smooth, germs. The complexification of an analytic germ $p \in \mathcal{E}_z^{\mathbb{D}_3}$ is given by $p(z_1, z_2)$ where z is replaced by z_1 and $\bar{z}$ by z_2. Similarly the complexification of an analytic map germ $f = pZ_1 + qZ_2 \in \vec{\mathcal{E}}_z^{\mathbb{D}_3}$ is given by

$$p(u^{\mathbb{C}}, v^{\mathbb{C}})\, Z_1^{\mathbb{C}}(z_1, z_2) + q(u^{\mathbb{C}}, v^{\mathbb{C}})\, Z_2^{\mathbb{C}}(z_1, z_2).$$

When F_0 is real analytic, we actually choose $F_0^{-1}(0)$, resp. Δ^{F_0}, as the real slices of the zero-set $(F_0^{\mathbb{C}})^{-1}(0)$, resp. of the discriminant $\Delta^{F_0^{\mathbb{C}}}$, of the complexification $F_0^{\mathbb{C}}$ of F_0, resp. of the projection $\pi_{F_0}^{\mathbb{C}}$. In coordinates,

$$\Delta^{F_0^{\mathbb{C}}} = \{\alpha \in (\mathbb{C}^N, 0) \,:\, \exists z \in (\mathbb{C}^2, 0), F_0^{\mathbb{C}}(z, \alpha) = 0 = \det(F_0^{\mathbb{C}})_z(z, \alpha))\}.$$

It is also the set of the singular values of the projection

$$\pi_{F_0}^{\mathbb{C}} : (F_0^{\mathbb{C}})^{-1}(0) \to \mathbb{C}^N,$$

restriction of the natural projection $(\mathbb{C}^{2+N}, 0) \to (\mathbb{C}^N, 0)$. In practice, those real slices may be larger than in the direct calculation for real objects, but they behave well under complexification. A good example is given by the swallowtail where it is known that the discriminant of the real projection and the real slice of the complex projection differ by a hairline, half the parabola of double crossing points (see [1]).

Let $\mathcal{I}(\Delta^{F_0^{\mathbb{C}}})$ be the ideal of germs vanishing on $\Delta^{F_0^{\mathbb{C}}}$. The **module of vector fields tangent to** $\Delta^{F_0^{\mathbb{C}}}$, the **geometric Derlog**, is given by

$$\mathrm{Derlog}(\Delta^{F_0^{\mathbb{C}}}) = \{\, \xi : (\mathbb{C}^N, 0) \to \mathbb{C}^N \;:\; \xi \cdot g_\alpha \in \mathcal{I}(\Delta^{F_0^{\mathbb{C}}}),\ \forall g \in \mathcal{I}(\Delta^{F_0^{\mathbb{C}}})\}.$$

We define $\mathrm{Derlog}^*(F_0)$ as the submodule of the real vector fields of the module $\mathrm{Derlog}^*(F_0^{\mathbb{C}})$ of liftable vector fields.

Similar ideas and techniques can be traced back to Martinet [27], Tessier [38] and have been used in various work, for instance, by Rieger in [32, 34, 33] in his $\mathcal{A}$-classifications of maps. In the non-equivariant case, $\mathrm{Derlog}(\Delta^{F_0})$ and $\mathrm{Derlog}^*(F_0)$ are equal free module ([26]). This is also the case here (see Corollary 4.4) but it is not always the case for other equivariant problems. For $\mathbb{D}_k$-equivariant cores, $k \geqslant 4$, $\mathrm{Derlog}^*(F_0)$ is free but is a proper Lie subalgebra of $\mathrm{Derlog}(\Delta^{F_0})$. The $\mathbb{D}_4$-equivariant core of lowest codimension has a modulus. Liftable vector fields must keep the moduli subvariety of Δ^{F_0} *pointwise* invariant, that is, vanish on it. This is an additional condition for the elements of $\mathrm{Derlog}^*(F_0)$. When $k \geqslant 5$, the reason is less clear but might be related to hidden conditions for the extension of vector fields from the fixed point subspaces to the full space. But the presence of moduli in the typical case does not always imply a difference between $\mathrm{Derlog}^*(F_0)$ and $\mathrm{Derlog}(\Delta^{F_0})$. For $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-equivariant maps, the two modules are the same, even if the core of lowest codimension has a modulus ([9]).

2.3. Path versus bifurcation equivalences: comparison of the two approaches

Both $\mathcal{K}_\lambda^{\mathbb{D}_3}$ and $\mathcal{K}^*$ are geometric subgroups of the contact group $\mathcal{K}$ in the sense of [10], so Damon's general theory about unfoldings and determinacy applies in both cases. We show in Theorem 5.1 that the two theories lead to the same results for our finite codimension problems. But they are not equal when we use them in practice. The path formulation differentiates between the singular behaviour attributable to the core and to the paths. Moreover, there is an important difference in the algebra. The algebraic techniques of bifurcation equivalence are well-known and involve modules over systems of rings ([10]). For instance,

$$\mathcal{T}_e\mathcal{K}_\lambda^{\mathbb{D}_3}(f) = < Tf, f_z Z >_{\mathcal{E}^{\mathbb{D}_3}_{(z,\lambda)}} + < f_\lambda >_{\mathcal{E}_\lambda} \tag{2.8}$$

is the extended tangent space of $f \in \vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\lambda)}$, which is a module over the system $\{\mathcal{E}_\lambda, \mathcal{E}^{\mathbb{D}_3}_{(z,\lambda)}\}$. The issue is that $\mathcal{T}_e\mathcal{K}_\lambda^{\mathbb{D}_3}(f)$ is not an $\mathcal{E}^{\mathbb{D}_3}_{(z,\lambda)}$-module and is not finitely generated as an $\mathcal{E}_\lambda$-module. But, in general, it behaves well-enough to generate an $\mathbb{R}$-vector subspace of $\vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\lambda)}$ of finite codimension, ensuring that the main results of singularity theory go through. When $\lambda \in (\mathbb{R}, 0)$, the **restricted tangent space** $< Tf, f_z Z >_{\mathcal{E}^{\mathbb{D}_3}_{(z,\lambda)}}$ is of finite codimension if and only if f is of finite codimension ([24]). With more parameters, this result is in general false and the calculations need a more sophisticated use of the Preparation Theorem ([10, 25]). On the other hand, $\mathcal{T}_e\mathcal{K}^*(\bar{\alpha})$ in (2.6) is an $\mathcal{E}_\lambda$-module, even for $\lambda \in (\mathbb{R}^l, 0)$. Moreover, the contribution of $\mathrm{Derlog}^*(F_0)$ in the tangent spaces does not depend on l.

The **recognition problem** for a normal form is the set of equalities and inequalities satisfied by the derivatives of the germs equivalent to that normal form. And so, it is usually simpler to solve recognition problems using the bifurcation equivalences of $\mathcal{K}^{\mathbb{D}_3}_\lambda$ because explicit simplification of the low order terms to their normal form is easier.

3. $\mathbb{D}_3$-equivariant cores

In this section we classify the cores up to topological codimension 3 under $\mathcal{K}^{\mathbb{D}_3}$-equivalence. If needed, we can adapt the results of [24], p. 178 and pp. 191-198, ignoring the λ-dependence of their formulas. For completeness, we recall some important steps and definitions. Let $f_0 \in \vec{\mathcal{E}}^{\mathbb{D}_3}_z$. Recall that $f_0 = [p, q]$ from (2.1). From [24], p. 178, the extended tangent space $\mathcal{T}_e\mathcal{K}^{\mathbb{D}_3}(f_0)$ is generated over $\mathcal{E}^{\mathbb{D}_3}_z$ by

(3.1) $[p, q], \quad [up + vq, 0], \quad [uq, p], \quad [u^2 q + vp, 0],$

(3.2) $[2up_u + 3vp_v, q + 2uq_u + 3vq_v], \quad [uq + 2vp_u + 3u^2 p_v, 2vq_u + 3u^2 q_v].$

The number $\mathrm{cod}(f_0) = \dim_{\mathbb{R}}(\mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}(f_0)) = \dim_{\mathbb{R}}(\vec{\mathcal{E}}^{\mathbb{D}_3}_z / \mathcal{T}_e\mathcal{K}^{\mathbb{D}_3}(f_0))$ is the $\mathcal{K}^{\mathbb{D}_3}$-codimension of f_0. When $\mathrm{cod}(f_0) = N < \infty$, an unfolding F_0 of f_0 is $\mathcal{K}^{\mathbb{D}_3}$-miniversal if and only if it has N parameters $\alpha \in (\mathbb{R}^N, 0)$ and the vector space $< (F_0)^o_{\alpha_1} \ldots (F_0)^o_{\alpha_N} >_{\mathbb{R}}$ projects onto a basis of $\mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}(f_0)$.

3.1. Low and high order terms

When solving the recognition problem for f_0 we need to control its low order coefficients when applying a $\mathcal{K}^{\mathbb{D}_3}$-equivalence. For our normal forms, it is actually enough to stop at second order. Consider $(T, Z) \in \mathcal{K}^{\mathbb{D}_3}$. From

Lemma 2.1, $T = cM_1 + dM_2 + eM_3 + gM_4$ and $Z = az + b\,\bar{z}^2$ where a, b, c, d, e, g are to be determined in $\mathcal{E}_z^{\mathbb{D}_3}$. Define

$$\tilde{u} = u(Z) = a^2 u + 2abv + b^2 u^2 \tag{3.3}$$

and

$$\tilde{v} = v(Z) = a^3 v + 3a^2 bu^2 + 3ab^2 uv + 2b^3 v^2 - b^3 u^3. \tag{3.4}$$

Given $[p, q] \in \vec{\mathcal{E}}_z^{\mathbb{D}_3}$, let $[p', q'] = [p, q](Z)$ and $[p'', q''] = T[p', q']$. And so,

$$p' = a\tilde{p} + 2b(au + bv)\tilde{q}, \qquad q' = b\tilde{p} + (a^2 - b^2 u)\tilde{q}, \tag{3.5}$$

where $\tilde{p}(u, v) = p(\tilde{u}, \tilde{v})$, $\tilde{q}(u, v) = q(\tilde{u}, \tilde{v})$, and

$$p'' = cp' + d(up' + vq') + euq' + g(vp' + u^2 q'), \quad q'' = cq' + ep'. \tag{3.6}$$

Develop in Taylor series expansions:

$$p(u, v) = A_1 u + B_1 v + D_1 u^2 + E_1 uv + F_1 v^2 + \mathcal{M}^3_{(u,v)}, \tag{3.7}$$

$$q(u, v) = A_0 + A_2 u + B_2 v + D_2 u^2 + E_2 uv + F_2 v^2 + \mathcal{M}^3_{(u,v)}. \tag{3.8}$$

In the appendix, we give the important coefficients for $[p'', q'']$, resp. $[p', q']$, in terms of the coefficients of $[p', q']$, resp. $[p, q]$. To eliminate the tail of the Taylor series of f_0, we use the following ideas of Gaffney [16]. The set $\mathcal{P}(f_0)$ of **higher order terms** of f_0 is

$$\mathcal{P}(f_0) = \{\ p \in \vec{\mathcal{E}}_z^{\mathbb{D}_3}\ :\ g_0 + p \sim_{\mathcal{K}^{\mathbb{D}_3}} g_0,\ \forall\, g_0 \sim_{\mathcal{K}^{\mathbb{D}_3}} f_0\ \}.$$

They are the terms that can be eliminated from any representative of the $\mathcal{K}^{\mathbb{D}_3}$-equivalence class of f_0. Before stating the result providing an estimate for $\mathcal{P}(f_0)$, we need the following definitions. The **unipotent subgroup** $\mathcal{UK}^{\mathbb{D}_3}$ is formed of $(T, Z) \in \mathcal{K}^{\mathbb{D}_3}$ with $T^o = Z_z^o = I$. It defines $\mathcal{TUK}^{\mathbb{D}_3}(f_0)$, the **unipotent tangent space** of f_0, formed of the t-derivatives, at the origin $t = 0$, of all the families $t \mapsto T(t) f_0(Z(t))$ where

$$T(t) = I + t\hat{T}, Z(t) = z + t\hat{Z}$$

with

$$\hat{T}^o = \hat{Z}^o = \hat{Z}_z^o = 0.$$

A subspace of $\vec{\mathcal{E}}_z^{\mathbb{D}_3}$ is **intrinsic** if it is globally invariant under $\mathcal{K}^{\mathbb{D}_3}$. Similarly, an ideal of $\mathcal{E}_z^{\mathbb{D}_3}$ is **intrinsic** if it is globally invariant with respect to any $\mathbb{D}_3$-equivariant change of co-ordinates $Z \in \vec{\mathcal{E}}_z^{\mathbb{D}_3}$ such that Z_z^o has positive entries. The **intrinsic part** $\operatorname{intr}(\mathcal{N})$ of a vector subspace $\mathcal{N}$ of $\vec{\mathcal{E}}_z^{\mathbb{D}_3}$ is its largest intrinsic subspace.

Theorem 3.1 ([16]). — *If $f_0 \in \vec{\mathcal{E}}_z^{\mathbb{D}_3}$ is of finite $\mathcal{K}^{\mathbb{D}_3}$-codimension, $\mathcal{P}(f_0)$ is an intrinsic sub-module of $\vec{\mathcal{E}}_z^{\mathbb{D}_3}$ and* $\operatorname{intr}(\mathcal{TUK}^{\mathbb{D}_3}(f_0)) \subset \mathcal{P}(f_0)$.

For explicit calculations we need the following.

Lemma 3.2. — (1) *The ideals* $<u,v>$ *and* $<u^2,v>$ *are intrinsic.*

(2) *Using the notation with the* $\mathbb{D}_3$*-invariants, the sub-module* $[\mathcal{I},\mathcal{J}] \subset \vec{\mathcal{E}}_z^{\mathbb{D}_3}$ *is intrinsic if and only if* $\mathcal{I}$ *and* $\mathcal{J}$ *are intrinsic ideals of* $\mathcal{E}_z^{\mathbb{D}_3}$ *and* $<u,v>\mathcal{J} \subset \mathcal{I} \subset \mathcal{J}$.

(3) *The following maps are generators for the unipotent tangent space:*

$$\mathcal{M}_{(u,v)}[p,q], \quad [up+vq,0], \quad [uq,p], \quad [u^2q+vp,0], \tag{3.9}$$

$$\mathcal{M}_{(u,v)}[2up_u+3vp_v, q+2uq_u+3vq_v], \tag{3.10}$$

$$[uq+2vp_u+3u^2p_v, 2vq_u+3u^2q_v]. \tag{3.11}$$

Proof. —

(1) This follows immediately from (3.3) and (3.4).

(2) Suppose that $\mathcal{I}$, $\mathcal{J}$ are intrinsic ideals and $<u,v>\mathcal{J} \subset \mathcal{I} \subset \mathcal{J}$. Apply to any $[p,q] \in [\mathcal{I},\mathcal{J}]$, any $(T,Z) \in \mathcal{K}^{\mathbb{D}_3}$. From (3.3,3.4), $\tilde{p} = p(Z) \in \mathcal{I}$ and $\tilde{q} = q(Z) \in \mathcal{J}$ and so, from (3.5), $p' \in \mathcal{I}$ and $q' \in \mathcal{J}$. Hence, $\mathcal{I}' \subset \mathcal{I}$ and $\mathcal{J}' \subset \mathcal{J}$. Finally, from (3.6) and $<u,v>\mathcal{J} \subset \mathcal{I} \subset \mathcal{J}$, $\mathcal{I}'' \subset \mathcal{I}$ and $\mathcal{J}'' \subset \mathcal{J}$. Hence, $[\mathcal{I},\mathcal{J}]$ is intrinsic. For the converse, let $[\mathcal{I},\mathcal{J}] \subset \vec{\mathcal{E}}_z^{\mathbb{D}_3}$ be intrinsic. Take any $Z(z) = az + b\bar{z}^2$ with $a^o \neq 0$. Choosing $c = 1$, $e = -\frac{b}{a}$ in (3.6), q'' becomes $(a^2 - 3b^2u - 2\frac{b^3}{a}v)\tilde{q}$. Because $a^o \neq 0$, the first factor is invertible and so $\tilde{q} = q(Z) \in \mathcal{J}$. Hence, $\mathcal{J}$ is intrinsic. Similarly, choosing $c = 1$, $g = 0$, $d = -\frac{2b^2}{a^2+b^2u}$ and $e = -\frac{2ab}{a^2+b^2u}$ in (3.6), we get

$$p'' = \left(a + adu - \frac{2b^2}{a^2+b^2u}(au+bv)\right)\tilde{p}$$

and so $\mathcal{I}$ is intrinsic. Finally, take $c = e = 1$ and $d = g = 0$, in (3.6), then $u\mathcal{J} \subset \mathcal{I} \subset \mathcal{J}$. From $c = d = 1$ and $e = g = 0$ in (3.6), $v\mathcal{J} \subset \mathcal{I}$. Combining them, we see that $<u,v>\mathcal{J} \subset \mathcal{I} \subset \mathcal{J}$.

(3) In the definition of $\mathcal{TUK}^{\mathbb{D}_3}(f_0)$, let $\hat{T}(z) = cM_1 + dM_2 + eM_3 + gM_4$, with $c^o = 0$, and $\hat{Z}(z) = az + b\,\bar{z}^2$, with $a^o = 0$. And so, the elements of the unipotent tangent space of f_0 are $\hat{T}f_0 + (f_0)_z\hat{Z}$. Explicitly, we find the generators (3.9-3.11).

□

3.2. Classification of $\mathbb{D}_3$-equivariant cores

For the classifications of one-parameter bifurcation germs up to topological $\mathcal{K}_\lambda^{\mathbb{D}_3}$-codimension 2, it is enough to list the cores of topological $\mathcal{K}^{\mathbb{D}_3}$-codimension up to 3. We define $\Delta_{u,v}(p,q) = p_u^o q_v^o - p_v^o q_u^o$.

PROPOSITION 3.3. — (1) *There are four classes of cores of topological $\mathcal{K}^{\mathbb{D}_3}$-codimension $i \leqslant 3$, denoted by C_i^3. For given $f_0 = [p,q]$, the normal forms and recognition problems are as follows.*

(a) *If $q^o \neq 0$, let $\epsilon = \operatorname{sign}(q^o)$, C_1^3 has normal form $[0,\epsilon]$.*

(b) *If $q^o = 0$, but $p_u^o \neq 0$. Let $\epsilon_1 = \operatorname{sign}(p_u^o)$.*

(i) *If $\Delta_{u,v}(p,q) \neq 0$, C_2^3 has normal form $[\epsilon_1 u, \epsilon_2 v]$ where $\epsilon_2 = \operatorname{sign}(p_u^o \cdot \Delta_{u,v}(p,q))$.*

(ii) *If $\Delta_{u,v}(p,q) = 0$, C_3^3 has normal form $[\epsilon_1 u, \epsilon_3 v^2]$ if*

$$\epsilon_3 = \operatorname{sign}\left((p_v^o)^2\Delta_{u,uu}(p,q) - 2p_u^o p_v^o \Delta_{u,uv}(p,q) + (p_u^o)^2\Delta_{u,vv}(p,q)\right) \neq 0,$$

(c) *If $q^o = p_u^o = 0$, but $\epsilon_4 = \operatorname{sign}(q_u^o) \neq 0$, let $\mu = p_v^o/|q_u^o|$. If $\mu \neq 0, -\epsilon_4, \frac{2}{3}\epsilon_4$, C_{3*}^3 has normal form $[\mu v, \epsilon_4 u]$. The asterisk * indicates that μ is a modal parameter. The $\mathcal{K}^{\mathbb{D}_3}$-codimension of C_{3*}^3 is 4 and its topological $\mathcal{K}^{\mathbb{D}_3}$-codimension is 3.*

(2) *Miniversal unfoldings of the cores of topological $\mathcal{K}^{\mathbb{D}_3}$-codimension up to 3 are as follows (the α_i's are the unfolding parameters):*

$$C_1^3 : \alpha_1 z + \epsilon\, \bar{z}^2, \tag{3.12}$$

$$C_2^3 : (\epsilon_1 u + \alpha_1)\, z + (\epsilon_2 v + \alpha_2)\, \bar{z}^2, \tag{3.13}$$

$$C_3^3 : (\epsilon_1 u + \alpha_1)\, z + (\epsilon_3 v^2 + \alpha_3 v + \alpha_2)\, \bar{z}^2, \tag{3.14}$$

$$C_{3*}^3 : \left((\mu + \alpha_0)v + \alpha_3 u + \alpha_1\right) z + (\epsilon_4 u + \alpha_2)\, \bar{z}^2. \tag{3.15}$$

Proof. — We follow the usual techniques of [24] along the tree of degeneracies for p and q. We show the computations for C_{3*}^3. Let $[p,q]$ satisfying $q^o = p_u^o = 0$ and $q_u^o \neq 0$. Set $\epsilon_4 = \operatorname{sign}(q_u^o)$ and $\mu = p_v^o/|q_u^o|$. First, we cast $[p,q]$ into $[\mu v + \mathcal{M}_{u,v}^2, \epsilon_4 u + \mathcal{M}_{u,v}^2]$. From (3.7-8) and (8.1-4) in Appendix 8.2, using that $A_0 = A_1 = 0$, we get

$$B_1'' = c_0 a_0^4 B_1, A_2'' = c_0 a_0^4 A_2$$

and

$$B_2'' = e_0 a_0^4 B_1 + c_0 a_0^3 b_0 (A_2 + B_1) + c_0 a_0^5 B_2.$$

Because $(\mu + \epsilon_4) \neq 0$, $(A_2 + B_1) \neq 0$ and so B_2'' can be put to 0 by choosing b_0. Then, we scale A_2'' to $\epsilon_4 = \operatorname{sign}(A_2)$ and so B_1'' is given by $\mu = B_1/|A_2|$. In a final step, we eliminate the terms of $\mathcal{M}_{u,v}^2$ using the unipotent tangent space of $[\mu v, \epsilon_4 u]$. From (3.9-11), it is generated by

$$[\mu uv, \epsilon_4 u^2], [\mu v^2, \epsilon_4 uv], [(\epsilon_4 + \mu)uv, 0], [\epsilon_4 u^2, \mu v], [\mu v^2, -\mu uv]$$

and

$$[(\epsilon_4 + 3\mu)u^2, 2\epsilon_4 v].$$

Some elementary algebra shows that if

$$\mu(\epsilon_4 + \mu)(2\epsilon_4 - 3\mu) \neq 0, \tag{3.16}$$

$\mathcal{TUK}^{\mathbb{D}_3}(f_0) = [< u, v >^2, < u^2, v >]$, which is intrinsic, showing that $\mathcal{P}(f_0)$ contains all the second order terms. Note that this last calculation confirms that B_2 is indeed irrelevant to the normal form.

We finish by calculating $\mathcal{T}_e\mathcal{K}^{\mathbb{D}_3}(f_0)$ using the generators in (3.1-2). Note that

$$\mathcal{T}_e\mathcal{K}^{\mathbb{D}_3}(f_0) = \mathcal{TUK}^{\mathbb{D}_3}(f_0) + < [p, q], [2up_u + 3vp_v, q + 2uq_u + 3vq_v] >_{\mathbb{R}} .$$

If (3.16) holds, the normal space $\mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}(f_0)$ is generated by $[1, 0]$, $[0, 1]$, $[u, 0]$ and the modal term $[v, 0]$. Hence $C^3_{3^*}$ is of $\mathcal{K}^{\mathbb{D}_3}$-codimension 4, topological $\mathcal{K}^{\mathbb{D}_3}$-codimension 3, with miniversal $\mathcal{K}^{\mathbb{D}_3}$-unfolding (3.15). □

4. Liftable vector fields

In this section we describe the discriminants and modules of liftable vector fields for the cores of Proposition 3.3.

4.1. Discriminants and algebraic Derlogs

4.1.1. Zero-sets

The lattice of isotropy subgroup of $\mathbb{D}_3$ is simple: $\mathbf{1} \to \mathbb{Z}_2^\kappa \to \mathbb{D}_3$. So the zero-set $P(u, v, \alpha)z + Q(u, v, \alpha)\bar{z}^2 = 0$ of a $\mathbb{D}_3$-equivariant miniversal unfolding $[P, Q]$ is formed of three types of solutions: O, R and T, distinguished by their isotropy. Their defining equations and non-degeneracy conditions are given in the following.

(1) The trivial solutions of type O, given by $z = 0$, of maximal isotropy $\mathbb{D}_3$. They bifurcate where $P(0, \alpha) = 0$.

(2) The solutions $(x, 0)$ of type R, of isotropy $\mathbb{Z}_2^\kappa$, given by the equation
$$P(x^2, x^3, \alpha) + Q(x^2, x^3, \alpha)x = 0.$$
They form three branches and bifurcate either where $P = 0$ or where $2P_u x + Q + 3Q_v x^3 + (3P_v + 2Q_u)x^2 = 0$.

(3) The solutions (x, y), $y \neq 0$, of type T, of trivial isotropy $\mathbf{1}$, given by the equations $P(u, v, \alpha) = Q(u, v, \alpha) = 0$. They bifurcate where $(P_uQ_v - P_vQ_u)(u, v, \alpha) = 0$.

4.1.2. Critical sets and discriminants

The discriminant is the projection on parameter space of the critical set, points $(z,\alpha) \in (\mathbb{R}^{2+N},0)$ where there is a local bifurcation for $F_0(z,\alpha)=0$. The critical set has four subsets corresponding to different bifurcations:

(1) The bifurcations from O, $\mathcal{S}_I$, of equations $z=0$ and $P=0$.
(2) The folds $\mathcal{F}_R$ on the R-branches, of equations

$$y=0,\ P+Qx=0,\ 2P_u x+Q+(3P_v+2Q_u)\,x^2+3Q_v x^3=0.$$

(3) The folds $\mathcal{F}_T$ on the T-branches, of equations

$$P=Q=P_uQ_v-P_vQ_u=0.$$

(4) The pitchforks $\mathcal{P}_R$ with a T-branch bifurcating from an R-branch, of equations $y=P(x^2,x^3,\alpha)=Q(x^2,x^3,\alpha)=0$.

For our cores, the discriminants are all principal but not irreducible varieties. The second and third cases are not quasi-homogeneous.

Proposition 4.1. — (1) *For C_1^3, S_I is $\alpha_1=0$.*
(2) *For C_2^3, S_I is $\alpha_1=0$, $\mathcal{P}_R$ is $\alpha_1^3+\epsilon_1\alpha_2^2=0$ and $\mathcal{F}_R$ is*

$$16\alpha_1-4\epsilon_1\alpha_2^2-128\epsilon_2\alpha_1^2+256\alpha_1^3+144\epsilon_1\epsilon_2\alpha_1\alpha_2^2-27\epsilon_2\alpha_2^4=0.$$

(3) *For C_3^3, S_I is $\alpha_1=0$, $\mathcal{F}_T$ is $4\alpha_2-\epsilon_3\alpha_3^2=0$, $\mathcal{F}_R$ is $4\alpha_1-\epsilon_1\alpha_2^2+\mathcal{M}_\alpha^5=0$ and $\mathcal{P}_R$ is*

$$\alpha_2^2-2\epsilon_1\epsilon_3\alpha_1^3\alpha_2+\epsilon_1\alpha_1^3\alpha_3^2+\alpha_1^6=0.$$

(4) *For $C_{3^*}^3$, S_I is $\alpha_1=0$, $\mathcal{P}_R$ is $\epsilon_4\mu^2\alpha_2^3+(\alpha_1-\epsilon_4\alpha_2\alpha_3)^2=0$ and $\mathcal{F}_R$ is*

$$27(\mu+\epsilon_4)^2\alpha_1^2+4(\mu+\epsilon_4)\alpha_2^3-18(\mu+\epsilon_4)\alpha_1\alpha_2\alpha_3+4\alpha_1\alpha_3^3-\alpha_2^2\alpha_3^2=0.$$

4.1.3. Analytic generators of the algebraic Derlogs

The algebraic structure of the analytic liftable vector fields for the cores of Proposition 3.3 is as follows.

Proposition 4.2. — *The following vector fields generate freely the analytic vector fields of* Derlog$^*(C_i^3)$, $i=1\dots 3^*$*:*

(1) *For C_1^3,* $\xi(\alpha_1)=\alpha_1$,

(2) *For* C_2^3, $\alpha \in (\mathbb{R}^2, 0)$, $\quad \xi_1 = (\xi_{11}, \xi_{12})$ *and* $\xi_2 = (\xi_{21}, \xi_{22})$ *where:*

$$\begin{aligned}
\xi_{11}(\alpha) &= 2\alpha_1(32\alpha_1 - 184\epsilon_2\alpha_1^2 + 16\epsilon_1\alpha_2^2 + 68\alpha_1^3 + 372\epsilon_1\epsilon_2\alpha_1\alpha_2^2 \\
&\qquad + 552\epsilon_2\alpha_1^4 + 234\epsilon_1\alpha_1^2\alpha_2^2 + 288\alpha_1^5 - 27\epsilon_1\epsilon_2\alpha_1^3\alpha_2^2), \\
\xi_{12}(\alpha) &= 3\alpha_1\alpha_2(32 - 184\epsilon_2\alpha_1 + 52\alpha_1^2 + 348\epsilon_2\alpha_1^3 \\
&\qquad + 90\epsilon_1\alpha_1\alpha_2^2 + 144\alpha_1^4 - 27\epsilon_1\epsilon_2\alpha_1^2\alpha_2^2), \\
\xi_{21}(\alpha) &= 2\alpha_1\alpha_2(-16\epsilon_2 + 112\alpha_1 - 156\epsilon_2\alpha_1^2 - 84\epsilon_1\alpha_2^2 \\
&\qquad - 44\alpha_1^3 + 60\epsilon_1\epsilon_2\alpha_1\alpha_2^2 + 48\epsilon_2\alpha_1^4 - 9\epsilon_1\alpha_1^2\alpha_2^2), \\
\xi_{22}(\alpha) &= -16\epsilon_2\alpha_2^2 + 32\epsilon_1\epsilon_2\alpha_1^3 + 192\alpha_1\alpha_2^2 - 144\epsilon_1\alpha_1^4 \\
&\qquad - 404\epsilon_2\alpha_1^2\alpha_2^2 - 108\epsilon_1\alpha_2^4 + 64\epsilon_1\epsilon_2\alpha_1^5 + 12\alpha_1^3\alpha_2^2 \\
&\qquad + 108\epsilon_1\epsilon_2\alpha_1\alpha_2^4 - 27\epsilon_1\alpha_1^2\alpha_2^4 + 72\epsilon_2\alpha_1^4\alpha_2^2.
\end{aligned}$$

(3) *For* C_3^3, $\alpha \in (\mathbb{R}^3, 0)$ *and we only state explicitly the jets of order 2:*

$$\begin{aligned}
\xi_1(\alpha) &= \big(4\alpha_1\alpha_2, 2\alpha_2^2, -3\alpha_2\alpha_3\big) + \vec{\mathcal{M}}_\alpha^3, \\
\xi_2(\alpha) &= \big(2\alpha_1^2, 6\alpha_1\alpha_2, 3\alpha_1\alpha_3\big) + \vec{\mathcal{M}}_\alpha^3, \\
\xi_3(\alpha) &= \big(2\alpha_1\alpha_3, \alpha_2\alpha_3, 10\epsilon_3\alpha_2 - 2\alpha_3^2\big) + \vec{\mathcal{M}}_\alpha^3.
\end{aligned}$$

(4) *For* $C_{3^*}^3$, $\alpha \in (\mathbb{R}^4, 0)$ *and we only state explicitly the lowest non zero order of the jets (α_4 corresponds to the modulus μ).*

$$\begin{aligned}
\xi_1(\alpha) &= (3\alpha_1, 2\alpha_2, \alpha_3, 0), \\
\xi_2(\alpha) &= (0, 0, 3\alpha_1, -2\epsilon_4\mu\alpha_2) + \vec{\mathcal{M}}_\alpha^2, \\
\xi_3(\alpha) &= (0, 0, 0, 3\alpha_1) + \vec{\mathcal{M}}_\alpha^2, \\
\xi_4(\alpha) &= (0, 18\epsilon_4\alpha_1^2, 3\epsilon_4(6\mu^2 - 7\epsilon_4\mu + 6)\alpha_1\alpha_2, \\
&\qquad 2\mu(3\mu - 2\epsilon_4)(\mu + 4\epsilon_4)\alpha_2^2 - 42\mu(\mu + \epsilon_4)(3\mu - 2\epsilon_4)\alpha_1\alpha_3) + \vec{\mathcal{M}}_\alpha^3.
\end{aligned}$$

Proof. — Because the real and complex algebra coincide, we can calculate $\mathrm{Derlog}^*(C_i^3)$ in real co-ordinates. From Theorem 4.3, the algebraic Derlogs are free modules. To find them, we go back to first principles. A vector field $\xi : (\mathbb{R}^i, 0) \to (\mathbb{R}^i, 0)$ is in $\mathrm{Derlog}^*(C_i^3)$, $i = 4$ for $C_{3^*}^3$, if

$$(F_0)_z(z, \alpha)\, Z(z, \alpha) + (F_0)_\alpha(z, \alpha)\, \xi(\alpha) = T(z, \alpha)\, F_0(z, \alpha)\,, \tag{4.1}$$

where F_0 is the unfolding of the core C_i^3, $Z \in \vec{\mathcal{E}}_{(z,\alpha)}^{\mathbb{D}_3}$ and $T \in \mathrm{M}_{(z,\alpha)}^{\mathbb{D}_3}$. The construction is algebraic, so we can write (4.1) directly in the algebra of invariants where $F_0 = [P, Q]$. We need to find a, b, c, d, e and g, functions of

(u, v, α), such that for some ξ,

$$\begin{aligned} a[P,Q] &+ b[uP+vQ,0] + c[uQ,P] \\ &+ d[u^2Q+vP,0] + e[2uP_u+3vP_v, Q+2uQ_u+3vQ_v] \\ &\quad + g[uQ+2vP_u+3u^2P_v, 2vQ_u+3u^2Q_v] + (F_0)_\alpha\xi(\alpha) = 0. \end{aligned}$$

From Theorem 4.3, we know that Derlog*(C_i^3) has $\mathrm{cod}(C_i^3)$ generators, and so we solved the system using Mathematica until we find the correct number of independent generators. We fully computed Derlog*(C_2^3). For C_3^3 and C_4^3, we computed only the first few terms of their Taylor series expansion that are needed for the proof of the classifications in Theorem 5.2 and 5.3. ☐

4.2. Free liftable and geometric Derlogs

Here we show that the algebraic Derlogs of the complexified cores in $\vec{\mathcal{O}}_z^{\mathbb{D}_3}$ are free modules and that the elements of the geometric Derlog are also liftable. We refer to Section 2.2.3 for the definitions and notations used in this section. Let $f_0^{\mathbb{C}} : (\mathbb{C}^2, 0) \to \mathbb{C}$ be a core of finite $\mathcal{K}^{\mathbb{D}_3}$-codimension N and let $F_0^{\mathbb{C}} : (\mathbb{C}^{2+N}, 0) \to \mathbb{C}$ be a $\mathcal{K}^{\mathbb{D}_3}$-miniversal unfolding given by

$$F_0^{\mathbb{C}}(z,\alpha) = f_0^{\mathbb{C}}(z) + \sum_{i=1}^{N} \alpha_i\, h_i(z),$$

where $\{h_i\}_{i=1}^N$ is a basis of the normal space $\mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}(f_0^{\mathbb{C}}) = \vec{\mathcal{O}}_z^{\mathbb{D}_3}/\mathcal{T}_e\mathcal{K}^{\mathbb{D}_3}(f_0^{\mathbb{C}})$. From the Malgrange Preparation Theorem,

$$\mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}(F_0^{\mathbb{C}}) = \vec{\mathcal{O}}_{(z,\alpha)}^{\mathbb{D}_3}/\mathcal{T}_e\mathcal{K}^{\mathbb{D}_3}(F_0^{\mathbb{C}})$$

is finitely generated as an $\mathcal{O}_\alpha$-module by $\{h_i\}_{i=1}^n$. From the results of Roberts [36, 35], $\mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}(F_0^{\mathbb{C}})$ is a coherent sheaf of modules. Let $p \in \mathcal{O}_\alpha^N$, the formula $\varphi(p)(z,\alpha) = \sum_{i=1}^N p_i(\alpha)\, h_i(z)$ defines a linear and surjective map

$$\mathcal{O}_\alpha^N \xrightarrow{\varphi} \mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}(F_0^{\mathbb{C}}) \to 0. \tag{4.2}$$

The kernel of φ, that we also described in [15], is Derlog*(F_0), because, if $\xi = (p_1, \ldots, p_N) \in \ker\varphi$, there exists (T, Z) such that

$$\sum_{i=1}^{N} p_i(\alpha)\, h_i(z) = (F_0^{\mathbb{C}})_z(z,\alpha)Z(z,\alpha) + T(z,\alpha)F_0^{\mathbb{C}}(z,\alpha).$$

Re-arranging, and noting that $h_i = (F_0^{\mathbb{C}})_{\alpha_i}(z,\alpha)$, $i = 1\dots N$, we get

$$-(F_0^{\mathbb{C}})_z(z,\alpha)Z(z,\alpha) + \sum_{i=1}^{N} p_i(\alpha)\,(F_0^{\mathbb{C}})_{\alpha_i}(z,\alpha) = T(z,\alpha)F_0^{\mathbb{C}}(z,\alpha).$$

Hence, the vector field $\zeta = (-Z,\xi)$, is tangent at the smooth points of $(F_0^{\mathbb{C}})^{-1}(0)$ because it is in the kernel of the derivative $(F_0^{\mathbb{C}})_{(z,\alpha)}(z,\alpha)$ when $F_0^{\mathbb{C}}(z,\alpha) = 0$, and ζ is the lift of ξ.

Theorem 4.3. — *The algebraic Derlog of a core of finite $\mathcal{K}^{\mathbb{D}_3}$-codimension is a free $\mathcal{O}_\alpha$-module of rank equal to the $\mathcal{K}^{\mathbb{D}_3}$-codimension N of the core.*

Proof. — We need to prove that $\ker\varphi$ is a free $\mathcal{O}_\alpha$-module. Because $\mathbb{D}_3$ is finite, the ring $\mathcal{O}_z^{\mathbb{D}_3}$ is Noetherian and Cohen-Macaulay ([17]). It is of dimension 2 and regular. Of interest for us is the zero-th Fitting ideal of $\ker\varphi$ in the exact sequence

$$0 \to \mathcal{O}_\alpha^N \to \mathcal{O}_\alpha^N \xrightarrow{\varphi} \mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}(F_0^{\mathbb{C}}) \to 0. \tag{4.3}$$

We use the argument of Tessier [38] as repeated several times in the literature (see [12]). Relation (4.3) is between coherent sheaves. The support is a well-defined notion for any sheaf; it is the set of points where the stalk is nonzero. The support of $\mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}(F_0^{\mathbb{C}})$ is the set of (z,α) where the germ of $F_0^{\mathbb{C}}$ is not infinitesimally stable. This corresponds to the critical set:

$$C_{F_0^{\mathbb{C}}} = \{(z,\alpha) \ \mid\ F_0^{\mathbb{C}}(z,\alpha) = 0 \text{ and } \operatorname{rank}(F_0^{\mathbb{C}})_z(z,\alpha) < 2\}. \tag{4.4}$$

We can decompose $C_{F_0^{\mathbb{C}}} = C_1 \cup C_2 \cup C_3$, differentiated by their isotropy, that is, $C_1 \subset \operatorname{Fix}\mathbf{1}$, $C_2 \subset \operatorname{Fix}\mathbb{Z}_2^\kappa$ and $C_3 \subset \operatorname{Fix}\mathbb{D}_3$, of equations

- C_1: $P = Q = P_uQ_v - P_vQ_u = 0$,
- C_2: $P(z^2,z^3) + zQ(z^2,z^3) = 0$,
  $2zP_u + 3z^2Q_v + Q + 2z^2Q_u + 3z^3Q_v = 0$,
- C_3: $z_1 = z_2 = 0$, $P(0,\alpha) = 0$.

Let $A_1 = (P, Q, P_uQ_v - P_vQ_u)$ and $I_1(A_1)$ be the ideal of $\mathcal{O}_{(z,\alpha)}^{\mathbb{D}_3}$ generated by the elements of A_1. From Theorem 3.19 of [29], $\dim(\mathcal{O}_{(z,\alpha)}^{\mathbb{D}_3}/I_1(A_1)) \geqslant N-1$ because $\dim\mathcal{O}_{(z,\alpha)}^{\mathbb{D}_3} = 2+N$ and, in the notation of [29], $n = 2+N$, $p = 1$, $q = 3$ and $r = 1$. The results hold similarly for A_2 and A_3 that define C_2 and C_3, respectively. Let $\pi_{F_0} : C_{F_0^{\mathbb{C}}} \subset (F_0^{\mathbb{C}})^{-1}(0) \to \mathbb{C}^N$ be the projection $(z,\alpha) \mapsto \alpha$. It is a finite map because it is the restriction of the corresponding projection for a $\mathcal{K}$-miniversal unfolding of f_0, that is a finite map (see Looijenga [26]) and $\dim C_{F_0^{\mathbb{C}}} = N-1$. We finish following the argument in Corollary 6.13 of [26] or Lemma 3.4 of [12]. Since $C_{F_0^{\mathbb{C}}} = \operatorname{supp}\mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}(F_0^{\mathbb{C}})$

and has dimension $N-1$, $\mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}(F_0^{\mathbb{C}})$ is a Cohen-Macaulay $\mathcal{O}^{\mathbb{D}_3}_{(z,\alpha)}$-module, hence a Cohen-Macaulay $\mathcal{O}_{C_{F_0^{\mathbb{C}}}}$-module. As π_{F_0} restricted to $C_{F_0^{\mathbb{C}}}$ is finite with $\Delta^{F_0^{\mathbb{C}}} = \pi_{F_0}(C_{F_0^{\mathbb{C}}})$, it follows that $\mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}(F_0^{\mathbb{C}})$ is Cohen-Macaulay as an $\mathcal{O}_{\Delta^{F_0^{\mathbb{C}}}}$-module and hence as an $\mathcal{O}_\alpha$-module. Because f_0 is of finite codimension, the annihilator of $\mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}(F_0^{\mathbb{C}})$ is non zero and so $\mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}(F_0^{\mathbb{C}})$ has projective dimension 1 and its zeroth Fitting ideal is principal. Hence $\ker\varphi$ is a free $\mathcal{O}_\alpha$-module of rank N. □

Liftable vector fields must be tangent to the discriminant by projecting down from $(F_0^{\mathbb{C}})^{-1}(0)$. Here, the converse is also true, like in the non-equivariant case.

Corollary 4.4. — *An analytic vector field is liftable over π_{F_0} if and only if it is tangent to the discriminant $\Delta^{F_0^{\mathbb{C}}}$.*

Proof. — The necessity is clear from projecting down π_{F_0}. For the converse, this problem has been classically tackled using Hartog's Theorem. We use Bruce [4] for a nice exposition of the fundamental ideas. Outside a set of codimension one in the critical set (4.4), the singularities on $\Delta^{F_0^{\mathbb{C}}}$ are of the type fold (no symmetry), pitchfork ($\mathbb{Z}_2$-symmetry) or S_I, the $\mathbb{D}_3$-equivariant germ of codimension 0. Any analytic vector field tangent to $\Delta^{F_0^{\mathbb{C}}}$ at those singularities lift locally (see [26], [14] and Proposition 4.2, respectively), therefore, it lifts everywhere. □

5. Fundamental theorems of path equivalence and classification

We can now state the fundamental abstract result showing that path formulation achieves the same classification as bifurcation equivalence for problems of finite codimension.

5.1. Fundamental theorem

In this section we shall use every group of equivalence we have considered. First we recall the set-up for path formulation we developed in Section 2.2.1. The core $f_0 \in \mathcal{E}_z^{\mathbb{D}_3}$ of a l-parameter bifurcation germ $f \in \vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\lambda)}$, with $\lambda \in (\mathbb{R}^l, 0)$, is defined by $f_0(z) = f(z, 0)$. Given a $\mathbb{D}_3$-equivariant a-parameter unfolding $F_0 : (\mathbb{R}^{2+a}, 0) \to \mathbb{R}^2$ of f_0, with $\alpha \in (\mathbb{R}^a, 0)$, a path is a smooth map $\bar{\alpha} : (\mathbb{R}^l, 0) \to (\mathbb{R}^a, 0)$ and it defines a bifurcation germ

$\bar{\alpha}^*F_0 \in \vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\lambda)}$ via $(\bar{\alpha}^*F_0)(z,\lambda) = F(z,\bar{\alpha}(\lambda))$. The set of such paths associated with F_0 is denoted by $\vec{\mathcal{P}}^a_{0,\lambda}$.

Theorem 5.1. — (a) *If $f \in \vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\lambda)}$ has a core f_0 of finite $\mathcal{K}^{\mathbb{D}_3}$-codimension, say N, with $\mathcal{K}^{\mathbb{D}_3}$-miniversal unfolding F_0, there exists a path $\bar{\alpha} \in \vec{\mathcal{P}}^N_{0,\lambda}$ such that f is $\mathbb{D}_3$-bifurcation equivalent to $\bar{\alpha}^*F_0$.*

(b) *In the situation of (a), the $\mathcal{K}^*$-codimension of $\bar{\alpha}$ is finite if and only if the $\mathcal{K}^{\mathbb{D}_3}_\lambda$-codimension of $\bar{\alpha}^*F_0$ is finite. In that case, a map A is a $\mathcal{K}^*$-miniversal unfolding of $\bar{\alpha}$ if and only if A^*F_0 is a $\mathcal{K}^{\mathbb{D}_3}_\lambda$-miniversal unfolding for $\bar{\alpha}^*F_0$.*

(c) *Let $\bar{\alpha}, \bar{\beta} \in \vec{\mathcal{P}}^N_{0,\lambda}$, $\bar{\alpha}$ is path equivalent to $\bar{\beta}$ if and only if $\bar{\alpha}^*F_0$ is $\mathbb{D}_3$-bifurcation equivalent to $\bar{\beta}^*F_0$ for finite codimension problems.*

Proof. —

(a) This follows from the unfolding theory for the core f_0 (see (2.4)).

(b) Our set-up corresponds to the algebraic path formulation of Theorem 3.3.2 of [15] where part (b) is proved in a general context by defining the module $M = \mathcal{T}_e\mathcal{K}^{\mathbb{D}_3}(F_0) \cap < h_1 \ldots h_N >_{\mathcal{E}_\alpha}$ of vector fields on $\mathbb{R}^N$, where $\{h_i\}_{i=1}^N$ is a basis of $\mathcal{N}_e\mathcal{K}^{\mathbb{D}_3}(f_0)$. The complexification of the situation using finite determinacy leads to the module $M^{\mathbb{C}}$, corresponding to $\mathrm{Derlog}^*(F_0^{\mathbb{C}})$, equal to $\ker\varphi$ where φ is defined in (4.2).

(c) Suppose that two paths $\bar{\alpha}_0$ and $\bar{\alpha}_1$ are $\mathcal{K}^*$-equivalent (see 2.5), that is, there exists (H, L) such that

$$\bar{\alpha}_1(\lambda) = H(\lambda, \bar{\alpha}_0(L(\lambda))).$$

And so, H can be taken to lift to a λ-family of diffeomorphisms

$$\Phi : (\mathbb{R}^{2+N+l}, 0) \to (\mathbb{R}^{2+N}, 0)$$

preserving $F_0^{-1}(0)$. Hence, Φ is a diffeomorphism between the sections over $\bar{\alpha}_0(L)$ and $\bar{\alpha}_1$. Their zero-set being diffeomorphic, $\bar{\alpha}_0(L)^*F_0$ and $\bar{\alpha}_1^*F_0$ are $\mathcal{K}^{\mathbb{D}_3}_\lambda$-equivalent using the usual construction of equivalences of contact type between germs with the same zero-set ([28]).

For the reverse implication, we assume that $\bar{\alpha}_0{}^*F_0$ and $\bar{\alpha}_1{}^*F_0$ are $\mathcal{K}^{\mathbb{D}_3}_\lambda$-equivalent, with equivalence (T, Z, L) say. Let $\bar{\beta} = \bar{\alpha}_1(L)$. Because $\mathbb{D}_3$ acts absolutely irreducibly (the only $\mathbb{D}_3$-equivariant matrices are multiple of the identity), we can define a smooth family $t \mapsto (T'(t), Z'(t), I)$, $t \in [0,1]$, of $\mathcal{K}^{\mathbb{D}_3}_\lambda$-equivalences where $T'(t) = (1-t)I + tT$ and $Z'(t) = (1-t)z + tZ$. Therefore the family $t \mapsto g(t) = (T'(t), Z'(t), I)\cdot(\bar{\alpha}_0{}^*F_0) \in \vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\lambda)}$, $t \in [0,1]$, connects $\bar{\alpha}_0{}^*F_0$ and $\bar{\beta}^*F_0$ by $\mathcal{K}^{\mathbb{D}_3}_\lambda$-equivalent germs. We want to associate to the family $t \mapsto g(t)$ a family of paths for F_0 connecting $\bar{\alpha}$ and $\bar{\beta}$ via $\mathcal{K}^*$-equivalent paths. For $\lambda = 0$, $(T'(z,0,t), Z'(z,0,t))$ defines an (invertible) family in $\mathcal{K}^{\mathbb{D}_3}$ and so, $h(t) \in \vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\lambda)}$, defined via the equation

$$g(t)(z,\lambda) = T'(t)(z,0)h(t)(Z'(t)(z,0),\lambda),$$

corresponds to a (λ,t)-unfolding of f_0 because $(\bar{\alpha}_0{}^*F_0)(z,0) = f_0(z)$. We use the $\mathcal{K}^{\mathbb{D}_3}$-unfolding theory with (t,λ) as unfolding parameters to derive the family of paths in $\vec{\mathcal{P}}^N_{0,\lambda}$ associated with g. Let $\bar{t}$ be arbitrary in $[0,1]$, we consider germs around $\bar{t}$ if necessary. Hence there is a smooth family of paths $\bar{\alpha}(t) : (\mathbb{R}^l \times \mathbb{R}, (0,\bar{t})) \to (\mathbb{R}^N, 0)$ such that

$$h(t)(z,\lambda) = \bar{T}(t)(z,\lambda)F_0(\bar{X}(t)(z,\lambda), \bar{\alpha}(t)(\lambda)),$$

for t close to $\bar{t}$. Thus, the family $t \mapsto g(t)$ satisfies

$$g(t) = (T'(t), X'(t), I)\cdot(\bar{T}(t), \bar{X}(t), I)\cdot(\bar{\alpha}(t)^*F_0) = (\hat{T}(t), \hat{X}(t), I)\cdot(\bar{\alpha}(t)^*F_0).$$

The family $t \mapsto g(t)$ is formed of $\mathcal{K}^{\mathbb{D}_3}_\lambda$-equivalent maps, so there exists a family of change of coordinates $t \mapsto (T'(t), X'(t), I) \in \mathcal{K}^{\mathbb{D}_3}_\lambda$ such that

$$g(t)(z,\lambda) = T'(t)(z,\lambda)F_0(X'(t)(z,\lambda), \bar{\alpha}_0(\lambda)). \tag{5.1}$$

Using (5.1), (5.2) defines $(T(t), X(t), I) \in \mathcal{K}^{\mathbb{D}_3}_\lambda$ such that

$$\begin{aligned} \bar{\alpha}(t)^*F_0 &= (\hat{T}(t), \hat{X}(t), I)^{-1}\cdot(T'(t), X'(t), I)\cdot(\bar{\alpha}_0{}^*F_0) \\ &= (T(t), X(t), I)\cdot(\bar{\alpha}_0{}^*F_0). \end{aligned} \tag{5.2}$$

Finally we need to show that the family $t \mapsto \bar{\alpha}(t)$ is locally $\mathcal{K}^*$-trivial and conclude using, if necessary, a partition of unity to join together our local information. Recall that, for all $t \in (\mathbb{R}, \bar{t})$, the $\mathbb{D}_3$-equivariant bifurcation germs $\bar{\alpha}(t)^*F_0$ are $\mathcal{K}^{\mathbb{D}_3}_\lambda$-equivalent and so, differentiating in t at $t = \bar{t}$,

$$\frac{d\bar{\alpha}}{dt}(\bar{t})^*F_0 \in \mathcal{T}_e\mathcal{K}^{\mathbb{D}_3}_\lambda(\bar{\alpha}(t)^*F_0).$$

Because the paths $\bar{\alpha}(t)$ are of finite codimension, the techniques of Lemma 3.2 of [30] show that the derivative $\frac{d\bar{\alpha}}{dt}$ is the pull-back by $\bar{\alpha}(t)$ of a liftable vector field ξ. Integrating this vector field, and its lift, gives the $\mathcal{K}^*$-equivalence of the members of the family $t \mapsto \bar{\alpha}(t)$. □

5.2. Classification of one-parameter bifurcation germs

We can now classify the one-parameter paths up to topological $\mathcal{K}^*$-codimension two corresponding to the cores (3.12-15). We use β's as our unfolding parameters for the paths so as not to create confusion with α, which we used for the miniversal unfoldings of the cores.

Theorem 5.2. — *There are 8 one-parameter paths of topological $\mathcal{K}^*$-codimension $k \leqslant 2$. Their $\mathcal{K}^*$-miniversal unfoldings A are given in the following list. To recover the paths, set the unfolding parameters β_i's to 0, $\bar{\alpha}(\lambda) = A(\lambda, 0)$.*

(1) *for C_1^3, there are three $\mathcal{K}^*$-miniversal unfoldings $A : (\mathbb{R}^{1+k}, 0) \to \mathbb{R}$:*

$$\mathrm{I}_3: \quad \delta_1\lambda, \qquad \mathrm{II}_3: \quad \delta_2\lambda^2 + \beta_1, \qquad \mathrm{III}_3: \quad \delta_3\lambda^3 + \beta_1\lambda + \beta_2,$$

(2) *for C_2^3, there are three $\mathcal{K}^*$-miniversal unfoldings $A : (\mathbb{R}^{1+k}, 0) \to \mathbb{R}^2$ (see Figure 5.1):*

$$\mathrm{X}_3: (\delta\lambda, \nu\lambda + \beta_1),$$
$$\mathrm{XI}_3: (\delta\lambda, \nu\lambda^2 + \beta_1 + \beta_2\lambda),$$
$$\mathrm{XII}_3: (\nu\lambda^2 + \beta_1, \delta\lambda + \beta_2).$$

(3) *for C_3^3, there is one $\mathcal{K}^*$-miniversal unfolding $A : (\mathbb{R}^{1+k}, 0) \to \mathbb{R}^3$:*

$$\mathrm{XIII}_3: \quad (\delta\lambda, \nu\lambda + \beta_1, \beta_2),$$

(4) *for $C_{3^*}^3$, there is one $\mathcal{K}^*$-miniversal unfolding $A : (\mathbb{R}^{1+k}, 0) \to \mathbb{R}^4$:*

$$\mathrm{XIV}_3: \quad (\delta_1\lambda, \delta_2\lambda + \beta_1, \beta_2, 0),$$

where the ν's are modal parameters for the paths. In Section 6 we give explicit formula for the sign constants $\delta^2 = \delta_1^2 = \delta_2^2 = 1$ and the moduli.

Proof. — To compute the codimension, the normal spaces and the higher order terms of the different paths, we use the definition of the extended tangent spaces to the paths given in (2.6). For instance, for case XIII_3, the path is $(\delta\lambda, \nu\lambda, 0)$. Replacing α_1 by $\delta\lambda$, α_2 by $\nu\lambda$ and α_3 by 0 in (2.6) and using the generators for $\mathrm{Derlog}^*(C_3^3)$ in Proposition 4.2, we get:

$$\bar{\alpha}^*\xi_1 = \left(4\delta\nu\lambda^2 + \mathcal{M}_\lambda^3, 2\nu^2\lambda^2 + \mathcal{M}_\lambda^3, \mathcal{M}_\lambda^3\right),$$
$$\bar{\alpha}^*\xi_2 = \left(2\lambda^2 + \mathcal{M}_\lambda^3, 6\delta\nu\lambda^2 + \mathcal{M}_\lambda^3, \mathcal{M}_\lambda^3\right),$$
$$\bar{\alpha}^*\xi_3 = \left(\mathcal{M}_\lambda^3, \mathcal{M}_\lambda^3, 10\epsilon_3\nu\lambda + \mathcal{M}_\lambda^3\right).$$

Using the ideas in [5], the higher order terms that can be removed belong to the intrinsic part of the unipotent subgroup of $\mathcal{K}^*$ generated by $\bar{\alpha}^*\xi_i$'s (they are quadratic or with upper triangular derivatives) and $\lambda^2\bar{\alpha}_\lambda = (\delta\lambda^2, \nu\lambda^2, 0)$. When $\nu \neq 0$, the terms ignored in the path are contained in that intrinsic part, so they can be removed. For the unfolding theory, the extended tangent space of $\bar{\alpha}$ is $\mathcal{T}_e\mathcal{K}^*(\bar{\alpha}) = <\bar{\alpha}^*\xi_1, \bar{\alpha}^*\xi_2, \bar{\alpha}^*\xi_3, \bar{\alpha}_\lambda>_{\mathcal{E}_\lambda}$. When $\nu \neq 0$, it contains $\vec{\mathcal{M}}_\lambda^2$ and so the normal space $\mathcal{N}_e\mathcal{K}^*(\bar{\alpha})$ is generated over $\mathbb{R}$ by $(0,0,1)$, $(0,1,0)$ and $(0,\lambda,0)$, hence ν is a modal parameter and $\bar{\alpha}$ is of topological codimension 2 with the given $\mathcal{K}^*$-miniversal unfolding. □

Path formulation allows us to identify where the modal parameter belongs to, the core or the path, controlling its relative position with respect to Δ^{F_0}. The following figure represents the situation for C_2^3. We have portrayed in Figure 5.1 the 3 paths of Theorem 5.2 that are a straight line for X_3 or parabolas for $\mathrm{XI}_3, \mathrm{XII}_3$ with their positions with respect to the discriminant, determined ultimately by the different moduli.

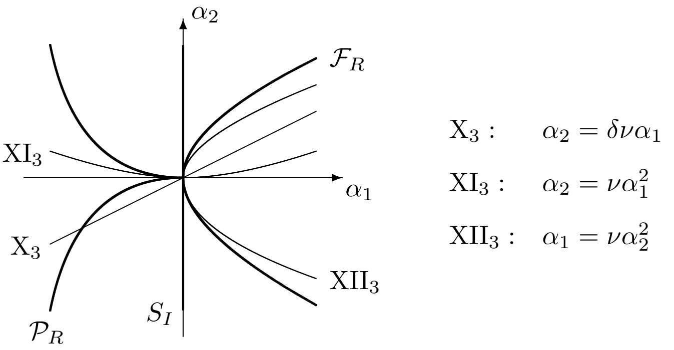

Figure 5.1. Discriminant related to the core C_2^3, $\epsilon_1 = 1$, and paths for X_3, XI_3 and XII_3. The coefficients $\delta\nu$ and ν are all positive.

5.3. Classification of two-parameter bifurcation germs

We classify two-parameter paths up to topological $\mathcal{K}^*$-codimension 1 to illustrate the power of path formulation in the multiparameter situation.

THEOREM 5.3. — *There are 5 two-parameter paths of topological $\mathcal{K}^*$-codimension $k \leqslant 1$. Their $\mathcal{K}^*$-miniversal unfoldings A are given in the following list. To recover the paths, set when needed the unfolding parameter β to 0, $\bar{\alpha}(\lambda) = A(\lambda_1, \lambda_2, 0)$. The sign constants are $\delta_i^2 = 1$, $i = 1, 2$, and μ, ν are modal parameters.*

(1) *for C_1^3, there are two $\mathcal{K}^*$-miniversal unfoldings $A : (\mathbb{R}^{2+k}, 0) \to \mathbb{R}$, namely, $A_{1,0}(\lambda) = \lambda_1$, of $\mathcal{K}^*$-codimension 0, and $A_{1,1}(\lambda, \beta) = \delta_1\lambda_1^2 + \delta_2\lambda_2^2 + \beta$, of $\mathcal{K}^*$-codimension 1.*
(2) *for C_2^3, there are two $\mathcal{K}^*$-miniversal unfoldings $A : (\mathbb{R}^{2+k}, 0) \to \mathbb{R}^2$, namely $A_{2,0}(\lambda) = (\lambda_1, \delta\lambda_2)$, of $\mathcal{K}^*$-codimension 0, and $A_{2,1}(\lambda, \beta) = (\delta_1\lambda_1, \mu\delta_1 + \delta_2\lambda_2^2 + \beta)$, of $\mathcal{K}^*$-codimension 1 when $\mu \neq 0$.*
(3) *for C_3^3, there is one $\mathcal{K}^*$-miniversal unfolding $A : (\mathbb{R}^{2+1}, 0) \to \mathbb{R}^3$, given by $A_{3,1}(\lambda, \beta) = (\delta_1\lambda_1, \delta_2\lambda_2, \nu\lambda_1 + \beta)$ when $\nu \neq 0$.*

Proof. — It is straightforward to see that the classification of the l-parameter paths for the core C_1^3 is given by the classical contact equivalence classes of $\bar{\alpha} : (\mathbb{R}^l, 0) \to (\mathbb{R}, 0)$. When $l = 2$, there are two such classes up to codimension 1, with miniversal unfoldings $A_{1,0}$ and $A_{1,1}$.

For the two other cores, the procedure is routine. For the path $A_{2,0}$ for the core C_2^3, if the determinant $(\bar{\alpha}_1)^o_{\lambda_1}(\bar{\alpha}_2)^o_{\lambda_2} - (\bar{\alpha}_1)^o_{\lambda_2}(\bar{\alpha}_2)^o_{\lambda_1} \neq 0$, a change of coordinate brings the path into the pre-normal form

$$\bar{\alpha} = (\lambda_1 + \mathcal{M}_\lambda^2, \delta\lambda_2 + \mathcal{M}_\lambda^2).$$

The extended tangent space is equal to $\vec{\mathcal{P}}_\lambda^2$ (the codimension is 0) and the tangent space is invariant to the second order terms as required to eliminate the higher order terms. For the path $A_{3,1}$ for the core C_3^3, under the same condition, a change of coordinates brings the path into the pre-normal form $(\delta_1\lambda_1, \delta_2\lambda_2, \nu\lambda_1)$ modulo $\mathcal{M}_\lambda^2$. We get the generators:

$$\begin{pmatrix} 4\delta_1\lambda_1\lambda_2 + \mathcal{M}_\lambda^3 \\ 2\delta_2\lambda_2^2 + \mathcal{M}_\lambda^3 \\ -3\nu\lambda_1\lambda_2 + \mathcal{M}_\lambda^3 \end{pmatrix}, \begin{pmatrix} 2\delta_1\lambda_1^2 + \mathcal{M}_\lambda^3 \\ \mathcal{M}_\lambda^3 \\ 3\nu\lambda_1^2 + \mathcal{M}_\lambda^3 \end{pmatrix}, \begin{pmatrix} \mathcal{M}_\lambda^2 \\ \mathcal{M}_\lambda^2 \\ \lambda_2 + \mathcal{M}_\lambda^2 \end{pmatrix}$$

and the derivatives $(\delta_1, 0, \nu)$, $(0, \delta_2, 0)$. When $\nu \neq 0$ we get that $\vec{\mathcal{M}}_\lambda^2 \subset \mathcal{T}_e\mathcal{K}^*(\bar{\alpha})$ which takes care of the higher order terms. The normal space $\mathcal{N}_e\mathcal{K}^*(\bar{\alpha})$ is generated over $\mathbb{R}$ by $(0, 1, 0)$ and $(0, \lambda, 0)$, hence ν is a modal parameter and $\bar{\alpha}$ is of topological $\mathcal{K}^*$-codimension 1 with the given $\mathcal{K}^*$-miniversal unfolding. □

6. Bifurcation diagrams

In this section we analyse the cases from Theorem 5.2 that we need for our application in Section 7.

6.1. Bifurcation diagrams and stability

The **bifurcation diagram** of $f \in \vec{\mathcal{E}}^{\mathbb{D}_3}_{(z,\lambda)}$ is its zero-set. We use the same terminology for any a-parameter unfolding $F = [P, Q]$ of f, with unfolding parameter $\alpha \in (\mathbb{R}^a, 0)$. We described in Section 4.1.1 the three types of solutions in the zero-set set of $Pz + Q\bar{z}^2 = 0$ with different classes of isotropy subgroups. Here, we specify the information on the eigenvalues of the linearisation of F.

(1) For O, the double eigenvalue of $F_z(0,\lambda,\alpha)$ at the trivial solution $z=0$ is equal to $P(0,0,\lambda,\alpha)$.

(2) For R, the eigenvalues of $F_z(x,0,\lambda,\alpha)$ at the $\mathbb{Z}_2$-symmetric points of equation

$$P(x^2,x^3,\lambda,\alpha)+x\,Q(x^2,x^3,\lambda,\alpha)=0$$

are $3P$ or $-3Qx$ for one, and

$$x\,(Q+2P_u x+(3P_v+2Q_u)x^2+3Q_v x^3)$$

for the other.

(3) For T, the sign of the determinant of F_z at the points without any symmetry, of equation

$$P(u,v,\lambda,\alpha)=Q(u,v,\lambda,\alpha)=0,$$

is equal to the sign of $(P_uQ_v-P_vQ_u)$ and the sign of the trace is the sign of

$$(2uP_u+3u^2Q_v+3vP_v+2vQ_u).$$

To keep track of the stability, we recall that the signs of the eigenvalues of the linearisation for the trivial and R-branches are invariant under $\mathcal{K}_\lambda^{\mathbb{D}_3}$-equivalence ([24], pp. 99-101). On the T-branches, the sign of $\det(F_z)$ is invariant. Moreover, from the previous data for solutions of the type T (which can exist only if $Q^o=0$), we can conclude that the signs of the real part of the eigenvalues are also invariant on the T-branches when $P_u^o\cdot(P_u^oQ_v^o-P_v^oQ_u^o)\neq 0$. Case XIV_3 in the one-parameter classification partially escapes the scope of those results. Because the hypotheses for the T-branches are not satisfied, a Hopf bifurcation may occur.

6.2. Transition varieties

The **transition varieties** are the hypersurfaces in the space of unfolding parameters delimiting open regions where the bifurcation diagrams of the unfolding are topologically equivalent. The transition varieties belong to two categories: the values of the parameters where there is a *multi-local singularity* (at least two generic singularities for the same value of λ) and the values where there is a *codimension 1 singularity.* In the first case we denote the transition variety $\mathcal{D}(X,Y)$ where X,Y are any of the 3 generic singularities of Section 4.1.2 arising at the same value of λ. In the second case there are the usual codimension 1 bifurcations on the R or T-branches, $\mathcal{B}_R$, $\mathcal{B}_T$ (symmetry preserving bifurcation point), or $\mathcal{H}_R$, $\mathcal{H}_T$

(hysteresis point), and the codimension one, corank one bifurcations with $\mathbb{Z}_2$-symmetry, $\mathcal{C}_R$, $\mathcal{J}_R$, and the codimension one, corank two bifurcations with $\mathbb{Z}_2$-symmetry, $\mathcal{Q}_R$, and, finally, S_{II} and S_X the $\mathbb{D}_3$-equivariant germs of $\mathcal{K}_\lambda^{\mathbb{D}_3}$-codimension one occurring on the trivial branch. Here, for the paths II_3 and XII_3, we only need explicitly S_{II}, S_X and $\mathcal{B}_R$. Their defining equations are $P(0,0,\lambda) = P_\lambda(0,0,\lambda) = 0$ for S_{II}, $P(0,0,\lambda) = Q(0,0,\lambda) = 0$ for S_X, and $P(x,0,\lambda) + Q(x,0,\lambda)\,x = 0$, $P_\lambda(x,0,\lambda) + Q_\lambda(x,0,\lambda)\,x = 0$ and $[2P_u x + Q + (3P_v + 2Q_u)\,x^2 + 3Q_v x^3](x,0,\lambda) = 0$ for $\mathcal{B}_R$.

6.3. Normal forms with $q^0 \neq 0$

Let $F = [P(u,v,\lambda,\beta), Q(u,v,\lambda,\beta)]$ be a miniversal unfoldings of a normal form, where β represents the unfolding parameters. Setting $\beta = 0$ recovers the normal form. The diagrams are basically of two types depending if the $\mathbb{O}(2)$-symmetry breaking term q^o is zero or not. When $q^o \neq 0$, we have a family of normal forms $[\delta_k \lambda^k, \epsilon]$, where $k \in \mathbb{N}$ and $\epsilon = \mathrm{sign}(q^o)$. The necessary conditions on $f = [p,q]$ are $p^o_{\lambda^j} = 0$, $j = 0 \dots k-1$. The non-degeneracy conditions are $q^o \cdot p^o_{\lambda^k} \neq 0$, with $\delta_k = \mathrm{sign}(p^o_{\lambda^k})$. A $\mathcal{K}_\lambda^{\mathbb{D}_3}$-miniversal unfolding is $[\delta_k \lambda^k + \sum_{i=0}^{k-2} \beta_{k-i-1}\lambda^i, \epsilon]$. The transitions varieties are, respectively, empty for I_3, $S_I : \beta_1 = 0$ for II_3 and $S_{II} :\ \beta_2^2 + \frac{4}{27}\delta_3\beta_1^3 = 0$ for III_3. All solutions are $\mathbb{Z}_2$-symmetric (type R) for some $\mathbb{Z}_2 \subset \mathbb{D}_3$, and so form 3 conjugate branches. Here the branches exhibit only a simple "snake-like" behaviour. In the figures, the given linearised stability corresponds to the dynamics of $\dot{z} = F(z,\lambda,\beta)$. Figure 6.1 illustrates that the typical $\mathbb{D}_3$-bifurcation is to saddle points. The central axis corresponds to the trivial solution, the vertical direction to the "z"-axis. The constants in the diagrams are such that the trivial solution is always stable for $\lambda < 0$. Thick lines represent stable solutions or transition varieties in parameter space.

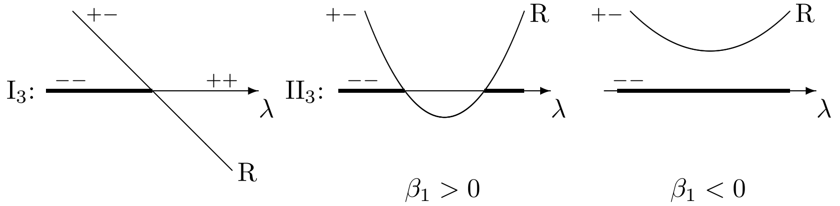

Figure 6.1. Bifurcation diagrams for I_3 *(*$\delta_1 = 1$*) and* II_3 *(*$\delta_2 = -1$*),* $\epsilon = 1$.

6.4. Normal form XII_3 with $q^0 = 0$

In the second type of diagrams there are solutions with trivial symmetry (type T). We only discuss the case XII_3 we use for our application. The necessary conditions $q^o = p^o_\lambda = 0$ and the nondegeneracy conditions are $p^o_u \cdot q^o_\lambda \cdot p^o_{\lambda\lambda} \cdot \Delta_{u,v}(p,q) \cdot (4p^o_u p^o_{\lambda\lambda} - q^{o2}_\lambda) \neq 0$. The normal form for the $\mathcal{K}^{\mathbb{D}_3}_\lambda$-miniversal unfolding is

$$[\beta_1 + \epsilon_1 u + \nu\lambda^2,\ \beta_2 + \epsilon_2 v + \delta\lambda]$$

where $\epsilon_2 = \operatorname{sign}(p^o_u \cdot \Delta_{u,v}(p,q))$, $\epsilon_1 = \operatorname{sign} p^o_u$, $\delta = \operatorname{sign} q^o_\lambda$ and $\nu = \mid p^o_u \mid \cdot p^o_{\lambda\lambda} \cdot (q^o_\lambda)^{-2}$. The modal parameter ν has critical values $\frac{1}{4}\epsilon_1$ and, for the unfolding, also 0. The transition varieties are $S_{II}:\ \beta_1 = 0$, $S_X:\ \beta_1 + \nu\beta_2^2 = 0$ and $\mathcal{B}_R:\ \beta_1 = \frac{\nu}{(4\epsilon_1\nu - 1)}\beta_2^2 + \ldots$. We can also solve the recognition problem for unfoldings. In the path formulation, the unfolding $(A_1(\lambda,\beta), A_2(\lambda,\beta))$ is miniversal if and only if $\beta \in (\mathbb{R}^2, 0)$ and the Jacobian matrix of the map $(\lambda, \beta_1, \beta_2) \mapsto (A_1, (A_1)_\lambda, A_2)$ is invertible at the origin.

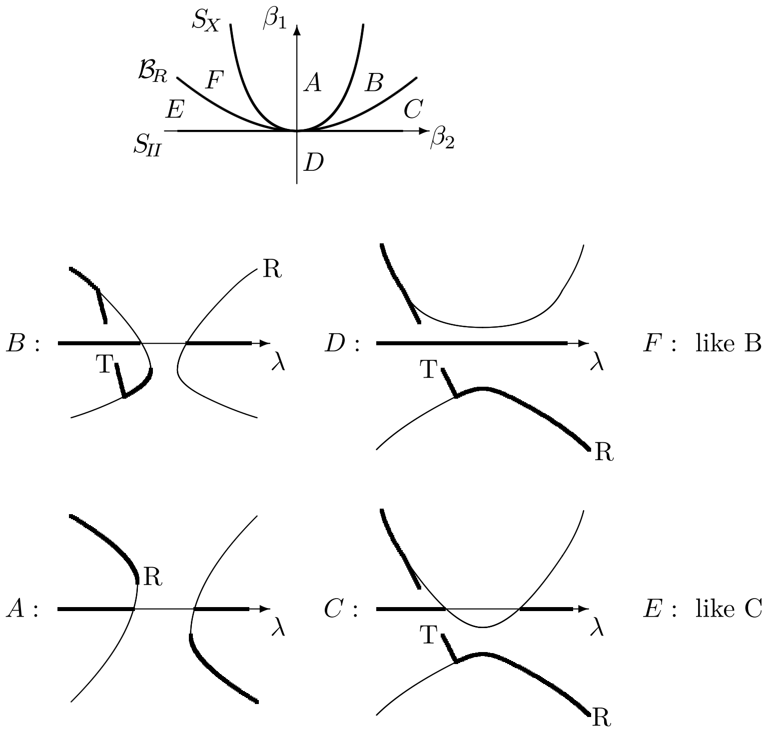

Figure 6.2. Bifurcation diagrams and transition varieties for $\mathrm{XII}_3,\ 0 < \ \nu,\ \delta = \epsilon_1 = \epsilon_2 = 1$.

The diagrams and transition varieties are represented in Figures 6.2-6.4.

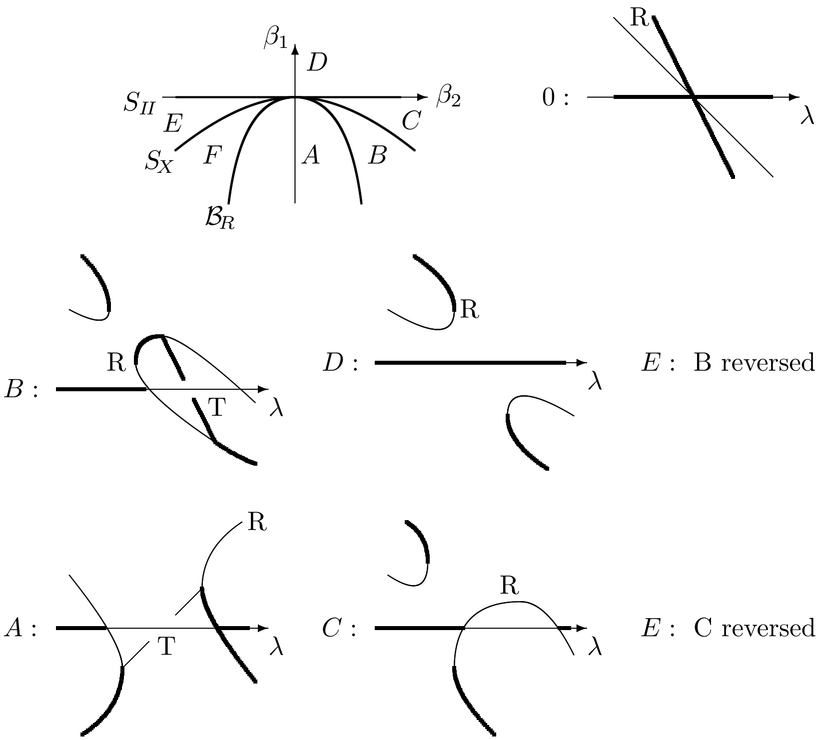

Figure 6.3. Bifurcation diagrams and transition varieties for XII_3, $-\frac{1}{4} < \nu < 0$, $\delta = \epsilon_1 = \epsilon_2 = -1$.

7. About the 1-1:resonance for period-3 points in reversible systems

Our set-up can be used to describe subharmonic bifurcations in reversible systems (Vanderbauwhede [39]). In that paper the generic bifurcations in reversible forced vector fields are considered when the kernel of the linearisation is two dimensional and irreducible. In [19], Gervais studied the same problem using the singularity theory approach to the bifurcation equations, although without analyzing new normal forms. In [3] the bifurcation of periodic points for reversible-symplectic maps was studied. When a multiplier associated with a fixed point crosses a root of unity $\exp(2\pi i\,\frac{m}{3})$, $m = 1, 2$,

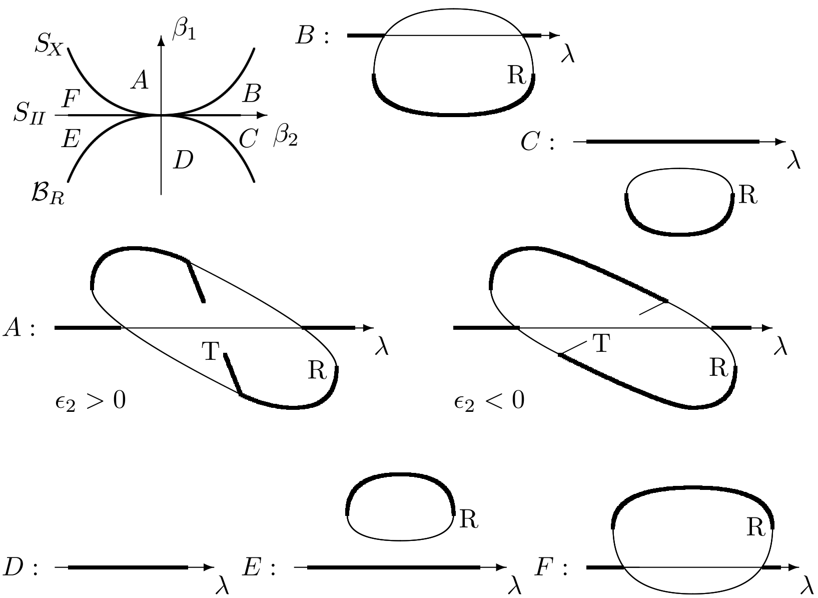

Figure 6.4. Bifurcation diagrams and transition varieties for XII_3, $\nu < -\frac{1}{4}$, $\epsilon_1 = \delta = -1$.

or when there is a collision of two multipliers at such a root of unity, we get $\mathbb{D}_3$-equivariant gradient bifurcation equations of corank 2 via a discrete Lagrangian formulation. For the bifurcation of subharmonic periodic orbits of reversible (possibly also symplectic) systems there are problems leading to each of our diagrams. The core depends essentially on the nonlinearity and the dependence of the multipliers of the linearisation on the parameter λ typically controls the path. The normal forms II_3 and XII_3 are important because they correspond to the bifurcation of periodic orbits from a *collision of multipliers* on the unit circle at the third root of unity. To get details of those collisions we use the recent approach of Ciocci [7, 8] on the bifurcation of periodic points in reversible maps building on ideas of Vanderbauwhede ([39]). This framework can also be used for periodically forced ODEs or autonomous systems via time-T or Poincaré maps ([39, 40]).

To facilitate comparison we follow the derivation and notation of [8]. Let $\Phi : (\mathbb{R}^{n+l}, 0) \to (\mathbb{R}^n, 0)$ be a family of R-reversible map, $R^2 = I$ and $R\Phi(Rx, \lambda) = \Phi^{-1}(x, \lambda)$. We assume that $A_0 = \Phi_x(x_0, \lambda_0)$ is invertible when there is an R-invariant fixed point (x_0, λ_0), $Rx_0 = x_0$. The Implicit

Function Theorem implies that we have a branch of those. With a change of co-ordinate, we can assume that the branch is the trivial branch, that is, $\Phi(0,\lambda) = 0$, $\forall\lambda \in (\mathbb{R}^l, 0)$. The spectrum of A_0 has single ± 1, complex conjugate pairs $\{\mu, \bar{\mu}\}$ on S^1 or quadruples $\{\mu, \mu^{-1}, \bar{\mu}, \bar{\mu}^{-1}\}$ outside of S^1. Of interest are the pairs on S^1. Without colliding, they are stuck on the circle as λ varies. They will encounter roots of unity and generically two periodic orbits will bifurcate ([7]). Suppose $n \geqslant 4$, we are looking at the bifurcation near the collision of two pairs of conjugate eigenvalues at a third root of unity. This collision corresponds in general to the splitting of the eigenvalues off the unit circle, creating a kernel of A_0 of geometric multiplicity 1 and algebraic 2. Clearly this collision can only occur in a two-parameter problem. Some of its aspect is parameter driven: how the parameters drive the eigenvalues. Here we shall pay more attention to the degeneracies in the nonlinear part.

The equation we would like to solve is $\Phi(x,\lambda)^3 = x$. Define the following matrices

$$J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, R_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, J_0 = \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix},$$

$$R = \begin{pmatrix} -R_1 & 0 \\ 0 & R_1 \end{pmatrix}, N_0 = \begin{pmatrix} 0 & I_2 \\ 0 & 0 \end{pmatrix}, M_0 = \begin{pmatrix} 0 & 0 \\ I_2 & 0 \end{pmatrix},$$

$$S_0 = \exp(\theta_0 J_0) = \begin{pmatrix} R_{\theta_0} & 0 \\ 0 & R_{\theta_0} \end{pmatrix}$$

where $R_{\theta_0} = \cos(\theta_0) I_2 + \sin(\theta_0) J_2$. Under the previous hypotheses, there exists a basis for $U = \ker(A_0^3 - I)$ such that $A_0 = S_0 \exp(N_0)$. The matrix unfolding of A_0 in the sense of Arnold is

$$\mathcal{A}(\vartheta, \sigma) = S_0 \exp(N_0 + B(\vartheta, \sigma))$$

where $B(\vartheta, \sigma) = \begin{pmatrix} \vartheta J_2 & 0 \\ \sigma I_2 & \vartheta J_2 \end{pmatrix}$, ϑ measuring the distance to the root of unity and σ representing the separation from the unit circle.

In [7] the Generalised Lyapounov-Schmidt (GLS) reduction is derived. There exists a reduced map $\Phi_r : (U \times \mathbb{R}^l, 0) \to U$ and an R-equivariant map $x^* : (U \times \mathbb{R}^l, 0) \to \mathbb{R}^n$ such that $\Phi_r(0,\lambda) = 0$, $(\Phi_r)_x(0,\lambda) = \mathcal{A}(\vartheta(\lambda), \sigma(\lambda))$, $x^*(0,\lambda) = 0$, $x^*_u(0,\lambda) = I$ with the following important property:

For all $\lambda \in (\mathbb{R}^l, 0)$, x is a period-q point for Φ if and only if $x = x^*(u,\lambda)$ where u is a period-q point for Φ_r if and only if $\Phi_r(u,\lambda) = S_0 u$.

Note that, for every k, $x^*(u,\lambda) = O(\|u\|^k)$ and Φ_r is the restriction to $U \times \mathbb{R}^l, 0)$ of the Birkhoff normal form of Φ. We can define a more convenient bifurcation map on U. In our case of the 1:1-resonance, U is four dimensional, so we identify it with $(z, w) \in U \equiv \mathbb{C}^2$, where z corresponds to the co-ordinates on the geometric kernel. On U, $(\Phi_r)(0, \lambda)$ is

$$\mathcal{A}(\vartheta(\lambda), \sigma(\lambda)) = \exp(\vartheta J_0)\exp(N_0 + \sigma M_0) - \exp(-\vartheta J_0)\exp(-(N_0 + \sigma M_0)),$$

where $\vartheta, \sigma : (\mathbb{R}^l, 0) \to (\mathbb{R}, 0)$. The equation $\Phi_r(u, \lambda) = S_0 u$ is equivalent to

$$\mathcal{B}(u, \lambda) = S_0^{-1}\Phi_r(u, \lambda) - S_0\Phi_r^{-1}(u, \lambda) = 0,$$

where $\mathcal{B} : (\mathbb{C}^2 \times \mathbb{R}^2, 0) \to (\mathbb{C}^2, 0)$ is S_0-equivariant (rotation) and R anti-equivariant (reflection) (hence $\mathbb{D}_3$-equivariant). Let

$$\mathcal{B}(z, w, \lambda) = (\mathcal{B}_1(z, w, \lambda), \mathcal{B}_2(z, w, \lambda)).$$

The linear part of $\mathcal{B}$ can be calculated. The dependence of the functions ϑ, σ on the parameters λ can be complicated and many scenarios are possible. It is here that path formulation is useful as it allows to consider most cases with similar calculations. The Taylor series expansion of $\mathcal{B}_{(z,w)}(0, 0, \lambda)$ is

$$\begin{pmatrix} 2\vartheta(1 + \frac{\sigma}{2} + \frac{\sigma^2}{24})J_2 + \dots & 2(1 - \frac{\vartheta^2}{2})(1 + \frac{\sigma}{6})I_2 + \dots \\ 2\sigma(1 - \frac{\vartheta^2}{2})(1 + \frac{\sigma}{6})I_2 + \dots & 2\vartheta(1 + \frac{\sigma}{2} + \frac{\sigma^2}{24})J_2 + \dots \end{pmatrix}.$$

We solve the first equation for $w = \hat{w}(z, \lambda)$ because the derivative $(\mathcal{B}_1)_w^o = 2I_2$. Replacing into $\mathcal{B}_2$, we get the $\mathbb{D}_3$-equivariant bifurcation equation whose normal form is II_3 when $q_0 \neq 0$ and XII_3 when $q_0 = 0$:

$$\mathcal{B}_3(z, \lambda) = \mathcal{B}_2(z, \hat{w}(z, \lambda), \lambda). \tag{7.1}$$

The symmetries are $\mathcal{B}_1(\bar{z}, \bar{w}) = -\overline{\mathcal{B}_1(z, w)}$, $\mathcal{B}_2(\bar{z}, -\bar{w}) = -\overline{\mathcal{B}_1(z, w)}$ and $\mathcal{B}_i(e^{i\theta_0}z, e^{i\theta_0}w) = e^{i\theta_0}\mathcal{B}_1(z, w)$, $i = 1, 2$. This means that $\hat{w}$ is $e^{i\theta_0}$-equivariant and R anti-equivariant, hence it is an imaginary valued $\mathbb{D}_3$-equivariant map of the form $[ip(u, v), iq(u, v)]$ where $[p, q] \in \vec{\mathcal{E}}_z^{\mathbb{D}_3}$. Note that $J_2 z = iz$. Therefore, $\mathcal{B}_3$ is $\mathbb{D}_3$-equivariant with respect to the standard action. The Taylor series expansion of $\hat{w}$ is $O(|z|^2)$ and so q^o for $\mathcal{B}_3$ depends only on the term in $\bar{z}^2$ of $\mathcal{B}_2$. If that term is non zero we get the normal form II_3 provided the main bifurcation parameter is ϑ and σ represents the unfolding parameter. In case $q^o = 0$, we have the normal form XII_3 with the same parameter structure. To understand the diagrams in our context, bifurcating orbits have period 3, of type R they consist of points in $\mathrm{Fix}(R)$, of type T of points without extra symmetry. When $q^o \neq 0$ we find the diagram of the generic collision with the two bifurcating families mentioned in [8]. When $q^o = 0$, the next branching structure is described in Figures 6.2-6.4. Note that the bifurcation to the T-branches is the symmetry breaking Rimmer bifurcation [40].

8. Final Remarks

8.1. Variational problems

When the map is symplectic, for instance because the underlying dynamical system is Hamiltonian, most reduction techniques will give rise to a gradient bifurcation equation. Our normal forms and their miniversal unfoldings are gradients of some $\mathbb{D}_3$-invariant potential because they satisfy the identity $3p_v(u,v) \equiv 2q_u(u,v)$. With additional structure, germs and their variational miniversal unfoldings should be constrained and produce fewer examples, but this is not very spectacular at low codimension. The only restriction is for $C^3_{3^*}$ where the modal parameter is fixed to $\mu = \frac{2}{3}\operatorname{sign} q^o_u$ and obviously precludes the existence of any possible tertiary Hopf bifurcation. The only one-parameter diagram affected is normal form XIV_3.

Theorem 8.1. — *The variational problems of topological codimension up to 2 correspond to all the normal forms* I_3 *up to* XIV_3 *(in that last case the modal parameter* $\mu = \frac{2}{3}\operatorname{sign} q^o_u$*). Their miniversal unfoldings are all gradients as well.*

Thus it is possible to treat the $\mathbb{D}_3$-equivariant gradient problems as dissipative up to codimension 1. At codimension 2, there is only one situation where additional care is needed. Ignoring a gradient structure for a particular diagram is of no consequence because the main qualitative features are not lost by change of coordinates, provided one does *not use genericity arguments* for its existence. But what might be very different are the *possible* perturbations as it is the case with corank 2 problems without symmetry ([3]). Here we need at least *three* parameters to see any difference.

8.2. The classification for the other dihedral groups

The more complex classification occurs for $k = 4$ and is due to [22], re-organised in [3] following the path formulation. Written in terms of the invariants $[p,q]$, the normal forms with the $\mathbb{O}(2)$-symmetry breaking $q^o = 0$ are the same for $k \geqslant 4$. Cases $\mathrm{X}_k, \mathrm{XI}_k, \mathrm{XII}_k, \mathrm{XIII}_k$ and XIV_k represent all the normal forms up to topological $\mathcal{K}_\lambda^{\mathbb{D}_k}$-codimension 2 when $q^o = 0$. The main difference arises when $q^o \neq 0$, because there is more complicated behaviour depending on k. Also, some germs have many moduli, often without influence on both the bifurcation diagrams and their deformations ([9]).

With an automatic computer procedure, additional normal forms of $\mathbb{D}_3$-equivariant bifurcation germs were calculated in [18], improving on [6]. But not all the normal forms we classify were found. We believe that path formulation would add more structure to such systematic research. In particular, it helps to split it into 2 steps: classify the cores and then study the paths for each core.

Appendix A

Explicit changes of co-ordinates

We refer to Section 3.1 for the details of the notation. The coefficients of the Taylor series expansion of $[p'', q'']$ in (3.6) are given by

$$A_1'' = c_0 A_1' + e_0 A_0', \;\; B_1'' = c_0 B_1' + d_0 A_0', \tag{8.1}$$

$$A_0'' = c_0 A_0', \;\; A_2'' = c_0 A_2' + e_0 A_1', \;\; B_2'' = e_0 B_1' + c_0 B_2', \tag{8.2}$$

for the first order terms, and, for the second order terms, by

$$\begin{aligned}
D_1'' &= (c_u + d_0) A_1' + c_0 D_1' + e_0 A_2', \\
E_1'' &= (c_v + g_0) A_1' + (c_u + d_0) B_1' + c_0 E_1' + e_0 B_2' + d_0 A_2', \\
F_1'' &= (c_v + g_0) B_1' + d_0 B_2' + c_0 F_1', \\
D_2'' &= e_0 D_1' + e_u A_1' + c_0 D_2' + c_u A_2', \\
E_2'' &= e_0 E_1' + e_u B_1' + e_v A_1' + c_0 E_2' + c_u B_2' + c_v A_2', \\
F_2'' &= e_0 F_1' + e_v B_1' + c_0 F_2' + c_v B_2'.
\end{aligned}$$

To keep the formulas simple we have ignored the A_0, A_0' terms on the coefficients of second order because they are not needed if $A_0 \neq 0$ as the normal forms simplify immediately.

The coefficients of the Taylor series expansion of $[p', q']$ in (3.5), in terms of the coefficients of $[p, q]$ in (3.7-8), are given by

$$A_1' = a_0^3 A_1 + 2a_0 b_0 A_0, \;\; B_1' = 2a_0^2 b_0 A_1 + a_0^4 B_1 + 2a_0 b_0^2 A_0, \tag{8.3}$$

$$A_0' = a_0^2 A_0, \;\; A_2' = a_0^2 b_0 A_1 + a_0^4 A_2 + 2a_0 a_u A_0 - b_0^2 A_0, \tag{8.4}$$

for the first order terms, and, for the second order, by

$$
\begin{aligned}
D_1' &= 3a_0^2 a_u A_1 + a_0 b_0^2 A_1 + 3a_0^3 b_0 B_1 + a_0^5 D_1 + 2a_0^3 b_0 A_2,\\
E_1' &= 3a_0^2 a_v A_1 + 4a_0 b_0 a_u A_1 + 2a_0^2 b_u A_1 + 4a_0^3 a_u B_1 + 3a_0^2 b_0^2 B_1\\
&\quad + 4a_0^4 b_0 D_1 + a_0^6 E_1 + 6a_0^2 b_0^2 A_2 + 2a_0^4 b_0 B_2,\\
F_1' &= 4a_0 b_0 a_v A_1 + 2a_0^2 b_v A_1 + 4a_0^3 a_v B_1 + 2a_0 b_0^3 B_1 + 4a_0 b_0^3 A_2\\
&\quad + 2a_0^3 b_0^2 B_2 + 4a_0^3 b_0^2 D_1 + 2a_0^5 b_0 E_1 + a_0^7 F_1,\\
B_2' &= 2a_0 b_0^2 A_1 + a_0^3 b_0 B_1 + 2a_0^3 b_0 A_2 + a_0^5 B_2 + 2a_0 a_v A_0,\\
D_2' &= 4a_0^3 a_u A_2 + 2a_0 b_0 a_u A_1 + b_0^3 A_1 + 3a_0^4 b_0 B_2 + a_0^2 b_u A_1\\
&\quad + 3a_0^2 b_0^2 B_1 + a_0^4 b_0 D_1 + a_0^6 D_2,\\
E_2' &= 4a_0^3 a_u A_2 + 6a_0^2 b_0 a_u A_2 - 2a_0 b_0^3 A_2 + 5a_0^4 a_u B_2 + 2a_0^3 b_0^2 B_2\\
&\quad + 2a_0^3 b_u A_2 + 2a_0 b_0 a_v A_1 + a_0^7 E_2 + 4a_0 b_0 b_u A_1 + 2b_0^2 a_u A_1\\
&\quad + a_0^2 b_v A_1 + a_0^3 b_u B_1 + 3a_0^2 b_0 a_u B_1 + 3a_0 b_0^3 B_1 + 4a_0^3 b_0^2 D_1\\
&\quad + a_0^5 b_0 E_1 + 4a_0^5 b_0 D_2,\\
F_2' &= 6a_0^2 b_0 a_v A_2 + 5a_0^4 a_v B_2 + 2a_0^3 b_v A_2 + 2a_0^2 b_0^3 B_2 + 4a_0^4 b_0^2 D_2\\
&\quad + 2a_0^6 b_0 E_2 + 4a_0 b_0 b_v A_1 + 2a_v b_0^2 A_1 + 3a_0^2 b_0 a_v B_1 + 2b_0^4 B_1\\
&\quad + 4a_0^2 b_0^3 D_1 + 2a_0^4 b_0^2 E_1 + a_0^6 b_0 F_1 + a_0^8 F_2 + a_0^3 b_v B_1.
\end{aligned}
$$

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Manuscrit reçu le 21 décembre 2008,
accepté le 6 juillet 2009.

Jacques-Élie FURTER
Brunel University
Department of Mathematical Sciences
Uxbridge UB8 3PH (United Kingdom)
mastjef@brunel.ac.uk

Angela Maria SITTA
Universidade Estadual Paulista - UNESP
Departamento de Matemática - IBILCE
Campus de São José do Rio Preto - SP (Brazil)
angela@ibilce.unesp.br