ANNALES

DE

L'INSTITUT FOURIER

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Tome 60, nº 4 (2010), p. 1233-1257.

<http://aif.cedram.org/item?id=AIF_2010__60_4_1233_0>

Ann. Inst. Fourier, Grenoble
60, 4 (2010) 1233-1257

WEAK MIXING AND PRODUCT RECURRENCE

by Piotr OPROCHA

Abstract. — In this article we study the structure of the set of weakly product recurrent points. Among others, we provide necessary conditions (related to topological weak mixing) which imply that the set of weakly product recurrent points is residual. Additionally, some new results about the class of systems disjoint from every minimal system are obtained.

Résumé. — Dans cet article nous étudions la structure de l'ensemble des points faiblement produit-récurrents. Nous donnons entre autres des conditions suffisantes (en rapport avec le mélange topologique faible) qui impliquent que l'ensemble des points faiblement produit-récurrents est résiduel. De plus, nous obtenons certains résultats nouveaux concernant la classe des systèmes disjoints de tous les systèmes minimaux.

1. Introduction

The notion of recurrence is one of the most important properties in the study of dynamical systems. Generally speaking, a point is recurrent if its orbit returns arbitrarily close to the initial state. An interesting question about the dynamics of such a point is whether its returns can be synchronized with returns of another recurrent point. It leads to the notion of product recurrence.

By a dynamical system we mean a pair consisting of a continuous map $f\colon X \to X$ and a compact metric space (X,d) on which the map f acts on. Recall that a point x is said to be *recurrent* if $x \in \omega(x,f)$ and *product recurrent* if given any recurrent point y in any dynamical system g and any neighborhoods U of x and V of y, the return time sets $R(x,U,f)$ and $R(y,V,g)$ intersect nontrivially. This definition was originally stated for homeomorphism, however since it relies only on the future of the orbit,

Keywords: Product recurrence, weak mixing, minimal system, disjointness.
Math. classification: 37B20, 37B05.

it can be stated the same way for arbitrary continuous map. Which is more important, most of properties of recurrent points which are valid for homeomorphisms, transfer to the more general setting of not necessary invertible maps. In particular, a point x is product recurrent if and only if it is a distal point (namely, because IP-sets are subsets of $\mathbb{N}$ and so [8, Thm. 9.11] has the same proof in the context of continuous maps).

A more mild condition, than product recurrence is the so-called *weak product recurrence* introduced by Haddad and Ott in [10]. We say that a point $x \in X$ is *weakly product recurrent* if given any uniformly recurrent point y in any dynamical system g and any neighborhoods U of x and V of y, the sets $R(x, U, f)$ and $R(y, V, g)$ intersect nontrivially. In particular, any product recurrent point is weakly product recurrent, since we consider a smaller class of possible points y in that definition. Haddad and Ott prove in [10] the following fact:

Theorem 1.1. — *A point $x \in X$ is weakly product recurrent if it has the following property: for every neighborhood V of x there exists n such that if $S \subset \mathbb{N}$ is any finite set satisfying $|s - t| > n$ for all distinct $s, t \in S$, then there exists $l \in \mathbb{N}$ such that $l + s \in R(x, V, f)$ for every $s \in S$.*

It is rather hard to verify this condition, so in practice it is much more natural to use the following condition, also introduced in [10] (see [10, Cor. 3.2]).

Theorem 1.2. — *A point $x \in X$ is weakly product recurrent if the following conditions hold:*

(1) *The orbit of x is dense in X.*
(2) *For any neighborhood V of x there exists N such that for any $k \in \mathbb{N}$, if $n_i \geqslant N$ for $1 \leqslant i \leqslant k$, then the intersection*

$$V \cap f^{-n_1}(V) \cap \ldots \cap f^{-(n_1+\ldots+n_k)}(V) \neq \emptyset.$$

That way, the authors of [10] answer in negative a long standing problem, stated in [1, p. 232] whether every weakly product recurrent point must be a distal point(1). Namely, it is easy to verify that every topologically exact map or map which verifies specification property fulfil assumptions of Theorem 1.2 (later we show in Theorem 3.2 that if assumptions of Theorem 1.2 are fulfilled then f must be at least mixing). But consider a map which admits a periodic decomposition, say $X = A_0 \cup A_1$, $f(A_i) = A_{i+1 (\mathrm{mod}\ 2)}$, and $A_0 \cap A_1 = \emptyset$. Certainly such a map doesn't fulfill sufficient conditions

(1) The authors write in [1]: "Another question (even for $\mathbb{Z}$ or $\mathbb{N}$ actions): If (x, y) is recurrent for all almost periodic points y, is x necessarily a distal point?"

from [10] (it is easy to verify that if the space admits a periodic decomposition into disjoint pieces then assumptions of Theorems 1.1 or 1.2 can't be fulfilled). But there can be a high degree of mixing in the system, e.g. $f^2|_{A_i}$ can be mixing. Furthermore, a single periodic orbit serves as an example of a distal systems, thus it is product recurrent (and so it is also weakly product recurrent). This suggests that there should exist sufficient conditions inducing weak product recurrence for maps admitting regular periodic decompositions (e.g. relatively mixing maps [2] or more generally, maps with weakly mixing subsets [4]).

The main question we try to answer is the following:

Question 1. — When a continuous map admits a weakly product recurrent point which is not product recurrent (and how large the set of such points can be)?

It is noteworthy that the solution to the above mentioned question by Auslander and Furstenberg stated in [1, p. 232] can be deduced from paper by Huang and Ye [11] published three years before [10], or even from Furstenberg paper from 1967 [7] (we will comment on it later in Section 4), however due to different terminology and different motivation it is not immediate to realize that the answer is really contained there. While Theorems 1.1 and 1.2 are essentially included in results of [11] (but not in [7]), techniques used in the proofs are completely different (the main technique in [10] is the van der Waerden theorem, while [11] relies on properties of disjoint systems and transitivity). It is also interesting, that results in [11] (and so also the present paper) are related to another question stated by Furstenberg many years before [1], about the structure of the set of systems disjoint from every minimal or distal system (see [7]).

In this paper we will provide sufficient conditions for weak product recurrence, much more general that these provided by Theorem 1.2. In particular, our condition works well for totally transitive maps with dense periodic points, while, as we show, assumptions of Theorem 1.2 imply that the map must be at least mixing (see Theorem 3.2). Our results do not give the full answer on Question 1 (which is a variation of [10, Question 5.1]), however some insight into the structure of maps with weakly product recurrent points is obtained and in a large class of transitive systems these points are successfully localized (e.g. in any totally transitive system with dense periodic points every point with dense orbit is weakly product recurrent but not product recurrent). As a consequence of our work, some small step towards the full characterization of the class of systems disjoint from any minimal system is also made (see Theorem 4.4).

2. Preliminaries

2.1. Sets of integers and recurrence

Let $\mathbb{N} = \{0, 1, \ldots\}$ denote the set of natural numbers. The cardinality of a set A is denoted $\#A$. A set $A \subset \mathbb{N}$ is *syndetic* if there exists a positive integer k such that

$$\{i, i+1, \ldots, i+k\} \cap A \neq \emptyset.$$

for every $i \in \mathbb{N}$. We say that A is *thick* if for every $n \in \mathbb{N}$ there is i such that

$$\{i, i+1, \ldots, i+n\} \subset A.$$

A set $A \subseteq \mathbb{N}$ is called an *IP-set* if there exists a sequence $\{p_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$ such that A consists exactly of numbers p_i together with all finite sums ($k \in \mathbb{N}$):

$$p_{n_1} + p_{n_2} + \ldots + p_{n_k} \text{ with } n_1 < n_2 < \ldots < n_k.$$

Let (X, d) be a compact metric space and let $f \colon X \to X$ be a continuous map. For every $x \in X$ we denote by $\mathrm{Orb}^+(x, f)$ the set $\{f^n(x) \,:\, n \in \mathbb{N}\}$ and call it the *(positive) orbit* of x. The *ω-limit set* (or *positive limit set*) of a point x is the set $\omega(x, f)$ of all limit points of positive orbit of x regarded as a sequence. By the *set of return times* of a point x to its open neighborhood U we mean the set

$$R(x, U, f) = \left\{i \in \mathbb{N} \,:\, f^i(x) \in U\right\}.$$

We also define the *set of hitting times* (V is an arbitrary open set):

$$N(x, V, f) = \left\{i \in \mathbb{N} \,:\, f^i(x) \in V\right\}.$$

When the map can be easily deduced from the context, we will simply write $R(x, U)$ and $N(x, U)$.

If U is an open neighborhood of x then $N(x, U, f) = R(x, U, f)$, however we decided to use both symbols because sometimes it is very convenient to stress the fact that we deal with recurrence, by putting $R(\cdot)$ instead of $N(\cdot)$. We will also write $N(U, V, f) = \left\{i \in \mathbb{N} \,:\, f^{-i}(V) \cap U \neq \emptyset\right\}$.

A subset M of X is *minimal* if it is closed, nonempty, invariant (that is $f(M) \subset M$) and contains no proper subset with these three properties. It is well known that if a set $M \subset X$ is minimal then the orbit of every point of M is dense in M. A point x is called *uniformly recurrent* (or *minimal*) if it belongs to a minimal set. It is also well known that if x is uniformly recurrent, then $R(x, U)$ is syndetic for any open neighborhood U of x.

Definition 2.1. — *Two points x and y in X are said to be* proximal *if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that* $\lim_{k\to\infty} d(f^{n_k}(x), f^{n_k}(y)) = 0$.

Definition 2.2. — *A point $x \in X$ is said to be* distal *if x is not proximal to any point in its orbit closure $\overline{\mathrm{Orb}^+(x,f)}$ other than itself.*

The following fact is well known (see [8, Thm. 9.11]). We state it for the readers convenience (note that if x is distal then it must be uniformly recurrent):

Theorem 2.3. — *A point x is product recurrent if and only if it is uniformly recurrent distal point.*

2.2. Symbolic dynamics and odometers

Let $\Sigma = \{0,1\}$ and denote $\Sigma_2^+ = \Sigma^{\mathbb{N}}$. By a *word*, we mean any element of the free monoid Σ^* with the set of generators equal Σ. If $x \in \Sigma_2^+$ and $0 \leqslant i < j$ then by $x_{[i,j]}$ we mean the sequence $x_i, x_{i+1}, \dots, x_j$. We may naturally identify $x_{[i,j]}$ with the word $x_{[i,j]} = x_i x_{i+1} \dots x_j \in \Sigma^*$. For simplicity, we use the following notation $x_{[i,j)} = x_{[i,j-1]}$.

We endow Σ_2^+ with a compact metric ρ defined by

$$\rho(x,y) = 2^{-k}, \text{ where } k = \min\left\{m \geqslant 0 : x_{[0,m]} \neq y_{[0,m]}\right\}$$

if $x \neq y$ and $\rho(x,y) = 0$ otherwise.

If $a_1 \dots a_m \in \Sigma^*$ then we define the so-called *cylinder set*:

$$[a_1 \dots a_m] \quad = \quad \left\{x \in \Sigma_2^+ \ : \ x_{[0,m)} = a_1 \dots a_m\right\}.$$

It is well known that cylinder sets form a neighborhood basis for the space Σ_2^+.

By 0^∞ we denote the sequence $x \in \Sigma_2^+$ such that $x_i = 0$ for all $i \in \mathbb{N}$. The usual map on Σ_n^+ is the shift map σ defined by $\sigma(x)_i = x_{i+1}$ for all i. Dynamical system (Σ_2^+, σ) is called the full (one-sided) shift over 2 symbols.

If $X \subset \Sigma_2^+$ is closed and invariant (i.e. $\sigma(X) \subset X$) then we say that X (together with the map $\sigma = \sigma|_X$) is a *shift* or *subshift*. For simplicity we write $C_X[u] = [u] \cap X$ where $u \in \Sigma^*$.

Let $\mathbf{s} = \{s_m\}_{m=1}^{\infty}$ be a sequence of positive integers such that s_m divides s_{m+1}. We call such a sequence *a scale*. If we endow the cyclic group $\mathbb{Z}_n$ with the discrete topology, and define $\pi_m \colon \mathbb{Z}_{s_{m+1}} \to \mathbb{Z}_{s_m}$ by $\pi_m(z) = z \pmod{s_m}$ then the inverse limit

$$G_{\mathbf{s}} = \varprojlim \{\mathbb{Z}_{s_m}, \pi_m\} = \{\{x_n\}_{n=1}^{\infty} \ : \ \pi_m(x_{m+1}) = x_m\}$$

is well defined, compact subset of the countable Cartesian product of $\mathbb{Z}_{s_m}$ with the product topology. Denote $\mathbf{0} = (0, 0, \ldots)$ and $\mathbf{1} = (1, 1, \ldots)$. By the *odometer on scale* $\mathbf{s}$ we mean $G_{\mathbf{s}}$ together with the map $R_1 \colon G_{\mathbf{s}} \to G_{\mathbf{s}}$ defined by $R_1(\mathbf{j}) = \mathbf{j} + \mathbf{1}$, where the addition is coordinatewise, modulo s_m on each coordinate m.

3. Non-minimal weakly mixing maps

We recall that f is *transitive* if for any pair of nonempty open sets $U, V \subset X$ there exists $n > 0$ such that $f^n(U) \cap V \neq \emptyset$; is *totally transitive* if f^k is transitive for $k = 1, 2, \ldots$; is *(topologically) weakly mixing* if $f \times f$ is transitive on $X \times X$; is *(topologically) mixing* if for any pair of nonempty open sets $U, V \subset X$ there exists $N > 0$ such that $f^n(U) \cap V \neq \emptyset$ for all $n > N$; is *(topologically) exact* if for every nonempty open set U there is $n > 0$ such that $f^n(U) = X$. Since we do not consider measure theoretic analogues of the above properties, the word "topologically" will be usually omitted. It is easy to verify that mixing implies weak mixing which implies transitivity and that exactness is stronger than (i.e. implies) mixing.

It is also well known that if f is transitive then the set of points with dense orbits is residual in X. Another fact is that if there is x such that $\omega(x, f) = X$ then f is transitive. In particular, this condition holds when x is a recurrent point with dense orbit. In other words, if assumptions of Theorem 1.2 are fulfilled then the map is transitive (using the assumptions one can easily show, that x is recurrent). As the first result (Theorem 3.2), we show that the dynamics must be much more complicated in the case of maps fulfilling the assumptions of Theorem 1.2.

Before we proceed further, we need one more definition. In [13] we introduced the following property, when studying conditions sufficient for spectral decomposition (a generalization of Smale's spectral decomposition theorem [14]).

Definition 3.1. — *Let f be a continuous map acting on a compact metric space (X, d). We say that f has the property* **(P)** *if for any nonempty open set U there are a point $x \in U$ and an integer $K > 0$ such that*

$$f^{nK}(x) \in U \quad \text{for all } n \in \mathbb{N}.$$

The property **(P)** is closely related to dense periodicity. In fact, the Reader can check that the condition **(P)** holds if and only if the map $\bar{f}$ induced (by the formula $\bar{f}(A) = f(A)$) on the hyperspace of nonempty compact subsets of X has dense periodic points (this result was recently announced by Héctor Méndez Lango at VII Iberoamerican Conference on Topology and its Applications (Valencia, June 2008) [12]).

THEOREM 3.2. — *Assume that there is a point $x \in X$ such that the following conditions hold:*

(1) *The orbit of x is dense in X.*
(2) *For any neighborhood V of x there exists N such that for any $k \in \mathbb{N}$, if $n_i \geqslant N$ for $1 \leqslant i \leqslant k$, then the intersection*

$$V \cap f^{-n_1}(V) \cap \ldots \cap f^{-(n_1+\ldots+n_k)}(V) \neq \emptyset.$$

*In that case f has the property **(P)** and is mixing.*

Proof. — Fix any open set U. There is $s > 0$ such that $f^s(x) \in U$. There is an open neighborhood V of x such that $f^s(\overline{V}) \subset U$ and there is $K > 0$ such that

$$V \cap f^{-K}(V) \cap f^{-2K}(V) \cap \ldots \cap f^{-nK}(V) \neq \emptyset \tag{3.1}$$

for every $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, denote by x_n a point in this intersection. We may assume that $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence. Denote $x^* = \lim_{n\to\infty} x_n$ and observe that $f^{nK}(x^*) \in \overline{V}$ for every $n \in \mathbb{N}$. But if we denote $y = f^s(x^*)$ then

$$f^{nK}(y) = f^s(f^{nK}(x^*)) \in U.$$

We have just proved that f has the property **(P)**. It remains to prove that f is mixing.

Fix any two nonempty open sets U, W. There are $U' \subset U$ and $s > 0$ such that $f^s(U') \subset W$. There is also an open neighborhood V of x and $m > 0$ such that $f^m(V) \subset U'$ (because x has dense orbit). Let N be a constant given by the assumptions for V. Put $K = N + s > 0$ and observe that $n - s > N$ for every $n > K$. In particular, for every $n > K$ there is $y = y(n) \in X$ such that $y \in V \cap f^{-n+s}(V)$. Denote $z = f^m(y)$ and observe that $z \in U' \subset U$ and

$$f^n(z) = f^{s+m}(f^{n-s}(y)) \in f^{s+m}(V) \subset f^s(U') \subset W.$$

Indeed, f is topologically mixing and the proof is finished. □

In [11] Huang and Ye introduced the following definition.

DEFINITION 3.3. — *Let f be a continuous map acting on a compact metric space (X, d). We say that f has* dense small periodic sets *if and only if for any nonempty open set U there exists nonempty compact set $A \subset U$ and k such that A is invariant for f^k.*

It follows just by the definition that every map with dense small periodic sets also has the property **(P)** and by [11, Prop. 3.2] we get the converse implication, thus these notions are equivalent.

We remark here that the property **(P)** (or equivalently density of small periodic sets) is much more general than dense periodicity. There is known an example of a weakly mixing systems with the property **(P)** but without periodic points [11, Example 3.7], however the construction is quite complex. Such systems can be constructed in a much simpler way even with a higher degree of mixing. The idea of this construction comes from [9].

Fix any $n > 0$ and consider words $u = 0^n 1, v = 0^{n+1} 1$ over the alphabet $\{0, 1\}$. Let K be the set of all infinite sequences obtained as all (infinite) concatenations of words u, v in any possible order. Denote

$$\Lambda_n = \overline{\bigcup_{i=0}^{\infty} \sigma^i(K)} = \bigcup_{i=0}^{n+1} \sigma^i(K).$$

It is easy to verify that $\sigma|_{\Lambda_n}$ has dense periodic points, however period of any periodic point in Λ_n is greater than n. Additionally $\sigma|_{\Lambda_n}$ is mixing (or even exact), since there is no restriction on concatenations in K and $|v| = |u| + 1$.

Example 3.4. — Consider the infinite Cartesian product $\Lambda = \prod_{n=1}^{\infty} \Lambda_n$ endowed with the product metric $d(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} 2^{-n} \rho(\mathbf{x}_n, \mathbf{y}_n)$, and the map $F \colon \Lambda \to \Lambda$ defined by $F(\mathbf{x})_n = \sigma(\mathbf{x}_n)$. It is easy to verify that Λ is compact and F is continuous. It is also easy to see that F is mixing and has the property **(P)** (or even dense regularly recurrent points), e.g. see [9, Lemma 3]. Finally, it obvious that F has no periodic point, as n-th coordinate cannot have period n.

In [4] the authors introduced a local analogue of the definition of weak mixing.

Definition 3.5. — *A set $A \subset X$, $\#A > 1$ is called a* weakly mixing set *if for any $m \in \mathbb{N}$, any choice of nonempty open subsets $V_1, \ldots, V_m, U_1, \ldots, U_m$ of X with $A \cap U_i \neq \emptyset$, $A \cap V_i \neq \emptyset$, $i = 1, 2, \ldots, m$, there exists $k \in \mathbb{N}$ such that $f^k(V_i \cap A) \cap U_i \neq \emptyset$ for each $1 \leqslant i \leqslant m$.*

In fact Definition 3.5 is an equivalent condition to the original definition of weakly mixing set stated in [4] (see [4, Prop. 4.2.]). But the advantage of the above formulation is that it is immediately seen, that if f is weakly mixing and $\#X > 1$, then X is a weakly mixing set (by Furstenberg theorem [7], if $f \times f$ is transitive, then so does any finite product $f \times \ldots \times f$).

We also stress the fact that weakly mixing sets are always perfect, i.e. they cannot contain isolated points (see comments at the top of page 287 in [4]).

In the same fashion, we can also state a local version of the property **(P)**.

Definition 3.6. — *We say that a set $A \subset X$ has the property* ***(P)*** *if for any nonempty open set $U \subset X$ with $U \cap A \neq \emptyset$ there exists a point $x \in U \cap A$ and $K > 0$ such that*

$$f^{nK}(x) \in U \quad \text{for all } n \in \mathbb{N}.$$

The following lemma is only a slight extension of [2, Thm 1.1] (namely, density of periodic points is replaced by the property **(P)**).

Proposition 3.7. — *If f has the property* ***(P)*** *and is totally transitive then it is weakly mixing.*

Proof. — Fix two nonempty open sets U, V. There is K such that $f^{nK}(U) \cap U \neq \emptyset$ for every $n > 0$. The map f^K is transitive, so there is m such that $f^{mK}(U) \cap V \neq \emptyset$. If we put $s = mK$ then $f^s(U) \cap U \neq \emptyset$ and $f^s(U) \cap V \neq \emptyset$. The result follows by [3, Thm. 4]. □

If we restrict our attention to transitive maps with the property **(P)** then by Proposition 3.7 we see that in most cases such maps have a weakly mixing set with the property **(P)** (it is enough to take any piece in the terminal periodic decomposition if such a decomposition exists [2]). Maps without weakly mixing subset behave similarly to odometers (see [2]) however they still can reveal complicated, nonminimal dynamics (e.g. see [4, Ex. 5.4]).

Theorem 3.8. — *Let A be a weakly mixing set and let U, V be open sets intersecting A. For any $K > 0$ there is $x \in U \cap A$ such that $f^{nK}(x) \in V$ for some integer $n > 0$.*

Proof. — Fix arbitrary open sets U, V such that $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$. Fix an integer $K > 0$. There are a point $x_1 \in U \cap A$ and $s_1 > K$ such that $f^{s_1}(x_1) \in V$. If $s_1 = 0 \pmod{K}$ then we are done ($n = s_1$), so suppose that this does not hold. We will perform an inductive construction.

Assume that there are $x_1, \dots, x_l \in U$ and positive integers $s_1, \dots, s_l > K$ such that $f^{s_j}(x_j) \in V$ and reminders $r_j = s_j \pmod{K}$ form a strictly increasing sequence $0 < r_1 < \dots < r_l$. The map f is continuous and A is a weakly mixing set, so there are $W_1, \dots W_l \subset U$ and $m > 0$ such that $f^m(W_j \cap A) \cap W_j \neq \emptyset$, $f^m(W_j \cap A) \cap V \neq \emptyset$ and $f^{s_j}(W_j) \subset V$ for $j = 1, \dots, l$. If $m \neq 0 \pmod{K}$ then reminders $r'_j = m + s_j \pmod{K}$ are all different, and are also different from m (mod K). The set A has no isolated points (see [4]), so there is $x'_{l+1} \in W_1 \cap (A \setminus \{x_1, \ldots, x_l\})$ such that $f^m(x'_{l+1}) \in V$. By the same arguments we get points $x'_j \in f^{-m}(W_j) \cap W_j \cap A$ such that x'_j are distinct for distinct indices j. Put $s'_j = m + s_j$, $s_{l+1} = m$, and then renumerate the sequence s'_j with respect to increasing values of reminders $r'_j = s'_j \,(\mathrm{mod}\ K)$ (note that in our case numbers r'_j form an increasing sequence of positive integers).

The process described above provides a method to extend cardinality of the sequence x_j, as long as $m \neq 0 \,(\mathrm{mod}\ K)$. Since there are at most K different reminders, we must get $m = 0 \,(\mathrm{mod}\ K)$ at some step. In that case it is enough to take $x \in f^{-m}(V) \cap A \cap W_1$ and put $n = m/K$. □

THEOREM 3.9. — *Let A be a weakly mixing set with the property **(P)**. In that case the set A contains a point which is not uniformly recurrent, in particular f is not minimal.*

Proof. — Assume that A is a weakly mixing set. There are nonempty open sets U, V such that $\overline{U} \cap \overline{V} = \emptyset$, $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$ (by the definition $\#A > 1$).

There are K and $x \in A$ such that $f^{nK}(x) \in U$ for all $n \in \mathbb{N}$. There is also a point $y \in A$ such that $\overline{\mathrm{Orb}^+(y, f)} \supset A$. Assume that y is uniformly recurrent. This implies that x is uniformly recurrent, since $x \in \overline{\mathrm{Orb}^+(y, f)}$. In that case $A \subset \overline{\mathrm{Orb}^+(x, f)} = \overline{\mathrm{Orb}^+(y, f)}$, in particular there is $s_1 > 0$ such that $f^{s_1}(x) \in V$. We obtain that there are $0 < r_1 < K$ and $i_1 \geqslant 0$ such that $s_1 = i_1 K + r_1$ (if $s_1 = 0 \,(\mathrm{mod}\ K)$ then $f^{s_1}(x) \in U \cap V$ which contradicts the assumptions).

Assume that there are constructed $s_j = i_j K + r_j$ for $j = 1, \ldots, n$ such that $0 < r_1 < r_2 < \ldots < r_n < K$ and $f^{s_j}(x) \in V$ for all j. There is an open neighborhood W of x such that $f^{s_j}(W) \subset V$. By Theorem 3.8 there is m such that $f^m(W \cap A) \cap V \neq \emptyset$ and $m = 0 \,(\mathrm{mod}\ K)$. Thus there is an open set $W' \subset W$, $W' \cap A \neq \emptyset$ and numbers $a_1, \ldots, a_{n+1}$, all a_i different modulo K, such that $f^{a_i}(W') \subset V$ for $i = 1, \ldots, n+1$ (namely $a_j = s_j$ for $j \leqslant n$ and $a_{n+1} = m$). But there is also $l > 0$ such that $f^l(x) \in W'$. Numbers $s'_j = a_j + l$, $j = 1, \ldots, n+1$ are also all different modulo K. Put $r'_j = s'_j \,(\mathrm{mod}\ K)$ and sort these numbers, such that $r'_1 < r'_2 < \ldots < r'_{n+1}$. Note that $r'_1 > 0$ because otherwise $s'_1 = 0 \,(\mathrm{mod}\ K)$ and so $f^{s'_1}(x) \in U \cap V$ which contradicts assumptions about U and V.

By the above inductive procedure we eventually construct a sequence $s_1, \ldots, s_K$. But then $0 < r_1 < \ldots < r_K < K$ which is impossible. We have just proved that y can't be uniformly recurrent and the proof is finished. □

It is not hard to prove, that if U, V are nonempty open sets and f is weakly mixing then the set $\{n : f^n(U) \cap V \neq \emptyset\}$ contains an IP-set. The following theorem shows that a stronger property holds. Namely, an IP-set can be induced by a point $x \in U$.

THEOREM 3.10. — *Let A be a weakly mixing set and let U, V be open sets such that $A \cap U \neq \emptyset$, $A \cap V \neq \emptyset$. In that case there is $x \in U \cap A$ such that the set $N(x, V, f)$ contains an IP-set.*

Proof. — Fix any nonempty open sets U, V intersecting A. We claim that there is a sequence of integers $\{p_i\}_{i=1}^{\infty} \subset \mathbb{N}$ and sequences of points $\{x_i\}_{i=1}^{\infty} \subset U \cap A, \{y_i\}_{i=1}^{\infty} \subset V \cap A$ such that

$$f^s(x_n) \in V \quad \text{and} \quad f^s(y_n) \in V$$

for every $s \in P(n) = \{p_{j_1} + \ldots + p_{j_k} : 1 \leqslant j_1 < \ldots < j_k \leqslant n\}$. To prove the claim (and construct appropriate sets), we use the mathematical induction.

To obtain p_1, x_1 and y_1 we apply the definition of weakly mixing set. Now assume that the desired sequences are constructed for $i = 1, \ldots, n$. There are open neighborhoods $U' \subset U$ and $V' \subset V$ of x_n and y_n respectively such that

$$f^s(U') \subset V \text{ and } f^s(V') \subset V$$

for every $s \in P(n)$. Obviously $U' \cap A \neq \emptyset$, $V' \cap A \neq \emptyset$ and so we can apply the definition of weakly mixing set, obtaining an integer k such that $f^k(U' \cap A) \cap V' \neq \emptyset$ and $f^k(V' \cap A) \cap V' \neq \emptyset$. Put $p_{n+1} = k$ and take any $x_{n+1} \in U' \cap A \cap f^{-k}(V')$ and $y_{n+1} \in V' \cap A \cap f^{-k}(V')$. We obtain that

$$f^s(x_{n+1}) \in V \quad \text{and} \quad f^s(y_{n+1}) \in V$$

for every $s \in P(n+1)$, because $P(n+1) = \{k\} \cup P(n) \cup (k + P(n))$.

Let $x = \lim_{n\to\infty} x_n$ (going to a subsequence, we may assume that this limit exists). Then $x \in \overline{U} \cap A$ and the set $N(x, \overline{V}, f)$ contains an IP-set generated by the sequence $\{p_i\}_{i=1}^{\infty}$. The proof is finished, by the fact that for every open set $U \cap A \neq \emptyset$ there is an open set $W \cap A \neq \emptyset$ such that $\overline{W} \subset U$. □

THEOREM 3.11. — *If A is a weakly mixing set with the property **(P)** then weakly product recurrent points which are not product recurrent form a residual subset of A.*

Proof. — Let $\{U_s\}_{s=1}^{\infty}$ generate a countable basis of the topology of A (i.e. U_s are nonempty open sets such that $\{A \cap U_s\}_{s=1}^{\infty}$ is a basis of the topology induced from X to A). By the property **(P)**, for every s there are $K_s > 0$ and $x_s \in U_s \cap A$ such that $f^{iK_s}(x_s) \in U_s$ for every $i \in \mathbb{N}$.

For every $s, n \geqslant 1$ we define a set $A_n^s \subset X$ in the following way: $x \in A_n^s$ if there is $j_s \geqslant 0$ such that

$$f^{(j_s+i)K_s}(x) \in U_s \quad \text{ for } i = 0, 1, \dots, n.$$

We claim that each set A_n^s is dense. Since f is continuous, we can find an open set $V \ni x_s$ such that $f^{iK_s}(V) \subset U_s$ for $i = 0, 1, \dots, n$. Now fix an arbitrary open set U such that $U \cap A \neq \emptyset$. By Theorem 3.8 there is j_s such that $f^{j_s K_s}(U \cap A) \cap V \neq \emptyset$. If we fix any $x \in f^{-j_s K_s}(V) \cap U \cap A$ then $f^{(i_s+i)K_s}(x) \in f^{iK_s}(V) \subset U_s$. Consequently, $x \in A_n^s$, and since U was arbitrary, the proof of the claim is finished.

But obviously A_n^s is also open, since f is continuous. Denote

$$D = \left(\bigcap_{n,s \geqslant 1} A_n^s \right) \cap \operatorname{Trans}(f, A)$$

where $\operatorname{Trans}(f, A) \subset A$ consists of points with orbits dense in A, that is $A \subset \overline{\operatorname{Orb}^+(x, f)}$ for every $x \in \operatorname{Trans}(f, A)$. Note that D is residual in A (the proof that $\operatorname{Trans}(f, A)$ is residual in A is the same as the proof that the set of points with dense orbits is residual for every transitive map), and if $x \in D$ then it is not product recurrent ($A \subset \overline{\operatorname{Orb}^+(x, f)}$ and so by Theorem 3.9 the point x is not uniformly recurrent, while product recurrent points are always uniformly recurrent by Theorem 2.3).

It remains to prove that each point $x \in D$ is weakly product recurrent. To see this, take arbitrary minimal system $g\colon Y \to Y$, a (uniformly recurrent) point $y \in Y$ and its open neighborhood $V \ni y$. Fix any open neighborhood U of x. There is l such that $x \in U_l \subset U$. Let p be a periodic point with the prime period K_l (in the trivial odometer T). Obviously p is uniformly recurrent distal point, and so by [8, Thm. 9.11] the pair (p, y) is uniformly recurrent point for the product map $T \times g$. In particular $R(y, V, g^{K_l})$ is syndetic. But $x \in \bigcap_{n \geqslant 1} A_n^l$ which implies that the set $N(x, U_l, f^{K_l}) \subset R(x, U, f^{K_l})$ is thick. This immediately implies that

$$R(x, U, f) \cap R(y, V, g) \neq \emptyset.$$

$\square$

If the assumptions of Theorem 1.2 are fulfilled then every point with dense orbit is weakly product recurrent. Theorems 3.11 says only that there exists a residual subset of weakly product recurrent points (in particular, we can't exclude the situation that there is a point with dense orbit which is not weakly product recurrent). Theorem 3.13 is another generalization of Theorem 1.2. First observe that we can extend the statement of Theorem 3.8 when weak mixing is global.

THEOREM 3.12. — *Assume that f is weakly mixing, U is a nonempty open set and $K > 0$. There is a nonempty open set $V \subset U$ and positive integers $j_0, \dots, j_{K-1}$ such that $j_i \,(\mathrm{mod}\, K) = i$ and $f^{j_i}(V) \subset U$.*

Proof. — Denote $V_0 = U$. The map f is weakly mixing, so there is $k > 0$ such that

$$f^{-k-i}(U) \cap U \neq \emptyset$$

for $i = 0, \dots, K$. In particular there is $j_0 > 0$ such that $j_0 \,(\mathrm{mod}\, K) = 0$ and

$$V_1 = f^{-j_0}(V_0) \cap V_0 \neq \emptyset.$$

By the same argument, for any nonempty open set $V_i \subset U$, where $i < K$ there is $j_i > 0$ such that $j_i \,(\mathrm{mod}\, K) = i$ and

$$V_{i+1} = f^{-j_i}(V_i) \cap V_i \neq \emptyset.$$

This provides an inductive procedure for the construction of a sequence of sets

$$V = V_K \subset V_{K-1} \subset \ldots \subset V_1 \subset V_0 = U$$

and times $j_0, \dots, j_{K-1}$ such that $j_i \,(\mathrm{mod}\, K) = i$ and $f^{j_i}(V_{i+1}) \subset V_i$. But then $f^{j_i}(V) \subset f^{j_i}(V_{i+1}) \subset V_i \subset U$. □

THEOREM 3.13. — *If f is weakly mixing ang has the property **(P)** then every point with dense orbit is weakly product recurrent.*

Proof. — There are a basis of the topology $\{U_s\}_{s=1}^{\infty}$ of X, positive integers K_s and points $x_s \in U_s$ such that $f^{iK_s}(x_s) \in U_s$ for every $i \in \mathbb{N}$.

By the proof of Theorem 3.11 it remains to show that if a point $x \in X$ has dense orbit then for every $s, n \geqslant 1$ there is $j \geqslant 0$ such that

$$f^{(j+i)K_s}(x) \in U_s \quad \text{for } i = 0, 1, \dots, n.$$

First note that there is an open set $V_n \ni x_s$ such that $f^{(j+i)K_s}(V_n) \subset U_s$ for $i = 0, 1, \dots, n$ and some $j > 0$. By Theorem 3.12 there is $V \subset V_n$ and integers k_i such that $f^{k_i}(V) \subset V_n$ and $k_i \,(\mathrm{mod}\, K_s) = i$, where $i = 0, \dots, K_s - 1$. For technical reasons denote $k_{K_s} = k_0$. There are integers $0 \leqslant r < K_s$ and $l > 0$ such that $f^l(x) \in V$ and $l \,(\mathrm{mod}\, K_s) = r$ (the orbit of x is dense in X). But then, if we put $t = l + k_{K_s - r}$ then $t = 0 \,(\mathrm{mod}\, K_s)$ and $f^t(x) \in V \subset V_n$. We may write $t = t'K_s$ for some positive integer t'. Then by the definition of V_n we obtain that

$$f^{(j+i+t')K_s}(x) \in U_s \quad \text{for } i = 0, 1, \dots, n$$

and the proof is completed. □

4. Systems disjoint from every minimal system

It is interesting that questions about weak product recurrence are closely related to questions about disjointness of systems (the notion of (topological) disjointness was introduced by Furstenberg in [7] and next widely studied by many other authors). Before going further we have to recall some basic definitions.

Let f, g be two continuous surjective maps acting on compact metric spaces X and Y respectively. We say that a nonempty closed set $J \subset X \times Y$ is a *joining* of (X, f) and (Y, g) if it is invariant (for the product map $f \times g$) and its projections on the first and second coordinate are X and Y respectively. If each joining is equal to $X \times Y$ then we say that (X, f) and (Y, g) are *disjoint* and denote this fact by $(X, f) \perp (Y, g)$ or simply by $f \perp g$.

Let $\mathscr{M}$ be the collection of all possible minimal systems. We denote by $\mathscr{M}^{\perp}$ the collection of all systems disjoint from any minimal system.

Proposition 4.1. — *Let f be a continuous map and let x be a recurrent point. Denote $\Lambda = \overline{\mathrm{Orb}^{+}(x, f)}$. If $(\Lambda, f|_{\Lambda}) \in \mathscr{M}^{\perp}$ then x is weakly product recurrent.*

Proof. — If x is periodic then we are done, so assume that it is not the case. Then, by the definition of recurrent point, x is not isolated in Λ. Let (Y, g) be a minimal system and fix any $y \in Y$. Then, if we take the closure J of the orbit of the point (x, y) under the product map $f \times g$ then projection of J onto the first and second coordinate are Λ and Y respectively. But then J is a joining and so $J = \Lambda \times Y$ since $f \perp g$. This implies that the orbit of (x, y) is dense in $\Lambda \times Y$, and since x is not isolated in Λ we see that (x, y) is recurrent. □

In [11, Theorem 3.4] the authors proved the following fact:

Theorem 4.2. — *If a continuous map f acting on a compact metric space (X, d) is totally transitive and has dense small periodic sets then $(X, f) \in \mathscr{M}^{\perp}$.*

Note that by Proposition 3.7, assumptions of Theorem 4.2 actually imply that f is weakly mixing. Additionally, by Proposition 4.1 we see that all points with dense orbit are weakly product recurrent, while they are not product recurrent by Theorem 3.9 provided that $\#X > 1$.

There are two essential consequences of the above observations. First of all, by Theorem 3.2, they show that the main result of [10] is essentially contained in [11]. Secondly, theorem proved by Furstenberg in 1967 in [7] saying that every totally transitive (in fact, weakly mixing) system with dense periodic points is in $\mathscr{M}^\perp$ answers the question stated later in [1, p. 232]. Namely, it is enough to take any mixing map on the unit interval. We also see that Theorem 4.2 provides another proof of Theorem 3.13, however from a completely different point of view.

Before we go further, we should justify that results of Theorem 3.11 cover an essentially wider class of systems that these covered by Theorem 4.2. The simplest example is to take a non-mixing but relatively mixing map on an infinite space. Such a map is not disjoint from any system defined by periodic point whose period is a multiple of the length of the terminal regular periodic decomposition defining relatively mixing system. Another example, which essentially involves the idea of weakly mixing set and property **(P)** is the following.

Example 4.3. — Let f be a mixing interval map and let g be a nontrivial odometer (i.e. other than periodic orbit). Let $F = f\times g\colon I\times G_{\mathbf{s}} \to I\times G_{\mathbf{s}}$ be the product map and denote $A = I \times \{\mathbf{0}\}$. Note that A is a weakly mixing set, since f is mixing and the set $\left\{i \,:\, \operatorname{dist}(F^i(A), A) < \delta\right\}$ is syndetic for any $\delta > 0$ (or even more, it contains an infinite progression). It is also easy to verify, that for any periodic point $p \in I$ and open neighborhood $U \ni (p, \mathbf{0})$ there is k (which is a multiple of the prime period of p) such that $k\mathbb{N} \subset N((p, \mathbf{0}), U, F)$. In particular, A has the property **(P)**.

The property **(P)** is somehow related to a special class of distal points: periodic points or more generally odometers. If we extend the rate of mixing in the system, then we are no more restricted to this particular class of distal points in Theorem 3.11. We can prove even more, that is a result analogous to Theorem 4.2.

Theorem 4.4. — *If f is weakly mixing and distal points are dense in X then $(X, f) \in \mathscr{M}^\perp$.*

Proof. — If $\#X = 1$ then we are done, so assume that $\#X > 1$. In particular, it implies that X is perfect [4].

The proof is similar to the proof of Theorem 3.11. Fix a countable basis $\{U_s\}_{s=1}^\infty$ of the topology of X and choose a distal point $x_s \in U_s$ for every $s \in \mathbb{N}$. For every $s, n \geqslant 1$ we define a set $A_n^s \subset X$ by

$$A_n^s = \left\{ y \,:\, d(f^{i+j}(y), f^{i+j}(x_s)) < \frac{1}{n} \text{ for some } i \in \mathbb{N} \text{ and } j = 0, \dots, n \right\}.$$

It is easy to verify that every set A_n^s is open. We will show that they are also dense.

Fix $n, s > 0$ an let U be an open neighborhood of x_s such that $d(f^j(x), f^j(y)) \leqslant \frac{1}{n}$ for every $x, y \in U$ and $j = 0, 1, \ldots, n$. Fix any open set V. If f is weakly mixing, then a finite Cartesian product of copies of f is also transitive [7]. In particular there is $\hat{\imath} > 0$ such that $f^{-\hat{\imath}-l}(U) \cap V \neq \emptyset$ for $l = 0, \ldots, m$, where m is as large as we want (and $\hat{\imath} = \hat{\imath}(m)$, i.e. $\hat{\imath}$ depends on U, V and m). But x_s is uniformly recurrent, so for some $m > 0$ the set $R(x_s, U) \cap \{k, k+1, \ldots, k+m\} \neq \emptyset$ for every $k \in \mathbb{N}$. This implies that there is $0 \leqslant \hat{\jmath} \leqslant m$ such that $f^{\hat{\imath}+\hat{\jmath}}(x_s) \in U$. By the definition of $\hat{\imath}$ there is also $y \in V$ such that $f^{\hat{\imath}+\hat{\jmath}}(y) \in U$. It is enough to put $i = \hat{\imath} + \hat{\jmath}$ and then $d(f^{i+j}(y), f^{i+j}(x_s)) < \frac{1}{n}$ for $j = 0, \ldots, n$. This proves that $A_n^s \cap V \neq \emptyset$ and so A_n^s is dense, since V was chosen arbitrary.

This immediately implies that if we denote

$$D = \left(\bigcap_{n,s \geqslant 1} A_n^s \right) \cap \operatorname{Trans}(f)$$

then D is a residual subset of X ($\operatorname{Trans}(f) = \operatorname{Trans}(f, X)$ is the set of points with dense orbits). But then D is dense, since X is perfect.

Fix any minimal system $(Y, g) \in \mathscr{M}$ and let $J \subset X \times Y$ be a joining. Additionally, fix any $x \in D$ and let z be such that $(x, z) \in J$. Let U be a neighborhood of x, and let $V \subset Y$ be any nonempty open set. There is $k \geqslant 0$ and an open neighborhood V' of z such that $f^k(V') \subset V$. There are also an open set W ($W \neq X$) and an integer s such that $x_s \in W$ and $f^k(W) \subset U$. There is an open set $W' \ni x_s$ such that $\overline{W'} \subset W$.

The point x_s is distal and so the pair (x_s, z) is uniformly recurrent (see [8, Thm. 9.11]). This implies that the set $R(x_s, W', f) \cap R(z, V, g)$ is syndetic. But $x \in D$ which implies that for every n there is i_n such that if $m \in R(x_s, W', f) \cap \{i_n, \ldots, i_n + n\}$ then $d(f^m(x_s), f^m(x)) < \frac{1}{n}$. Note that if n is large enough (in particular $\frac{1}{n} < \operatorname{dist}(\overline{W'}, X \setminus W)$), then $f^m(x) \in W$ and also $R(x_s, W', f) \cap R(z, V', g) \cap \{i_n, \ldots, i_n + n\} \neq \emptyset$. This implies that $N(x, W, f) \cap R(z, V', g) \neq \emptyset$ and as a consequence $N(x, U, f) \cap N(z, V, g) \neq \emptyset$. In other words $\operatorname{Orb}^+((x, z), f \times g) \cap (U \times V) \neq \emptyset$ which immediately implies that $J = X \times Y$, since the orbit of x is dense in X. Indeed, $f \perp g$ and so the proof is finished. ☐

Remark 4.5. — Note that if the system in the assumptions of Theorem 4.4 is nontrivial (i.e. X is not singleton) then it is not minimal. Namely, any nontrivial weakly mixing system contains at least one pair of distinct proximal points. In particular, all points with dense orbits are weakly product recurrent but not product recurrent.

We don't know any example of a map which fulfills the assumptions of Theorem 4.4 and does not fulfill the assumptions of Theorem 4.2. It is noteworthy that every system in $\mathscr{M}^{\perp}$ has dense minimal points by [11, Theorem 4.3], which makes the problem even harder. We state the following question as a problem for further research:

QUESTION 2. — *Does there exist a weakly mixing map with dense distal points but without the property* ***(P)****?*

In our opinion, even an example of a weakly mixing map with dense distal points but without regularly recurrent points would be interesting (recall that a point x is *regularly recurrent* if for every U, the set $R(x,U)$ contains an infinite arithmetic progression).

It would be also nice to state an extended version of Theorem 4.4 in terms of weakly mixing sets (similarly to Theorem 3.11), obtaining another, (more general than Theorem 4.4) condition for weak product recurrence. However, in that case sets $N(U,V)$ are no longer thick, while it is the main tool used in the proof for synchronization with return times of distal points. In other words, presently we don't know how to prove that sets A_n^s are dense when weak mixing is replaced by weakly mixing set.

5. Semicocyle extensions with one point of discontinuity

In [10] the authors provided an example of a uniformly recurrent point x which is not weakly product recurrent. In their example there was a point $y \in \overline{\mathrm{Orb}^+(x)}$ such that (x,y) was not recurrent. The aim of this section is to show that such a situation is to some extent common. We adopt a technique from [6] (see also [5]).

Let us consider a metric space (X,d) without isolated points and a continuous map $f\colon X \to X$. Assume that x_0 is a point with dense orbit and denote $\Theta = \mathrm{Orb}^+(x_0,f)$. There is a natural bijection $\mathbb{N} \ni n \to f^n(x_0) \in \Theta$, so we can identify Θ with $\mathbb{N}$ and endow (both spaces) with the topology induced from X.

DEFINITION 5.1. — *Let K be a compact metric space. Every continuous function $\phi\colon \Theta \to K$ is called a* semicocycle *on X.*

According to the above mentioned identification of Θ, we may interpret every semicocycle as a sequence $\phi \in K^{\mathbb{N}}$. We endow $K^{\mathbb{N}}$ with the Tychonov topology and define the shift map $\sigma\colon K^{\mathbb{N}} \to K^{\mathbb{N}}$ in the standard way, putting $\sigma(x)_i = x_{i+1}$.

Definition 5.2. — *By a* semicocycle extension *of f determined by a semicocycle ϕ we will understand the orbit closure X_f of the pair (x_0, ϕ) under the product map $f \times \sigma$, together with the map $T_f = (f \times \sigma)|_{X_f}$, that is the map $T_f\colon X_f \to X_f$.*

By *the fiber* of a point x we mean the set $F(x) = \{\xi \in K\ ;\ (x, \xi) \in X_f\}$. The set of points with singleton fibers is denoted $C = \{x \in X\ :\ \#F(x) = 1\}$. The set C can be interpreted as the maximal set onto which ϕ can be extended continuously, while the set $D = X \setminus C$ is the set of "discontinuities" of ϕ.

Theorem 5.3. — *Every minimal system $f\colon X \to X$ acting on a Cantor set X (i.e. X is homeomorphic to the standard middle-third Cantor set) has a minimal extension T_f which is the orbit closure of a uniformly recurrent point which is not weakly product recurrent (or even more, this point in pair with some other point in its orbit closure is not recurrent for $T_f \times T_f$).*

Proof. — For simplicity assume that X is the middle third Cantor set on $[0, 1]$. For $i = 0, 1, \ldots$ denote $A_i = \left(1/3^{i+1}, 1/3^i\right] \cap X$. Note that each set A_i is a compact set with nonempty interior and that

$$\bigcup_{i=0}^{\infty} A_i = X \setminus \{0\}\,.$$

Denote $A = \bigcup_{i=0}^{\infty} A_{2i}$, $B = \bigcup_{i=0}^{\infty} A_{2i+1}$. These sets form a decomposition of the set $X \setminus \{0\}$ with additional property that the sets $A \cap (0, \varepsilon)$ and $B \cap (0, \varepsilon)$ have nonempty interior for every $\varepsilon > 0$.

Now, fix any $x_0 \in X$ such that $0 \notin \mathrm{Orb}^+(x_0, f)$. Such a point exists, since X is infinite; orbit of x is dense in X because f is minimal. Denote $\Theta = \mathrm{Orb}^+(x_0, f)$ and define a map $\phi\colon \Theta \to \{0, 1\}$ by $\phi(y) = 0$ if $y \in A$ and $\phi(y) = 1$ if $y \in B$. It is easy to verify that ϕ is continuous (thus is a semicocycle) and that the set of singleton fibers equals $C = X \cap (0, 1]$. The semicocycle extension T_f defined by ϕ is a minimal map because f is minimal (see [6, Thm. 3.3]). Observe that $\#F(0) > 1$, because there are increasing sequences n_i, k_i such that $\lim_{i\to\infty} f^{n_i}(x_0) = \lim_{i\to\infty} f^{k_i}(x_0) = 0$ and $f^{n_i}(x_0) \in A$, $f^{k_i}(x_0) \in B$. The set X has no isolated points, so it contains no periodic orbit. In particular $f^n(0) \in (0, 1]$ for every $n > 0$ and so, if we fix any $z_1, z_2 \in F(0)$ then $T_f^n(z_1) = T_f^n(z_2)$ for every $n > 0$.

Take any distinct $z_1, z_2 \in F(0)$. By the above arguments the pair (z_1, z_2) is not recurrent for the product system $T_f \times T_f$ and so none of the points

z_1, z_2 is weakly product recurrent (both points are uniformly recurrent). The proof is finished, since every semicocycle extension of a minimal system is minimal [6, Theorem 3.3] □

6. A weakly mixing minimal system

In this section we provide an example of a minimal weakly mixing dynamical system without weakly product recurrent points. It shows, that if a map is weakly mixing then assumptions of Theorem 3.11 cannot be weakened too much, that is there must be a degree of nonminimality included in the dynamics.

The main aim of this section is to show that there exists a subshift $X \subset \Sigma_2^+$ such that:

(6.1) $\sigma = \sigma|_X$ is a weakly mixing minimal map

(6.2) for every $y \in X$ there is $z \in X$ and $j > 0$ such that if $y_{[i,i+j]} = y_{[0,j]}$ then $z_{[i,i+j]} \neq z_{[0,j]}$ for all $i > 0$.

Note that condition 6.2 implies that (y, z) is not recurrent in the product system $\sigma \times \sigma$. The point z is minimal, thus y is not a weakly product recurrent point (and so there is no product recurrent point in X). In fact we obtain much more than we need to prove that x is not weakly product recurrent. It is not even 'internally' product recurrent, that is additional assumption in the definition of weak product recurrence that $f = g$ and $y \in X$ does not change anything.

Now we can start our construction. For every n we will construct a set of words P_n obtaining a sequence $\{P_n\}_{n=1}^{\infty}$ and next use it to define a point x. Finally, the shift X will be obtained as the closure of the orbit of x under σ.

Denote $a = 001101$, $b = 001011$, $c = 001111$, $d = 00101101$, and observe that the set $\mathcal{C} = \{a, b, c, d\}$ is a code, that is if a word u is a concatenation of a sequence of words form $\mathcal{C}$, then a decomposition of u into elements of $\mathcal{C}$ is unique. Additionally, if u is a concatenation of words of $\mathcal{C}$ and there are words v, v' such that $u = vav'$ then v and v' are also concatenations of words from $\mathcal{C}$. Generally speaking, the word a can appear in u only as an element of the decomposition (cannot appear as a subword of vw for some $v, w \in \mathcal{C}$). It is because the word 001 can appear only as a prefix of a word from $\mathcal{C}$.

Given a number n let $D(n) = \{1, 2, \ldots, n+1\}$. Denote $P_0 = \mathcal{C}$ and $Q_0 = \{uvw \ : \ u, v, w \in \mathcal{C}\} = \{q_1, \ldots, q_s\}$ where $s = 4^3 = 64$.

We will construct sets P_n, Q_n for $n > 0$ inductively. Given P_n we always define Q_n by putting $Q_n = \{uvw \,:\, u, v, w \in P_n\}$, so the only problem is to construct the set P_n. The main aim of the construction is that each P_n has the following properties ($n > 0$):

(6.3) There is $k > 0$ and there are $u_i \in P_n$ such that $|u_i| = k + i$ for $i = 0, 1, \dots, n$,

(6.4) Assume that $Q_{n-1} = \{q_1, \dots, q_s\}$. Then $u \in P_n$ if and only if there is an onto map $\pi\colon D(s) \to D(s-1)$ such that $u = q_{\pi(1)}q_{\pi(2)}\dots, q_{\pi(s+1)}$.

(6.5) Assume that $u \in P_n$ and that $u = u_1 \dots u_m$ is its unique decomposition into elements of $\mathcal{C}$. Fix arbitrary index i such that $u_i = a$. There are words $v = v_1 \dots v_m, w = w_1 \dots w_m \in P_n$, where $v_i, w_i \in \mathcal{C}$, $|u_i| = |v_i| = |w_i|$ and such that:
 (a) $v_i = b$ and if $j \neq i$ and $u_j = a$ then $v_j \neq b$,
 (b) if $v_j = a$ then $w_j \neq b$ and if $w_j = a$ then $v_j \neq b$ for every j,
 (c) if $u_j = a$ then $w_j \neq b$ and if $w_j = a$ then $u_j \neq b$ for every j.

To start the construction we have to define a set P_1 which has all the properties specified above. Let F_0 be the set of all surjections $D(64) \to D(63)$. Note that every function $\eta \in F_0$ can be defined in the following way. First we assign $\eta(j) \in D(63)$ for some $j \in D(64)$, and next define a bijection between the sets $D(64) \setminus \{j\}$ and $D(63)$.

In the first step of our inductive procedure, we put

$$P_1 = \{f(D(64)) \,:\, f \in F_0\}\,.$$

We have to check that P_1 has all the properties we claimed. This will finish the first step of induction. Conditions 6.3 and 6.4 follow just from the definition of F_0, because $|aaa|+1 = |aad|$ and $q_1 \dots q_s aaa, q_1 \dots q_s aad \in P_1$, where $Q_0 = \{q_1, \dots, q_s\}$.

To prove that also condition 6.5 holds, fix any $u \in P_1$. There is a function $\pi \in F_0$ such that

$$u = q_{\pi(1)}q_{\pi(2)}\dots, q_{\pi(s+1)}.$$

The unique decomposition of u into elements of $\mathcal{C}$ is induced by the decomposition of each $q_{\pi(i)}$. Fix any j such that $q_{\pi(j)} = axy$ for some $x, y \in \mathcal{C}$ (the proof for the cases xay and xya is the same). There are k, l such that $q_{\pi(k)} = bxy$ and $q_{\pi(l)} = cxy$. Now,define two permutations $\xi, \psi\colon D(s) \to D(s)$ by putting

$$\xi(i) = j, \xi(j) = k, \xi(k) = i, \quad \psi(i) = k, \psi(k) = i$$

and the identity everywhere else. Permutations ξ and ψ define two new elements $\eta, \theta \in F_0$ by $\eta = \pi \circ \xi$ and $\theta = \pi \circ \psi$. We use these two maps to define words $v = q_{\eta(1)}q_{\eta(2)}\dots,q_{\eta(s+1)}$ and $w = q_{\theta(1)}q_{\theta(2)}\dots,q_{\theta(s+1)}$. Observe that $v, w \in P_1$. We will show that these words (together with u) fulfill the condition 6.5. The only difference between symbols in u, v, w can appear along words $q_{\pi(i)}$, $q_{\pi(j)}$ and $q_{\pi(k)}$ because all other elements in the decomposition are the same for all that three words u, v, w. Assume for simplicity, that $i < j < k$. Then we see the following

$$\begin{array}{rcccccc}
u & = & \dots\ q_{\pi(i)}\ \dots\ q_{\pi(j)}\ \dots\ q_{\pi(k)}\ \dots & = & \dots\ axy\ \dots\ bxy\ \dots\ cxy\ \dots \\
v & = & \dots\ q_{\eta(i)}\ \dots\ q_{\eta(j)}\ \dots\ q_{\eta(k)}\ \dots & = & \dots\ bxy\ \dots\ cxy\ \dots\ axy\ \dots \\
w & = & \dots\ q_{\theta(i)}\ \dots\ q_{\theta(j)}\ \dots\ q_{\theta(k)}\ \dots & = & \dots\ cxy\ \dots\ bxy\ \dots\ axy\ \dots
\end{array}$$

The first step of our construction is done.

Now assume that the set P_n has already been defined. Define P_{n+1} to consists of all the words which follow the scheme of 6.4 (and denote by F_n the set of all maps $\pi\colon D(s) \to D(s-1)$, where $s = |Q_n|$). To be consistent with the previous considerations, assume that $Q_n = \{q_1, \dots, q_s\}$ (note that q_i and s here are different from q_i and s in the first step). There is an integer k and words $u_i \in P_n$ such that $|u_i| = k + i$, $i = 0, \dots, n$. It is enough to put $u_i' = q_1 \dots q_s u_0 u_0 u_i \in P_{n+1}$ for $i = 0, \dots, n$, and $u_{n+1}' = q_1 \dots q_s u_0 u_1 u_n \in P_{n+1}$. Simple calculations show immediately that $|u_i'| = |u_0'| + i$ for $i = 0, \dots, n+1$ which gives 6.3.

To prove that also 6.5 holds it is enough to follow the same rule as in the first step. Namely, if u decomposes into $u = q_{\pi(1)}q_{\pi(2)}\dots,q_{\pi(s+1)}$, then decomposition of u into elements of $\mathcal{C}$ must agree with it, i.e. it induces (the unique) decomposition of each q_j. But then, if we chose any occurrence of a in u then it must fall into some $q_{\pi(i)}$, let say at a position l. As previously, we may assume that $q_{\pi(i)} = AXY$, $A, X, Y \in P_n$ and that this "special" occurrence of a is placed somewhere in A. Now we use condition 6.5 for A and a at position l (the word A is an element of P_n so condition 6.5 is guaranteed by the induction) obtaining words $B, C \in P_n$. Words A, B, C decompose into elements of $\mathcal{C}$ in the same way (i.e. if we considers words from $\mathcal{C}$ in the decomposition of A, B or C then the number of words and their lengths are the same which follows directly from 6.5). We can arrange the words from decompositions into pairs. If we consider A and B then the pair (a, b) can be seen exactly once as l-th pair. If we consider A, C or B, C (or these words in reverse order) then the pair (a, b) never appears (see 6.5). Now it is enough to follow the same rule as in the first step. The only difference is that we should put AXY in the place of axy, BXY in the place of bxy etc., e.g.

$$\begin{array}{rcl} u &=& \ldots AXY \ldots BXY \ldots CXY \ldots \\ v &=& \ldots BXY \ldots CXY \ldots AXY \ldots \\ w &=& \ldots CXY \ldots BXY \ldots AXY \ldots \end{array}$$

The construction is finished.

Now, we can return to our main goal: the construction of a special minimal dynamical system. Fix any $u_1 \in P_1$. If $u_n \in P_n$ is defined, then by the condition 6.4 there is $u_{n+1} \in P_{n+1}$ such that u_n is its prefix. Denote

$$x = \lim_{n\to\infty} u_n 0^\infty \quad \text{and} \quad X = \overline{\mathrm{Orb}^+(x,\sigma)}.$$

Theorem 6.1. — *The shift map $\sigma = \sigma|_X$ is minimal and weakly mixing.*

Proof. — Note that for every $i, n \in \mathbb{N}$, if $u \in P_{n+i+1}$ then u is a concatenation of words from P_{n+1} (it is a consequence of 6.4). But each word in P_{n+1} contains all the words from P_n, which proves that occurrences of u_n in x are syndetic, which is equivalent to the fact that x is uniformly recurrent.

If we fix any three open sets $U, V, W \subset X$, then there are $n > 1$ and words $u, v, w \in P_n$ such that $C_X[u] \subset U$, $C_X[v] \subset V$, $C_X[w] \subset W$. Denote $l = |v|$ and observe that $uvw \in Q_n$. Fix any $k > l$ and any word $A \in P_k$. Since $k > l$ the word uvw is present somewhere in A. There are also words $B_0, \ldots, B_l \in P_k$ such that $|B_i| = |B_0| + i$ and words x, y such that $A = xuvwy$. Let z_i be an element of X which has AB_iA as a prefix. Note that $\sigma^{|x|}(z_i) \in U$ for all $i = 0, \ldots, l$. Now, denote $s = |A| + |B_0| + |x| + |u| + l$ and observe that w is a prefix of $\sigma^s(z_0)$, while $\sigma^s(z_l)$ has v as its prefix. This implies that $\sigma^s(z_0) \in W$ and $\sigma^s(z_l) \in V$. Since $\sigma^{|x|}(z_0), \sigma^{|x|}(z_l) \in U$ we obtain that $\sigma^{-s+|x|}(V) \cap U \neq \emptyset$, $\sigma^{-s+|x|}(W) \cap U \neq \emptyset$. Sets U, V, W were arbitrary, thus the map $\sigma|_X$ is weakly mixing (see [3, Thm. 4]). ☐

Theorem 6.2. — *For every $y \in X$ there is $z \in X$ such that the pair (y, z) is not recurrent in the product system $\sigma \times \sigma$. In particular there is no weakly product recurrent point in X.*

Proof. — Fix any $y \in X$. There is an increasing sequence n_k, such that $\lim_{k\to\infty} \sigma^{n_k}(x) = y$. For every k there are words $A_k, B_k \in P_k$ and indices i_k, j_k, l_k, such that $x_{[i_k, j_k)} = A_k$, $x_{[j_k, l_k)} = B_k$, where $j_k = i_k + |A_k|$, $l_k = j_k + |B_k|$ and $i_k \leqslant n_k < j_k$. Now, assume that the first occurrence of word a in y starts at the position s, i.e. $y_{[s, s+|a|)} = a$ and s is the minimal number with this property. The word $y_{[s, s+|a|)}$ is rewritten somewhere in word $x_{[n_k, j_k)}$ or $x_{[j_k, l_k)}$ and furthermore it is exactly an element of the decomposition of one of this words into words from $\mathcal{C}$ (the word 001 can appear only as a prefix of a word in $\mathcal{C}$). For simplicity assume that a is a subword of $x_{[n_k,j_k)}$. This fix a word a in the decomposition of A_k, let say $A_k = u_1 \dots u_m$ and $u_p = a$. By 6.5 there is a word $A'_k \in P_k$ such that $A_k = v_1 \dots v_m$, $v_p = b$ (where $|v_i| = |u_i|$ for all i) and if $u_i = a$ then $v_i \neq b$ for $i \neq p$. Generally speaking, if we write A_k over A'_k then we can see a over b only once.

Put $z_k = \sigma^{n_k - i_k}(A'_k B_k 0^\infty)$ and let $z = \lim_{k\to\infty} z_k$ (we go to a subsequence if necessary). Note that $z \in X$ because $\lim_{k\to+\infty} |B_k| = +\infty$. But again, if we write y over z then we can see a over b only at one position. This implies, that there is an open set U (defined by a sufficiently large prefix of y) and an open neighborhood V of z, such that if $\sigma^n(y) \in U$ and $n > 0$ then $\sigma^n(z) \notin V$. This immediately implies that the pair (y, z) is not recurrent (for $\sigma \times \sigma$) and so y is not weakly product recurrent. □

When a minimal point x is not weakly product recurrent, there is a point y such that the pair (x, y) is not recurrent in a product system. However it does not immediately imply that y belongs to the same system that x does, i.e. it may happen that $x \times X$ is recurrent in the product system $f \times f$. So even, if [10, Question 5.3] has positive answer, that is weak product recurrent point which is uniformly recurrent must be distal (in particular cannot belong to a weakly mixing system), the presented construction goes a step further. Presented system is not even 'internally' product recurrent in the sense, that every point $x \in X$ has its counterpart $y \in X$ making the pair (x, y) not recurrent for $f \times f$.

7. Acknowledgements

The author thank the (unknown) referee of the paper for many valuable suggestions, in particular for pointing out relations between weak product recurrence and results on disjointness of systems.

The author want to express his gratitude to Professor Tomasz Downarowicz for numerous discussions on symbolic dynamics. Among others, he guided the author during his studies on minimal systems, and suggested how the system in Section 6 can be constructed.

The author express many thanks to Professors Francisco Balibrea Gallego and Juan Luis García Guirao for providing him with [8] and [10] and also for interesting discussions on dynamical properties of cartesian products of maps.

The research leading to presented results has received funding from the European Community's Seventh Framework Programme (FP7/2007-2013) under grant agreement No. 219212, belonging to a Marie Curie Intra-European Fellowship for Career Development.

This research was also supported by the Polish Ministry of Science and Higher Education (2009). The financial support of these institutions is widely acknowledged.

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Manuscrit reçu le 24 mars 2009,
accepté le 8 septembre 2009.

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