Katsutoshi YAMANOI

On fundamental groups of algebraic varieties and value distribution theory


<http://aif.cedram.org/item?id=AIF_2010__60_2_551_0>
ON FUNDAMENTAL GROUPS OF ALGEBRAIC VARIETIES AND VALUE DISTRIBUTION THEORY

by Katsutoshi YAMANOI

Dedicated to Professor Junjiro Noguchi on his 60th birthday

Abstract. — If a smooth projective variety $X$ admits a non-degenerate holomorphic map $\mathbb{C} \to X$ from the complex plane $\mathbb{C}$, then for any finite dimensional linear representation of the fundamental group of $X$ the image of this representation is almost abelian. This supports a conjecture proposed by F. Campana, published in this journal in 2004.

Résumé. — Si une variété $X$ projective lisse admet une application holomorphe non-dégénérée $\mathbb{C} \to X$ du plan complexe $\mathbb{C}$, alors pour chaque représentation linéaire de dimension finie du groupe fondamental de $X$ l’image de cette représentation est presque abélienne. Cela soutient une conjecture proposée par F. Campana, parue dans ce même journal en 2004.

1. Main results

Let $X$ be a smooth projective variety. We say that a holomorphic map $f: \mathbb{C} \to X$ is non-degenerate if the image $f(\mathbb{C})$ is Zariski dense in $X$. A group $G$ is called almost abelian if $G$ has a finite index subgroup which is abelian. In this paper, we prove the following theorem.

Theorem 1.1. — Let $X$ be a smooth projective variety which admits a non-degenerate holomorphic map $f: \mathbb{C} \to X$. Then for any representation $\varrho: \pi_1(X) \to \text{GL}_n(\mathbb{C})$, the image $\varrho(\pi_1(X))$ is almost abelian.

This theorem shows that the following conjecture proposed by F. Campana [4, Conjecture 9.8] is true in the special case that $\pi_1(X)$ is linear.

Keywords: Value distribution theory, holomorphic map, fundamental group, algebraic variety.
Math. classification: 32H30, 14F35.
**Conjecture 1.2.** — Let $X$ be a smooth projective variety which admits a non-degenerate holomorphic map $f : \mathbb{C} \to X$. Then the fundamental group $\pi_1(X)$ is almost abelian.

This conjecture comes from Campana’s theory of “special” variety (cf. [4]). A complex manifold $X$ which admits a holomorphic map $f : \mathbb{C} \to X$ with metrically dense image has vanishing Kobayashi pseudo-metric. It is Campana’s view that a smooth projective variety $X$ would have vanishing Kobayashi pseudo-metric if and only if $X$ is “special” (cf. [4, Conjecture 9.2]), and that the fundamental group of a “special” variety would be almost abelian (cf. [4, Conjecture 7.1]). For more discussion about Conjecture 1.2, we refer the reader to [4].

A representation $\varrho : \pi_1(X) \to \text{GL}_n(\mathbb{C})$ is called big if the following condition is satisfied (cf. [11]):

If $Z \subset X$ is a positive dimensional subvariety containing a very general point of $X$, then the image $\varrho(\text{Im}(\pi_1(Z)) \to \pi_1(X)))$ is infinite. Here $\tilde{Z}$ is a desingularization of $Z$.

For example, if there exists an unramified Galois covering $\tilde{X} \to X$ such that $\tilde{X}$ is a Stein space and its Galois transformation group $\Gamma$ is a linear group, then the corresponding surjection $\pi_1(X) \to \Gamma$ is a big representation.

**Corollary 1.3.** — Let $X$ be a smooth projective variety with a big representation $\varrho : \pi_1(X) \to \text{GL}_n(\mathbb{C})$. If $X$ admits a non-degenerate holomorphic map $f : \mathbb{C} \to X$, then there exists a finite unramified covering $X' \to X$ which is birationally equivalent to an Abelian variety.

The strategy of the proof of Theorem 1.1 is roughly as follows. Based on results of [4] and [20], Campana proved the following ([4]): If there exists a representation $\varrho : \pi_1(X) \to \text{GL}_n(\mathbb{C})$ such that the image $\varrho(\pi_1(X))$ is not almost abelian, then there exist a finite unramified covering $X' \to X$ and a dominant rational map $X' \dashrightarrow Z$ with $Z$ of general type and positive dimensional. The proof of this result shows that $Z$ is not only of general type, but has more precise structure. Thanks to this precise structure, we can show that every holomorphic map $g : \mathbb{C} \to Z$ is degenerate, i.e. the image $g(\mathbb{C})$ is not Zariski dense in $Z$. This implies our theorem.

2. A reduction of the proof of Theorem 1.1

The proof of Theorem 1.1 is based on the following proposition, which is a special case of the theorem.
Proposition 2.1. — Let $X$ be a smooth projective variety, and let $G$ be an almost simple algebraic group defined over the complex number field. Assume that there exists a representation $\varrho: \pi_1(X) \to G(\mathbb{C})$ whose image $\varrho(\pi_1(X))$ is Zariski dense in $G$. Then every holomorphic map $f: \mathbb{C} \to X$ is degenerate.

Proposition 2.1 implies Theorem 1.1. — Let $X$ be a smooth projective variety which admits a non-degenerate holomorphic map $f: \mathbb{C} \to X$. Let $\varrho: \pi_1(X) \to \text{GL}_n(\mathbb{C})$ be a representation. We shall prove that $\varrho(\pi_1(X))$ is almost abelian.

Let $H \subset \text{GL}_n(\mathbb{C})$ be the Zariski closure of the image $\varrho(\pi_1(X))$. Let $H_0 \subset H$ be the connected component of $H$ containing the identity element of $H$. Then $\Gamma = \varrho^{-1}(\varrho(\pi_1(X)) \cap H_0)$ is a finite index subgroup of $\pi_1(X)$. Replacing $X$ by the finite unramified covering $X' \to X$ which corresponds to $\Gamma$, we may assume that $H$ is connected.

Let $R(H) \subset H$ be the radical of $H$, i.e. $R(H)$ is the maximal connected solvable closed normal subgroup of $H$. Put $H_{s.s.} = H/R(H)$. We first prove that $H_{s.s.}$ is trivial.

Assume, for the sake of contradiction, that $H_{s.s.}$ is not trivial. Then $H_{s.s.}$ is a semi-simple algebraic group. Hence $H_{s.s.}$ is an almost direct product of almost simple algebraic groups $G_1, \ldots, G_l$. Let $H_{s.s.} \to G_1$ be a projection, and let $\varrho': \pi_1(X) \to G_1$ be the composition of $\varrho$ and the two projections $H \to H_{s.s.} \to G_1$. Since the image $\varrho'(\pi_1(X))$ is Zariski dense in $G_1$, we may apply Proposition 2.1 to conclude that every holomorphic map $\mathbb{C} \to X$ is degenerate. This contradicts to our assumption that $X$ admits a non-degenerate holomorphic map $f: \mathbb{C} \to X$. Hence we have proved that $H_{s.s.}$ is trivial, i.e. $H = R(H)$.

Now the image $\varrho(\pi_1(X))$ is a solvable group. We note that every finite unramified covering $X'$ of $X$ admits a non-degenerate holomorphic map $f': \mathbb{C} \to X'$ coming from a lifting of $f: \mathbb{C} \to X$. Hence by [14, Theorem 6.4.1], the Albanese map of $X'$ is surjective for every finite unramified covering $X'$. Hence by [3, Théorème 2.9], there exists a finite unramified covering $X'$ of $X$ such that $\varrho$ factors the induced group homomorphism $\pi_1(X') \to \pi_1(\text{Alb}(X'))$. From this, we conclude that $\varrho(\pi_1(X'))$ is abelian. Hence $\varrho(\pi_1(X))$ is almost abelian. \hfill $\square$

3. Representations over non-archimedean local fields

Let $K$ be a number field, and let $\mathcal{O}_K$ be the ring of integers in $K$. Given a prime ideal $p$ from $\mathcal{O}_K$, we denote by $K_p$ the completion of $K$ with
respect to the natural discrete valuation defined by \( p \). Let \( G \) be an almost simple algebraic group defined over \( K_p \), and let \( \rho: \pi_1(X) \to G(K_p) \) be a \( p \)-adic representation. We say that \( \rho \) is \( p \)-bounded if the image \( \rho(\pi_1(X)) \) is contained in a maximal compact subgroup of \( G(K_p) \). If \( \rho \) is not \( p \)-bounded, then we say that \( \rho \) is \( p \)-unbounded.

In this section, we prove the following:

**Proposition 3.1. —** Let \( X \) be a smooth projective variety. Let \( G \) be an almost simple algebraic group defined over the \( p \)-adic field \( K_p \). Assume that there exists a \( p \)-unbounded representation \( \rho: \pi_1(X) \to G(K_p) \) whose image is Zariski dense in \( G \). Then every holomorphic map \( f: \mathbb{C} \to X \) is degenerate.

The proof of this proposition is based on the consideration of the spectral covering \( \pi: X^s \to X \). We follow the exposition of [20, Section 1]. The construction of \( X^s \) is based on the theory of equivariant harmonic maps to buildings due to Gromov and Schoen [8]; Since \( \rho \) is reductive, there exists a non-constant \( \rho \)-equivariant pluriharmonic map \( u: \hat{X} \to \Delta(G) \) from the universal covering of \( X \) to the Bruhat-Tits building of \( G \). Considering the complexified differential of \( u \), we get a multi-valued holomorphic one form \( \omega \) on \( X \). We consider a finite ramified Galois covering \( \pi: X^s \to X^s \) such that \( \pi^* \omega \) splits into single-valued holomorphic one forms \( \omega_1, \ldots, \omega_l \). All the forms \( \omega_1, \ldots, \omega_l \) are contained in the space \( H^0(X^s, \pi^*\Omega^1_X) \). The covering \( \pi: X^s \to X \) is unramified outside \( \bigcup_{\omega_i \neq \omega_j} (\omega_i - \omega_j) \) where \( \omega_i - \omega_j \) are considered as forms from \( H^0(X^s, \pi^*\Omega^1_X) \) (cf. [9, Lemma 2.1]). For more detail about the construction of the spectral covering, we refer the reader to [20], [21], [5] and [10].

We construct the Albanese map \( \Phi: X^s \to A \) with respect to \( \omega_1, \ldots, \omega_l \) as follows (cf. [21, p. 64]): Let \( a: \hat{X}^s \to A(\hat{X}^s) \) be the Albanese map, where \( \psi: \hat{X}^s \to X^s \) is a desingularization of \( X^s \). For \( i = 1, \ldots, l \), let \( \tilde{\omega}_i \) be the holomorphic one form on \( A(\hat{X}^s) \) such that \( \psi^*(\omega_i) = a^*\tilde{\omega}_i \). Let \( B \subset A(\hat{X}^s) \) be the maximal Abelian subvariety such that all \( \tilde{\omega}_i \) vanish on \( B \). We set \( A = A(\hat{X}^s)/B \). Then since \( X^s \) is normal, the composition of \( \hat{X}^s \to A(\hat{X}^s) \to A \) factors through \( \psi: \hat{X}^s \to X^s \). This induces the desired map \( \Phi: X^s \to A \).

We summarize the needed properties of the spectral covering from [20, Section 1].

**Proposition 3.2. —** Assume furthermore that \( \rho \) is big. Then:

1. \( \Phi: X^s \to A \) is generically finite.
2. \( X^s \) is of general type.
The proof of (1) can be found in [20, p. 148]. Indeed the following stronger result is proved in [20, p. 148]: The Stein factorization of $\Phi: X^s \to A$ is a Shafarevich map for the pull-back representation $\pi^*\varrho: \pi_1(X^s) \to G(K_p)$. The implication of (1) is immediate; Since $\varrho$ is big, $\pi^*\varrho$ is also big. Hence the Shafarevich map for the representation $\pi^*\varrho$ is birational, which implies (1). The proof of (2) can be found in [20, p. 151].

**Notation.** Before going to prove Proposition 3.1, we introduce the notations of Nevanlinna theory (cf. [14], [13]). Let $Y$ be a Riemann surface with a proper surjective holomorphic map $p_Y: Y \to \mathbb{C}$. For $r > 0$, we set $Y(r) = p_Y^{-1}\{z; |z| < r\}$. We put

$$N_{\text{ram}} p_Y(r) = \frac{1}{\deg p_Y} \int_1^r \left[ \sum_{y \in Y(t)} \text{ord}_y \text{ram } p_Y \right] \frac{dt}{t},$$

where $\text{ram } p_Y$ is the ramification divisor of $p_Y$.

Let $X$ be a projective variety and let $Z$ be a closed subscheme of $X$. Let $g: Y \to X$ be a holomorphic map with Zariski dense image. Since $Y$ is one dimensional, the pull-back $g^*Z$ is a divisor on $Y$. We set

$$N(r, g, Z) = \frac{1}{\deg p_Y} \int_1^r \left[ \sum_{y \in Y(t)} \text{ord}_y g^* Z \right] \frac{dt}{t},$$

$$\tilde{N}(r, g, Z) = \frac{1}{\deg p_Y} \int_1^r \left[ \sum_{y \in Y(t)} \min\{1, \text{ord}_y g^* Z\} \right] \frac{dt}{t}.$$

Let $\psi: \tilde{X} \to X$ be a desingularization, let $\tilde{g}: Y \to \tilde{X}$ be the lifting of $g$. Let $M$ be a line bundle on $X$. Let $|\cdot|$ be a smooth Hermitian metric on $\psi^* M$, let $\Omega$ be the curvature form of $(M, |\cdot|)$. We define

$$T(r, g, M) = \frac{1}{\deg p_Y} \int_1^r \left[ \int_{Y(t)} \tilde{g}^* \Omega \right] \frac{dt}{t} + O(1).$$

This definition is independent of the choice of desingularization and Hermitian metric up to bounded function $O(1)$. Given a divisor $D \in H^0(X, M)$, we have the following Nevanlinna inequality (cf. [14, p. 180], [12, p. 269]):

$$(3.1) \quad N(r, g, D) \leq T(r, g, M) + O(1).$$

Let $M$ be an ample line bundle on $X$. Let $\omega \in H^0(X, \Omega_X^1)$ be a holomorphic one form. Set $\eta = g^* \omega/p_Y^*(dz)$. Then $\eta$ is a meromorphic function on $Y$. We set

$$m(r, \eta) = \frac{1}{\deg p_Y} \int_{\partial Y(r)} \max\{\log |\eta(y)|, 0\} \frac{d\arg p_Y(y)}{2\pi}.$$
Then by the lemma on logarithmic derivative ([13, Lemma 1.6]), we have
\[ m(r, \eta) = o(T(r, g, M)) \] ||.
Here the symbol || means that the stated estimate holds for \( r > 0 \) outside some exceptional interval with finite Lebesgue measure. By the first main theorem (cf. [12, p. 269]), we have
\[ T(r, \eta, \mathcal{O}_{\mathbb{P}^1}(1)) = N(r, \eta, \infty) + m(r, \eta) + O(1), \]
where we consider \( \eta \) as a holomorphic map from \( Y \) into \( \mathbb{P}^1 \). Thus we have
\[ (3.2) \quad T(r, \eta, \mathcal{O}_{\mathbb{P}^1}(1)) \leq N(r, \eta, \infty) + o(T(r, g, M)) ||. \]

**Proof of Proposition 3.1.** — First we shall reduce to the case that \( \varrho \) is big. Put \( H = \ker \rho \) and consider the \( H \)-Shafarevich map \( \text{sh}^H_{X}: X \rightarrow \text{Sh}^H(X) \) ([11, p. 185]). We remark that \( \text{Sh}^H(X) \) is only defined up to birationally equivalent class. Replacing \( X \) and \( \text{sh}^H_{X} \) by suitable models, we may assume that \( \text{sh}^H_{X}: X \rightarrow \text{Sh}^H(X) \) is a morphism. Let \( F \) be a general fiber of \( \text{sh}^H_{X} \) and let \( \pi_1(F)_X \) be the image of the natural map \( \pi_1(F) \rightarrow \pi_1(X) \). Then by the definition of the \( H \)-Shafarevich map, the image \( \varrho(\pi_1(F)_X) \subset G(K_p) \) is finite. We apply [21, Lemma 2.2.3]. The conclusion is as follows: After passing to a blowing-up and a finite unramified covering \( e: X' \rightarrow X \), denoting \( s: X' \rightarrow \Sigma \) the Stein factorization of \( \text{sh}^H_{X} \circ e \), there exists a representation \( \varrho_{\Sigma}: \pi_1(\Sigma) \rightarrow G(K_p) \) such that the pullback representation \( e^*\varrho: \pi_1(X') \rightarrow G(K_p) \) factors through \( \varrho_{\Sigma} \). Replacing \( X' \) and \( \Sigma \) by suitable models, we may assume that \( \Sigma \) is smooth. By the construction of \( \Sigma \), we remark that the representation \( \varrho_{\Sigma} \) is big and Zariski dense (cf. [21, Proposition 2.2.2]). Given a holomorphic map \( f: \mathbb{C} \rightarrow X \), we may take a lifting \( f': \mathbb{C} \rightarrow X' \) of \( f \). If the composite holomorphic map \( s \circ f': \mathbb{C} \rightarrow \Sigma \) is degenerate, then \( f \) is also degenerate. Thus replacing \( X \) by \( \Sigma \), \( \varrho \) by \( \varrho_{\Sigma} \) and \( f \) by \( s \circ f' \), we have reduced to the case when \( \varrho \) is big.

Now assume, for the sake of contradiction, that there exists non-degenerate holomorphic map \( f: \mathbb{C} \rightarrow X \). Then we may construct a Riemann surface \( Y \) with proper, surjective holomorphic map \( p_Y: Y \rightarrow \mathbb{C} \) such that:

- the lifting \( g: Y \rightarrow X^s \) of \( f \) exists, and
- \( p_Y \) is unramified outside the discrete set \( g^{-1}(R) \subset Y \), where \( R \) is the ramification divisor of \( \pi: X^s \rightarrow X \). Hence we have
  \[ (3.3) \quad N_{\text{ram}, p_Y}(r) \leq (\deg p_Y) \bar{N}(r, g, R). \]

Since we are assuming that \( f \) is non-degenerate, we remark that
  \[ (3.4) \quad \text{the image } g(Y) \text{ is Zariski dense in } X^s. \]
For $\omega_i \neq \omega_j$, we set $\Xi_{ij} = (\omega_i - \omega_j)_0$, where $\omega_i - \omega_j$ is considered as a form from $H^0(X^s, \pi^*\Omega^1_X)$. We have

\begin{equation}
R \subset \cup_{i,j} \Xi_{ij}.
\end{equation}

Let $M$ be an ample line bundle on $X^s$.

**Claim.** — $\bar{N}(r, g, \Xi_{ij}) \leq \varepsilon T(r, g, M) \mid|$ for all $\varepsilon > 0$.

**Proof of Claim.** — We prove the claim in the two possible cases:

**Case 1.** — $g^*\omega_i \neq g^*\omega_j$. Since $\omega_i \in H^0(X^s, \pi^*\Omega^1_X)$, we may consider $g^*\omega_i$ as a holomorphic section of $p_Y^*\Omega^1_C$. Thus $\eta_i = g^*\omega_i/p_Y^*(dz)$ is a holomorphic function on $Y$. Since $g^*\omega_i \neq g^*\omega_j$, we have $\eta_i \neq \eta_j$. Note that if $g(y) \in \Xi_{ij}$, we have $\eta_i(y) = \eta_j(y)$. Hence using the Nevanlinna inequality (3.1), we have

\[\bar{N}(r, g, \Xi_{ij}) \leq N(r, \eta_i - \eta_j, 0) \leq T(r, \eta_i - \eta_j, \mathcal{O}_{\mathbb{P}^1}(1)) + O(1).\]

Since $\eta_i - \eta_j$ has no poles, we have $N(r, \eta_i - \eta_j, \infty) = 0$. Thus, applying (3.2) to $\eta_i - \eta_j = g^*(\omega_i - \omega_j)/p_Y^*(dz)$, we have

\[T(r, \eta_i - \eta_j, \mathcal{O}_{\mathbb{P}^1}(1)) = o(T(r, g, M)) \mid|.
\]

We conclude that

\[\bar{N}(r, g, \Xi_{ij}) = o(T(r, g, M)) \mid|.
\]

**Case 2.** — $g^*\omega_i = g^*\omega_j$. Let $b: X^s \to B$ be the Albanese map with respect to $\omega_i - \omega_j$, which is constructed as follows: Let $\Phi: X^s \to A$ be the Albanese map with respect to $\omega_1, \ldots, \omega_l$. For $k = 1, \ldots, l$, let $\tilde{\omega}_k$ be the holomorphic one form on $A$ such that $\Phi^*\tilde{\omega}_k = \omega_k$. Let $C \subset A$ be the maximal Abelian subvariety such that $\tilde{\omega}_i - \tilde{\omega}_j$ vanishes on $C$. Put $B = A/C$. We define the map $b: X^s \to B$ by the composition of $\Phi: X^s \to A$ and the quotient $A \to B$.

Let $\Xi'_{ij}$ be an irreducible component of $\Xi_{ij}$. Since there are only finitely many irreducible components of $\Xi_{ij}$, it is enough to prove

\[\bar{N}(r, g, \Xi'_{ij}) \leq \varepsilon T(r, g, M) \mid| \quad \text{for all } \varepsilon > 0.
\]

Since $\omega_i - \omega_j$ vanishes on $\Xi'_{ij}$, we see that $b(\Xi'_{ij})$ is a point on $B$. We take an open subset $U \subset B$ and a holomorphic function $\varphi$ on $U$ such that $\varphi(b(\Xi'_{ij})) = 0$ and $\omega_i - \omega_j = b^*(d\varphi)$ on $b^{-1}(U)$.

Let $S \to b(X^s)$ be the normalization. Since $X^s$ is normal, $b$ factors as

\[X^s \xrightarrow{\psi} S \xrightarrow{\psi} B.
\]
Since $\psi$ is finite, $c(Ξ_{ij})$ is a point on $S$: We denote this point by $P$. Let $O_{S,P}^n$ be the stalk at $P$ in the sense of analytic space, and let $m \subset O_{S,P}^n$ be the maximal ideal. Since $S$ is normal, $O_{S,P}^n$ is integral. We remark that $ϕ \circ ψ \in O_{S,P}^n$ is neither zero nor a unit, which follows from $ω_i - ω_j = b^*(dϕ)$ and $ϕ(b(Ξ_{ij})) = 0$. Hence we have

\begin{equation}
(3.6) \quad \dim O_{S,P}^n/(ϕ \circ ψ) = \dim S - 1.
\end{equation}

Set

$$V_n = \operatorname{Spec} O_{S,P}^n/(\langle ϕ \circ ψ \rangle + m^n).$$

Then $V_n$ is a closed subscheme of $S$ with $\operatorname{supp} V_n = P$.

Let $L$ be an ample line bundle on $S$. Using (3.6), we have

$$h^0(V_n, O_{V_n} \otimes L^\otimes \ell) = h^0(V_n, O_{V_n}) = O(n^{\dim S - 1}).$$

On the other hand, there are positive constants $c > 0$ and $\ell_0 > 0$ such that

$$h^0(S, L^\otimes \ell) > c\ell^{\dim S}$$

for $\ell > \ell_0$. Thus we may take a positive integer $\ell(n)$ such that $\ell(n) = o(n)$ as $n \to \infty$, and that $h^0(V_n, O_{V_n} \otimes L^\otimes(n)) < h^0(S, L^\otimes(n))$. For example, $\ell(n) \sim n^{1 - \frac{1}{2\dim S}}$. Thus we may take a divisor $D_n$ from $H^0(S, L^\otimes(n))$ such that $V_n \subset D_n$.

Now we claim that if $c \circ g(y) = P$ for $y \in Y$, then $\operatorname{ord}_y(c \circ g)^*D_n \geq n$. To see this, we take $y \in Y$ such that $c \circ g(y) = P$. Let $O \subset Y$ be the connected component of $(b \circ g)^{-1}(U)$ containing $y$. By the assumption $g^*(ω_i - ω_j) = 0$, we have $ϕ \circ b \circ g = 0$ on $O$. Hence $(ϕ \circ ψ) \circ (c \circ g) = 0$ on $O$. Thus by the construction of $V_n$, we have $\operatorname{ord}_y(c \circ g)^*V_n \geq n$. Hence by $V_n \subset D_n$, we have $\operatorname{ord}_y(c \circ g)^*D_n \geq n$.

By (3.4), $c \circ g(Y)$ is Zariski dense in $S$. Hence we have

$$n\bar{N}(r, g, Ξ_{ij}) \leq n\bar{N}(r, c \circ g, P) \leq N(r, c \circ g, D_n) \leq l(n)T(r, c \circ g, L) + O(1),$$

where the last estimate follows from the Nevanlinna inequality (3.1). Thus, by $l(n) = o(n)$ and $T(r, c \circ g, L) = O(T(r, g, M))$, we have

$$\bar{N}(r, g, Ξ_{ij}) \leq \varepsilon T(r, g, M) \mid \mid$$

for all $\varepsilon > 0$. We have proved our claim.

Now we go back to the proof of Proposition 3.1. By (3.3) and (3.5), we have

$$N_{ram,pV}(r) \leq \varepsilon T(r, g, M) \mid \mid$$
for all $\varepsilon > 0$. Hence by Proposition 3.2 and Proposition 3.3 below, we conclude that the image $g(Y)$ is not Zariski dense in $X^*$, which contradicts to (3.4). This conclude the proof of Proposition 3.1.

\[ \text{Proposition 3.3. — Let } X \text{ be a smooth projective variety such that} \]
(1) the Albanese map is generically finite, and (2) $X$ is of general type.
Let $M$ be an ample line bundle on $X$. Let $g: Y \to X$ be a holomorphic map from a Riemann surface $Y$ with a proper surjective holomorphic map $p_Y: Y \to \mathbb{C}$. Assume that
\[
N_{\text{ram}}(r) \leq \varepsilon T(r,g,M) \]
for all $\varepsilon > 0$. Then the image of $g$ is not Zariski dense in $X$.

This is a generalization of [18, Corollary 3.1.14]. The proof is parallel to that of [18, Corollary 3.1.14]. See also [17] for a generalization of Proposition 3.3.

4. Proof of Proposition 2.1

In this section, we prove Proposition 2.1. A representation of the fundamental group into an algebraic group $G$ is called rigid if every nearby representation is conjugate to it. A representation which is not rigid is called non-rigid. The proof of Proposition 2.1 divides into two cases according to whether the representation $\rho: \pi_1(X) \to G$ is rigid or non-rigid.

4.1. Case 1: $\rho$ is rigid

In this case $\rho$ is defined over some number field $K$. Given a prime ideal $p$ from $\mathcal{O}_p$, we denote by $\rho_p: \pi_1(X) \to G(K_p)$ the composition of $\rho: \pi_1(X) \to G(K)$ and the inclusion $G(K) \subset G(K_p)$. If there exists a prime ideal $p$ such that $\rho_p$ is $p$-unbounded, then Proposition 2.1 is a direct consequence of Proposition 3.1. Hence in the following, we consider the case that $\rho_p$ is $p$-bounded for every prime ideal $p$.

In this case, we remark that $\rho^{-1}(G(\mathcal{O}_K))$ is of finite index in $\pi_1(X)$ (cf. [19, p. 120]). This can be proved as follows: Since $\pi_1(X)$ is finitely generated, there are only finite prime ideals $p_1, \ldots, p_k$ such that $\rho(\pi_1(X))$ is not contained in $G(\mathcal{O}_{K_{p_i}})$. Since $\rho(\pi_1(X))$ is $p_i$-bounded for all $p_i$, the image of $\rho(\pi_1(X))$ in $G(K_p)/G(\mathcal{O}_{K_p})$ is finite for all $p_i$. This shows our assertion.
Thus, after passing to a finite unramified covering, we may assume that \( \varrho(\pi_1(X)) \subset G(\mathcal{O}_K) \).

Now by a result of Simpson (cf. [16, p. 58]), \( \varrho \) is a complex direct factor of a \( \mathbb{Z} \)-variation of Hodge structure. In particular, there is the period mapping \( c : X \to \Gamma \setminus \mathcal{D} \) of this variation of Hodge structure (cf. [7, p. 57]). Here \( \mathcal{D} \) is the classifying space and \( \Gamma \) is the arithmetic group which preserves the polarization and the lattice of the variation of Hodge structure. Then \( c \) is a horizontal locally liftable holomorphic map (cf. [7, 3.13]). Hence, for a holomorphic map \( f : \mathbb{C} \to X \), \( c \circ f : \mathbb{C} \to \Gamma \setminus \mathcal{D} \) is also a horizontal locally liftable holomorphic map. Since \( \mathcal{D} \) has negative curvature in the horizontal direction, \( c \circ f \) is constant (cf. [6, Corollary 9.7]). Since \( \Gamma \setminus \mathcal{D} \) has the structure of a normal analytic space (cf. [7, p. 56]), the fibers of \( c \) are Zariski closed subsets on \( X \). This shows that \( f \) is degenerate. Hence we have proved Proposition 2.1 when \( \varrho \) is rigid.

4.2. Case 2: \( \varrho \) is non-rigid

It suffices to prove the following:

**Lemma 4.1.** — Let \( G \) be an almost simple algebraic group defined over the complex number field. Assume that there exists a Zariski dense, non-rigid representation \( \varrho : \pi_1(X) \to G(\mathbb{C}) \). Then every holomorphic map \( f : \mathbb{C} \to X \) is degenerate.

**Proof.** — We remark that \( G \) is defined over some number field \( K \) after some conjugations. Since \( \pi_1(X) \) is finitely presented, there exists an affine scheme \( R \) over \( K \) such that

\[
R(L) = \text{Hom}(\pi_1(X), G(L))
\]

for every field extension \( L/K \). This space is defined as follows: We choose generators \( \gamma_1, \ldots, \gamma_k \) for \( \pi_1(X) \). Let \( \mathcal{R} \) be the set of relations among the generators \( \gamma_i \). Then

\[
R \subset G \times \cdots \times G
\]

is the closed subscheme defined by the equations \( r(m_1, \ldots, m_k) = 1 \) for \( r \in \mathcal{R} \). A representation \( \tau : \pi_1(X) \to G(L) \) corresponds to the point \( (m_1, \ldots, m_k) \in R(L) \) with \( m_i = \tau(\gamma_i) \). Note that \( R \) is an affine scheme, since it is a closed subscheme of an affine variety. Let \( R_{Z.D.} \subset R \) be the space of Zariski dense representations. Then by [1, Proposition 8.2], \( R_{Z.D.} \).
is a Zariski open subset of \( R \). The group \( G \) acts on \( R \) by simultaneous conjugation. Put \( M = R/G \), and let \( p: R \to M \) be the quotient map. Then \( M \) is an affine scheme defined over \( K \). Let \([q] \in R_{Z.D.}(\mathbb{C})\) be the point which correspond to the Zariski dense representation \( \varrho: \pi_1(X) \to G(\mathbb{C}) \).

Since \( R_{Z.D.}(\mathbb{Q}) \) is dense in \( R_{Z.D.}(\mathbb{C}) \), by deforming \( \varrho \) slightly and replacing \( K \) by its finite extension, we may assume that \( \varrho \) is defined over \( K \). Let \( p \) be a prime ideal from \( \mathcal{O}_K \) and let \( K_p \) be the completion. In the following, we shall work over this \( K_p \).

Since \( \varrho \) is non-rigid, we have \( \dim M > 0 \). Hence there exists a morphism \( \psi: M \to \mathbb{A}^1 \) such that the image \( \psi(M) \) is Zariski dense in \( \mathbb{A}^1 \). Since the image \( \psi \circ p(R_{Z.D.}(\mathbb{Q})) \) is also Zariski dense in \( \mathbb{A}^1 \), there exists an affine curve \( C \subset R_{Z.D.}(\mathbb{Q}) \) such that the restriction \( \psi \circ p|_C: C \to \mathbb{A}^1 \) is generically finite. We may take a Zariski open subset \( U \subset \mathbb{A}^1 \) such that \( \psi \circ p|_C \) is finite over \( U \). Let \( x \in U(K_p) \) be a point, and let \( y \in C(K_p) \) be a point over \( x \). Then \( y \) is defined over some extension of \( K_p \) whose extension degree is bounded by the degree of \( \psi \circ p|_C: C \to \mathbb{A}^1 \). Note that there are only finitely many such field extensions. Hence there exists a finite extension \( L/K_p \) such that the points over \( U(K_p) \) are all contained in \( C(L) \). Since \( U(K_p) \subset \mathbb{A}^1(L) \) is unbounded, the image \( \psi \circ p(R_{Z.D.}(\mathbb{Q})) \subset \mathbb{A}^1(L) \) is unbounded.

Let \( R_0 \subset R(L) \) be the subset whose points correspond to \( p \)-bounded representations. Let \( M_0 \subset M(L) \) be the image of \( R_0 \) under the quotient \( p: R \to M \). Then by Lemma 4.2 below, \( M_0 \) is compact. Hence \( \psi(M_0) \) is compact. In particular it is bounded. On the other hand, \( \psi \circ p(R_{Z.D.}(\mathbb{Q})) \subset \mathbb{A}^1(L) \) is unbounded. Hence we have \( R_{Z.D.}(\mathbb{Q}) \not\subset R_0 \). Thus we may take a Zariski dense, \( p \)-unbounded representation \( \tilde{\varrho}: \pi_1(X) \to G(L) \). By Proposition 3.1, every holomorphic map \( f: \mathbb{C} \to X \) is degenerate.

**Lemma 4.2.** — \( M_0 \) is compact.

**Proof.** — Note that there are only finitely many conjugacy classes of maximal compact subgroups in \( G(L) \). Hence all maximal compact subgroups are conjugate to one of maximal compact subgroups \( H_1, \ldots, H_k \subset G(L) \). Hence given a \( p \)-bounded representation \( \tau: \pi_1(X) \to G(L) \), there is a \( G(L) \)-conjugation \( \tilde{\tau}: \pi_1(X) \to G(L) \) of \( \tau \) such that the image \( \tilde{\tau}(\pi_1(X)) \) is contained in one of \( H_1, \ldots, H_k \).

Now take a sequence \([\tau_1], [\tau_2], \ldots \in M_0 \). Then we may take representations \( \tau_1, \tau_2, \ldots \) from \( R_0 \) such that \( \tau_j(\pi_1(X)) \) is contained in one of \( H_1, \ldots, H_k \). By taking subsequence, we may assume that \( \tau_j(\pi_1(X)) \subset H_i \) for all \( j \). Now since \( H_i \) is compact, some subsequence \( \tau_j \) should converge to \( \tau_\infty: \pi_1(X) \to H_i \). Then the sequence \([\tau_j]\) converges to \([\tau_\infty] \in M_0 \). This shows that \( M_0 \) is compact.

TOME 60 (2010), FASCICULE 2
5. Proof of Corollary 1.3

Let $X$ be a smooth projective variety with a big representation $\rho: \pi_1(X) \to \text{GL}_n(\mathbb{C})$. Assume that $X$ admits a non-degenerate holomorphic map $f: \mathbb{C} \to X$. Then by Theorem 1.1, the image $\rho(\pi_1(X))$ is almost abelian. Hence after passing to a finite unramified covering $X'$ of $X$ we may assume that $\rho(\pi_1(X))$ is a free abelian group, i.e. $\rho$ factors the Albanese map $a_X: X \to \text{Alb}(X)$. We shall prove that the Albanese map $a_X$ is birational.

Since $X$ admits a non-degenerate holomorphic map, the Albanese map $a_X$ is surjective ([14, Theorem 6.4.1]) and has connected fibers ([15]). Let $F$ be a general fiber of $a_X$. Then $\rho(\text{Im}(\pi_1(F) \to \pi_1(X)))$ is trivial. Since $\rho$ is big, $F$ should be a point. Hence the Albanese map $a_X$ is birational. This conclude the proof of the corollary.

BIBLIOGRAPHY


Katsutoshi YAMANOI
Kumamoto University
Graduate School of Science and Technology
Kurokami, Kumamoto 860-8555 (Japan)
yamanoi@kumamoto-u.ac.jp