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DYNAMICS ON CHARACTER VARIETIES AND MALGRANGE IRREDUCIBILITY OF PAINLEVÉ VI EQUATION

by Serge CANTAT & Frank LORAY

ABSTRACT. — We consider representations of the fundamental group of the four punctured sphere into $\mathrm{SL}(2,\mathbb{C})$. The moduli space of representations modulo conjugacy is the character variety. The Mapping Class Group of the punctured sphere acts on this space by symplectic polynomial automorphisms. This dynamical system can be interpreted as the monodromy of the Painlevé VI equation. Infinite bounded orbits are characterized: they come from $\mathsf{SU}(2)$-representations. We prove the absence of invariant affine structure (and invariant foliation) for this dynamical system except for special explicit parameters. Following results of Casale, this implies that Malgrange's groupoid of the Painlevé VI foliation coincides with the symplectic one. This provides a new proof of the transcendence of Painlevé solutions.

RÉSUMÉ. — Nous étudions l'action du groupe modulaire sur l'espace des représentations du groupe fondamental de la sphère privée de quatre points dans $\mathrm{SL}(2,\mathbb{C})$. Ce système dynamique peut être interprété comme la monodromie de l'équation de Painlevé VI. Nous caractérisons les orbites infinies bornées : elles proviennent des représentations dans $\mathsf{SU}(2)$. Nous démontrons l'absence de struture affine invariante (excepté pour des paramètres spéciaux) puis déduisons, en nous appuyant sur des travaux de Casale, que le groupoïde de Malgrange associé est le groupoïde symplectique. Ceci permet de donner une preuve de l'irréductibilité de l'équation de Painlevé VI, c'est-à-dire de la forte transcendance de ses solutions, par une approche galoisienne, dans l'esprit de la tentative de Drach et Painlevé.

1. Introduction

This is the first part of a series of two papers (see [6]), the aim of which is to describe the dynamics of a polynomial action of the group

$$\Gamma_2^{\pm} = \{M \in \mathsf{PGL}(2,\mathbf{Z}) \mid \quad M = \mathrm{Id} \bmod(2)\} \tag{1.1}$$

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on the family of affine cubic surfaces

$$x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D, \tag{1.2}$$

where A, B, C, and D are complex parameters. This dynamical system appears in several different mathematical areas, like the monodromy of the sixth Painlevé differential equation, the geometry of hyperbolic threefolds, and the spectral properties of certain discrete Schrödinger operators. One of our main goals here is to classify parameters (A, B, C, D) for which $\Gamma_2^{\pm}$ preserves a holomorphic geometric structure, and to apply this classification to provide a galoisian proof of the irreducibility of the sixth Painlevé equation.

1.1. Character variety

Let $\mathbb{S}^2_4$ be the four punctured sphere. Its fundamental group is isomorphic to a free group of rank 3; if α, β, γ and δ are the four loops which are depicted on figure 1.1, then

$$\pi_1(\mathbb{S}^2_4) = \langle \alpha, \beta, \gamma, \delta \,|\, \alpha\beta\gamma\delta = 1 \rangle.$$

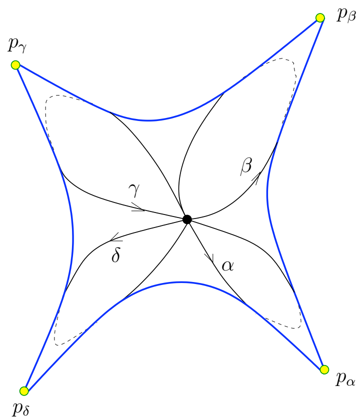

Figure 1.1. The four punctured sphere.

Let $\mathsf{Rep}(\mathbb{S}^2_4)$ be the set of representations of $\pi_1(\mathbb{S}^2_4)$ into $\mathsf{SL}(2, \mathbf{C})$. Such a representation ρ is uniquely determined by the 3 matrices $\rho(\alpha)$, $\rho(\beta)$, and $\rho(\gamma)$, so that $\mathsf{Rep}(\mathbb{S}^2_4)$ can be identified with the affine algebraic variety

$(\mathsf{SL}(2,\mathbf{C}))^3$. Let us associate the 7 following traces to any element ρ of $\mathsf{Rep}(\mathbb{S}^2_4)$:

$$\begin{array}{ccccccc} a = \mathsf{tr}(\rho(\alpha)) & ; & b = \mathsf{tr}(\rho(\beta)) & ; & c = \mathsf{tr}(\rho(\gamma)) & ; & d = \mathsf{tr}(\rho(\delta)) \\ x = \mathsf{tr}(\rho(\alpha\beta)) & ; & y = \mathsf{tr}(\rho(\beta\gamma)) & ; & z = \mathsf{tr}(\rho(\gamma\alpha)). & & \end{array}$$

The polynomial map $\chi : \mathsf{Rep}(\mathbb{S}^2_4) \to \mathbf{C}^7$ defined by

$$\chi(\rho) = (a, b, c, d, x, y, z) \tag{1.3}$$

is invariant under conjugation, by which we mean that $\chi(\rho') = \chi(\rho)$ if ρ' is conjugate to ρ by an element of $\mathsf{SL}(2,\mathbf{C})$. Moreover,

(1) the algebra of polynomial functions on $\mathsf{Rep}(\mathbb{S}^2_4)$ which are invariant under conjugation is generated by the components of χ;

(2) the components of χ satisfy the quartic equation

$$x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D, \tag{1.4}$$

in which the variables A, B, C, and D are given by

$$\begin{gathered} A = ab + cd, \quad B = bc + ad, \quad C = ac + bd, \\ \text{and} \quad D = 4 - a^2 - b^2 - c^2 - d^2 - abcd. \end{gathered} \tag{1.5}$$

(3) the algebraic quotient $\mathsf{Rep}(\mathbb{S}^2_4)//\mathsf{SL}(2,\mathbf{C})$ of $\mathsf{Rep}(\mathbb{S}^2_4)$ by the action of $\mathsf{SL}(2,\mathbf{C})$ by conjugation is isomorphic to the six-dimensional quartic hypersurface of $\mathbf{C}^7$ defined by equation (1.4).

The affine algebraic variety $\mathsf{Rep}(\mathbb{S}^2_4)//\mathsf{SL}(2,\mathbf{C})$ will be denoted $\chi(\mathbb{S}^2_4)$ and called the *character variety of* $\mathbb{S}^2_4$. For each choice of four complex parameters A, B, C, and D, we will denote by $S_{(A,B,C,D)}$ (or S if there is no obvious possible confusion) the cubic surface of $\mathbf{C}^3$ defined by the equation (1.4). The family of these surfaces $S_{(A,B,C,D)}$ will be denoted Fam. Viewed as a family of cubic surfaces of $\mathbb{P}^3$, it is universal in the sense that Fam contains all smooth projective cubics of $\mathbb{P}^3$ up to linear tranformation.

Singular points of the surface $S_{(A,B,C,D)}$ arise from semi-stable points of $\mathsf{Rep}(\mathbb{S}^2_4)$, that is to say either from reducible representations, or from those representations for which one of the matrices $\rho(\alpha)$, $\rho(\beta)$, $\rho(\gamma)$ or $\rho(\delta)$ is in the center $\pm I$ of $\mathsf{SL}(2,\mathbf{C})$. The smooth part of the surface $S_{(A,B,C,D)}$ is equipped with the holomorphic volume form

$$\Omega = \frac{dx \wedge dy}{2z + xy - C} = \frac{dy \wedge dz}{2x + yz - A} = \frac{dz \wedge dx}{2y + zx - B} \tag{1.6}$$

After minimal resolution of the singular points by blowing-up, the 2-form extends as a global holomorphic volume form on the smooth surface (see [19]).

1.2. Automorphisms and modular groups

The extended mapping class group $\mathsf{Mod}^{\pm}(\mathbb{S}^2_4)$ is the group of isotopy classes of self-homeomorphisms of the four punctured sphere $\mathbb{S}^2_4$; the usual mapping class group $\mathsf{Mod}^{+}(\mathbb{S}^2_4)$ is the index 2 subgroup consisting only in orientation preserving homeomorphisms. Given such a homeomorphism h, one can check that the induced action on the character variety

$$\begin{array}{ccc} \chi(\mathbb{S}^2_4) & \rightarrow & \chi(\mathbb{S}^2_4) \\ \chi(\rho) & \mapsto & \chi(\rho\circ h^{-1}) \end{array} \tag{1.7}$$

is well-defined and depends only on the isotopy class of h. By the way, we get a polynomial action induced by the representation

$$\mathsf{Mod}^{\pm}(\mathbb{S}^2_4) \quad \rightarrow \quad \mathsf{Aut}[\chi(\mathbb{S}^2_4)] \tag{1.8}$$

of the mapping class group on the character variety $\chi(\mathbb{S}^2_4)$. This action is not faithfull: it factors through an index 4 subgroup isomorphic to $\mathsf{PGL}(2,\mathbf{Z})$, the stabilizer of p_δ (see section 2.2).

Let $\Gamma_2^{\pm}$ be the subgroup of $\mathsf{PGL}(2,\mathbf{Z})$ whose elements coincide with the identity modulo 2. This group has index 6 and coincides with the stabilizer of the fixed points of the four punctures. Consequently, $\Gamma_2^{\pm}$ acts polynomially on $\chi(\mathbb{S}^2_4)$ and preserves the fibers of the projection

$$(a,b,c,d,x,y,z)\mapsto(a,b,c,d).$$

The group $\Gamma_2^{\pm}$ is the free product of 3 involutions, s_x, s_y, and s_z, acting on each member $S_{(A,B,C,D)}$ as follows.

$$s_x=\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}:\begin{cases} x & \mapsto & -x-yz+A\\ y & \mapsto & y\\ z & \mapsto & z\end{cases} \tag{1.9}$$

$$s_y=\begin{pmatrix}1 & 0\\ 2 & -1\end{pmatrix}:\begin{cases} x & \mapsto & x\\ y & \mapsto & -y-xz+B\\ z & \mapsto & z\end{cases} \tag{1.10}$$

$$s_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}:\begin{cases} x & \mapsto & x\\ y & \mapsto & y\\ z & \mapsto & -z-xy+C\end{cases} \tag{1.11}$$

The following result is essentially due to Èl'-Huti (see [12], and §3.1).

THEOREM A. — *For any choice of the parameters A, B, C, and D, the morphism*

$$\Gamma_2^{\pm} \to \mathsf{Aut}[S_{(A,B,C,D)}]$$

is injective and the index of its image is bounded by 24. *For a generic choice of the parameters, this morphism is an isomorphism.*

As a consequence of this result, it suffices to understand the action of $\Gamma_2^{\pm}$ on the surfaces $S_{(A,B,C,D)}$ in order to get a full understanding of the action of $\mathsf{Mod}^{\pm}(\mathbb{S}_4^2)$ on $\chi(\mathbb{S}_4^2)$.

The volume form Ω defined in (1.6) is almost invariant under the action of $\mathsf{Aut}[S_{(A,B,C,D)}]$, by which we mean that $f^*\Omega = \pm\Omega$ for any automorphism f. The action of the standard modular group $\Gamma_2 = \Gamma_2^{\pm} \cap \mathsf{PSL}(2, \mathbf{Z})$ is volume preserving.

1.3. Painlevé VI equation

The dynamics of $\Gamma_2^{\pm}$ on the varieties $S_{(A,B,C,D)}$ is also related to the monodromy of a famous ordinary differential equation. The sixth Painlevé equation $P_{VI} = P_{VI}(\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta)$ is the second order non linear ordinary differential equation

$$P_{VI}\begin{cases} \frac{d^2q}{dt^2} & = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right) \\ & \quad + \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left(\frac{(\theta_\delta-1)^2}{2} - \frac{\theta_\alpha^2}{2}\frac{t}{q^2} + \frac{\theta_\beta^2}{2}\frac{t-1}{(q-1)^2} + \frac{1-\theta_\gamma^2}{2}\frac{t(t-1)}{(q-t)^2}\right). \end{cases}$$

the coefficients of which depend on complex parameters

$$\theta = (\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta).$$

The main property of this equation is the absence of movable singular points, the so-called Painlevé property: all essential singularities of all solutions $q(t)$ of the equation only appear when $t \in \{0, 1, \infty\}$; in other words, any solution $q(t)$ extends analytically as a meromorphic function on the universal cover of $\mathbb{P}^1(\mathbf{C}) \setminus \{0, 1, \infty\}$.

Another important property, expected by Painlevé himself, is the irreducibility. Roughly speaking, the general solution is more transcendental than solutions of linear, or first order non linear, ordinary differential equations with rational coefficients. Painlevé proved that any irreducible second order polynomial differential equation without movable singular point falls after reduction into the 4-parameters family P_{VI} or one of its degenerations $P_V, \ldots, P_I$. The fact that Painlevé equations are indeed irreducible was actually proved by Nishioka and Umemura for P_I (see [28, 36]) and by Watanabe in [37] for P_{VI}. Another notion of irreducibility, related with transcendence of first integrals, was developped by Malgrange and Casale in [25, 8] and then applied to the first of Painlevé equation (see §6 for more details).

A third important property, discovered by R. Fuchs, is that solutions of P_{VI} parametrize isomonodromic deformations of rank 2 meromorphic connections over the Riemann sphere having simple poles at $\{0, t, 1, \infty\}$, with respective set of local exponents $(\pm\frac{\theta_\alpha}{2}, \pm\frac{\theta_\beta}{2}, \pm\frac{\theta_\gamma}{2}, \pm\frac{\theta_\delta}{2})$. From this point of view, the good space of initial conditions at, say, t_0, is the moduli space $\mathcal{M}_{t_0}(\theta)$ of those connections for $t = t_0$ (see [19])[(1)]; it turns to be a convenient semi-compactification of the naive space of initial conditions $\mathbf{C}^2 = \{(q(t_0), q'(t_0))\}$ (compare [30]). The Riemann-Hilbert correspondance

$$RH \ : \ \mathcal{M}_{t_0}(\theta) \to S_{(A,B,C,D)}$$

which to a connection associates its monodromy representation, provides an analytic diffeomorphism of the moduli space of connections onto the cubic surface with parameters

$$a = 2\cos(\pi\theta_\alpha), \ b = 2\cos(\pi\theta_\beta), \ c = 2\cos(\pi\theta_\gamma), \ d = 2\cos(\pi\theta_\delta) \tag{1.12}$$

as far as $S_{(A,B,C,D)}$ is smooth. In the singular case, the analytic map RH is a minimal resolution of singularities. Along irreducible components of the exceptional divisor, P_{VI} equation restricts to a Riccati equation: this is the locus of Riccati-type solutions. The (non linear) monodromy of P_{VI}, obtained after analytic continuation around 0 and 1 of local P_{VI} solutions at $t = t_0$, induces a representation

$$\pi_1(\mathbb{P}^1(\mathbf{C}) \setminus \{0, 1, \infty\}, t_0) \to \mathsf{Aut}[S_{(A,B,C,D)}]$$

whose image coincides with the action of $\Gamma_2 \subset \mathsf{PSL}(2, \mathbf{Z})$ (see [11, 19]).

1.4. Symmetries, Okamoto correspondance and coverings

There is an order 24 group of symmetries acting on the character variety $\chi(\mathbb{S}^2_4)$, preserving our family Fam, that can be used to relate dynamical properties of $\Gamma_2^\pm$ on different surfaces $S_{(A,B,C,D)}$. This group comes from the combination of two natural actions: one of them is the symmetric group Sym_4, acting by homeomorphisms on $\mathbb{S}^2_4$, freely permuting the punctures; the other one is the group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ acting by even sign changes

(1) In the resonant case, i.e. when one $\theta_\omega \in \mathbf{Z}$, $\mathcal{M}_{t_0}(\theta)$ is the moduli space of parabolic connections.

on the generators (see section 2.4), relating those $\mathsf{SL}(2,\mathbf{C})$-representations that induce the same $\mathsf{PSL}(2,\mathbf{C})$-representation. This yields an order 192 group acting on variables (a,b,c,d,x,y,z) (compare [2]) but the induced action on variables (A,B,C,D,x,y,z) has a kernel Q of order 8.

In fact, the map

$$\Pi \; : \; \begin{array}{ccc} \mathbf{C}^4 & \to & \mathbf{C}^4 \; ; \\ (a,b,c,d) & \mapsto & (A,B,C,D) \end{array} \tag{1.13}$$

defined by (1.5) is a non Galois ramified cover of degree 24 and Q is the Galois group. This finite correspondance arise from well known Okamoto symmetries between Painlevé VI equations (compare [18]). In this paper, we pay a particular attention to the study of fibers $\Pi^{-1}(A,B,C,D)$. The reason is that certain point $m \in S_{(A,B,C,D)}$ give rise to representations of very different nature depending on the choice of (a,b,c,d) in the fiber. The image of the representation can be reducible or irreducible, finite or infinite, discrete or dense, in $\mathsf{SU}(2)$ or $\mathsf{SL}(2,\mathbf{R})$, depending on this choice. For instance, when (a,b,c,d) are real parameters, then so are (A,B,C,D) and the real part $S_{(A,B,C,D)}(\mathbf{R})$ of the surface has at most one bounded component and at most four unbounded ones (see [2]); unbounded components always stand for $\mathsf{SL}(2,\mathbf{R})$-representations. For bounded components, we prove

Theorem B. — *Let (A,B,C,D) be real parameters. If the smooth part of $S_{(A,B,C,D)}(\mathbf{R})$ has a bounded component, then all parameters (a,b,c,d) in the fiber $\Pi^{-1}(A,B,C,D)$ are real. Moreover, the bounded component stands for either $\mathsf{SU}(2)$ or $\mathsf{SL}(2,\mathbf{R})$-representations depending on the choice of (a,b,c,d) in the fiber: both cases occur for a given surface $S_{(A,B,C,D)}(\mathbf{R})$.*

Finally, there are also some natural endomorphisms between certain surfaces $S_{(A,B,C,D)}$ semi-conjugating the $\Gamma_2^{\pm}$-action. Precisely, one can construct a two-fold ramified cover from the cubic surface with parameter $(a,b,c,d) = (a,b,0,0)$ onto the cubic surface with parameter (a,a,b,b): this is done by considering the two-fold cover of the sphere ramified over p_γ and p_δ and pulling-back representations having order 4 "local monodromy" around those two points. We skip this construction in this paper (see [7], Appendix B). Combining this with the "Okamoto correspondance" above, one can relate dynamics on cubic surfaces associated to

$$(a,b,c,d) = (d,d,d,d) \quad \text{and} \quad (2,2,2,4d-d^2-2);$$

this has been used in [11].

In the same spirit, for special parameters $A = B = C = 0$, the one parameter family $S_{(0,0,0,D)}$ also stands for the character variety of the once punctured torus (see [14]). This can be naturally deduced from the elliptic cover of the four punctured sphere (see [7], section 2.4).

1.5. The Cayley cubic

One very specific choice of the parameters will play a central role in this paper, namely $(A,B,C,D) = (0,0,0,4)$. The surface $S_{(0,0,0,4)}$ is the unique surface in our family having four singularities, the maximal possible number of isolated singularities for a cubic surface: we shall call it the Cayley cubic. From the point of view of character varieties, this surface appears in the very special case $(a,b,c,d) = (0,0,0,0)$ consisting only of solvable representations, most of them being dihedral. They lift to diagonal representations on the torus cover $\mathbb{T}^2 \to \mathbb{S}^2$, that can be viewed as representations $\pi_1(\mathbb{T}^2) \to \mathbf{C}^*$. This yields a natural degree 2 orbifold cover

$$\mathbf{C}^* \times \mathbf{C}^* \to S_{(0,0,0,4)}\ ;\ (u,v) \mapsto \left(-u - \frac{1}{u}, -v - \frac{1}{v}, -uv - \frac{1}{uv}\right) \tag{1.14}$$

that semi-conjugates the action of $\mathsf{PGL}(2,\mathbf{Z})$ on the character surface $S_{(0,0,0,4)}$ to the monomial action of $\mathsf{GL}(2,\mathbf{Z})$ on $\mathbf{C}^* \times \mathbf{C}^*$ defined by

$$M\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^{m_{11}}v^{m_{12}} \\ u^{m_{21}}v^{m_{22}} \end{pmatrix}, \tag{1.15}$$

for any element $M = (m_{i,j})$ of $\mathsf{GL}(2,\mathbf{Z})$. On the universal cover $\mathbf{C} \times \mathbf{C} \to \mathbf{C}^* \times \mathbf{C}^*$, the lifted dynamics is the standart action of the affine group $\mathsf{GL}(2,\mathbf{Z}) \ltimes \mathbf{Z}^2$ on the complex plane $\mathbf{C}^2$.

From the Painlevé point of view, the Cayley cubic $S_{(0,0,0,4)}$ corresponds to the Picard parameter $(\theta_1,\theta_2,\theta_3,\theta_4) = (0,0,0,1)$. The singular foliation defined by the corresponding Painlevé equation $P_{VI}(0,0,0,1)$ is transversely affine (see [8]); as was shown by Picard himself, it admits explicit first integrals by means of elliptic functions (see §6). Moreover, this specific equation has countably many agebraic solutions, that are given by finite order points on the Legendre family of elliptic curves (see §6). They correspond to those finite orbits given by those (u,v) whose entries are roots of the unity.

The Cayley cubic has also the "maximal number of automorphisms": the order 24 symmetric group (see §1.4) stabilizes the Cayley cubic, so that the maximal index of theorem A is obtained in this case.

1.6. Dynamics

Dynamics on character varieties have been popularized by W. Goldman and studied by many authors (see [15] and references therein). For example, in [13], Goldman proves that the mapping class group of a compact orientable surface M acts ergodically on the character variety corresponding to representations of $\pi_1(M)$ into the compact Lie group $\mathsf{SU}(2)$ (see §1.7 and [14] for similar results in the case of the four-punctured sphere). On the other hand, one can use properties such as discreteness, finiteness, or the presence of parabolic elements in the image of representations $\rho : \pi_1(\mathbb{S}^2_4) \to \mathsf{SL}(2, \mathbf{C})$ to construct invariant sets through the action of the mapping class group $\mathsf{Mod}^{\pm}(\mathbb{S}^2_4)$ (see [14, 27]).

One can also study the action of automorphisms on a fixed surface $S_{(A,B,C,D)}$. They can be classified into three types: elliptic, parabolic and hyperbolic. This classification is compatible with the description of mapping classes, Dehn twists corresponding to parabolic transformations, and pseudo-Anosov mappings to hyperbolic automorphisms. The most striking result in that direction is summarized in the following theorem (see [23, 7]).

Theorem. — *Let A, B, C, and D be four complex numbers. Let M be an element of $\Gamma_2^{\pm}$, and f_M be the automorphism of $S_{(A,B,C,D)}$ which is determined by M. The topological entropy of $f_M : S_{(A,B,C,D)}(\mathbf{C}) \to S_{(A,B,C,D)}(\mathbf{C})$ is equal to the logarithm of the spectral radius of M.*

This statement is obtained in [7] by a deformation argument: the topological entropy does not depend on the parameters (A, B, C, D); it suffices to compute it in the case of the Cayley cubic by looking at the birational action at infinity. This improves the previous work [23] where the authors provided an algorithm to compute the topological entropy for smooth cubics.

1.7. Bounded orbits

Section 4 is devoted to the study of parabolic elements (or Dehn twists), and bounded orbits of $\Gamma_2^{\pm}$. For instance, given a representation $\rho : \mathbb{S}^2_4 \to \mathsf{SU}(2) \subset \mathsf{SL}(2, \mathbf{C})$, the $\Gamma_2^{\pm}$-orbit of the correponding point $\chi(\rho)$ will be bounded, contained in the cube $[-2, 2]^3$. A converse is given by the following result.

Theorem C. — *Let m be a point of $S_{(A,B,C,D)}$ with a bounded infinite $\Gamma_2^{\pm}$-orbit. Then, the parameters (A, B, C, D) are real numbers and the orbit* *is contained and dense in the unique bounded connected component of the smooth part of* $S_{(A,B,C,D)}(\mathbf{R})$.

Following Theorem B, infinite bounded orbit always arise from $\mathsf{SU}(2)$-representations, after conveniently choosing parameters (a, b, c, d). Theorem C should be compared with results of Goldman [13], Previte and Xia [33], concerning the dynamics on the character variety for $\mathsf{SU}(2)$-representations(2). We note that an infinite bounded orbit may also correspond to $\mathsf{SL}(2, \mathbf{R})$-representations for an alternate choice of parameters (a, b, c, d) (see Theorem B).

This theorem stresses the particular role played by the real case, when all the parameters A, B, C, and D are real numbers; in that case, $\Gamma_2^{\pm}$ preserves the real part of the surface and we have two different, but closely related, dynamical systems: the action on the complex surface $S_{(A,B,C,D)}(\mathbf{C})$ and the action on the real surface $S_{(A,B,C,D)}(\mathbf{R})$. The link between those two dynamical systems will be studied in [6].

The classification of finite orbits has been closed only recently by O. Lisovyy and Yu. Tykhyy in [24]: the list of known orbits given in [4] is actually complete. Fixed points of $\Gamma_2^{\pm}$ are precisely the singular points of $S_{(A,B,C,D)}$ (see [19]); they correspond to one-parameter Riccati solutions for P_{VI} equation. Finite orbits of length $\geqslant 2$ correspond to algebraic solutions of P_{VI} equation (see also [22]). Up to symmetries, there are 3 continuous families of finite orbits, of respective length 2, 3 and 4; the other finite orbits have length > 4 and the corresponding (A, B, C, D, x, y, z)-entries are real algebraic numbers (see [7]); they are rigid. Apart from the infinite discrete family of finite orbits on the Cayley cubic, there are 45 sporadic finite orbits up to symmetries, the larger one having length 72 (see [4, 24]). All but one correspond to representations into finite subgroups of $\mathsf{SU}(2)$ (maybe conveniently choosing parameter (a, b, c, d)).

For instance, when $A = B = 0$, a length 2 orbit is given by $(x, y, z) = (0, 0, z)$ where z runs over the roots of $z^2 = Cz + D$. The corresponding P_{VI}-solution is $q(t) = 1 + \sqrt{1-t}$. In this case, the trace parameters are either of the form $(a, 0, c, 0)$, or of the form $(a, b, a, -b)$ (see [7], example 5.5). The representation is dihedral in the first case; in general, it is Zariski dense in the later case(3).

(2) After reading the first version [7] of our paper, professor Iwasaki informed us that theorem C was already announced in [21].

(3) This representation was already considered in [32]: for convenient choice of parameters a and b, the image of the representation is a dense subgroup of $\mathsf{SU}(2)$.

1.8. Dynamics, affine structures, and the irreducibility of P_{VI}

The main result that we shall prove concerns the classification of parameters (A, B, C, D) for which $S_{(A,B,C,D)}$ admits a $\Gamma_2^{\pm}$-invariant holomorphic geometric structure.

THEOREM D. — *The group $\Gamma_2^{\pm}$ does not preserve any holomorphic curve of finite type, any (singular) holomorphic foliation, or any (singular) holomorphic web. The group $\Gamma_2^{\pm}$ does not preserve any meromorphic affine structure, except in the case of the Cayley cubic, i.e. when $(A, B, C, D) = (0, 0, 0, 4)$, or equivalently when*

$$(a, b, c, d) = (0, 0, 0, 0) \text{ or } (2, 2, 2, -2),$$

up to multiplication by -1 and permutation of the parameters.

Following [9], the same strategy shows that the Galois groupoid of Painlevé VI equation is the whole symplectic pseudo-group except in the Cayley case (see section 6), and we get

THEOREM E. — *The sixth Painlevé equation is irreducible in the sense of Malgrange and Casale except when $(A, B, C, D) = (0, 0, 0, 4)$, i.e. except in one of the following cases:*

- $\theta_\omega \in \frac{1}{2} + \mathbf{Z}$, $\forall\, \omega = \alpha, \beta, \gamma, \delta$,
- $\theta_\omega \in \mathbf{Z}$, $\forall\, \omega = \alpha, \beta, \gamma, \delta$, *and* $\sum_\omega \theta_\omega$ *is even.*

Following [10], Malgrange-Casale irreducibility also implies Nishioka-Umemura irreducibility, so that theorem 1.8 indeed provides a galoisian proof of the irreducibility in the spirit of Drach and Painlevé.

1.9. Aknowledgement

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2. The family of surfaces

As explained in 1.1, we shall consider the family Fam of complex affine surfaces which are defined by the following type of cubic equations

$$x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D,$$

in which A, B, C, and D are four complex parameters. Each choice of (A, B, C, D) gives rise to one surface S in our family; if necessary, S will also be denoted $S_{(A,B,C,D)}$. When the parameters are real numbers, $S(\mathbf{R})$ will denote the real part of S.

This section contains preliminary results on the geometry of the surfaces $S_{(A,B,C,D)}$, and the automorphisms of these surfaces. Most of these results are well known to algebraic geometers and specialists of Painlevé VI equations.

2.1. The Cayley cubic

In 1869, Cayley proved that, up to projective transformations, there is a unique cubic surface in $\mathbb{P}^3(\mathbf{C})$ with four isolated singularities. One of the nicest models of the Cayley cubic is the surface $S_{(0,0,0,4)}$, whose equation is

$$x^2 + y^2 + z^2 + xyz = 4.$$

The four singular points of S_C are rational nodes located at

$$(-2, -2, -2),\ (-2, 2, 2),\ (2, -2, 2) \quad \text{and} \quad (2, 2, -2).$$

This specific member of our family of surfaces will be called *the Cayley cubic* and denoted S_C. This is justified by the following corollary of Cayley's Theorem (see [7], Appendix A).

Theorem 2.1 (Cayley). — *If S is a member of the family* Fam *with four singular points, then S coincides with the Cayley cubic S_C.*

The Cayley cubic is isomorphic to the quotient of $\mathbf{C}^* \times \mathbf{C}^*$ by the involution $\eta(u,v) = (u^{-1}, v^{-1})$. The map

$$\pi_C(u,v) = \left(-u - \frac{1}{u}, -v - \frac{1}{v}, -uv - \frac{1}{uv}\right)$$

gives an explicit isomorphism between $(\mathbf{C}^* \times \mathbf{C}^*)/\eta$ and S_C. The four fixed points

$$(1,1), \quad (1,-1), \quad (-1,1) \quad \text{and} \quad (-1,-1)$$

of η respectively correspond to the singular points of S_C above.

The real surface $S_C(\mathbf{R})$ contains the four singularities of S_C, and the smooth locus $S_C(\mathbf{R}) \setminus \mathsf{Sing}(S_C)$ is made of five components : a bounded one, the closure of which coincides with the image of $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbf{C}^* \times \mathbf{C}^*$ by π_C, and four unbounded ones, corresponding to images of $\mathbf{R}^+ \times \mathbf{R}^+$, $\mathbf{R}^+ \times \mathbf{R}^-$, $\mathbf{R}^- \times \mathbf{R}^+$, and $\mathbf{R}^- \times \mathbf{R}^-$.

As explained in section 1.5, the group $\mathsf{GL}(2,\mathbf{Z})$ acts on $\mathbf{C}^* \times \mathbf{C}^*$ by monomial transformations, and this action commutes with the involution η, permuting its fixed points. As a consequence, $\mathsf{PGL}(2,\mathbf{Z})$ acts on the quotient S_C. Precisely, the generators

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of $\mathsf{PGL}(2,\mathbf{Z})$ respectively send the triple (x,y,z) to

$$(x, -z-xy, y), \quad (z, y, -x-yz) \quad \text{and} \quad (x, y, -z-xy).$$

As we shall see, the induced action of $\mathsf{PGL}(2,\mathbf{Z})$ on S_C coincides with the action of the extended mapping class group of $\mathbb{S}^2_4$ considered in §1.2.

The group $\mathsf{PGL}(2,\mathbf{Z})$ preserves the real part of S_C ; for example, the product $\mathbf{C}^* \times \mathbf{C}^*$ retracts by deformation on the real 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, and the monomial action of $\mathsf{GL}(2,\mathbf{Z})$ preserves this torus (it is the standard one under the parametrization $(s,t) \mapsto (e^{2i\pi s}, e^{2i\pi t})$).

2.2. Mapping class group action

The group of automorphisms $\mathsf{Aut}(\pi_1(\mathbb{S}^2_4))$ acts on $\mathsf{Rep}(\mathbb{S}^2_4)$ by composition: $(\Phi, \rho) \mapsto \rho \circ \Phi^{-1}$. Since inner automorphisms act trivially on $\chi(\mathbb{S}^2_4)$, we get a morphism from the group of outer automorphisms $\mathsf{Out}(\pi_1(\mathbb{S}^2_4))$ into the group of polynomial diffeomorphisms of $\chi(\mathbb{S}^2_4)$:

$$\begin{cases} \mathsf{Out}(\pi_1(\mathbb{S}^2_4)) & \to & \mathsf{Aut}[\chi(\mathbb{S}^2_4)] \\ \Phi & \mapsto & f_\Phi \end{cases} \tag{2.1}$$

such that $f_\Phi(\chi(\rho)) = \chi(\rho\circ\Phi^{-1})$ for any representation ρ.

The extended mapping class group $\mathsf{Mod}^{\pm}(\mathbb{S}^2_4)$ embeds in the group of outer automorphisms of $\pi_1(\mathbb{S}^2_4)$ in the following way. Fix a base point $p_0 \in \mathbb{S}^2_4$. In any isotopy class, one can find a homeomorphism h fixing p_0 and thus inducing an automorphism of the fundamental group

$$h_* : \pi_1(\mathbb{S}^2_4, p_0) \to \pi_1(\mathbb{S}^2_4, p_0)\ ;\ \gamma \mapsto h\circ\gamma.$$

The class of h_* modulo inner automorphisms does not depend on the choice of the representative h in the homotopy class and we get a morphism

$$\mathsf{Mod}^{\pm}(\mathbb{S}^2_4) \to \mathsf{Out}(\pi_1(\mathbb{S}^2_4)) \tag{2.2}$$

which turns out to be injective. Its image coincides with the subgroup of those outer automorphisms that preserve the peripheral structure of the fundamental group (see [20]).

Now, the action of $\mathsf{Out}(\pi_1(\mathbb{S}^2_4))$ on $\chi(\mathbb{S}^2_4)$ gives rise to a morphism

$$\begin{cases} \mathsf{Mod}^{\pm}(\mathbb{S}^2_4) & \to & \mathsf{Aut}[\chi(\mathbb{S}^2_4)] \\ \qquad [h] & \mapsto & \{\chi(\rho)\mapsto\chi(\rho\circ h^{-1})\} \end{cases} \tag{2.3}$$

into the group of polynomial diffeomorphisms of $\chi(\mathbb{S}^2_4)$. (here, we use that $\rho\circ(h_*)^{-1} = \rho\circ h^* = \rho\circ h^{-1}$). Our goal in this section is to give explicit formulae for this action of $\mathsf{Mod}^{\pm}(\mathbb{S}^2_4)$ on $\chi(\mathbb{S}^2_4)$, and to describe the subgroup of $\mathsf{Mod}^{+}(\mathbb{S}^2_4)$ which stabilizes each surface $S_{(A,B,C,D)}$.

Consider the two-fold ramified cover

$$\pi_T : \mathbb{T}^2 = \mathbf{R}^2/\mathbf{Z}^2 \to \mathbb{S}^2 \tag{2.4}$$

with Galois involution $\sigma : (x,y)\mapsto(-x,-y)$ sending its ramification points

$$(0,0),\quad (1/2,0),\quad (0,1/2)\quad \text{and}\quad (1/2,1/2)$$

respectively to the four punctures

$$p_\alpha,\quad p_\beta,\quad p_\gamma\quad \text{and}\quad p_\delta$$

(see figure 3 in [7]). The mapping class group of the torus, and also of the once punctured torus $\mathbb{T}^2_1 = \mathbb{T}^2\setminus\{(0,0)\}$, is isomorphic to $\mathsf{GL}(2,\mathbf{Z})$. This group acts by linear homeomorphisms on the torus, fixing $(0,0)$, and permuting the other three ramification points of π_T. This action provides a section of the projection $\mathsf{Diff}(\mathbb{T}^2)\to\mathsf{Mod}^{\pm}(\mathbb{T}^2)$. Since this action commutes with the involution σ (which generates the center of $\mathsf{GL}(2,\mathbf{Z})$), we get a morphism from $\mathsf{PGL}(2,\mathbf{Z})$ to $\mathsf{Mod}^{\pm}(\mathbb{S}^2_4)$. This morphism is one to one and its image is contained in the stabilizer of p_δ in $\mathsf{Mod}^{\pm}(\mathbb{S}^2_4)$.

The subset $H\subset\mathbb{T}^2$ of ramification points of π coincides with the 2-torsion subgroup of $(\mathbb{T}^2,+)$; H acts by translation on $\mathbb{T}^2$ and commutes with the involution σ as well. This provides an isomorphism (see section 4.4 in [3])

$$\mathsf{PGL}(2,\mathbf{Z}) \ltimes H \to \mathsf{Mod}^{\pm}(\mathbb{S}^2_4). \tag{2.5}$$

The proof of the following Lemma is straightforward (see [7], pages 15-16 for details).

Lemma 2.2. — *The subgroup of* $\mathsf{Aut}(\chi(\mathbb{S}^2_4))$ *obtained by the action of the subgroup* $\mathsf{PGL}(2,\mathbf{Z})$ *of* $\mathsf{Mod}^{\pm}(\mathbb{S}^2_4)$ *is generated by the three polynomial automorphisms* B_1, B_2 *and* T_3 *of equations 2.8, 2.9, and 2.10 below. The 4-order translation group* H *acts trivially on parameters* (A,B,C,D,x,y,z), *permuting parameters* (a,b,c,d) *as follows*

$$P_1 = (1/2,0) : (a,b,c,d) \mapsto (d,c,b,a) \tag{2.6}$$

$$P_2 = (0,1/2) : (a,b,c,d) \mapsto (b,a,d,c) \tag{2.7}$$

The $\mathsf{PSL}(2,\mathbf{Z})$-action on $\chi(\mathbb{S}^2_4)$ is given by the generators B_1 and B_2:

$$B_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} : \begin{cases} a & \mapsto & b \\ b & \mapsto & a \\ c & \mapsto & c \\ d & \mapsto & d \end{cases} \quad \text{and} \quad \begin{cases} x & \mapsto & x \\ y & \mapsto & -z-xy+ac+bd \\ z & \mapsto & y \end{cases} \tag{2.8}$$

$$B_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \begin{cases} a & \mapsto & a \\ b & \mapsto & c \\ c & \mapsto & b \\ d & \mapsto & d \end{cases} \quad \text{and} \quad \begin{cases} x & \mapsto & z \\ y & \mapsto & y \\ z & \mapsto & -x-yz+ab+cd \end{cases} \tag{2.9}$$

In order to generate $\mathsf{PGL}(2,\mathbf{Z})$, we have to add the involution:

$$T_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \begin{cases} a & \mapsto & c \\ b & \mapsto & b \\ c & \mapsto & a \\ d & \mapsto & d \end{cases} \quad \text{and} \quad \begin{cases} x & \mapsto & y \\ y & \mapsto & x \\ z & \mapsto & z \end{cases} \tag{2.10}$$

Remark 2.3. — The formulae 2.8, 2.9, and 2.10 for B_1, B_2 and T_3 specialize to the formulae of section 2.1 when $(A,B,C,D)=(0,0,0,4)$.

Remark 2.4. — The Artin Braid Group $\mathcal{B}_3 = \langle \beta_1,\beta_2 \mid \beta_1\beta_2\beta_1 = \beta_2\beta_1\beta_2 \rangle$ is isomorphic to the group of isotopy classes of the thrice punctured disk fixing its boundary. There is therefore a morphism from $\mathcal{B}$ into the subgroup of $\mathsf{Mod}^+(\mathbb{S}^2_4)$ that stabilizes p_δ. This morphism gives rise to the following well known exact sequence

$$I \to \langle(\beta_1\beta_2)^3\rangle \to \mathcal{B}_3 \to \mathsf{PSL}(2,\mathbf{Z}) \to 1,$$

where generators β_1 and β_2 are respectively sent to B_1 and B_2, and the group $\langle(\beta_1\beta_2)^3\rangle$ coincides with the center of $\mathcal{B}_3$. In particular, the action of $\mathcal{B}_3$ on $\chi(\mathbb{S}^2_4)$ coincides with the action of $\mathsf{PSL}(2,\mathbf{Z})$. We note that $\mathsf{PSL}(2,\mathbf{Z})$ is the free product of the trivolution B_1B_2 and the involution $B_1B_2B_1$. In $\mathsf{PGL}(2,\mathbf{Z})$, we also have relations

$$T_3^2 = I, T_3B_1T_3 = B_2^{-1} \quad \text{and} \quad T_3B_2T_3 = B_1^{-1}.$$

2.3. The modular groups $\Gamma_2^\pm$ and Γ_2

Since the action of $M \in \mathsf{GL}(2,\mathbf{Z})$ on the set H of points of order 2 depends only on the equivalence class of M modulo 2, we get an exact sequence

$$I \to \Gamma_2^\pm \to \mathsf{PGL}(2,\mathbf{Z}) \ltimes H \to \mathsf{Sym}_4 \to 1$$

where $\Gamma_2^\pm \subset \mathsf{PGL}(2,\mathbf{Z})$ is the subgroup defined by those matrices $M \equiv I$ modulo 2. This group acts on the character variety, and since it preserves the punctures, it fixes a, b, c, and d. The group $\Gamma_2^\pm$ is the free product of the 3 involutions, s_x, s_y, and s_z, given in §1.2: (1.9), (1.10) and (1.11) respectively. We note that $s_x = B_2B_1^{-1}B_2^{-1}T_3$, $s_y = B_2B_1B_2^{-1}T_3$ and $s_z = B_2B_1B_2T_3$. The standard modular group $\Gamma_2 \subset \mathsf{PSL}(2,\mathbf{Z})$ is generated by

$$\begin{cases} g_x = s_zs_y = B_1^2 & = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\ g_y = s_xs_z = B_2^2 & = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ g_z = s_ys_x = B_1^{-2}B_2^{-2} & = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \end{cases}$$

(we have $g_zg_yg_x = I$); as we shall see, this corresponds to Painlevé VI monodromy (see [19] and section 6). The following proposition is now a direct consequence of lemma 2.2.

Proposition 2.5. — *Let $\mathsf{Mod}_0^\pm(\mathbb{S}^2_4)$ (resp. $\mathsf{Mod}_0^+(\mathbb{S}^2_4)$) be the subgroup of $\mathsf{Mod}^\pm(\mathbb{S}^2_4)$ (resp. $\mathsf{Mod}^+(\mathbb{S}^2_4)$) which stabilizes the four punctures of $\mathbb{S}^2_4$.*

This group coincides with the stabilizer of the projection $\pi : \chi(\mathbb{S}^2_4) \to \mathbf{C}^4$ which is defined by

$$\pi(a, b, c, d, x, y, z) = (a, b, c, d).$$

Its image in $\mathsf{Aut}(\chi(\mathbb{S}^2_4))$ *coincides with the image of* $\Gamma_2^{\pm}$ *(resp.* Γ_2*) and is therefore generated by the three involutions* s_x, s_y *and* s_z *(resp. the three automorphisms* g_x, g_y, g_z*).*

As we shall see in sections 3.1 and 3.2, this group is of finite index in $\mathsf{Aut}(\chi(\mathbb{S}^2_4))$.

Remark 2.6. — Let us consider the exact sequence

$$I \to \Gamma_2^{\pm} \to \mathsf{PGL}(2, \mathbf{Z}) \to \mathsf{Sym}_3 \to 1,$$

where $\mathsf{Sym}_3 \subset \mathsf{Sym}_4$ is the stabilizer of p_δ, or equivalently of d, or D. A splitting $\mathsf{Sym}_3 \hookrightarrow \mathsf{PGL}(2, \mathbf{Z})$ is generated by the transpositions $T_1 = T_3B_1B_2$ and $T_2 = B_1B_2T_3$. They act as follows on the character variety.

$$T_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} : \begin{cases} a & \mapsto b \\ b & \mapsto a \\ c & \mapsto c \\ d & \mapsto d \end{cases} \quad \text{and} \quad \begin{cases} x & \mapsto \&x \\ y & \mapsto \&z \\ z & \mapsto \&y \end{cases}$$

$$T_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} : \begin{cases} a & \mapsto a \\ b & \mapsto c \\ c & \mapsto b \\ d & \mapsto d \end{cases} \quad \text{and} \quad \begin{cases} x & \mapsto z \\ y & \mapsto y \\ z & \mapsto x \end{cases}$$

2.4. Symmetries

There are other symmetries between surfaces $S_{(A,B,C,D)}$ that do not arise from the action of the mapping class group. Indeed, given any 4-uple $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in \{\pm 1\}^4$ with $\prod_{i=1}^4 \epsilon_i = 1$, the ϵ-twist of a representation $\rho \in \mathsf{Rep}(\mathbb{S}^2_4)$ is the new representation $\otimes_\epsilon \rho$ generated by

$$\begin{cases} \tilde{\rho}(\alpha) & = \epsilon_1 \rho(\alpha) \\ \tilde{\rho}(\beta) & = \epsilon_2 \rho(\beta) \\ \tilde{\rho}(\gamma) & = \epsilon_3 \rho(\gamma) \\ \tilde{\rho}(\delta) & = \epsilon_4 \rho(\delta) \end{cases}$$

This provides an action of $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ on the character variety given by

$$\otimes_\epsilon : \begin{cases} a & \mapsto & \epsilon_1 a \\ b & \mapsto & \epsilon_2 c \\ c & \mapsto & \epsilon_3 b \\ d & \mapsto & \epsilon_4 d \end{cases} \quad \begin{cases} A & \mapsto & \epsilon_1\epsilon_2 A \\ B & \mapsto & \epsilon_2\epsilon_3 B \\ C & \mapsto & \epsilon_1\epsilon_3 C \\ D & \mapsto & D \end{cases} \quad \text{and} \quad \begin{cases} x & \mapsto & \epsilon_1\epsilon_2 x \\ y & \mapsto & \epsilon_2\epsilon_3 y \\ z & \mapsto & \epsilon_1\epsilon_3 z \end{cases}$$

The action on (A, B, C, D, x, y, z) is trivial iff $\epsilon = \pm(1, 1, 1, 1)$. The "Benedetto-Goldman symmetry group" of order 192 acting on (a, b, c, d, x, y, z) which is described in [2] (§3C) is precisely the group generated by ϵ-twists and the symmetric group $\mathsf{Sym}_4 = \langle T_1, T_2, P_1, P_2\rangle$. The subgroup Q acting trivially on (A, B, C, D, x, y, z) is of order 8 generated by

$$Q = \langle P_1, P_2, \otimes_{(-1,-1,-1,-1)}\rangle. \tag{2.11}$$

2.5. Okamoto correspondances

Many kinds of conjugacy classes of representations ρ with

$$\chi(\rho) = (a, b, c, d, x, y, z)$$

give rise to the same (A, B, C, D, x, y, z)-point ; in order to underline this phenomenon, we would like to understand the ramified cover

$$\Pi \ : \ \begin{cases} \mathbf{C}^4 & \to & \mathbf{C}^4 \\ (a, b, c, d) & \mapsto & (A, B, C, D) \end{cases}$$

defined by equation (1.5).

Lemma 2.7. — *The degree of the covering map* Π*, that is the number of points* (a, b, c, d) *giving rise to a given generic* (A, B, C, D)*-point, is* 24.

Proof. — We firstly assume $B \neq \pm C$ so that $a \neq \pm b$. Then, solving $B = bc + ad$ and $C = ac + bd$ in c and d yields

$$c = \frac{aC - bB}{a^2 - b^2} \quad \text{and} \quad d = \frac{aB - bC}{a^2 - b^2}.$$

Subsituting in $A = ab + cd$ and $D = 4 - a^2 - b^2 - c^2 - d^2 - abcd$ gives $\{P = Q = 0\}$ with

$$\begin{aligned} P &= -ab(a^2 - b^2)^2 + A(a^2 - b^2)^2 + (B^2 + C^2)ab - BC(a^2 + b^2) \\ \text{and } Q &= (a^2 + b^2)(a^2 - b^2)^2 + (D - 4)(a^2 - b^2)^2 \\ &\quad + (B^2 + C^2)(a^2 - a^2b^2 + b^2) + BCab(a^2 + b^2 - 4). \end{aligned}$$

These two polynomials have both degree 6 in (a, b) and the corresponding curves must intersect in 36 points. However, one easily check that they intersect along the line at infinity with multiplicity 4 at each of the two points $(a : b) = (1 : 1)$ and $(1 : -1)$; moreover, they also intersect along the forbidden lines $a = \pm b$ at $(a, b) = (0, 0)$ with multiplicity 4 as well, provided that $BC \neq 0$. As a consequence, the number of preimages of (A, B, C, D) is $36 - 4 - 4 - 4 = 24$ (counted with multiplicity). $\square$

Remark 2.8. — Π is not a Galois cover: the group of deck transformations is the order 8 group $Q = \langle P_1, P_2, \otimes_{(-1,-1,-1,-1)} \rangle$ (see §2.4).

To understand the previous remark, it is convenient to introduce the Painlevé VI parameters, which are related to (a, b, c, d) by the map

$$\begin{array}{ccc} \mathbf{C}^4 & \to & \mathbf{C}^4 \\ (\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta) & \mapsto & (a, b, c, d) \end{array} \quad \text{with} \quad \begin{cases} a &= 2\cos(\pi\theta_\alpha) \\ b &= 2\cos(\pi\theta_\beta) \\ c &= 2\cos(\pi\theta_\gamma) \\ d &= 2\cos(\pi\theta_\delta) \end{cases}$$

The composite map $(\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta) \mapsto (A, B, C, D)$ has been studied in [18]: It is an infinite Galois ramified cover whose deck transformations coincide with the group G of so called Okamoto symmetries. Those symmetries are "birational transformations" of Painlevé VI equation; they have been computed directly on the equation by Okamoto in [31] (see [29] for a modern presentation). Let $\mathsf{Bir}(P_{VI})$ be the group of all birational symmetries of Painlevé sixth equation. The Galois group G is the subgroup of $\mathsf{Bir}(P_{VI})$ generated by the following four kind of affine transformations.

(1) Even translations by integers

$$\oplus_n : \begin{cases} \theta_\alpha & \mapsto \theta_\alpha + n_1 \\ \theta_\beta & \mapsto \theta_\beta + n_2 \\ \theta_\gamma & \mapsto \theta_\gamma + n_3 \\ \theta_\delta & \mapsto \theta_\delta + n_4 \end{cases} \quad \text{with} \quad \begin{cases} n = (n_1, n_2, n_3, n_4) \in \mathbf{Z}^4, \\ n_1 + n_2 + n_3 + n_4 \in 2\mathbf{Z}. \end{cases}$$

Those symmetries also act on the space of initial conditions of P_{VI} in a non trivial way, but the corresponding action on (x, y, z) is very simple: we recover the twist symmetries $\otimes_\epsilon$ of section 2.4 by considering n modulo $2\mathbf{Z}^4$.

(2) An action of Sym_4 permuting $(\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta)$. This corresponds to the action of Sym_4 on (a, b, c, d, x, y, z) permuting (a, b, c, d) in the same way. This group is generated by the four permutations T_1, T_2, P_1 and P_2 (see lemma 2.2 and remark 2.6).

(3) Twist symmetries on Painlevé parameters

$$\otimes_\epsilon : \begin{cases} \theta_\alpha & \mapsto \ \epsilon_1\theta_\alpha \\ \theta_\beta & \mapsto \ \epsilon_2\theta_\beta \\ \theta_\gamma & \mapsto \ \epsilon_3\theta_\gamma \\ \theta_\delta & \mapsto \ \epsilon_4\theta_\delta \end{cases} \quad \text{with} \quad \epsilon = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in \{\pm 1\}^4.$$

The corresponding action on (a, b, c, d, x, y, z) is trivial.

(4) The special Okamoto symmetry (called s_2 in [29])

$$\mathrm{Ok} : \begin{cases} \theta_\alpha & \mapsto \ \frac{\theta_\alpha - \theta_\beta - \theta_\gamma - \theta_\delta}{2} + 1 \\ \theta_\beta & \mapsto \ \frac{-\theta_\alpha + \theta_\beta - \theta_\gamma - \theta_\delta}{2} + 1 \\ \theta_\gamma & \mapsto \ \frac{-\theta_\alpha - \theta_\beta + \theta_\gamma - \theta_\delta}{2} + 1 \\ \theta_\delta & \mapsto \ \frac{-\theta_\alpha - \theta_\beta - \theta_\gamma + \theta_\delta}{2} + 1 \end{cases}$$

The corresponding action on (A, B, C, D, x, y, z) is trivial (see [18]), but the action on (a, b, c, d) is rather subbtle, as we shall see.

The ramified cover $(\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta) \mapsto (a, b, c, d)$ is also a Galois cover: its Galois group K is the subgroup of G generated by those translations $\oplus_n$ with $n \in (2\mathbf{Z})^4$ and the twists $\otimes_\epsilon$. One can check that $[G : K] = 24$ but K is not a normal subgroup of G: it is not Ok-invariant. In fact, K is normal in the subgroup $G' \subset G$ where we omit the generator Ok and $Q = G'/K$ coincides with the order 8 group of symmetries fixing (A, B, C, D). Therefore, G/K may be viewed as the disjoint union of left cosets

$$G/K = Q \cup \mathrm{Ok} \cdot Q \cup \widetilde{\mathrm{Ok}} \cdot Q$$

where $\widetilde{\mathrm{Ok}}$ is the following symmetry (called $s_1 s_2 s_1$ in [29])

$$\widetilde{\mathrm{Ok}} : \begin{cases} \theta_\alpha & \mapsto \ \frac{\theta_\alpha - \theta_\beta - \theta_\gamma + \theta_\delta}{2} \\ \theta_\beta & \mapsto \ \frac{-\theta_\alpha + \theta_\beta - \theta_\gamma + \theta_\delta}{2} \\ \theta_\gamma & \mapsto \ \frac{-\theta_\alpha - \theta_\beta + \theta_\gamma + \theta_\delta}{2} \\ \theta_\delta & \mapsto \ \frac{\theta_\alpha + \theta_\beta + \theta_\gamma + \theta_\delta}{2} \end{cases}$$

Let us now go back to the description of the fiber $\Pi^{-1}(A, B, C, D)$. Given a (a, b, c, d)-point, we would like to describe explicitly all other parameters (a', b', c', d') in the same Π-fibre, i.e. giving rise to the same parameter (A, B, C, D). We already know that the Q-orbit

$$\begin{cases} (a, b, c, d) \quad (-a, -b, -c, -d) \ (d, c, b, a) \ (-d, -c, -b, -a) \\ (b, a, d, c) \quad (-b, -a, -d, -c) \ (c, d, a, b) \ (-c, -d, -a, -b) \end{cases}$$

which generically is of length 8, is contained in the fibre. In order to describe the remaining part of the fibre, let us choose $(a_\epsilon, b_\epsilon, c_\epsilon, d_\epsilon) \in \mathbf{C}^4$, $\epsilon = 0, 1$, such that

$$\begin{cases} a_0 &= \frac{\sqrt{2+a}}{2} \\ b_0 &= \frac{\sqrt{2+b}}{2} \\ c_0 &= \frac{\sqrt{2+c}}{2} \\ d_0 &= \frac{\sqrt{2+d}}{2} \end{cases} \quad \text{and} \quad \begin{cases} a_1 &= \frac{\sqrt{2-a}}{2} \\ b_1 &= \frac{\sqrt{2-b}}{2} \\ c_1 &= \frac{\sqrt{2-c}}{2} \\ d_1 &= \frac{\sqrt{2-d}}{2} \end{cases}$$

If θ_α is such that $(a_0, a_1) = (\cos(\pi\frac{\theta_\alpha}{2}), \sin(\pi\frac{\theta_\alpha}{2}))$, then $a = 2\cos(\pi\theta_\alpha)$; therefore, the choice of (a_0, a_1) is equivalent to the choice of a P_{VI}-parameter θ_α modulo $2\mathbf{Z}$, i.e. of $\frac{\theta_\alpha}{2}$ modulo $\mathbf{Z}$. Then, looking at the action of the special Okamoto symmetry Ok on Painlevé parameters $(\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta)$, we derive the following new point (a', b', c', d') in the Π-fibre

$$\begin{cases} a' &= -2\sum(-1)^{\frac{-\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}{2}} a_{\epsilon_1} b_{\epsilon_2} c_{\epsilon_3} d_{\epsilon_4} \\ b' &= -2\sum(-1)^{\frac{\epsilon_1-\epsilon_2+\epsilon_3+\epsilon_4}{2}} a_{\epsilon_1} b_{\epsilon_2} c_{\epsilon_3} d_{\epsilon_4} \\ c' &= -2\sum(-1)^{\frac{\epsilon_1+\epsilon_2-\epsilon_3+\epsilon_4}{2}} a_{\epsilon_1} b_{\epsilon_2} c_{\epsilon_3} d_{\epsilon_4} \\ d' &= -2\sum(-1)^{\frac{\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4}{2}} a_{\epsilon_1} b_{\epsilon_2} c_{\epsilon_3} d_{\epsilon_4} \end{cases}$$

where sums $\sum$ are taken over all $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in (\{0,1\})^4$ for which $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4$ is even. One can check that the different choices for (a_0, b_0, c_0, d_0) and (a_1, b_1, c_1, d_1) lead to 16 distinct possible (a', b', c', d'), namely 2 distinct Q-orbits, which together with the Q-orbit of (a, b, c, d) above provide the whole Π-fibre.

Example 2.9. — When $(a, b, c, d) = (0, 0, 0, d)$, we have $(A, B, C, D) = (0, 0, 0, D)$ with $D = 4 - d^2$. The Π-fibre is given by the Q-orbits of the 3 points

$$(0,0,0,d) \quad \text{and} \quad (\tilde{d}, \tilde{d}, \tilde{d}, -\tilde{d}) \quad \text{where} \quad \tilde{d} = \sqrt{2 \pm \sqrt{4 - d^2}}$$

(only the sign of the square root inside is relevant up to Q). The fibre has length 24 except in the Cayley case $d = 0$ where it has length 9, consisting of the two Q-orbits of

$$(0,0,0,0) \quad \text{and} \quad (2,2,2,-2)$$

(note that $(0, 0, 0, 0)$ is Q-invariant) and in the Markov case $d = 2$ where it has length 16, consisting of the two Q-orbits of

$$(0,0,0,2) \quad \text{and} \quad (\sqrt{2}, \sqrt{2}, \sqrt{2}, -\sqrt{2}).$$

3. Geometry and Automorphisms

This section is devoted to a geometric study of the family of surfaces $S_{(A,B,C,D)}$, and to the description of the groups of polynomial automorphisms $\mathsf{Aut}[S_{(A,B,C,D)}]$. Section 3.3 introduces the concept of elliptic, parabolic, and hyperbolic automorphisms of $S_{(A,B,C,D)}$.

3.1. The triangle at infinity and automorphisms

Let S be any member of the family Fam. The closure $\overline{S}$ of S in $\mathbb{P}^3(\mathbf{C})$ is given by a cubic homogeneous equation

$$w(x^2+y^2+z^2)+xyz = w^2(Ax+By+Cz)+Dw^3.$$

The intersection of $\overline{S}$ with the plane at infinity does not depend on the parameters and coincides with the triangle Δ given by the equation

$$\Delta \quad : \quad xyz = 0;$$

moreover, one easily checks that the surface $\overline{S}$ is smooth in a neighborhood of Δ (all the singularities of $\overline{S}$ are contained in S).

The three involutions s_x, s_y and s_z generating the $\Gamma_2^{\pm}$-action (see §1.2) now admit the following geometric description. Since the equation defining S is of degree 2 with respect to the x variable, each point (x,y,z) of S gives rise to a unique second point (x',y,z). This procedure determines a holomorphic involution of S, namely the involution s_x defined by formula (1.9). Geometrically, the involution s_x corresponds to the following: if m is a point of $\overline{S}$, the projective line which joins m and the vertex $v_x=[1;0;0;0]$ of the triangle Δ intersects $\overline{S}$ on a third point; this point is $s_x(m)$. The two other involutions s_y and s_z are obtained similarly, by changing y and z root respectively.

From section 2.3, we deduce that, *for any member S of the family* Fam, *the group $\mathcal{A}=\langle s_x,s_y,s_z\rangle$ generated by the three involutions above coincides with the image of $\Gamma_2^{\pm}$ into* $\mathsf{Aut}[S]$, *which is obtained by the action of $\Gamma_2^{\pm}\subset \mathsf{Mod}^{\pm}(\mathbb{S}^2_4)$ on $\chi(\mathbb{S}^2_4)$* (see §1.2).

Theorem 3.1. — *Let $S=S_{(A,B,C,D)}$ be any member of the family of surfaces* Fam. *Then*

- *there is no non-trivial relation between the three involutions s_x, s_y and s_z, and $\mathcal{A}$ is therefore isomorphic to the free product $(\mathbf{Z}/2\mathbf{Z})\star(\mathbf{Z}/2\mathbf{Z})\star(\mathbf{Z}/2\mathbf{Z})$;*
- *the index of $\mathcal{A}$ in* $\mathsf{Aut}[S]$ *is bounded by* 24.

Moreover, for a generic choice of the parameters (A, B, C, D)*,* $\mathcal{A}$ *coincides with* $\mathsf{Aut}[S]$*.*

This result is almost contained in Èl'-Huti's article [12] and is more precise than Horowitz's main theorem (see [16], [17]).

Proof. — Since $\overline{S}$ is smooth in a neighborhood of the triangle at infinity and the three involutions are the reflexions with respect to the vertices of that triangle, we can apply the main theorems of Èl'-Huti's article:

- $\mathcal{A}$ is isomorphic to the free product
$$(\mathbf{Z}/2\mathbf{Z}) \star (\mathbf{Z}/2\mathbf{Z}) \star (\mathbf{Z}/2\mathbf{Z}) = \langle s_x \rangle \star \langle s_y \rangle \star \langle s_z \rangle;$$
- $\mathcal{A}$ is of finite index in $\mathsf{Aut}[S]$;
- $\mathsf{Aut}[S]$ is generated by $\mathcal{A}$ and the group of projective transformations of $\mathbb{P}^3(\mathbf{C})$ which preserve $\overline{S}$ and Δ (i.e., by affine transformations of $\mathbf{C}^3$ that preserve S).

We already know that $\mathcal{A}$ and the image of $\Gamma_2^{\pm}$ in $\mathsf{Aut}[S]$ coincide. We now need to study the index of $\mathcal{A}$ in $\mathsf{Aut}[S]$. Let f be an affine invertible transformation of $\mathbf{C}^3$, that we decompose as the composition of a linear part M and a translation of vector T. Let S be any member of Fam. If f preserves S, then the equation of S is multiplied by a non zero complex number when we apply f. Looking at the cubic terms, this means that M is a diagonal matrix composed with a permutation of the coordinates. Looking at the quadratic terms, this implies that T is the nul vector, so that $f = M$ is linear. Coming back to the equation of S, we now see that M is one of the 24 linear transformations of the type $\sigma \circ \epsilon$ where ϵ either is the identity or changes the sign of two coordinates, and σ permutes the coordinates. If (A, B, C, D) are generic, $S_{(A,B,C,D)}$ is not invariant by any of these linear maps. Moreover, one easily verifies that the subgroup $\mathcal{A}$ is a normal subgroup of $\mathsf{Aut}[S]$: if such a linear transformation $M = \sigma \circ \epsilon$ preserves S, then it normalizes $\mathcal{A}$. This shows that $\mathcal{A}$ is a normal subgroup of $\mathsf{Aut}[S]$, the index of which is bounded by 24. □

3.2. Consequences and notations

As a corollary of theorem 3.1 and proposition 2.5, we get the following result: *The mapping class group* $\mathsf{Mod}_0^{\pm}(\mathbb{S}_4^2)$ *acts on the character variety* $\chi(\mathbb{S}_4^2)$*, preserving each surface* $S_{(A,B,C,D)}$*, and its image in* $\mathsf{Aut}[S_{(A,B,C,D)}]$ *coincides with the image of* $\Gamma_2^{\pm}$*, and therefore with the finite index subgroup* $\mathcal{A}$ *of* $\mathsf{Aut}[S_{(A,B,C,D)}]$. In other words, up to finite index subgroups, describing the dynamics of $\mathsf{Mod}^{\pm}(\mathbb{S}^2_4)$ on the character variety $\chi(\mathbb{S}^2_4)$ or of the group $\mathsf{Aut}[S]$ on S for any member S of the family Fam is one and the same problem.

In the following, we shall identify the subgroup $\Gamma_2^{\pm}$ of $\mathsf{PGL}(2,\mathbf{Z})$ and the subgroup $\mathcal{A}$ of $\mathsf{Aut}[S_{(A,B,C,D)}]$: if f is an element of $\mathcal{A}$, M_f will denote the associated element of $\Gamma_2^{\pm}$ (either viewed as a matrix or an isometry of $\mathbb{H}$), and if M is an element of $\Gamma_2^{\pm}$, f_M will denote the automorphism associated to M (for any surface S of the family Fam). Recall that, as a subgroup of $\mathsf{PGL}(2,\mathbf{R})$, the group $\Gamma_2^{\pm}$ acts by (anti-)conformal isometries on the Poincaré half plane $\mathbb{H}$.

3.3. Elliptic, Parabolic, Hyperbolic

Non trivial isometries of $\mathbb{H}$ are classified into three different species. Let M be an element of $\mathsf{PGL}(2,\mathbf{R}) \setminus \{\mathrm{Id}\}$, viewed as an isometry of $\mathbb{H}$. Then,

- M is elliptic if M has a fixed point in the interior of $\mathbb{H}$. Ellipticity is equivalent to $\det(M) = 1$ and $|\mathsf{tr}(M)| < 2$ (in which case M is a rotation around a unique fixed point) or $\det(M) = -1$ and $\mathsf{tr}(M) = 0$ (in which case M is a reflexion around a geodesic of fixed points).
- M is parabolic if M has a unique fixed point, which is located on the boundary of $\mathbb{H}$; M is parabolic if and only if $\det(M) = 1$ and $\mathsf{tr}(M) = 2$ or -2;
- M is hyperbolic if it has exactly two fixed points which are on the boundary of $\mathbb{H}$; this occurs if and only if $\det(M) = 1$ and $|\mathsf{tr}(M)| > 2$, or $\det(M) = -1$ and $\mathsf{tr}(M) \neq 0$.

An element f of $\mathcal{A}\setminus\{\mathrm{Id}\}$ will be termed elliptic, parabolic, or hyperbolic, according to the type of M_f. Examples of elliptic elements are given by the three involutions s_x, s_y and s_z. Examples of parabolic elements are given by the three automorphisms g_x, g_y and g_z (see section 2.3). The dynamics of these automorphisms will be described in details in §4.1. Let us just mention the fact that g_x (resp. g_y, g_z) preserves the conic fibration $\{x = c^{ste}\}$ (resp. $\{y = c^{ste}\}$, $\{z = c^{ste}\}$) of any member S of Fam. The following Proposition is standart; we skip the proof.

Proposition 3.2. — *Let S be one of the surfaces in the family* Fam *(S may be singular). An element f of $\mathcal{A}$ is*

- *elliptic if and only if f is conjugate to one of the involutions s_x, s_y or s_z, if and only if f is periodic;*
- *parabolic if and only if f is conjugate to a non trivial power of one of the automorphisms g_x, g_y or g_z;*
- *hyperbolic if and only if f is conjugate to a cyclically reduced composition which involves the three involutions s_x, s_y, and s_z.*

3.4. Singularities, fixed points, and orbifold structure

The singularities of the elements of Fam will play an important role in this article. In this section, we collect a few results regarding these singularities.

LEMMA 3.3. — *Let S be a member of* Fam. *A point m of S is singular if and only if m is a fixed point of the group $\mathcal{A}$.*

Proof. — This is a direct consequence of the fact that m is a fixed point of s_x if and only if $2x + yz = Ax$, if and only if the partial derivative of the equation of S with respect to the x-variable vanishes. □

Example 3.4. — The family of surfaces with parameters

$$(4 + 2d, 4 + 2d, 4 + 2d, -(8 + 8d + d^2)) \quad \text{with} \quad \in \mathbf{C}$$

is a deformation of the Cayley cubic, that corresponds to $d = -2$, and any of these surfaces has 3 singular points (counted with multiplicity).

LEMMA 3.5. — *If m is a singular point of S, there exists a neighborhood of m which is isomorphic to the quotient of the unit ball in $\mathbf{C}^2$ by a finite subgroup of* $\mathsf{SU}(2)$.

Proof. — Any singularity of a cubic surface is a quotient singularity, except when the singularity is isomorphic to $x^3 + y^3 + z^3 + \lambda xyz = 0$, for at least one parameter λ (see [5]). Since the second jet of the equation of S never vanishes when S is a member of Fam, the singularities of S are quotient singularities. Since S admits a global volume form Ω, the finite group is conjugate to a subgroup of $\mathsf{SU}(2, \mathbf{C})$. □

As a consequence, any member S of Fam is endowed with a well defined orbifold structure. If S is singular, the group $\mathcal{A}$ fixes each of the singular points and preserves the orbifold structure. We shall consider this action in the orbifold category, but we could as well extend the action of A to a smooth desingularization of S.

LEMMA 3.6. — *The complex affine surface S is simply connected. When S is singular, the fundamental group of the complex surface $S \setminus \mathsf{Sing}(S)$ is normally generated by the local finite fundamental groups around the singular points.*

Proof. — Recall that a smooth cubic surface in $\mathbb{P}^3(\mathbf{C})$ may be viewed as the blow-up of $\mathbb{P}^2(\mathbf{C})$ at 6-points in general position. To construct our family Fam by this way, we consider the triangle $XYZ = 0$ of $\mathbb{P}^2(\mathbf{C})$ and blow-up twice each line. The resulting surface $\tilde{S}$ embeds in $\mathbb{P}^3(\mathbf{C})$ as a smooth member of our family by sending the strict transform of the triangle at infinity. Singular cubics arise when 3 of the 6 points lie on a line, or all of them lie on a conic. In this case, the corresponding line(s) and/or conic have negative self-intersection in $\tilde{S}$; after blow them down to singular points, one obtain the embedding in our family Fam.

Our claim is that the quasi-projective surface $\tilde{S}'$ obtained by deleting the strict transform of the triangle $XYZ = 0$ from $\tilde{S}$ is simply connected. Indeed, the fundamental group of $\mathbb{P}^2 - \{XYZ = 0\}$ is isomorphic to $\mathbf{Z}^2$, generated by two loops, say one turning around $X = 0$, and the other one around $Y = 0$. After blowing-up one point lying on $X = 0$, and adding the exceptional divisor (minus $X = 0$), the first loop becomes homotopic to 0; after blowing-up the 6 points and adding all exceptional divisors, the two generators become trivial and the resulting surface $\tilde{S}'$ is simply connected. The affine surface S is obtained after blowing-down some rational curves in $\tilde{S}$ and is therefore simply connected as well.

The second assertion of the lemma directly follows from Van Kampen Theorem. □

3.5. Reducible representations versus singularities.

THEOREM 3.7 ([2, 19]). — *The surface $S_{(A,B,C,D)}$ is singular if, and only if, we are in one of the following cases*

- $\Delta(a, b, c, d) = 0$ *where*

$$\Delta = (2(a^2 + b^2 + c^2 + d^2) - abcd - 16)^2 - (4 - a^2)(4 - b^2)(4 - c^2)(4 - d^2),$$

- *at least one of the parameters a, b, c or d equals ± 2.*

More precisely, a representation ρ is sent to a singular point if, and only if, we are in one of the following cases :

- *the representation ρ is reducible and then $\Delta = 0$,*

- *one of the generators $\rho(\alpha)$, $\rho(\beta)$, $\rho(\gamma)$ or $\rho(\delta)$ equals $\pm I$ (the corresponding trace parameter is then equal to ± 2).*

In fact, it is proved in [2] that the set Z of parameters (A,B,C,D) for which $S_{(A,B,C,D)}$ is singular is defined by $\delta = 0$ where δ is the discriminant of the polynomial

$$P_z = z^4 - Cz^3 - (D+4)z^2 + (4C - AB)z + 4D + A^2 + B^2$$

defined in section 4.1: P_z has a multiple root at each singular point. Now, consider the ramified cover

$$\Pi : \mathbf{C}^4 \to \mathbf{C}^4; (a,b,c,d) \mapsto (A,B,C,D)$$

defined by (1.5). One can check by direct computation that

$$\delta \circ \Pi = \frac{1}{16}(a^2-4)(b^2-4)(c^2-4)(d^2-4)\Delta^2.$$

One also easily verifies that the locus of reducible representations is also the ramification locus of Π:

$$\mathrm{Jac}(\Pi) = -\frac{1}{2}\Delta.$$

It is a well known fact (see [19]) that Okamoto symmetries permute the two kinds of degenerate representations given by Theorem 3.7. For instance, a singular point is defined by the following equations:

$$A = 2x + yz, \quad B = 2y + xz, \quad C = 2z + xy$$
$$\text{and} \quad x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D.$$

Now, a compatible choice of parameters (a,b,c,d) is provided by

$$(a,b,c,d) = (y,z,x,2)$$

and one easily check that the corresponding representations satisfy $\rho(\delta)= I$.

3.6. $\mathsf{SU}(2)$-representations versus bounded components.

When a, b, c, and d are real numbers, A, B, C, and D are real as well. In that case, the real part $S_{(A,B,C,D)}(\mathbf{R})$ stands for $\mathsf{SU}(2)$ and $\mathsf{SL}(2,\mathbf{R})$-representations; precisely, each connected component of the smooth part of $S_{(A,B,C,D)}(\mathbf{R})$ is either purely $\mathsf{SU}(2)$, or purely $\mathsf{SL}(2,\mathbf{R})$, depending on the choice of (a,b,c,d) fitting to (A,B,C,D).

Moving into the parameter space $\{(a,b,c,d)\}$, when we pass from $\mathsf{SU}(2)$ to $\mathsf{SL}(2,\mathbf{R})$-representations, we must go through a representation of the group $\mathsf{SU}(2) \cap \mathsf{SL}(2,\mathbf{R}) = \mathsf{SO}(2,\mathbf{R})$. Since representations into $\mathsf{SO}(2,\mathbf{R})$

are reducible, they correspond to singular points of the cubic surface (see §3.5). In other words, any bifurcation between $\mathsf{SU}(2)$ and $\mathsf{SL}(2,\mathbf{R})$-representations creates a real singular point of $S_{(A,B,C,D)}$.

Since $\mathsf{SU}(2)$-representations are contained in the cube $[-2,2]^3$, they always form a bounded component of the smooth part of $S_{(A,B,C,D)}(\mathbf{R})$: Unbounded components always correspond to $\mathsf{SL}(2,\mathbf{R})$-representations, whatever the choice of parameters (a,b,c,d) is.

The topology of $S_{(A,B,C,D)}(\mathbf{R})$ is studied in [2] when (a,b,c,d) are real numbers. There are at most four singular points, and the smooth part has at most one bounded and at most four unbounded components. On the other hand, if A, B, C, and D are real numbers, then a, b, c, and d are not necessarily real.

Example 3.8. — If a, b, c, and d are purely imaginary numbers, then A, B, C, and D are real numbers. In this specific example, there are representations $\rho:\pi_1(\mathbb{S}^2_4)\to\mathsf{SL}(2,\mathbf{C})$ with trace parameters

$$(a,b,c,d,x,y,z)\in(i\mathbf{R})^4\times(\mathbf{R})^3,$$

the image of which are Zariski dense in the (real) Lie group $\mathsf{SL}(2,\mathbf{C})$. Such a representation correspond to a point (x,y,z) on $S_{(A,B,C,D)(\mathbf{R})}$ which is not realized by a representation into $\mathsf{SL}(2,\mathbf{R})$.

The goal of this section is to prove Theorem B which partly extends the above mentionned results of Benedetto and Goldman [2].

Denote by $Z\subset\mathbf{R}^4$ the subset of those parameters (A,B,C,D) for which the corresponding surface $S_{(A,B,C,D)}(\mathbf{R})$ is singular (see section 3.5). Over each connected component of $\mathbf{R}^4\setminus Z$, the surface $S_{(A,B,C,D)}(\mathbf{R})$ is smooth and has constant topological type. Let $\mathcal{B}$ be the union of connected components of $\mathbf{R}^4\setminus Z$ over which the smooth surface has a bounded component.

The ramified cover $\Pi:\mathbf{C}^4\to\mathbf{C}^4;(a,b,c,d)\mapsto(A,B,C,D)$ has degree 24; Okamoto correspondences, defined in section 2.5, “act” transitively on fibers (recall that Π is not Galois). Because of their real nature, these correspondences permute real parameters (a,b,c,d): therefore, Π restricts to a degree 24 ramified cover $\Pi|_{\mathbf{R}^4}:\mathbf{R}^4\to\Pi(\mathbf{R}^4)$. Following [2], we have

$$\Pi^{-1}(\mathcal{B})\cap\mathbb{R}^4=(-2,2)^4\setminus\{\Delta=0\}.$$

Using again that $\mathsf{SU}(2)\cap\mathsf{SL}(2,\mathbf{R})=\mathrm{SO}(2)$ is abelian, and therefore corresponds to reducible representations, we promptly deduce that, along each connected component of $(-2,2)^4\setminus\{\Delta=0\}$, the bounded component of the corresponding surface $S_{(A,B,C,D)}(\mathbf{R})$ constantly stands either for $\mathsf{SU}(2)$-representations, or for $\mathsf{SL}(2,\mathbf{R})$-representations. We shall denote by $\mathcal{B}^{\mathsf{SU}(2)}$

and $\mathcal{B}^{\mathsf{SL}(2,\mathbf{R})}$ the corresponding components of $\mathcal{B}$. Theorem B may now be rephrased as the following equalities:

$$\mathcal{B} = \mathcal{B}^{\mathsf{SU}(2)} = \mathcal{B}^{\mathsf{SL}(2,\mathbf{R})}.$$

To prove these equalities, we first note that $\mathcal{B}^{\mathsf{SU}(2)} \cup \mathcal{B}^{\mathsf{SL}(2,\mathbf{R})} \subset \Pi([-2,2]^4)$ is obviously bounded by $-8 \leqslant A, B, C \leqslant 8$ and $-20 \leqslant D \leqslant 28$ (this bound is not sharp !).

Lemma 3.9. — *The set $\mathcal{B}$ is bounded, contained into $-8 \leqslant A, B, C \leqslant 8$ and $-56 \leqslant D \leqslant 68$.*

Proof. — The orbit of any point p belonging to a bounded component of $S_{(A,B,C,D)}(\mathbf{R})$ is bounded. Applying the tools involved in section 4, we deduce that the bounded component is contained into $[-2,2]^3$. Therefore, for any $p = (x, y, z)$ and $s_x(p) = (x', y, z)$ belonging to the bounded component, we get $A = x + x' + yz$ and then $-8 \leqslant A \leqslant 8$. Using s_y and s_z, we get the same bounds for B and C. Since p is in the surface, we also get $D = x^2 + y^2 + z^2 + xyz - Ax - By - Cz$. □

The order 24 group of Benedetto-Goldman symmetries act on the parameters (A, B, C, D) by freely permutting the triple (A, B, C), and freely changing sign for two of them. This group acts on the set of connected components of $\mathbf{R}^4 \setminus Z$, $\mathcal{B}$, $\mathcal{B}^{\mathsf{SU}(2)}$ and $\mathcal{B}^{\mathsf{SL}(2,\mathbf{R})}$. The crucial Lemma is

Lemma 3.10. — *Up to Benedetto-Goldman symmetries, $\mathbf{R}^4 \setminus Z$ has only one bounded component.*

Sketch of proof. — Up to Benedetto-Goldman symmetries, one can always assume $0 \leqslant A \leqslant B \leqslant C$. This fact is easily checked by looking at the action of symmetries on the projective coordinates $[A : B : C] = [X : Y : 1]$: the triangle $T = \{0 \leqslant X \leqslant Y \leqslant 1\}$ happens to be a fundamental domain for this group action. One can show by standart computation (see [7], proof of Lemma 9.8) that $\mathbf{R}^4 \setminus Z$ has exactly one bounded component over the cone

$$\mathcal{C} = \{(A, B, C) \ ; \ 0 \leqslant A \leqslant B \leqslant C\}$$

with respect to the projection $(A, B, C, D) \mapsto (A, B, C)$.

□

We thus conclude that $\mathcal{B} = \mathcal{B}^{\mathsf{SU}(2)} = \mathcal{B}^{\mathsf{SL}(2,\mathbf{R})}$ and Theorem B is proved in the case the real surface $S_{(A,B,C,D)}(\mathbf{R})$ is smooth. The general case follows from the following lemma, the proof of which is left to the reader.

Lemma 3.11. — *Let (A, B, C, D) be real parameters such that the smooth part of the surface $S_{(A,B,C,D)}(\mathbf{R})$ has a bounded component. Then,* *there exist an arbitrary small real perturbation of (A,B,C,D) such that the corresponding surface is smooth and has a bounded component.*

We would like now to show that there is actually only one bounded component in $\mathbf{R}^4 \setminus Z$ (up to nothing).

Inside $[-2,2]^4$, the equation Δ splits into the following two equations

$$2(a^2+b^2+c^2+d^2) - abcd - 16 = \pm\sqrt{(4-a^2)(4-b^2)(4-c^2)(4-d^2)}.$$

Those two equations cut-off the parameter space $[-2,2]^4$ into many connected components and we have[(4)]

Theorem 3.12 (Benedetto-Goldman [2]). — *When a, b, c and d are real and $S_{(A,B,C,D)}(\mathbf{R})$ is smooth, then $S_{(A,B,C,D)}(\mathbf{R})$ has a bounded component if, and only if, a, b, c and d both lie in $(-2,2)$. In this case, the bounded component corresponds to $\mathsf{SL}(2,\mathbf{R})$-representations if, and only if,*

$$2(a^2+b^2+c^2+d^2) - abcd - 16 > \sqrt{(4-a^2)(4-b^2)(4-c^2)(4-d^2)}.$$

When we cross the boundary

$$2(a^2+b^2+c^2+d^2) - abcd - 16 = \sqrt{(4-a^2)(4-b^2)(4-c^2)(4-d^2)}$$

inside $(-2,2)^4$, we pass from $\mathsf{SL}(2,\mathbf{R})$ to $\mathsf{SU}(2)$-representations: at the boundary, the bounded component must degenerate down to a singular point.

We now prove the

Proposition 3.13. — *The set $(-2,2)^4 \setminus \{\Delta = 0\}$ has 24 connected components, 8 of them corresponding to $\mathsf{SL}(2,\mathbf{R})$-representations, the other 16 to $\mathsf{SU}(2)$-representations. Okamoto correspondances permute transitively those components.*

Recall that the group of cover transformations Q has order 8 and does not change the nature of the representation: the image $\rho(\pi_1(\mathbb{S}^2_4))$ remains unchanged in $\mathsf{PGL}(2,\mathbb{C})$. Therefore, up to this tame action, Okamoto correspondence provides, to any smooth point (A,B,C,D,x,y,z) of the character variety, exactly 3 essentially distinct representations, two of them in $\mathsf{SU}(2)$, and the third one in $\mathsf{SL}(2,\mathbf{R})$. It may happens (see [32]) that one of the two $\mathsf{SU}(2)$-representations is dihedral, while the other one is dense!

Proof. — We shall prove that the $\mathsf{SL}(2,\mathbf{R})$-locus, i.e. the real semi-algebraic set X of $[-2,2]^4$ defined by

$$2(a^2+b^2+c^2+d^2) - abcd - 16 > \sqrt{(4-a^2)(4-b^2)(4-c^2)(4-d^2)},$$

(4) In [2], the connected components of $[-2,2]^4$ standing for $\mathsf{SL}(2,\mathbf{R})$-representations are equivalently defined by $\Delta > 0$ and $2(a^2+b^2+c^2+d^2) - abcd - 16 > 0$.

consist in connected neighborhoods of those 8 vertices corresponding to the Cayley surface

$$(a,b,c,d) = (\epsilon_1 \cdot 2, \epsilon_2 \cdot 2, \epsilon_3 \cdot 2, \epsilon_4 \cdot 2), \quad \epsilon_i = \pm 1,\ \epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1.$$

Benedetto-Goldman symmetries act transitively on those components. On the other hand, the Cayley surface also arise for $(a,b,c,d) = (0,0,0,0)$, which is in the $\mathsf{SU}(2)$-locus: the Okamoto correspondence therefore sends any of the 8 components above into the $\mathsf{SU}(2)$-locus, thus proving the theorem.

By abuse of notation, still denote by Z the discriminant locus defined by $\{\Delta = 0\} \subset (-2,2)^4$. The restriction $Z_{a,b}$ of Z to the slice

$$\Pi_{a,b} = \{(a,b,c,d)\ ;\ c,d \in (-2,2)\}, \quad (a,b) \in (-2,2)^2,$$

is the union of two ellipses, namely those defined by

$$c^2 + d^2 - \delta cd + \delta^2 - 4 = 0, \quad \text{where } \delta = \frac{1}{2}\left(ab \pm \sqrt{(4-a^2)(4-b^2)}\right).$$

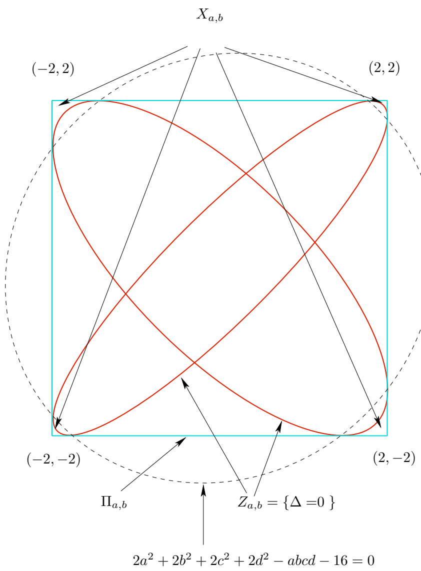

Figure 3.1. Z restricted to the slice $\Pi_{a,b}$.

Those two ellipses are circumscribed into the square $\Pi_{a,b}$ (see figure 3.1) and, for generic parameters a and b, cut the square into 13 connected components. One easily verify that $\mathsf{SL}(2,\mathbf{R})$-components (namely those connected components of $X_{a,b} = X \cap \Pi_{a,b}$ defined by the inequality of the previous theorem) are those 4 neighborhoods of the vertices of the square.

This picture degenerates precisely when $a = \pm 2$, $b = \pm 2$ or $a = \pm b$. We do not need to consider the first two cases, since they are on the boudary of $(-2,2)^4$. Anyway, in these cases, the two ellipses coincide; they moreover degenerate to a double line when $a = \pm b$.

In the last case $a = \pm b$, the picture bifurcates. When $a = b$, one of the ellipses degenerates to the double line $c = d$, and the two components of $X_{a,b}$ near the vertices $(2,2)$ and $(-2,-2)$ collapse. When $a = -b$, the components of $X_{a,b}$ near the two other vertices collapse as well. This means that each component of $X_{a,b}$ stands for exactly two components of X: we finally obtain 8 connected components for the $\mathsf{SL}(2,\mathbf{R})$-locus $X \subset (-2,2)^4$. One easily verify that there are sixteen $\mathsf{SU}(2)$-components in $(-2,2)^4 \setminus Z$. $\square$

4. Bounded Orbits

4.1. Dynamics of parabolic elements

Parabolic elements will play an important role in the proof of theorem 1.8. In this section, we describe the dynamics of these automorphisms, on any member S of our family of cubic surfaces. Since any parabolic element is conjugate to a power of g_x, g_y or g_z, we just need to study one of these examples.

Once the parameters A, B, C, and D have been fixed, the automorphism g_z is given by

$$g_z \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} A - x - zy \\ B - Az + zx + (z^2-1)y \\ z \end{pmatrix}.$$

This defines a global polynomial diffeomorphism of $\mathbf{C}^3$, that preserves each horizontal plane $\Pi_{z_0} = \{(x,y,z_0),\ x \in \mathbf{C},\ y \in \mathbf{C}\}$. On each of these planes, g_z induces an affine transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -1 & -z_0 \\ z_0 & z_0^2 - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} A \\ B - Az_0 \end{pmatrix},$$

which preserves the conic $S_{z_0} = S \cap \Pi_{z_0}$. The trace of the linear part of this affine transformation is $z_0^2 - 2$ while the determinant is 1.

PROPOSITION 4.1. — *Let S be any member of the family of cubic surfaces* Fam*. Let g_z be the automorphism of S defined by the composition $s_y \circ s_x$. On each fiber S_{z_0} of the fibration*

$$\pi_z : S \to \mathbf{C}, \quad \pi_z(x, y, z) = z,$$

g_z induces a homographic transformation $\overline{g_{z_0}}$, and

- *$\overline{g_{z_0}}$ is an elliptic homography if and only if $z_0 \in (-2, 2)$; this homography is periodic if and only if z_0 is of type $\pm 2\cos(\pi\theta)$ with θ rational;*
- *$\overline{g_{z_0}}$ is parabolic (or the identity) if and only if $z_0 = \pm 2$;*
- *$\overline{g_{z_0}}$ is loxodromic if and only if z_0 is not in the interval $[-2, 2]$.*

If z_0 is different from 2 and -2, g_z has a unique fixed point inside Π_{z_0}, the coordinate of which are (x_0, y_0, z_0) where

$$x_0 = \frac{Bz_0 - 2A}{{z_0}^2 - 4}, \quad y_0 = \frac{Az_0 - 2B}{{z_0}^2 - 4}.$$

This fixed point is contained in the surface S if and only if z_0 satisfies the quartic equation $P_z(z_0) = 0$ where

$$P_z = z^4 - Cz^3 - (D+4)z^2 + (4C - AB)z + 4D + A^2 + B^2. \tag{4.1}$$

In that case, the union of the two g_z-invariant lines of Π_{z_0} which go through the fixed point coincides with S_{z_0}; moreover, the involutions s_x and s_y permute those two lines. If the fixed point is not contained in S, the conic S_{z_0} is smooth, and the two fixed points of the (elliptic or loxodromic) homography $\overline{g_{z_0}}$ are at infinity.

If $z_0 = 2$, the affine transformation induced by g_z on Π_{z_0} is

$$\overline{g_{z_0}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} A \\ B - 2A \end{pmatrix}.$$

Either $\overline{g_{z_0}}$ has no fixed point, or $A = B$ and there is a line of fixed points, given by $x+y = A/2$. This line of fixed points intersects the surface S if and only if S_{z_0} coincides with this (double) line. In that case the involutions s_x and s_y also fix the line pointwise. When the line does not intersect S, the conic S_{z_0} is smooth, with a unique point at infinity; this point is the unique fixed point of the parabolic transformation $\overline{g_{z_0}}$. In particular, any point of S_{z_0} goes to infinity under the action of g_z.

If $z_0 = -2$, then

$$\overline{g_{z_0}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} A \\ B + 2A \end{pmatrix}.$$

Either g_z does not have any fixed point in Π_{z_0}, or $A = -B$ and g_{z_0} has a line of fixed points given by $x - y = A/2$. This line intersects S if and only if S_{z_0} coincides with this (double) line. In that case the involutions s_x and s_y fixe the line pointwise.

Lemma 4.2. — *With the notation that have just been introduced, the homographic transformation $\overline{g_{z_0}}$ induced by g_z on S_{z_0} has a fixed point in S_{z_0} if and only if z_0 satisfies equation (4.1). Moreover*

- *when $z_0 \neq 2, -2$, S_{z_0} is a singular conic, namely a union of two lines that are permuted by s_x and s_y, and the unique fixed point of $\overline{g_{z_0}}$ is the point of intersection of these two lines, with coordinates*
$$x_0 = \frac{Bz_0 - 2A}{{z_0}^2 - 4}, \quad y_0 = \frac{Az_0 - 2B}{{z_0}^2 - 4};$$
- *when $z_0 = 2$, then $A = B$, S_{z_0} is the double line $x + y = A/2$, and this line is pointwise fixed by $\overline{g_{z_0}}$, s_x and s_y;*
- *when $z_0 = -2$, then $A = -B$, S_{z_0} is the double line $x - y = A/2$, and this line is pointwise fixed by $\overline{g_{z_0}}$, s_x and s_y;*

The dynamics of g_z on S is now easily described. Let $p_0 = (x_0, y_0, z_0)$ be a point of S. If z_0 is in the interval $(-2, 2)$, the orbit of p_0 under g_z is bounded, and it is periodic if, and only if, either p_0 is a fixed point, or z_0 is of type $\pm 2\cos(\pi\theta)$, where θ is a rational number. If $z_0 = \pm 2$, and if p_0 is not a fixed point, $g^n(p_0)$ goes to infinity when n goes to $+\infty$ and $-\infty$. If z_0 is not contained in the interval $[-2, 2]$, for instance if the imaginary part of z_0 is not 0, either p_0 is fixed or $g^n(p_0)$ goes to infinity when n goes to $-\infty$ or $+\infty$. Of course, the same kind of results are valid for g_x and g_y, with the appropriate permutation of variables and parameters.

4.2. Bounded Orbits

The goal of this setion is to prove Theorem C. We fix a point p in one of the surfaces S and denote its $\Gamma_2^{\pm}$-orbit by $\mathrm{Orb}(p)$.

Lemma 4.3. — *If $\mathrm{Orb}(p)$ is bounded and $\#\mathrm{Orb}(p) > 4$, then A, B, C, and D are real and $p \in S(\mathbf{R})$.*

Proof. — Let $p_0 = (x_0, y_0, z_0)$ be a point of the orbit. If the third coordinate $z_0 \notin (-2, 2)$, the homography induced by g_z on the conic S_{z_0} is parabolic or hyperbolic. Since the orbit of p_0 is bounded, this implies that p_0 is a fixed point of g_z, s_x and s_y (see lemma 4.2). Since $\mathrm{Orb}(p_0)$

has length > 4, $s_z(p_0)$ is different from p_0, so that p_0 is not fixed by g_x, nor by g_y either ; this implies that $x_0, y_0 \in (-2,2)$. Moreover, the point $p_1 := s_z(p_0) = (x_0, y_0, z_1)$ is not fixed by g_z, otherwise the orbit would have length 2, so that $z_1 \in (-2,2)$ and $p_1 \in (-2,2)^3$. This argument shows the following: if one of the coordinates of p_0 is not contained in $(-2,2)$, then p_0 is fixed by two of the involutions s_x, s_y and s_z while the third one maps p_0 into $(-2,2)^3$.

Let now p be a point of the orbit with coordinates in $(-2,2)^3$; if the three points $s_x(p)$, $s_y(p)$ and $s_z(p)$ either escape from $(-2,2)^3$ or coincide with p, then the orbit reduces to $\{p, s_x(p), s_y(p), s_z(p)\}$, and has length $\leqslant 4$. From this we deduce that the orbit contains at least two distinct points $p_1, p_2 \in (-2,2)^3$, which are, say, permuted by s_x. Let (x_i, y_1, z_1) be the coordinates of p_i, $i = 1,2$. Then, $A = x_1 + x_2 + y_1 z_1 \in \mathbb{R}$. If B and C are also real, then p_1 is real and satisfies the equation of S, so that D is real as well and $\mathrm{Orb}(m) = \mathrm{Orb}(p_1) \subset S(\mathbf{R})$.

Now, assume by contradiction that $B \notin \mathbf{R}$. Then,

$$q_i := s_y(p_i) = (x_i, B - y_1 - x_i z_1, z_1) \notin (-2,2)$$

and is therefore fixed by s_x (otherwise the orbit would not be bounded): we thus have

$$2x_i + (B - y_1 - x_i z_1) z_1 = A.$$

Since B is the unique imaginary number of this equation, z_1 must vanish, and we get $x_1 = x_2 (= \frac{A}{2})$, a contradiction. A similar argument shows that C must be real as well. □

Remark 4.4. — By the same way, replacing $(-2,2)$ and $\mathbf{R}$ respectively by $(-2,2) \cap 2\cos(\pi \mathbf{Q})$ and $\mathbf{R} \cap \overline{\mathbf{Q}}$ in the previous proof shows that finite orbits of length > 4 occur only for real algebraic (A, B, C, D, x, y, z). This was noticed in [7], but now follows from the complete classification of finite orbits in [24].

LEMMA 4.5. — *Let S be an element of the family* Fam *and p a point of S. There exists a positive integer N such that, if p' is a point of the orbit of p with a coordinate of the form*

$$2\cos(\pi \frac{k}{n}), \quad k \wedge n = 1,$$

then n divides N.

Proof. — The point p is an element of the character variety $\chi(\mathbb{S}^2_4)$. Let us choose a representation $\rho : \pi_1(\mathbb{S}^2_4) \to \mathsf{SL}(2, \mathbf{C})$ in the conjugacy class that is determined by p. Since $\pi_1(\mathbb{S}^2_4)$ is finitely generated, Selberg's lemma (see [1]) implies the existence of a torsion free, finite index subgroup G of $\rho(\pi_1(\mathbb{S}^2_4))$. If we define N to be the cardinal of the quotient $\rho(\pi_1(\mathbb{S}^2_4))/G$, then the order of any torsion element in $\rho(\pi_1(\mathbb{S}^2_4))$ divides N.

If p' is a point of the orbit of p, the coordinates of p' are traces of elements of $\rho(\pi_1(\mathbb{S}^2_4))$. Assume that the trace of an element M in $\rho(\pi_1(\mathbb{S}^2_4))$ is of type $2\cos(\pi\theta)$. If $\theta = \frac{k}{n}$ and k and n are relatively prime integers, then M is a cyclic element of $\rho(\pi_1(\mathbb{S}^2_4))$ of order n, so that n divides N. $\square$

Proof of Theorem C. — Let $\mathrm{Orb}(p)$ be an infinite and bounded $\Gamma_2^{\pm}$-orbit in $S = S_{(A,B,C,D)}$. Following Lemma 4.3, A, B, C and D are real and $\mathrm{Orb}(p) \subset S(\mathbf{R})$. We want to prove that the closure $\overline{\mathrm{Orb}(p)}$ is open in $S(\mathbf{R}) \setminus \{\mathsf{Sing}(S(\mathbf{R}))\}$; since $\overline{\mathrm{Orb}(p)}$ is closed, it will therefore coincide with the (unique) bounded connected component of $S \setminus \{\mathsf{Sing}(S)\}$, thus proving the theorem.

We first claim that there exists an element (actually infinitely many) $p_0 = (x_0, y_0, z_0)$ of the orbit which is contained in $(-2,2)^3$ and for which at least one of the Möbius transformations $\overline{g_{x_0}}$, $\overline{g_{y_0}}$ or $\overline{g_{z_0}}$ is (elliptic) non periodic. Indeed, if a point p_0 of the orbit is such that $\overline{g_{z_0}}$ is not of the form above, then we are in one of the following cases

- $P_z(z_0) = 0$ and p_0 is a fixed point of $\overline{g_{z_0}}$,
- $z_0 = 2\cos(\pi\frac{k}{n})$ with $k \wedge n = 1$, $n|N$ and $\overline{g_{z_0}}$ is periodic of period n

(where N is given by Lemma 4.5). This gives us finitely many possibilities for z_0; we also get finitely many possibilities for x_0 and y_0 and the claim follows.

Let p_0 be a point of $\mathrm{Orb}(p)$, with, say, $\overline{g_{x_0}}$ elliptic and non periodic, so that the closure $\overline{\mathrm{Orb}(p)}$ contains the "circle" $\overline{\mathrm{Orb}_{g_x}(p_0)} = S_{x_0}(\mathbf{R})$. Let us first prove that $\overline{\mathrm{Orb}(p)}$ contains an open neighborhood of p_0 in $S(\mathbf{R}) \setminus \{\mathsf{Sing}(S(\mathbf{R}))\}$.

Since the point p_0 is not fixed by $g_x = s_z \circ s_y$, then either s_y or s_z does not fix p_0, say s_z; this means that the point p_0 is not a critical point of the projection

$$\pi_x \times \pi_y \ : \ S(\mathbf{R}) \to \mathbf{R}^2 \ ; \ (x,y,z) \mapsto (x,y).$$

Therefore, there exists some $\varepsilon > 0$ and a neighborhood V_ε of p_0 in $S(\mathbf{R})$ such that $\pi_x \times \pi_y$ maps V_ε diffeomorphically onto the square

$$(x_0 - \varepsilon, x_0 + \varepsilon) \times (y_0 - \varepsilon, y_0 + \varepsilon).$$

By construction, we have

$$\pi_x \times \pi_y(\overline{\mathrm{Orb}(p)}) \supset \pi_x \times \pi_y(\overline{\mathrm{Orb}_{g_x}(p_0)}) \supset \{x_0\} \times (y_0 - \varepsilon, y_0 + \varepsilon).$$

For each $y_1 \in (y_0-\varepsilon, y_0+\varepsilon)$ of irrational type, that is to say not of the form $2\cos(\pi\theta)$ with θ rational, there exists $p_1 = (x_0, y_1, z_1) \in \overline{\mathrm{Orb}(p)}$ (namely, the preimage of (x_0, y_1) by $\pi_x \times \pi_y$) and

$$\overline{\mathrm{Orb}(p)} \supset \overline{\mathrm{Orb}_{g_y}(p_1)} = S_{y_1}(\mathbf{R});$$

in other words, for each $y_1 \in (y_0 - \varepsilon, y_0 + \varepsilon)$ of irrational type, we have

$$\pi_x \times \pi_y(\overline{\mathrm{Orb}(p)}) \supset \pi_x \times \pi_y(\overline{\mathrm{Orb}_{g_y}(p_1)}) \supset (x_0 - \varepsilon, x_0 + \varepsilon) \times \{y_0\}.$$

Since those coordinates y_1 of irrational type are dense in $(y_0 - \varepsilon, y_0 + \varepsilon)$, we deduce that $V_\varepsilon \subset \overline{\mathrm{Orb}(p)}$, and $\overline{\mathrm{Orb}(p)}$ is open at p_0.

It remains to prove that $\overline{\mathrm{Orb}(p)}$ is open at any point $q \in \overline{\mathrm{Orb}(p)}$ which is not a singular point of $S(\mathbf{R})$. Let $q = (x_0, y_0, z_0)$ be such a point and assume that $q \notin \mathrm{Orb}(p)$ (otherwise we have already proved the assertion).

Since q is not a singular point of $S(\mathbf{R})$, one of the projections, say $\pi_x \times \pi_y : S(\mathbf{R}) \to \mathbf{R}^2$, is regular at q and we consider a neighborhood V_ε like above, $\pi_x \times \pi_y(V_\varepsilon) = (x_0-\varepsilon, x_0+\varepsilon)\times(y_0-\varepsilon, y_0+\varepsilon)$. By assumption, $\mathrm{Orb}(p)\cap V_\varepsilon$ is infinite (accumulating q) and, applying once again Lemma 4.5, one can find one such point $p_1 = (x_1, y_1, z_1) \in \mathrm{Orb}(p)\cap V_\varepsilon$ such that either x_1 or y_1 has irrational type, say x_1. Now, reasonning with p_1 like we did above with p_0, we eventually conclude that $V_\varepsilon \supset \overline{\mathrm{Orb}(p)}$, and $\overline{\mathrm{Orb}(p)}$ is open at q. □

5. Invariant geometric structures

In this section, we study the existence of $\mathcal{A}$-invariant geometric structures on surfaces S of the family Fam. An example of such an invariant structure is given by the area form Ω, defined by (1.6) in section 1.1. Another example occurs for the Cayley cubic: S_C is covered by $\mathbf{C}^* \times \mathbf{C}^*$ and the action of $\mathcal{A}$ on S_C is covered by the monomial action of $\mathsf{GL}(2, \mathbf{Z})$, that is also covered by the linear action of $\mathsf{GL}(2, \mathbf{Z})$ on $\mathbf{C} \times \mathbf{C}$ if we use the covering mapping

$$\pi : \mathbf{C} \times \mathbf{C} \to \mathbf{C}^* \times \mathbf{C}^*, \quad \pi(\theta, \phi) = (\exp(\theta), \exp(\phi));$$

as a consequence, there is an obvious $\mathcal{A}$-invariant affine structure on S_C.

Remark 5.1. — The surface S_C is endowed with a natural orbifold structure, the analytic structure near its singular points being locally isomorphic to the quotient of $\mathbf{C}^2$ near the origin by the involution $\sigma(x, y) = (-x, -y)$. The affine structure can be understood either in the orbifold language, or as an affine structure defined only outside the singularities (see below).

5.1. Invariant curves, foliations and webs

We start with

LEMMA 5.2. — *Whatever the choice of S in the family* Fam, *the group $\mathcal{A}$ does not preserve any (affine) algebraic curve on S.*

Of course, invariant curves appear if we blow up singularities. This is important for the study of special (Riccati) solutions of Painlevé VI equation (see section 6).

Proof. — Let C be an algebraic curve on S. Either C is contained in a fiber of π_z, or the projection $\pi_z(C)$ covers $\mathbf{C}$ minus at most finitely many points. If C is not contained in a fiber, we can choose $m_0 = (x_0, y_0, z_0)$ in C and a neighborhood U of m_0 such that z_0 is contained in $(0,2)$ and, in U, C intersects each fiber S_z of the projection π_z in exactly one point. Let $m' = (x', y', z')$ be any element of $C \cap U$ such that z' is an element of $(0,2)$. Then $\overline{g_z}$ is an elliptic transformation of $S_{z'}$ that preserves $C \cap S_{z'}$; since the intersection of C and $S_{z'}$ contains an isolated point m', this point is g_z periodic. As a consequence, z' is of the form $2\cos(\pi p/q)$ (see proposition 4.1). Since any $z' \in (0,2)$ sufficiently close to z_0 should satisfy an equation of this type, we obtain a contradiction.

Since no curve can be simultaneously contained in fibers of π_x, π_y and π_z, the lemma is proved. □

A (singular) web on a surface X is given by a hypersurface in the projectivized tangent bundle $\mathbb{P}TX$; for each point, the web determines a finite collection of directions tangent to X through that point. The number of directions is constant on an open subset of X but it may vary along the singular locus of the web. Foliations are particular cases of webs, and any web is locally made of a finite collection of foliations in the complement of its singular locus.

PROPOSITION 5.3. — *Whatever the choice of S in the family* Fam, *the group $\mathcal{A}$ does not preserve any web on S.*

Proof. — Let us suppose that there exists an invariant web W on one of the surfaces S. Let k and l be coprime positive integers and $m = (x, y, z)$ be a periodic point of g_z of period l, with

$$z = 2\cos(\pi k/l).$$

Let $L_1, \ldots, L_d$ be the directions determined by W through the point m, and $C_1, \ldots, C_d$ the local leaves of W which are tangent to these directions. The automorphism g_z^s, with $s = l(d!)$, fixes m, preserves the web and fixes each of the directions L_i; it therefore preserves each of the C_i. The proof of lemma 5.2 now shows that $d = 1$ and that the curves C_i are contained in the fiber of π_z through m. Since the set of points m which are g_z-periodic is Zariski dense in S, this argument shows that the web is the foliation by fibers of π_z. The same argument shows that the web should also coincide with the foliations by fibers of π_x or π_y, a contradiction. □

5.2. Invariant Affine Structures

A holomorphic affine structure on a complex surface M is given by an atlas of charts $\Phi_i : U_i \to \mathbf{C}^2$ for which the transition functions $\Phi_i \circ \Phi_j^{-1}$ are affine transformations of the plane $\mathbf{C}^2$. A local chart $\Phi : U \to \mathbf{C}^2$ is said to be affine if, for all i, $\Phi \circ \Phi_i^{-1}$ is the restriction of an affine transformation of $\mathbf{C}^2$ to $\Phi_i(U_i) \cap \Phi(U)$. A subgroup G of $\mathsf{Aut}(M)$ preserves the affine structure if elements of G are given by affine transformations in local affine charts.

Theorem 5.4. — *Let S be an element of* Fam. *Let G be a finite index subgroup of* $\mathsf{Aut}(S)$. *The group G preserves an affine structure on $S \setminus$* $\mathsf{Sing}(S)$, *if, and only if S is the Cayley cubic S_C.*

In what follows, S is a cubic of the family Fam and G will be a finite index subgroup of $\mathcal{A}$ preserving an affine structure on S.

Before giving the proof of this statement, we collect a few basic results concerning affine structures. Let X be a complex surface with a holomorphic affine structure. Let $\pi : \widetilde{X} \to X$ be the universal cover of X; the group of deck transformations of this covering is isomorphic to the fundamental group $\pi_1(X)$. Gluing together the affine local charts of X, we get a developping map

$$\mathbf{dev} : \widetilde{X} \to \mathbf{C}^2,$$

and a monodromy representation $\mathbf{Mon} : \pi_1(X) \to \mathsf{Aff}(\mathbf{C}^2)$ such that

$$\mathbf{dev}(\gamma(m)) = \mathbf{Mon}(\gamma)(\mathbf{dev}(m))$$

for all γ in $\pi_1(X)$ and all m in $\widetilde{X}$. The map $\mathbf{dev}$ is a local diffeomorphism but, a priori, it is neither surjective, nor a covering onto its image.

Let f be an element of $\mathsf{Aut}(X)$ that preserves the affine structure of X. Let m_0 be a fixed point of f, let $\widetilde{m_0}$ be an element of the fiber $\pi^{-1}(m_0)$, and let $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ be the lift of f that fixes $\widetilde{m_0}$. Since f is affine, there exists a unique affine automorphism $\mathbf{Aff}(f)$ of $\mathbf{C}^2$ such that

$$\mathbf{dev} \circ \widetilde{f} = \mathbf{Aff}(f) \circ \mathbf{dev}.$$

5.3. Proof of theorem 5.4: step 1.

In this first step, we show that $S \setminus \mathsf{Sing}(S)$ cannot be simply connected, and deduce from this fact that S is singular. Then we study the singularities of S and the fundamental group of $S \setminus \mathsf{Sing}(S)$.

Simple connectedness. — Assume that $S \setminus \mathsf{Sing}(S)$ is simply connected. The developping map **dev** is therefore defined on $S \setminus \mathsf{Sing}(S) \to \mathbf{C}^2$. Let N be a positive integer for which g_x^N is contained in G. Choose a fixed point m_0 of g_x as a base point. Since g_x^N preserves the affine structure, there exists an affine transformation $\mathbf{Aff}(g_x^N)$ such that

$$\mathbf{dev} \circ g_x^N = \mathbf{Aff}(g_x^N) \circ \mathbf{dev}.$$

In particular, **dev** sends periodic points of g_x^N to periodic points of $\mathbf{Aff}(g_x^N)$. Let m be a nonsingular point of S with its first coordinate in the interval $(-2, 2)$, and let U be an open neighborhood of m. Section 4.1 shows that periodic points of g_x^N form a Zariski-dense subset of U, by which we mean that any holomorphic functions $\Psi : U \to \mathbf{C}$ which vanishes on the set of periodic points of g_x^N vanishes everywhere. Since **dev** is a local diffeomorphism, periodic points of $\mathbf{Aff}(g_x^N)$ are Zariski-dense in a neighborhood of $\mathbf{dev}(m)$, and therefore $\mathbf{Aff}(g_x^N) = \mathrm{Id}$. This provides a contradiction, and shows that $S \setminus \mathsf{Sing}(S)$ is not simply connected.

Consequently, lemma 3.6 implies that S is singular and that the fundamental group of $S \setminus \mathsf{Sing}(S)$ is generated, as a normal subgroup, by the local fundamental groups around the singularities.

Orbifold structure. — We already explained in section 3.4 that the singularities of S are quotient singularities. If q is a singular point of S, S is locally isomorphic to the quotient of the unit ball $\mathbb{B}$ in $\mathbf{C}^2$ by a finite subgroup H of $\mathsf{SU}(2)$.

The local affine structure around q can therefore be lifted into a H-invariant affine structure on $\mathbb{B} \setminus \{(0,0)\}$, and then extended up to the origin by Hartogs theorem. In particular, **dev** lifts to a local diffeomorphism between $\mathbb{B}$ and an open subset of $\mathbf{C}^2$. This remark shows that the affine structure is compatible with the orbifold structure of S defined in section 3.4.

Let h be an element of the local fundamental group H. Let us lift the affine structure on $\mathbb{B}$ and assume that the monodromy action of h is trivial, i.e. $\mathbf{dev} \circ h = \mathbf{dev}$. Since **dev** is a local diffeomorphism, the singularity is isomorphic to a quotient of $\mathbb{B}$ by a proper quotient of H, namely the quotient of H by the smallest normal subgroup containing h. This provides a contradiction and shows that (i) H embeds in the global fundamental group of $S\setminus\mathsf{Sing}(S)$ and (ii) the universal cover of S in the orbifold sense is smooth (it is obtained by adding points to the universal cover of $S\setminus\mathsf{Sing}(S)$ above singularities of S).

In what follows, we denote the orbifold universal cover by $\pi:\widetilde{S}\to S$, and the developing map by $\mathbf{dev}:\widetilde{S}\to\mathbf{C}^2$.

Singularities. — Let q be a singular point of S. Let $\widetilde{q}$ be a point of the fiber $\pi^{-1}(q)$. Since the group $\mathcal{A}$ fixes all the singularities of S, it fixes q and one can lift the action of $\mathcal{A}$ on S to an action of $\mathcal{A}$ on $\widetilde{S}$ that fixes $\widetilde{q}$. If f is an element of $\mathcal{A}$, $\widetilde{f}$ will denote the corresponding holomorphic diffeomorphism of $\widetilde{S}$. Then we compose $\mathbf{dev}$ by a translation of the affine plane $\mathbf{C}^2$ in order to assume that

$$\mathbf{dev}(\widetilde{q})=(0,0).$$

By assumption, $\mathbf{dev}\circ\widetilde{g}=\mathbf{Aff}(g)\circ\mathbf{dev}$ for any element g in G, from which we deduce that the affine transformation $\mathbf{Aff}(g)$ are in fact linear. Since $\mathcal{A}$ almost preserves an area form, $\mathbf{Aff}(g)$ is an element of $\mathsf{GL}(2,\mathbf{C})$ with determinant $+1$ or -1; passing to a subgroup of index 2 in G, we shall assume that the determinant is 1. Since $\mathbf{dev}$ realizes a local conjugation between the action of G near $\widetilde{q}$ and the action of $\mathbf{Aff}(G)$ near the origin, the morphism

$$\begin{cases} G & \to \ \mathsf{SL}(2,\mathbf{C}) \\ g & \mapsto \ \mathbf{Aff}(g) \end{cases}$$

is injective.

Since G is a finite index subgroup of $\mathsf{Aut}(S)$, G contains a non abelian free group of finite index and is not virtually solvable. Let H be the finite subgroup of $\pi_1(S\setminus\mathsf{Sing}(S))$ that fixes the point $\widetilde{q}$. This group is normalized by the action of $\mathcal{A}$ on $\widetilde{S}$. Consequently, using the local affine chart determined by $\mathbf{dev}$, the group $\mathbf{Aff}(G)$ normalizes the monodromy group $\mathbf{Mon}(H)$. If $\mathbf{Mon}(H)$ is not contained in the center of $\mathsf{SL}(2,\mathbf{C})$, the eigenlines of the elements of $\mathbf{Mon}(H)$ determine a finite, non empty, and $\mathbf{Aff}(G)$-invariant set of lines in $\mathbf{C}^2$, so that $\mathbf{Aff}(G)$ is virtually solvable. This would contradict the injectivity of $g\mapsto\mathbf{Aff}(g)$. From this we deduce that any element of $\mathbf{Mon}(H)$ is a homothety with determinant 1. Since the monodromy representation is injective on H, we conclude that H "coincides" with the subgroup $\{+\mathrm{Id},-\mathrm{Id}\}$ of $\mathsf{SU}(2)$.

Linear part of the monodromy. — By lemma 3.6, the fundamental group of $S\setminus\mathsf{Sing}(S)$ is generated, as a normal subgroup, by the finite fundamental groups around the singularities of S. Since $\pm\mathrm{Id}$ is in the center of $\mathsf{GL}(2,\mathbf{C})$, the linear part of the monodromy $\mathbf{Mon}(\gamma)$ of any element γ in $\pi_1(S\setminus\mathsf{Sing}(S))$ is equal to $+\mathrm{Id}$ or $-\mathrm{Id}$.

5.4. Proof of theorem 5.4: step 2

We now study the dynamics of the parabolic elements of G near the fixed point q.

Linear part of automorphisms. — Let g be an element of the group G. Let m be a fixed point of g and $\widetilde{m}$ a point of the fiber $\pi^{-1}(m)$. Let $\widetilde{g}_{\widetilde{m}}$ be the unique lift of g to $\widetilde{S}$ fixing $\widetilde{m}$ (with the notation used in step 1, $\widetilde{g}_{\widetilde{q}}=\widetilde{g}$). Since g preserves the affine structure, there exists an affine transformation $\mathbf{Aff}(\widetilde{g}_{\widetilde{m}})$ such that

$$\mathbf{dev}\circ\widetilde{g}_{\widetilde{m}}=\mathbf{Aff}(\widetilde{g}_{\widetilde{m}})\circ\mathbf{dev}.$$

Note that $\mathbf{Aff}(\widetilde{g}_{\widetilde{m}})$ depends on the choice of m and $\widetilde{m}$, but that $\mathbf{Aff}(\widetilde{g}_{\widetilde{m}})$ is uniquely determined by g up to composition by an element of the monodromy group $\mathbf{Mon}(\pi_1(S\setminus\mathsf{Sing}(S))$. Since the linear parts of the monodromy are equal to $+\mathrm{Id}$ or $-\mathrm{Id}$, we get a well defined morphism

$$\begin{cases} G & \to \quad \mathsf{PSL}(2,\mathbf{C})\\ g & \mapsto \quad \mathbf{Lin}(g)\end{cases}$$

that determines the linear part of $\mathbf{Aff}(\widetilde{g}_{\widetilde{m}})$ modulo $\pm\mathrm{Id}$ for any choice of m and $\widetilde{m}$.

Parabolic elements. — Since the linear part $\mathbf{Lin}(g)$ does not depend on the fixed point m, it turns out that $\mathbf{Lin}$ preserves the type of g: we now prove and use this fact in the particular case of the parabolic elements g_x, g_y and g_z .

Let N be a positive integer such that g_x^N is contained in G. For m, we choose a regular point of S which is periodic of period l for g_x^N and which is not a critical point of the projection π_x. Then g_x^{Nl} fixes the fiber S_x of π_x through m pointwise. Since g_x is not periodic and preserves the fibers of π_x, this implies that the differential of g_x^{Nl} at m is parabolic. Let $\widetilde{m}$ be a point of $\pi^{-1}(m)$ and $(\widetilde{g_x^{Nl}})_{\widetilde{m}}$ the lift of g_x^{Nl} fixing that point. The universal cover π provides a local conjugation between g_x^{Nl} and $(\widetilde{g_x^{Nl}})_{\widetilde{m}}$ around m and $\widetilde{m}$, and the developping map provides a local conjugation between $(\widetilde{g_x^{Nl}})_{\widetilde{m}}$ and $\mathbf{Lin}(g_x^{Nl})$. As a consequence, $\mathbf{Lin}(g_x^{Nl})$ is a parabolic element of $\mathsf{PSL}(2,\mathbf{C})$.

Since a power of $\mathbf{Lin}(g_x^N)$ is parabolic, $\mathbf{Lin}(g_x^N)$ itself is parabolic. In particular, the dynamics of $\widetilde{g_x^N}$ near $\widetilde{q}$ is conjugate to a linear upper triangular transformation of $\mathbf{C}^2$ with diagonal entries equal to 1.

As a consequence, the lift $\widetilde{g_x}$ is locally conjugate near $\widetilde{q}$ to a linear parabolic transformation with eigenvalues ± 1. The eigenline of this transformation corresponds to the fiber S_z through q. Since the local fundamental group H coincides with $\pm Id$, this eigenline is mapped to a curve a fixed point by the covering π. In particular, the fiber S_z through q is a curve of fixed points for g_x.

Of course, a similar study holds for g_y and g_z.

Fixed points and coordinates of the singular point. — The study of fixed points of g_x, g_y and g_z (see lemma 4.2) now shows that the coordinates of the singular point q are equal to ± 2. Let ϵ_x, ϵ_y and ϵ_z be the sign of the coordinates of q, so that

$$q = (2\epsilon_x, 2\epsilon_y, 2\epsilon_z).$$

Recall from section 3.4 that the coefficients A, B, C, and D are uniquely determined by the coordinates of any singular point of S. If the product $\epsilon_x\epsilon_y\epsilon_z$ is positive, then, up to symmetry, $q = (2, 2, 2)$ and S is the surface

$$x^2 + y^2 + z^2 + xyz = 8x + 8y + 8z - 28;$$

in this case, q is the unique singular point of S, and this singular point is not a node: the second jet of the equation near q is $(x + y + z)^2 = 0$. This contradicts the fact that q has to be a node (see section 5.3). From this we deduce that the product $\epsilon_x\epsilon_y\epsilon_z$ is equal to -1, and that S is the Cayley cubic.

6. Irreducibility of Painlevé VI Equation.

The goal of this section is to apply the previous section to the irreducibility of Painlevé VI equation.

6.1. The Riemann-Hilbert correspondance and P_{VI}-monodromy

The naive phase space of Painlevé VI equation is parametrized by coordinates $(t, q(t), q'(t)) \in (\mathbb{P}^1 \setminus \{0, 1, \infty\}) \times \mathbf{C}^2$; the "good" phase space is a convenient semi-compactification still fibering over the three punctured sphere

$$\mathcal{M}(\theta) \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

whose fibre $\mathcal{M}_{t_0}(\theta)$, at any point $t_0 \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, is the Hirzebruch surface $\mathbb{F}_2$ blown-up at 8-points minus some divisor, a union of 5 rational curves (see [30]). The analytic type of the fibre, namely the position of the 8 centers and the 5 rational curves, only depends on Painlevé parameters $\theta = (\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta) \in \mathbf{C}^4$ and t_0. This fibre bundle is analytically (but not algebraically!) locally trivial: the local trivialization is given by the Painlevé foliation (see [34]) which is transversal to the fibration. The monodromy of Painlevé equation is given by a representation

$$\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, t_0) \to \mathsf{Diff}(\mathcal{M}_{t_0}(\theta))$$

into the group of analytic diffeomorphisms of the fibre.

On the other hand, the space of initial conditions $\mathcal{M}_{t_0}(\theta)$ may be interpreted as the moduli space of rank 2, trace free meromorphic connections over $\mathbb{P}^1$ having simple poles at $(p_\alpha, p_\beta, p_\gamma, p_\delta) = (0, t_0, 1, \infty)$ with prescribed residual eigenvalues $\pm\frac{\theta_\alpha}{2}$, $\pm\frac{\theta_\beta}{2}$, $\pm\frac{\theta_\gamma}{2}$ and $\pm\frac{\theta_\delta}{2}$. The Riemann-Hilbert correspondance therefore provides an analytic diffeomorphism

$$\mathcal{M}_{t_0}(\theta) \to \hat{S}_{(A,B,C,D)}$$

where $\hat{S}_{(A,B,C,D)}$ is the minimal desingularization of $S = S_{(A,B,C,D)}$, the parameters (A, B, C, D) being given by formulae (1.12) and (1.5). From this point of view, the Painlevé VI foliation coincides with the isomonodromic foliation: leaves correspond to universal isomonodromic deformations of those connections. The monodromy of Painlevé VI equation correspond to a morphism

$$\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, t_0) \to \mathsf{Aut}(S_{(A,B,C,D)})$$

and coincides with the Γ_2-action described in section 2.3. For instance, g_x (resp. g_y) is the Painlevé VI monodromy when t_0 turns around 0 (resp. 1) in the obvious simplest way. All this is described with much detail in [19].

6.2. Classical solutions versus periodic orbits

When $S_{(A,B,C,D)}$ is singular, the exceptional divisor in $\hat{S}_{(A,B,C,D)}$ is a finite union of rational curves in restriction to which Γ_2 acts by Möbius transformations. To each such rational curve corresponds a rational hypersurface $\mathcal{H}$ of the phase space $\mathcal{M}(\theta)$ invariant by the Painlevé VI foliation. On $\mathcal{H}$, the projection $\mathcal{M}(\theta) \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ restricts to a regular rational fibration and the Painlevé equation restricts to a Riccati equation of hypergeometric type: we get a one parameter family of Riccati solutions. See [37, 35, 19] for a classification of singular points of $S_{(A,B,C,D)}$ and their link with Riccati solutions; they occur precisely when either one of the θ-parameter is an integer, or when the sum $\sum \theta_i$ is an integer. Since

$S_{(A,B,C,D)}$ is affine, there are obviously no other complete curve in $\mathcal{M}_{t_0}(\theta)$ (see section 5.1).

The complete list of algebraic solutions of Painlevé VI equation has been conjectured in [4], and closed only recently by Lisovyy and Tykhyy in [24]. Apart from those solutions arising as special cases of Riccati solutions, that are well known, they correspond to finite Γ_2-orbits on the smooth part of $S_{(A,B,C,D)}$ (see [22]). Following section 4.2, apart from the three well-known families of 2, 3 and 4-sheeted algebraic solutions, other algebraic solutions are countable and the cosines of the corresponding θ-parameters are rational numbers. In the particular Cayley case $S_C = S_{(0,0,0,4)}$, periodic Γ_2-orbits arise from pairs of roots of unity (u, v) on the two-fold cover $(\mathbf{C}^*)^2$ (see 2.1); there are infinitely many periodic orbits in this case and they are dense in the real bounded component of $S_C \setminus \{\mathsf{Sing}(S_C)\}$. The corresponding algebraic solutions were discovered by Picard in 1889 (before Painlevé discovered the general P_{VI}-equation !); see [26] and below.

6.3. Nishioka-Umemura and Malgrange irreducibility

In 1998, Watanabe proved in [37] the irreducibility of Painlevé VI equation in the sense of Nishioka-Umemura for any parameter θ: the generic solution of $P_{VI}(\theta)$ is non classical, and classical solutions are

- Riccati solutions (like above),
- algebraic solutions.

Non classical roughly means “very transcendental” with regards to the XIXth century special functions: the general solution cannot be expressed in an algebraic way by means of solutions of linear, or first order non linear differential equations. A precise definition can be found in [10].

Another notion of irreducibility was introduced by Malgrange in [25]: he defines the Galois groupoid of an algebraic foliation to be the smallest algebraic Lie-pseudo-group that contains the tangent pseudo-group of the foliation (hereafter referred to as the "pseudo-group"); this may be viewed as a kind of Zariski closure for the pseudo-group of the foliation. Larger Galois groupoids correspond to more complicated foliations. From this point of view, it is natural to call irreducible any foliation whose Galois groupoid is as large as possible, *i.e.* coincides with the basic pseudo-group.

For Painlevé equations, a small restriction has to be taken into account: it has been known since Malmquist that Painlevé foliations may be defined as kernels of closed meromorphic 2-forms. The pseudo-group, and the Galois groupoid, both preserve the closed 2-form. The irreducibility conjectured by Malgrange is that the Galois groupoid of Painlevé equations coincide with the algebraic Lie-pseudo-group of those transformations on the phase space preserving ω. This was proved for Painlevé I equation by Casale in [9].

For a second order polynomial differential equation $P(t, y, y', y'') = 0$, like Painlevé equations, Casale proved in [10] that Malgrange-irreducibility implies Nishioka-Umemura-irreducibility; the converse is not true as we shall see.

6.4. Invariant geometric structures

Restricting to a transversal, e.g. the space of initial conditions $\mathcal{M}_{t_0}(\theta)$ for Painlevé VI equations, the Galois groupoid defines an algebraic geometric structure which is invariant under monodromy transformations; reducibility would imply the existence of an extra geometric structure on $\mathcal{M}_{t_0}(\theta)$, additional to the volume form ω, preserved by all monodromy transformations. In that case, a well known result of Cartan, adapted to our algebraic setting by Casale in [9], asserts that monodromy transformations

- either preserve an algebraic foliation,
- or preserve an algebraic affine structure.

Here, "algebraic" means that the object is defined over an algebraic extension of the field of rational functions, or equivalently, becomes well-defined over the field of rational functions after some finite ramified cover. For instance, "algebraic foliation" means polynomial web. As a corollary of proposition 5.3 and Theorem 5.4, we shall prove the following

THEOREM 6.1. — *The sixth Painlevé equation is irreducible in the sense of Malgrange, except in one of the following cases:*

- $\theta_\omega \in \frac{1}{2} + \mathbf{Z}$, $\omega = \alpha, \beta, \gamma, \delta$,
- $\theta_\omega \in \mathbf{Z}$, $\omega = \alpha, \beta, \gamma, \delta$, *and* $\sum_\omega \theta_\omega$ *is even.*

All these special parameters are equivalent, modulo Okamoto symmetries, to the case $\theta = (0, 0, 0, 1)$. *The corresponding cubic surface is the Cayley cubic.*

Of course, in the Cayley case, the existence of an invariant affine structure shows that the Painlevé foliation is Malgrange-reducible (see [8]). This will be made more precise in section 6.6.

Before proving the theorem, we need a stronger version of Lemma 5.2

LEMMA 6.2. — *Let S be an element of the family* Fam. *There is no $\mathcal{A}$-invariant curve of finite type in S.*

By "curve of finite type" we mean a complex analytic curve in S with a finite number of irreducible components C_i, such that the desingularization of each C_i is a Riemann surface of finite type.

Proof. — Let $C \subset S$ be a complex analytic curve of finite type. Since S is embedded in $\mathbf{C}^3$, C is not compact. In particular, C is not isomorphic to the projective line and the group of holomorphic diffeomorphisms of C is virtually solvable. Since $\mathcal{A}$ contains a non abelian free subgroup, there exists an element f in $\mathcal{A} \setminus \{\mathrm{Id}\}$ which fixes C pointwise. From this we deduce that C is contained in the algebraic curve of fixed points of f. This shows that the Zariski closure of C is an $\mathcal{A}$-invariant algebraic curve, and we conclude by Lemma 5.2. □

6.5. Proof of theorem 6.1

In order to prove that Painlevé VI equation, for a given parameter $\theta \in \mathbf{C}^4$ is irreducible, it suffices, due to [9] and the discussion above, to prove that the space of initial conditions $\mathcal{M}_{t_0}(\theta)$ does not admit any monodromy-invariant web or algebraic affine structure. Via the Riemann-Hilbert correspondance, such a geometric structure will induce a similar Γ_2-invariant structure on the corresponding character variety $S_{(A,B,C,D)}$. But we have to be carefull: the Riemann-Hilbert map is not algebraic but analytic. As a consequence, the geometric structures we have now to deal with on $S_{(A,B,C,D)}$ are not rational anymore, but meromorphic (on a finite ramified cover). Anyway, the proof of proposition 5.3 is still valid in this context and exclude the possibility of Γ_2-invariant analytic web.

Multivalued affine structures. — We now explain more precisely what is a Γ_2-invariant multivalued meromorphic affine structure in the above sense. First of all, a meromorphic affine structure is an affine structure in the sense of section 5.2 defined on the complement of a proper analytic subset Z, having moderate growth along Z in a sense that we do not need to consider here. This structure is said to be Γ_2-invariant if both Z, and the regular affine structure induced on the complement of Z, are Γ_2-invariant. Now, a multivalued meromorphic affine structure is a meromorphic structure (with polar locus Z') defined on a finite analytic ramified cover $\pi' : S' \to S$; the ramification locus X is an analytic set. This structure is said to be Γ_2-invariant if both X and $Z = \pi'(Z')$ are invariant and, over the complement of $X \cup Z$, Γ_2 permutes the various regular affine structures induced by the various branches of π'.

Let us prove that the multivalued meromorphic affine structure induced on S by a reduction of Painlevé VI Galois groupoid has actually no pole, and no ramification apart from singular points of S. Indeed, let C be the union of Z and R; then C is analytic in S but comes from an algebraic curve in $\mathcal{M}_{t_0}(\theta)$ (the initial geometric structure is algebraic in $\mathcal{M}_{t_0}(\theta)$), so that the 1-dimensional part of C is a curve of finite type. Lemma 6.2 then show that C is indeed a finite set. In other words, C is contained in $\mathsf{Sing}(S)$, R itself is contained in $\mathsf{Sing}(S)$ and Z is empty.

Singularities of S. — Since the ramification set R is contained in $\mathsf{Sing}(S)$, the cover π' is an étale cover in the orbifold sense (singularities of S' are also quotient singularities). Changing the cover $\pi' : S' \to S$ if necessary, we may assume that π' is a Galois cover.

If S is simply connected, then of course π' is trivial, the affine structure is univalued, and theorem 5.4 provides a contradiction. We can therefore choose a singularity q of S, and a point q' in the fiber $(\pi')^{-1}(q)$. Since $\pi_1(S;q)$ is finitely generated, the number of subgroups of index $\deg(\pi')$ in $\pi_1(S;q)$ is finite. As a consequence, there is a finite index subgroup G in Γ_2 which lifts to S' and preserves the univalued affine structure defined on S'.

We now follow the proof of theorem 5.4 for G, S' and its affine structure. First, we denote $\pi : \tilde{S} \to S'$ the universal cover of S', we choose a point $\tilde{q}$ in the fiber $\pi^{-1}(q')$, and we lift the action of G to an action on the universal cover $\tilde{S}$ fixing $\tilde{q}$. Then we fix a developping map $\mathbf{dev} : \tilde{S} \to \mathbf{C}^2$ with $\mathbf{dev}(\tilde{q}) = 0$; these choices imply that $\mathbf{Aff}(g)$ is linear for any g in G. Section 5.3 shows that the singularities of S and S' are simple nodes.

Now comes the main difference with sections 5.3 and 5.4: *a priori*, the fundamental group of S' is not generated, as a normal subgroup, by the local fundamental groups around the singularities of $\mathsf{Sing}(S')$. It could be the case that S' is smooth, with an infinite fundamental group. So, we need a new argument to prove that g_x (*resp.* g_y and g_z) has a curve of fixed points through the singularity q.

Parabolic dynamics. — Let $g = g_x^n$ be a non trivial iterate of g_x that is contained in G. The affine transformation $\mathbf{Aff}(g)$ is linear, with determinant 1 ; we want to show that this transformation is parabolic.

Let $\tilde{U}$ be an open subset of $\tilde{S}$ on which both $\mathbf{dev}$ and the universal cover $\pi' \circ \pi$ are local diffeomorphisms, and let U be the projection of $\tilde{U}$ on S by $\pi' \circ \pi$. We choose $\tilde{U}$ in such a way that U contains points $m = (x, y, z)$ with x in the interval $[-2, 2]$. The fibration of U by fibers of the projection π_x is mapped onto a fibration $\mathcal{F}$ of $\mathbf{dev}(\tilde{U})$ by the local diffeomorphism $\mathbf{dev} \circ (\pi' \circ \pi)^{-1}$. Let us prove, first, that $\mathcal{F}$ is a foliation by parallel lines.

Let m be a point of U which is g-periodic, of period l. Then, the fiber of π_x through m is a curve of fixed point for g^l. If $\tilde{m}$ is a lift of m in $\tilde{S}$, one can find a lift $\gamma \circ \tilde{g}^l$ of g to $\tilde{S}$ (γ in $\pi_1(S,q) = \mathsf{Aut}(\pi)$) that fixes pointwise the fiber through $\tilde{m}$. As a consequence, the fiber of $\mathcal{F}$ through $\mathbf{dev}(\tilde{m})$ coincides locally with the set of fixed points of the affine transformation $\mathbf{Aff}(g^l) \circ \mathbf{Mon}(\gamma)$. As such, the fiber of $\mathcal{F}$ through $\mathbf{dev}(\tilde{m})$ is an affine line.

This argument shows that an infinite number of leaves of $\mathcal{F}$ are affine lines, or more precisely coincide with the intersection of affine lines with $\mathbf{dev}(\tilde{U})$. Since g preserves each fiber of π_x, the foliation $\mathcal{F}$ is leafwise $(\mathbf{Aff}(g^l) \circ \mathbf{Mon}(\gamma))$-invariant. Assume now that L is a line which coincides with a leaf of $\mathcal{F}$ on $\mathbf{dev}(\tilde{U})$. If L is not parallel to the line of fixed points of $\mathbf{Aff}(g^l) \circ \mathbf{Mon}(\gamma)$, then the affine transformation $\mathbf{Aff}(g^l) \circ \mathbf{Mon}(\gamma)$ is a linear map (since it has a fixed point), with determinant ± 1, and with two eigenlines, one of them, the line of fixed points, corresponding to the eigenvalue 1. This implies that $\mathbf{Aff}(g^l) \circ \mathbf{Mon}(\gamma)$ has finiter order. Since g is not periodic, we conclude that L is parallel to the line of fixed points of $\mathbf{Aff}(g^l) \circ \mathbf{Mon}(\gamma)$, and that the foliation $\mathcal{F}$ is a foliation by parallel lines.

By holomorphic continuation, we get that the image by $\mathbf{dev}$ of the fibration $\pi_x \circ \pi$ is a foliation of the plane by parallel lines.

Let us now study the dynamics of $\tilde{g}$ near the fixed point $\tilde{q}$. Using the local chart $\mathbf{dev}$, $\tilde{g}$ is conjugate to the linear transformation $\mathbf{Aff}(g)$. Since g preserves each fiber of π_x, $\mathbf{Aff}(g)$ preserves each leaf of the foliation $\mathcal{F}$. Since g is not periodic, $\mathbf{Aff}(g)$ is not periodic either, and $\mathbf{Aff}(g)$ is a linear parabolic transformation. As a consequence, g has a curve of fixed points through q.

Conclusion. — We can now apply the arguments of the end of section 5.4 word by word to conclude that S is the Cayley cubic.

6.6. Picard parameters of Painlevé VI equation and the Cayley cubic

Let us now explain in more details why the Cayley case is so special with respect to Painlevé equations. Consider the universal cover

$$\pi_t : \mathbf{C} \to \{y^2 = x(x-1)(x-t)\} \; ; \; z \mapsto (x(t,z), y(t,z))$$

of the Legendre elliptic curve with periods $\mathbf{Z} + \tau\mathbf{Z}$ - this makes sense at least on a neighborhood of $t_0 \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. The functions $\tau = \tau(t)$ and π_t are analytic in t.

The following theorem, obtained by Picard in 1889, shows that the Painlevé equations corresponding to the Cayley cubic have (almost) classical solutions.

Theorem 6.3 (Picard, see [8] for example). — *The general solution of the Painlevé sixth differential equation $P_{VI}(0, 0, 0, 1)$ is given by*

$$t \mapsto x(t, c_1 + c_2 \cdot \tau(t)), \quad c_1, c_2 \in \mathbf{C}.$$

Moreover, the solution is algebraic if, and only if c_1 and c_2 are rational numbers.

Note that $c_1, c_2 \in \mathbf{Q}$ exactly means that $\pi_t(c_1 + c_2 \cdot \tau(t))$ is a torsion point of the elliptic curve.

Finally, $P_{VI}(0, 0, 0, 1)$-equation can actually be integrated by means of elliptic functions, but in a way that is non classical with respect to Nishioka-Umemura definition. Coming back to Malgrange's point of view, the corresponding polynomial affine structure on the phase space $\mathcal{M}(0, 0, 0, 1)$ has been computed by Casale in [8], thus proving the reducibility of P_{VI} $(0, 0, 0, 1)$ equation (and all its birational Okamoto symmetrics) in the sense of Malgrange.

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