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POLYNOMIAL BOUNDS FOR THE OSCILLATION OF SOLUTIONS OF FUCHSIAN SYSTEMS

by Gal BINYAMINI & Sergei YAKOVENKO

Abstract. — We study the problem of placing effective upper bounds for the number of zeroes of solutions of Fuchsian systems on the Riemann sphere. The principal result is an explicit (non-uniform) upper bound, polynomially growing on the frontier of the class of Fuchsian systems of a given dimension n having m singular points. As a function of n, m, this bound turns out to be double exponential in the precise sense explained in the paper.

As a corollary, we obtain a solution of the so-called restricted infinitesimal Hilbert 16th problem, an explicit upper bound for the number of isolated zeroes of Abelian integrals which is polynomially growing as the Hamiltonian tends to the degeneracy locus. This improves the exponential bounds recently established by A. Glutsyuk and Yu. Ilyashenko.

Résumé. — Nous étudions le problème d'une borne supérieure effective sur le nombre des racines isolées des solutions de systèmes de type Fuchs sur la sphère de Riemann. Le résultat principal est une borne explicite non uniforme à croissance polynômiale sur la frontière de l'ensemble des systèmes fuchsiens de dimension n quelconque ayant m singularités. Comme une fonction de n, m, la borne est doublement exponentielle dans le sens précis décrit dans le manuscrit.

Comme corollaire, nous obtenons la solution à croissance polynômiale du problème d'Hilbert infinitésimal restreint, qui améliore les bornes exponentielles récemment obtenues par A. Glutsyuk et Yu. Ilyashenko

1. Zeros of solutions of Fuchsian systems and restricted infinitesimal Hilbert 16th problem

1.1. Fuchsian systems and zeros of their solutions

Let Ω be a meromorphic (rational) $n \times n$-matrix 1-form on the Riemann sphere $\mathbb{C}P^1$ with a singular (polar) locus $\Sigma = \{\tau_1, \ldots, \tau_m\}$ consisting of m

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distinct points. The linear system of Pfaffian equations

$$\mathrm{d}x - \Omega x = 0, \qquad \Omega = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \tag{1.1}$$

is said to be *Fuchsian*, if all points $\tau_i \in \Sigma$ are first order poles of Ω. We will denote the collection of all Fuchsian systems of rank n with m singular points by $\mathscr{F}_{n,m}$. Later it will be identified with an $m(n^2+1)$-dimensional semialgebraic variety. A Fuchsian system can be always expanded as the sum of its *principal Fuchsian parts*,

$$\Omega = \sum_{j=1}^{m} \Omega_j, \qquad \Omega_j = \frac{A_j \mathrm{d}t}{t-\tau_j}, \quad \tau_j \in \Sigma, \quad \sum_{j=1}^{m} A_j = 0, \tag{1.2}$$

with the *residue matrices* $A_1, \ldots, A_m$ (the point $\tau = \infty$, singular for each Ω_j, is nonsingular for the sum; if $\infty = \tau_m \in \Sigma$, then one has $m-1$ terms corresponding to the finite points $\tau_1, \ldots, \tau_{m-1}$ of Σ, yet with a nontrivial residue A_m corresponding to the point at infinity).

Solutions of a Fuchsian system (1.1) are multivalued holomorphic vector-functions on $\mathbb{C}P^1 \smallsetminus \Sigma$, ramified over the singular locus, and growing moderately (no faster than polynomially in the reciprocal distance to Σ) when approaching points of Σ along non-spiraling paths.

We are interested in *explicit global upper bounds* on the maximal number of isolated zeros of linear combinations of the form $y(t) = \sum_{i=1}^{n} c_i\, x_i(t)$, $t \in \mathbb{C}P^1$. The initial motivation for this question came from the attempts to solve the so called infinitesimal Hilbert 16th problem, see [23, 11] and §1.4, but the subject is interesting in itself as well, as linear systems provide a vast and essentially unique source of transcendental functions which still admit global finite bounds for their zeros [23, §3].

Because of the multivaluedness of the solutions, some disambiguation is required. Assume that the punctured Riemann sphere $\mathbb{C}P^1 \smallsetminus \Sigma$ is covered by finitely many *simply connected polygonal domains* U_α (polygonality here and below means that each domain U_α is bounded by finitely many circular arcs and line segments). In each such domain one can consistently choose a holomorphic branch of any solution $x(t)$ to (1.1) and form all scalar linear combinations $y(t)$ as above. Clearly, the number

$$N_\alpha(\Omega) = \sup_{x(\cdot),\ c \in \mathbb{C}^n} \#\{t \in U_\alpha\ c_1 x_1(t) + \cdots + c_n x_n(t) = 0\} \leqslant +\infty,$$

where #{a set} denotes the number of *isolated* points of the set and the supremum is taken over all solutions $x(\cdot)$ of the system (1.1), is independent of the choice of the branches.

Let $N(\Omega, U)$ be the total number of the zeros counted in this way in all domains of a given covering $U = \{U_\alpha\}$, and $\mathcal{N}(\Omega)$ the infimum of this total number, taken over all polygonal coverings U,

$$N(\Omega, U) = \sum_\alpha N_\alpha(\Omega) \leqslant +\infty, \qquad \mathcal{N}(\Omega) = \inf_U N(\Omega, U). \tag{1.3}$$

By abuse of language, we say that the number $\mathcal{N}(\Omega)$, if finite, gives a *global bound for the number of zeros of solutions of the Fuchsian system* (1.1). Considered as a function $\mathcal{N} : \mathscr{F}_{n,m} \to \mathbb{N} \cup \{+\infty\}$ on the space of all Fuchsian systems (or its subspaces), it will be called the *counting function*.

Without additional assumptions the "bound" $\mathcal{N}(\Omega)$ may well be infinite, since even individual linear combinations may have infinitely many isolated zeros accumulating to the singular locus Σ. It turns out that the finiteness of the bound is closely related to the spectral properties of the *residue matrices*

$$A_i = \operatorname{Res}_{\tau_i} \Omega \in \operatorname{Mat}_{n\times n}(\mathbb{C}), \qquad \tau_i \in \Sigma, \quad i = 1, \dots, m.$$

Example 1.1. — The function $y(t) = t^{\mathrm{i}} + t^{-\mathrm{i}} = 2\cos\ln t$ is a linear combination corresponding to the *Euler system* with $\Omega = t^{-1}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\mathrm{d}t$ in the standard affine chart on $\mathbb{C}P^1$. This combination has infinitely many zeros accumulating to the two singular points $\tau_1 = 0$ and $\tau_2 = \infty$ with the residues $A_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = -A_\infty$.

However, under additional assumptions on the spectra of the residue matrices A_i one can guarantee finiteness of the upper bound. The following result can easily be obtained by the tools developed in [14, 15].

Theorem 1.2. — *Assume that all residue matrices $A_i = \operatorname{Res}_{\tau_i} \Omega$ of the Fuchsian system* (1.1) *have only real eigenvalues,*

$$\operatorname{Spec} A_i \subseteq \mathbb{R}, \qquad i = 1, \dots, m.$$

Then the corresponding bound for the number of isolated zeros of solutions is finite, $\mathcal{N}(\Omega) < \infty$.

We will not give the proof of this result since it is purely existential and does not allow any explicit estimate for the bound $\mathcal{N}(\Omega)$. Formally it follows from Theorem 1.5 below, whose proof is independent of Theorem 1.2.

1.2. Unboundedness of the counting function

Denote by $\mathscr{S}_{n,m} \subset \mathscr{F}_{n,m}$ the subspace of the space of Fuchsian systems with real spectra of all residues:

$$\mathscr{S}_{n,m} = \{\Omega \in \mathscr{F}_{n,m} : \forall \tau \in \varSigma,\ \operatorname{Spec} \operatorname{Res}_\tau \Omega \subseteq \mathbb{R}\}. \tag{1.4}$$

By Theorem 1.2, the counting function $\mathcal{N}$ takes finite values on $\mathscr{S}_{n,m}$. Yet since $\mathscr{S}_{n,m}$ is non-compact, this finiteness does not prevent the counting function from being *unbounded* on $\mathscr{S}_{n,m}$. The following examples show that this is indeed the case.

Example 1.3. — The counting function may grow to infinity together with the norms of residue matrices of Ω.

Indeed, for the Euler system, $\Omega = t^{-1}\left(\begin{smallmatrix} N & 0 \\ 0 & 0\end{smallmatrix}\right)\mathrm{d}t$ with $N \in \mathbb{N}$, the linear combination $y(t) = t^N - 1$ has N isolated zeros uniformly distributed on the unit circle. In this example N is the norm of the residues at $t = 0$ and $t = \infty$.

Somewhat less expected is the fact that the counting function may grow to infinity as some of the singular points collide with each other even without explosion (growth to infinity) of the residues.

Example 1.4. — Consider the family of Fuchsian systems Ω_ε, $\varepsilon \in \mathbb{C}$, $\varepsilon \neq 0$, with the four-point singular locus $\varSigma = \{0, \varepsilon, 1/\varepsilon, +\infty\}$ and four residue matrices, each with the real spectrum,

$$A_0 = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix},\ A_\varepsilon = \begin{pmatrix} 0 & \\ -1 & 0 \end{pmatrix}, \qquad \begin{aligned} A_{1/\varepsilon} &= -A_\varepsilon, \\ A_\infty &= -A_0. \end{aligned} \tag{1.5}$$

As the complex parameter $\varepsilon \neq 0$ tends to zero, two pairs of singularities, $\{0, \varepsilon\}$ and $\{1/\varepsilon, \infty\}$, collide. The limit is the Euler system described in Example 1.1 with the residue matrices $A_0 + A_\varepsilon$ and $-(A_\infty + A_{1/\varepsilon})$ respectively, which has infinitely many isolated zeros accumulating to both singularities.

This obviously means that $\limsup_{|\varepsilon|\to 0} \mathcal{N}(\Omega_\varepsilon) = +\infty$. Indeed, otherwise there would exist a finite upper bound N for $\mathcal{N}(\Omega_\varepsilon)$ uniform over all sufficiently small $|\varepsilon|$. Consider a triangle $K \Subset \mathbb{C}P \smallsetminus \varSigma$ which contains at least $N + 1$ isolated zeros of the solution $y(t) = t^{\mathrm{i}} + t^{-\mathrm{i}}$ of the limit system Ω_0 *strictly inside* (*cf.* with Example 1.1). By semicontinuity, for all sufficiently small values of $|\varepsilon|$, the corresponding combination of solutions of the system with the matrix Ω_ε will have at least $N+1$ zeros, since the restriction of Ω_ε on K depends holomorphically on ε. Hence $\mathcal{N}(\Omega_\varepsilon) \geqslant N+1$ in contradiction with the assumption.

1.3. Polynomial bounds

The two examples above show that the counting function $\mathcal{N}(\Omega)$ grows to infinity as Ω approaches the frontier of the subclass $\mathscr{S}_{n,m}$ of Fuchsian systems, at least along some parts of this frontier. The main positive result of the current work is an *explicit upper bound* for $\mathcal{N}(\Omega)$ which grows *polynomially* as the Fuchsian system (1.1), identified with its matrix Ω, approaches the frontier of the class $\mathscr{S}_{n,m}$.

Here and everywhere in this paper we define the norm of a polynomial $p \in \mathbb{C}[x_1, \ldots, x_n]$ in several variables as follows,

$$\|p\| = \sum |c_\alpha|, \qquad \text{if} \quad p(x) = \sum c_\alpha x^\alpha, \quad \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}_+^r. \tag{1.6}$$

The norm $|M|$ of a matrix $M \in \operatorname{Mat}_{n\times n}(\mathbb{C})$ is taken to be the standard Euclidean (Hermitian) matrix norm (though any other norm would of course be essentially equivalent).

Let $R_\flat : \mathscr{F}_{n,m} \to \mathbb{R}_+$ be the function defined as follows,

$$R_\flat(\Omega) = 2 + \sum_{\tau' \neq \tau'' \in \Sigma} \operatorname{dist}^{-1}(\tau', \tau'') + \sum_{\tau \in \Sigma} |\operatorname{Res}_\tau \Omega|, \tag{1.7}$$

where $\operatorname{dist}(\cdot,\cdot)$ is the Fubini–Study distance on $\mathbb{C}P^1$. The function $R_\flat$ serves to measure the (reciprocal) distance to the frontier of the set $\mathscr{F}_{n,m}$. Here and below $\mathbb{R}_+$ is a closed subset of *nonnegative* real numbers.

Theorem 1.5. — *The counting function $\mathcal{N}(\cdot)$ for the number of isolated zeros of solutions of Fuchsian systems is explicitly bounded on the subset $\mathscr{S}_{n,m} \subset \mathscr{F}_{n,m}$ by a power of the function $R_\flat(\cdot)$ as follows:*

$$\mathcal{N}\big|_{\mathscr{S}_{n,m}} \leqslant R_\flat^\nu, \qquad \nu = \nu_{n,m} \leqslant 2^{O(n^2 m)}. \tag{1.8}$$

The constant in $O(n^2 m)$ is explicit and computable.

Remark 1.6. — Note that the Fubini-Study distance on $\mathbb{C}P^1$ is not *conformally* invariant, whereas the counting function $\mathcal{N}$ obviously is. This may be used to improve the bounds: for instance, if $m = 3$, then any three points can be placed at a distance 1 from each other by a suitable conformal change of variable t.

Remark 1.7. — The term $O(n^2 m)$ appearing in the bound above is asymptotically equivalent to the dimension of the variety $\mathscr{S}_{n,m}$. A careful inspection of the proof will reveal that this quantity in fact plays the key role in determining the magnitude of the upper bound. On suitably defined subsets one may apply the same techniques to establish similar bounds depending on the dimension of the set, though we don't explore this direction here.

The proof of Theorem 1.5 is based on the controlled process of reduction of the system (1.1) to a scalar nth order differential equation along the lines suggested by A. Grigoriev [8, 9], see also [24]. As a result, we obtain an explicit uniform upper bound for the variation of argument of all linear combinations along arcs sufficiently distant from the singular locus of this equation.

By the argument principle, this implies an explicit upper bound for the number of isolated zeros away from the singular locus Σ of the initial system. As for the zeros arbitrarily close to Σ, we use the method developed in [20] for Fuchsian equations.

1.4. Applications for the infinitesimal Hilbert 16th problem

The principal motivation for Theorem 1.5 was the so called *restricted infinitesimal Hilbert sixteenth problem* as it was formulated in [6, 12].

Let $H \in \mathbb{R}[x, y]$ be a real polynomial of degree $n+1$ and $\omega = p\,\mathrm{d}x + q\,\mathrm{d}y$ a real polynomial 1-form of degree $n = \max(\deg p, \deg q)$. The problem is to find explicit upper bound for the number $I_{\omega,H}$ of real ovals $\delta_t \subset \{H = t\}$ on the *nonsingular* affine level curves of H, such that the Abelian integral

$$y_{\omega,H}(t) = \oint_{\delta_t} \omega, \qquad t \in (a, b), \tag{1.9}$$

vanishes (assuming that such ovals are isolated). The supremum

$$\mathcal{I}(H) = \sup_{\deg \omega \leqslant n} I_{\omega,H} \tag{1.10}$$

depends only on H and is always finite, so that $\mathcal{I}$ can be considered as a function on the linear space $\mathscr{H}_n = \{\deg H \leqslant n+1\}$ of real polynomials of the specified degree.

In [14, 21] it was shown that the value $\mathcal{I}(H)$ admits a uniform bound,

$$\sup_{H \in \mathscr{H}_n} \mathcal{I}(H) = \mathcal{I}_n < +\infty, \tag{1.11}$$

yet the proof is non-constructive and the growth of $\mathcal{I}_n$ as a function of n is completely uncontrollable.

All known explicit bounds for $\mathcal{I}(H)$ are non-uniform and finite only for a residual (open dense) subset of polynomials H for $n \geqslant 3$. The only case where a uniform bound was known to be extendable to a class of degenerate Hamiltonians, is that of hyperelliptic integrals, see [18]. Yet the bound in this case is given by a tower function (iterated exponent) of unspecified height (iteration depth).

As a corollary of Theorem 1.5 we obtain a *polynomial* global upper bound for the number of zeros of Abelian integrals on the same set $\mathscr{M}_n$. More precisely, consider the expansion of H as the sum of the principal homogeneous parts,

$$H = H_{n+1} + \cdots + H_1 + H_0, \qquad H_k(\lambda x, \lambda y) = \lambda^k H_k(x, y).$$

Let $\mathscr{M}_n$ be a semialgebraic subset in $\mathscr{H}_n$, defined by the following conditions,

(1) the principal homogeneous part H_{n+1} is *nonzero*:

$$H_{n+1}(x, y) = \sum_{i+j=n+1} c_{ij} x^i y^j \not\equiv 0;$$

(2) the principal part is square-free, *i.e.*, H_{n+1} factors as a product of $n+1$ pairwise different linear terms corresponding to different points $\zeta_1, \ldots, \zeta_{n+1}$ of the projective line $\mathbb{C}P^1$;

(3) the polynomial H itself has exactly n^2 different critical points in $\mathbb{C}^2$ with pairwise different critical values $\tau_1, \ldots, \tau_{n^2} \in \mathbb{C}$ (the corresponding critical points will automatically be nondegenerate, *i.e.*, the Hessian of H is nonvanishing at each of them).

Obviously, $\mathscr{H}_n$ is a complex affine space and $\mathscr{M}_n$ a semialgebraic subset in this space. Let $R_\natural : \mathscr{M}_n \to \mathbb{R}_+$, $H \mapsto R_\natural(H)$ be the function defined in terms of the homogeneous decomposition of H as follows,

$$R_\natural(H) = 2 + \frac{\|H\|}{\|H_{n+1}\|} + \sum_{i\neq j}^{n+1} \operatorname{dist}^{-1}(\zeta_i, \zeta_j) + \sum_{j=1}^{n^2} |\tau_j| + \sum_{i\neq j}^{n^2} |\tau_i - \tau_j|^{-1}. \tag{1.12}$$

This is a continuous semialgebraic function on the set $\mathscr{M}_n$ with the infinite limit on the frontier $\partial\mathscr{M}_n$.

Theorem 1.8. — *The number of isolated zeros of Abelian integrals admits a computable semialgebraic bound on $\mathscr{M}_n$:*

$$\mathfrak{I}(H)\big|_{\mathscr{M}_n} \leqslant R_\natural^\nu, \qquad \nu = \nu_n \leqslant 2^{O(n^4)}. \tag{1.13}$$

As in Theorem 1.5, the constant in $O(n^4)$ is explicit and computable.

Remark 1.9. — Very recently A. Glutsyuk and Yu. Ilyashenko in a series of papers [6, 5, 4, 3] achieved another bound for the number of zeros of Abelian integrals, based on completely different ideas. Their bound is finite *on the same residual subset $\mathscr{M}_n$*, yet for any fixed n it grows *exponentially* near some parts of the boundary $\partial\mathscr{M}_n \subset \mathscr{H}_n$.

Theorem 1.8 is a corollary to Theorem 1.5 and follows immediately from the fact that Abelian integrals satisfy a hypergeometric Picard–Fuchs system of linear ordinary differential equations explicitly constructed in [19]. The interested reader may easily restore the details of the derivation.

We do not give a detailed proof of this Theorem here, since in a separate publication [2] we establish a *uniform* explicit bound for the number of zeros of Abelian integrals, thus solving the *unrestricted* infinitesimal Hilbert 16th problem.

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2. On the general nature of non-uniform bounds

The form of the bound established in Theorem 1.5 begs for further analysis, as one can question the form in which the numerous different parameters determining the Fuchsian system (1.1) are incorporated into a single function $R_\flat(\Omega)$. In this section we discuss the general form of explicit *non-uniform* bounds on semialgebraic parameter spaces, of which (1.8) is a particular case, and study their universality.

This allows us to conclude that the double exponential dependence of the bound (1.8) is independent of the particular form chosen for $R_\flat$, as long as certain "natural" conditions are satisfied. Furthermore, we show that the freedom of choosing the "natural" inverse distance $R_\flat$ to the frontier will affect the powers ν_n, changing them by a polynomial in n and m. Thus the exponential bound for the powers $\nu_{n,m}$ (double exponential with respect to $R_\flat$) *cannot be substantially improved in the class of non-uniform bounds on the whole class of Fuchsian systems*. We explain the precise meaning of the "natural bounds" in this section.

2.1. Semialgebraic sets and their carpeting functions

Recall that a subset $Z \subseteq \mathbb{R}^N$ of the real affine space is called *semialgebraic*, if it is a finite union of subsets each defined by finitely many polynomial equalities and inequalities of the form $\{f_\alpha(x) = 0,\ g_\beta(x) < 0\}$, where $f_\alpha, g_\beta \in \mathbb{R}[x_1, \dots, x_N]$ are real polynomials. A projective subset $Z \subseteq \mathbb{R}P^N$ is semialgebraic, if it is semialgebraic in some (hence any) affine chart. A subset in $\mathbb{C}^N$ is semialgebraic, if it is semialgebraic in the corresponding "realification" $\mathbb{R}^{2N}$; *etc.* .

The category of semialgebraic sets is stable by set theoretic operations (finite union, intersection, complement) as well as by projections (as asserted by the celebrated Tarski–Seidenberg principle). As a result, we conclude that any formula involving quantifiers and polynomial expressions, defines a semialgebraic set. For the same reasons the image of a semialgebraic set by a polynomial or semialgebraic map is again semialgebraic (a map is semialgebraic if its graph is a semialgebraic subset of the Cartesian product of the domain and the range).

The space $\mathscr{F}_{n,m}$ of Fuchsian systems can be identified in several ways with semialgebraic subsets in suitable affine spaces, see §2.2 and Remark 2.4. Following the ideas of A. Khovanskii [15], we introduce the notion of a *carpeting function* for *noncompact* semialgebraic subsets. Loosely speaking, the carpeting function plays the role of the "reciprocal distance to the frontier of the set".

Definition 2.1. — *A carpeting function of a noncompact semialgebraic subset Z is a continuous semialgebraic positive function $R : Z \to \mathbb{R}_+$ which is proper as a map from Z to $\mathbb{R}_+$. For technical reasons we will always require in addition that $R \geqslant 2$ on Z.*

In other words, a continuous semialgebraic function R is carpeting for a non-compact set Z if $R(z)$ tends to infinity along any sequence $\{z_k\}_{k=1}^\infty$ without accumulation points in Z ("converging to a frontier of Z").

Example 2.2. — The function $R_\flat$ is "almost carpeting" on $\mathscr{F}_{n,m}$: it is proper, yet not semialgebraic, since the Fubini–Study distance is not algebraic. Yet the latter circumstance is purely technical: one can easily construct a semialgebraic distance on $\mathbb{C}P^1$ and use it in (1.7) instead of the Fubini–Study distance.

Any two carpeting functions on the same set are related by a two-sided Łojasiewicz-type inequality.

Lemma 2.3. — *For any two carpeting functions R_1, R_2 on the same semialgebraic set Z, there exist a finite positive constant s such that*

$$\forall z \in Z \qquad R_2^{1/s}(z) \leqslant R_1(z) \leqslant R_2^s(z). \tag{2.1}$$

Proof. — Consider the joint graph $\{(z,u,v) \in \mathbb{R}^{n+2} : z \in Z, u = R_1(z), v = R_2(z)\}$, which is a semialgebraic set by construction. Its projection $S \subseteq \mathbb{R}^2$ on the (u,v)-plane parallel to the z-direction is semialgebraic by the Tarski–Seidenberg principle.

The function $\varphi(u) = \sup\{v \in \mathbb{R}_+ : (u,v) \in S\}$ takes only finite values. Indeed, the continuous function R_2 attains its maximum on the compact subset $\{z \in Z : \mathbb{R}_1(z) = u\}$. Since the function R_1 is carpeting, it assumes arbitrarily large values, hence the function $\varphi(u)$ is defined for all sufficiently large $u \in [2, +\infty)$. Being semialgebraic by construction, it grows no faster than polynomially, $\varphi(u) \leqslant Cu^s$, $C, s \in \mathbb{R}_+$. The constant C can be absorbed into the increased power, $\varphi(u) \leqslant u^{s+c}$, $c = \log_2 C$, for all $u \geqslant 2$.

The inequality in the other direction is obtained analogously. ☐

In the future any two functions R_1, R_2 constrained by the inequalities (2.1) on their common domain, will be referred to as *polynomially equivalent*.

Remark 2.4. — The class of Fuchsian systems (or, more precisely, its interior) can be alternatively described as follows. Consider the rational matrix function

$$A(t,\lambda) = \mathbf{P}(t)/Q(t), \quad \mathbf{P}(t) = \sum_0^{m-1} M_k\, t^k, \quad Q(t) = t^m + \sum_0^{m-1} c_k\, t^k, \tag{2.2}$$

with the $(n \times n)$-matrix coefficients $M_0, \dots, M_{m-1}$ and scalar coefficients $c_0, \dots, c_{m-1} \in \mathbb{C}$ denoted by $\lambda \in \mathbb{C}^d$, $d = (m-1)(n^2+1)$. Assume further that $\mathbf{P}(t)$ and $Q(t)$ are coprime. The corresponding linear system will be Fuchsian provided that the denominator has no multiple roots, that is, the discriminant $\Delta \in \mathbb{C}[\lambda]$ of the polynomial Q (an explicit polynomial in λ) does not vanish (the point at infinity is always Fuchsian).

There is an obvious carpeting function on the space of the parameters,

$$R_\sharp(\lambda) = 2 + \frac{1}{|\Delta(\lambda)|} + \|\mathbf{P}\| + \|Q\|. \tag{2.3}$$

One can easily establish a two-sided inequality between the functions $R_\flat$ and $R_\sharp$ and show that they are polynomially equivalent (one has to express the discriminant via differences of the roots of the polynomial Q). Moreover, one can easily show that the in upper bound (1.8) the function $R_\flat$ can be replaced by $R_\sharp$ without changing the growth rate of the exponent ν, *cf.* with Corollary 2.5 below.

2.2. Product structure of the space $\mathscr{F}_{n,m}$ of Fuchsian systems

Representation (1.2) together with the special role played by the spectra of the residues and geometry of the singular locus, suggests that the space $\mathscr{F}_{n,m}$ of Fuchsian systems is naturally (bijectively) parameterized by a semialgebraic variety having a rather special *product structure*:

$$\begin{aligned}
&\mathscr{F}_{n,m} \simeq \mathscr{P}_m \times \mathscr{R}_{n,m} \subseteq \left(\mathbb{C}P^1\right)^m \times \left(\mathbb{C}^{n^2}\right)^m,\\
&\mathscr{P}_m = \{(\tau_1,\dots,\tau_m) : \tau_i \in \mathbb{C}P^1,\ \tau_i \neq \tau_j \text{ for } i \neq j\} \subseteq \left(\mathbb{C}P^1\right)^m,\\
&\mathscr{R}_{n,m} = \{(A_1,\dots,A_m) : A_i \neq 0,\ A_i \in \operatorname{Mat}_n(\mathbb{C})\} \subseteq \left(\mathbb{C}^{n^2}\right)^m.
\end{aligned} \tag{2.4}$$

This parametrization is continuous (even biholomorphic) in the sense that convergence of the tuples $(\tau_i, A_i) \in \mathscr{P}_m \times \mathscr{R}_{n,m}$ in the natural (geometric) sense implies the uniform convergence of the corresponding rational matrix 1-forms on $\mathbb{C}P^1$ on compact sets disjoint with the singular loci of the forms (but not necessarily vice versa).

From now on we will identify the variety $\mathscr{F}_{n,m}$ of Fuchsian matrix 1-forms on $\mathbb{C}P^1$ having m distinct poles, with points ($2m$-tuples) from the space $\left(\mathbb{C}P^1\right)^m \times \left(\mathbb{C}^{n^2} \smallsetminus \{0\}\right)^m$. The set $\mathscr{S}_{n,m}$ becomes then a relatively closed subset of $\mathscr{F}_{n,m}$.

The compactification of the space $\mathscr{F}_{n,m}$ preserving the aforementioned product structure is the product of projective spaces:

$$\overline{\mathscr{F}}_{n,m} = \left(\mathbb{C}P^1\right)^m \times \left(\mathbb{C}P^{n^2}\right)^m. \tag{2.5}$$

The frontier $\partial\mathscr{F}_{n,m} = \overline{\mathscr{F}}_{n,m} \smallsetminus \mathscr{F}_{n,m}$ of the variety of Fuchsian systems consists of components of three types,

$$\partial\mathscr{F}_{n,m} \subseteq \mathscr{V}_{n,m} \cup \mathscr{E}_{n,m} \cup \mathscr{C}_{n,m}, \tag{2.6}$$

where:

1. the *vanishing frontier* $\mathscr{V}_{n,m}$ consists of the $2m$-tuples with one or more zero residue matrix A_i equal to zero,
2. the *explosion frontier* $\mathscr{E}_{n,m}$ is the subset corresponding to one or more "*infinite*" residues A_i, and
3. $\mathscr{C}_m =$ is the *collision frontier* corresponding to the union of the diagonals $\bigcup_{i\neq j}\{\tau_i = \tau_j\}$.

The parametrization of Fuchsian 1-forms by poles and residues extends continuously on the vanishing and the collision frontier, but not on the explosion frontier. Note that while the passage to limit as Ω tends to $\mathscr{E}_{n,m}$ is impossible, the *simultaneous limit* on the intersection $\mathscr{E}_{n,m}\cap\mathscr{C}_{n,m}$ may well make sense in the class of *rational* systems. One can argue that generically such "collision with exploding residues" results in creation of an *irregular linear system*, thus the intersection $\mathscr{E}_{n,m}\cap\mathscr{C}_{n,m}$ may well be considered as an *irregularity frontier.*

Examples 1.3 and 1.4 suggest that the counting function $\mathcal{N}:\mathscr{S}_{n,m}\to\mathbb{R}_+$ necessarily grows to infinity near the explosion and the collision frontier components. On the contrary, Theorem 1.5 shows that the counting function is bounded near the vanishing frontier.

Together with Theorem 1.2 this allows to argue that, as long as the problem of counting zeros of solutions is considered as an *algorithmic mass problem formulated for the entire class of general Fuchsian systems*, the natural domain for the counting function should be the partial closure of the set $\mathscr{S}_{n,m}$, obtained by adjoining the vanishing frontier component $\mathscr{V}_{n,m}$,

$$\begin{aligned}\mathscr{S}^*_{n,m} &= \{(\tau_1,\dots,\tau_m):\tau_i\neq\tau_j\}\times\{(A_1,\dots,A_m):\operatorname{Spec}A_i\subseteq\mathbb{R}\}\\ &\subseteq\mathscr{F}^*_{n,m}=\mathscr{P}_m\times\left(\mathbb{C}^{n^2}\right)^m.\end{aligned}\tag{2.7}$$

The partial closures $\mathscr{F}^*_{n,m}$ and $\mathscr{S}^*_{n,m}$ are semialgebraic varieties "parameterizing" the class of Fuchsian systems with m singularities and eventually zero residue matrices (resp., the class of such systems satisfying the spectral condition (1.4)).

The function $R_\flat$ defined by the formula (1.7), is clearly a positive function on $\mathscr{S}^*_{n,m}$ which tends to infinity polynomially as Ω approaches the frontier of $\mathscr{F}_{n,m}$. This function is not semialgebraic because the Fubini–Study distance $\operatorname{dist}:\mathbb{C}P^1\times\mathbb{C}P^1\to\mathbb{R}_+$ is not semialgebraic, yet this failure is purely technical; clearly, there are semialgebraic distance functions on the projective line, all of them equivalent to each other; this would transform (1.7) into a genuine carpeting function. However, this choice is by no means unique.

Since all carpeting functions on the same semialgebraic set are equivalent, Theorem 1.5 admits the following reformulation.

Corollary 2.5. — *For any carpeting function $R:\mathscr{S}^*_{n,m}\to\mathbb{R}_+$ the counting function $\mathcal{N}(\Omega)$ admits an explicit polynomial upper bound of the form*

$$\mathcal{N}(\Omega)|_{\mathscr{S}^*_{n,m}}\leqslant R^\nu(\Omega)\qquad\text{for some finite }\nu.$$

The constant ν depends on n,m and the choice of R.

2.3. Product spaces and natural carpeting functions on them

There is no single distinguished (or preferred) carpeting function on a semialgebraic variety, and it is even less clear how one may compare carpeting functions defined on different semialgebraic varieties. Thus it is rather difficult to analyze the asymptotic behavior (in n, m) of different bounds in invariant terms.

However, in the particular case of the varieties of Fuchsian systems $\mathscr{F}_{n,m}$ one may argue that symmetry considerations distinguish a certain class of *natural* carpeting functions. More precisely, we introduce two special classes of *product semialgebraic spaces*, provisionally called *Fermi and Bose types* (respectively, with or without the explicit prohibition of coincident terms in the product) of several copies of the same semialgebraic variety Z. Then we show that on such product spaces one can introduce "anonymous" carpeting functions that do not depend *explicitly* (in the accurate sense introduced below) on the number of the copies. These natural carpeting functions depend on the product structure types and are defined modulo *parametric polynomial equivalence* (as described in Definition 2.10).

Let Z be an arbitrary semialgebraic set.

Definition 2.6. — *A Bose product space Z^n is the (Cartesian) power of Z, the space of all tuples of points $(z_1, \dots, z_n)$, $z_i \in Z$.*

A Fermi product Z^{*n} is the space of all pairwise different tuples of points $(z_1, \dots, z_n)$, $z_i \in Z$:

$$Z^{*n} = Z^n \smallsetminus \bigcup_{i \neq j} \{z_i = z_j\}.$$

Assume now that $R : Z \to \mathbb{R}_+$ is a carpeting functions on Z and R' is a symmetric carpeting function on the Fermi square Z^{*2}, $R'(z_1, z_2) = R'(z_2, z_1)$. Starting from these two functions, we can construct carpeting functions on arbitrary Bose/Fermi products, aggregating these "basic" functions by an arbitrary *semialgebraic operation*.

Definition 2.7. — *A continuous commutative and associative binary operation $\odot : \mathbb{R}_+ \times \mathbb{R}_+$ will be called semialgebraic, if its graph is a semialgebraic subset in $\mathbb{R}^3_+$, and carpeting, if all sets $\{(x, y) \in \mathbb{R}^2_+ : x \odot y \leqslant C\}$ are compact and exhaust $\mathbb{R}_+ \times \mathbb{R}_+$ as $C \to +\infty$ (i.e., if the function $f(x, y) = (x \odot y) + 2$ is carpeting in the sense of Definition 2.1).*

Because of the commutativity and associativity, the expressions $\bigodot_{i=1}^n c_i$ make sense for any unordered collection of positive numbers c_i.

Example 2.8. — The sum $x+y$, maximum $\max(x,y)$, “radius” $\sqrt{x^2+y^2}$ are semialgebraic carpeting operations. The product is semialgebraic albeit not carpeting, yet the “shifted product” $x \odot y = (2+x)(2+y)$ is.

More generally, let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be any semialgebraic homeomorphism of $\mathbb{R}_+$ and $\odot$ any semialgebraic carpeting binary operation (*e.g.*, one from the above list). Then the binary operation $(x,y) \mapsto h^{-1}\big(h(x) \odot h(y)\big)$ is again carpeting.

Definition 2.9. — *A natural family of carpeting functions on the Bose products Z^n (resp., Fermi products Z^{*n}) is a family of carpeting functions defined by the formulas*

$$R(z_1,\dots,z_n) = \bigodot_{i=1}^{n} R(z_i), \quad \textit{resp.,}\ R'(z_1,\dots,z_n) = \bigodot_{i\neq j} R'(z_i,z_j), \tag{2.8}$$

where R (resp., R') is an arbitrary semialgebraic carpeting function on Z^1 (resp., on Z^{*2}) and $\odot$ any semialgebraic carpeting commutative associative binary operation on the nonnegative ray $\mathbb{R}_+$.

Combining arbitrary *natural* carpeting functions for the Bose powers $\mathscr{R}_{n,m} \simeq \big(\mathbb{C}^{n^2}\big)^m$ and Fermi powers $\mathscr{P}_m = (\mathbb{C}P^1)^{*m}$, by the carpeting operation $\odot$, we obtain the class of *natural carpeting functions* on the product spaces $\mathscr{F}_{n,m} = \mathscr{P}_m \times \mathscr{R}_{n,m}$ as in (2.4).

2.4. Polynomial equivalence of natural carpeting functions

The degree of freedom used to introduce the natural carpeting functions on the product spaces is considerable, as one can vary both the basic functions R, R' as well as the semialgebraic operation $\odot$. However, these variations result in controllable change of the growth rate.

Definition 2.10. — *Two families of carpeting functions $R_1, R_2 : Z_n \to \mathbb{R}_+$ defined on the same family of semialgebraic sets Z_n indexed by a natural parameter n (in general, the dimensions $\dim Z_n$ grow to infinity with n), are said to be polynomially equivalent, if there exists a sequence of finite positive constants $s_n \in \mathbb{R}_+$, such that,*

$$R_2^{1/s_n}(z) \leqslant R_1(z) \leqslant R_2^{s_n}(z) \qquad \forall z \in Z_n; \tag{2.9}$$

cf. with (2.1), *and this sequence grows at most polynomially as $n \to \infty$:*

$$\exists c, r < +\infty \quad \textit{such that}\ \ s_n \leqslant cn^r, \quad \forall n = 1,2,3,\dots \tag{2.10}$$

An obvious modification allows one to also speak of polynomially equivalent carpeting functions for families Z_α indexed by a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$. In this case the respective exponents $s_\alpha \in \mathbb{R}_+$ should grow no faster than $c\,|\alpha|^r$ for some finite $c, r \in \mathbb{R}$.

Clearly, there is no reason why two arbitrary families of carpeting functions be polynomially equivalent: even if this happens, one can involve one of the functions into a sufficiently quickly growing sequence of powers and destroy the initial equivalence. Yet any two *natural* parametric families of carpeting functions on a family of product spaces turn out to be *always equivalent* in the sense of Definition 2.10. This can be considered as a parametric version of the Łojasiewicz inequality.

In complete analogy with Lemma 2.3, one can show that any two carpeting operations are equivalent, in particular, for any such operation $\odot$,

$$\exists\gamma < +\infty,\ \forall a, b \geqslant 2, \qquad (ab)^{1/\gamma} \leqslant a \odot b \leqslant (ab)^\gamma. \tag{2.11}$$

Iterating this inequality, we obtain the following result.

LEMMA 2.11. — *For any carpeting operation $\odot$ as in Definition 2.7, there exist two finite positive constants c, r such that for any $n \in \mathbb{N}$ and any nonnegative numbers $x_1, \dots, x_n \geqslant 2$*

$$x_1 \odot \cdots \odot x_n \leqslant (x_1 \cdots x_n)^{cn^r}. \tag{2.12}$$

Proof. — It is sufficient to prove the inequality for the number of terms being a power of two (for the intermediate cases one can replace some of the extra terms by units).

For $n = 2^m$ the proof goes by induction, iterating (2.11). Indeed, if $\gamma < 2^r$, then the $\odot$-product of 2^{m+1} terms is bounded by the product as follows,

$$(X_1^{c\,2^{mr}} X_2^{c\,2^{mr}})^\gamma \leqslant (X_1X_2)^{\gamma c\,2^{mr}} \leqslant (X_1X_2)^{c\,2^{(m+1)r}},$$

where $X_1 = x_1 \cdots x_{2^m}$ and $X_2 = x_{2^m+1} \cdots x_{2(m+1)}$ stand for the products of the first and the last 2^m terms. The base of the induction can always be guaranteed by a sufficiently large $c > 0$. □

LEMMA 2.12. — *Any two natural carpeting families on Bose (resp., Fermi) powers Z^n (resp., Z^*n) of the same semialgebraic set $Z = Z^1$ are polynomially equivalent to each other.*

Proof. — The proof follows immediately from Lemma 2.3 and Lemma 2.11. □

The function $R_\flat$ defined by formula (1.7), represents the class of natural carpeting functions on the product spaces $\mathscr{F}^*_{n,m}$ and its relatively closed subset $\mathscr{S}^*_{n,m}$. From Lemma 2.12 we conclude that the assertion of Theorem 1.5 does not depend on the specific choice of the expression (1.7); the counting function $\mathcal{N}(\cdot)$ for zeros of solutions of Fuchsian systems admits semialgebraic bound which is double exponential in the complex dimension $d = \dim \mathscr{F}^*_{n,m} \sim n^2 m$ of the corresponding variety of Fuchsian systems.

The following result is the most general form of the polynomial bound on zeros of solutions of Fuchsian systems.

Theorem 2.13. — *For any natural carpeting function $R(\cdot)$ on the class of Fuchsian systems considered as a product space $\mathscr{F}^*_{n,m}$,*

$$\mathcal{N}\big|_{\mathscr{S}^*_{n,m}} \leqslant R^{2^{O(d)}}, \qquad d = \dim_{\mathbb{C}} \mathscr{F}^*_{n,m}. \tag{2.13}$$

The constant in $O(d)$ is explicit, though it depends on the choice of the natural carpeting function.

Proof. — This follows immediately from Theorem 1.5 since the carpeting functions (1.7) form a natural carpeting family, and the polynomially growing terms $s_{n,m}$ of the polynomial equivalence are obviously absorbed by the exponential $2^{O(m^2 n)}$. □

Remark 2.14. — Note that the carpeting function $R_\sharp$ introduced in (2.3), is *not natural* on the space $\mathscr{F}_{n,m}$ with respect to the product structure of the latter as it was introduced in Definition 2.9. Despite that, the function $R_\sharp$ is polynomially equivalent to the function $R_\flat$ in the sense of Definition 2.10.

2.5. Near optimality of the double exponential bounds

The proof of Theorem 2.13 suggests that the double exponential bound asserted in (2.13) is *almost optimal*. Indeed, a more moderate growth rate (say, with the exponent ν in (1.8) growing polynomially in n, m, *i.e.,* just one notch slower than the exponent) would already be highly dependent on the choice of carpeting function. This observation suggests that, at least *as long as the entire Fuchsian class $\mathscr{S}^*_{n,m}$ is concerned*, one cannot expect very significant improvements to the (non-uniform) upper bounds of Theorem 2.13.

Of course, the situation changes dramatically if instead of the entire class $\mathscr{F}_{n,m}$ we consider its various subclasses. In [2] we show that for a *isomonodromic regular quasiunipotent classes* of linear systems on $\mathbb{C}P^1$ one can produce a uniform upper bound for the number of zeros of isolated solutions. This bound also turns out to be double exponential, yet because of the uniformity one can easily ask about its possible improvement.

The rest of the paper is devoted to the detailed proof of Theorem 1.5.

3. From Fuchsian systems to Fuchsian equations with a complexity control

It is now well understood that there is a substantial difference between systems of first order linear ordinary differential equations and (scalar) high order linear ODE's with respect to studying zeros of solutions and their linear combinations, see [17] and references therein. In this section we prove that any linear combination of solutions of a Fuchsian system (1.1) from the class $\mathscr{F}^*_{n,m}$ satisfies a linear equation of order n with *explicitly bounded coefficients*.

LEMMA 3.1. — *An arbitrary linear combination of coordinates of a solution of a Fuchsian system with the matrix 1-form $\Omega \in \mathscr{F}^*_{n,m}$, in a suitable affine chart t on $\mathbb{C}P^1$, satisfies a linear ordinary differential equation with polynomial coefficients,*

$$\begin{gathered} a_0(t)\, y^{(n)} + \cdots + a_{n-1}(t)\, y' + a_n(t)\, y = 0, \\ a_0, \dots, a_n \in \mathbb{C}[t], \qquad \gcd(a_0, \cdots, a_n) = 1 \end{gathered} \tag{3.1}$$

and the following qualifications :

(a) $\deg a_j \leqslant d = n^2 m$,

(b) $\|a_j\| \leqslant \|a_0\| \cdot R^\nu$, *with* $R = R_\flat(\Omega)$ *and* $\nu \leqslant 2^{O(d)}$,

(c) *solutions of the equation* (3.1) *are holomorphic outside the finite locus* $\Sigma = \{\tau_1, \dots, \tau_m\} \subset \mathbb{C}$ *such that*

$$|\tau_i| \leqslant R, \qquad |\tau_i - \tau_j| \geqslant 1, \tag{3.2}$$

in particular, solutions are holomorphic in the disk $\{|t| > R\}$.

The proof of this Lemma is largely parallel to the exposition from [24, §2]. For convenience of the reader we reproduce it with the required modifications. In what follows we will refer to the ratio

$$\angle D = \max_{j=1,\dots,n} \frac{\|a_j\|}{\|a_0\|} \tag{3.3}$$

as the *slope* of the differential operator D as in the left hand side (3.1) (and the corresponding linear ordinary homogeneous equation). Thus Lemma 3.1

asserts an explicit global upper bound on the slope of the equation (3.1) derived from the Pfaffian system $\mathrm{d}x = \Omega x$, in terms of the carpeting function $R_\flat(\Omega)$.

3.1. Preparatory remarks

We start with choosing a convenient affine chart. From elementary geometric arguments involving spherical areas, it follows that for any m points on the Riemann sphere, there always exists a point at least $O(1/\sqrt{m})$-distant from them in the sense of Fubini–Study distance. We choose the affine chart $z \in \mathbb{C}$ in which this point corresponds to infinity and normalize it so that the Fubini–Study metric is given by the form $\frac{|dz|}{1+|z|^2}$. Such map is defined uniquely modulo the Euclidean rotation, and we have an obvious inequality $\operatorname{dist}(z_1, z_2) \leqslant |z_1 - z_2|$.

By construction, in the chosen chart z the affine coordinates $\tau_1, \dots, \tau_m$ of the singular points of a system (1.1) with $R_\flat(\Omega) = R$, satisfy the inequality $|z_j| \leqslant O(\sqrt{m})$ for all $j = 1, \dots, m$. Since all terms in the expression (1.7) are nonnegative and the residues are invariant, we conclude that in this chart the inequalities

$$|z_i| \leqslant O(\sqrt{m}), \quad j = 1, \dots, m, \qquad |z_i - z_j| \geqslant 1/R, \quad i \neq j \tag{3.4}$$

hold simultaneously. By a suitable affine rescaling $t = cz$ we can normalize the singular locus of the system to satisfy the inequalities (3.2). In the affine chart thus constructed, the Pfaffian linear system (1.1) is equivalent to a system of linear ordinary differential equations of the form

$$\begin{gathered} \frac{\mathrm{d}x}{\mathrm{d}t} = A(t) \cdot x, \quad A(t) = \sum_{j=1}^{m} \frac{A_j}{t - \tau_j}, \\ |A_j| \leqslant R, \quad |\tau_j| \leqslant R, \qquad |\tau_i - \tau_j| \geqslant 1. \end{gathered} \tag{3.5}$$

Finally, performing a linear transformation $x \mapsto Ux$ with a unitary (constant) matrix U, we may without loss of generality assume that the linear combination in question is just the first component, $y(t) = x_1(t)$. This transformation does not affect the norms of the residues, hence the inequalities (3.4) are preserved.

3.2. Objects defined over $\mathbb{Q}$ and their complexity

If we consider the entries of the residue matrices A_i and the affine coordinates $\tau_i \in \mathbb{C}$ of the singular points as coordinates on $\mathscr{F}^*_{n,m}$, denoting a general point of this space by λ, then the coefficients of the linear system (3.5) become *integer numbers*, more precisely, only 0 and ± 1. In other words, the entries of the matrix 1-form Ω_λ are defined over the field $\mathbb{Q}$ on the space with coordinates (t, λ). In sharp contrast with the fields of real or complex numbers, for objects defined over the field $\mathbb{Q}$ one can define their complexity (bitsize, or, more precisely, the exponent of the bitsize). This definition may be introduced by many equivalent ways. For instance, as follows:

(1) Complexity of a polynomial $p \in \mathbb{Z}[\lambda, t]$ is its norm $\|p\| \in \mathbb{Z}_+$, the sum of all absolute values of its integer coefficients;
(2) Complexity of a rational function $R \in \mathbb{Q}(\lambda, t)$ is the minimal value of sum $\|p\| + \|q\|$ over all representations $R = p/q$ with $p, q \in \mathbb{Z}[\lambda, t]$; in particular, complexity of an irreducible rational number $r = p/q$, $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$, is $|p| + |q|$;
(3) Complexity of any object constructible over $\mathbb{Q}(\lambda, t)$ is the sum of complexities of all rational functions explicitly occurring in the construction. For instance, complexity of a rational matrix 1-form Ω is the sum of complexities of all its entries; complexity of a differential operator with rational coefficients is the sum of complexities of its coefficients *etc.* .

This definition (as most constructions from the complexity theory) in fact applies not to the object in question, but rather to a specific *representation* of this object. It will be important in this paper that the objects under discussion are described by their *explicit* representation.

Whenever two representations are related by a simple transformation (*e.g.*, expansion or reduction to common denominator), their complexity is usually comparable. More generally, *any explicit construction over $\mathbb{Q}$ admits explicit control over the complexity growth of all intermediate objects.* This observation is the primary reason for introducing the objects over $\mathbb{Q}$ in the context where the initial field is $\mathbb{C}$ or $\mathbb{R}$. We will apply this technique to the straightforward process of derivation of a scalar differential equation of high order from a system of first order equations (consecutive derivations and elimination).

3.3. Derivation of the scalar Fuchsian equation over $\mathbb{Q}(\lambda, t)$

By direct inspection of the formula (3.5) one can see that the matrix product $\mathbf{P} = \prod_1^m (t - \tau_i) A(t)$ is a polynomial matrix with entries in $\mathbb{Z}[\lambda, t]$

of degree $\leqslant m$ and norm not exceeding $m\cdot 2^{m-1}\leqslant 2^m$ (here and below we use the fact that the norm is multiplicative, $\|pq\|\leqslant\|p\|\cdot\|q\|$ for any two polynomials p,q). For brevity, we call elements of the ring $\mathbb{Z}[\lambda,t]$ the *lattice polynomials*.

Lemma 3.2. — *The first component of any solution of the system* (1.1) *considered as a linear system with coefficients from the field* $\mathbb{Q}(\lambda,t)$*, satisfies a linear* n*th order differential equation with lattice polynomial coefficients of the form*

$$a_0(\lambda,t)\,y^{(n)}+a_1(\lambda,t)\,y^{(n-1)}+\cdots+a_n(\lambda,t)\,y=0,\qquad a_0,\dots,a_n\in\mathbb{Z}[\lambda,t],\qquad a_0\neq 0. \tag{3.6}$$

The coefficients a_j *are lattice polynomials in the variables* λ,t *having degrees* $\leqslant n^2m$ *and explicitly bounded norms,* $\|a_j\|\leqslant 2^{O(n^2m)}$.

Note that the norms in the formulation of Lemma 3.2 are computed with respect to the ring $\mathbb{Z}[\lambda,t]$, hence they are *nonnegative integer numbers*.

Proof of the Lemma. — By virtue of the system (3.5), the higher derivatives of the function $y(t)$ can be expressed inductively as linear combinations of the dependent variables $x=(x_1,\dots,x_n)$,

$$y^{(k)}(t)=\xi_k\cdot x,\qquad \xi_k\in\big(\mathbb{Q}(\lambda,t)\big)^m, \tag{3.7}$$

with the n-vectors ξ_k over the field $\mathbb{Q}(\lambda,t)$ determined by the recursive rule

$$\begin{aligned}\xi_0&=(1,0,\dots,0),\\ \xi_{k+1}&=\partial_t\xi_k+\xi_k\cdot A,\end{aligned}\qquad A\in\operatorname{Mat}_n\big(\mathbb{Q}(\lambda,t)\big),\ \partial_t=\partial/\partial t \tag{3.8}$$

(we treat x as a column vector and ξ_k as row vectors from the dual space).

The structure and complexity of the n-covectors ξ_k can easily be established by the inspection of the formulas (3.8). Denote

$$Q(\lambda,t)=\prod_{i=1}^m(t-\tau_i)\in\mathbb{Z}[\lambda,t],\qquad \|Q\|=2^m,\ \deg Q=m,$$

the *discriminant*, so that $A=\mathbf{P}/Q$ in (3.8).

The n-vectors $\xi_0,\dots,\xi_n$ defined by (3.8), satisfy the following properties:

1. Their multiples $\alpha_k=Q^k\xi_k$ are lattice n-vector polynomials for all $k=0,\dots,n$, of degree no greater than km, and the norms $\|\alpha_k\|\leqslant(3nm)^n2^{nm}$.
2. The wedge product of any n vectors from this collection, the polynomial n-form

$$Q^{n(n+1)/2}\xi_0\wedge\cdots\wedge\xi_{i-1}\wedge\xi_{i+1}\wedge\cdots\wedge\xi_n,$$

has as its only coefficient a lattice polynomial of degree at most n^2m and norm bounded by $n!(3nm)^{n^2}2^{2n^2m}$.

(3) The wedge product $\xi_0 \wedge \cdots \wedge \xi_{n-1}$ does not vanish identically on $\mathscr{F}^*_{n,m}$.

Indeed, the polynomial 1-forms $\alpha_k = Q^k\xi_k$ satisfy the recurrence relations

$$\alpha_{k+1} = Q \cdot \partial_t\alpha_k - (k\,\partial_t Q - \mathbf{P}) \cdot \alpha_k, \tag{3.9}$$

which follow directly from (3.8). The bound for the degrees $\deg\alpha_k$ is obvious since $\deg Q, \deg \mathbf{P} \leqslant m$. The bound for the norms $\|\alpha_k\|$ can also be derived from the recursive equations,

$$\begin{aligned}\|\alpha_{k+1}\| &= \|Q \cdot \partial_t\alpha_k - (k\partial_t Q - \mathbf{P})\alpha_k\| \\ &\leqslant \|Q\|\,(\deg\alpha_k)\,\|\alpha_k\| + k(\deg Q)\,\|Q\|\,\|\alpha_k\| + n\,\|P\|\,\|\alpha_k\| \\ &\leqslant (2^m km + 2^m km + 2^m nm)\,\|\alpha_k\| \\ &\leqslant 3 \cdot 2^m \cdot nm\,\|\alpha_k\|,\end{aligned}$$

so that an easy induction gives $\|\alpha_k\| \leqslant (3nm)^n 2^{nm}$. To establish the bound for the wedge products, we expand the $n\times n$ determinants using the Laplace expansion involving $n!$ summands, each having the degree $\leqslant \frac{1}{2}n(n-1)m$ and norm $\leqslant (3nm)^{n^2}2^{n^2m} \leqslant 2^{O(n^2m)}$.

Finally, to establish (3), we note that for *specific* Fuchsian systems (1.1) and for specific solutions $y(t)$ we can apriori guarantee that the functions $y^{(0)}(t), \dots, y^{(n-1)}(t)$ are linearly independent. For instance, the function $y(t) = 1+t+\cdots+t^{n-1}$ is obviously linearly independent with its derivatives of all orders $\leqslant n-1$. This function is a linear combination of solutions of an Euler system with two singularities at the origin and at infinity, which is a particular case of a Fuchsian system. For the corresponding value of the parameters ł, the Wronskian $\xi_0 \wedge \cdots \wedge \xi_{n-1}$ is not identically zero as a function of t.

To derive the equation (3.6), it remains to note that any $n+1$ vectors in the n-space $\big(\mathbb{Q}(\lambda,t)\big)^n$ are linearly dependent over this field. In our particular case the vectors $\xi_0, \dots, \xi_{n-1}$ are linearly independent, so the vector ξ_n expands as a linear combination, $\xi_n = c_0\xi_0 + \cdots + c_{n-1}\xi_{n-1}$, of the previous vectors with coefficients $c_i \in \mathbb{Q}(\lambda, t)$. These coefficients can be found using the Cramer rule, as ratios of the determinants,

$$c_i = \frac{\xi_0 \wedge \cdots \wedge \xi_{i-1} \wedge \xi_{i+1} \wedge \cdots \wedge \xi_n}{\xi_0 \wedge \cdots \wedge \xi_{n-1}}, \qquad i = 0, \dots, n-1. \tag{3.10}$$

The numerator and the denominator of any such fraction are rational functions defined over $\mathbb{Q}$ with the denominators being known powers of the discriminant Q. Multiplying the linear identity by the common denominator, we obtain a linear identity between the n-vectors $\xi_0, \dots, \xi_n$ with *lattice* polynomial coefficients, which by (3.7) produces a linear nth order differential equation for the function y.

The leading coefficient a_0 of this equation is a nonzero lattice polynomial, therefore its norm is a positive integer number which is necessarily $\geqslant 1$. $\square$

3.4. Explicit bounds on coefficients of the scalar equation over $\mathbb{C}(t)$

Specializing the equation (3.6) for each particular value of the parameters λ would produce an equation of the type (3.1) with coefficients from $\mathbb{C}[t]$. The bound for the degrees of these coefficients will obviously be the same as for (3.6). However, the condition $\|a_0\|_{\mathbb{Z}[\lambda,t]} = 1$ does not prevent the specialization $a_0(\lambda, \cdot)$ of the leading coefficient from having an arbitrarily small norm $\|a_0(\lambda, \cdot)\|_{\mathbb{C}[t]}$ or simply vanishing. In this subsection we prove, following [9, 24], that this vanishing may happen if and only if all other (nonleading) coefficients also vanish. Moreover, we show that the ratios $\|a_j(\lambda, \cdot)\|_{\mathbb{C}[t]} \,/\, \|a_0(\lambda, \cdot)\|_{\mathbb{C}[t]}$ are explicitly bounded as functions of λ. (Lemma 3.4 below).

Consider a linear parametric differential equation (3.6) with lattice polynomial coefficients $a_0, \dots, a_n \in \mathbb{Z}[t, \lambda]$. Denote by $\varLambda \subset \mathbb{C}^\nu$ the singular locus corresponding to the parameters λ for which the leading coefficient $a_0(\lambda, \cdot)$ vanishes identically as a univariate polynomial.

Solutions of the equation (3.6) may be considered as holomorphic functions of t, λ by fixing an initial condition at some point $t = t_0$ and treating λ as parameters of the equation varying near some value λ_0, provided that the bidisk $\{|t - t_0| < a,\ |\lambda - \lambda_0| < b\}$ is free from singular points of the equation. Yet in general these solutions will exhibit singularities on the locus $\{(\lambda, t) : a_0(\lambda, t) = 0\} \subset \mathbb{C} \times \mathbb{C}^\nu$ which contains some exceptional lines $\{\lambda_*\} \times \mathbb{C}$ for $\lambda_* \in \varLambda$, while intersecting the other (generic) lines $\{\lambda\} \times \mathbb{C}$, $\lambda \notin \varLambda$, only by isolated points.

Definition 3.3. — *The family of equations* (3.6) *has an apparent singularity at a point $\lambda_* \in \varLambda \subset \mathbb{C}^n$, if there exist:*

(1) *a small neighborhood $U \subset \mathbb{C}^\nu$ of λ_*,*
(2) *an open disk $D \subset \mathbb{C}$,*
(3) *a fundamental system of solutions $y_1(\lambda, t), \dots, y_n(\lambda, t)$ of the family,*

such that each function $y_i(\lambda, \cdot)$ analytically depends on λ in the disk D as long as $\lambda \in U \smallsetminus \varLambda$, and remains bounded in D as $\lambda \to \lambda_*$.

By the removable singularity theorem, this means that a fundamental system of solutions admits uniform analytic limit in D as $\lambda \to \lambda_*$, though the linear independence of the functions $y_i(\lambda_*, \cdot)$ may fail at the limit.

Denote by D_λ the linear differential operator corresponding to the linear homogeneous equation (3.6) after specializing the value $\lambda \notin \Lambda$. Denote by

$$\varphi(\lambda) = \angle D_\lambda = \max_{j=1,\dots,n} \frac{\|a_j(\lambda,\cdot)\|}{\|a_0(\lambda,\cdot)\|}, \qquad \lambda \in \mathbb{C}^\nu \smallsetminus \Lambda, \tag{3.11}$$

the slope of this operator. Without loss of generality we may assume that the polynomials $a_0(\lambda,\cdot),\dots,a_n(\lambda,\cdot)$ are mutually prime in the ring $\mathbb{C}[t]$: nontrivial common divisor may occur only on a proper algebraic subset of $\mathbb{C}^\nu$ which can be combined with Λ.

Lemma 3.4. — *Assume that the family of equations* (3.6) *with lattice polynomial coefficients of known degree $\leqslant d$ and bounded norms $\|a_j\| \leqslant M$, depending on the complex ν-dimensional parameter $\lambda \in \mathbb{C}^\nu$, has only apparent singularities at all points of the locus Λ which is a proper algebraic subset of $\mathbb{C}^\nu$. Then*

$$\forall \lambda \text{ such that } \quad |\lambda| \leqslant M, \qquad \varphi(\lambda) \leqslant M^{d^{O(\nu)}} \tag{3.12}$$

with an explicit constant in $O(\cdot)$.

Sketch of the proof. — The proof of the Lemma almost literally reproduces that of Lemma 5 from [24] with minimal modifications.

More specifically, we start with the observation that the slope φ is locally bounded everywhere on $\mathbb{C}^\nu \smallsetminus \Lambda$. This follows from the assumption that the family (3.6) has only apparent singularities, in the same way as in the proof of Lemma 4 from [24]. The only required change is to replace the unit disk by the disk D from the Definition 3.3.

Next, we consider the positive subgraph X of the function $\varphi(\cdot)$ over the ball of radius M. This subgraph is a semialgebraic subset in $X \subseteq \mathbb{C}^{\nu+1}$, defined by lattice polynomial equalities and inequalities. Since the function φ is locally bounded near each point of the compact ball $\{|\lambda| \leqslant M\}$, the semialgebraic set X is globally bounded.

The lattice polynomials defining X are of degrees not exceeding $O(d)$ in $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$. In the same way the norms of these polynomials are no greater than $O(M)$. Finally, the dimension of the space in which X resides, is $O(\nu)$. By the quantitative version of the Tarski–Seidenberg quantifier elimination theorem [7, 10] and, most recently, [1, Theorem 7.2], the diameter of X does not exceed $M^{d^{O(\nu)}}$, as asserted in (3.12). □

Proof of Lemma 3.1. — The proof directly follows from Lemmas 3.2 and 3.4: we first derive the scalar equation (3.6) over $\mathbb{Q}(\lambda, t)$ and transform it to the form with lattice polynomial coefficients by eliminating the denominators.

The resulting family of scalar equations (3.6) has only apparent singularities, since for any parameter λ solutions of the initial system (3.5) depend analytically on λ in any disk free from singular points $\Sigma = \Sigma_\lambda \subseteq \mathbb{C}$ of the latter. By Lemma 3.4, we conclude with the inequality asserted by the Lemma, for all values λ outside from the vanishing locus $\Lambda = \{\lambda : a_0(\lambda, \cdot) \equiv 0\}$. This locus is a proper algebraic hypersurface of the complex affine space, hence by continuity the inequality $\|a_j\| \leqslant \|a_0\| \cdot R^\nu$ holds everywhere on the ball $\{|\lambda| \leqslant R\}$. □

4. Counting zeros of solutions of Fuchsian equations

In this section we complete the proof of the main Theorem 1.5. It is based on a complex counterpart of the de la Vallée Poussin theorem [22], which asserts that the variation of argument of any (complex) solution of a *monic* linear differential equation (with the leading coefficient identically equal to 1) along any rectilinear segment can be explicitly bounded from above in terms of the magnitude of the non-principal coefficients of this equation.

4.1. Lower bounds of polynomials away from their zeros

We start with an observation which relates the norm (1.6) on univariate polynomials with the supremum-norm on disks of finite radius centered at the origin. Clearly, for any finite d the two norms on the finite-dimensional space of polynomials of degree $\leqslant d$ are equivalent. However, we shall also be interested in the explicit constants involved in this equivalence.

The first side of this equivalence is easy to describe explicitly. Indeed, $\|p\| = 1$ implies an easy upper bound $|p(t)| \leqslant R^d$ for $|t| \leqslant R$. We will prove the inequality in the opposite direction, establishing a *lower* bound for $|p(t)|$ for all $t \in \mathbb{C}$ distant from the null locus of p, provided that $\|p\|_{\mathbb{C}[t]} = 1$.

Lemma 4.1. — *Let $p = \sum_0^d p_j\, t^j$ be a polynomial of degree d and unit norm $\|p\|_{\mathbb{C}[t]} = 1$, and $\Sigma = \{t \in \mathbb{C} : p(t) = 0\}$ the zero locus of p.*

Then for an arbitrary point $t \notin \Sigma$ in the disk of radius $R \geqslant 2$, we have an inequality

$$|p(t)| \geqslant 2^{-O(d)} (r/R)^d, \qquad r = \operatorname{dist}(t, \Sigma),\ |t| \leqslant R. \tag{4.1}$$

Proof. — For a polynomial $p = \sum_{j=0}^{d} p_j t^j$ of unit norm and degree d, at least one of its $d+1$ coefficients must be at least $1/(d+1)$ in the absolute value. Writing the Cauchy formula for the corresponding derivative at the origin, we conclude that

$$p_j = \frac{1}{2\pi \mathrm{i}} \oint_{|t|=1} \zeta^{-j}\, p(\zeta)\mathrm{d}\zeta, \qquad \text{hence} \quad \frac{1}{2\pi} \int_{|t|=1} |p(\zeta)|\, \mathrm{d}s \geqslant \frac{1}{d+1}.$$

By the mean value theorem there exists a point t_* on the unit circle, such that $|p(t_*)| \geqslant \frac{1}{d+1}$.

Let $K \subset \mathbb{C}$ be the disk of radius $2R$, centered at the origin. The polynomial p can be written as the product,

$$p(t) = \alpha \prod_{|t_j|>2R} (t - t_j) \prod_{|t_j|\leqslant 2R} \left(1 - \tfrac{t}{t_j}\right), \qquad \alpha \in \mathbb{C}, \tag{4.2}$$

where t_j are roots of the polynomial, counted with their multiplicity. Denote by $d_\pm$ the number of roots of Σ inside (resp., outside) the disk $\{|t| \leqslant 2R\}$, counted with multiplicities so that $d_- + d_+ = d$.

A lower bound for $|\alpha|$ follows from the inequality $|p(t_*)| = 1/(d+1)$. For $t = t_*$ the first product in (4.2) is *majorized* by $(2R+1)^{d_-}$, while the second product does not exceed $(4/3)^{d_+}$, since $|t|/|t_j| \leqslant 1/(2R-1) \leqslant 1/3$ for all terms in it. Altogether we conclude that

$$|\alpha| \geqslant \frac{1}{d+1} \cdot \frac{(3/4)^{d_+}}{(2R+1)^{d_-}}.$$

This implies the lower bound for $|p(t)|$ with $|t| \leqslant R$:

$$|p(t)| \geqslant |\alpha|\, r^{d_-} (1/2)^{d_+}.$$

Combining these two estimates, we conclude that

$$|p(t)| \geqslant \frac{1}{d+1} \cdot \left(\frac{r}{2R+1}\right)^{d_-} (3/8)^{d_+} \geqslant 2^{-O(d)} \left(\frac{r}{R}\right)^{d}$$

for all $2 \leqslant R < +\infty$ and $R > r > 0$. □

Remark 4.2. — One can give a different bound using the Cartan lemma [16]. This lemma establishes an explicit lower bound for a monic polynomial outside a union of disks of given total diameter. For our purposes it is less convenient, since the polynomial in question is not monic (and can actually be very far from it).

4.2. Variation of argument of solutions along arcs

For the next step, we give an upper bound for the variation of argument of complex-valued solutions of a Fuchsian equation along a circular arc (or line segment) in terms of the slope of the equation and the normalized length of the arc.

LEMMA 4.3. — *Let D be a differential operator of order k with coefficients of degree $\leqslant d$ and slope $S = \angle D$, and γ a closed circular arc or line segment disjoint with the singular locus Σ, which belongs to the disk of radius R centered at the origin.*

Then the variation of argument of any nonzero solution of the homogeneous equation $Dy = 0$ along the arc γ is explicitly bounded,

$$\operatorname{Var}\operatorname{Arg} y(t)|_\gamma \leqslant kSL(R/r)^{O(d)}. \tag{4.3}$$

where L is the length of the arc γ and $r = \operatorname{dist}(\gamma, \Sigma)$.

Proof. — Without loss of generality we may assume that the polynomial coefficients of the operator D are normalized as follows,

$$\|a_0\| = 1, \qquad \|a_j\| \leqslant S, \quad j = 1, \dots, k,$$

where $S = \angle D$.

On the disk of radius R the absolute values of the non-principal coefficients are bounded, $|a_j(t)| \leqslant SR^d$. Restricting the equation $Dy = 0$ on the arc γ parameterized by the arc-length, we obtain a homogeneous linear ordinary differential equation. Dividing by $a_0|_\gamma$ and applying Lemma 4.1 to this polynomial, this equation can be brought into the monic form with explicitly bounded complex valued coefficients,

$$\begin{aligned} y^{(k)} + A_1(s)\, y^{(k-1)} + \cdots + A_k(s) = 0, &\qquad s \in [0, L], \\ |A_j(s)| \leqslant 2^{O(d)} \cdot SR^{2d}/r^d \leqslant S(R/r)^{O(d)}, &\qquad j = 1, \dots, k. \end{aligned} \tag{4.4}$$

Applying the complex generalization of the de la Vallée Poussin nonoscillation theorem [22, Corollary 2.7] to (4.4), we conclude that the variation of argument admits the bound

$$\operatorname{Var}\operatorname{Arg} y(\cdot)|_\gamma \leqslant O(1) \cdot kLA, \qquad A = \max_{j=1,\dots,k} \max_{s \in \gamma} |A_j(s)|. \tag{4.5}$$

Substituting the bounds (4.4) into this inequality, we conclude with the bound (4.3). □

The assertion of Lemma 4.3 allows one to place an explicit upper bound for the number of isolated zeros of any solution to a linear equation with rational coefficients in a simply connected polygonal domain away from the singular locus. Indeed, one can estimate the variation of argument of any solution along each arc bounding the polygonal domain, and apply the argument principle.

In addition to the explicit bound on the variation of argument along arcs distant from the singular points, one can slightly modify the argument above to give a bound on the variation of argument along a small circle centered at a *Fuchsian singular point* as well.

Lemma 4.4. — *Let $t_0 \in \Sigma$ be a Fuchsian singular point of a linear operator with polynomial coefficients of order k, degree d and slope S, which belongs to the disk of radius $R \geqslant 2$ and is at least r-distant from all other points $t_1, \dots, t_d \in \Sigma$.*

Then the variation of argument of any solution $y(t)$ along the circle $\gamma_\rho = \{|t - t_0| = \rho\}$ is bounded,

$$\operatorname{Var}\operatorname{Arg} y(\cdot)|_{\gamma_\rho} \leqslant kS(R/r)^{O(d)} \tag{4.6}$$

uniformly over all sufficiently small $\rho > 0$.

Proof. — Note that the parallel translation $p(\cdot) \mapsto (\operatorname{Tr}_R p)(\cdot) = p(\,\cdot + R)$ is a linear operator of explicitly bounded norm on the space of polynomials of degree $\leqslant d$. Indeed, by the Taylor formula,

$$\|\operatorname{Tr}_R p\| \leqslant \sum_{i=0}^{d} \frac{R^i \, \|p^{(i)}\|}{i!} \leqslant \|p\| \cdot \sum_0^d \frac{(Rd)^i}{i!} \leqslant 2^{O(d)} R^d \cdot \|p\|.$$

Thus without loss of generality we may assume that the Fuchsian singularity $t_0 = 0$ is at the origin, and the slope of the new differential equation is at most $2^{O(d)} S R^d$, where S is the original slope.

Since the point is Fuchsian, the equation can be written in the form

$$t^k q_0(t)\, y^{(k)}(t) + t^{k-1} q_1(t)\, y^{(k-1)}(t) + \cdots + q_k(t) y(t) = 0, \tag{4.7}$$

where $q_0, q_1, \dots, q_k$ are polynomials of degrees $\leqslant d - k, d - k + 1, \dots, d$ respectively, see [13, Proposition 19.19]. Since the norm is multiplicative, and $\|t^j\| = 1$ we conclude that $\|q_j\|/\|q_0\| \leqslant 2^{O(d)} S R^d$ (see above). By Lemma 4.1, $|q_0(t)| \geqslant 2^{-O(d)} r^d$ and, obviously, $|q_j(t)| \leqslant \|q_j\|$ for all sufficiently small $|t| = \rho$.

Letting $t = \rho s$ and noting that $\frac{\mathrm{d}^k}{\mathrm{d}t^k} = \rho^{-k} \frac{\mathrm{d}^k}{\mathrm{d}s^k}$, we rewrite the equation above in the form

$$s^k q_0(\rho s)\, y^{(k)}(s) + s^{k-1} q_1(\rho s)\, y^{(k-1)}(s) + \cdots + q_k(\rho s) y(s) = 0, \tag{4.8}$$

with $|s| = 1$. Notice that the rescaling of $\frac{\mathrm{d}^k}{\mathrm{d}t^k}$ precisely cancels out with the rescaling of the vanishing terms t^k in the coefficients. This circumstance is unique to Fuchsian differential operators.

After dividing the equation above by the leading term (nonzero and bounded from below), we obtain a monic linear ordinary differential equation on the unit circle $|s| = 1$ with coefficients not exceeding an explicit constant $A = 2^{O(d)} S(R/r)^d$. The bound for the variation of argument along the circle again follows from (4.5). □

4.3. Counting zeros in topological annuli

In this section we recall an "*argument principle*" for multivalued functions. This principle asserts that the number of zeros of a solution of a Fuchsian differential operator in an annulus is explicitly bounded, provided that the monodromy of the operator has all eigenvalues on the unit circle and the annulus is at a positive distance from the singular locus. The proof is based on an idea due to Petrov (the so called "Petrov's trick"). The result we need appears (in slightly less general form) in [20].

Let K be an *annulus* bounded by two disjoint circles $C_{1,2}$ with centers on the real axis, and D a monic linear operator with coefficients holomorphic in the closure $\overline{K}$ and real on the real axis $K \cap \mathbb{R}$ (we will refer to such operators as *real*). In this case Lemma 4.3 asserts that there exists an upper bound $B = B(D, K)$ for the variation of argument of any solution of the homogeneous equation $Dy = 0$ in K along each of the boundary arcs of ∂K,

$$\operatorname{Var}\operatorname{Arg} y(\cdot)|_{C_{1,2}} \leqslant B = B(D, K). \tag{4.9}$$

Lemma 4.5. — *If the monodromy of a real differential operator D along the equator of an annulus K symmetric around the real axis has all eigenvalues on the unit circle, then the number of zeros of any solution in K is explicitly bounded,*

$$\mathcal{N}(D, K) \leqslant (2k + 1)(2B + 1), \tag{4.10}$$

where k is the order of the operator and $B = B(D, K)$ the bound from (4.9).

Proof. — This Lemma almost literally coincides with Theorem 2 from [20]. To simplify the exposition, we first make a real conformal automorphism of the complex line $\mathbb{C}P^1$, which transforms the given annulus K into an annulus K' bounded by two *concentric* circles centered at the origin, and then pass to the logarithmic chart ζ on the universal covering over K'. As a result, we arrive at the following problem.

Let $\widehat{\Pi}$ be the "vertical" strip $c_- \leqslant \operatorname{Re}\zeta \leqslant c_+$ and D a linear ordinary operator with coefficients holomorphic in $\widehat{\Pi}$, $2\pi\mathrm{i}$-periodic in this strip and real on $\widehat{\Pi} \cap \mathbb{R}$. Let B be the upper bound for the variation of argument of any solution of the equation $Dy = 0$ along the segment of length 2π on any of the lines $\operatorname{Re}\zeta = c_\pm$. Because of the periodicity, the shift operator $\mathbf{M} : y(\cdot) \mapsto y(\cdot + 2\pi\mathrm{i})$ is an automorphism of the space of solutions of the equation, corresponding to the monodromy of the original equation in the annulus.

We need to show that if all eigenvalues of $\mathbf{M}$ are of unit modulus, $|\mu| = 1$, then the number of zeros of any solution in the rectangle $\Pi = \widehat{\Pi} \cap \{|\operatorname{Im}\zeta| \leqslant 2\pi\}$ is at most $(k+1)(2B+1)$; this is equivalent to the assertion of the Lemma.

By the argument principle, it is sufficient to place an upper bound on the variation of argument of any solution f of the equation $Df = 0$ along the perimeter of the rectangle, which consists of two vertical segments on the lines $\operatorname{Re}\zeta = c_\pm$ and two horizontal segments

$$S_\pm = \{\operatorname{Im}\zeta = \pm 2\pi,\ c_- \leqslant \operatorname{Re}\zeta \leqslant c_+\}.$$

The contribution of the vertical segments is explicitly given by the assumptions of the lemma; it does not exceed $2B$. It remains to estimate the variation of arguments along the horizontal segments. Each increment of $\operatorname{Arg} f(\zeta)$ by an angle π or more on a connected interval of ζ implies, by an intermediate value theorem, that at some point ζ_0 on this interval the imaginary part $\operatorname{Im}\big(\mu f(\zeta_0)\big)$ vanishes, for an arbitrary choice of the complex number $\mu \neq 0$. This observation implies the inequality

$$\operatorname{Var}\operatorname{Arg} f|_{S_\pm} \leqslant \pi(\nu_\pm + 1), \qquad \nu_\pm = \{\zeta \in S_\pm : \operatorname{Im}\big(\mu^{\mp 1} f(\zeta)\big) = 0\}$$

valid for any complex number $\mu \neq 0$. In general, one has no explicit control over the imaginary part of a solution along the segments $S_\pm$. However, if f is real on the real segment $S_0 = \Pi \cap \mathbb{R}$, then by the Schwarz symmetry principle, $f(\zeta + 2\pi\mathrm{i}) = \overline{f(\zeta - 2\pi\mathrm{i})}$ for $\zeta \in \Sigma_0$. Therefore for any μ with $|\mu| = 1$,

$$\operatorname{Im}(\mu^{\mp 1} f)|_{S_\pm} = \operatorname{Im}(\mu^{\mp 1}\mathbf{M}^{\pm 1} f)|_{S_0} = \mathbf{P}_\mu f|_{S_0},$$

where $\mathbf{P}_\mu$ is the *Petrov operator*

$$\mathbf{P}_\mu = \frac{1}{2\mathrm{i}}\,(\mu^{-1}\mathbf{M} - \mu\mathbf{M}^{-1}). \tag{4.11}$$

This yields an inequality relating the number $N(f, \Pi)$ of *complex* zeros of a real solution f in the rectangle Π to the number of *real* zeros of the function $f' = \mathbf{P}_\mu f$ on $S_0 \subset \Pi$: for any μ with $|\mu| = 1$,

$$N(f,\Pi) \leqslant \frac{1}{2\pi}\bigl(2B + 2\pi\Bigl(N(f',S_0)\bigr) + 1\Bigr) \leqslant (2B+1) + N(f',\Pi). \tag{4.12}$$

This inequality is a complex analog for the Rolle inequality relating the number of zeros of a real smooth function and its derivative. The inequality remains valid also in the case when $f' = \mathbf{P}_\mu f \equiv 0$, if we set $N(0,\Pi) = 0$. This follows from the symmetry of the function f (the contributions of $\operatorname{Var}\operatorname{Arg} f$ along the segments $S_\pm$ cancel each other).

To complete the proof of the Lemma, we show that the product (composition) of operators $\mathbf{P}_D = \prod_i \mathbf{P}_{\mu_i}^{m_i}$ (where μ_i range over the eigenvalues of the monodromy operator $\mathbf{M}$ and m_i over the corresponding multiplicities) vanishes on the linear space of solutions of the operator $Dy = 0$ in $\widehat{\Pi}$. This assertion is an analog of the Hamilton–Cayley theorem for the Petrov operators.

Indeed, suppose $\mathbf{M}$ is a Jordan block of size m with an eigenvalue μ and let $0 \subset V_1 \subset \cdots \subset V_{\mu_i} \subset 0$ denote the corresponding flag. Then $\mathbf{M}$ acts on V_{j+1}/V_j as multiplication by μ, and $\mathbf{P}_\mu$ acts by zero. Thus $\mathbf{P}_\mu V_{j+1} \subset V_j$ and it follows that $\mathbf{P}_\mu^m = 0$. It remains to note that Petrov operators with different numbers μ commute with each other.

Now the proof of the Lemma for functions real on $\mathbb{R}$ is straightforward: if $Df = 0$, then $\mathbf{P}_D f = 0$, and by (4.12), the number of isolated zeros of f does not exceed $k(2B+1)$, where k is the order of D, equal to the number of the Petrov operators forming the product $\mathbf{P}$.

Finally consider the case where f is a general complex valued solution. By the same argument as above, the total number of zeros of f in $\Pi \cap \{|\operatorname{Im}\zeta| \leqslant 2\pi\}$ does not exceed

$$B + \frac{1}{2\pi}\bigl(\operatorname{Var}\operatorname{Arg} f|_{S_+} + \operatorname{Var}\operatorname{Arg} f|_{S_-}\bigr). \tag{4.13}$$

Assume that $\operatorname{Im} f|_{S_\pm} \not\equiv 0$ (if this assumption fails, consider $\operatorname{Re} f|_{S_\pm}$ instead). Let $f_\pm$ denote the function which agrees with $\operatorname{Im}(\mathbf{M}^\pm f)$ on S_0 and is extended analytically onto Π. The values of $f^\pm$ on S_0 correspond to the imaginary part of f on $S^\pm$ respectively.

Since D is a real operator, $f^\pm$ is again a solution of the equation $Dy = 0$. These functions are real on S_0, the previous considerations apply, and we conclude that

$$\operatorname{Var}\operatorname{Arg} f|_{S_\pm} \leqslant 2\pi N(f^\pm, S_0) \leqslant k(2B+1).$$

Combining this with (4.13) we finally obtain the bound (4.10). $\square$

4.4. Symmetrization

In order to apply Lemma 4.5, to a given annulus, one needs the assumption that the coefficients of the operator D are real on the real axis, eventually after the rotation of the axis. In this section we show that this additional assumption can in fact always be achieved, if the operator D is obtained from a system of linear differential equations (1.1). This construction will be referred to as the *symmetrization* of the operator relative to a given axis.

Let $f \in \mathscr{O}(U)$ be a function holomorphic in an open domain $U \subset \mathbb{C}$. The function $f^\dagger$, defined in the domain $U^\dagger$ by the formula

$$f^\dagger(t) = \overline{f(\bar{t})}, \qquad t \in U^\dagger = \{\bar{t} : t \in U\}, \tag{4.14}$$

(the bar stands for the complex conjugacy) will be called a *reflection* of f in the real axis. Obviously, $f^\dagger$ is holomorphic, $f^\dagger \in \mathscr{O}(U^\dagger)$. For polynomials the reflection consists replacing all (constant) coefficients by their complex conjugates:

$$p(t) = \sum_{k=0}^{m} c_k\, t^k \iff p^\dagger(t) = \sum_{k=0}^{m} \bar{c}_k\, t^k, \qquad c_0, \dots, c_m \in \mathbb{C},$$

and it extends to the rational functions in the obvious way.

Let $U = U^\dagger \subseteq \mathbb{C}$ be a domain symmetric with respect to the real axis $\mathbb{R} \subset \mathbb{C}$. A $\mathbb{C}$-linear subspace $L \subseteq \mathscr{O}(U)$ of functions defined in U, is called *symmetric* (with respect to the real line), if together with each function $f \in \mathscr{O}(U)$ it contains the function $f^\dagger$. A symmetric subspace is closed under taking the real/imaginary part on $U \cap \mathbb{R}$: for any $f \in L$ the functions $\operatorname{Re} f|_{U\cap\mathbb{R}}$ and $\operatorname{Im} f_{U\cap\mathbb{R}}$ can be analytically continued from the subset of real points of U onto the entire domain U using the formulas $\operatorname{Re} f = \frac{1}{2}(f(t) + f^\dagger(t))$, $\operatorname{Im} f = \frac{1}{2\mathrm{i}}(f(t) + f^\dagger(t))$ respectively. A symmetric linear space (in case it is finite-dimensional) always admits a basis of functions real on $U \cap \mathbb{R}$. A monic linear ordinary differential operator $D \in \mathscr{O}(U)[\partial]$ defines a symmetric space of solutions, if and only if its coefficients are real on $\mathbb{R}$, *i.e.*, $D \in \mathscr{O}_{\mathbb{R}}(U \cap \mathbb{R})[\partial]$. Such operators will be referred to as *real* (on $\mathbb{R}$).

Any finite-dimensional subspace $L \subset \mathscr{O}(U)$ of functions can be embedded into the symmetric completion $L^\ominus = L + L^\dagger$ by simply adjoining the reflections $f^\dagger$ of all elements $f \in L$. As a consequence, each linear operator $D \in \mathscr{O}(U)[\partial]$ can be considered as a "right factor" of a real operator $D^\ominus \in \mathscr{O}_{\mathbb{R}}(U \cap \mathbb{R})[\partial]$ of higher order with real coefficients, which vanishes on all functions from $L^\ominus$. However, the slope $\angle\, D^\ominus$ of the operator may increase uncontrollably (relatively to $\angle\, D$) after such symmetrization.

Example 4.6. — The function $f(t) = \mathrm{i} + \frac{1}{100}t^{100}$ satisfies a first order equation

$$y' - \frac{t^{99}}{\mathrm{i} + \frac{1}{100}t^{100}}\, y = 0$$

which has very small (in the absolute value) coefficient in the disk $|t| < \frac{1}{2}$ and the slope $\angle D$ very close to 1. On the other hand, the symmetrization $L^{\ominus}$ is generated by the functions 1 and $t^{100}/100$ and is defined by the second order equation

$$y'' - \frac{99}{t}\, y' = 0.$$

This equation has uniformly large coefficients in the same disk (and a pole at the origin). Even the slope of the corresponding second order operator is relatively large, $\angle D^{\ominus} = 99$.

However, at the level of linear first order *systems* the symmetrization can be achieved very easily. Let $\Omega^{\dagger} \in \mathscr{F}_{n,m}$ be the rational matrix 1-form built from the 1-form $\Omega \in \mathscr{F}_{n,m}$ on $\mathbb{C}$ by the reflection in the real axis as above (replacing all coefficients of the entries ω_{ij} by their conjugates). Then each solution of the reflected system $\mathrm{d}X^{\dagger} = \Omega^{\dagger} \cdot X^{\dagger}$ can be obtained by reflection of the corresponding solution of the initial system $\mathrm{d}x = \Omega x$ (1.1) and vice versa. Therefore the symmetric linear space $L^{\ominus}$ spanned by all components of a solution of the system

$$\mathrm{d}y = \Omega^{\ominus} y, \qquad \Omega^{\ominus} = \begin{pmatrix} \Omega & \\ & \Omega^{\dagger} \end{pmatrix}, \; y \in \mathbb{C}^{2n}, \tag{4.15}$$

includes among its solutions the symmetrization of the solution of the initial system. Reducing the doubled system (4.15) to a scalar equation, one obtains an explicit expression for the operator $D^{\ominus}$.

Clearly, one can replace the real axis $\mathbb{R} \subseteq \mathbb{C}$ by any (real one-dimensional) line ℓ in the complex plane $\mathbb{C}$. Any such line can be mapped to the real axis by a suitable affine transformation $\varphi : \mathbb{C} \to \mathbb{C}$ (in fact, rotation and translation suffice). The corresponding reflection operator replacing (4.14), can be explicitly written as follows,

$$f \mapsto f_{\ell}^{\dagger} = (f \circ \varphi)^{\dagger} \circ \varphi^{-1}.$$

Let now U be an arbitrary concentric annulus free from singular points of a Fuchsian system $\Omega \in \mathscr{F}_{n,m}$. Among all systems obtained by reflection of the matrix 1-form Ω in different axes ℓ passing through the center, one can always find a system (denote it again by $\Omega^{\dagger}$) whose singular points (except for the singular point at the center) will be sufficiently far away from the singularities of Ω. More precisely, we can always guarantee that the matrix form $\Omega^{\ominus}$ of the symmetrized system (4.15) satisfy the inequality

$$R_\flat(\Omega^{\ominus}) \leqslant R_\flat(\Omega)^\nu, \qquad \nu = O(n+m), \tag{4.16}$$

with an explicit constant in $O(\cdot)$. Indeed, the norms of the residues of the symmetrized system are majorized by the norms of residues of Ω (residues of the matrix form $\Omega^\dagger$ have the same norm as those of Ω). As for the distances between the singular locus $\Sigma = \{\tau_j\}$ and its mirror image $\Sigma^\dagger = \Sigma_\ell^\dagger$, we can show that that they can be offset by a suitable rotation of the axis of the symmetry $\ell \subset \mathbb{C}$. Indeed, rotation of ℓ by an angle α results in the rotation of $\Sigma_\ell^\dagger$ by the angle 2α. Yet for any two m-point sets $A, B \subset \{\tau = 1\}$ on the unit circle, one can always find a rotation ρ of the circle so that the arclength distance between the points of A and $\rho(B)$ will not exceed $2\pi/m^2$ (a simple consequence of the pigeonhole principle). This shows that even in the worst case when the singular locus belongs to a single circle, the distance between Σ and $\Sigma^\dagger$ can always be assumed bounded from below. The general case of arbitrary singular locus can be immediately reduced to this case by controlling the angular distance between the loci and their distance from the center.

4.5. Demonstration of Theorem 1.5

Consider an arbitrary Fuchsian linear system (1.1) from the Fuchsian class $\mathscr{F}_{n,m}$ with the matrix form Ω the corresponding system of linear equations in the affine chart t as in (3.5).

This system after reduction to a linear differential equation of order n takes the form (3.6) with the coefficients $a_0, \dots, a_n \in \mathbb{C}[t]$ of the corresponding differential operator D of degree $d \leqslant n^2 m$ and the slope not exceeding $R^{2^{O(n^2 m)}}$, $R = R_\flat$, by Lemma 3.1. Solutions of this equation are analytic outside the singular locus $\Sigma = \{\tau_0, \dots, \tau_m\}$, though the leading coefficient a_0 vanishes at some other points as well (apparent singularities of the equation).

Moreover, for any singular point $\tau \in \Sigma$, one can find a real line $\tau \in \ell \subset \mathbb{C}$ ("axis of symmetry") passing through this point, such that the linear operator $D^{\ominus}$ obtained by symmetrization of D in this axis has the order $2n$ and the slope bounded by the same double exponential expression $R^{2^{O(n^2 m)}}$. Indeed, one has to apply Lemma 3.1 to the symmetrized system (4.15), in which $\Omega^\dagger$ is the result of reflection of Ω in ℓ and note that the carpeting function will be replaced by a polynomially equivalent one (4.16).

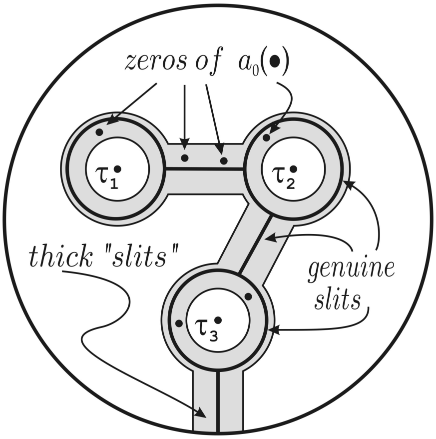

Figure 4.1. "Thick slits" and genuine slits.

Abusing the language, we will say that the operator D admits an axis of symmetry.

Consider the system of "thick slits" of the disk $K_R = \{|t| \leqslant \mathbb{R}\}$ as follows (see Fig. 4.5):

1. annuli $\{\frac{1}{4} \leqslant |t - \tau_j| \leqslant \frac{1}{2}\}$ around each singular point $\tau_j \in \Sigma$;
2. rectangular strips of width $\frac{1}{2}$ along all segments $[\tau_i, \tau_j]$ (these segments should be axes of symmetry for these rectangles);
3. rectangular strips of with $\frac{1}{2}$ along the segment connecting τ_j with the exterior circle $\partial K_R = \{|t| = R\}$.

By elementary geometric considerations it is clear that one can construct a system of genuine slits of $K_R \smallsetminus \Sigma$ as follows:

1. these slits are only circular arcs and line segments;
2. they subdivide $K_R \smallsetminus \Sigma$ into simply connected polygonal domains and punctured disks;
3. each slit (circular or straight) is at least r-distant from the null locus of the leading coefficient $V = \{a_0 = 0\}$, where $1/r = O(n^2 m)$;
4. the total length of the slits is bounded by $O(m^2 R)$.

Indeed, even if all $d = m^2 n$ points from V belong to a given annulus or a rectangular strip of width $O(1)$, one can always find a circle (resp., line segment) which is at least $O(1/d)$-distant from V.

We can now assemble all bounds given by Lemma 3.1, Lemmas 4.3 in common terms relevant to the Fuchsian class $\mathscr{F}_{n,m}$ and the carpeting function $R = R_\flat$ on it. The variation of arguments of any linear combination of solutions of the system is explicitly bounded along all slits constructed above, and using the argument principle for simply connected pieces (resp., Lemma 4.5 for annuli), we arrive at the explicit bounds for the counting function.

Obviously, in all products (4.3), (4.6) and hence in (4.10), the double exponential term $S = R^{2^{O(d)}}$ absorbs all other terms without changing the asymptotic behavior of the entire products, which ultimately proves the bound asserted in Theorem 1.5. $\square$

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