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ON DELIGNE-MALGRANGE LATTICES, RESOLUTION OF TURNING POINTS AND HARMONIC BUNDLES

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ABSTRACT. — In this short survey, we would like to overview the recent development of the study on Deligne-Malgrange lattices and resolution of turning points for algebraic meromorphic flat bundles. We also explain their relation with wild harmonic bundles. The author hopes that it would be helpful for access to his work on wild harmonic bundles.

RÉSUMÉ. — Nous donnons un sommaire du développement récent de l'étude sur les réseaux de Deligne-Malgrange et la résolution des tournants pour les connexions méromorphes plates algébriques. Nous expliquons également leur relation avec les fibrés harmoniques sauvages. L'auteur espère que ce sera utile pour l'accès à son travail sur les fibrés harmoniques sauvages.

1. Introduction

In this introduction, let us briefly describe an outline of the study on Deligne-Malgrange lattices, resolution of turning points and their relation with wild harmonic bundles. The precise definitions and statements will be given later.

Deligne-Malgrange lattice. Let X be a complex manifold, and let D be a simple normal crossing hypersurface with the irreducible decomposition $D = \bigcup D_i$. Let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle on (X, D), i.e., $\mathcal{E}$ is a locally free $\mathcal{O}_X(*D)$-module, and ∇ is a flat connection of $\mathcal{E}$. If $(\mathcal{E}, \nabla)$ has regular singularity along D, it is well known (see [5]) that there uniquely exists a locally free $\mathcal{O}_X$-submodule $E \subset \mathcal{E}$ with $\mathcal{O}_X(*D)E = \mathcal{E}$, such that (i) ∇ is logarithmic with respect to E in the sense $\nabla(E) \subset E \otimes \Omega^1(\log D)$,

(ii) the eigenvalues α of the residues $\mathrm{Res}_{D_i}(\nabla)$ satisfy $0 \leqslant \mathrm{Re}(\alpha) < 1$. Such E is called the *Deligne lattice* of $(\mathcal{E}, \nabla)$.

It is natural to ask what happens in the irregular case. B. Malgrange [12] established the existence of a lattice E which *generically* has the property generalizing (i) and (ii). (It will be reviewed in Subsection 2.3.) We call it the *Deligne-Malgrange lattice* of $(\mathcal{E}, \nabla)$. The existence of such a lattice is quite significant in the study of meromorphic flat bundles. For example, it makes possible to apply GAGA to meromorphic flat bundles on projective varieties, as already remarked in [12].

Resolution of turning points. However, in general, there may exist some points of D, called *turning points*, around which the Deligne-Malgrange lattice does not have the good property. As observed by C. Sabbah [20], the existence of turning points is a major obstacle in the asymptotic analysis for meromorphic flat bundles ([11], [20]). So he proposed a conjecture which claims the existence of resolution of turning points, *i.e.*, a projective birational morphism $\varphi : (X', D') \longrightarrow (X, D)$ such that there does not exist any turning points for the Deligne-Malgrange lattice of $\varphi^*(\mathcal{E}, \nabla)$. It seems quite significant in the study of algebraic meromorphic flat bundles and algebraic holonomic D-modules. It might be compared with resolution of singularity of algebraic varieties. He showed it in the case $\dim X = 2$ and $\mathrm{rank}\,\mathcal{E} \leqslant 5$. Relatedly, Y. André [1] showed a conjecture of Malgrange on the non-existence of confluence, motivated by Sabbah's conjecture.

Recently, there have been major developments in the study of such resolution of turning points. In [17], the author proved the existence of a resolution of turning points for a meromorphic flat bundles on projective *surfaces*. In [14], it is shown for those on a smooth proper algebraic variety of *arbitrary dimension*. (See Theorem 2.12 below.) So we have a satisfactory existence theorem in the *algebraic case*. The more general case, including the non-algebraic case, has been studied by K. Kedlaya with a completely different method. He established it in [8] for those on a general complex (not necessarily algebraic) surface.

Remark 1.1. — See [13] for the asymptotic analysis in view of good Deligne-Malgrange lattices or more generally good lattices, where we put a stress on Stokes filtrations.

Harmonic bundle and a characterization of semisimplicity of meromorphic flat bundles. Let (V, ∇) be a flat bundle on a complex

manifold. Let h be a hermitian metric of V. Then, we have the decomposition $\nabla = \nabla^u + \Phi$, where ∇^u is a unitary connection and Φ is a self-adjoint section of $\mathrm{End}(E) \otimes \Omega^1$, with respect to h. We have the decompositions $\nabla^u = \partial_V + \overline{\partial}_V$ and $\Phi = \theta + \theta^\dagger$ into the $(1,0)$-part and the $(0,1)$-part. We say that h is a *pluri-harmonic metric* of (V, ∇), if $(V, \overline{\partial}_V, \theta)$ is a Higgs bundle. In that case, (V, ∇, h) is called a *harmonic bundle*.
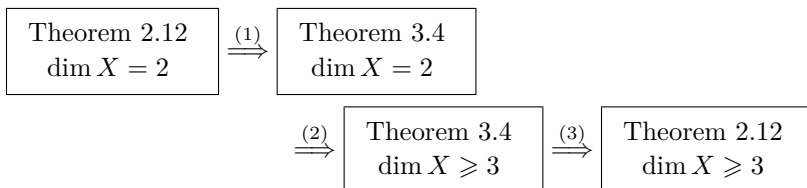
According to K. Corlette [4], a flat bundle on a smooth projective variety has a pluri-harmonic metric if and only if it is semisimple, *i.e.*, a direct sum of simple ones. Moreover, such a pluri-harmonic metric is essentially unique. His result was generalized to the case of meromorphic flat bundles with regular singularity. Namely, let X be a smooth proper algebraic variety, and let D be a normal crossing hypersurface. Let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle on (X, D) with regular singularity. Then, it is semisimple, if and only if $(\mathcal{E}, \nabla)_{|X-D}$ has a pluri-harmonic metric satisfying some condition around D. Moreover, such a pluri-harmonic metric is essentially unique. In the curve case, it was due to C. Simpson [22] with Sabbah's observation [19] that semisimplicity is related with parabolic polystability. In the higher dimensional case, it can be shown by two methods. One is given by J. Jost and K. Zuo [6] with a small and technically minor complement by the author [16]. The other is given by using Kobayashi-Hitchin correspondence for tame harmonic bundles [18].

We can establish such a characterization even in the irregular case by the method of Kobayashi-Hitchin correspondence. (See Theorem 3.2 below.) In the curve case, it was due to Sabbah [19]. (See also the related work due to O. Biquard-P. Boalch [3].) In the higher dimensional case, it was done in [14]. However, we should remark that there is a big difference between regular and irregular meromorphic flat bundles on higher dimensional varieties, that is the existence of *turning points*, which prevents us from applying Kobayashi-Hitchin correspondence directly.

Let us describe how it obscures us. For the construction of pluri-harmonic metrics, at least in the surface case, we would like to use a general framework in global analysis, *i.e.*, (i) take an appropriate initial metric, (ii) deform it along the flow given by a heat equation, (iii) the limit of the heat flow should be a Hermitian-Einstein metric, and it should be pluri-harmonic under some condition. Simpson [21] established a nice general theory for (ii) and (iii), which is valid once an appropriate initial metric is taken in (i). To construct an appropriate initial metric, we would like to know the local normal form of meromorphic flat bundles, which is obscured by the existence of turning points. This is the main motivation for the author to

study resolution of turning points. (More precisely, it is difficult to use the above framework directly in constructing a pluri-harmonic metric, even if there are no turning points. Some difficulty is caused by the nilpotent parts of the residues. We can overcome it by using a perturbation of parabolic structure, which is explained in the introductions of [15] and [18].)

Outline of the proof. Actually, our proof proceeds as follows:

$$
\boxed{\begin{array}{c} \text{Theorem 2.12} \\ \dim X = 2 \end{array}} \overset{(1)}{\Longrightarrow} \boxed{\begin{array}{c} \text{Theorem 3.4} \\ \dim X = 2 \end{array}}
$$

$$
\overset{(2)}{\Longrightarrow} \boxed{\begin{array}{c} \text{Theorem 3.4} \\ \dim X \geqslant 3 \end{array}} \overset{(3)}{\Longrightarrow} \boxed{\begin{array}{c} \text{Theorem 2.12} \\ \dim X \geqslant 3 \end{array}}
$$

The steps (1)–(3) are done in [14]. The proof of Theorem 2.12 in the surface case is given in [17]. (We can also apply the result due to Kedlaya in [8].)

As mentioned above, once we know the existence of resolution of turning points, we can use Kobayashi-Hitchin correspondence to establish the characterization of semisimplicity, which is (1). And, (2) is rather easy to show by using Mehta-Ramanathan type theorem, which is mentioned in Subsection 2.3.2 below.

As explained above, the author studied Theorem 2.12 to show Theorem 3.4. Interestingly, we can deduce Theorem 2.12 by assuming Theorem 3.4, which is the argument used in (3). The brief idea is as follows. It is rather easy to observe that we have only to consider any simple meromorphic flat bundle $(\mathcal{E}, \nabla)$ to show Theorem 2.12. Then, Theorem 3.4 implies that $(\mathcal{E}, \nabla)$ is equipped with a nice pluri-harmonic metric h. Hence, we obtain the Higgs field θ of $(\mathcal{E}, \nabla)_{|X-D}$, whose "characteristic polynomial" is meromorphic on (X, D). It can be shown that the turning points of $(\mathcal{E}, \nabla)$ coincide with the "turning points" of θ. (See Remark 3.5 for turning points of θ.) Thus, we can replace the problem to find a resolution of turning points of $(\mathcal{E}, \nabla)$, with the problem to find a resolution of the turning points of θ. The latter is much easier, for which we can use classical techniques in algebraic or complex geometries. See [14] for more details.

For the proof of Theorem 2.12 in the surface case, we use in [17] mod p-reduction and p-curvature, instead of the Higgs field associated to a harmonic bundle. (See [7] for p-curvature.) The brief idea is similar to the above. We reduce the problem to a control of the spectral manifold of the p-curvature for the mod p-reductions, *uniformly in* p. We use an observation due to J. Bost–Y. Laszlo–C. Pauly [9], *i.e.,* the spectral manifold of the

p-curvature is obtained as the pull back of some subvariety of the cotangent bundle via Frobenius morphism. Since the family of such subvarieties is bounded, we can control them uniformly in p. See [17] for more details.

Remark 1.2. — More precisely, we use the above strategy in the case that X is projective. To generalize the result in the case that X is not necessarily projective but proper algebraic, we apply the Hard Lefschetz theorem for polarized wild pure twistor D-modules. (Theorem 21.1.1 of [14].) By Chow's lemma, we can take a projective birational morphism $\pi : X' \longrightarrow X$ such that X' is smooth projective. If $(\mathcal{E}, \nabla)$ on X is simple, $\pi^*(\mathcal{E}, \nabla)$ is also simple, and it is equipped with a $\sqrt{-1}\boldsymbol{R}$-good wild pluri-harmonic metric h'. By using the Hard Lefschetz theorem for polarized wild pure twistor D-modules, we obtain a $\sqrt{-1}\boldsymbol{R}$-good wild pluri-harmonic metric h for $(\mathcal{E}, \nabla)$. See Section 22.4 of [14] for more details.

I hope that this work would please Bernard Malgrange, the father of the study on Deligne-Malgrange lattices and many other attractive subjects.

Notation. When we are given a complex manifold X with a hypersurfaces D, we will freely use the following notation.

$M(X, D)$**:** the space of meromorphic functions on X whose poles are contained in D.

$H(X)$**:** the space of holomorphic functions on X.

$\widehat{D}$**:** the completion of a complex manifold X along a closed analytic subspace D. (See [2].)

$\mathcal{F}_{|\widehat{D}}$**:** $\iota^{-1}\mathcal{F} \otimes_{\iota^{-1}\mathcal{O}_X} \mathcal{O}_{\widehat{D}}$ for any $\mathcal{O}_X$-module $\mathcal{F}$, where ι denote the inclusion $D \longrightarrow X$.

2. Deligne-Malgrange lattice

2.1. One dimensional case

We explain what is Deligne-Malgrange lattice in the curve case. Let $X := \{z \,|\, |z| < 1\}$ and $D := \{0\}$. Let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle on

(X, D). According to the classical Hukuhara-Turrittin theorem, there exist an appropriate ramified covering $\varphi : (X, D) \longrightarrow (X, D)$ given by $\varphi(z) = z^e$ for some $e > 0$, a finite subset $\mathrm{Irr}(\nabla') \subset M(X', D')/H(X')$, and a formal decomposition

$$(2.1) \qquad \varphi^*(\mathcal{E}, \nabla)_{|\widehat{D}'} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\varphi^*\nabla)} (\widehat{\mathcal{E}}'_{\mathfrak{a}}, \widehat{\nabla}'_{\mathfrak{a}}),$$

such that each $\widehat{\nabla}^{\mathrm{reg}}_{\mathfrak{a}} := \widehat{\nabla}'_{\mathfrak{a}} - d\widetilde{\mathfrak{a}}$ has regular singularity, where $\widetilde{\mathfrak{a}} \in M(X', D')$ is a lift of each $\mathfrak{a}$. We call $\mathrm{Irr}(\varphi^*\nabla)$ the set of irregular values of $\varphi^*(\mathcal{E}, \nabla)$. We implicitly assume that $\widehat{\mathcal{E}}'_{\mathfrak{a}} \neq 0$ for each $\mathfrak{a} \in \mathrm{Irr}(\varphi^*\nabla)$. It is well known that the decomposition (2.1) and $\mathrm{Irr}(\varphi^*\nabla)$ are uniquely determined for $\varphi^*(\mathcal{E}, \nabla)$. If we do not have to take a ramified covering, i.e., $(\mathcal{E}, \nabla)_{|\widehat{D}}$ has such a decomposition, it is called unramified.

Remark 2.1. — In this paper, we follow [11] to call the existence of the decomposition (2.1) Hukuhara-Turrittin theorem.

We can take the Deligne lattices $\widehat{E}'_{\mathfrak{a}}$ for meromorphic flat bundles with regular singularity $(\widehat{\mathcal{E}}'_{\mathfrak{a}}, \widehat{\nabla}^{\mathrm{reg}}_{\mathfrak{a}})$. We obtain the formal lattice:

$$\bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\varphi^*\nabla)} \widehat{E}'_{\mathfrak{a}} \subset \varphi^*(\mathcal{E})_{|\widehat{D}'} .$$

It determines the lattice $E' \subset \varphi^*\mathcal{E}$ with the following property:

- We have the decomposition $(E', \varphi^*\nabla)_{|\widehat{D}'} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\varphi^*\nabla)} (\widehat{E}'_{\mathfrak{a}}, \widehat{\nabla}'_{\mathfrak{a}})$.
- $\widehat{\nabla}'_{\mathfrak{a}} - d\widetilde{\mathfrak{a}}$ are logarithmic with respect to $\widehat{E}'_{\mathfrak{a}}$ for any $\mathfrak{a}$, and the eigenvalues α of the residue satisfy $0 \leqslant \mathrm{Re}(\alpha) < 1$.

It is easy to observe that the lattice E' is invariant under the action of the Galois group of the ramified covering φ. Hence, we obtain the lattice $E \subset \mathcal{E}$ as the descent of E'. This is the Deligne-Malgrange lattice in the one dimensional case.

Any morphism of meromorphic flat bundles $(\mathcal{E}_1, \nabla_1) \longrightarrow (\mathcal{E}_2, \nabla_2)$ on (X, D) induces a morphism of their Deligne-Malgrange lattices, which can be easily checked by using the argument for the uniqueness of unramifiedly Deligne-Malgrange lattice in [10].

Example 1. Let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle on $\mathbb{P}^1$ given as follows:

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(*0) \cdot v_1 \oplus \mathcal{O}_{\mathbb{P}^1}(*0) \cdot v_2, \quad \nabla(v_1, v_2) = (v_1, v_2) \cdot \begin{pmatrix} 0 & 1 \\ z^{-1} & 0 \end{pmatrix} \cdot d\Big(\frac{1}{z}\Big).$$

It has the singularity only at 0. (We regard it as a meromorphic flat bundle on $\mathbb{P}^1$ for use in Example 2.)

Let $\varphi(\zeta) = \zeta^2$ be a ramified covering $\boldsymbol{C}_\zeta \longrightarrow \boldsymbol{C}_z$. We set

$$
\boldsymbol{w} := \varphi^* \boldsymbol{v} \cdot \begin{pmatrix} \zeta & \zeta \\ 1 & -1 \end{pmatrix}.
$$

We have the following:

$$
\varphi^* \nabla \boldsymbol{w} = \boldsymbol{w} \cdot \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} d\left(\frac{2}{3}\zeta^{-3}\right) + \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{d\zeta}{\zeta} \right).
$$

By a well-established argument, which is explained in [10] very clearly, we can show that $\varphi^*(\mathcal{E}, \nabla)$ is unramified, and it has the formal decomposition:

$$
(2.2) \qquad \varphi^*(\mathcal{E}, \nabla)_{|\widehat{\{\zeta=0\}}} \simeq \left(L\left(\frac{2}{3}\zeta^{-3}, \frac{1}{2}\right) \oplus L\left(-\frac{2}{3}\zeta^{-3}, \frac{1}{2}\right) \right)_{|\widehat{\{\zeta=0\}}}.
$$

Here, $L(\mathfrak{a}, \alpha)$ is a meromorphic line bundle $\mathcal{O}_{\boldsymbol{C}_\zeta}(*0) \cdot e$ with the connection $\nabla e = e \cdot \left(d\mathfrak{a} + \alpha \cdot d\zeta/\zeta \right)$ for $\mathfrak{a} \in \boldsymbol{C}[\zeta^{-1}]$ and $\alpha \in \boldsymbol{C}$. The set of irregular values of $\varphi^*(\mathcal{E}, \nabla)$ is

$$
\mathrm{Irr}(\varphi^* \nabla) = \left\{ \frac{2}{3}\zeta^{-3}, \; -\frac{2}{3}\zeta^{-3} \right\}.
$$

Let $E' \subset \varphi^* \mathcal{E}$ be the lattice generated by $\boldsymbol{w}$, or equivalently $\left(\zeta \cdot \varphi^* v_1, \varphi^* v_2 \right)$. It is easy to show that E' is Deligne-Malgrange lattice of $\varphi^*(\mathcal{E}, \nabla)$. Hence, the Deligne-Malgrange lattice $E \subset \mathcal{E}$ is generated by $\left(z \cdot v_1, v_2 \right)$ around 0.

It is easy to observe that the isomorphism (2.2) is just formal but non-convergent. Note that the monodromy of $(\mathcal{E}, \nabla)$ is trivial, and that the right hand side of (2.2) has non-trivial monodromy.

2.2. Good Deligne-Malgrange lattice

In the higher dimensional case, the existence of such a lattice is proved by Malgrange. But, before recalling his result, we explain what is an *ideal* generalization of the property, that is *good Deligne-Malgrange lattice*. We remark in advance that a meromorphic flat bundle does not have a good Deligne-Malgrange lattice, in general.

2.2.1. Good set of irregular values

We recall the notion of good set of irregular values. We use the partial order $\leqslant_{\mathbb{Z}^n}$ of $\mathbb{Z}^n$ given by $\boldsymbol{a} \leqslant_{\mathbb{Z}^n} \boldsymbol{b} \iff a_i \leqslant b_i \, (\forall i)$. Let $\boldsymbol{0}$ denote the zero in $\mathbb{Z}^n$. We also use $\boldsymbol{0}_n$ when we distinguish the dependence on n.

Let Y be a complex manifold. Let $X := \Delta^\ell \times Y$, $D_i := \{z_i = 0\} \times Y$ and $D := \bigcup_{i=1}^\ell D_i$. We also put $D_{\underline{\ell}} = \bigcap_{i=1}^\ell D_i$, which is naturally identified with Y. For any $\boldsymbol{m} = (m_i \,|\, i = 1, \ldots, \ell) \in \mathbb{Z}^\ell$, we set $\boldsymbol{z}^{\boldsymbol{m}} := \prod_{i=1}^\ell z_i^{m_i}$. For any $f \in M(X, D)$, we have the Laurent expansion:

$$f = \sum_{\boldsymbol{m} \in \mathbb{Z}^\ell} f_{\boldsymbol{m}}(y) \cdot \boldsymbol{z}^{\boldsymbol{m}}$$

Here $f_{\boldsymbol{m}}$ are holomorphic functions on $D_{\underline{\ell}}$. We often use the following identification implicitly:

$$(2.3) \qquad M(X, D)/H(X) \simeq \left\{ f \in M(X, D) \,\middle|\, f_{\boldsymbol{m}} = 0, \ \forall \boldsymbol{m} \geqslant \boldsymbol{0} \right\}.$$

For any $f \in M(X, D)$, let $\operatorname{ord}(f)$ denote the minimum of the set

$$\left\{ \boldsymbol{m} \in \mathbb{Z}^\ell \,\middle|\, f_{\boldsymbol{m}} \neq 0 \right\} \cup \{\boldsymbol{0}\}$$

with respect to $\leqslant_{\mathbb{Z}^\ell}$, if it exists. It is always contained in $\mathbb{Z}^\ell_{\leqslant 0}$, if it exists.

For any $\mathfrak{a} \in M(X, D)/H(X)$, we take any lift $\widetilde{\mathfrak{a}}$ to $M(X, D)$, and we set $\operatorname{ord}(\mathfrak{a}) := \operatorname{ord}(\widetilde{\mathfrak{a}})$, if the right hand side exists. If $\operatorname{ord}(\mathfrak{a})$ exists in $\mathbb{Z}^\ell - \{\boldsymbol{0}\}$, $\widetilde{\mathfrak{a}}_{\operatorname{ord}(\mathfrak{a})}$ is independent of the choice of a lift $\widetilde{\mathfrak{a}}$, which is denoted by $\mathfrak{a}_{\operatorname{ord}(\mathfrak{a})}$.

A finite subset $\mathcal{I} \subset M(X, D)/H(X)$ is called a good set of irregular values on (X, D), if the following conditions are satisfied:

- $\operatorname{ord}(\mathfrak{a})$ exists for each $\mathfrak{a} \in \mathcal{I}$, and $\mathfrak{a}_{\operatorname{ord}(\mathfrak{a})}$ is nowhere vanishing on $D_{\underline{\ell}}$ for $\mathfrak{a} \neq 0$.
- For any two distinct $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$, $\operatorname{ord}(\mathfrak{a} - \mathfrak{b})$ exists in $\mathbb{Z}^\ell_{\leqslant 0} - \{\boldsymbol{0}\}$, and $(\mathfrak{a} - \mathfrak{b})_{\operatorname{ord}(\mathfrak{a} - \mathfrak{b})}$ is nowhere vanishing on $D_{\underline{\ell}}$.
- The set $\mathcal{T}(\mathcal{I}) := \left\{ \operatorname{ord}(\mathfrak{a} - \mathfrak{b}) \,\middle|\, \mathfrak{a}, \mathfrak{b} \in \mathcal{I} \right\}$ is totally ordered with respect to the partial order on $\mathbb{Z}^\ell$.

The conditions do not depend on the choice of a holomorphic coordinate such that $D = \bigcup_{i=1}^\ell \{z_i = 0\}$.

2.2.2. Good Deligne-Malgrange lattice

Let X be a complex manifold. Let D be a simple normal crossing hypersurface of X with the irreducible decomposition $D = \bigcup_{j \in \Lambda} D_j$, i.e., $D_J := \bigcap_{j \in J} D_j$ are smooth for any $J \subset \Lambda$. Let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle on (X, D). Let E be a lattice of $(\mathcal{E}, \nabla)$. We say that E is an unramifiedly good Deligne-Malgrange lattice if the following holds at any $P \in D$.

- Let X_P be a small coordinate neighbourhood of P in X. Let $I(P) :=$ $\{i \mid P \in D_i\}$. We set $D_P := X_P \cap D$ and $D_{I(P)} := X_P \cap \bigcap_{i \in I(P)} D_i$. Then, we have a good set of irregular values

$$\operatorname{Irr}(\nabla, P) \subset M(X_P, D_P)/H(X_P)$$

and a decomposition

$$(2.4) \qquad (E, \nabla)_{|\widehat{D}_{I(P)}} = \bigoplus_{\mathfrak{a} \in \operatorname{Irr}(\nabla, P)} (\widehat{E}_{\mathfrak{a}}, \widehat{\nabla}_{\mathfrak{a}})$$

such that (i) $\widehat{\nabla}_{\mathfrak{a}} - d\widetilde{\mathfrak{a}}$ are logarithmic, where $\widetilde{\mathfrak{a}} \in M(X_P, D_P)$ are lifts of $\mathfrak{a}$, (ii) the eigenvalues α of the residues $\operatorname{Res}_{D_i}(\widehat{\nabla}_{\mathfrak{a}} - d\widetilde{\mathfrak{a}})$ satisfy $0 \leqslant \operatorname{Re}(\alpha) < 1$. The decomposition (2.4) is called the irregular decomposition at P.

We say that E is a good Delinge-Malgrange lattice, if the following holds for any $P \in D$:

- If we take an appropriate ramified covering

$$\varphi : (X'_P, D'_P) \longrightarrow (X_P, D_P),$$

there exists an unramifiedly good Deligne-Malgrange lattice E' of $\varphi^*(\mathcal{E}, \nabla)$, and $E_{|X_P}$ is the descent of E'.

(See Sections 5.1 and 5.7 of [14].) *We should emphasize that $(\mathcal{E}, \nabla)$ does not have a good Deligne-Malgrange lattice, in general.* (See Example 2, below.) But, if they exist, they are uniquely determined. In the one dimensional case, a Deligne-Malgrange lattice in Section 2.1 is always a good Deligne-Malgrange lattice.

Remark 2.2. — We have the more general notions of unramifiedly good lattice and good lattice for meromorphic flat bundles. See Sections 5.1 and 5.7 of [14]. In this paper, we adopt a simplified definition of good lattice. See [13] for a review on good lattices and the related asymptotic analysis.

Remark 2.3. — In [20], Sabbah studied a closely related notion, *i.e.,* good formal structure. If $(\mathcal{E}, \nabla)$ has the good Deligne-Malgrange lattice, it clearly has the good formal structure. At this moment, the converse is not clear to the author except for the surface case. (See Section 5.6 of [14].) However, we do not have to care it at least in the algebraic case, because we can take resolutions (Theorem 2.12 below).

2.2.3. Formal decompositions

For simplicity, let $X := \Delta^n$, $D_i := \{z_i = 0\}$, $D = \bigcup_{i=1}^{\ell} D_i$ and $O = (0, \dots, 0) \in X$. We set $\underline{\ell} := \{1, \dots, \ell\}$. For any $I \subset \underline{\ell}$, we set $I^c := \underline{\ell} - I$. We put $D(I) := \bigcup_{i \in I} D_i$ and $D_I := \bigcap_{i \in I} D_i$. Let $p_I : M(X, D)/H(X) \longrightarrow M(X, D)/M(X, I^c)$ be the naturally defined projection. For any $P \in D$, let X_P, D_P, $I(P)$ and $D_{I(P)}$ be as in Section 2.2.2.

Let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle on (X, D) with the unramifiedly good Deligne-Malgrange lattice E. We set $\mathrm{Irr}(\nabla) := \mathrm{Irr}(\nabla, O)$. We set $\mathrm{Irr}(\nabla, I) := p_I\big(\mathrm{Irr}(\nabla)\big)$.

PROPOSITION 2.4. — *There exist a neighbourhood U of O in X, and formal flat decompositions for any $I \subset \underline{\ell}$*

$$(2.5) \qquad (E, \nabla)_{|\widehat{D}_I \cap U} = \bigoplus_{\mathfrak{b} \in \mathrm{Irr}(\nabla, I)} \big({}^I\widehat{E}_{\mathfrak{b}}, {}^I\widehat{\nabla}_{\mathfrak{b}}\big)$$

satisfying the following:

- *Let $\widetilde{\mathfrak{b}} \in M(X, D)$ be any lift of $\mathfrak{b} \in \mathrm{Irr}(\nabla, I)$. Then, the following holds:*

$$(2.6) \qquad \big({}^I\widehat{\nabla}_{\mathfrak{b}} - d\widetilde{\mathfrak{b}}\big) \cdot {}^I\widehat{E}_{\mathfrak{b}} \subset {}^I\widehat{E}_{\mathfrak{b}} \otimes \Big(\Omega^1_X\big(\log D(I)\big) + \Omega^1_X\big({*}D(I^c)\big)\Big)$$

- *For any $I \subset J \subset \underline{\ell}$, let $p_{J,I} : \mathrm{Irr}(\nabla, J) \longrightarrow \mathrm{Irr}(\nabla, I)$ denote the naturally defined map. Then,*

$$(2.7) \qquad {}^I\widehat{E}_{\mathfrak{b}|\widehat{D}_J \cap U} = \bigoplus_{\mathfrak{c} \in p_{J,I}^{-1}(\mathfrak{b})} {}^J\widehat{E}_{\mathfrak{c}}.$$

- *The naturally defined map*

$$M(X, D)/M\big(X, D(I(P)^c)\big) \longrightarrow M(X_P, D_P)/H(X_P)$$

induces the bijection $\mathrm{Irr}(\nabla, I(P)) \simeq \mathrm{Irr}(\nabla, P)$ for any $P \in D \cap U$, and the restriction of (2.5) for $I = I(P)$ to $\widehat{D}_{I(P)}$ gives the irregular decomposition at $P \in U \cap D$.

Proof. — See Section 5.7 of [14]. Note that we adopt the simplified definition of good lattice in this paper. In contrast, in [14], we define the notion of *good lattice* by the existence of the decompositions (2.5) satisfying (2.6), from which the second and third conditions follow by the uniqueness. □

Remark 2.5. — By gluing the decompositions (2.5) with the relation (2.7), we may take decompositions on $\widehat{D}(I)$ in several levels. See Section 5.1.2 of [14] for more details. It is useful in the study of asymptotic

analysis. However, we may NOT have a decomposition on $\widehat{D}$ whose restriction to D_ℓ is the same as (2.5) with $I = \ell$, in general, as remarked by Sabbah [20].

2.3. Deligne-Malgrange lattice

2.3.1. Existence theorem due to Malgrange

Let X and D be as in Section 2.2.2. As already mentioned, a meromorphic flat bundle $(\mathcal{E}, \nabla)$ on (X, D) may not have the good Deligne-Malgrange lattice, in general. Instead, Malgrange proved the following.

THEOREM 2.6 ([12]). — *There exists a unique lattice $E \subset \mathcal{E}$ such that (i) $E_{|X-Z}$ is a good Deligne-Malgrange lattice of $(\mathcal{E}, \nabla)_{|X-Z}$ for some analytic closed subset $Z \subset D$ with $\mathrm{codim}_X(Z) \geqslant 2$, (ii) E is coherent and reflexive as an $\mathcal{O}_X$-module.*

He called this lattice the *canonical lattice*. But, we would like to call it the *Deligne-Malgrange lattice*.

DEFINITION 2.7. — *Let E be the Deligne-Malgrange lattice of $(\mathcal{E}, \nabla)$. Let Z_0 be the minimum among the closed analytic subsets of D such that $E_{|X-Z_0}$ is the good Deligne-Malgrange lattice of $(\mathcal{E}, \nabla)_{|X-Z_0}$. Each point of Z_0 is called a turning point of $(\mathcal{E}, \nabla)$.*

Example 2. Let ψ be the rational map from $X := \boldsymbol{C}^2$ to $\mathbb{P}^1$ given by $\psi(z, w) := [z : w]$. We set $D := \{z = 0\}$. Let $(\mathcal{E}, \nabla)$ be as in Example 1. We obtain the meromorphic flat bundle $(\mathcal{E}_1, \nabla_1) := \psi^*(\mathcal{E}, \nabla)$ on (X, D):

$$\nabla_1 \psi^* \boldsymbol{v} = \psi^* \boldsymbol{v} \begin{pmatrix} 0 & 1 \\ w/z & 0 \end{pmatrix} \cdot d\left(\frac{w}{z}\right).$$

Let $X' := \boldsymbol{C}^2 = \{(\zeta, w)\}$, and let $\overline{\varphi} : X' \longrightarrow X$ be given by $\overline{\varphi}(\zeta, w) = (\zeta^2, w)$. We set $D' := \{\zeta = 0\}$.

Let $E_1 \subset \mathcal{E}_1$ be the lattice generated by $z \cdot \psi^* v_1$ and $\psi^* v_2$. Let $E'_1 \subset \overline{\varphi}^* \mathcal{E}_1$ be the lattice generated by $\zeta \cdot \overline{\varphi}^* \psi^* v_1$ and $\overline{\varphi}^* \psi^* v_2$. Let us observe that E_1 and E'_1 are the Deligne-Malgrange lattices for $(\mathcal{E}_1, \nabla_1)$ and $\overline{\varphi}^*(\mathcal{E}_1, \nabla_1)$, respectively. Since E_1 is the descent of E'_1, we have only to check the claim for E'_1.

Let $P \in D' - \{(0,0)\}$, and X'_P be a small neighbourhood of P in X'. We set $D'_P := X'_P \cap D'$. For $\mathfrak{a} \in M(X', D')$ and $\alpha \in \boldsymbol{R}$, let $L(\mathfrak{a}, \alpha)$ denote $\mathcal{O}_{X'}(*D') \cdot e$ with a flat connection $\nabla e = e \cdot (d\mathfrak{a} + \alpha \cdot dz/z)$. By choosing a

branch of $w^{1/2}$ on X'_P, we can take a morphism $\overline{\psi}_P : X'_P \longrightarrow \boldsymbol{C}_\zeta$ such that $\varphi \circ \overline{\psi}_P = \psi \circ \overline{\varphi}$ given by $\overline{\psi}_P(\zeta, w) = \zeta \cdot w^{-1/2}$. Then, we have the following flat decomposition obtained as the pull back of (2.2) via $\overline{\psi}_P$:

$$(2.8) \qquad \overline{\varphi}^*(\mathcal{E}_1)_{|\widehat{D}'_P} \simeq \left(L\left(\frac{2}{3}\zeta^{-3}w^{3/2}, \frac{1}{2}\right) \oplus L\left(-\frac{2}{3}\zeta^{-3}w^{3/2}, \frac{1}{2}\right) \right)_{|\widehat{D}'_P}.$$

And, it is easy to see that $E'_{1|X'_P} = \overline{\psi}^*_P E'$ gives the unramifiedly good Deligne-Malgrange lattice of $\overline{\varphi}^*(\mathcal{E}_1, \nabla_1)_{|X'_P}$. Thus, we can conclude that E'_1 is the Deligne-Malgrange lattice for $\overline{\varphi}^*(\mathcal{E}_1, \nabla_1)$.

Let us observe that $(0,0)$ is a turning point, and that the Deligne-Malgrange lattice is not good in this case. The irregular values of $\overline{\varphi}^*(\mathcal{E}, \nabla)$ are $\pm \zeta^{-3} \cdot w^{3/2}$. When we go around $w = 0$, the irregular values are exchanged, and hence the direct summands in (2.8) are exchanged. Hence, we can conclude that $(0,0)$ is a turning point.

Deligne-Malgrange filtered sheaf. Let us explain a slightly refined notion. We use the notation in Section 2.2.2. If $(\mathcal{E}, \nabla)$ has the unramifiedly good Deligne-Malgrange lattice, we can take the lattice $_{\boldsymbol{a}}E^{DM} \subset \mathcal{E}$ for any $\boldsymbol{a} = (a_i \,|\, i \in \Lambda) \in \boldsymbol{R}^\Lambda$, such that we have a decomposition for each $P \in D$

$$(_{\boldsymbol{a}}E^{DM}, \nabla)_{|\widehat{D}_{I(P)}} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla, P)} (_{\boldsymbol{a}}\widehat{E}^{DM}_{\mathfrak{a}}, \widehat{\nabla}_{\mathfrak{a}}),$$

such that (i) $\widehat{\nabla}_{\mathfrak{a}} - d\widetilde{\mathfrak{a}}$ are logarithmic with respect to $_{\boldsymbol{a}}\widehat{E}^{DM}_{\mathfrak{a}}$, (ii) the eigenvalues α of the residues $\mathrm{Res}_{D_i}(\widehat{\nabla}_{\mathfrak{a}} - d\mathfrak{a})$ satisfy $a_i \leqslant \mathrm{Re}(\alpha) < a_i + 1$. It is called unramifiedly good $\boldsymbol{a}$-Deligne-Malgrange lattice in this paper. If $(\mathcal{E}, \nabla)$ has the good Deligne-Malgrange lattice, we can take the lattice $_{\boldsymbol{a}}E^{DM} \subset \mathcal{E}$ for any $\boldsymbol{a} \in \boldsymbol{R}^\Lambda$, such that the following holds for any $P \in D$:

- Take an appropriate ramified covering $\varphi : (X'_P, D'_P) \longrightarrow (X_P, D_P)$ such that $\varphi^*(\mathcal{E}, \nabla)$ has the unramifiedly good Deligne-Malgrange lattice. Let e_i denote the ramification index of φ along D_i for $i \in I(P)$. Let $\varphi^*\boldsymbol{a} := (e_i \cdot a_i \,|\, i \in I(P))$. Then, $_{\boldsymbol{a}}E^{DM}_{|X'_P}$ is the descent of unramifiedly good $\varphi^*\boldsymbol{a}$-Deligne-Malgrange lattice of $\varphi^*(\mathcal{E}, \nabla)$.

In the general case, we have the lattice $_{\boldsymbol{a}}E^{DM} \subset \mathcal{E}$ for each $\boldsymbol{a} \in \boldsymbol{R}^\Lambda$ such that (i) $_{\boldsymbol{a}}E^{DM}_{|X-Z}$ is the good $\boldsymbol{a}$-Deligne-Malgrange lattice of $(\mathcal{E}, \nabla)_{|X-Z}$, where Z is a closed analytic subset of D with $\mathrm{codim}_X(Z) \geqslant 2$, (ii) $_{\boldsymbol{a}}E^{DM}$ is coherent and reflexive as an $\mathcal{O}_X$-module. In the case $\boldsymbol{a} = (0, \ldots, 0)$, we prefer the symbol $^\diamond E^{DM}$, which is the same as the Deligne-Malgrange lattice in the previous sense. The increasing sequence of $\mathcal{O}_X$-modules $\boldsymbol{E}^{DM}_* := (_{\boldsymbol{a}}E^{DM} \,|\, \boldsymbol{a} \in \boldsymbol{R}^\Lambda)$ is associated to $(\mathcal{E}, \nabla)$, which is called the *Deligne-Malgrange filtered sheaf*. (See also Section 5.1.5 of [14].)

If we are given a morphism of meromorphic flat connections $(\mathcal{E}_1, \nabla_1) \longrightarrow (\mathcal{E}_2, \nabla_2)$ on (X, D), we have the induced morphisms of $\boldsymbol{a}$-Deligne-Malgrange lattices, and hence $\boldsymbol{E}_{1*}^{DM} \longrightarrow \boldsymbol{E}_{2*}^{DM}$.

Remark 2.8. — In general, it is important to consider this kind of filtered objects. As remarked below, the stability condition is defined for such filtered objects on projective varieties, and Kobayashi-Hitchin correspondence is formulated as the equivalence between stability and existence of a nice metric adapted to this filtrations. See [15], [18] and [14] for more details.

2.3.2. Some applications of the existence of Deligne-Malgrange lattice

Algebraicity of meromorphic flat bundles. One of the most important immediate consequences is the algebraicity of meromorphic flat bundles on a projective variety. Namely, let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle on a projective variety X in the sense of complex analytic geometry. By GAGA, the Deligne-Malgrange lattice E of $(\mathcal{E}, \nabla)$ is $\mathcal{O}_X$-coherent in the senses of both complex analytic geometry and algebraic geometry. Hence, we can conclude that $(\mathcal{E}, \nabla)$ is algebraic, as was observed in [12].

Semisimplicity and poly-stability. Let X be a smooth projective variety, and let D be a simple normal crossing hypersurface with the irreducible decomposition $D = \bigcup_{j \in \Lambda} D_j$. Let L be any ample line bundle on X. In general, we have the notion of μ_L-stability, μ_L-semistability, and μ_L-polystability for filtered flat sheaf $(\boldsymbol{E}_*, \nabla)$ on (X, D). Namely, the parabolic slope $\mu_L(\boldsymbol{E}_*)$ is attached to any filtered sheaf, and $(\boldsymbol{E}_*, \nabla)$ is called μ_L-stable, if and only if $\mu_L(\boldsymbol{F}_*) < \mu_L(\boldsymbol{E}_*)$ holds for any subobject $(\boldsymbol{F}_*, \nabla) \subset (\boldsymbol{E}_*, \nabla)$ such that $0 < \operatorname{rank} \boldsymbol{F}_* < \operatorname{rank} \boldsymbol{E}_*$. (See Section 17.1 of [14], for example.)

We can observe $\mu_L(\boldsymbol{E}_*^{DM}) = 0$ for any meromorphic flat bundles on projective varieties. Hence, $(\boldsymbol{E}_*^{DM}, \nabla)$ is μ_L-stable, if and only if $(\mathcal{E}, \nabla)$ is simple, *i.e.*, there does not exist any non-trivial $(\mathcal{E}', \nabla') \subset (\mathcal{E}, \nabla)$ such that $0 < \operatorname{rank} \mathcal{E}' < \operatorname{rank} \mathcal{E}$. (We can find this observation in Sabbah's work [19].) Note that we have the Mehta-Ramanathan type theorem for μ_L-stability condition, and hence simplicity. (See Section 17.2 of [14].)

PROPOSITION 2.9. — $(\mathcal{E}, \nabla)$ *is* μ_L*-stable if and only if the restriction* $(\mathcal{E}, \nabla)_{|Y}$ *is* μ_L*-stable for a sufficiently ample generic hypersurface* Y. *Hence,* $(\mathcal{E}, \nabla)$ *is simple if and only if the restriction* $(\mathcal{E}, \nabla)_{|Y}$ *is simple.*

Remark 2.10. — In this paper $(\mathcal{E}, \nabla)$ is called simple if there are no non-trivial meromorphic flat sub-connections. In the context of Mehta-Ramanathan type theorem, "simplicity" is also often used in the sense that any non-trivial automorphism is a constant multiplication.

2.4. Resolution of turning points

To explain the motivation for the problem, we briefly recall classical asymptotic analysis of meromorphic flat bundles on curves. Let $X := \Delta$ and $D := \{0\}$. Let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle on (X, D). Assume that the Deligne-Malgrange lattice E is unramified, and hence we have the formal decomposition $(E, \nabla)_{|\widehat{D}} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\nabla)} (\widehat{E}_{\mathfrak{a}}, \widehat{\nabla}_{\mathfrak{a}})$. This is just formal and not convergent, in general. So we may not extend it to the decomposition in a neighbourhood of D. But, we can lift it to a flat decomposition on small sectors of $X - D$. Such a lifting is not unique, and the ambiguity leads us to Stokes structure. (See a standard textbook [23], for example.)

Majima [11] initiated the systematic study on asymptotic analysis for meromorphic flat bundles on higher dimensional varieties, and Sabbah [20] revisited it with a different formulation. Briefly speaking, they established the higher dimensional generalization of the lifting of formal decomposition to flat decompositions on multi-sectors, and they studied the classification of meromorphic flat bundles. However, we need a formal decomposition in the beginning. In this sense, the existence of turning points prevents us from applying their general result. To deal with this obstacle, Sabbah proposed a conjecture which we state in a slightly generalized form.

CONJECTURE 2.11. — *There exists a resolution of turning points for* $(\mathcal{E}, \nabla)$. *Namely, there exists a birational morphism* $\varphi : (\widetilde{X}, \widetilde{D}) \longrightarrow (X, D)$ *such that*

(i) $\widetilde{X} - \widetilde{D} \simeq X - D$,

(ii) $\varphi^*(\mathcal{E}, \nabla)$ *has a good Deligne-Malgrange lattice.*

Let us look at Example 2. We take a blow up $\widetilde{X} = \widetilde{\boldsymbol{C}}^2$ of $X = \boldsymbol{C}^2$ at $(0, 0)$. Then, we obtain the morphism $\widetilde{\psi} : \widetilde{X} \longrightarrow \mathbb{P}^1$, and the pull back $\widetilde{\psi}^*(\mathcal{E}, \nabla)$. Since ψ is only a rational morphism which is not determined at $(0, 0)$, the pull back $\psi^*(\mathcal{E}, \nabla)$ may have a bad property at $(0, 0)$. But, $\widetilde{\psi}$ is a morphism, and it is easy to observe that $\widetilde{\psi}^*(\mathcal{E}, \nabla)$ has the good Deligne-Malgrange lattice.

One of the main theorems in this survey is the following.

THEOREM 2.12 (Theorem 19.2.1 and Corollary 22.5.2 of [14]). — *If X is proper algebraic, we can take a resolution of turning points for $(\mathcal{E}, \nabla)$.*

Remark 2.13. — Kedlaya [8] established it without the algebraicity assumption in the case $\dim X = 2$.

3. Harmonic bundle

Let us recall the classical theorem due to Corlette.

THEOREM 3.1 ([4]). — *Let X be a complex projective manifold. Let (E, ∇) be a flat bundle. Then, the following conditions are equivalent.*

- *(E, ∇) is semisimple in the category of flat bundles on X.*
- *There exists a pluri-harmonic metric h of (E, ∇).*

This is generalized to a characterization of semisimplicity for meromorphic flat bundles on projective varieties by the existence of pluri-harmonic metric with some nice property around singularity, which we would like to explain.

$\sqrt{-1}\boldsymbol{R}$**-good wild harmonic bundle.** Let X and D be as in Subsection 2.2.2. Let (E, ∇, h) be a harmonic bundle on $X - D$. We have the associated Higgs field θ. We say that (E, ∇, h) is an unramifiedly $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle on (X, D), if the following holds for any $P \in D$.

- Let $(X_P, z_1, \ldots, z_n)$ be a small coordinate neighbourhood of P in X such that $D \cap X_P = \bigcup_{i=1}^{\ell} \{z_i = 0\} =: D_P$. There exist a good set of irregular values $\mathrm{Irr}(\theta, P) \subset M(X_P, D_P)/H(X_P)$ and a decomposition

$$(3.1) \qquad (E, \theta)_{|X_P - D_P} = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\theta, P)} (E_{\mathfrak{a}}, \theta_{\mathfrak{a}}),$$

  such that the eigenvalues of $\theta_{\mathfrak{a}} - d\widetilde{\mathfrak{a}}$ are of the following form

$$\sum_{i=1}^{\ell} \alpha_i \frac{dz_i}{z_i} + \tau,$$

  where $\widetilde{\mathfrak{a}} \in M(X_P, D_P)$ is a lift of $\mathfrak{a}$, $\alpha_i \in \sqrt{-1}\boldsymbol{R}$, and τ is a multivalued holomorphic one form.

We say that (E, ∇, h) is a $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle, if the following holds for any $P \in D$:

- There exist a small neighbourhood X_P of P and a ramified covering $\varphi_P : (X'_P, D'_P) \longrightarrow (X_P, D_P)$ such that $\varphi_P^*(E, \nabla, h)$ is an unramifiedly $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle on (X'_P, D'_P).

See Section 11.1 of [14] for a different formulation.

Prolongment of $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle. Let X and D be as above. Let E be a holomorphic vector bundle on $X - D$ with a hermitian metric h. For any holomorphic local coordinate $(U, z_1, \ldots, z_n)$ of X such that $U \cap D = \bigcup_{i=1}^{\ell} \{z_i = 0\}$, we put

$$\mathcal{P}E(U) := \left\{ f \in E(U \setminus D) \,\middle|\, |f|_h = O\Big(\prod_{i=1}^{\ell} |z_i|^{-N}\Big) \ \exists N > 0 \right\}$$

$$\mathcal{P}_0 E(U) := \left\{ f \in E(U \setminus D) \,\middle|\, |f|_h = O\Big(\prod_{i=1}^{\ell} |z_i|^{-\epsilon}\Big) \ \forall \epsilon > 0 \right\}.$$

By taking the sheafification, we obtain an $\mathcal{O}_X(*D)$-module $\mathcal{P}E$ and an $\mathcal{O}_X$-module $\mathcal{P}_0 E$. Good Deligne-Malgrange lattice naturally appears in the study of wild harmonic bundles, due to the following theorem.

THEOREM 3.2. — *Let (E, ∇, h) be a $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle.*

- *$(\mathcal{P}E, \nabla)$ is a meromorphic flat bundle on (X, D).*
- *$\mathcal{P}_0 E$ is the good Deligne-Malgrange lattice of $\mathcal{P}E$.*

Proof. — See Section 11.4 of [14] for the claims (i) $\mathcal{P}_0 E$ is locally free, (ii) ∇ is meromorphic, (iii) $(\mathcal{P}_0 E, \nabla)$ is a good lattice of $(\mathcal{P}E, \nabla)$, which particularly imply the first claim. By using the comparison of the KMS-spectra in Section 12.2 of [14], we can show that $\mathcal{P}_0 E$ is the good Deligne-Malgrange lattice. $\qquad\qquad\square$

We give a refined statement. Let $D = \bigcup_{j \in \Lambda} D_j$ be the irreducible decomposition. Let $(U, z_1, \ldots, z_n)$ be any holomorphic local coordinate of X such that $U \cap D = \bigcup_{i=1}^{\ell} \{z_i = 0\}$. For $i = 1, \ldots, \ell$, let $j(i) \in \Lambda$ be determined by $D_{j(i)} \cap U = \{z_i = 0\}$. Let E be a holomorphic vector bundle with a hermitian metric h. For any $\boldsymbol{a} = (a_j \,|\, j \in \Lambda) \in \boldsymbol{R}^\Lambda$, we put

$$\mathcal{P}_{\boldsymbol{a}} E(U) := \left\{ f \in E(U \setminus D) \,\middle|\, |f|_h = O\Big(\prod_{i=1}^{\ell} |z_i|^{-\epsilon - a_{j(i)}}\Big) \ \forall \epsilon > 0 \right\}.$$

By taking the sheafification, we obtain an $\mathcal{O}_X$-module $\mathcal{P}_{\boldsymbol{a}} E$.

THEOREM 3.3. — *If (E, ∇, h) is a $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle, the filtered sheaf $\mathcal{P}_* E = \big(\mathcal{P}_{\boldsymbol{a}} E \,\big|\, \boldsymbol{a} \in \boldsymbol{R}^\Lambda\big)$ is equal to the Deligne-Malgrange*

filtered sheaf associated to the meromorphic flat bundle $(\mathcal{P}E, \nabla)$. Each $\mathcal{P}_{\boldsymbol{a}}E$ is the $\boldsymbol{a}$-good Deligne-Malgrange lattice.

Proof. — See Sections 11.4 and 12.2 of [14]. □

Characterization of semisimplicity. Let us assume moreover that X is proper and algebraic. Let $(\mathcal{E}, \nabla)$ be a meromorphic flat bundle on (X, D). According to Theorem 2.6, there exists a closed subset $Z \subset D$ with $\mathrm{codim}_X(Z) \geqslant 2$ such that $(\mathcal{E}, \nabla)_{|X-Z}$ has a good Deligne-Malgrange lattice. The following theorem is the other main result in this survey.

THEOREM 3.4. — *The following conditions are equivalent.*

- $(\mathcal{E}, \nabla)$ *is semisimple in the category of meromorphic flat bundles.*
- $(E, \nabla) := (\mathcal{E}, \nabla)_{|X-D}$ *has a pluri-harmonic metric h such that (i) (E, ∇, h) is a $\sqrt{-1}\boldsymbol{R}$-good wild harmonic bundle on $(X-Z, D-Z)$, (ii) $\mathcal{P}E_{|X-Z} = \mathcal{E}_{|X-Z}$.*
- $(E, \nabla) := (\mathcal{E}, \nabla)_{|X-D}$ *has a pluri-harmonic metric h such that $\mathcal{P}_*E_{|X-Z} = \boldsymbol{E}^{DM}_{*|X-Z}$, where $\boldsymbol{E}^{DM}_*$ denotes the Deligne-Malgrange filtered sheaf associated to $(\mathcal{E}, \nabla)$.*

Such a pluri-harmonic metric is unique up to positive constant multiplication on each simple summand.

Proof. — We give only some remarks. We can easily observe that the third condition follows from the second condition, by using Theorem 3.3. Note that the semisimplicity is preserved by birational morphism. As mentioned in Subsection 2.3.2, the semisimplicity of $(\mathcal{E}, \nabla)$ on (X, D) is equivalent to the polystability of $(\boldsymbol{E}^{DM}_*, \nabla)$, if X is projective. Hence, we obtain the first condition from the third one, by using Chow's lemma and the result in Section 17.6.1 of [14].

We give a complementary remark on an argument to show that the first condition implies the second condition. Assume that $(\mathcal{E}, \nabla)$ is semisimple. By using Theorem 19.2.3 and Proposition 22.5.1 of [14], we can take a closed analytic subset $Z' \subset D$ and a pluri-harmonic metric h of $(\mathcal{E}, \nabla)$ such that (i) (E, ∇, h) is $\sqrt{-1}\boldsymbol{R}$-good wild on $(X-Z', D-Z')$, (ii) $\mathcal{P}E_{|X-Z'} = \mathcal{E}_{|X-Z'}$. Let us show that we may have $Z = Z'$. We set $Z_0 := Z \cap Z'$.

For any point $P \in Z \setminus Z'$, (E, ∇, h) is $\sqrt{-1}\boldsymbol{R}$-good wild around P by our choice of Z'. For any point $P \in Z' \setminus Z$, we can conclude that (E, ∇, h) is $\sqrt{-1}\boldsymbol{R}$-good wild around P by using the results in Section 17.5.1 of [14]. Hence, $(\mathcal{P}E_{|X-Z_0}, \nabla)$ is a meromorphic flat bundle with the good Deligne-Malgrange lattice $\mathcal{P}_0E_{|X-Z_0}$. By using the reflexivity of the Deligne-Malgrange lattice, we may conclude that $\mathcal{P}_0E_{|X-Z_0}$ is the Deligne-Malgrange

lattice of $\mathcal{E}_{|X-Z_0}$, and $\mathcal{P}E_{|X-Z_0} = \mathcal{E}_{|X-Z_0}$. Thus, we obtain the second
condition from the first condition. $\square$

Remark 3.5. — Let X and D be as above. Let $(\mathcal{E}, \nabla)$ be a meromorphic
flat bundle on (X, D). If it is semisimple, we have the pluri-harmonic metric
h as in the above theorem, and hence the corresponding Higgs field θ. We
say that a point $P \in D$ is a turning point of θ, if (E, ∇, h) does not have
a decomposition as in (3.1) around P. By the argument in the proof of the
above theorem, we can observe that the turning points of $(\mathcal{E}, \nabla)$ coincide
with the turning points for θ.

BIBLIOGRAPHY

[1] Y. ANDRÉ, "Structure des connexions méromorphes formelles de plusieurs variables et semi-continuité de l'irrégularité", *Invent. Math.* **170** (2007), p. 147-198.

[2] J. BINGENER, "Über Formale Komplexe Räume", *Manuscripta Math.* **24** (1978), p. 253-293.

[3] O. BIQUARD & P. BOALCH, *Wild non-abelian Hodge theory on curves*, vol. 140, Compos. Math., 2004.

[4] K. CORLETTE, "Flat G-bundles with canonical metrics", *J. Differential Geom.* **28** (1988), p. 361-382.

[5] P. DELIGNE, "Équation différentielles à points singuliers réguliers", in *Lectures Notes in Maths.*, vol. 163, Springer, 1970.

[6] J. JOST & K. ZUO, "Harmonic maps of infinite energy and rigidity results for representations of fundamental groups of quasiprojective varieties", *J. Differential Geom.* **47** (1997), p. 469-503.

[7] N. KATZ, "Nilpotent connections and the monodromy theorem; applications of a result of Turrittin", *Publ. Math. I.H.E.S.* **39** (1970), p. 175-232.

[8] K. KEDLAYA, "Good formal structures for flat meromorphic connections, I; Surfaces", math:0811.0190.

[9] Y. LASZLO & C. PAULY, *On the Hitchin morphism in positive characteristic*, Intenat. Math. Res. Notices, 2001, 129–143, math.AG/0005044.

[10] A. LEVELT, "Jordan decomosition for a class of singular differential operators", *Ark. Math.* **13** (1975), p. 1-27.

[11] H. MAJIMA, "Asymptotic analysis for integrable connections with irregular singular points", in *Lecture Notes in Mathematics*, vol. 1075, Springer-Verlag, Berlin, 1984.

[12] B. MALGRANGE, "Connexions méromorphies 2, Le réseau canonique", *Invent. Math.* **124** (1996), p. 367-387.

[13] T. MOCHIZUKI, "Asymptotic behaviour of variation of pure polarized TERP structures", math:0811.1384, Version 1.

[14] ———, "Wild harmonic bundles and wild pure twistor D-modules", math:0803.1344, Version 3.

[15] ———, "Kobayashi-Hitchin correspondence for tame harmoinc bundles and an application", in *Astérisque*, vol. 309, Société Mathématique de France, 2006.

[16] ———, *Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules, I, II*, vol. 185, Mem. Amer. Math. Soc., 2007.

[17] ———, "Good formal structure for meromorphic flat connections on smooth projective surfaces", in *Algebraic Analysis and Around*, vol. 54, Advanced Studies in Pure Mathematics, 2009, math:0803.1346, p. 223-253.

[18] ———, "Kobayashi-Hitchin correspondence for tame harmonic bundles II", *Geometry & Topology* **13** (2009), p. 359-455, math.DG/0602266.

[19] C. SABBAH, "Harmonic metrics and connections with irregular singularities", *Ann. Inst. Fourier (Grenoble)* **49** (1999), p. 1265-1291.

[20] ———, "Équations différentielles à points singuliers irréguliers et phénomène de Stokes en dimension 2", in *Astérisque*, vol. 263, Société Mathématique de France, Paris, 2000.

[21] C. SIMPSON, "Constructing variations of Hodge structure using Yang-Mills theory and application to uniformization", *J. Amer. Math. Soc.* **1** (1988), p. 867-918.

[22] ———, "Harmonic bundles on non-compact curves", *J. Amer. Math. Soc.* **3** (1990), p. 713-770.

[23] W. WASOW, "Asymptotic expansions for ordinary equations", 1987, Reprint of 1976 edition. Dover Publications, Inc., New York.

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