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ON A GENERAL DIFFERENCE GALOIS THEORY I

by Shuji MORIKAWA

Abstract. — We know well difference Picard-Vessiot theory, Galois theory of linear difference equations. We propose a general Galois theory of difference equations that generalizes Picard-Vessiot theory. For every difference field extension of characteristic 0, we attach its Galois group, which is a group of coordinate transformation.

Résumé. — La théorie de Picard-Vessiot aux différences, la théorie de Galois des équations aux différences linéaires, est bien connue. Nous proposons une théorie de Galois des équations aux différences générales qui généralise la théorie de Picard-Vessiot. Pour toute extension de corps aux différences de caractéristique 0, nous attachons son groupe de Galois qui est un groupe de transformations de coordonnées.

Introduction

This is the first part of our general Galois theory, the second part being collaboration [8].

In the first part, we propose a general Galois theory of difference equations, which includes in particular Galois theory of q-difference equations and as a geometric counterpart, Galois theory of discrete dynamical systems on algebraic varieties. Our theory generalizes Picard-Vessiot theory, Galois theory of linear difference equations ([3], [9]). For a general Galois theory of differential equations, we have the differential Galois theories of Malgrange and Umemura ([7], [11], [10], [13], [12]). We attach to an arbitrary difference field extension L/k of characteristic 0, a formal group Inf-gal (L/k) of infinite dimension in general and of particular type called a Lie-Ritt functor (Definition 2.19).

Keywords: General difference Galois theory, dynamical system, integrable dynamical system, Galois groupoid.
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In section 1, we recall the definition of the Lie-Ritt functor, which is a group functor of coordinate transformations defined by a system of partial differential equations.

In section 2, we canonically construct a difference-differential ring, the Galois hull $\mathcal{L}/\mathcal{K}$ for a given difference field extension L/k (Definition 2.19). Our Galois group Inf-gal (L/k) is the infinitesimal automorphism group of the difference-differential ring extension $\mathcal{L}/\mathcal{K}$.

In section 3, we show that for a Picard-Vessiot extension L/k, our Galois group Inf-gal (L/k) is isomorphic to the Lie algebra Lie Gal (L/k) of the Galois group Gal (L/k) in Picard-Vessiot theory (Theorem 3.3).

In section 4, we give a conjecture that the canonical morphism Lie (Inf-gal (L/k)) $\to$ Lie (Inf-gal (M/k)) is surjective for a tower of difference field extensions $L/M/k$ (Conjecture 4.2). We believe that this conjecture is true.

In section 5, we give a few examples. The reader finds extensive examples in the second part [8].

We explain in the second part [8], how our Galois group Inf-gal (L/k) is related with Malgrange's Galois D-groupoid ([1], [2], [4], [7]). We expect that we can apply our method to establish a general Galois theory for differential equations in [5] and [6].

I would like to express my thanks to my advisor Prof. H. Umemura for giving me the opportunity to carry out these studies. My thanks should also go to Dr. F. Heiderich who carefully read the first version of the paper, pointed out an error in the proof of Theorem 2.15 and improved English.

All the rings except for Lie algebras are assumed to be commutative, to have a unit element and to contain the rational number field $\mathbb{Q}$. We denote by $\mathbb{N}$ the set of natural numbers and 0. Namely, $\mathbb{N} := \{0, 1, 2, \ldots\}$.

1. Lie-Ritt functor

Let us recall the framework in sections 1, 2 of [10]. Let A be a ring , $N(A)$ be the ideal that consists of all nilpotent elements of A. For indeterminates $x_1, x_2, \ldots, x_n$, $A[[x_1, x_2, \ldots, x_n]]$ is the ring of formal power series with coefficients in A. We define $\Gamma_n(A)$ by setting

$$\Gamma_n(A) = \{\Phi = (\phi_1, \phi_2, \ldots, \phi_n) | \phi_i \in A[[x_1, x_2, \ldots, x_n]] \text{ for all } 1 \leqslant i \leqslant n, \\ \Phi \equiv (x_1, x_2, \ldots, x_n) \bmod N(A)[[x_1, x_2, \ldots, x_n]]^n\}.$$

Lemma 1.1. — *Let* $\Phi = (\phi_1, \phi_2, \ldots, \phi_n)$, $\Psi = (\psi_1, \psi_2, \ldots, \psi_n)$ *be elements of* $\Gamma_n(A)$. *We define the multiplication of* Φ *and* Ψ *as the composite*

$$\Phi \circ \Psi = (\psi_1(\phi_1, \phi_2, \ldots, \phi_n), \ldots, \psi_n(\phi_1, \phi_2, \ldots, \phi_n)).$$

Then $\Gamma_n(A)$ *is a group.*

Proof. — See Proposition 1.6 of [10]. □

Let R be a $\mathbb{Q}$-algebra. The category $(\mathrm{Alg}\,/R)$ is the category of R-algebras and the category (Group) is the category of groups.

Definition 1.2. — *The group functor*

$$\Gamma_{nR} : (\mathrm{Alg}\,/R) \to (\mathrm{Group}), \quad A \mapsto \Gamma_n(A)$$

is called the Lie-Ritt functor of all the infinitesimal coordinate transformations of n*-variables defined over* R*.*

$R[[x]] := R[[x_1, x_2, \ldots, x_n]]$ is a differential ring with derivations $\{\partial/\partial x_i\}_{1 \leqslant i \leqslant n}$.

Let $\{\phi_1, \phi_2, \ldots, \phi_n\}$ be independent variables or symbols. For $J = (j_1, j_2, \ldots, j_n) \in \mathbb{N}^n$ and $1 \leqslant l \leqslant n$, we introduce the symbols

$$\frac{\partial^{|J|}\phi_l}{\partial^J x} := \frac{\partial^{j_1+j_2+\cdots+j_n}\phi_l}{\partial^{j_1}x_1\partial^{j_2}x_2\cdots\partial^{j_n}x_n}.$$

So if we define

$$\frac{\partial}{\partial x_i}\left(\frac{\partial^{|J|}\phi_l}{\partial^J x}\right) := \frac{\partial^{|J+1_i|}\phi_l}{\partial^{J+1_i}x},$$

where

$$1_i = (0, \ldots, \overset{i}{\check{1}}, \ldots, 0),$$

then we get a partial differential ring $R[[x]][\partial^{|J|}\phi_l/\partial^J x]_{1 \leqslant l \leqslant n,\ J \in \mathbb{N}^n}$ with derivations $\{\partial/\partial x_i\}_{1 \leqslant i \leqslant n}$. For a differential ideal I of

$$R[[x]][\partial^{|J|}\phi_l/\partial^J x]_{1 \leqslant l \leqslant n,\ J \in \mathbb{N}^n},$$

let I_A be the differential ideal of

$$A[[x]][\partial^{|J|}\phi_l/\partial^J x]_{1 \leqslant l \leqslant n,\ J \in \mathbb{N}^n}$$

generated by I.

Definition 1.3. — *A subgroup functor* G *of* Γ_{nR} *is called a Lie -Ritt functor if there exists a differential ideal* I *of* $R[[x]][\partial^{|J|}\phi_l/\partial^J x]_{1 \leqslant l \leqslant n,\ J \in \mathbb{N}^n}$ *such that*

$$G(A) = \{\Phi \in \Gamma_{nR}(A) \mid F(\Phi) = 0 \text{ for all } F \in I_A\}$$

for every R*-algebra* A*.*

Let k be a field of characteristic 0. Let $G : (\mathrm{Alg}/k) \to (\mathrm{Group})$ be a Lie-Ritt functor from the category (Alg/k) of k-algebras to the category (Group) of groups. For a k-vector space V, we define an algebra structure in the k-vector space $k \oplus V$ by defining the multiplication

$$(a, u)(b, v) = (ab, av + bu) \quad \text{for } (a, u),\ (b, v) \in k \oplus V.$$

We denote the ring $k \oplus V$ by $D(V)$.

The natural projection $p : D(V) \to k,\ (a, u) \mapsto a$ induces a group morphism

$$G(p) : G(D(V)) \to G(k),$$

so that we get a functor

$$L_G : (\mathrm{Vect}/k) \to (\mathrm{Set}), \quad V \mapsto L_G(V) = \ker(G(p))$$

from the category (Vect/k) of k-vector spaces to the category (Set) of sets.

Definition 1.4. — *Let $A = k[\varepsilon]/(\varepsilon^2)$ and $p_A : A \to k,\ a + b\varepsilon \mapsto a$ be the natural projection. Then for a Lie-Ritt functor G, we denote $\ker(G(p_A))$ by $\mathrm{Lie}\, G := L_G(A)$.*

Lemma 1.5. — *If the functor $L_G : (\mathrm{Vect}/k) \to (\mathrm{Set})$ transforms products to products, namely $L_G(V_1 \times V_2) \cong L_G(V_1) \times L_G(V_2)$ for arbitrary two k-vector spaces V_1, V_2, then $\mathrm{Lie}\, G$ is a k-Lie algebra.*

Proof. — See Lemma 2.3 and Proposition 2.14 of [10]. □

2. Definition of infinitesimal Galois group of difference field extensions

Definition 2.1. — *A difference ring (R, σ) is a ring R equipped with a morphism $\sigma : R \to R$.*

Remark 2.2. — In the Picard-Vessiot theory [9], a difference ring is a ring equipped with an automorphism. We have to treat a more general case so that we can apply our theory to concrete equations.

For a ring A, we consider the set

$$F(\mathbb{N}, A) := \{u : \mathbb{N} \to A\}$$

of functions on $\mathbb{N}$ with values in the ring A. Since A is the ring, $F(\mathbb{N}, A)$ has a natural ring structure. In fact, for $u, v \in F(\mathbb{N}, A)$ and $n \in \mathbb{N}$, we define the ring structure by

$$(u+v)(n) := u(n)+v(n),$$
$$(u\cdot v)(n) := u(n)\cdot v(n).$$

Moreover, $F(\mathbb{N}, A)$ is a difference ring with shift operator $\Sigma : F(\mathbb{N}, A) \to F(\mathbb{N}, A)$. Namely, for a function $u \in F(\mathbb{N}, A)$, we define

$$\Sigma(u)(n) = u(n+1) \quad \text{for} \quad n \in \mathbb{N}.$$

Thus we get a difference ring $(F(\mathbb{N}, A), \Sigma)$.

Definition 2.3. — *Let (R, σ) be a difference ring and A a ring. An Euler morphism is a morphism*

$$(R, \sigma) \to (F(\mathbb{N}, A), \Sigma)$$

of difference algebras.

Proposition 2.4. — *Let $j : A \to B$ be a ring morphism. Then*

$$\tilde{j} : F(\mathbb{N}, A) \to F(\mathbb{N}, B), \quad u \mapsto (n \mapsto j(u(n)))$$

is a difference morphism. Namely

$$(\mathrm{Alg}) \to (\text{Difference Alg}), \quad A \mapsto F(\mathbb{N}, A)$$

is a functor from the category (Alg) *of algebras to the category (*Difference Alg) of difference algebras.

Proof. — This is an evident consequence of the definition. □

We often denote the difference ring (R, σ) simply by R if there is no danger of confusion of the choice of the operator σ. In this case, the abstract ring R of (R, σ) will be denoted by $R^\natural$.

For an element a of a difference ring (R, σ), we can consider a function

$$u_a : \mathbb{N} \to R^\natural, \quad n \mapsto \sigma^n(a)$$

on $\mathbb{N}$ so that $u_a \in F(\mathbb{N}, R^\natural)$.

Proposition 2.5. — *The map*

$$\iota_R : (R, \sigma) \to (F(\mathbb{N}, R^\natural), \Sigma), \quad a \mapsto u_a$$

is an injective Euler morphism.

Proof. — The map ι_R is a difference morphism because σ is a difference morphism. Let $\iota_R(a) = \iota_R(b)$. Then $a = \iota_R(a)(0) = \iota_R(b)(0) = b$. Therefore ι_R is injective. □

By Proposition 2.5, we have an isomorphism $(R, \sigma) \cong (\iota_R(R), \Sigma)$. So we may identify the difference operator σ with Σ.

Definition 2.6. — *We call the Euler morphism* $\iota_R : (R,\sigma) \to (F(\mathbb{N}, R^\natural), \Sigma)$ *the universal Euler morphism. See Proposition 2.7.*

We show in the next proposition that in fact the Euler morphism ι_R is universal among the Euler morphisms.

Proposition 2.7. — *For a difference ring* R *and a ring* S,

$$\mathrm{Hom}_\sigma(R, F(\mathbb{N}, S)) \cong \mathrm{Hom}\,(R^\natural, S).$$

Proof. — Let $j \in \mathrm{Hom}(R^\natural, S)$. Then by the method of Proposition 2.4, we have a difference morphism $\tilde{j} : F(\mathbb{N}, R^\natural) \to F(\mathbb{N}, S)$. Since $\iota_R : R \to F(\mathbb{N}, R^\natural)$ is a difference morphism, $\tilde{j} \circ \iota_R$ is a difference morphism from R to $F(\mathbb{N}, S)$. So we have a map

$$\alpha : \mathrm{Hom}\,(R^\natural, S) \to \mathrm{Hom}_\sigma(R, F(\mathbb{N}, S)), \quad j \mapsto \tilde{j} \circ \iota_R.$$

Conversely, the map

$$f_0 : F(\mathbb{N}, S) \to S, \quad u \mapsto u(0)$$

is a ring morphism. So for $\psi \in \mathrm{Hom}_\sigma(R, F(\mathbb{N}, S))$, we have a ring morphism $f_0 \circ \psi \in \mathrm{Hom}\,(R^\natural, S)$. So we have

$$\beta : \mathrm{Hom}_\sigma(R, F(\mathbb{N}, S)) \to \mathrm{Hom}\,(R^\natural, S), \quad \psi \mapsto f_0 \circ \psi.$$

These maps α, β are an inverse of each other. □

Condition 2.8. — *Let* (k,σ) *be a difference field of characteristic 0 and* (L,σ) *be a difference extension field over* (k,σ)*. The abstract field* L *is finitely generated over* k *as an abstract field and* L *is algebraic over* $\sigma(L)$*.*

A Picard-Vessiot extension is a Galois extension for linear difference equations [9]. We need a more general definition of a difference Galois extension. In fact, we replace Galois extensions by the Galois hull (Definition 2.10).

Let

$$\mathrm{Der}\,(L^\natural/k^\natural) := \{\partial : L^\natural \to L^\natural \mid \partial \text{ is a derivation with } \partial(k^\natural) = 0\}.$$

So $\mathrm{Der}\,(L^\natural/k^\natural)$ is a vector space over $L^\natural$ and the dimension of the $L^\natural$-vector space $\mathrm{Der}\,(L^\natural/k^\natural)$ coincides with the transcendence degree L/k. Namely,

$$\dim_{L^\natural} \mathrm{Der}\,(L^\natural/k^\natural) = \mathrm{tr.d.}\,[L^\natural : k^\natural].$$

Let d be $\dim_{L^\natural} \mathrm{Der}\,(L^\natural/k^\natural)$ and $D_1, D_2, \dots, D_d$ be a mutually commutative basis of the $L^\natural$-vector space $\mathrm{Der}\,(L^\natural/k^\natural)$ so that $D_iD_j = D_jD_i$ for $1 \leqslant i, j \leqslant d$. Thus we have a partial differential field

$$L^\sharp := (L^\natural, \{D_1, D_2, \dots, D_d\}).$$

For a function $u \in F(\mathbb{N}, L^\sharp)$ and $1 \leqslant i \leqslant d$, we define $D_i u \in F(\mathbb{N}, L^\sharp)$ by setting

$$(D_i u)(n) := D_i(u(n)) \quad \text{for} \quad n \in \mathbb{N}.$$

Thus $F(\mathbb{N}, L^\sharp)$ has the difference operator Σ and the differential operators $D_1, D_2, \ldots, D_d$. We denote the difference-differential ring $F(\mathbb{N}, L^\natural)$ by $F(\mathbb{N}, L^\sharp)$.

Proposition 2.9. — *The operators $\Sigma, D_1, D_2, \ldots, D_d$ of the ring $F(\mathbb{N}, L^\sharp)$ commute mutually.*

Proof. — We only have to prove the commutatively of Σ and D_i for $1 \leqslant i \leqslant d$, because the differential operators $D_1, D_2, \ldots, D_d$ commute mutually by the assumption.

In fact, for $u \in F(\mathbb{N}, L^\sharp)$ and $n \in \mathbb{N}$,

$$D_i(\Sigma(u))(n) = D_i(\sigma(u(n))) = D_i(u(n+1)) = (D_i u)(n+1) = \Sigma(D_i u)(n).$$

So the proposition is proved. □

Definition 2.10. — *Let L/k be a difference field extension satisfying condition 2.8. We denote by $\mathcal{L}$ the $\{\Sigma, D_1, D_2, \ldots, D_d\}$-invariant subring of $F(\mathbb{N}, L^\sharp)$ generated by the image $\iota_L(L)$ of L by the universal Euler morphism ι_L and by the ring of constant functions $L^\sharp$ of the difference ring $F(\mathbb{N}, L^\sharp)$. We also denote by $\mathcal{K}$ the $\{\Sigma, D_1, D_2, \ldots, D_d\}$-invariant subring of $F(\mathbb{N}, L^\sharp)$ generated by the image $\iota_L(k)$ and $L^\sharp$. So $\mathcal{L}/\mathcal{K}$ is a difference-differential ring extension with difference operator Σ and derivations $D_1, D_2, \ldots, D_d$. We call the extension $\mathcal{L}/\mathcal{K}$ the Galois hull of L/k.*

Remark 2.11. — Since the image $\iota_L(k)$ of the universal Euler morphism and the constant field $L^\sharp$ of the difference ring $F(\mathbb{N}, L^\sharp)$ are closed by $D_1, D_2, \ldots, D_d$ and Σ, the difference-differential ring $\mathcal{K}$ is generated by $\iota_L(k)$ and $L^\sharp$ as a ring.

Definition 2.12. — *Let $(R, \{D_1, D_2, \ldots, D_d\})$ be a partial differential ring. The universal Taylor morphism is the map*

$$\begin{aligned} \tau : R &\to R[[W_1, W_2, \ldots, W_d]]. \\ a &\mapsto \sum_{(m_1, m_2, \ldots, m_d) \in \mathbb{N}^d} \frac{1}{m_1! m_2! \cdots m_d!} D_1^{m_1} D_2^{m_2} \cdots D_d^{m_d}(a) W_1^{m_1} W_2^{m_2} \cdots W_d^{m_d} \end{aligned}$$

from the partial differential ring R to the formal power series ring $R[[W_1, W_2, \ldots, W_d]]$. The universal Taylor morphism is a partial differential ring morphism compatible with the derivations D_i and $\partial/\partial W_i$ for $1 \leqslant i \leqslant d$.

Since $(L^\natural, \{D_1, D_2, \ldots, D_d\})$ is a partial differential field, we have the universal Taylor morphism

(2.1)
$$\begin{aligned} L^\sharp \to {} & L^\natural[[W]] = L^\natural[[W_1, W_2, \ldots, W_d]] \\ a \mapsto {} & \sum_{m=(m_1,m_2,\ldots,m_d)\in\mathbb{N}^d} \frac{1}{m_1!m_2!\cdots m_d!} D_1^{m_1} D_2^{m_2}\cdots D_d^{m_d}(a) W_1^{m_1} W_2^{m_2}\cdots W_d^{m_d}. \end{aligned}$$

By the universal Taylor morphism (2.1) and Proposition 2.4, we have the following difference-differential morphism

$$\mathcal{L} \hookrightarrow F(\mathbb{N}, L^\sharp) \to F(\mathbb{N}, L^\natural[[W]]).$$

Let A be an $L^\natural$-algebra, so that we have the structural morphism $j : L^\natural \to A$. Then we have a difference-differential morphism

$$\mathcal{L} \to F(\mathbb{N}, L^\natural[[W]]) \to F(\mathbb{N}, A[[W]]).$$

This morphism is denoted by $\mathrm{id}_A : \mathcal{L} \to F(\mathbb{N}, A[[W]])$. Now let $N(A)$ be the ideal of all the nilpotent elements of the $L^\natural$-algebra A.

Definition 2.13. — *Let L/k be the difference field extension satisfying Condition 2.8. For an $L^\natural$-algebra A, we set*

$$\begin{aligned} \mathcal{F}(L/k)(A) := \{ f \in \mathrm{Hom}_{\Sigma, D_1, D_2, \ldots, D_d}(\mathcal{L}, F(\mathbb{N}, A[[W]])) | f|_{\mathcal{K}} = \mathrm{id_A}|_{\mathcal{K}}, \\ f(a) \equiv \mathrm{id_A(a)} \mod \mathrm{F(\mathbb{N}, N(A)[[W]])} \textit{ for all } \mathrm{a} \in \mathcal{L} \}, \end{aligned}$$

so that $\mathcal{F}(L/k) : (\mathrm{Alg}\, /L^\natural) \to (\mathrm{Set})$ is a functor from the category $(\mathrm{Alg}\, /L^\natural)$ of $L^\natural$-algebras to the category (Set) of sets. Here $f|_{\mathcal{K}}$ means the restriction of the morphism f to $\mathcal{K}$.

We show that the functor $\mathcal{F}(L/k)$ is independent of the choice of a basis of the $L^\natural$-vector space $\mathrm{Der}\,(L/k)$.

Let $D_1, D_2, \ldots, D_d$ and $D'_1, D'_2, \ldots, D'_d$ be two bases of the $L^\natural$-vector space $\mathrm{Der}\,(L^\natural/k^\natural)$. Let $\mathcal{F}(L/k)$, $\mathcal{F}'(L/k)$ be the functors defined by using each bases.

Lemma 2.14. — *The functor $\mathcal{F}(L/k)$ is isomorphic to $\mathcal{F}'(L/k)$.*

Proof. — Let $\{x_1, x_2, \ldots, x_d\}$,$\{y_1, y_2, \ldots, y_d\}$ be two transcendence bases of $L^\natural$ over $k^\natural$. Then $\{\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_d\}, \{\partial/\partial y_1, \partial/\partial y_2, \ldots, \partial/\partial y_d\}$ are commutative bases of the $L^\natural$-vector space $\mathrm{Der}\,(L^\natural/k^\natural)$.

Here we have

$$\partial/\partial y_i = \sum_{j=1}^{d} a_{i,j} \partial/\partial x_j$$

for some $a_{i,j} \in L^{\natural}$ for $1 \leqslant i,j \leqslant d$ with determinant $|a_{i,j}| \neq 0$.

Let τ_1, τ_2 be the universal Taylor morphisms with respect to $\{\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_d\}, \{\partial/\partial y_1, \partial/\partial y_2, \dots, \partial/\partial y_d\}$. Namely,

$$\begin{array}{rcl} \tau_1 : \left(L^{\natural}, \left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_d}\right\}\right) & \rightarrow & L^{\natural}[[W_1, W_2, \dots, W_d]] = L^{\natural}[[W]] \\ \tau_2 : \left(L^{\natural}, \left\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \dots, \frac{\partial}{\partial y_d}\right\}\right) & \rightarrow & L^{\natural}[[Z_1, Z_2, \dots, Z_d]] = L^{\natural}[[Z]]. \end{array}$$

We define $L^{\natural}$-morphisms

$$\alpha : L^{\natural}[[W]] \to L^{\natural}[[Z]], \quad \beta : L^{\natural}[[Z]] \to L^{\natural}[[W]]$$

by

$$\alpha(W_i) = \tau_2(x_i) - x_i, \quad \beta(Z_i) = \tau_1(y_i) - y_i.$$

Then we have

$$\alpha \circ \beta = \mathrm{Id}_{L^{\natural}[[Z]]}, \quad \beta \circ \alpha = \mathrm{Id}_{L^{\natural}[[W]]},$$

where Id is the identity map.

Define

$$\alpha\left(\frac{\partial}{\partial W_i}\right) = \sum_{j=1}^{d} \tau_2(a_{ij}) \frac{\partial}{\partial Z_j},$$

then α is a differential $L^{\natural}$-isomorphism. So the differential morphism α induces a difference-differential $L^{\natural}$-isomorphism

$$\Psi_\alpha : F(\mathbb{N}, L^{\natural}[[W]]) \to F(\mathbb{N}, L^{\natural}[[Z]])$$

and a natural difference-differential A-isomorphism

$$\widetilde{\Psi}_\alpha : F(\mathbb{N}, A[[W]]) \to F(\mathbb{N}, A[[Z]]).$$

We must prove that

$$\mathcal{F}(L/k)(A) \to \mathcal{F}'(L/k)(A), \quad f \mapsto \widetilde{\Psi}_\alpha \circ f$$

is an isomorphism for any $L^{\natural}$-algebra A.

Since $\widetilde{\Psi}_\alpha$ is A-isomorphism, we have $\widetilde{\Psi}_\alpha \circ \mathrm{id}_1 = \mathrm{id}_2$. Here $\mathrm{id}_1 = \mathrm{id}_A \in \mathcal{F}(L/k)(A)$ and $\mathrm{id}_2 = \mathrm{id}_A \in \mathcal{F}'(L/k)(A)$. Therefore the above map is well-defined and $\mathcal{F}(L/k) \cong \mathcal{F}'(L/k)$. □

We denote a function $u \in F(\mathbb{N}, A)$ by $u(\underline{n})$ where $\underline{n}$ is a variable on $\mathbb{N}$ in order to distinguish it from its particular value $u(l)$ at a number $l \in \mathbb{N}$. The value $u(l)$ will also be denoted by $u|_{\underline{n}=l}$. The image $\iota_L(a) \in F(\mathbb{N}, L^{\sharp})$ of an element $a \in L$ by the universal Euler morphism ι_L is a function on $\mathbb{N}$ that we denote by $\iota_L(a\,;\underline{n})$. For an $L^{\natural}$-algebra A and $\mathrm{id}_A \in \mathcal{F}(L/k)(A)$, the image $\mathrm{id}_A(\iota_L(a\,;\underline{n})) \in F(\mathbb{N}, A[[W]])$ is a function on $\mathbb{N}$ with values in the $A[[W]]$ that we denote by $\iota_L(a\,;\underline{n}, W)$. Namely, for $a \in L$,

$$\begin{aligned} \iota_L(a\,;\underline{n}) &:= \iota_L(a), \\ \iota_L(a\,;\underline{n}, W) &:= \mathrm{id}_A(\iota_L(a\,;\underline{n})). \end{aligned} \tag{2.2}$$

Lemma 2.15. — *For an $L^\natural$-algebra A and $f \in \mathcal{F}(L/k)(A)$, there exists a coordinate transformation*

$$W = (W_1, W_2, \dots, W_d) \mapsto \Phi(W) = (\phi_1(W), \phi_2(W), \dots, \phi_d(W))$$

satisfying the following conditions.

(i) $\phi_i(W) \in A[[W]]$ *for all* $1 \leqslant i \leqslant d$,
(ii) $\phi_i(W) \equiv W_i \mod N(A)[[W]]$ *for all* $1 \leqslant i \leqslant d$,
(iii) $f(\iota_L(a\,;\underline{n})) = \iota_L(a\,;\underline{n}, \Phi(W))$ *for all* $a \in L$.

To prove Lemma 2.15, we will use the next lemmas.

Lemma 2.16. — *Let L/k be an algebraic extension of a field of characteristic 0. Let M be a commutative ring over k. If k-morphisms $p, q : L \to M$ satisfy $(p-q)^2 = 0$, then p coincides with q over L.*

Proof of Lemma 2.16. — Let $\theta \in L$ be algebraic over k and

$$P(X) = X^m + a_{m-1}X^{m-1} + \cdots + a_1 X + a_0 \quad (a_i \in k)$$

be the minimal polynomial over k of θ. Since $(p-q)^2 = 0$, $(p(\theta)-q(\theta))^2 = 0$ *i.e.*, there exists a nilpotent element n of A such that $n^2 = 0$ and $p(\theta) - q(\theta) = n$.

$$\begin{aligned} 0 &= p(P(\theta)) \\ &= p(\theta)^m + p(a_{m-1})p(\theta)^{m-1} + \cdots + p(a_0) \\ &= (q(\theta)+n)^m + q(a_{m-1})(g(\theta)+n)^{m-1} + \cdots + q(a_0) \\ &= q(P(\theta)) + nq\left(\frac{\partial P}{\partial X}(\theta)\right) \\ &= nq\left(\frac{\partial P}{\partial X}(\theta)\right). \end{aligned}$$

Since $\partial P/\partial X(\theta)^\natural e0$, we have $(\partial P/\partial X(\theta))^{-1} \in L$. So we have $n = 0$, $p = q$ over L. $\square$

Lemma 2.17. — *Let L/k be an algebraic extension of a field of characteristic 0. Let S be a Noether ring over k and $N(S)$ be the ideal of all the nilpotent elements of S. For k-morphisms $p, q : L \to S$, if $p(a) \equiv q(a)$* mod $N(S)$ *for all $a \in L$, then p coincides with q over L.*

Proof of Lemma 2.17. — First, we prove

$$p(a) \equiv q(a) \mod N(S)^s \tag{2.3}$$

for all $s \in \mathbb{N}$ and $a \in L$ by induction on s.

By assumption, equation (2.3) holds for $s = 1$.

We assume that $p(a) \equiv q(a) \mod N(S)^s$ for all $a \in L$.

Let π_{s+1} be the natural projection $\pi_{s+1} : S \to S/N(S)^{s+1}$, and $p_{s+1} = \pi_{s+1} \circ p$, $q_{s+1} = \pi_{s+1} \circ q$. Since $p(a) - q(a) \in N(S)^s$ for $a \in L$, we have $(p_{s+1} - q_{s+1})^2 = 0$. Therefore by Lemma 2.16, we get $p_{s+1} = q_{s+1}$. So we have

$$p(a) \equiv q(a) \mod N(S)^{s+1}$$

for $a \in L$. By induction, we have equation (2.3). Therefore we have

$$p(a) - q(a) \in \bigcap_{s \in \mathbb{N}} N(S)^s$$

for $a \in L$. Since S is a Noether ring, we have $N(S)^n = \{0\}$ for sufficiently big integer $n \in \mathbb{N}$ and hence $\cap_{s \in \mathbb{N}} N(S)^s = \{0\}$. So we get $p = q$. □

Proof of Lemma 2.15. — Let $x_1, x_2, \ldots, x_d$ be a transcendence basis of L over k. Then $D_1 = \partial/\partial x_1, D_2 = \partial/\partial x_2, \ldots, D_d = \partial/\partial x_d$ form a basis of the $L^\natural$-vector space $\mathrm{Der}\,(L^\natural/k^\natural)$. We prove the lemma using this basis in the definition of $\mathcal{F}(L/k)$.

Since $\iota_L(x_i\,; 0) = x_i$, we have

$$D_j(\iota_L(x_i\,; 0)) = \delta_{i,j} \quad \text{for} \quad 1 \leqslant i, j \leqslant d.$$

So we have

$$\iota_L(x_i\,; 0, W) = x_i + W_i.$$

So for $f \in \mathcal{F}(L/k)(A)$, there exists $\phi_i(W) \in A[[W]]$ such that

$$f(\iota_L(x_i\,; \underline{n}))|_{\underline{n}=0} = x_i + \phi_i(W).$$

Then $\Phi(W) := (\phi_1(W), \phi_2(W), \ldots, \phi_d(W))$ satisfy the first and second conditions. So we have to prove the third condition.

First, we will prove $f(\iota_L(a\,; \underline{n}))|_{\underline{n}=0} = \iota_L(a\,; 0, \Phi(W))$ for all $a \in L$.

For an intermediate field $M := k(x_1, x_2, \ldots, x_d)$ of L/k, we know

$$f(\iota_L(x_i\,; \underline{n}))|_{\underline{n}=0} = \iota_L(x_i\,; 0, \Phi(W)).$$

Let F and G be morphisms

$$F : L \to A[[W]], \quad a \mapsto f(\iota_L(a\,; \underline{n}))|_{\underline{n}=0}$$

and

$$G : L \to A[[W]], \quad a \mapsto \iota_L(a\,; 0, \Phi(W)).$$

Then we have

$$F|_M = G|_M$$

and

$$F(a) \equiv G(a) \mod N(A)[[W]]$$

for all $a \in L$.

We fix an integer $l \in \mathbb{N}$ and an element $a \in L$. Moreover we may identify M with $\mathrm{id}_A \circ \iota_L(M)|_{\underline{n}=0}$. We consider an M-algebra S defined by

$$S := M[F(a), G(a)][[W]]/(W)^l.$$

Here to be precise, $M[F(a),G(a)][[W]]$ should be understood as $(\mathrm{id}_A \circ \iota_L(M))$ $[F(a),G(a)][[W]] \subset A[[W]]$. Then S is a Noether ring over M. Moreover the morphisms F and G induce M-morphisms

$$F_{l,a} : M[a] \to S$$

and

$$G_{l,a} : M[a] \to S.$$

Since $a \in L$ is algebraic over M, by Lemma 2.17 we have

$$F_{l,a} = G_{l,a}. \tag{2.4}$$

Let π_l be the natural projection $\pi_l : A[[W]] \to A[[W]]/(W)^l$ and $F_l := \pi_l \circ F$, $G_l := \pi_l \circ G$. Then since $F_l(a) = G_l(a)$ by (2.4), we have

$$F_l = G_l$$

for all $l \in \mathbb{N}$. Since $\cap_{l\in\mathbb{N}}(W)^l = \{0\}$, we have

$$F = G.$$

Namely, we have

$$f(\iota_L(a\,;\underline{n}))|_{\underline{n}=0} = \iota_L(a\,;0,\Phi(W))$$

for all $a \in L$.

Next, for $l \in \mathbb{N}$ and $a \in L$,

$$\begin{aligned} f(\iota_L(a\,;\underline{n}))|_{\underline{n}=l} &= \Sigma^l(f(\iota_L(a\,;\underline{n})))|_{\underline{n}=0} \\ &= f(\iota_L(\sigma^l(a)\,;\underline{n}))|_{\underline{n}=0} \\ &= \iota_L(\sigma^l(a)\,;0,\Phi(W)) \\ &= \iota_L(a\,;\underline{n},\Phi(W))|_{\underline{n}=l}. \end{aligned}$$

Therefore we have $f(\iota_L(a\,;\underline{n})) = \iota_L(a\,;\underline{n},\Phi(W))$ for all $a \in L$. □

By Lemma 2.15, for $f \in \mathcal{F}(L/k)(A)$, there exists $\mu(W) \in (N(A)[[W]])^d$ such that

$$f(\iota_L(a\,;\underline{n})) = \iota_L(a\,;\underline{n}, W + \mu(W))$$

for all $a \in L$. By Taylor expansion,

$$\begin{aligned} f(\iota_L(a\,;\underline{n})) &= \iota_L(a\,;\underline{n}, W + \mu(W)) \\ &= \sum \frac{1}{m!}\left(\frac{\partial^{|m|}}{\partial W^m}\iota_L(a\,;\underline{n}, W)\right)(\mu(W))^m \\ &= \sum \frac{1}{m!}\,\mathrm{id}_A(D^{|m|}(\iota_L(a\,;\underline{n})))(\mu(W))^m. \end{aligned}$$

Therefore we regard $f(\iota_L(a\,;\underline{n}))$ as an element of $\mathrm{id}_A(\mathcal{L})\hat{\otimes}_{L^\sharp}A[[W]]$ for $a \in L$ and $f \in \mathcal{F}(L/k)(A)$. Here $\hat{\otimes}$ is the completion of the ring $\mathrm{id}_A(\mathcal{L}) \otimes_{L^\sharp} A[[W]]$ with respect to the W-adic topology. We may identify $\mathrm{id}_A(\mathcal{L})$ with $\mathcal{L}$.

Therefore we get

$$f(\mathcal{L}) \subset \mathcal{L}\hat{\otimes}_{L^\sharp}A[[W]] \tag{2.5}$$

for all $f \in \mathcal{F}(L/k)(A)$.

Remark 2.18. — $\mathcal{L}$ and $A[[W]]$ are linearly disjoint over $L^\sharp$ in $F(\mathbb{N}, A[[W]])$.

Proof. — Let us take elements $a_1, a_2, \ldots, a_t \in A[[W]]$ that are linear independent over $L^\sharp$. Assume that we have $\sum_{i=1}^t c_i a_i = 0$ with $u_i \in \mathcal{L}$. Then $u_i \in \mathcal{L}$ is a function from $\mathbb{N}$ to $L^\sharp$, so we have $u_i(n) \in L^\sharp$ for $n \in \mathbb{N}$. Since

$$\sum_{i=1}^t u_i(n)a_i = 0$$

for all $n \in \mathbb{N}$ and $a_1, a_2, \ldots a_t$ are linear independent over $L^\sharp$, we get

$$u_i(n) = 0$$

for all $n \in \mathbb{N}$. Therefore we have $u_i = 0$ for $1 \leqslant i \leqslant t$. So $a_1, a_2, \ldots, a_t$ are linearly independent over $\mathcal{L}$. □

Now we can define the Galois group.

Definition 2.19. — *For an $L^\natural$-algebra A, we set*

$$\begin{aligned} \text{Inf-gal}\,(L/k)(A) := \{\varphi \in \mathrm{Aut}_{\Sigma, D_1, D_2, \ldots, D_d}(\mathcal{L}\hat{\otimes}_{L^\sharp}A[[W]]/\mathcal{K}\hat{\otimes}_{L^\sharp}A[[W]]) \mid \\ \varphi \equiv \mathrm{Id} \mod \mathcal{L}\hat{\otimes}_{L^\sharp}N(A)[[W]]\}. \end{aligned}$$

Here Id *is the identity map on $\mathcal{L}\hat{\otimes}_{L^\sharp}A[[W]]$. Then the infinitesimal Galois group*

$$\text{Inf-gal}\,(L/k) : (\mathrm{Alg}/L^\natural) \to (\mathrm{Group})$$

is a functor from the category $(\mathrm{Alg}/L^\natural)$ *of* $L^\natural$*-algebras to the category* (Group) *of groups.*

We will see that the infinitesimal Galois group $\operatorname{Inf-gal}(L/k)$ is a Lie-Ritt functor (Theorem 2.22).

Theorem 2.20. — *Let A be an $L^\natural$-algebra, then the infinitesimal Galois group* $\operatorname{Inf-gal}(L/k)\ (A)$ *operates on the set* $\mathcal{F}(L/k)(A)$*. Moreover the operation* $(\operatorname{Inf-gal}(L/k)(A), \mathcal{F}(L/k)(A))$ *is a principle homogeneous space. In other words,* $(\operatorname{Inf-gal}(L/k), \mathcal{F}(L/k))$ *is a principle homogeneous space.*

To prove this theorem we need the following lemma.

Lemma 2.21. — *For all $f \in \mathcal{F}(L/k)(A)$, we have in $F(\mathbb{N}, A[[W]])$*

$$\overline{L^\sharp[\mathcal{L}, A[[W]]]} = \overline{L^\sharp[f(\mathcal{L}), A[[W]]]}.$$

Here overlines express the closure with respect to the W-adic topology.

Proof. — We use the notation of (2.2). Then for all $f \in \mathcal{F}(L/k)(A)$, we have to prove

$$f(\iota_L(a\,;\underline{n})) \in \overline{L^\sharp[\mathcal{L}, A[[W]]]} \quad \text{and} \quad \mathrm{id_A}(\iota_L(a\,;\underline{n})) \in \overline{\mathrm{L^\sharp[f(\mathcal{L}), A[[W]]]}}.$$

There exists $\mu(W) \in N(A)(W)^d$ such that $f(\iota_L(a\,;\underline{n})) = \iota_L(a\,;\underline{n}, W + \mu(W))$ for all $a \in L$ by Lemma 2.15. By Taylor expansion,

$$f(\iota_L(a\,;\underline{n})) = \sum \frac{1}{m!}\,\mathrm{id}_A(D^{|m|}(\iota_L(a\,;\underline{n})))(\mu(W))^m \in \overline{L^\sharp[\mathcal{L}, A[[W]]]},$$

and

$$\mathrm{id}_A(\iota_L(a\,;\underline{n})) = \sum \frac{1}{m!} f(D^{|m|}(\iota_L(a\,;\underline{n})))(-\mu(W))^m \in \overline{L^\sharp[f(\mathcal{L}), A[[W]]]}.$$

Therefore we get

$$\overline{L^\sharp[\mathcal{L}, A[[W]]]} = \overline{L^\sharp[f(\mathcal{L}), A[[W]]]}.$$

□

Proof of Theorem 2.20. — For all $f \in \mathcal{F}(L/k)(A)$, we have $f(\mathcal{L}) \subset \mathcal{L}\hat{\otimes}_{L^\sharp} A[[W]]$ by (2.5). So by composing with $\varphi \in \operatorname{Inf-gal}(L/k)(A)$, we have $\varphi \circ f \in \mathcal{F}(L/k)(A)$. Thus $\operatorname{Inf-gal}(L/k)(A)$ operates on $\mathcal{F}(L/k)(A)$.

Next we prove that the operation $(\operatorname{Inf-gal}(L/k)(A), \mathcal{F}(L/k)(A))$ is a principle homogeneous space. So we have to prove that for all $f \in \mathcal{F}(L/k)(A)$ there exists exactly one $\varphi \in \operatorname{Inf-gal}(L/k)(A)$ such that $f = \varphi \circ \mathrm{id}_A$.

For $f \in \mathcal{F}(L/k)(A)$, we define an $A[[W]]$-morphism

$$\varphi_f\ :\ \overline{L^\sharp[\mathcal{L}, A[[W]]]} \to \overline{L^\sharp[\mathcal{L}, A[[W]]]}$$

as $\varphi_f(u) = f(u)$ for all $u \in \mathcal{L}$. Then the following diagram is commutative.

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{\mathrm{id}_A} & \mathcal{L}\hat{\otimes}_{L^\sharp}A[[W]] & \xrightarrow{\cong} & \overline{L^\sharp[\mathcal{L}, A[[W]]]} \\ \| & & & & \downarrow{\varphi_f} \\ \mathcal{L} & \xrightarrow{f} & \mathcal{L}\hat{\otimes}_{L^\sharp}A[[W]] & \xrightarrow{\cong} & \overline{L^\sharp[\mathcal{L}, A[[W]]]} \end{array} \tag{2.6}$$

The automorphism of the second arrow of (2.6) is the morphism

$$\mathcal{L}\hat{\otimes}_{L^\sharp}A[[W]] \to \overline{L^\sharp[\mathcal{L}, A[[W]]]}, \quad u \otimes a \mapsto ua.$$

Since both $L^\sharp$ and $A[[W]]$ are constants of Σ and f is a difference-differential morphism, φ_f is a difference-differential morphism. Therefore we get $\varphi_f \in \text{Inf-gal}\,(L/k)(A)$.

We assume $\varphi \circ \mathrm{id}_A = \psi \circ \mathrm{id}_A$ for $\varphi, \psi \in \text{Inf-gal}\,(L/k)(A)$. Then for all $u \otimes 1 \in \mathcal{L}\hat{\otimes}_{L^\sharp}A[[W]]$, we have

$$\varphi(u \otimes 1) = \varphi \circ \mathrm{id}_A(u) = \psi \circ \mathrm{id}_A(u) = \psi(u \otimes 1).$$

Since φ and ψ are $A[[W]]$-morphisms, we have $\varphi = \psi$. □

Theorem 2.22. — *The infinitesimal Galois group* $\text{Inf-gal}\,(L/k)$ *is a Lie-Ritt functor.*

Proof. — Let I be a differential ideal generated by constraints of $\mathcal{L}$ over $\mathcal{K}$.

Since $(\text{Inf-gal}\,(L/k), \mathcal{F}(L/k))$ is a principle homogeneous space by Theorem 2.20, we have

$$\text{Inf-gal}\,(L/k)(A) \cong \mathcal{F}(L/k)(A).$$

For the function $u \in \mathcal{L}$, $\mathrm{id}_A(u) \in F(\mathbb{N}, A[[W]])$ is the function on $\mathbb{N}$ with values in $A[[W]]$ that we denote by $u(\underline{n}, W)$. By Lemma 2.15, we get

$$\begin{aligned} \mathcal{F}(L/k)(A) \cong \{\Phi(W) \in A[[W]]^d \mid & \Phi(W) \equiv W \mod (N(A)[[W]])^d, \\ & u(\underline{n}, \Phi(W)) = u(\underline{n}, W) \quad \text{for all } u \in I\ \}. \end{aligned}$$

Thus we can regard the infinitesimal Galois group $\text{Inf-gal}\,(L/k)(A)$ as coordinate transforms defined by a differential ideal. Therefore $\text{Inf-gal}\,(L/k)$ is a Lie-Ritt functor. □

3. Equivalence with Picard-Vessiot theory

In this section, we assume that for every difference field (F, σ), $\sigma : F \to F$ is an automorphism, and for the base field (k, σ), the field C of constants of k is algebraically closed.

We consider

$$\sigma(y) = Ay \tag{3.1}$$

a linear difference system with $A \in GL_n(k)$ so $y = (y_{i,j})$ is an unknown $(n \times n)$-matrix. Let $X = (X_{i,j})$ be an $(n \times n)$-matrix of indeterminates over k. We define the extension of σ to the k-algebra $k[X_{i,j}, (\det X)^{-1}]$ as $\sigma(X) = AX$. If I is a maximal σ-invariant ideal of $k[X_{i,j}, (\det X)^{-1}]$, we call the difference ring $k[X_{i,j}, (\det X)^{-1}]/I$ a Picard-Vessiot ring. We know that the Picard-Vessiot ring is determined up to automorphism ([3], [9]). The total Picard-Vessiot ring L is the total ring of fractions of the Picard-Vessiot ring. Let $\mathrm{Gal}\,(L/k)$ be the group of automorphisms of the ring L over k that commute with the action of σ. The group $\mathrm{Gal}\,(L/k)$ is called the difference Galois group of equation (3.1) over the field k. The following theorem is well-known.

Theorem 3.1. — *The difference Galois group* $\mathrm{Gal}\,(L/k)$ *has a natural structure of linear algebraic group over* C.

See Theorem 1.13 of [9].

Theorem 3.2. — *Let L/k be the total Picard-Vessiot ring. Then there exist idempotents $e_0, e_1, \dots, e_{t-1} \in L$ with $e_i e_j = 0$ for $0 \leqslant i \neq j \leqslant t-1$ such that*

(i) *$L = L_0 \oplus L_1 \oplus \cdots \oplus L_{t-1}$ where $L_i = Le_i$,*
(ii) *$\sigma(e_i) = e_{i+1 \bmod t}$,*
(iii) *$(L_0, \sigma^t)/(k, \sigma^t)$ is a Picard-Vessiot field.*

Moreover there exists an exact sequence

$$0 \to \mathrm{Gal}\,(L_0/k) \to \mathrm{Gal}\,(L/k) \to \mathbb{Z}/t\mathbb{Z} \to 0. \tag{3.2}$$

See Corollary 1.16 and Corollary 1.17 of [9].

By exact sequence (3.2), we get an isomorphism

$$\mathrm{Lie}\,(\mathrm{Gal}\,(L_0/k)) \cong \mathrm{Lie}\,(\mathrm{Gal}\,(L/k))$$

of C-Lie algebras. As we are interested in the Lie algebra of the Galois group, we may assume that the Picard-Vessiot ring R is a domain so that the total Picard-Vessiot ring L is a field. So we consider a difference field extension $(L, \sigma)/(k, \sigma)$ such that

(1) $L = k(Z_{i,j})_{1 \leqslant i,\, j \leqslant n}$,
(2) $\sigma(Z) = AZ$, with $Z = (Z_{i,j}) \in \mathrm{GL}_n, (L)$,
(3) $C_L = C_K$.

We take a basis $D_1, D_2, \dots, D_d$ of the $L^\natural$-vector space $\mathrm{Der}(L^\natural/k^\natural)$. We can state equivalence between the difference Galois group $\mathrm{Gal}\,(L/k)$ and the infinitesimal Galois group $\text{Inf-gal}\,(L/k)$ in the following way.

THEOREM 3.3. — *Let $L = k(z_{i,j})/k$ be a Picard-Vessiot field of a linear difference equation $\sigma(y) = Ay$ so that $A = (a_{i,j}) \in GL_d(k)$ and $\sigma((z_{i,j})) = A(z_{i,j})$. Let $\mathrm{Gal}\,(L/k)$ be the Galois group of difference Picard-Vessiot theory. Then we have an isomorphism*

$$\mathrm{Lie}\,(\text{Inf-gal}\,(L/k)) \cong \mathrm{Lie}\,(\mathrm{Gal}\,(L/k)) \otimes_C L^\natural$$

of $L^\natural$-Lie algebras.

Proof. — We define a matrix $\langle A\rangle = (\langle a_{i,j}\rangle_{k,l}) \in F(\mathbb{N}, M_d(L^\sharp))$ by

$$\langle A\rangle(n) := \sigma^{n-1}(A)\sigma^{n-2}(A)\cdots\sigma(A)A \tag{3.3}$$

for $n \in \mathbb{N}$. Then the image of $Z = (z_{i,j})$ by the universal Euler morphism $\iota_L : L \to F(\mathbb{N}, L^\sharp)$ is

$$\iota_L(Z) = \langle A\rangle Z.$$

Since $D(\sigma^n(a_{i,j})) = 0$ for any $D \in \mathrm{Der}\,(L^\natural/k^\natural)$ and $n \in \mathbb{N}$, we have

$$D(\langle A\rangle) = 0.$$

Moreover we have

$$\Sigma(\langle A\rangle) = \iota_L(A)\langle A\rangle.$$

Therefore a ring generated by $\mathcal{K}$ and $\langle a_{i,j}\rangle_{k,l}$ is closed under the operators of Σ, D, so that $\mathcal{L}$ is the ring generated by $\mathcal{K}$ and $\langle a_{i,j}\rangle_{k,l}$. Let us consider natural group morphisms

$$p \ :\ \text{Inf-gal}\,(L/k)(L^\natural[\varepsilon]/(\varepsilon^2)) \to \text{Inf-gal}\,(L/k)(L^\natural),$$

$$q \ :\ \mathrm{Aut}_\sigma((L\otimes_C L^\natural[\varepsilon]/(\varepsilon^2))/(k\otimes_C L^\natural[\varepsilon]/(\varepsilon^2))) \to \mathrm{Aut}_\sigma(L\otimes_C L^\natural/k\otimes_C L^\natural).$$

Since $\ker q \cong \mathrm{Lie}\,(\mathrm{Gal}\,(L/k)) \otimes_C L^\natural$, by Lemma 1.5 it is sufficient to prove the group isomorphism

$$\ker p \cong \ker q.$$

An element $\mu \in \ker q$ defines a $k \otimes_C L^\natural[\varepsilon]/(\varepsilon^2)$-automorphism of $L \otimes_C L^\natural[\varepsilon]/(\varepsilon^2)$ so that there exists a matrix $M_\mu \in M_d(L^\natural)$ such that

$$\mu : L\otimes_C L^\natural[\varepsilon]/(\varepsilon^2) \to L\otimes_C L^\natural[\varepsilon]/(\varepsilon^2), \quad (x_{i,j})\otimes 1 \mapsto (x_{i,j})\otimes(I_d + \varepsilon M_\mu).$$

Here I_d is an unit matrix of degree d. Now we need the next lemma.

LEMMA 3.4. — *Let L/k be a Picard-Vessiot field. Then the ring $\mathcal{K}$ and the image $\iota_L(L)$ by the universal Euler morphism ι_L are linearly disjoint over the image $\iota_L(k)$ in the ring $\mathcal{L}$.*

We assume Lemma 3.4 and we continue to prove Theorem 3.3. After the proof of Theorem 3.3, we prove Lemma 3.4.

By Lemma 3.4, we have a ring isomorphism

$$\mathcal{K} \otimes_{\iota_L(k)} \iota_L(L) \to \mathcal{L}, \quad a \otimes b \mapsto ab. \tag{3.4}$$

For the derivation $D_1, D_2, \ldots, D_d$ over $\mathcal{L}$, we define a differential structure over $\mathcal{K} \otimes_{\iota_L(k)} \iota_L(L)$ by

$$D_i(a \otimes 1) = D_i(a) \otimes 1,$$
$$D_i(1 \otimes \iota_L((z_{i,j}))) = \iota_L((z_{i,j}))(z_{i,j})^{-1} D_i((z_{i,j})) \in \mathcal{L} \cong \mathcal{K} \otimes_{\iota_L(k)} \iota_L(L)$$

for $a \in \mathcal{K}$. And the difference structure over $\mathcal{K} \otimes_{\iota_L(k)} \iota_L(L)$ is

$$\sigma(a \otimes b) = \sigma(a) \otimes \sigma(b).$$

Then the isomorphism (3.4) is a difference-differential morphism.

The difference automorphism μ induces a difference-differential morphism

$$\begin{array}{cccc} \rho_\mu & : & \mathcal{L} \cong \mathcal{K} \otimes_{\iota_L(k)} \iota_L(L) & \to & \mathcal{L} \hat{\otimes}_{L^\sharp} L^\natural[\varepsilon]/(\varepsilon^2)[[W]]. \\ & & 1 \otimes \iota_L((z_{i,j})) & \mapsto & (z_{i,j}) \otimes (I_d + \varepsilon M_\mu) \\ & & a \otimes 1 & \mapsto & a \otimes 1 \end{array}$$

Since $\rho_\mu \in \mathcal{F}(L/k)(L^\natural[\varepsilon]/(\varepsilon^2))$, by Theorem 2.20 there exists $\tau_\mu \in$ Inf-gal $(L/k)(L^\natural[\varepsilon]/(\varepsilon^2))$ such that $\rho_\mu = \tau_\mu \circ \mathrm{id}_{L^\natural[\varepsilon]/(\varepsilon^2)}$. Therefore we have a map

$$\alpha \ : \ \ker q \to \ker p, \quad \mu \mapsto \tau_\mu. \tag{3.5}$$

Conversely, since $\tau \in \ker p$ is a difference-differential morphism, there exists a matrix $M_\tau \in M_d(L^\natural)$ such that

$$\tau((z_{i,j}) \otimes 1) = (z_{i,j}) \otimes (1 + \varepsilon M_\tau).$$

So we have a difference $k \otimes_C L^\natural[\varepsilon]/(\varepsilon^2)$-automorphism

$$\mu_\tau \ : \ L \otimes_C L^\natural[\varepsilon]/(\varepsilon^2) \to L \otimes_C L^\natural[\varepsilon]/(\varepsilon^2), \quad \mu_\tau((z_{i,j}) \otimes 1) = (z_{i,j}) \otimes (1 + \varepsilon M_\tau).$$

Therefore we have a map

$$\beta \ : \ \ker p \to \ker q, \quad \tau \mapsto \mu_\tau. \tag{3.6}$$

Since maps (3.5) and (3.6) commute with each other, we get

$$\ker p \cong \ker q.$$

□

Proof of Lemma 3.4. — Let $a_1, a_2, \ldots, a_n \in L \setminus 0$ be linearly independent over k and linearly dependent over $\mathcal{K}$. So there exist $u_i \in \mathcal{K}, 1 \leqslant i \leqslant n$ such that

$$u_1 a_1 + u_2 a_2 + \cdots + u_n a_n = 0$$

and at least one of the u_i's is not equal to 0.

Since L/k is a Picard-Vessiot field, we have $\mathcal{K} = k \cdot L^\sharp$. Therefore for all $1 \leqslant i \leqslant n$, there exist $\alpha_{i,j} \in k$ and $\beta_j \in L^\sharp$ such that

$$u_i = \sum_j \alpha_{i,j} \beta_j.$$

Therefore one of the $\alpha^{\natural}_{i,j} e0$. So

$$\sum_i u_i a_i = \sum_i \Big(\sum_j \alpha_{i,j} \beta_j \Big) a_i = \sum_j \Big(\sum_i \alpha_{i,j} a_i \Big) \beta_j.$$

Since $a_1, a_2, \ldots, a_n \in L$ are linearly independent over k, we have $\sum_i \alpha_{i,j} a^{\natural}_i e0$ for a certain j. So there exist $d_1, d_2, \ldots, d_m \in L$ such that linearly independent over k and

$$c_1 d_1 + c_2 d_2 + \cdots + c_m d_m = 0 \tag{3.7}$$

for certain $c_i \in L^\sharp \setminus 0$. Here m is a minimum number satisfying the above condition. Since L and $L^\sharp$ are fields, we have from (3.7)

$$1 + c_2 d_2 + \cdots + c_m d_m = 0$$

for $c_i \in L^\sharp$ and $d_i \in L$. Therefore we have

$$c_2 (d_2 - \phi(d_2)) + \cdots + c_m (d_m - \phi(d_m)) = 0.$$

By the minimality of m, we get

$$d_i \in L \cap L^\sharp = C_L = C_k \subset k.$$

This is contradiction to $d_1, d_2, \ldots, d_m$ are linearly independent over k. $\square$

4. Surjection conjecture

In our theory, we can not expect Galois correspondence. In Galois theory of algebraic equations, the following surjection theory is as useful as Galois correspondence, for example to prove that we can not solve general quintic equation by extraction of radicals and the arithmetic operations $\pm, \times, \div$.

Theorem 4.1. — *Let $L/M/k$ be a tower of Galois extensions. Then we have a surjective group morphism* $\mathrm{Gal}\,(L/k) \to \mathrm{Gal}\,(M/k)$.

For a tower of difference field extensions $L/M/k$, it is natural to expect an analogous result for $\text{Inf-gal}\,(L/k) \to \text{Inf-gal}\,(M/k)$.

Let $\mathcal{M}_1$(resp. $\mathcal{K}_1$) be the difference-differential ring of $F(\mathbb{N}, M^\sharp)$ generated by the image of the universal Euler morphism $\iota_M(M)$ (resp. $\iota_M(k)$) and $M^\sharp$. Let $D_1, D_2, \ldots, D_n$ be a basis of the $M^\natural$-vector space $\mathrm{Der}\,(M^\natural/k^\natural)$. Let $D_1, D_2, \ldots, D_n, D_{n+1}, \ldots, D_d$ be a basis of the $L^\natural$-vector space $\mathrm{Der}\,(L^\natural/k^\natural)$. By these derivations, we can construct the Galois hull $\mathcal{L}/\mathcal{K}$ of L/k. We denote by $\mathcal{M}$ the difference-differential subring of $\mathcal{L}$ generated by the image $\iota_L(M)$ of the Euler morphism and $L^\sharp$. Then we get the difference-differential isomorphism

$$\mathcal{M} \cong \mathcal{M}_1 \otimes_{M^\sharp} L^\sharp.$$

Therefore for an $L^\natural$-algebra A and a morphism $f \in \mathcal{F}(L/k)(A)$ so that $f : \mathcal{L} \to \mathcal{L} \otimes_{L^\sharp} A[[W]]$, the restriction $f|_{\mathcal{M}_1 \otimes_{m^\sharp} L^\sharp}$ to $\mathcal{M}_1 \otimes_{M^\sharp} L^\sharp$ is considered as an element of $\mathcal{F}(M/k)(A)$. So we have a map $\mathcal{F}(L/k)(A) \to \mathcal{F}(M/k)(A)$. By Theorem 2.20, $(\text{Inf-gal}\,(L/k)(A), \mathcal{F}(L/k)(A))$ is a principal homogeneous space. Therefore we can define a functorial group morphism

$$\text{Inf-gal}\,(L/k)(A) \to \text{Inf-gal}\,(M/k)(A).$$

Conjecture 4.2. — *The canonical morphism* $\mathrm{Lie}\,(\text{Inf-gal}\,(L/k)) \to \mathrm{Lie}\,(\text{Inf-gal}\,(M/k))$ *is surjective.*

5. Examples

Definition 5.1. — *Let A be a set and $u \in F(\mathbb{N}, A)$ so that u is a function on $\mathbb{N}$ with values in A. We denote the function $u : \mathbb{N} \to A$ by a matrix*

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & n & \cdots \\ u(0) & u(1) & u(2) & \cdots & u(n) & \cdots \end{pmatrix}.$$

We denote the constant function $v : \mathbb{N} \to A$ so that $v(n) = a$ for all $n \in \mathbb{N}$ by a. Namely

$$a = \begin{pmatrix} 0 & 1 & 2 & \cdots & n & \cdots \\ a & a & a & \cdots & a & \cdots \end{pmatrix}.$$

We give three examples here. The reader finds extensive examples in section 5 of the second part [8].

Example 5.2. — Let L be the rational function field $\mathbb{C}(x)$ of 1-variable x over the complex number field $\mathbb{C}$. For a non-zero complex number a satisfying $a^n \neq 1$ for all $n \geqslant 2$, we consider a $\mathbb{C}$-morphism

$$\sigma : L \to L, \quad x \mapsto ax.$$

So $(L,\sigma)/(k,\mathrm{Id}_k) = (\mathbb{C},\mathrm{Id}_{\mathbb{C}})$ is a difference field extension. In particular, the extension is a Picard-Vessiot extension. We calculate the Galois group of $(L,\sigma)/(k,\mathrm{Id}_k)$. Since transcendence degree of L/k is 1, we choose $d/dx \in \mathrm{Der}\,(L/k)$ that spands the L-vector space $\mathrm{Der}\,(L/k)$.

First, we determine the Galois hull $\mathcal{L}/\mathcal{K}$. Since $\mathcal{K}$ is generated by the image $\iota_L(k)$ of k by the universal Euler morphism ι_L and the field $L^\sharp$ of constant functions of the difference ring $F(\mathbb{N},L^\sharp)$, we get $\mathcal{K} = \mathbb{C}(x)^\sharp$. Moreover since $\mathcal{L}$ is a difference-differential subring of $F(\mathbb{N},L^\natural)$ generated by the image $\iota_L(L)$ and the constant filed $L^\sharp$, we investigate the image of x by the universal Euler morphism.

The image of $x\in L$ by the universal Euler morphism ι_L is

$$\begin{aligned}\iota_L(x) &= \begin{pmatrix} 0 & 1 & 2 & \cdots & n & \cdots \\ x & ax & a^2x & \cdots & a^n x & \cdots \end{pmatrix}\\ &= \begin{pmatrix} 0 & 1 & 2 & \cdots & n & \cdots \\ 1 & a & a^2 & \cdots & a^n & \cdots \end{pmatrix}\cdot x\\ &= \langle a\rangle x.\end{aligned}$$

Here x means the constant function with value x and $\cdot$ is the product in the ring $F(\mathbb{N},L^\sharp)$ and the function $\langle a\rangle \in F(\mathbb{N},L^\sharp)$ is $\langle a\rangle(n) = a^n$ for $n\in\mathbb{N}$. Since $\sigma(\langle a\rangle) = a\cdot\langle a\rangle$ and $d/dx\,\langle a\rangle = 0$, a ring generated by $L^\sharp$ and $\langle a\rangle$ is closed by σ and d/dx. Therefore the Galois hull $\mathcal{L}$ is generated by $\langle a\rangle$ and $\mathcal{K}$. So $\mathcal{K} = \mathbb{C}(x)^\sharp$ and $\mathcal{L} = \mathbb{C}(x)^\sharp\iota_L(L)$. We calculate the Lie algebra Lie (Inf-gal (L/k)). By Theorem 2.20, (Inf-gal $(L/k), \mathcal{F}(L/k)$) is a principle homogeneous space. So we can determine Inf-gal (L/k) by considering $\mathcal{F}(L/k)$. The Lie algebra of Lie (Inf-gal (L/k)) is the kernel of

$$\text{Inf-gal}\,(L/k)(L^\natural[\varepsilon]/(\varepsilon^2)) \to \text{Inf-gal}\,(L/k)(L^\natural)$$

by section 1.

Let $f:\mathcal{L}\to F(\mathbb{N},L^\natural[\varepsilon]/(\varepsilon^2)[[W]])$ be an element of $\mathcal{F}(L/k)(L^\natural[\varepsilon]/(\varepsilon^2))$. Then since f commutes with σ, we have

$$\begin{aligned}\sigma(\iota_L(x)^{-1}f(\iota_L(x))) &= (\sigma(\iota_L(x)))^{-1}f(\sigma(\iota_L(x))\\ &= (a\iota_L(x))^{-1}f(a\iota_L(x))\\ &= \iota_L(x)^{-1}f(\iota_L(x)).\end{aligned}$$

Therefore $(\iota_L(x))^{-1}f(\iota_L(x))$ is a constant of σ. So there exists a constant map $\alpha:\mathbb{N}\to A[[W]]$ such that $f(\iota_L(x)) = \alpha\iota_L(x)$.

Moreover f commutes with the derivation d/dx. So we get

$$\frac{d}{dW}f(\iota_L(x)) = f\left(\frac{d}{dx}\iota_L(x)\right) = f\left(\frac{1}{x}\iota_L(x)\right) = \frac{1}{x}\alpha\iota_L(x). \tag{5.1}$$

On the other hand, we have

$$\frac{d}{dW} f(\iota_L(x)) = \frac{d}{dW}(\alpha \iota_L(x)) = \frac{d\alpha}{dW}\,\iota_L(x) + \alpha \left(\frac{d}{dW}\iota_L(x)\right) = \frac{d\alpha}{dW}\,\iota_L(x) + \alpha \frac{1}{x}\iota_L(x). \tag{5.2}$$

By (5.1) and (5.2), we get

$$\frac{d\alpha}{dW}\,\iota_L(x) = 0$$

and since $\iota_L(x)$ is invertible, we have

$$\frac{d\alpha}{dW} = 0.$$

Namely α is independent of W. Since $f \equiv \mathrm{id}_{L^\natural[\varepsilon]/(\varepsilon^2)} \mod F(\mathbb{N}, \varepsilon L^\natural[[W]])$, there exists $\mu \in L^\natural$ such that $f(\iota_L(x)) = (1+\varepsilon\mu)\iota_L(x)$. This means the Lie algebra of the infinitesimal Galois group of $(L,\sigma)/(k, \mathrm{Id}_k)$ is 1-dimensional.

Example 5.3. — Let L be the rational function field $\mathbb{C}(x,y)$ in two variables x, y. We consider the $\mathbb{C}$-morphism $\sigma : L \to L$ given by $x \mapsto 2x$ and $y \mapsto xy$, so $(L,\sigma)/(k,\mathrm{Id}_k) := (\mathbb{C}, \mathrm{Id}_\mathbb{C})$ is the difference extension field. Since the transcendence degree of L/k is 2, we choose $d/dx, d/dy \in \mathrm{Der}\,(L/k)$ so that $[d/dx, d/dy] = 0$ and $\{d/dx, d/dy\}$ spans the $L^\natural$-vector space $\mathrm{Der}\,(L^\natural/k^\natural)$.

Now, we consider the image of x, y by the universal Euler morphism ι_L to seek the Galois hull $\mathcal{L}/\mathcal{K}$ of $(L,\sigma)/(k,\mathrm{Id}_k)$. The image of x, y are

$$\iota_L(x) = \begin{pmatrix} 0 & 1 & 2 & \cdots & n & \cdots \\ x & 2x & 2^2 x & \cdots & 2^n x & \cdots \end{pmatrix}$$

and

$$\iota_L(y) = \begin{pmatrix} 0 & 1 & 2 & \cdots & n & \cdots \\ y & xy & 2x^2 y & \cdots & 2^{\frac{1}{2}n(n-1)} x^n y & \cdots \end{pmatrix}.$$

So we have the following conditions.

$$\begin{cases} x\frac{d}{dx}(\iota_L(x)) = \iota_L(x), \\ y\frac{d}{dy}(\iota_L(x)) = 0, \end{cases} \quad \begin{cases} x\frac{d}{dx}(\iota_L(y)) = 1_A \cdot \iota_L(y), \\ y\frac{d}{dy}(\iota_L(y)) = \iota_L(y), \end{cases} \quad \begin{cases} \sigma(\iota_L(x)) = 2 \cdot \iota_L(x), \\ \sigma(\iota_L(x)) = \iota_L(x) \cdot \iota_L(y). \end{cases} \tag{5.3}$$

Here $1_A \in F(\mathbb{N}, L^\natural)$ denotes the function $1_A(n) = n$ for $n \in \mathbb{N}$. So we get $(d/dx)\,1_A = (d/dy)\,1_A = 0$ and $\sigma(1_A) = 1_A + 1$. The Galois hull $\mathcal{L}$ is the difference-differential subring of $F(\mathbb{N}, L^\sharp)$ generated by $\iota_L(x), \iota_L(y), 1_A, \mathcal{K}$. Thus we have $\mathcal{K} = \mathbb{C}(x,y)^\sharp$ and $\mathcal{L} = \mathcal{K}(\iota_L(x), \iota_L(y), 1_A)$.

As in Example 5.2, we consider $\mathcal{F}(L/k)$ to calculate the Lie algebra of Inf-gal (L/k).

Let $f \in \mathcal{F}(L/k)(L^{\natural}[\varepsilon]/(\varepsilon^2))$. Then by constraints (5.3) and $f \equiv \mathrm{id} \mod F(\mathbb{N}, \varepsilon L^{\natural}[[W]])$, we get

$$\begin{cases} f(\iota_L(x)) = \iota_L(x)(1+\varepsilon\mu), \\ f(\iota_L(y)) = \iota_L(y)(1+\varepsilon(\eta + 1_A\mu +^{\natural} u(x-1)\log(1+\frac{X}{x}))), \\ f(1_A) = 1_A + \varepsilon^{\natural} u. \end{cases}$$

Here $\mu, ^{\natural}u, \eta$ are elements of $L^{\natural}$. Since (Inf-gal $(L/k), \mathcal{F}(L/k)$) is a principle homogeneous space by Theorem 2.20, we know that the dimension of Lie algebra of Inf-gal (L/k) is 3.

Let M be the rational function field $\mathbb{C}(x)$ of 1-variable x. Since the restriction of σ to M is a k-morphism over M, (M,σ) is the difference fields. So we have the difference field extension $(M,\sigma)/(k,\mathrm{Id_k}) = (\mathbb{C}, \mathrm{Id}_{\mathbb{C}})$ and $(L,\sigma)/(M,\sigma)$. Since both difference extensions are Picard-Vessiot extensions, we can show that the difference Galois groups are Gal $(M/k) \cong \mathbb{G}_m$ and Gal $(L/M) \cong \mathbb{G}_m$. So by Theorem 3.3, we have dim(Lie Inf-gal (M/k)) $= 1$ and dim(Lie Inf-gal (L/M)) $= 1$. Therefore we have an inequality

$$3 = \dim(\mathrm{Lie\,Inf\text{-}gal}\,(L/k))^{\natural} e \dim(\mathrm{Lie\,Inf\text{-}gal}\,(L/M)) \\ + \dim(\mathrm{Lie\,Inf\text{-}gal}\,(M/k)) = 2.$$

Example 5.4. — The reader finds this example in Example 5.3 of the second part [8]. Let L be the rational function field $\mathbb{C}(x)$ in one variable x. We define a $\mathbb{C}$-morphism $\sigma : L \to L$ by $\sigma(x) = x^2$. Then we consider the difference field extension $(L,\sigma)/(k,\mathrm{Id}_k) := (\mathbb{C}, \mathrm{Id}_{\mathbb{C}})$. We choose the derivation d/dx as a basis of the L-vector space Der $(L/\mathbb{C})$. The image of x by the universal Euler morphism ι_L is

$$\iota_L(x) = \begin{pmatrix} 0 & 1 & 2 & \cdots & n & \cdots \\ x & x^2 & x^{2^2} & \cdots & x^{2^n} & \cdots \end{pmatrix}.$$

So we get $x(d/dx)\iota_L(x) = 2^G\iota_L(x)$ and $\sigma(\iota_L(x)) = (\iota_L(x))^2$, where $2^G \in F(\mathbb{N}, L^{\sharp})$ is the function such that $2^G(n) = 2^n$ for $n \in \mathbb{N}$. Since $(d/dx)2^G = 0$ and $\sigma(2^G) = 2 \cdot 2^G$, the Galois hull $\mathcal{L}/\mathcal{K}$ is given by $\mathcal{K} = \mathbb{C}(x)^{\sharp}$ and $\mathcal{L} = \mathcal{K}(\iota_L(x), 2^G)$.

Let $f \in \mathcal{F}(L/k)(L^{\natural}[\varepsilon]/(\varepsilon^2))$. Then by the constraints of $\iota_L(x)$ and 2^G, we get

$$\begin{cases} f(\iota_L(x)) = \iota_L(x)(1+\varepsilon 2^G(\mu +^{\natural} u\log(1+\frac{X}{x}))), \\ f(2^G) = 2^G(1+\varepsilon^{\natural} u) \end{cases}$$

with $\mu, {}^{\natural}u \in L^{\natural}$.

Since the Lie algebra of the infinitesimal Galois group Inf-gal$\,(L/k)$ is 2-dimensional non-commutative Lie algebra, the Galois group is the 1-dimensional affine transformation group.

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