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<http://aif.cedram.org/item?id=AIF_2009__59_6_2329_0>
SPHERICAL CONJUGACY CLASSES
AND THE BRUHAT DECOMPOSITION

by Giovanna CARNOVALE (*)

ABSTRACT. — Let $G$ be a connected, reductive algebraic group over an algebraically closed field of zero or good and odd characteristic. We characterize spherical conjugacy classes in $G$ as those intersecting only Bruhat cells in $G$ corresponding to involutions in the Weyl group of $G$.

RÉSUMÉ. — Soit $G$ un groupe algébrique réductif connexe, sur un corps algébriquement clos de caractéristique zéro ou bonne et impaire. Nous caractérisons les classes de conjugaison sphériques de $G$ comme celles ayant une intersection seulement avec des cellules de Bruhat de $G$ correspondantes à des involutions dans le groupe de Weyl de $G$.

Introduction

The Bruhat decomposition of a connected reductive algebraic group $G$ over an algebraically closed field states that the two-sided cosets of $G$ with respect to a Borel subgroup $B$ (Bruhat cells) are naturally parametrized by the elements in the Weyl group of $G$ and have a well-understood geometrical behaviour. It is a fundamental tool in the theory of algebraic groups, as it is relevant for the comprehension of the geometry of the flag variety $G/B$, for instance, in the computation of its cohomology. Besides, intersection of Bruhat cells corresponding to opposite Borel subgroups (double Bruhat cells) play a significant role in the description of the symplectic leaves of a natural Poisson structure on $B$ ([10]). New interest has been raised by Bruhat cells and double Bruhat cells for their applications to total positivity

Keywords: Conjugacy class, spherical homogeneous space, Bruhat decomposition.

(*) The author is grateful to the referee for suggesting improvements in the presentation of the manuscript and to Mauro Costantini for useful comments on a previous version of this paper.
and to the theory of cluster algebras. For instance, as it has been very recently shown, double Bruhat cells serve as a geometric model for cluster algebras of finite type, since every cluster algebra of finite type with principal coefficients at an arbitrary acyclic initial cluster can be realized as the coordinate ring of a certain reduced double Bruhat cell ([26]).

The interplay between conjugacy classes in an algebraic group and the Bruhat decomposition has been successfully exploited in the past. Probably the first results in this sense are in [24] where the Bruhat decomposition of a semisimple algebraic group $G$ is used for the construction of a cross-section for the collection of regular conjugacy classes of $G$.

More recently, [11] and [12] have provided an analysis of the intersection of conjugacy classes in a Chevalley group with Bruhat cells corresponding to generalized Coxeter elements and their conjugates.

If we consider spherical conjugacy classes, that is, those conjugacy classes of a group $G$ on which $B$ acts with finitely many orbits, it is natural to inquire about their intersection with Bruhat cells. A characterization of spherical conjugacy classes has been given in terms of a formula involving the dimension of the class $O$ and the maximal element $w$ in the Weyl group $W$ of $G$ for which $O \cap BwB$ is non-empty. This is obtained in [5] over the complex numbers and in [6] over an arbitrary algebraically closed field of zero or odd good characteristic. The motivation in [5] was the proof - in the spherical case - of a conjecture due to De Concini, Kac and Procesi on the dimension of irreducible representations of quantum groups at the roots of unity ([9]). The proof relied on the classifications of spherical nilpotent orbits ([19]) and of reductive spherical pairs ([4]) and on geometric properties of spherical homogeneous spaces in the complex setting ([4],[18]). In [6] a different approach was developed and a crucial step in the argument was that every spherical conjugacy class intersects only Bruhat cells $BwB$ for $w$ an involution in $W$. The aim of the present paper is to show that this property fully characterizes spherical conjugacy classes.

**Theorem.** — Let $G$ be a connected reductive algebraic group over an algebraically closed field of zero or good, odd characteristic. A conjugacy class $O$ in $G$ is spherical if and only if $O$ intersects only Bruhat cells corresponding to involutions in the Weyl group of $G$.

The paper is structured as follows: after fixing notation and recalling basic facts about spherical homogeneous spaces and conjugacy classes in §1, we analyse the case of $G$ simple of type $G_2$ in full detail in §2. The reason for doing so is twofold. On the one hand we would like to give an
idea of the techniques involved through an example, and on the other hand it would not be more efficient to treat the case of $G_2$ together with the others because separate descriptions for behaviour of roots with different length ratios are needed.

In §3 we restrict our attention to those conjugacy classes intersecting only Bruhat cells corresponding to involutions. For such a class $O$ we consider the maximal element $w \in W$ for which $O \cap BwB$ is non-empty and the set of $B$-orbits in $O$ that are contained in $BwB$, the so-called maximal $B$-orbits. The properties of a special class of representatives $x$ of maximal $B$-orbits are analyzed, allowing a description of the centralizer $B_x$ in $B$. This is achieved by using the same strategy as in [6]. The proofs therein are rather laborious and need a case-by-case analysis but they apply also to the present situation so we use them referring to [6]. The hypothesis on the class $O$ imposes restrictions on the representatives $x$ in maximal $B$-orbits: for instance, if $x = uv \in N(T)U$ then $v$ lies in the subgroup generated by the root subgroups $X_\alpha$ for which $w\alpha = -\alpha$. This condition is powerful for a general $w$ but it is empty when $w$ is the longest element $w_0$ in $W$ and it acts as $-1$ in the geometric representation. For this reason we deal with this situation separately and an unpleasant case-by-case analysis is needed in the doubly-laced case. This is done in §4, where the theorem in this case is proved by showing the sufficient condition that the maximal $B$-orbits are finitely-many.

The rest of the paper is devoted to an estimate of the centralizer $G_x$ in $G$ of a representative $x$ in a maximal $B$-orbit. Indeed, since $O$ is parted into finitely many $B$-orbits if and only if it has a dense $B$-orbit ([3, 15, 17, 25]), we may conclude that $O$ is spherical once we prove that the dimension of a maximal $B$-orbit equals the dimension of $O$. In §5 we consider the general case and we construct some families of elements contained in $G_x \cap X_\alpha s_\alpha B$ for different roots $\alpha$. We need different strategies according to the behaviour of $\alpha$ with respect to $w$. In particular, when $w\alpha = -\alpha$ we apply the results in §4. Once we have constructed enough elements in $G_x$ we show using the intersections $G_x \cap B\sigma B$ and induction on the length of $\sigma$ that the image of $G_x$ through the projection of $G$ on $G/B$ is dense in the flag variety obtaining the sought equality of dimensions.

1. Preliminaries

Unless otherwise stated $G$ will denote a connected, reductive algebraic group over an algebraically closed field $k$ of characteristic 0 or odd and
good ([23, §1.4]). When we write an integer as an element in $k$ we shall mean the image of that integer in the prime field of $k$.

Let $B$ be a Borel subgroup of $G$, let $T$ be a maximal torus contained in $B$ and let $B^{-}$ be the Borel subgroup opposite to $B$. Let $U$ (respectively $U^{-}$) be the unipotent radical of $B$ (respectively $B^{-}$).

We shall denote by $\Phi$ the root system relative to $(B, T)$; by $\Delta = \{ \alpha_1, \ldots, \alpha_n \}$ the corresponding set of simple roots and by $\Phi^+$ the corresponding set of positive roots. We shall use the numbering of the simple roots in [2, Planches I-IX].

We shall denote by $W$ the Weyl group associated with $G$ and by $s_\alpha$ the reflection corresponding to the root $\alpha$. By $\ell(w)$ we shall denote the length of the element $w \in W$ and by $\text{rk}(1 - w)$ we shall mean the rank of $1 - w$ in the geometric representation of the Weyl group. By $w_0$ we shall denote the longest element in $W$ and $\vartheta$ will be the automorphism of $\Phi$ given by $-w_0$. By $\Pi$ we shall always denote a subset of $\Delta$ and $\Phi(\Pi)$ will indicate the corresponding root subsystem of $\Phi$. We shall denote by $W_\Pi$ the parabolic subgroup of $W$ generated by the $s_\alpha$ for $\alpha$ in $\Pi$.

Given an element $w \in W$ we shall denote by $\dot{w}$ a representative of $w$ in the normalizer $N(T)$ of $T$. For any root $\alpha$ in $\Phi$ we shall write $x_\alpha(t)$ for the elements in the corresponding root subgroup $X_\alpha$ of $G$. Moreover, we choose $x_\alpha(1)$ and $x_\alpha(-1)$ so that $x_\alpha(1)x_{-\alpha}(1)x_\alpha(-1) = n_\alpha \in s_\alpha T$ so that the properties in [22, Lemma 8.1.4] hold.

If $\Pi \subset \Delta$ we shall indicate by $P_\Pi$ the standard parabolic subgroup of $G$ whose Levi component contains the root subgroups corresponding to roots in $\Phi(\Pi)$ and by $P^u_\Pi$ its unipotent radical. If $\Pi = \{ \alpha \}$ we shall simply write $P_\alpha$ and $P^u_\alpha$.

For $w \in W$, we will put

\begin{equation}
\Phi_w := \{ \alpha \in \Phi^+ \mid w^{-1} \alpha \in -\Phi^+ \}
\end{equation}

\begin{equation}
U_w = \langle X_\alpha \mid \alpha \in \Phi_w \rangle, \quad U^w = \langle X_\alpha \mid \alpha \in \Phi^+ \setminus \Phi_w \rangle
\end{equation}

so that $BwB = U^w\dot{w}B$ for any choice of $\dot{w} \in N(T)$. We shall denote by $T^w$ the subgroup of $T$ that is centralized by any representative $\dot{w}$ of $w$.

We shall make extensive use of Chevalley’s commutator formula ([7, Theorem 5.2.2]): for $\alpha$ and $\beta$ linearly independent roots and $a, b \in k$ there are structure constants $c^{ij}_{i\alpha, j\beta}$ in the prime field of $k$ such that

\begin{equation}
x_\alpha(a)x_\beta(b) = x_\beta(b)x_\alpha(a) \prod_{i, j > 0} x_{i\alpha + j\beta}(c^{ij}_{i\alpha, j\beta} a^ib^j)
\end{equation}
where the product is taken over all \((i, j)\) such that \(i\alpha + j\beta \in \Phi\) and in any order for which \(i + j\) is increasing. Moreover, \(c_{ij}^{i\alpha, j\beta} \in \{\pm 1, \pm 2, \pm 3\}\) and 3 occurs only if \(\Phi\) has a component of type \(G_2\), so \(c_{ij}^{i\alpha, j\beta} \neq 0\).

Given an element \(x \in G\) we shall denote by \(O_x\) the conjugacy class of \(x\) in \(G\) and by \(G_x\) (resp. \(B_x\), resp. \(T_x\)) the centralizer of \(x\) in \(G\) (resp. \(B\), resp. \(T\)). For a conjugacy class \(O = O_x\) we shall denote by \(V\) the set of \(B\)-orbits into which \(O\) is parted.

**Definition 1.1.** — Let \(K\) be a connected algebraic group. A homogeneous \(K\)-space is called spherical if it has a dense orbit for some Borel subgroup of \(K\).

It is well-known ([3], [25] in characteristic 0, [15], [17] in positive characteristic) that \(X\) is a spherical homogeneous \(G\)-space if and only if the set of \(B\)-orbits in \(X\) is finite.

2. \(B\)-orbits and Bruhat decomposition

Let \(V\) be the set of \(B\)-orbits in a conjugacy class \(O\) in \(G = \bigcup_{w \in W} BwB\). There is a natural map \(\phi: V \to W\) associating to \(v \in V\) the element \(w\) in the Weyl group of \(G\) for which \(v \subset BwB\).

It is shown in [6] for \(G\) simple that if \(O\) is spherical as a homogeneous space then the image of \(\phi\) consists of involutions. The same proof holds for \(G\) reductive. This motivates the following definition.

**Definition 2.1.** — Let \(G\) be a connected reductive algebraic group. A conjugacy class \(O\) in \(G\) is called quasi-spherical if the image of \(\phi\) consists of involutions.

**Remark 2.2.** — Regular conjugacy classes in simple algebraic groups of rank greater than 1 cannot be quasi-spherical. Indeed, by [24, Theorem 8.1] regular classes meet Bruhat cells corresponding to Coxeter elements.

2.1. The case of \(G_2\)

We aim at showing that every quasi-spherical conjugacy class is spherical. In order to illustrate this result explicitly, we analyze quasi-spherical conjugacy classes for \(G\) simple of type \(G_2\) by inspection, making use of the classification of unipotent conjugacy classes (see, for instance, [16, Section 7.12]) and of the commutator formula (1.3).
Theorem 2.3. — Let $G$ be simple of type $G_2$. Then every quasi-spherical conjugacy class of $G$ is spherical.

Proof. — Let $\alpha$ and $\beta$ denote the short and long simple roots, respectively, and let $O$ be a conjugacy class in $G$. We first assume that $O$ is unipotent so it is either of type $A_1$, $A_1$, subregular or regular ([16, Section 7.18]). If $O$ is of type $A_1$ or $A_1$ then $O$ is spherical, hence quasi-spherical by [5, Proposition 6, Proposition 11] which hold for arbitrary $k$. Alternatively, one may use [19, Theorem 3.2] and [14, Theorem 4.14]. If $O$ is regular it cannot be quasi-spherical by Remark 2.2. The element $u = x_\beta(1)x_{3\alpha+\beta}(1) \in G$ does not lie in the regular unipotent conjugacy class by [24, Lemma 3.2(c)]. Its conjugate $w_0uw_0^{-1} = x_{-\beta}(a)x_{-3\alpha-\beta}(b)$ with $ab \neq 0$ lies in $Bs_\beta Bs_{3\alpha+\beta}B = Bs_\beta s_{3\alpha+\beta}B$ by [22, Lemma 8.1.4(i), Lemma 8.3.7] and $s_\beta s_{3\alpha+\beta}$ is not an involution. Then its class is not quasi-spherical and by exclusion it is the subregular unipotent conjugacy class, so the statement holds for unipotent conjugacy classes.

Let us now consider a representative $x \in O \cap B$ with Jordan decomposition $x = su \in TU$ with $s \neq 1$. Then $G_s$ is connected and reductive ([16, Theorem 2.2, Theorem 2.11]). We shall analyze the different cases according to the semisimple rank srk of $G_s$.

If srk$G_s = 0$ then $O$ is regular, hence it is not quasi-spherical by Remark 2.2.

If srk$G_s = 1$ and $u \neq 1$ then $O$ is regular, hence it is not quasi-spherical. Let us assume $u = 1$. Up to conjugation by an element in $N(T)$ we may assume that $G_s$ is either $H_1 = \langle T, X_{\pm\beta} \rangle$ or $H_2 = \langle T, X_{\pm\alpha} \rangle$.

If $G_s = H_1$ conjugation of $s$ by $x_{-\alpha}(1)x_{-\alpha-\beta}(1)$ yields

$$s_1 = sx_{-\alpha}(a)x_{-\alpha-\beta}(b)x_{-2\alpha-\beta}(c)x_{-3\alpha-\beta}(d)x_{-3\alpha-2\beta}(e) \in O$$

for $a, b, c, d, e \in k$ with $ab \neq 0$. Conjugation by a suitable element in $X_{-2\alpha-\beta}$ gives

$$s_2 = sx_{-\alpha}(a)x_{-\alpha-\beta}(b)x_{-3\alpha-\beta}(d')x_{-3\alpha-2\beta}(e') \in O$$

for $d', e' \in k$. Conjugation by a suitable element in $X_{-3\alpha-\beta}$ gives

$$s_3 = sx_{-\alpha}(a)x_{-\alpha-\beta}(b)x_{-3\alpha-2\beta}(e') \in O$$

and conjugation by a suitable element in $X_{-3\alpha-2\beta}$ gives

$$s_4 = sx_{-\alpha}(a)x_{-\alpha-\beta}(b) \in O \cap Bs_\alpha s_{\alpha+\beta}B$$

so $O$ is not quasi-spherical.

Let $G_s = H_2$. Conjugation of $s$ by $x_{-3\alpha-\beta}(1)x_{-\beta}(1)$ yields

$$s_1 = sx_{-\beta}(a)x_{-3\alpha-\beta}(b)x_{-3\alpha-2\beta}(c) \in O$$
for some \(a, b, c \in k\) with \(ab \neq 0\). Conjugation by a suitable element in \(X_{-3\alpha - 2\beta}\) gives

\[
s_2 = sx_{-\beta}(a)x_{-3\alpha - \beta}(b) \in \mathcal{O} \cap Bs_3 s_{3\alpha + \beta}B
\]
so \(\mathcal{O}\) is not quasi-spherical, concluding the analysis if \(\text{srk} \, G_s = 1\).

Let \(\text{srk} \, G_s = 2\) with \(s \neq 1\). Up to conjugation by an element in \(N(T)\) we may assume that \(G_s\) is either

\[
H_3 = \langle T, X_{\pm \beta}, X_{\pm (3\alpha + \beta)}, X_{\pm (3\alpha + 2\beta)} \rangle \quad \text{or} \quad H_4 = \langle T, X_{\pm \beta}, X_{\pm (2\alpha + \beta)} \rangle.
\]

If \(G_s = H_3\) of type \(A_2\) and \(u = 1\) then \(\mathcal{O}\) is spherical by [5, Proposition 6, Theorem 16] whose proofs hold in arbitrary good odd characteristic. Let us assume that \(u \neq 1\). If \(u\) is regular in \(H_3\) then \(\mathcal{O}\) is regular by [24, Corollary 3.7], hence it is not quasi-spherical. It remains to analyze the class of \(x = sx_{-\beta}(1)\). Conjugating by \(x_{-\alpha}(1)\) and reordering the terms gives

\[
x_1 = sx_{-\beta}(1)x_{-\alpha - \beta}(b)x_{-2\alpha - \beta}(c)x_{-3\alpha - \beta}(d)x_{-3\alpha - 2\beta}(e)x_{-\alpha}(f) \in \mathcal{O}
\]
for some \(b, c, d, e, f \in k\) with \(f \neq 0\). We can get rid of the term in \(X_{-\alpha - \beta}\) conjugating by a suitable element in \(X_{-\alpha - \beta}\). Then we can get rid of the term in \(X_{-2\alpha - \beta}\) conjugating by a suitable element in \(X_{-2\alpha - \beta}\) and, finally, we can get rid of the term in \(X_{-3\alpha - 2\beta}\) by conjugating by a suitable element in \(X_{-3\alpha - 2\beta}\) obtaining

\[
x_2 = sx_{-\beta}(1)x_{-3\alpha - \beta}(b_1)x_{-\alpha}(f) \in \mathcal{O}
\]
for some \(b_1 \in k\). If \(b_1 = 0\) then \(\mathcal{O} \cap Bs_3 s_{3\alpha}B \neq \emptyset\) so \(\mathcal{O}\) is not quasi-spherical.

If \(b_1 \neq 0\) we have, for some \(h \in T\) and some nonzero \(a_i \in k:\)

\[
x_2 = shx_\beta(a_1)n_\beta x_\beta(a_2)x_{3\alpha + \beta}(a_3)n_{3\alpha + \beta}x_{3\alpha + \beta}(a_4)x_{-\alpha}(f)
\]

\[
= shx_\beta(a_1)n_\beta x_{3\alpha + \beta}(a_3)x_\beta(a_2)x_{3\alpha + 2\beta}(a_5)n_{3\alpha + \beta}x_{3\alpha + \beta}(a_4)x_{-\alpha}(f)
\]

\[
\in T X_\beta X_{3\alpha + 2\beta} n_\beta n_{3\alpha + \beta} X_{3\alpha + 2\beta} X_\beta x_{-\alpha}(f)
\]

\[
\subset Bn_\beta n_{3\alpha + \beta} P_\alpha x_{-\alpha}(f) \subset Bn_\beta n_{3\alpha + \beta} x_{-\alpha}(f) \subset BX_{2\alpha + \beta} n_\beta n_{3\alpha + \beta} U
\]
so \(\mathcal{O} \cap Bs_3 s_{3\alpha + \beta}B \neq \emptyset\) and \(\mathcal{O}\) is not quasi-spherical.

Let \(G_s = H_4\) be of type \(A_1 \times \tilde{A}_1\). If \(u = 1\) then \(\mathcal{O}\) is spherical by the argument in [5, Theorem 16]. If \(u\) has nontrivial components both in \(A_1\) and in \(\tilde{A}_1\) then \(\mathcal{O}\) is regular, hence it is not quasi-spherical. We are left with the analysis of the classes of \(y = sx_{-\beta}(1)\) and \(z = sx_{-2\alpha - \beta}(1)\).

Conjugating \(y\) by \(x_{-3\alpha - \beta}(1)\) we get \(y_1 = sx_{-3\alpha - \beta}(a)x_{-\beta}(1)x_{-3\alpha - 2\beta}(b)\) for some \(a, b \in k\) with \(a \neq 0\). Conjugation by a suitable element in \(X_{-3\alpha - 2\beta}\) yields

\[
y_2 = sx_{-3\alpha - \beta}(a)x_{-\beta}(1) \in \mathcal{O} \cap Bs_{3\alpha + \beta} s_{3\beta} B
\]
hence $O_y$ is not quasi-spherical.

Conjugating $z$ by $x_-(1)$ we get $z_1 = sx_{-\alpha}(a)x_{-2\alpha-\beta}(1)x_{-3\alpha-\beta}(c)$ for some $a, c \in k$ with $a \neq 0$. Conjugating $z_1$ by a suitable element in $X_{-\alpha-\beta}$ we get the element $z_2 = sx_{-\alpha}(a)x_{-\alpha-\beta}(d)x_{-3\alpha-\beta}(c_1)x_{-3\alpha-2\beta}(c_2)$ for some $c_1, c_2, d \in k$ with $d \neq 0$. We can get rid of the term in $X_{-3\alpha-\beta}$ conjugating by a suitable element in $X_{-2\alpha-\beta}$ and then we can get rid of the term in $X_{-3\alpha-2\beta}$ conjugating by a suitable element in $X_{-3\alpha-2\beta}$.

Thus $z_3 = sx_-(1)x_{-\alpha-\beta}(d) \in O \cap B s_\alpha s_{\alpha+\beta} B$ hence $O_z$ is not quasi-spherical. This exhausts the list of conjugacy classes for $G$ of type $G_2$ and we have verified that all quasi-spherical conjugacy classes are spherical. □

3. Maximal $B$-orbits

Let $O$ be a conjugacy class of $G$. Since $O$ is an irreducible variety there exists a unique element in $W$ for which $O \cap B w B$ is dense in $O$. We shall denote this element by $z_O$. Denoting by $\overline{X}$ the Zarisky closure of $X$ in $Y$ we have

$$O \subset \overline{O}^G = \overline{O} \cap B z_O B^G \subset B z_O B^G = \bigcup_{\sigma \leq z_O} B \sigma B$$

so the element $z_O$ is maximal in the image of $\phi$ (cf. [5, Section 1]). We will call maximal orbits the elements $v$ in $V$ for which $\phi(v) = z_O$ and we shall denote by $V_{\text{max}}$ the set of maximal $B$-orbits in $O$.

**Lemma 3.1.** — The following are equivalent for a conjugacy class $O$ in $G$.

1. $O$ is spherical.
2. $V_{\text{max}}$ contains only one element.
3. $V_{\text{max}}$ is a finite set.

**Proof.** — It follows from [5, Corollary 26], [6, Corollary 4.11] that if $O$ is spherical then $V_{\text{max}}$ contains only one element, namely the dense $B$-orbit so 1 implies 2 and 2 trivially implies 3. Let us show that 3 implies 1. Since $\bigcup_{v \in V_{\text{max}}} v = O \cap B z_O B$ is dense in $O$ we have $O \subset \bigcup_{v \in V_{\text{max}}} v O$ with $O$ irreducible ([16, Proposition 1.5]) and $V_{\text{max}}$ a finite set. Then there necessarily exists $v_0 \in V_{\text{max}}$ which is dense in $O$. □

Let us analyze the maximal $B$-orbits in quasi-spherical conjugacy classes.

**Lemma 3.2.** — Let $O$ be a quasi-spherical conjugacy class with $w = z_O$. Let $v \in V_{\text{max}}$ and let $x = w v w \in v$ with $u \in U^w$, $w \in N(T)$ and $v \in U$. Then for every $\alpha \in \Delta$ such that $w s_\alpha > w$ in the Bruhat order we have:
(1) $s_{\alpha}w = ws_{\alpha} \text{ so } w\alpha = \alpha$;
(2) $v \in P^u_{\alpha}$, the unipotent radical of $P_{\alpha}$;
(3) $X_{\pm\alpha}$ commutes with $\dot{w}$.

Proof. — This is proved as [6, Lemma 3.4], since the proof therein uses only maximality of $w$ and that $O$ is quasi-spherical.

**Lemma 3.3.** — Let $O$ be a quasi-spherical conjugacy class with $w = z_O$, let $\Pi = \{\alpha \in \Delta \mid w(\alpha) = \alpha\}$ and let $w_{\Pi}$ be the longest element in $W_{\Pi}$. Then $w = w_{\Pi}w_0$.

Proof. — By Lemma 3.2 if $\alpha \in \Delta$ and $w\alpha \in \Phi^+$ then $w\alpha = \alpha$. The statement follows from [21, Proposition 3.5].

The Lemmas above show that maximal $B$-orbits in quasi-spherical conjugacy classes behave similarly to the dense $B$-orbit $v_0$ in a spherical conjugacy class. The analysis of $z_O$ given in [6] applies.

**Proposition 3.4.** — The following properties hold for a quasi-spherical conjugacy class $O$ with $w = z_O = w_0w_{\Pi}$.

1. $\Pi$ is invariant with respect to $\vartheta = -w_0$;
2. The restriction of $w_0$ to $\Phi(\Pi)$ coincides with $w_{\Pi}$;
3. $\Phi_w = \Phi^+ \setminus \Phi(\Pi)$, notation as in (1.1);
4. $U_w = \langle X_\gamma \mid \gamma \in \Phi(\Pi) \cap \Phi^+ \rangle$ and it normalizes $U^w$, notation as in (1.2);
5. $U_w$ commutes with $\dot{w}$ if $x = u\dot{w}v \in O \cap U^wN(T)U$.

Proof. — The proof is as in [6, Section 3].

In [6] an analysis of the possible $\Pi$ for which $\phi(v_0) = w_0w_{\Pi} = z_O$ for the dense $B$-orbit $v_0$ of a spherical conjugacy class in a simple algebraic group was given. The proof of [6, Lemma 4.1] can be adapted to the case of maximal $B$-orbits in quasi-spherical conjugacy classes, yielding the following statement.

**Lemma 3.5.** — Let $O$ be a quasi-spherical conjugacy class and let $w = w_0w_{\Pi} = z_O$. Let $\alpha$ and $\beta$ be simple roots with the following properties: $(\beta, \beta) = (\alpha, \alpha); w_0(\beta) = -\beta; \beta \perp \alpha; \beta \perp \alpha'$ for every $\alpha' \in \Pi \setminus \{\alpha\}$.

Then $\{\alpha\}$ cannot be a connected component of $\Pi$. In particular, the list of the possible subsets $\Pi$ for which $z_O = w_0w_{\Pi}$ for $G$ simple coincides with the list given in [6, Corollary 4.2].

Proof. — The proof follows as in [6, Lemma 4.1] since it only uses maximality of $w$ and that $O$ is quasi-spherical. There, the proof is given for $G$ simple but it holds for $G$ reductive, too.

**TOME 59 (2009), FASCICULE 6**
Let $O$ be quasi-spherical with $w = z_O = w_0 w_\Pi$ and let $\Phi_1 = \Phi \cap \text{Ker}(1 + w)$. Then $\Phi_1$ is a root subsystem of $\Phi$ and we put $\Phi_1^+ = \Phi^+ \cap \Phi_1$. If we write $w = \prod_j s_{\gamma_j}$ as a product of reflections with respect to mutually orthogonal roots then each $\gamma_j$ lies in $\Phi_1$. We shall denote by $W(\Phi_1)$ the subgroup of $W$ generated by reflections with respect to roots in $\Phi_1$, so $w \in W(\Phi_1)$.

**Lemma 3.6.** — Let notation be as above and let $\beta \in \Phi$. Then $\beta \in \Phi_1$ if and only if $\beta \perp \Pi$ and $\vartheta \beta = \beta$.

**Proof.** — If $\beta \perp \Pi$ and $\vartheta \beta = \beta$ then $w_\Pi \beta = \beta$ and $w_0 \beta = -\beta$ thus $\beta \in \Phi_1$.

Conversely, if $w\beta = -\beta$ then for every $\alpha \in \Pi$ we have $\beta \perp \alpha$ because $\alpha$ and $\beta$ lie in distinct eigenspaces of the orthogonal transformation $w$. Let now $\alpha \in \Phi$ and $w\beta = -\beta$. We have

$$(\vartheta \beta, \alpha) = -(w_0 \beta, \alpha) = -(w_\Pi \beta, \alpha) = -(w\beta, \alpha) = (\beta, \alpha)$$

and since this holds for every $\alpha$, we have the statement. \hfill \Box

Let us denote by $G(\Phi_1)$ the subgroup of $G$ generated by $T$ and the root subgroups $X_{\pm \beta}$ with $\beta \in \Phi_1$. Let $U_{\Phi_1} = \langle X_\beta, \beta \in \Phi_1^+ \rangle$.

The following Lemma is an analogue of [6, Lemma 4.8, Remark 4.9] for quasi-spherical conjugacy classes.

**Lemma 3.7.** — Let $O$ be a quasi-spherical conjugacy class and let $z_O = w_0 w_\Pi$. Let $\dot{w} \in N(T)$ be a representative of $w$. Then for every $x = \dot{w} tv \in \dot{w} B \cap O$ we have $v \in U_{\Phi_1}$, $w \in W(\Phi_1)$ and $x$ commutes with $(T^w)^o U_w$.

**Proof.** — The proof when $G$ is simple follows exactly as in [6, Lemmas 4.5, 4.6, 4.7, 4.8, 4.9]. Indeed, for their proofs we only need $w$ to be maximal, $O$ to be quasi-spherical and the list in [6, Corollary 4.2]. The general case is a consequence of the case of $G$ simple. \hfill \Box

**Lemma 3.8.** — Let $O$ be a quasi-spherical conjugacy class and let $w = w_0 w_\Pi = z_O$. Then $\langle X_{-\alpha}, \alpha \in \Pi \rangle$ commutes with every $x = \dot{w} tv \in \dot{w} B \cap O$.

**Proof.** — It is not restrictive to assume $G$ to be simple. By Lemmas 3.2 and 3.7 it is enough to show that $X_{-\alpha}$ commutes with $v$ for every $\alpha \in \Pi$. If this were not the case, by (1.3) there would occur in the expression of $v$ at least one root subgroup $X_\gamma$ with nontrivial coefficient and with $\gamma - \alpha \in \Phi$. We consider such a $\gamma$ of minimal height. By Lemma 3.7 and [2, Chapitre 6, §1.3] this could happen only if $\Phi$ is doubly-laced and $\alpha$ is a short root. Then we would also have $\alpha + \gamma \in \Phi$, which is impossible because $X_\alpha$ commutes with $v$ by Lemma 3.7. \hfill \Box

A consequence of Lemma 3.7 is the following result.

**ANNALES DE L’INSTITUT FOURIER**
Proposition 3.9. — Let $O$ be a quasi-spherical conjugacy class, let $w = z_O = w_0w_1$ and let $v \in V_{\text{max}}$. Then $\dim(v) = \ell(w) + \text{rk}(1 - w)$.

Proof. — Let $n$ be the rank of $G$ and let $x = \dot{w}v \in v$. By Lemma 3.7 the centralizer $B_x$ of $x$ in $B$ contains $(T^w)^{\circ}U_w$. On the other hand, if $b = u^w_{w_0}tw \in U^wU_wT$ commutes with $x$ we have
\[
\dot{w}vu^w_{w_0}tv = u^w_{w_0}tu \dot{w}v = u^w_{w_0}tu(\dot{w}^{-1}t\dot{w})v = u^w_{w_0}tu(\dot{w}^{-1}t\dot{w})v
\]
where for the last equality we used Lemma 3.2. By uniqueness in the Bruhat decomposition we have $u^w_{w_0} = 1$ so $B_x \subset T_xU_w$ because $U_w \subset B_x$. Moreover, if $t \in T_x$ we have
\[
\dot{w}(\dot{w}^{-1}t\dot{w})v = t\dot{w}v = \dot{w}vt \in \dot{w}TU
\]
and uniqueness of the decomposition in $TU$ gives $t \in T^w$. Therefore $(T^w)^{\circ}U_w \subset B_x \subset T^wU_w$ and $\dim v = |\Phi^+| + n - (|\Phi^+| - \ell(w)) - (n - \text{rk}(1 - w)) = \ell(w) + \text{rk}(1 - w)$. \hfill \qed

4. The case $z_O = w_0 = -1$

In this section $\Phi$ is such that $w_0$ acts as $-1$ in the geometric representation of $W$. If $O$ is a quasi-spherical conjugacy class intersecting the big Bruhat cell $Bw_0B$ then $\Phi_1 = \Phi$ and $\Pi = \emptyset$ so Lemma 3.7 gives no restriction to a representative $x = \dot{w}v \in O \cap \dot{w}U$. For this reason we use a different approach for such classes.

By Lemma 3.1 if a conjugacy class has finitely-many maximal $B$-orbits then it is spherical. The aim of this Section is to show that every quasi-spherical conjugacy class $O$ intersecting $Bw_0B$ has only finitely-many maximal $B$-orbits. This will be achieved by counting the possible representatives of a maximal $B$-orbit lying in $\dot{w}_0U$ for a fixed $\dot{w}_0 \in N(T)$. Next Lemma shows that every maximal $B$-orbit meets $\dot{w}_0U$.

Lemma 4.1. — Let $G$ be simple and let $O$ be a quasi-spherical conjugacy class with $z_O = w_0 = -1$. For any $v \in V_{\text{max}}$ and any representative $\dot{w}_0$ of $w_0$ in $N(T)$ we have $v \cap \dot{w}_0U \neq \emptyset$.

Proof. — Let $x = \dot{w}_0tv \in v \cap U\dot{w}_0B$. Then for every $s \in T$ we have $x_s = s^{-1}u^{-1}\dot{w}_0tvus = \dot{w}_0s^2tu' \in v \cap \dot{w}_0TU$ and since the map $s \mapsto s^2 \in T$ is onto ([1, III.8.9]) we may choose $s$ so that $x_s \in v \cap \dot{w}_0U$. \hfill \qed

Lemma 4.2. — Let $O$ be a quasi-spherical conjugacy class with $z_O = w_0 = -1$. Let $\dot{w}_0$ be a representative of $w_0$ and let $x = \dot{w}_0v \in O \cap \dot{w}_0U$,
with \( v = \prod_{\gamma \in \Phi^+} x_\gamma(c_\gamma) \) in a fixed ordering. Let \( \alpha \) and \( \beta \) be adjacent simple roots of the same length. Then the number of possibilities for \( c_\alpha \) and \( c_\beta \) is finite and \( c_{\alpha + \beta} \) is completely determined by the ordering, \( c_\alpha \) and \( c_\beta \).

Proof. — Let \( P = P_{(\alpha, \beta)} \) with unipotent radical \( P^u \). Let us assume that \( \alpha \) precedes \( \beta \) in the ordering. We have:

\[
x = \dot{w}_0 v \in \dot{w}_0 x_\alpha(c_\alpha)x_\beta(c_\beta)x_{\alpha + \beta}(c_{\alpha + \beta})P^u.
\]

For \( h \in k \) we put \( y(h) := n_\alpha x_\alpha(h)x_\alpha(-h)n_\alpha^{-1} \). Then, for some nonzero structure constants \( \theta_1, \theta_2, \theta_3, c_{\alpha \beta} \) and some \( t_1 \in T \) we have

\[
y(h) = n_\alpha \dot{w}_0 x_{-\alpha}(\theta_1 h)x_\alpha(c_\alpha - h)x_\alpha(h)x_\beta(c_\beta)x_{\alpha + \beta}(c_{\alpha + \beta})x_\alpha(-h)n_\alpha^{-1}P^u
\]

\[
= \dot{w}_0 t_1 x_\alpha(\theta_1 \theta_2 h)x_{-\alpha}(\theta_3(c_\alpha - h))n_\alpha x_\beta(c_\beta)x_{\alpha + \beta}(c_{\alpha + \beta} + hc_\beta c_{\alpha \beta})n_\alpha^{-1}P^u.
\]

Let \( h_1 \) and \( h_2 \) be the solutions of

\[
X^2(\theta_1 \theta_2 \theta_3) - c_\alpha \theta_1 \theta_2 \theta_3 X - 1 = 0
\]

so that \( -(\theta_1 \theta_2 h_i)^{-1} = (c_\alpha - h_i) \theta_3 \). By [22, Lemma 8.1.4 (i)] we have

\[
y(h_i) \in \dot{w}_0 t_1 n_\alpha t_2 x_{\beta + \alpha}(c_\beta')x_\beta(\theta_4(c_{\alpha + \beta} + h_i c_\beta c_{\alpha \beta}^{11}))P^u
\]

\[
\subset \dot{w}_0 n_\alpha t_3 x_\beta(\theta_4(c_{\alpha + \beta} + h_i c_\beta c_{\alpha \beta}^{11}))P^u \subset \mathcal{O} \cap Bw_0 s_{\alpha + \beta} B
\]

for some \( t_2, t_3 \in T \), some \( c_\beta' \in k \) and some nonzero structure constant \( \theta_4 \). Since \( w_0 s_{\alpha + \beta} = \alpha \in \Phi^+ \), conjugation of \( y(h_i) \) by \( n_\beta \) would yield an element in \( \mathcal{O} \cap Bw_0 s_{\alpha + \beta} s_{\beta} B \) unless

\[
(4.1) \quad c_{\alpha + \beta} + h_i c_\beta c_{\alpha \beta}^{11} = 0.
\]

As \( s_{\alpha + \beta} s_{\beta} \) is not an involution, (4.1) must hold for both \( i = 1, 2 \) thus we have either \( h_1 = h_2 \) so that

\[
(4.2) \quad \Delta_\alpha = \theta_1^2 \theta_2^2 \theta_3^2 c_\alpha^2 + 4 \theta_1 \theta_2 \theta_3 = 0, \quad \text{or}
\]

\[
(4.3) \quad c_\beta = c_{\alpha + \beta} = 0.
\]

Let us now consider, for \( l \in k \), the element

\[
z(l) = n_\beta x_\beta(l)x_\beta(-l)n_\beta^{-1}
\]

\[
\in n_\beta x_\beta(l)\dot{w}_0 x_\beta(c_\beta)x_\alpha(c_\alpha)x_{\alpha + \beta}(c_{\alpha + \beta} + c_\alpha c_\beta c_{\alpha \beta}^{11})x_\beta(-l)n_\beta^{-1}P^u.
\]

Repeating the same argument for \( \beta \) we see that there are nonzero structure constants \( \eta_1, \eta_2, \eta_3, \eta_4 \) so that if \( l_j \) is a solution of

\[
\eta_1 \eta_2 \eta_3 X^2 - c_\beta \eta_1 \eta_2 \eta_3 X - 1 = 0
\]

then

\[
z(l_j) \in \mathcal{O} \cap \dot{w}_0 n_\beta Tx_\alpha(\eta_4(c_{\alpha + \beta} + c_\alpha c_\beta c_{\alpha \beta}^{11} - l_j c_\alpha c_{\alpha \beta}^{11}))P^u
\]
so conjugation by $n_\alpha$ would yield an element in $\mathcal{O} \cap Bw_0 s_{\alpha + \beta} s_\alpha B$ unless
\begin{equation}
(4.4) \quad c_{\alpha + \beta} + c_\alpha c_\beta c_{\alpha \beta}^{11} - l_j c_\alpha c_{\alpha \beta}^{11} = 0
\end{equation}
for both $j = 1, 2$. This forces either $l_1 = l_2$ and therefore
\begin{equation}
(4.5) \quad \Delta_\beta = \eta_1^2 \eta_2^2 \eta_3^2 c_\beta^2 + 4 \eta_1 \theta_2 \theta_3 = 0, \quad \text{or}
\end{equation}
\begin{equation}
(4.6) \quad c_\alpha = c_{\alpha + \beta} = 0.
\end{equation}
If (4.2) holds then $c_\alpha \neq 0$ so (4.5) holds. Then the possibilities for $c_\alpha$ and $c_\beta$ are finite. Besides, by (4.4) the coefficient $c_{\alpha + \beta}$ is completely determined by $c_\alpha$, $c_\beta$, and the structure constants.

If (4.3) holds then (4.5) cannot hold so $c_\alpha = c_\beta = c_{\alpha + \beta} = 0$, whence the statement. \hfill \Box

\textbf{Lemma 4.3.} — Let $\mathcal{O}$, $z_\mathcal{O}$, $x$, $v$ be as in Lemma 4.2. Let $\alpha$ and $\beta$ be adjacent simple roots with $2(\alpha, \alpha) = (\beta, \beta)$. Then $c_{2\alpha + \beta}$ and $c_{\alpha + \beta}$ are completely determined by the ordering, $c_\alpha$ and $c_\beta$.

\textbf{Proof.} — It is not restrictive to assume that $\alpha$ precedes $\beta$ and $\alpha + \beta$ in the ordering. Let $P = P_{\{\alpha, \beta\}}$ and $P^u$ be its unipotent radical. Then
\begin{equation*}
x \in \hat{w}_0 x_\alpha(c_\alpha)x_\beta(c_\beta)x_{\alpha + \beta}(c_{\alpha + \beta})x_{2\alpha + \beta}(c_{2\alpha + \beta})P^u.
\end{equation*}

Conjugation by $n_\alpha x_\alpha(h)$ for $h \in k$ yields an element $y(h)$ lying in
\begin{equation*}
\hat{w}_0 t_1 x_\alpha(\eta_1 h)x_{\alpha - \alpha}(\eta_2(c_\alpha - h))x_{\beta}(\eta_3(c_{2\alpha + \beta} + h c_{\alpha + \beta} c_{\alpha,\alpha + \beta}^{11} + c_\beta h^2 c_{\alpha,\beta}^{21})^2)P^u
\end{equation*}
for some $t_1 \in T$ and some nonzero structure constants $\eta_1, \eta_2, \eta_3$. If $h_1, h_2$ are the solutions of
\begin{equation*}
\eta_1 \eta_2 X^2 - c_\alpha \eta_1 \eta_2 X - 1 = 0
\end{equation*}
then $y(h_1)$, $y(h_2)$ lie in $\mathcal{O} \cap Bw_0 s_\alpha B$ and $n_\beta y(h_1)n_\beta^{-1} \in \mathcal{O} \cap Bw_0 s_{\alpha + \beta} B$ unless
\begin{equation}
(4.7) \quad c_{2\alpha + \beta} + h_1 c_{\alpha + \beta} c_{\alpha,\alpha + \beta}^{11} + c_\beta h_1^2 c_{\alpha,\beta}^{21} = 0
\end{equation}
for both $i = 1, 2$. Besides, $h_1 + h_2 = c_\alpha$ and $h_1 h_2 = -(\eta_1 \eta_2)^{-1}$. Thus we have either
\begin{equation}
(4.8) \quad \Delta_\alpha = (\eta_1 \eta_2 c_\alpha)^2 + 4 \eta_1 \eta_2 = 0 \quad \text{and} \quad c_{2\alpha + \beta} = a c_\alpha c_{\alpha + \beta} + b c_\beta c_\alpha^2
\end{equation}
\begin{equation}
(4.9) \quad \text{or} \quad c_{2\alpha + \beta} = c_\beta \quad \text{and} \quad c_{\alpha + \beta} = d c_\beta c_\alpha
\end{equation}
for $a, b, c, d \in k$. On the other hand, reordering terms we have:
\begin{equation*}
x \in \hat{w}_0 x_\beta(c_\beta)x_\alpha(c_\alpha)x_{\alpha + \beta}(c_{\alpha + \beta} + c_\alpha c_\beta c_{\alpha \beta}^{11})x_{2\alpha + \beta}(c_{2\alpha + \beta} + c_\alpha c_\beta c_{\alpha \beta}^{11}+ l c_\alpha c_{\alpha \beta}^{11})P^u.
\end{equation*}

Conjugation by $n_\beta x_\beta(l)$ for $l \in k$ gives an element
\begin{equation*}
z(l) \in \hat{w}_0 t_2 x_\beta(\theta_1 h)x_{\beta}(\theta_2(c_\beta - l))x_{\alpha}(\theta_3(c_{\alpha + \beta} + c_\alpha c_\beta c_{\alpha \beta}^{11} - l c_\alpha c_{\alpha \beta}^{11}))P^u.
\end{equation*}
for some $t_2 \in T$ and some nonzero structure constants $\theta_1, \theta_2, \theta_3, c_{\alpha\beta}^{11}$. For the solutions $l_1, l_2$ of

$$\theta_1 \theta_2 X^2 - c_\beta \theta_1 \theta_2 X - 1 = 0$$

the corresponding elements $z(l_1), z(l_2)$ lie in $\mathcal{O} \cap B w_0 s_\beta B$ and $n_\alpha z(l_i) n_\alpha^{-1} \in B w_0 s_{2\alpha + \beta} s_\alpha B$ unless

$$(4.10) \quad c_{\alpha + \beta} + c_\alpha c_\beta c_{\alpha \beta}^{11} - l_i c_\alpha c_{\alpha \beta}^{11} = 0$$

for $i = 1, 2$. It follows that we have either

$$(4.11) \quad \Delta_\beta = (\theta_1 \theta_2 c_\beta)^2 + 4 \theta_1 \theta_2 = 0 \quad \text{and} \quad c_{\alpha + \beta} = d c_\alpha c_\beta$$

$$(4.12) \quad \text{or} \quad c_{\alpha + \beta} = c_\alpha = 0.$$ 

Arguing as in Lemma 4.2 we see that $c_{\alpha + \beta}$ and $c_{2\alpha + \beta}$ are completely determined by the ordering, $c_\alpha$ and $c_\beta$. □

**Lemma 4.4.** — Let $\Phi$ be a simply- or doubly-laced root system for which $w_0 = -1$. Let $\mathcal{O}$, $z_\mathcal{O}$, $x$, $v$ be as in Lemma 4.2. Then, for every $\gamma = \sum_{j \in J} n_j \alpha_j \in \Phi^+$ with $J \subset \{1, \ldots, n\}$ there is a polynomial $p_\gamma(X) \in k[x_j, j \in J]$ depending only on the fixed ordering of the positive roots and the structure constants of $G$ such that the coefficient $c_\gamma$ in the expression of $v$ is the evaluation of $p_\gamma(X)$ at $x_j = c_{\alpha_j}$ for every $j \in J$.

**Proof.** — We shall proceed by induction on the height $h$ of the root $\gamma$. Let us assume that the claim holds for all $\gamma$ with $ht(\gamma) \leq m - 1$. By Lemmas 4.2 and 4.3 the statement holds for $m = 1, 2$ so we assume $m \geq 3$.

Let $\nu \in \Phi^+$ with $ht(\nu) = m$. Then there exists $\beta \in \Delta$ for which $ht(s_\beta \nu) \leq m - 1$. We put

$$(4.13) \quad y = n_\beta x n_\beta^{-1} = \dot{w}_0 t \prod_{\gamma \in \Phi^+} x_{s_\beta \gamma}(\theta_\gamma c_\gamma)$$

for some nonzero structure constants $\theta_\gamma$. Here the products have to be intended in the fixed ordering of the $\gamma$’s. We have:

$$y = \dot{w}_0 t \left( \prod_{\gamma <_\alpha \beta} x_{s_\beta \gamma}(\theta_\gamma c_\gamma) x_{-\beta}(\theta_\beta c_\beta) \prod_{\gamma >_\alpha \beta} x_{s_\beta \gamma}(\theta_\gamma c_\gamma) \right)$$

where $<_\alpha$ indicates that a root precedes another in the fixed ordering and the expression makes sense also if $c_\beta = 0$. Then

$$y = \dot{w}_0 t x_{-\beta}(\theta_\beta c_\beta) x_{-\beta}(-\theta_\beta c_\beta) \prod_{\gamma <_\alpha \beta} x_{s_\beta \gamma}(\theta_\gamma c_\gamma) x_{-\beta}(\theta_\beta c_\beta) \prod_{\gamma >_\alpha \beta} x_{s_\beta \gamma}(\theta_\gamma c_\gamma) = x_\beta(\eta c_\beta) \dot{w}_0 t x_{-\beta}(-\theta_\beta c_\beta) \prod_{\gamma <_\alpha \beta} x_{s_\beta \gamma}(\theta_\gamma c_\gamma) x_{-\beta}(\theta_\beta c_\beta) \prod_{\gamma >_\alpha \beta} x_{s_\beta \gamma}(\theta_\gamma c_\gamma)$$
for some nonzero structure constant $\eta$. Conjugation by $x_\beta(-\eta c_\beta)$ yields
\[
    z = \dot{w}_0 t(x_{-\beta}(-\theta_\beta c_\beta)( \prod_{\gamma < \alpha} x_{s_\beta \gamma}(\theta_\gamma c_\gamma)x_{-\beta}(\theta_\beta c_\beta))( \prod_{\gamma > \alpha} x_{s_\beta \gamma}(\theta_\gamma c_\gamma))x_\beta(\eta c_\beta) \\
    = \dot{w}_0 t \prod_{\gamma \in \Phi^+} x_\gamma(d_\gamma) \in \dot{w}_0 t U \cap \mathcal{O}
\]
where the last equality indicates reordering of root subgroups. By the induction hypothesis applied to $z$ and $s_\beta \nu$, the coefficient $d_\beta \nu$ is evaluation at the $d_\alpha$ for $\alpha$ in the support of $s_\beta \nu$ of a polynomial $q_{s_\beta \nu}(X)$. Besides, each $d_\mu$ differs from $\theta_{s_\beta \mu} c_{s_\beta \mu}$ by a (possibly trivial) sum of monomials in the $\theta_{\mu'} c_{\mu'}$, $c_\beta$ and the structure constants coming from application of (1.3) when reordering root subgroups. More precisely, we have
\[
    d_\mu = \theta_{s_\beta \mu} c_{s_\beta \mu} + \sum \star(\prod_{l=1}^{p} c_{s_\beta \mu} i_l) c_{s_\beta \mu}^j
\]
where $\star$ denotes a coefficient depending on the structure constants and the sum is taken over the possible decompositions $\mu = \sum_{l=1}^{p} i_l s_\beta \nu_l + j \beta$ for $i_l > 0$ and $j \geq 0$. In particular, if $\mu$ is simple there is no such decomposition: in this case $d_\mu = \theta_{s_\beta \mu} c_{s_\beta \mu}$ and by Lemmas 4.2 and 4.3 the coefficient $c_{s_\beta \mu}$ is evaluation of a polynomial at the $c_\alpha$ for $\alpha$ in the support of $s_\beta \mu$, and such support is contained in \{\beta, \mu\}. Thus, by the induction hypothesis $d_\beta \nu$ is evaluation of a polynomial $q(X)$ at the $c_\alpha$ for $\alpha$ in the support of $\nu$. We wish to prove that the same holds for $c_{\nu_l}$. Contribution to $d_\beta \nu$ as in (4.14) may occur when
\[
    s_\beta \nu = \sum_{l=1}^{p} i_l s_\beta \nu_l + j \beta
\]
for $i_l > 0$ and $j \geq 0$. Then $\text{ht}(s_\beta \nu_l) < \text{ht}(s_\beta \nu) \leq m - 1$. We wish to show that $\text{ht}(\nu_l) \leq m - 1$ for every $l$ so we may apply the induction hypothesis to $c_{\nu_l}$. Suppose that there is a decomposition (4.15) and an $l$ for which $\text{ht}(\nu_l) \geq m$ for some $l$. Since
\[
    m \leq \text{ht}(\nu_l) \leq \text{ht}(s_\beta \nu_l) + 2 \leq \text{ht}(s_\beta \nu) - 1 + 2 \leq m
\]
we would necessarily have $\text{ht}(\nu) = \text{ht}(\nu_l) = m$; $\text{ht}(s_\beta \nu_l) = m - 2$; $\text{ht}(s_\beta \nu) = m - 1$ thus $s_\beta \nu = s_\beta \nu_l + \alpha$ for some $\alpha \in \Delta$. Applying $s_\beta$ to this equality we would have $\nu = \nu_l + s_\beta \alpha$ contradicting $\text{ht}(\nu) = \text{ht}(\nu_l)$.

Thus induction applies and
\[
    c_\nu = \star q(c_\alpha)_{\alpha \in \text{supp}(\nu)} + \sum \star(\prod_{l=1}^{p} (p_{\nu_l}(c_\alpha)_{\alpha \in \text{supp}(\nu_l)}) i_l c_{s_\beta \mu}^j
\]

TOME 59 (2009), FASCICULE 6
Remark 4.5. — The proof of Lemma 4.4 can be adapted to show that if $\Phi$ is simply-laced and $v \in P^u_\alpha$ for some $\alpha \in \Delta$ then $v = 1$ so $x = \tilde{w}_0$ and $O$ is a symmetric space.

**Proposition 4.6.** — Let $\Phi$ be irreducible, simply-laced, with $w_0 = -1$ and let $O$ be a quasi-spherical conjugacy class with $z_O = w_0$. Then $O$ is spherical.

**Proof.** — If $\Phi = \{\pm \alpha\}$ is of type $A_1$ the statement follows from even dimensionality of conjugacy classes ([20, Prop. 4.3]). Indeed, $\dim O \leq 2$ and for a representative $x$ of a maximal $B$-orbit in $O$ we have $\dim B.x = \ell(s_\alpha) + \rk(1 - s_\alpha) = 2$ by Proposition 3.9. Thus, $B.x$ is dense in $O$.

We assume now that the rank of $\Phi$ is at least two. By Lemma 4.1 every $v \in V_{\max}$ meets $\tilde{w}_0 U$ for every choice of $\tilde{w}_0$ and by Lemmas 4.2 and 4.4 there is only a finite number of elements in $O \cap \tilde{w}_0 U$. We conclude using Lemma 3.1. □

**Proposition 4.7.** — Let $G$ be simple of type $F_4$ and let $O$ be a quasi-spherical conjugacy class with $z_O = w_0$. Then $O$ is spherical.

**Proof.** — By Lemma 3.1 we need to show that there are only finitely many maximal $B$-orbits. By Lemma 4.1 it is enough to show that there are only finitely many elements in $\tilde{w}_0 U \cap O$ for a fixed $\tilde{w}_0 \in N(T)$. By Lemma 4.4 it is enough to show that there are only finitely many possibilities for $c_\alpha$ for $\alpha \in \Delta$. Applying Lemma 4.2 to the pair $\alpha_1, \alpha_2$ we see that for $x = \tilde{w}_0 v \in O \cap \tilde{w}_0 U$ there is a finite number of possibilities for the coefficients $c_{\alpha_1}$ and $c_{\alpha_2}$ in $v$. Applying Lemma 4.2 to the pair $\alpha_3, \alpha_4$ we see that there is a finite number of possibilities for $c_{\alpha_3}$ and $c_{\alpha_4}$, concluding the proof. □

**Proposition 4.8.** — Let $\Phi$ be irreducible of type $C_n$ and let $O$ be a quasi-spherical conjugacy class with $z_O = w_0$. Then $O$ is spherical.

**Proof.** — If $n = 2$ by Proposition 3.9 we have $6 = \dim B = \dim v$ for every maximal $B$-orbit $v$. On the other hand $\dim O < 2|\Phi^+| = 8$ because $O$ cannot be regular (see Remark 2.2). It follows from even dimensionality of conjugacy classes ([20, Prop. 4.3]) that $\dim O = \dim v$ so $v$ is dense and $O$ is spherical.

Let us now assume that $n \geq 3$. Let $\tilde{w}_0 \in N(T)$ be fixed and let $x = \tilde{w}_0 v = \tilde{w}_0 \prod_{\gamma \in \Phi^+} x_\gamma(c_{\gamma})$ be as in Lemma 4.2. By Lemmas 3.1, 4.1 and 4.4 it is enough to prove that there is a finite number of possibilities for $c_\alpha$.
for $\alpha \in \Delta$. By Lemma 4.2 we have either $\Delta_{\alpha_i} = 0$ for $i = 1, \ldots, n - 1$ or $c_{\alpha_i} = 0$ for $i = 1, \ldots, n - 1$. In the first case, Lemma 4.3 with $\alpha = \alpha_{n-1}$ and $\beta = \alpha_n$ gives $\Delta_{\alpha_n} = 0$ so there are finitely many possibilities for all $c_{\gamma}$. We shall thus focus on the case $c_{\alpha_i} = 0$ for $i \leq n - 1$. Then Lemma 4.3 with $\alpha = \alpha_{n-1}$ and $\beta = \alpha_n$ gives $c_{\alpha+\beta} = 0$. We claim that $c_{\gamma} = 0$ for every short root. We proceed by induction as in Lemma 4.4 and we look at the possible contribution as in (4.14) to $c_{\nu}$ with $\nu = s_{\beta} \mu$ and $\text{ht}(\mu) < \text{ht}(\nu)$. This would correspond to a decomposition of the short root $s_{\beta} \nu = \sum i_j s_{\beta} \nu_j + i \beta$ with $i_j > 0$ and $i \geq 0$. If $i > 0$ we have nontrivial contribution only if $\beta$ is a long root, for $c_{\beta} = 0$ if $\beta$ is short. Thus, both for $i = 0$ and $i > 0$ there is at least one $\nu_j$ which is short and then $\text{ht}(s_{\beta} \nu_j) \leq \text{ht}(s_{\beta} \nu) - 1 = m - 2$ so $\text{ht}(\nu_j) \leq m - 2 + 1$. By the induction hypothesis $c_{\nu_j} = 0$ and there is no contribution coming from this decomposition, so the claim is proved.

In other words, putting $\gamma_n = \alpha_n$ and $\gamma_i = s_i s_{i+1} \cdots s_{n-1} \alpha_n$ for $i = 1, \ldots, n - 1$ we have $x = \hat{w}_0 \prod_{i=1}^n x_{\gamma_i}(a_i)$ for some $a_i \in k$. We claim that there can be only finitely many elements of this type in a fixed class $O$. It is not restrictive to assume that $G = Sp_{2n}(k)$. Then, $G$ is the subgroup of $GL_{2n}(k)$ of matrices preserving the bilinear form associated with the matrix $egin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ with respect to the canonical basis of $k^{2n}$. We choose $B$ as the subgroup of $G$ of matrices of the form $egin{pmatrix} X & XA \\ 0 & X^{-1} \end{pmatrix}$ where $X$ is an invertible upper triangular matrix, $X^{-1}$ is its inverse transpose and $A$ is a symmetric matrix. Then the computations above translate into:

$$x = x(D, A) = \hat{w}_0 v = \begin{pmatrix} 0 & D \\ -D^{-1} & 0 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & D \\ -D^{-1} & -D^{-1} A \end{pmatrix}$$

for some diagonal matrices $D$ and $A$ with $D$ fixed and invertible. It is immediate to verify that for two diagonal matrices $A$ and $A'$, the characteristic polynomials of $x(D, A)$ and $x(D, A')$ are the same only if $D^{-1} A$ and $D^{-1} A'$ coincide up to a permutation of the diagonal entries. Therefore there are only finitely many matrices of the form $x(A, D)$ in a single conjugacy class $O$. Since by Lemma 4.1 each maximal $B$-orbit in $O$ contains some $x(A, D)$ or some of the finitely many representatives with $\Delta_{\alpha_j} = 0$ for every $j$, there are only finitely many maximal $B$-orbits in $O$ and we may conclude by using Lemma 3.1.

**Proposition 4.9.** — Let $\Phi$ be irreducible of type $B_n$ and let $O$ be a quasi-spherical conjugacy class with $z_O = w_0$. Then $O$ is spherical.

TOME 59 (2009), FASCICULE 6
Proof. — The case \( n = 2 \) is dealt with in Proposition 4.8 so we may assume \( n \geq 3 \). Let \( \dot{w}_0 \in N(T) \) be fixed and let \( x = \dot{w}_0v \) be as in Lemma 4.2. We shall show that there is a finite number of possibilities for \( c_\alpha \) for \( \alpha \in \Delta \). It follows from Lemma 4.2 that we have either \( c_\alpha = 0 \) for every long simple root \( \alpha \) or \( \Delta_\alpha = 0 \) for every long simple root \( \alpha \). In the first case, Lemma 4.3 with \( \alpha = \alpha_n \) and \( \beta = \alpha_{n-1} \) shows that (4.11) cannot be satisfied so \( c_{\alpha_n} = 0 \) as well. Hence, there is no freedom for the \( c_\alpha \) in this case and we shall focus on the second case. Let \( \Delta' = \{ \alpha_1, \ldots, \alpha_{n-1} \} \) and \( P = P_{\Delta'} \). Then \( x = \dot{w}_0v_1v_2 \) with \( v_1 \in \langle X_\alpha, \alpha \in \Delta' \rangle \) and \( v_2 \in Pu \) and we might assume that the fixed ordering of the roots is compatible with this decomposition. By Lemma 4.4 the factor \( v_1 \) is completely determined by the \( c_\alpha \) for \( \alpha \in \Delta' \) so there are finitely many possibilities for it. If there were infinitely many elements in \( O \cap \dot{w}_0v \), there would be infinitely many elements in \( O \cap \dot{w}_0v_1Pu \) for some \( v_1 \). We shall show that this cannot be the case. It is not restrictive to assume that \( G = SO_{2n+1}(k) \). We describe \( G \) as the subgroup of \( SL_{2n+1}(k) \) of matrices preserving the bilinear form associated with \[
    \begin{pmatrix}
      1 & 0 & 0 \\
      0 & 0 & I
    \end{pmatrix}
\]
with respect to the canonical basis of \( k^{2n+1} \). We may choose \( B \) to be the subgroup of matrices of the form
\[
    \begin{pmatrix}
      1 & 0 & t\gamma \\
      -X\gamma & X & XA \\
      0 & 0 & tX^{-1}
    \end{pmatrix}
\]
where \( X \) is an invertible \( n \times n \) upper triangular matrix, \( tX^{-1} \) is its inverse transpose, \( \gamma \) is a column in \( k^n \), \( t\gamma \) is its transpose and the symmetric part of \( A \) is \(-2^{-1} \gamma t\gamma \). The above discussion and Lemma 4.1 translate into the assumption that there would be infinitely many conjugate matrices of the form
\[
    x(V, \lambda) = \dot{w}_0v_1v_2 = \begin{pmatrix}
      (1)^n & 0 & 0 \\
      0 & 0 & I \\
      0 & I & 0
    \end{pmatrix}
    \begin{pmatrix}
      1 & 0 & 0 \\
      0 & V & 0 \\
      0 & 0 & tV^{-1}
    \end{pmatrix}
    \begin{pmatrix}
      1 & 0 & t\gamma \\
      -\gamma & I & A \\
      0 & 0 & I
    \end{pmatrix}
\]
where: \( \gamma \) is a vector in \( k^n \); \( V \) is a fixed upper triangular unipotent matrix; \( A \) is a matrix whose symmetric part is \(-2^{-1} \gamma t\gamma \) and by Lemma 4.4 the coefficients of \( A \) and \( \gamma \) depend polynomially on \( \lambda = \gamma_n \) and the coefficients of \( V \). The characteristic polynomial \( q_\lambda(T) \) of \( x(V, \lambda) \) depends polynomially on \( \lambda \) thus \( q_\lambda(T) = q_\mu(T) \) for at most finitely many \( \mu \) in \( k \) unless \( q_\lambda(T) \) is independent of \( \lambda \). We claim that this is not the case. In order to prove this,
we need a more explicit description of $V$.

Using Lemma 4.2 one can show that, up to conjugation in $SO_{2n+1}(k)$ by diagonal matrices of type $\text{diag}(1, -I_j, I_{n-j}, -I_j, I_{n-j})$ the matrix $V$ is an upper triangular unipotent matrix with all 2’s in the first off-diagonal. Inductively as in Lemma 4.4, using Lemma 4.2 one sees that $V$ is the upper triangular unipotent matrix with only 2’s above the diagonal. Thus it is enough to exhibit two matrices $x_1$ and $x_2$ of shape

$$\begin{pmatrix}
(-1)^n & 0 & \cdots & \cdots & t \\
0 & 0 & V^{-1} & & \gamma \\
-V & V & M
\end{pmatrix}$$

with $V$ as above, lying in quasi-spherical conjugacy classes and with distinct characteristic polynomials.

Let $n$ be even. For $\zeta$ a square root of 2 in $k$ we take

$$\gamma_1 = 2\zeta(1,-1,1,\ldots,-1) \text{ and } M_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & -A_1 & 0 \\
0 & 0 & -t(A_1)
\end{pmatrix}$$

where $A_1$ is the upper triangular unipotent matrix with

$$\begin{pmatrix}
1 & 0 & 0 \\
0 & A & 0 \\
0 & 0 & -tA
\end{pmatrix}$$

on the first upper off-diagonal and zero elsewhere. Using the Jordan decomposition of $a_1$ and the formulas in [8, §13.1] which hold in good characteristic by [16, Theorem 7.8] we see that the dimension of the conjugacy class $O_{a_1}$ of $a_1$ is $n^2 + n = \ell(w_0) + \text{rk}(1-w_0)$. Since $x_1$ lies in $\tilde{w}_0U$ we deduce from [5, Theorem 5], [6, Corollary 4.10] that $O_{a_1}$ is spherical, hence quasi-spherical. Therefore $x = x_1(V, 2\zeta)$. Let us now consider

$$M_2 = \begin{pmatrix}
4 & -2 & 2 & -2 & \cdots \\
2 & 0 & 2 & -2 & \cdots \\
\cdots & 2 & -2 & \cdots & \cdots \\
2 & \cdots & \cdots & \cdots & 0
\end{pmatrix}.$$
hence quasi-spherical. It follows that $x_2 = x(V, 0)$ and the characteristic polynomials of $x_1$ and $x_2$ are different.

Let now $n$ be odd and let $\xi$ be a square root of $-2$ in $k$. We may consider $\gamma_3 = -2\xi(1, -1, 1, \ldots, -1, 1)$ and $M_3$, constructed as $M_2$, and the corresponding matrix $x_3$. One verifies that $x_3$ is unipotent and lies in the conjugacy class associated with the Young diagram $(3, 2^{n-1})$, whose dimension is $n^2 + n$. As above, this class is spherical, hence quasi-spherical so $x_3 = x(V, -2\xi)$. On the other hand, taking $\gamma = 0$ and $M_4$ constructed as $M_1$ we get a matrix $x_4$ which is conjugate to

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & -A_2 & 0 \\
0 & 0 & -tA_2
\end{pmatrix}
$$

where $A_2$ is the upper triangular unipotent matrix with $(1, 0, \ldots, 1, 0)$ on the first upper off-diagonal and zero elsewhere. As for $n$ even we see that the dimension of $O_{x_4}$ is $n^2 + n$ and since $x_4$ lies in $w_0U$ we deduce as above that $O_{x_4}$ is spherical, hence quasi-spherical. Thus, $x_4 = x(V, 0)$ and the characteristic polynomials of $x_3$ and $x_4$ are distinct. It follows that in a fixed class $O$ there can only be finitely many elements of type $x(V, \lambda)$. By Lemma 4.1 each maximal $B$-orbit contains an element of type $x(V, \lambda)$ or a representative with all $c_\alpha = 0$, so there are only finitely many of them. We may conclude using Lemma 3.1. $\square$

**Remark 4.10.** — If $H$ is a connected reductive algebraic group the radical $R(H)$ of $H$ is a central torus ([22, Proposition 7.3.1]) contained in all Borel subgroups of $H$. Thus, a conjugacy class $O$ in $H$ is spherical (resp. quasi-spherical) if and only if its projection into the semisimple group $K = H/R(H)$ is spherical (resp. quasi-spherical). Moreover, a conjugacy class in $K$ is spherical (resp. quasi-spherical) if and only if its projection into each simple factor of $K$ is spherical (resp. quasi-spherical). Thus, the results we obtained so far apply also to the case of reductive groups. In particular, if $G$ is connected, reductive with $w_0 = -1$ and $O$ is quasi-spherical in $G$ with $z_O = w_0$ then $O$ is spherical.

5. Curves in the centralizer

In this section we aim at the understanding of $G_x$ for $x \in \mathfrak{w}U \cap O$ with $O$ quasi-spherical and $w = z_O$. We shall focus on searching suitable families of elements in $G_x$ by making use of the particular form of the chosen representative $x$ guaranteed by Lemma 3.7. By Lemma 3.8 if $w\alpha = \alpha$ then $X_{-\alpha} \subset X_\alpha s_\alpha B \cap G_x$. Now we aim at finding elements in $X_\gamma s_\gamma B \cap G_x$ for
the remaining roots $\gamma \in \Phi^+$, namely those such that $w\gamma \in -\Phi^+$. We shall first analyze those $\gamma$ for which $w\gamma = -\gamma$, that is, $\gamma \in \Phi_1$, by looking at $G_x \cap G(\Phi_1) = G(\Phi_1)x$. By Lemma 3.7, $x \in G(\Phi_1)$ so since $w$ is the longest element in $W(\Phi_1)$ we may use the results obtained in §4.

**Lemma 5.1.** — Let $\mathcal{O}$ be a quasi-spherical conjugacy class in $G$, let $w = z_\mathcal{O}$ and let $x \in \dot{w}U \cap \mathcal{O}$. Let $\alpha \in \Phi_1 \cap \Delta$. Then for all but finitely many $c \in k$ there exists $b_c \in B$ such that $x_\alpha(c)n_\alpha b_c \in G_x \cap G(\Phi_1)$.

**Proof.** — We have $x \in G(\Phi_1)$ by Lemma 3.7. The element $w$ is the longest element in $W(\Phi_1)$ and its restriction to $W(\Phi_1)$ is $-1$. Let us consider the conjugacy class $\mathcal{O}'$ of $x$ in $G(\Phi_1)$. It is quasi-spherical because $B_1 = B \cap G(\Phi_1)$ is a Borel subgroup of $G(\Phi_1)$ containing $T$. Therefore $\mathcal{O}'$ is spherical in $G(\Phi_1)$ by Remark 4.10.

Let $P$ be the minimal parabolic subgroup of $G(\Phi_1)$ associated with $\alpha$ and let $P^u$ be its unipotent radical. Let $x = \dot{w}x_\alpha(a)v$ with $v \in P^u$ and, for any nonzero $c \in k$, let $y_c = n_\alpha^{-1}x_\alpha(-c)x_\alpha(c)n_\alpha \in \mathcal{O}'$. We have, for some nonzero structure constants $\theta_1$, $\theta_2$, $\theta_3$ and for $t_1$, $t_2 \in T$:

$$y_c \in n_\alpha^{-1}(\dot{w}x_\alpha(-\theta_1c)x_\alpha(a+c)P^u)n_\alpha^{-1}$$

$$= \dot{w}t_1x_\alpha(-\theta_2\theta_1c)x_\alpha(\theta_3(a+c))P^u$$

$$= \dot{w}t_1x_\alpha(-\theta_2\theta_1c)x_\alpha((\theta_2\theta_1c)^{-1})x_\alpha(-(\theta_2\theta_1c)^{-1} + \theta_3(a+c))P^u$$

$$= \dot{w}n_\alpha t_2x_\alpha(\theta_2\theta_1c)x_\alpha(-\theta_2\theta_1c)^{-1} + \theta_3(a+c))P^u$$

where for the last equality we have used [22, Lemma 8.1.4 (i)]. Then, if

$$-(\theta_2\theta_1c)^{-1} + \theta_3(a+c) \neq 0$$

we have $y_c \in B_1ws_\alpha B_1s_\alpha B_1 = B_1wB_1$ because $ws_\alpha < w$ in the Bruhat ordering. Thus, for all but finitely many $c \in k$ the element $y_c$ lies in $B_1wB_1 \cap \mathcal{O}'$. By Lemma 3.1 applied to $\mathcal{O}'$ this intersection is the dense $B_1$-orbit $B_1x$ so for all but finitely many $c \in k$ there is $b_c \in B_1$ such that $b_c^{-1}y_c b_c = x$, that is, $x_\alpha(c)n_\alpha b_c \in G_x \cap G(\Phi_1)$. \qed

Next we shall consider the roots $\gamma$ for which $w\gamma \neq \pm \gamma$. The set of all such roots is $\Phi_2 = \Phi \setminus (\Phi_1 \cup \Phi(\Pi))$. Let $\Phi_2^+ = \Phi_2 \cap \Phi^+$. Every $\alpha \in \Phi_2^+$ determines the following subsets of $\Phi$:

$$\Phi^+(\alpha) = \bigcup_{j > 0} (j\alpha + \text{Ker}(1+w)) \cap \Phi^+ \subset \Phi^+(\Pi) \cup \Phi_2^+;$$

$$\Phi^-(\alpha) = \bigcup_{j > 0} (j\alpha + \text{Ker}(1+w)) \cap (-\Phi^+) \subset (\Phi(\Pi) \cup \Phi_2) \cap (-\Phi^+)$$
and $\Phi(\alpha) = \Phi^+(\alpha) \cup \Phi^-(\alpha)$. Fixing an ordering of the roots, we define:

$$U_\alpha = \prod_{\beta \in \Phi^+(\alpha)} X_\beta \subset U; \quad U^-_\alpha = \prod_{\beta \in \Phi^-(\alpha)} X_\beta \subset U^-.$$

**Lemma 5.2.** — Let $\Phi$ be a simply or doubly-laced irreducible root system, let $w = w_0w_\Pi$ and let $\alpha \in \Phi^+_\Sigma$. We have:

1. The subsets $U_\alpha$ and $U^-_\alpha$ are subgroups of $U$ and $U^-$, respectively.
2. The subgroups $U_\alpha$ and $U^-_\alpha$ are independent of the chosen ordering of the roots.
3. $U_\alpha \cap U^-_\alpha = 1$.
4. If $X^-_\beta \in U^-_\alpha$ then $X_\beta \notin (U^-_\alpha, U_\alpha) = U^-_\alpha, U_\alpha$.
5. $U_\alpha U^-_\alpha = U^-_\alpha U_\alpha$.
6. $w(\Phi(\alpha)) \subset \Phi(\alpha)$
7. If $\Phi$ is simply-laced and $\omega\alpha + \alpha \notin -\Phi^+$ then $wU^-_\alpha \omega^{-1} \subset U_\alpha$.
8. If $\Phi$ is doubly-laced and $\omega\alpha + \alpha \notin -(\Phi^+ \cup 2\Phi^+)$ then $wU^-_\alpha \omega^{-1} \subset U_\alpha$.
9. If $\Phi$ is doubly-laced and $\omega\alpha + \alpha = 2\beta \in -2\Phi^+$ then $X_\beta \subset U^-_\alpha$ and $wU^-_\alpha \omega^{-1} \subset U_\alpha X_\beta$. Besides, $X_\beta$ commutes with $U_\alpha$.
10. If $\Phi$ is simply- or doubly-laced and $\omega\alpha + \alpha = \beta \in -\Phi^+$ then $wU^-_\alpha \omega^{-1} \subset U_\alpha X_\beta$. Besides, $X_\beta$ commutes with $U_\alpha$.

**Proof.** — The first two assertions follow from iterated application of (1.3). Statement 3 follows directly from the definition of $\Phi(\alpha)$. Statement 4 is easily seen by looking at the coefficient of $\alpha$ in the expression of $\beta$. Statement 5 follows from 4 and (1.3). The sixth statement follows once we write $w = \prod_\gamma s_\gamma$ for mutually orthogonal roots $\gamma \in \Phi_\Pi$. Let us prove 7 and 8. If $wU^-_\alpha \omega^{-1} \subset U$ then $wU^-_\alpha \omega^{-1} \subset U_\alpha$ because $w$ is the product of reflections with respect to roots in $\Phi_\Pi$. Hence, it is enough to show that $w\mu \in \Phi^+$ for all $\mu \in \Phi^-(\alpha)$. If we had $w\mu \in -\Phi^+$ for $\mu = j\alpha + y$ with $j > 0$ and $y \in \text{Ker}(1 + u)$ we would have $\mu \in \Phi(\Pi)$ so $w\mu = \mu$, that is

$$2\mu = \mu + w\mu = j\alpha + y + j\omega\alpha - y = j(\alpha + w\alpha) \in -2\Phi.$$

Thus $\alpha + w\alpha \notin \Phi$ for it could neither be a positive nor a negative root, so $(\alpha, w\alpha) \geq 0$. Taking $(2\mu, 2\mu)$ we would have

$$2(\mu, \mu) = j^2((\alpha, \alpha) + (\alpha, w\alpha)) \geq j^2(\alpha, \alpha).$$

If $(\alpha, \alpha) = (\mu, \mu)$ then $j = 1$ and $(\alpha, w\alpha) = (\alpha, \alpha)$ which is impossible proving statement 7. If $(\alpha, \alpha) = 2(\mu, \mu)$ we have again $j = 1$ and (5.2) gives $2\mu = \alpha + w\alpha$ contradicting our assumption in the doubly-laced case. If $2(\alpha, \alpha) = (\mu, \mu)$ we have $j^2 \leq 4$ so $j \leq 2$. Then either $j = 2$ and $\mu = \alpha + w\alpha \in -\Phi^+$ against our assumptions, or $j = 1$ and $3(\alpha, \alpha) = 4(\alpha, w\alpha)$. Since this can never happen, $\mu \notin \Phi(\Pi)$ and statement 8 holds.
Let us prove 9. Let \( \mu = j \alpha + y \in \Phi^-(\alpha) \), with \( y \in \text{Ker}(1 + w) \) and \( j > 0 \) and let us assume that \( w \mu \in -\Phi^+ \). It follows from the proof of 7 and 8 that we have \( 2 \mu = j(\alpha + w \alpha) = 2j \beta \). Hence \( j = 1 \) and \( \beta = \mu \) so \( X_\beta \) is the only root subgroup in \( U_\alpha^- \) that is mapped onto a negative root subgroup under conjugation by \( w \), and it is mapped onto itself. Moreover, for every \( \gamma = i \alpha + y' \in \Phi^+(\alpha) \) with \( i > 0 \) and \( y' \in \text{Ker}(1 + w) \) we have \( 2(\beta, \gamma) = (\alpha + w \alpha, i \alpha + y') = i(\alpha, \alpha) + i(\alpha, w \alpha) \) because \( \alpha + w \alpha \) is orthogonal to \( \text{Ker}(1 + w) \).

Since \( (\alpha, \alpha) = (w \alpha, w \alpha) \) we have \( s_\alpha(w \alpha) \in \{ w \alpha - \alpha, w \alpha, w \alpha + \alpha \} \) so \( 2 \frac{(\alpha, w \alpha)}{(\alpha, \alpha)} \in \{ 0, \pm 1 \} \). Thus \( (\beta, \gamma) > 0 \) and therefore \( \beta + \gamma \notin \Phi \) so \( X_\beta \) commutes with \( X_\gamma \) and \( wU_\alpha^-w^{-1} \subset U_\alpha X_\beta U_\alpha = U_\alpha X_\beta \).

Let us prove the last assertion. Let us assume that \( \beta = \alpha + w \alpha \in -\Phi \).

If for some root \( \nu = j \alpha + y \in \Phi^-(\alpha) \) we had \( w \nu \in -\Phi^+ \) we would have, as before, \( w \nu = \nu \) and \( 2 \nu = \nu + w \nu = j(\alpha + w \alpha) = j \beta \in 2 \Phi \) so \( j = 2 \) and \( \beta = \nu \). Thus \( wU_\alpha^-w^{-1} \subset U_\alpha X_\beta U_\alpha \). As in the proof of 9 we verify that \( \beta + \gamma \notin \Phi \) for every \( \gamma \in \Phi^+(\alpha) \) whence \( X_\beta \) commutes with \( U_\alpha \) and \( wU_\alpha^-w^{-1} \subset U_\alpha X_\beta \). \( \square \)

**Lemma 5.3.** — Let \( G \) be a simple algebraic group, let \( \mathcal{O} \) be a quasi-spherical conjugacy class with \( w = z_\mathcal{O} = w_0 w_1 \) and let \( x = wv \in \mathcal{O} \cap \mathcal{w}U \).

Let \( \alpha \in \Phi_2 \) be such that \( \alpha + w \alpha \notin -\Phi^+ \). Let us also assume, if \( \Phi \) is doubly-laced, that \( \alpha + w \alpha \notin -2 \Phi^+ \). Then for every \( c \in k \) there exists an element in \( x_{w \alpha}(c)U^w \cap \mathcal{G}_x \).

**Proof.** — Since \( w \alpha \neq \alpha \) we have \( w \alpha \in -\Phi^+ \). For every \( c \in k \) we consider the elements

\[
y(c) = x_{\alpha}(c)x_{\alpha}(-c) = w x_{\alpha}(\theta c)v x_{\alpha}(-c) \in \mathcal{O}
\]

where \( \theta \) is a nonzero structure constant, and the elements

\[
z(c) = x_{w \alpha}(\theta c)w x_{w \alpha}(\theta c)v x_{\alpha}(-c) x_{w \alpha}(\theta c) = w x_{\alpha}(\eta \theta c)(x_{w \alpha}(\theta c) x_{\alpha}(-c) x_{w \alpha}(\theta c)) \in \mathcal{O}
\]

where \( \eta \) is a nonzero structure constant. By making use of (1.3) we shall show that for a suitable \( u_c \in U_\alpha \), possibly trivial, we have

\[u_c^{-1}z(c)u_c = x \prod_{\gamma \in \Phi^+ \setminus \Phi_1} x_{\gamma}(c_{\gamma}) \in \mathcal{w}U \cap \mathcal{O}.
\]

Lemma 3.7 will force \( c_{\gamma} = 0 \) for every \( \gamma \in \Phi^+ \setminus \Phi_1 \) so \( u_c^{-1}z(c)u_c = x \) and we have \( u_c^{-1}x_{w \alpha}(\theta c)x_{\alpha}(c) = u_c'x_{\alpha}(c)x_{w \alpha}(\theta c) \in U_\alpha x_{w \alpha}(\theta c) \in \mathcal{G}_x \). Taking inverses will give the statement because \( c \) is arbitrary, \( U_\alpha \subset U = U_w U^w \) and \( U_w \subset \mathcal{G}_x \) by Lemma 3.7.
By hypothesis $w\alpha + \alpha$ is either in $\Phi^+$ or it is not a root. Therefore we have $x_{w\alpha}(\theta c)x_{\alpha}(-c)x_{w\alpha}(-\theta c) = v' \in U_\alpha$. Besides, we have $v = \prod_{\gamma_i \in \Phi^+} x_{\gamma_i}(c_i)$ so that

$$x_{w\alpha}(\theta c)v x_{w\alpha}(-\theta c) = \prod_{i=1}^r (x_{\gamma_i}(c_i) \prod_{a_i,b_i > 0} x_{a_i\gamma_i+b_iw\alpha}(d_{abi}))$$

where we intend $d_{abi} = 0$ if $a_i\gamma_i + b_iw\alpha \notin \Phi$. We proceed as follows: if $w\alpha + \gamma_1 \in -\Phi^+$ we apply (1.3) in order to move the term in $X_{w\alpha+\gamma_1}$ to the left of $x_{\gamma_1}(c_1)$ whereas if $w\alpha + \gamma_1 \in \Phi^+$ we apply (1.3) in order to move the term in $X_{w\alpha+\gamma_1}$ to the right of $x_{\gamma_1}(c_r)$. At each step we might get extra factors either in $U_\alpha$ or in $U_\alpha^-$ and we repeat the procedure. Formula (1.3) can always be applied because we need never to interchange factors in $X_\beta$ with factors in $X_{-\beta}$ (cf. Lemma 5.2(4)). Therefore we have:

$$x_{w\alpha}(\theta c)v x_{w\alpha}(-\theta c) = u^-v \in U_\alpha^-vU_\alpha$$

because the coefficients of the terms in $v$ are never modified. By Lemma 5.2 (5) we have $x_{\alpha}(\eta \theta c)u^- = u_-\eta u_+ \in U_\alpha^-$ and thus

$$z(c) = \hat{w}u_-u_+vu' = \hat{w}u_-vu' = u_\alpha \hat{w}vU_\alpha$$

where for the second equality and the inclusions we have used Lemma 5.2 (4,7,8). Conjugation by $u^{-1}_c$ yields $u^{-1}_c z(c)u_c \in \mathcal{O} \cap \hat{w}U_\alpha$ hence the term in $U_\alpha$ vanishes by Lemma 3.7. Then $u^{-1}_c x_{w\alpha}(\theta c)x_{\alpha}(c) = u'_cx_{\alpha}(c)x_{w\alpha}(\theta c) \in G_\alpha \cap U_\alpha x_{w\alpha}(\theta c)$ and we have the statement.

Lemmas 5.4. — Let $\mathcal{O}$ be a quasi-spherical conjugacy class, let $w = z_\mathcal{O} = w_0w_\Pi$. If for some $\gamma \in \Phi^+ \setminus \Phi(\Pi)$ and for every scalar $c$ there is an element in $x_{-\gamma}(c)U^w$ centralizing $x \in \mathcal{O} \cap \hat{w}U$ then for every $\gamma' \in W_\Pi\gamma$ and for every $d \in k$ there is an element in $x_{-\gamma'}(d)U^w$ centralizing $x$.

Proof. — By Lemmas 3.7 and 3.8, the centralizer of $x$ contains $X_{\pm\alpha}$ hence $n_\alpha$, for every $\alpha \in \Pi$. Conjugation by $n_\alpha$ preserves $U^w$ and $U_w$ and maps $X_{-\gamma}$ onto $X_{-s_\alpha(\gamma)}$, whence the statement.

Lemmas 4.5. — Let $G$ be a simple algebraic group with $\Phi$ doubly-laced. Let $\mathcal{O}$ be a quasi-spherical conjugacy class with notation as above. Let $\alpha \in \Phi^+_2$ be such that $w\alpha + \alpha = 2\beta \in -2\Phi^+$. Then for $x \in \hat{w}U_\alpha \cap \mathcal{O}$ and for every $c \in k$ we have $x_{w\alpha}(c)U^w \cap G_\alpha \neq \emptyset$.

Proof. — Let $z(c)$ be defined as in the proof of Lemma 5.3. We have again

$$x_{w\alpha}(\theta c)v x_{w\alpha}(-\theta c) = u^-v \in U_\alpha^-vU_\alpha.$$
Let us first assume that $\beta \in -\Pi$. Then $u^- = x_\beta(a)u_-$ with $u_- \in U^-_\alpha \cap \dot{w}^{-1}U_\alpha\dot{w}$ by Lemma 5.2 (2,9). We have

$$z(c) = \dot{w}x_\alpha(\theta\eta c)x_\beta(a)u_-vux_\alpha(-c) = \dot{w}x_\beta(a)x_\alpha(\theta\eta c)u_-vux_\alpha(-c)$$

by Lemma 5.2 (9). Applying repeatedly (1.3) and Lemma 5.2 (5) we have for some $u' \in U_\alpha$, $u'_-, v_- \in U^-_\alpha$, and $a' \in k$

$$z(c) = \dot{w}x_\beta(a)u'_-vu' = \dot{w}x_\beta(a + a')v_-vu'$$

with $u_c = \dot{w}v_-\dot{w}^{-1} \in U_\alpha$. We claim that $a + a' \neq 0$. Otherwise, for some nonzero structure constant $\theta'$ we would have, by Lemma 5.2 (9),

$$z(c) = x_\beta((a + a')\theta')u_c\dot{w}vu' \in Bs_\beta B\dot{w}B = Bs_\beta \dot{w}B$$

with $s_\beta \dot{w} > \dot{w}$ contradicting maximality of $w$. Thus, $a + a' = 0$ and we may proceed as in the proof of Lemma 5.3. Moreover, $u_cX_\alpha \subset U^w$.

If $\beta \notin -\Pi$ then there is $\sigma \in W_\Pi$ such that $\sigma\beta \in -\Pi$ and $\sigma\alpha \in \Phi^+$ because the support of $\alpha$ contains at least one simple root outside $\Pi$. Since $w$ is the identity on $\Phi(\Pi)$ it commutes with $\sigma$ and we have $\sigma w\alpha \in -\Phi^+$ and $\sigma\alpha + w\sigma\alpha \in -2\Pi$. By the first part of the proof for every $c \in k$ there is an element in $x_{\sigma w\alpha}(c)U^w$ centralizing $x$ and we may apply Lemma 5.4 to get the statement. \hfill $\square$

**Lemma 5.6.** — Let $G$ be a simple algebraic group with $\Phi$ simply or doubly-laced. Let $O$ be a quasi-spherical conjugacy class with notation as above. Let $\alpha \in \Phi^+$ be such that $w\alpha + \alpha = \beta \in -\Phi$. Then for $x \in \dot{w}U \cap O$ and for every $c \in k$ we have $x_{w\alpha}(c)U^w \cap G_x \neq \emptyset$.

**Proof.** — Let us first assume that $\beta \in -\Pi$. Then $X_\beta \subset U^-_\alpha$ and $\dot{w}U^-_\alpha \dot{w}^{-1} \subset X_\beta U_\alpha$ by Lemma 5.2(10). We have $\beta + w\alpha = 2w\alpha + \alpha = w(w\alpha + 2\alpha) \notin \Phi$. This follows as in the proof of Lemma 5.2 (9,10).

Let $z(c)$ be as in the as in the proofs of Lemmas 5.3 and 5.5. As above we have:

$$x_{w\alpha}(\theta c)v_{w\alpha}(-\theta c) = u^-vu \in U^-_\alpha \dot{v} U_\alpha.$$

We may apply (1.3) to move $x_\alpha(\eta c)$ to the right of $v$ in the expression of $z(c)$. Then we have

$$z(c) = \dot{w}u_-vux_\alpha(\theta c)x_\alpha(-c)x_{w\alpha}(-\theta c)$$

with $u_- = x_\beta(h)u'_- \in X_\beta U^-_\alpha$, $u_c = \dot{w}u_-\dot{w}^{-1} \in U_\alpha \cap U^w$ and $u^+ \in U_\alpha$. Applying once more (1.3) to $x_{w\alpha}(\theta c)x_\alpha(-c)$ gives only a nontrivial extra term in $X_\beta$ by Lemma 5.2 (10). Then, for some $h_1, h_2 \in k$ and some $u' \in U_\alpha$ we have $z(c) = x_\beta(h_1)u_c\dot{w}vu'x_\beta(h_2)$. Conjugation by $u_c^{-1}x_\beta(-h_1)$ yields an element $z'(c)$ in $xU_\alpha X_\beta U_\alpha \cap O \subset B\dot{w}BX_\beta B \subset B\dot{w}s_\beta B \cup B\dot{w}B$. 

**TOME 59 (2009), FASCICULE 6**

2353
Maximality of $w$ forces $h_1 + h_2 = 0$ so the $X_\beta$-factor in $z'(c)$ is trivial. Lemma 3.7 implies that $z'(c) = x$ so, using that $\beta + \alpha$, $\beta + w\alpha \notin \Phi$ we have
\[ u_c^{-1}x_\beta(-h_1)x_{w\alpha}(\theta c)x_{\alpha}(c) = u_c^{-1}x_{\alpha}(c)x_{w\alpha}(\theta c)x_\beta(h) \in G_x \]
with $u_c \in U^w$. By Lemma 3.8 we have $x_\beta(h) \in G_x$ so $u_c^{-1}x_{\alpha}(c)x_{w\alpha}(\theta c) \in G_x$ and taking the inverse yields the statement for $\beta \in -\Pi$.
If $\beta \notin -\Pi$ we may apply Lemma 5.4 as we did in Lemma 5.5. \hfill \Box

We have constructed enough elements in $G_x$ and we are ready to prove the main result of this paper.

**Theorem 5.7.** — Let $G$ be a connected, reductive algebraic group over an algebraically closed field $k$ of characteristic zero or good and odd. Then every quasi-spherical conjugacy class $O$ in $G$ is spherical.

**Proof.** — By Remark 4.10 it is enough to prove the statement for $G$ simple. Type $G_2$ has already been discussed in Section 2.1 so we only need to consider $\Phi$ simply or doubly-laced. Moreover, when $z_O = w_0 = -1$ the statement has been proved in Propositions 4.6, 4.7, 4.8, 4.9 so we shall prove the remaining cases. Let $v$ be a maximal $B$-orbit in $O$. We will prove the statement by showing that $\dim(O) = \dim(v)$, so that $v$ is dense in $O$. To this end, we need to show that for some $x \in v$ we have $\dim G_x = \dim B_x + |\Phi^+|$. We will do so by using $x \in wU \cap O$ for $w = z_O$.

Let us consider the restriction $\pi_x$ to $G_x$ of the natural projection $\pi$ of $G$ onto the flag variety $G/B$. Let $gB$ be in the image of $\pi_x$. We may assume that $g \in G_x$ and then it is not hard to verify that $\pi_x^{-1}(gB) = gB_x$ so each non-empty fiber has dimension equal to $\dim B_x$. Since $\dim G/B = |\Phi^+|$ it is enough to prove that $\pi_x$ is dominant and use [22, Theorem 5.1.6]. We shall prove that $\pi_x(G_x) \cap \pi(B\sigma B)$ is dense in $\pi(B\sigma B)$ for every $\sigma \in W$. In particular, this is true for $\sigma = w_0$ so $\pi_x(G_x) \cap \pi(Bw_0B)$ is dense in $\pi(Bw_0B)$ thus $\pi_x(G_x)$ is dense in $G/B$.

More precisely, if we identify $\pi(B\sigma B) = \pi(U^\sigma B)$ with the affine space $A^{\ell(\sigma)}$ through the map $\pi(u\sigma B) = \pi(\prod_{\gamma \in \Phi_\sigma} x_\gamma(c_\gamma)\sigma B) \mapsto (c_\gamma)_{\gamma \in \Phi_\sigma}$, we will show by induction on $\ell(\sigma)$ that $\pi_x(G_x) \cap A^{\ell(\sigma)}$ contains the complement in $A^{\ell(\sigma)}$ of finitely many hyperplanes.

For $\sigma = 1$ there is nothing to say. Suppose that the statement holds for $\ell(\sigma) \leq s$ and let us consider $\tau \in W$ with $\ell(\tau) = s + 1$. Then $\tau = \sigma s_\alpha$ for some $\sigma \in W$ with $\ell(\sigma) = s$ and some $\alpha \in \Delta$ with $\sigma\alpha \in \Phi^+$. Besides, $\Phi_\sigma = \Phi_\tau \cup \{\sigma\alpha\}$ so $U^\tau = U^\sigma X_{\sigma\alpha}$. By the induction hypothesis the set $U'$ of elements $u$ in $U^\sigma$ for which $u\sigma b$ lies in $G_x$ for some $b \in B$ contains the complement of finitely many hyperplanes in $U^\sigma \cong A^{\ell(\sigma)}$. 

Annales de l'Institut Fourier
There are three possibilities: \( \alpha \in \Pi, \alpha \in \Delta \cap \Phi_1 \) and \( \alpha \in \Delta \cap \Phi_2 \).

If \( \alpha \in \Pi \) we have \( X_\alpha \cap \alpha \subset G_x \) by Lemma 3.7. Then for every \( u \in U' \) and every \( c \in k \) there is \( b \in B \) for which \( (u \sigma b)(x_\alpha(c) n_\alpha) \in G_x \). Let \( b = x_\alpha(r) v \) for \( r \in k \) and \( v \in P^u_\alpha \). Then for some \( v' \in P^u_\alpha \) and for some nonzero structure constant \( \eta \) we have

\[
(u \sigma b)(x_\alpha(c) n_\alpha) = u \sigma x_\alpha(r + c) n_\alpha v' = wx_\sigma x_\alpha(\eta(r + c)) v' = G_x.
\]

Since \( c \) is arbitrary and \( \eta \neq 0 \), if \( \alpha \in \Pi \) then \( \pi_x(G_x) \cap \pi(B \tau B) \) contains \( \pi(U' X_\sigma \tau B) \) so it contains the complement of finitely many hyperplanes in \( \mathbb{A}^{\ell(\tau)} \).

Let now \( \alpha \in \Delta \cap \Phi_1 \). By Lemma 5.1 for all but finitely many \( c \in k \) there is \( b_c \in B \) such that \( x_\alpha(c) n_\alpha b_c \in G_x \). Thus, for every \( u \in U' \) and for those \( c \) there is \( b \in B \) for which \( (u \sigma b)(x_\alpha(c) n_\alpha b_c) \in G_x \). Let \( b = x_\alpha(r) v \) for \( r \in k \) and \( v \in P^u_\alpha \). Then for some \( v' \in P^u_\alpha \) and for some nonzero structure constant \( \eta \) we have

\[
(u \sigma b)(x_\alpha(c) n_\alpha b_c) = u \sigma x_\alpha(r + c) n_\alpha v' b_c = wx_\sigma x_\alpha(\eta(r + c)) v' b_c \subset G_x.
\]

Since all but finitely many \( c \) were possible and \( \eta \neq 0 \), also in this case \( \pi_x(G_x) \cap \pi(B \tau B) \) contains \( \pi(U' x_\sigma \tau B) \) for all but finitely many \( c \), thus it contains the complement of finitely many hyperplanes in \( \mathbb{A}^{\ell(\tau)} \).

Finally, let \( \alpha \in \Phi_2 \cap \Delta \). Then by Lemmas 5.3, 5.5, and 5.6 for every \( c \in k \) there exists \( u_c \in U \) such that \( x_{-\alpha}(c) u_c \in G_x \). For \( c \neq 0 \) this element is equal to \( x_\alpha(c^{-1}) t_c n_\alpha x_\alpha(c^{-1}) u_c \) for some \( t_c \in T \) by [22, Lemma 8.1.4 (i)]. Thus, for every \( u \in U' \) and \( c \neq 0 \) there is \( b \in B \) for which \( (u \sigma b)(x_\alpha(c^{-1}) t_c n_\alpha x_\alpha(c^{-1}) u_c) \in G_x \). Let \( b = x_\alpha(r) v \) for \( r \in k \) and \( v \in P^u_\alpha \). Then for some \( v' \in P^u_\alpha \), \( t'_c \in T \) and for some nonzero structure constant \( \eta \) we have

\[
(u \sigma b)(x_\alpha(c^{-1}) t_c n_\alpha x_\alpha(c^{-1}) u_c) = u \sigma x_\alpha(r + c^{-1}) n_\alpha t'_c v' x_\alpha(c^{-1}) u_c = wx_\sigma x_\alpha(\eta(r + c^{-1})) v' b_c \subset G_x.
\]

Then again, \( \pi_x(G_x) \cap \pi(B \tau B) \) contains the complement of finitely many hyperplanes in \( \mathbb{A}^{\ell(\tau)} \) and we have the statement. \( \square \)

As a consequence of Theorem 5.7 we get the sought characterization.

**Theorem 5.8.** — Let \( \mathcal{O} \) be a conjugacy class in a connected reductive algebraic group \( G \) over a field of zero or good odd characteristic. Then \( \mathcal{O} \) is spherical if and only if \( \mathcal{O} \subset \bigcup_{w^2 = 1} B w B \).

**Proof.** — This is obtained combining Theorem 5.7 with [6, Theorem 2.7], whose proof holds also for \( G \) connected and reductive. \( \square \)
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Manuscrit reçu le 15 septembre 2008,
révisé le 8 janvier 2009,
accepté le 30 janvier 2009.

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