



# ANNALES

DE

# L'INSTITUT FOURIER

Abdelmalek ABDESSELAM & Jaydeep CHIPALKATTI

**The higher transvectants are redundant**

Tome 59, n° 5 (2009), p. 1671-1713.

[http://aif.cedram.org/item?id=AIF\\_2009\\_\\_59\\_5\\_1671\\_0](http://aif.cedram.org/item?id=AIF_2009__59_5_1671_0)

© Association des Annales de l'institut Fourier, 2009, tous droits réservés.

L'accès aux articles de la revue « Annales de l'institut Fourier » (<http://aif.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://aif.cedram.org/legal/>). Toute reproduction en tout ou partie cet article sous quelque forme que ce soit pour tout usage autre que l'utilisation à fin strictement personnelle du copiste est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

cedram

*Article mis en ligne dans le cadre du*  
*Centre de diffusion des revues académiques de mathématiques*  
<http://www.cedram.org/>

# THE HIGHER TRANSVECTANTS ARE REDUNDANT

by Abdelmalek ABDESSELAM & Jaydeep CHIPALKATTI

---

ABSTRACT. — Let  $A, B$  denote generic binary forms, and let  $u_r = (A, B)_r$  denote their  $r$ -th transvectant in the sense of classical invariant theory. In this paper we classify all the quadratic syzygies between the  $\{u_r\}$ . As a consequence, we show that each of the higher transvectants  $\{u_r : r \geq 2\}$  is redundant in the sense that it can be completely recovered from  $u_0$  and  $u_1$ . This result can be geometrically interpreted in terms of the incomplete Segre imbedding. The calculations rely upon the Cauchy exact sequence of  $SL_2$ -representations, and the notion of a 9-j symbol from the quantum theory of angular momentum.

We give explicit computational examples for  $SL_3, \mathfrak{g}_2$  and  $\mathfrak{S}_5$  to show that this result has possible analogues for other categories of representations.

RÉSUMÉ. — Pour deux formes binaires génériques  $A, B$ , notons  $u_r = (A, B)_r$  leur transvectant d'ordre  $r$ , tel que défini en théorie classique des invariants. Dans cet article, nous obtenons une classification complète des syzygies quadratiques entre les  $\{u_r\}$ . Il en résulte que les transvectants d'ordre supérieur  $\{u_r : r \geq 2\}$  sont redondants, en ce sens qu'ils peuvent être exprimés à partir de  $u_0$  et  $u_1$ . Ce résultat peut s'interpréter géométriquement en termes du plongement incomplet de Segre. Les calculs utilisés reposent sur la suite exacte de Cauchy en théorie des représentations de  $SL_2$ , ainsi que sur la notion de symbole 9-j de la théorie quantique du moment angulaire.

Nous donnons des exemples de calculs explicites concernant  $SL_3, \mathfrak{g}_2$  et  $\mathfrak{S}_5$  afin d'indiquer l'existence possible de résultats analogues pour d'autres catégories de représentations.

## 1. Introduction

Transvectants were introduced into algebra more or less independently by Cayley and Aronhold (see [11, 13]). The German school of classical invariant theorists used them dexterously in the symbolical treatment of algebraic forms (for instances, see [22, 41]). In their modern formulation, they encode

---

*Keywords:* Angular momentum in quantum mechanics, binary forms, Cauchy exact sequence, 9-j symbols, representations of  $SL_2$ , transvectants.

*Math. classification:* 13A50, 22E70.

the decomposition of the tensor product of two finite-dimensional  $SL_2$ -representations over a field of characteristic zero.

We begin by giving an elementary definition of transvectants. In §1.3-1.5 we describe their reformulation in the language of  $SL_2$ -representations. An outline of the main results is given in §1.9 (on page 1676) after the required notation is available.

We will use [20, 23] as standard references for classical invariant theory, and in particular the symbolic calculus. Modern accounts of this subject may be found in [15, 31, 35]. The reader is referred to [19, Lecture 6], [40, Ch. 3] and [42, Ch. 4] for the basic theory of  $SL_2$ -representations.

1.1.

Let

$$A = \sum_{i=0}^m \binom{m}{i} a_i x_1^{m-i} x_2^i, \quad B = \sum_{i=0}^n \binom{n}{i} b_i x_1^{n-i} x_2^i;$$

denote binary forms of orders  $m, n$  in the variables  $\mathbf{x} = \{x_1, x_2\}$ . (The coefficients are assumed to be in a field of characteristic zero.) Let  $r$  denote an integer such that  $0 \leq r \leq \min(m, n)$ . The  $r$ -th transvectant of  $A$  and  $B$  is defined to be the binary form

$$(1.1) \quad (A, B)_r = \frac{(m-r)!(n-r)!}{m!n!} \sum_{i=0}^r (-1)^i \binom{r}{i} \frac{\partial^r A}{\partial x_1^{r-i} \partial x_2^i} \frac{\partial^r B}{\partial x_1^i \partial x_2^{r-i}}$$

of order  $m+n-2r$ . In particular  $(A, B)_0$  is the product of  $A, B$ , and  $(A, B)_1$  is (up to a multiplicative factor) their Jacobian. By construction,

$$(1.2) \quad (B, A)_r = (-1)^r (A, B)_r.$$

The process of transvection commutes with a change of variables in the following sense. Let  $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  denote a matrix of indeterminates. Define

$$A' = \sum_{i=0}^m \binom{m}{i} a_i (\alpha x_1 + \beta x_2)^{m-i} (\gamma x_1 + \delta x_2)^i,$$

and similarly for  $B'$  etc. Then we have an identity

$$(A', B')_r = (\det g)^r [(A, B)_r]'$$

In classical terminology,  $(A, B)_r$  is a joint covariant of  $A, B$ .

1.2.

Now let  $A, B$  denote generic forms of orders  $m, n$ , that is to say, their coefficients are assumed to be independent indeterminates. Write  $u_r = (A, B)_r$  for the  $r$ -th transvectant.<sup>(1)</sup> Broadly speaking, the main result of this paper is that the higher transvectants  $\{u_r : r \geq 2\}$  are redundant in the sense that each of them can be recovered from the knowledge of  $u_0$  and  $u_1$ . We begin with an illustration.

*Example 1.1.* — Assume  $m = 5, n = 3$ . Then we have an identity

$$(1.3) \quad \frac{21}{8} (u_0, u_0)_2 + \frac{21}{16} (u_0, u_1)_1 + \frac{315}{256} u_1^2 = u_0 u_2,$$

which gives a formula for  $u_2$  in terms of  $u_0, u_1$ . (This is an instance of general formulae to be proved below.) Similarly,

$$(1.4) \quad \frac{20}{3} (u_0, u_1)_2 + \frac{20}{9} (u_0, u_2)_1 + \frac{25}{14} u_1 u_2 = u_0 u_3,$$

which indirectly expresses  $u_3$  in terms of  $u_0, u_1$ . Our result shows the existence of such formulae in general.

**THEOREM 1.2.** — Assume  $m, n, r \geq 2$ . With notation as above, there exist constants  $c_{i,j} \in \mathbf{Q}$  such that we have an identity

$$u_r = \frac{1}{u_0} \sum_{0 \leq i \leq j < r} c_{i,j} (u_i, u_j)_{r-i-j}.$$

Since the right hand side depends only on  $\{u_0, \dots, u_{r-1}\}$ , it follows by induction that  $u_0, u_1$  determine the rest of the higher transvectants. In fact, more generally we will exhibit explicit formulae for all the quadratic syzygies between the  $\{u_i\}$ , of which (1.3) and (1.4) are special cases.

The title of the paper should not be understood to mean that ‘higher transvection’ is redundant. Notice, for instance, that the formula for  $u_2$  itself involves  $(u_0, u_0)_2$ .

1.3.  $SL_2$ -representations

Throughout this paper we work over an arbitrary field  $\mathbb{k}$  of characteristic zero. Let  $V$  denote a two-dimensional  $\mathbb{k}$ -vector space with basis  $\mathbf{x} = \{x_1, x_2\}$ . For  $m \geq 0$ , the symmetric power  $S_m = \text{Sym}^m V$  is the space of binary  $m$ -ics, with an action of the linearly reductive group

<sup>(1)</sup> ‘Überschiebung’ in German.

$SL(V) = \{\varphi \in \text{End}(V) : \det \varphi = 1\}$ . The  $\{S_m : m \geq 0\}$  are a complete set of irreducible  $SL(V)$ -representations, and any finite-dimensional representation decomposes as a direct sum of irreducibles. By Schur's lemma, if a linear map  $S_m \rightarrow S_m$  is  $SL(V)$ -equivariant, then it is necessarily a scalar multiplication.

Henceforth,  $V$  will not be explicitly mentioned if no confusion is likely; for instance,  $S_m(S_n)$  will stand for  $\text{Sym}^m(\text{Sym}^n V)$  etc.

1.4.

It will be convenient to introduce several pairs of variables

$$\mathbf{y} = (y_1, y_2), \quad \mathbf{z} = (z_1, z_2), \dots$$

all on equal footing with  $\mathbf{x}$ . Then, for instance, an element of the tensor product  $S_m \otimes S_n$  can be represented as a bihomogeneous form  $F(\mathbf{x}, \mathbf{y})$  of orders  $m, n$  in  $\mathbf{x}, \mathbf{y}$  respectively. Define Cayley's Omega operator

$$\Omega_{\mathbf{x}\mathbf{y}} = \frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial x_2 \partial y_1},$$

and the polarisation operator

$$\mathbf{y}\partial_{\mathbf{x}} = y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2}.$$

If  $c_{\mathbf{x}}$  stands for the symbolic linear form  $c_1 x_1 + c_2 x_2$ , then

$$(\mathbf{y}\partial_{\mathbf{x}})^\ell c_{\mathbf{x}}^m = \frac{m!}{(m-\ell)!} c_{\mathbf{x}}^{m-\ell} c_{\mathbf{y}}^\ell.$$

The operators  $\Omega_{\mathbf{x}\mathbf{z}}, \mathbf{y}\partial_{\mathbf{z}}$  etc. are similarly defined. The symbolic bracket  $(\mathbf{x}\mathbf{y})$  stands for  $x_1 y_2 - x_2 y_1$ , and likewise for  $(\mathbf{x}\mathbf{z})$  etc.

1.5.

We have a direct sum decomposition of the tensor product

$$(1.5) \quad S_m \otimes S_n \simeq \bigoplus_{r=0}^{\min(m,n)} S_{m+n-2r},$$

usually called the Clebsch-Gordan decomposition. Let

$$\pi_r : S_m \otimes S_n \rightarrow S_{m+n-2r}$$

denote the projection map, which acts by the recipe

$$(1.6) \quad F(\mathbf{x}, \mathbf{y}) \xrightarrow{\pi_r} \mathfrak{f}(m, n; r) [\Omega_{\mathbf{x}\mathbf{y}}^r F(\mathbf{x}, \mathbf{y})]_{\mathbf{y} \rightarrow \mathbf{x}};$$

where

$$f(m, n; r) = \frac{(m - r)! (n - r)!}{m! n!}.$$

We have written  $\mathbf{y} \rightarrow \mathbf{x}$  for the substitution of  $x_1, x_2$  for  $y_1, y_2$  respectively, so that the right hand side of (1.6) is of order  $m + n - 2r$  in  $\mathbf{x}$  as required.

In particular if  $A(\mathbf{x}) \in S_m, B(\mathbf{x}) \in S_n$ , then a straightforward binomial expansion shows that the image  $\pi_r(A(\mathbf{x}) B(\mathbf{y}))$  coincides with the transvectant  $(A, B)_r$  as defined in (1.1). In symbols, if  $A = \alpha_{\mathbf{x}}^m, B = \beta_{\mathbf{x}}^n$ , then we have the formula

$$(1.7) \quad (A, B)_r = (\alpha \beta)^r \alpha_{\mathbf{x}}^{m-r} \beta_{\mathbf{x}}^{n-r}.$$

The initial scaling factor in (1.6) is so chosen that (1.7) has the simplest possible form.

### 1.6.

The map  $\pi_r$  is a split surjection, let  $\iota_r : S_{m+n-2r} \rightarrow S_m \otimes S_n$  denote its section. For  $c_{\mathbf{x}}^{m+n-2r} \in S_{m+n-2r}$ , it is given by

$$c_{\mathbf{x}}^{m+n-2r} \xrightarrow{\iota_r} \mathbf{g}(m, n; r) (\mathbf{x} \mathbf{y})^r c_{\mathbf{x}}^{m-r} c_{\mathbf{y}}^{n-r},$$

where

$$(1.8) \quad \mathbf{g}(m, n; r) = \frac{\binom{m}{r} \binom{n}{r}}{\binom{m+n-r+1}{r}}.$$

Define

$$(1.9) \quad \mathbf{h}(m, n; r) = f(m, n; r) \mathbf{g}(m, n; r) = \frac{(m + n - 2r + 1)!}{(m + n - r + 1)! r!}.$$

Now observe that by the formula on [23, p. 54],

$$\{ \Omega_{\mathbf{x} \mathbf{y}}^r [(\mathbf{x} \mathbf{y})^r c_{\mathbf{x}}^{m-r} c_{\mathbf{y}}^{n-r}] \}_{\mathbf{y} \rightarrow \mathbf{x}} = \frac{1}{\mathbf{h}(m, n; r)} c_{\mathbf{x}}^{m+n-2r},$$

which verifies that  $\pi_r \circ \iota_r$  is the identity map on  $S_{m+n-2r}$  (also see [17] and [30, §18.2]).

### 1.7. Angular momenta

There is a process analogous to transvection in the quantum theory of angular momentum. In brief, the eigenvectors (of the Casimir element for the Lie algebra  $\mathfrak{su}_2$ ) can exist in any of the states  $j$  labelled by the non-negative half-integers  $\{0, 1/2, 1, 3/2, \dots\}$ . The coupling of two states  $j_1, j_2$  produces a finite set of angular momentum states

$$|j_1 - j_2|, |j_1 - j_2| + 1, |j_1 - j_2| + 2, \dots, j_1 + j_2.$$

If we let  $m = 2j_1, n = 2j_2$ , then this reduces to the Clebsch-Gordan decomposition. (The standard account of this theory may be found in [6, 16].) At a crucial place in our study of transvectant syzygies we will need the concept of 9-j symbol which arises from the possible couplings of four angular momentum states. This is further explained in §7, where an introduction to the relevant notions from the quantum theory of angular momentum will be given.

### 1.8. Self-duality

The map  $S_m \otimes S_m \rightarrow \mathbb{k}$  establishes a canonical isomorphism of  $S_m$  with its dual representation  $S_m^\vee = \text{Hom}(S_m, \mathbb{k})$ . It identifies  $A \in S_m$  with the functional

$$S_m \rightarrow \mathbb{k}, \quad B \rightarrow (A, B)_m.$$

Consequently, every finite-dimensional  $SL_2$ -representation is canonically isomorphic to its own dual.<sup>(2)</sup> We have a canonical trace element in  $S_m \otimes S_m$  which corresponds to the form  $(\mathbf{x} \mathbf{y})^m$ .

### 1.9. Results

We can now state the main results of this paper. Let the  $\{\mathbf{u}_i\}$  be as in §1.2. For an integer  $r$  such that  $2 \leq r \leq \min(m, n)$ , define a (quadratic) syzygy of weight  $r$  to be an identity

$$(1.10) \quad \sum \vartheta_{i,j} (\mathbf{u}_i, \mathbf{u}_j)_{r-i-j} = 0, \quad \vartheta_{i,j} \in \mathbf{Q}$$

where the summation is quantified over all pairs  $(i, j)$  such that

$$0 \leq i \leq j, \quad i + j \leq r.$$

Notice that only one summand in (1.10) involves  $\mathbf{u}_r$ , namely  $\vartheta_{0,r} \mathbf{u}_0 \mathbf{u}_r$ .

Let  $\mathbb{K}(m, n; r)$  denote the vector space of weight  $r$  syzygies. In §2.3–2.4 we will show that there is a natural isomorphism of  $\mathbb{K}(m, n; r)$  with the space of equivariant morphisms

$$\text{Hom}_{SL(V)}(S_{2(m+n-r)}, \wedge^2 S_m \otimes \wedge^2 S_n).$$

This will imply that  $\mathbb{K}(m, n; r)$  has a basis which is in natural bijection with the set of integral points

$$\Pi(m, n; r) = \left\{ (a, b) \in \mathbf{N}^2 : a + b \leq \frac{r-2}{2} \right\}.$$

---

<sup>(2)</sup>This is no longer true of  $SL_N$ -representations when  $N > 2$ . In some contexts this self-duality leads to simplification, and in some others to confusion.

Since  $(a, b) = (0, 0)$  is such a point, there exist nontrivial syzygies of any weight  $r \geq 2$ . For an arbitrary  $p = (a, b) \in \Pi(m, n; r)$ , let  $\vartheta_{i,j}^{(p)}$  denote the corresponding syzygy coefficients.

In §2.10 we will give an explicit formula for the rational number  $\vartheta_{i,j}^{(p)}$ . It will follow that if we specialise to  $p = (0, 0)$ , then  $\vartheta_{0,r}^{(p)} \neq 0$ . We can then rewrite identity (1.10) in the form

$$u_r = \frac{1}{u_0} \sum -\frac{\vartheta_{i,j}^{(p)}}{\vartheta_{0,r}^{(p)}} (u_i, u_j)_{r-i-j},$$

and thereby complete the proof of Theorem 1.2. In Theorem 3.1 we prove the thematically related result that the morphism

$$\mathbf{P}S_m \times \mathbf{P}S_n \longrightarrow \mathbf{P}(S_{m+n} \oplus S_{m+n-2})$$

which sends a pair of forms  $(A, B)$  to  $(AB, (A, B)_1)$ , is an imbedding of algebraic varieties.

Of course it would be of interest to find similar redundancy theorems for other categories of representations. In sections 4,5 and 6, we give one example each of this phenomenon respectively for representations of  $SL_3, \mathfrak{g}_2$  and  $\mathfrak{S}_5$ .

## 2. The Cauchy exact sequence

In this section we establish the basic set-up which leads to the characterisation of quadratic syzygies between transvectants.

### 2.1.

Given any two finite-dimensional vector spaces  $W_1, W_2$ , we have a short exact sequence (see [4, §III.1]) of  $GL(W_1) \times GL(W_2)$ -representations

$$(2.1) \quad 0 \longrightarrow \underbrace{\wedge^2 W_1 \otimes \wedge^2 W_2}_c \xrightarrow{\delta} S_2(W_1 \otimes W_2) \xrightarrow{\epsilon} S_2(W_1) \otimes S_2(W_2) \longrightarrow 0,$$

which we may call the Cauchy exact sequence. (The corresponding formula on characters is due to Cauchy – see [19, Appendix A].)

Let the dot stand for symmetrised tensor product, i.e., we write  $g \cdot h$  instead of  $\frac{1}{2}(g \otimes h + h \otimes g)$ . With this notation,  $\epsilon$  is the ‘regrouping’ map

$$(g_1 \otimes g_2) \cdot (h_1 \otimes h_2) \longrightarrow (g_1 \cdot h_1) \otimes (g_2 \cdot h_2),$$

and  $\delta$  is the map

$$(g_1 \wedge h_1) \otimes (g_2 \wedge h_2) \longrightarrow (g_1 \otimes g_2) \cdot (h_1 \otimes h_2) - (g_1 \otimes h_2) \cdot (h_1 \otimes g_2).$$



The exactness of (2.1) is an instance of a general result about Schur functors (see *loc. cit.*), but it is elementary to check in this case. Indeed, it is immediate that  $\epsilon \circ \delta = 0$ , implying  $\text{im } \delta \subseteq \ker \epsilon$ . Now write  $w_i = \dim W_i$ , and observe that the dimensions of the first and the third vector space add up to the second:

$$\binom{w_1}{2} \binom{w_2}{2} + \binom{w_1 + 1}{2} \binom{w_2 + 1}{2} = \binom{w_1 w_2 + 1}{2},$$

hence  $\text{im } \delta = \ker \epsilon$ .

**2.2.**

Consider the Segre imbedding

$$\mathbf{P}S_m \times \mathbf{P}S_n \longrightarrow \mathbf{P}(S_m \otimes S_n), \quad [(A, B)] \longrightarrow [A \otimes B]$$

with image  $X$ , and ideal sheaf  $\mathcal{I}_X$ . Since  $X$  is projectively normal, we have an exact sequence

$$(2.2) \quad 0 \longrightarrow H^0(\mathcal{I}_X(2)) \xrightarrow{g} H^0(\mathcal{O}_P(2)) \xrightarrow{h} H^0(\mathcal{O}_X(2)) \longrightarrow 0.$$

Let us introduce a series of generic forms

$$(2.3) \quad A = \sum_{k=0}^m \binom{m}{k} a_k z_1^{m-k} z_2^k, \quad B = \sum_{k=0}^n \binom{n}{k} b_k z_1^{n-k} z_2^k,$$

of orders  $m, n$ , and

$$(2.4) \quad U_\ell = \sum_{k=0}^{m+n-2\ell} \binom{m+n-2\ell}{k} q_{k,\ell} z_1^{m+n-2\ell-k} z_2^k,$$

of orders  $m + n - 2\ell$  for  $0 \leq \ell \leq \min(m, n)$ . (That is to say, the  $\underline{a}, \underline{b}, \underline{q}$  are assumed to be sets of distinct indeterminates.) Consider the polynomial algebras

$$Q = \mathbb{k}[\{q_{k,\ell}\}], \quad R = \mathbb{k}[a_0, \dots, a_m; b_0, \dots, b_n].$$

The former is graded by  $\mathbf{N}$ , and the latter by  $\mathbf{N} \times \mathbf{N}$ . If we write  $U_\ell = (A, B)_\ell^{\mathbf{z}}$  (where the transvectant is taken with respect to  $\mathbf{z}$  variables) and equate coefficients in  $\mathbf{z}$ , then each  $q_{k,\ell}$  is given by a polynomial expression in  $\underline{a}, \underline{b}$ . This defines a ring morphism  $Q \longrightarrow R$ . Now, we have isomorphisms of graded (respectively bigraded) rings

$$Q \xrightarrow{\sim} \bigoplus_{e \geq 0} S_e([S_m \otimes S_n]^\vee),$$

$$R \xrightarrow{\sim} \bigoplus_{e, e' \geq 0} S_e(S_m^\vee) \otimes S_{e'}(S_n^\vee)$$

defined as follows: observe that

$$(-1)^k \times (U_\ell, z_2^{m+n-2\ell-k} z_1^k)_{m+n-2\ell}^{\mathbf{z}} = q_{k,\ell},$$

hence we identify  $q_{k,\ell}$  with the functional in  $[S_m \otimes S_n]^\vee$  which sends the biform  $\alpha_{\mathbf{x}}^m \beta_{\mathbf{y}}^n \in S_m \otimes S_n$  to

$$(-1)^k \times ((\alpha \beta)^\ell \alpha_{\mathbf{z}}^{m-\ell} \beta_{\mathbf{z}}^{n-\ell}, z_2^{m+n-2\ell-k} z_1^k)_{m+n-2\ell}^{\mathbf{z}}.$$

This extends to give an isomorphism of  $Q$  with the symmetric algebra on the space  $[S_m \otimes S_n]^\vee$ . The second isomorphism is defined similarly. The induced map  $Q_2 \rightarrow R_{2,2}$  on vector spaces may be naturally identified with the map  $h$  from (2.2).

**2.3.**

Consider a formal expression

$$\Psi = \sum_{i,j} \vartheta_{i,j} (U_i, U_j)_{r-i-j}^{\mathbf{z}},$$

where  $\vartheta_{i,j}$  are arbitrary elements in  $\mathbf{Q}$ . We should like to determine whether  $\Psi$  corresponds to a weight  $r$  syzygy. Now, the datum  $\Psi$  is equivalent to a morphism of  $SL(V)$ -representations

$$f_\Psi : S_{2(m+n-r)} \rightarrow Q_2, \quad H(\mathbf{z}) \rightarrow (H(\mathbf{z}), \Psi)_{2(m+n-r)}^{\mathbf{z}}.$$

This is to be interpreted as follows:  $\Psi, H(\mathbf{z})$  are both forms of order  $2(m+n-r)$  in the  $\mathbf{z}$ -variables. Hence after transvection the right hand side has no  $\mathbf{z}$ -variables remaining, and we get a quadratic expression in the  $\{q_{k,\ell}\}$ .

Now  $\Psi$  represents a *bona fide* weight  $r$  syzygy iff the following condition is satisfied: if we substitute  $(A, B)_i$  for  $U_i$ , then  $\Psi$  vanishes. This is equivalent to the requirement that  $h \circ f_\Psi = 0$ , i.e.,  $f_\Psi$  factor through  $\ker h$ . Hence we have proved the following:

PROPOSITION 2.1. — *The vector space  $\mathbb{K}(m, n; r)$  of weight  $r$  syzygies is naturally isomorphic to  $\text{Hom}_{SL(V)}(S_{2(m+n-r)}, H^0(\mathcal{I}_X(2)))$ .  $\square$*

**2.4.**

Now, by specialising (2.1) we have the exact sequence

$$(2.5) \quad 0 \rightarrow \underbrace{\wedge^2 S_m \otimes \wedge^2 S_n}_{\mathcal{C}} \xrightarrow{\delta} \underbrace{S_2(S_m \otimes S_n)}_{\mathcal{D}} \xrightarrow{\epsilon} \underbrace{S_2(S_m) \otimes S_2(S_n)}_{\mathcal{E}} \rightarrow 0.$$

By self-duality (see §1.8) we can identify  $H^0(\mathbf{P}(S_m \otimes S_n), \mathcal{O}_P(2))$  and  $H^0(\mathcal{O}_X(2))$  respectively with  $\mathcal{D}$  and  $\mathcal{E}$ , inducing an isomorphism of  $H^0(\mathcal{I}_X(2))$  with  $\mathcal{C}$ .

2.5.

We have isomorphisms

$$\wedge^2 S_m \simeq S_2(S_{m-1}) \simeq \bigoplus_{a=0}^{\lfloor \frac{m-1}{2} \rfloor} S_{2(m-1)-4a},$$

and similarly for  $\wedge^2 S_n$ . Hence, for each pair  $p = (a, b)$  in the set

$$(2.6) \quad \Pi(m, n; r) = \{(a, b) \in \mathbf{N}^2 : 2(a + b + 1) \leq r\},$$

we have a morphism  $\phi_{a,b}$  defined to be the composite

$$S_{2(m+n-r)} \xrightarrow{\theta_1} S_{2(m-1)-4a} \otimes S_{2(n-1)-4b} \xrightarrow{\theta_2} S_2(S_{m-1}) \otimes S_2(S_{n-1}) \xrightarrow{\theta_3} \wedge^2 S_m \otimes \wedge^2 S_n.$$

Here  $\theta_1$  is dual to the  $(r - 2a - 2b - 2)$ -th transvectant map,  $\theta_2$  is dual to the tensor product of  $2a$ -th and  $2b$ -th transvectant maps, and  $\theta_3$  is an isomorphism.

By construction the  $\{\phi_{a,b} : (a, b) \in \Pi\}$  form a basis of the space of  $SL(V)$ -equivariant morphisms  $S_{2(m+n-r)} \rightarrow \mathcal{C}$ . Let  $K^{(a,b)}$  denote the corresponding weight  $r$  syzygy, written as

$$(2.7) \quad \sum \kappa_{i,j} (\mathbf{u}_i, \mathbf{u}_j)_{r-i-j} = 0,$$

where the sum is quantified over pairs  $(i, j)$  such that  $0 \leq i, j \leq r$  and  $i + j \leq r$ . (We have not yet imposed the condition  $i \leq j$ .) In order to extract the coefficient  $\kappa_{i,j}$ , we will construct a sequence of morphisms

$$S_2(S_m \otimes S_n) \xrightarrow{\eta_1} (S_m \otimes S_n)^{\otimes 2} \xrightarrow{\eta_2} \left(\bigoplus_i S_{m+n-2i}\right) \otimes \left(\bigoplus_j S_{m+n-2j}\right) \xrightarrow{\eta_3} S_{m+n-2i} \otimes S_{m+n-2j} \xrightarrow{\eta_4} S_{2(m+n-r)},$$

where  $\eta_1$  is the natural inclusion

$$v_1 \cdot v_2 \longrightarrow \frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1),$$

$\eta_2$  is an isomorphism,  $\eta_3$  is the tensor product of natural projections, and  $\eta_4$  is the  $(r - i - j)$ -th transvection map.

In §2.6 – 2.7 below, we will give precise symbolic formulae for these maps. Once this is done, the following proposition is immediate.

PROPOSITION 2.2. — For any  $p = (a, b) \in \Pi(m, n; r)$ , the endomorphism

$$\underbrace{\eta_4 \circ \eta_3 \circ \eta_2 \circ \eta_1 \circ \delta \circ \theta_3 \circ \theta_2 \circ \theta_1}_{\xi} : S_{2(m+n-r)} \longrightarrow S_{2(m+n-r)}$$

is the multiplication by  $\kappa_{i,j}^{(a,b)}$ .

2.6.

In order to describe  $\theta_1$  we will realise  $S_{2(m+n-r)}$  as the space of order  $2(m+n-r)$  forms in  $\mathbf{z}$ , and  $S_{2m-2-4a} \otimes S_{2n-2-4b}$  as the space of bihomogeneous forms of orders  $(2m-2-4a, 2n-2-4b)$  in  $\mathbf{x}, \mathbf{y}$  respectively. Then

$$\theta_1 : f(\mathbf{z}) \longrightarrow \frac{(\mathbf{x} \mathbf{y})^{r-2a-2b-2}}{(2m+2n-2r)!} [(\mathbf{x} \partial_{\mathbf{z}})^{2m-2a+2b-r} (\mathbf{y} \partial_{\mathbf{z}})^{2n+2a-2b-r} f(\mathbf{z})].$$

We realise  $S_2(S_{m-1}) \otimes S_2(S_{n-1})$  as the space of quadrihomogeneous forms of orders  $(m-1, m-1, n-1, n-1)$  respectively in  $\mathbf{p}, \mathbf{q}, \mathbf{u}, \mathbf{v}$ , which are symmetric in the variable pairs  $\mathbf{p}, \mathbf{q}$  and  $\mathbf{u}, \mathbf{v}$ . Then

$$\theta_2 : g(\mathbf{x}, \mathbf{y}) \longrightarrow \frac{(\mathbf{p} \mathbf{q})^{2a} (\mathbf{u} \mathbf{v})^{2b}}{(2m-4a-2)!(2n-4b-2)!} \times [(\mathbf{p} \partial_{\mathbf{x}})^{m-2a-1} (\mathbf{q} \partial_{\mathbf{x}})^{m-2a-1} (\mathbf{u} \partial_{\mathbf{y}})^{n-2b-1} (\mathbf{v} \partial_{\mathbf{y}})^{n-2b-1} g(\mathbf{x}, \mathbf{y})].$$

2.7.

Now realise  $S_2(S_m \otimes S_n)$  as the space of forms of orders  $(m, n, m, n)$  respectively in  $\mathbf{p}, \mathbf{u}, \mathbf{q}, \mathbf{v}$  which are symmetric with respect to the simultaneous exchange of variable pairs  $\mathbf{p} \leftrightarrow \mathbf{q}, \mathbf{u} \leftrightarrow \mathbf{v}$ . Inside this space, the image of  $\delta$  consists of those forms which are antisymmetric in each of the pairs  $\mathbf{p}, \mathbf{q}$  and  $\mathbf{u}, \mathbf{v}$ . Then

$$\delta \circ \theta_3 : h(\mathbf{p}, \mathbf{q}, \mathbf{u}, \mathbf{v}) \longrightarrow (\mathbf{p} \mathbf{q})(\mathbf{u} \mathbf{v}) h(\mathbf{p}, \mathbf{q}, \mathbf{u}, \mathbf{v}).$$

Realising  $S_{m+n-2i} \otimes S_{m+n-2j}$  as biforms in  $\mathbf{x}, \mathbf{y}$ , the composite morphism  $\eta_3 \circ \eta_2 \circ \eta_1$  sends  $Q(\mathbf{p}, \mathbf{u}, \mathbf{q}, \mathbf{v})$  to

$$h(m, n; i) h(m, n; j) [\Omega_{\mathbf{p}\mathbf{u}}^i \Omega_{\mathbf{q}\mathbf{v}}^j Q(\mathbf{p}, \mathbf{u}, \mathbf{q}, \mathbf{v})],$$

followed by the substitutions  $\mathbf{p}, \mathbf{u} \rightarrow \mathbf{x}$  and  $\mathbf{q}, \mathbf{v} \rightarrow \mathbf{y}$ . The multiplier  $h$  is as in §1.6. Finally,

$$\eta_4 : R(\mathbf{x}, \mathbf{y}) \longrightarrow h(m+n-2i, m+n-2j; r-i-j) [\Omega_{\mathbf{x}\mathbf{y}}^{r-i-j} R(\mathbf{x}, \mathbf{y})]_{\mathbf{x}, \mathbf{y} \rightarrow \mathbf{z}}.$$

**2.8.**

The  $h$  factors are introduced to ensure that if  $\Psi = (u_i, u_j)_{r-i-j}$ , then the map (see §2.3)

$$\eta_4 \circ \dots \circ \eta_1 \circ f_\Psi : S_{2(m+n-r)} \longrightarrow S_{2(m+n-r)}$$

is the identity. By contrast, the normalising factors appearing in  $\theta_i$  are not so crucial; their purpose is merely to simplify some intermediate expressions. Their omission would have the harmless effect of multiplying each syzygy coefficient by the same factor.

**2.9.**

To recapitulate, for each  $(a, b) \in \Pi(m, n; r)$ , the endomorphism of  $S_{2(m+n-r)}$  defined by the composite

$$\begin{array}{ccccccc}
 S_{2(m+n-r)} & \longrightarrow & S_{2m-2-4a} & \longrightarrow & S_2(S_{m-1}) & \longrightarrow & \wedge^2 S_m \\
 & & \otimes & & \otimes & & \otimes \\
 & & S_{2n-2-4b} & & S_2(S_{n-1}) & & \wedge^2 S_n \\
 & & & & & & \downarrow \\
 S_{2(m+n-r)} & \longleftarrow & S_{m+n-2i} & \longleftarrow & (\bigoplus_i S_{m+n-2i}) & \longleftarrow & (S_m \otimes S_n) \\
 & & \otimes & & \otimes & & \otimes \\
 & & S_{m+n-2j} & & (\bigoplus_j S_{m+n-2j}) & & (S_m \otimes S_n) \\
 & & & & & & \longleftarrow S_2(S_m \otimes S_n)
 \end{array}$$

is the multiplication by  $\kappa_{i,j}^{(a,b)}$ .

**2.10.**

This reduces the calculation of  $\kappa_{i,j}^{(a,b)}$  to the task of chasing a long succession of symbolically defined morphisms. Here we record only the outcome of this calculation, and defer the proof to §7.12. Define

$$\begin{aligned}
 \mathcal{N}_1 &= (m+n-2i+1)! (m+n-2j+1)! (2m-2a)! \times \\
 &\quad (2a+1)! (m-2a-1)! (n-2b-1)! (2m-r-2a+2b)! \times \\
 &\quad (2n-r+2a-2b)! (2m+2n-r-2a-2b-1)!,
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{N}_2 &= j! (m-i)! (m-j)! (m+n-j+1)! (m+n-r+i-j)! \times \\
 &\quad (m+n-r-i+j)! (2m+2n-r-i-j+1)! \times \\
 &\quad (2m-4a-2)! (2n-4b-2)!.
 \end{aligned}$$

Let  $\Lambda = \Lambda(m, n, r; a, b)$  denote the set of integer triples  $(x, y, z)$  satisfying the inequalities

$$\begin{aligned} 0 &\leq x \leq \min(n - 2b - 1, n - j), \\ \max(0, n - r + 2a + 1 - x) &\leq y \leq \min(2a + 1, 2n - r + 2a - 2b), \\ \max(0, r - m - i - x) &\leq z \leq \min(n - i, r - i - j, n - i + 2a + 1 - y). \end{aligned}$$

For  $(x, y, z) \in \Lambda(m, n, r; a, b)$ , let

$$\begin{aligned} \mathbb{T}_1 &= (n - x)! (m - j + x)! (n - 2b - 1 + x)! \times \\ &\quad (m - 2a - 1 + y)! (r - 2a - 2b - 2 + y)! (m + n - 2i - z)! \times \\ &\quad (m + n - r + i - j + z)! (n - i + 2a + 1 - y - z)!, \\ \mathbb{T}_2 &= x! y! z! (n - j - x)! (n - 2b - 1 - x)! (2a + 1 - y)! \times \\ &\quad (2m - 4a - 1 + y)! (2n - r + 2a - 2b - y)! (n - i - z)! (r - i - j - z)! \times \\ &\quad (m + n - i + 1 - z)! (m - r + i + x + z)! (-n + r - 2a - 1 + x + y)!, \end{aligned}$$

and now define

$$(2.8) \quad \Gamma = (-1)^{n-j} \sum_{(x,y,z) \in \Lambda} (-1)^{x+y+z} \frac{\mathbb{T}_1}{\mathbb{T}_2}.$$

Then we have the formula

$$(2.9) \quad \kappa_{i,j}^{(p)} = \frac{\mathcal{N}_1}{\mathcal{N}_2} \Gamma.$$

From the definition of  $\kappa$  (but certainly not from its formula), it is clear that

$$\kappa_{j,i}^{(p)} = (-1)^{r-i-j} \kappa_{i,j}^{(p)}.$$

Now use the sign rule (1.2) to enforce  $i \leq j$ , and let

$$\vartheta_{i,j}^{(p)} = \begin{cases} 2 \kappa_{i,j}^{(p)} & \text{if } i \neq j, \\ \kappa_{i,j}^{(p)} & \text{if } i = j. \end{cases}$$

Then one can immediately rewrite (2.7) as a syzygy

$$(2.10) \quad \sum_{0 \leq i \leq j \leq r} \vartheta_{i,j}^{(p)} (\mathbf{u}_i, \mathbf{u}_j)_{r-i-j} = 0,$$

for every  $p \in \Pi(m, n; r)$ .

**2.11.**

The numerical restrictions on  $i, j, a, b$  ensure that only factorials of non-negative numbers appear in  $\mathcal{N}_1, \mathcal{N}_2$ , in particular the  $\mathcal{N}_i$  are always nonzero. Similarly, each lattice point  $(x, y, z) \in \Lambda$  is such that only factorials of non-negative integers appear in each  $\mathbb{T}_i$ . The rational number  $\Gamma$  is (up to a

factor) a 9-j symbol in the sense of the quantum theory of angular momentum; this will be further explained in §7.8.

If  $(i, j) = (0, r)$  and  $p = (0, 0)$ , then  $\Lambda$  reduces to the single triple  $(x, y, z) = (n - r, 1, 0)$ , which forces  $\Gamma \neq 0$ . As we remarked before, this implies Theorem 1.2.

Of course it will often happen that  $\vartheta_{0,r}^{(a,b)} \neq 0$  for values of  $(a, b)$  other than  $(0, 0)$ . E.g., for  $(m, n, r) = (8, 6, 5)$  we have  $\vartheta_{0,5}^{(1,0)} = -2/63$ . Hence, in general  $\mathbf{u}_r$  can be expressed in terms of  $\mathbf{u}_0, \dots, \mathbf{u}_{r-1}$  in more than one way.

**2.12.**

It is evident that the formula for the syzygy coefficients is very complicated, hence one would like some reassurance that it is indeed correct. To this end, we programmed it in MAPLE. E.g., let  $(m, n, r) = (7, 5, 4)$ , and choose  $(a, b) = (0, 1)$ . Then it gives the syzygy

$$\begin{aligned} & (\mathbf{u}_0, \mathbf{u}_0)_4 + \frac{8}{3} (\mathbf{u}_0, \mathbf{u}_1)_3 + \frac{54}{55} (\mathbf{u}_0, \mathbf{u}_2)_2 - \frac{1}{6} (\mathbf{u}_0, \mathbf{u}_3)_1 - \frac{10}{63} \mathbf{u}_0 \mathbf{u}_4 \\ & - \frac{7}{12} (\mathbf{u}_1, \mathbf{u}_1)_2 + \frac{63}{55} (\mathbf{u}_1, \mathbf{u}_2)_1 + \frac{49}{72} \mathbf{u}_1 \mathbf{u}_3 - \frac{1512}{3025} \mathbf{u}_2^2 = 0, \end{aligned}$$

which, as another MAPLE calculation shows, is indeed true of generic  $A$  and  $B$ . The formula has met the test in scores of such cases, in particular we are quite confident that it involves no typographical errors.

**2.13. Formulae for  $\mathbf{u}_2, \mathbf{u}_3$**

For  $r = 2, 3$ , we get  $\Pi(m, n; r) = \{(0, 0)\}$ . This gives a unique syzygy in either case, leading to the formulae below.

$$\mathbf{u}_0 \mathbf{u}_2 = z_1 (\mathbf{u}_0, \mathbf{u}_0)_2 + z_2 \mathbf{u}_1^2 + z_3 (\mathbf{u}_0, \mathbf{u}_1)_1,$$

where

$$\begin{aligned} z_1 &= \frac{(m - 2 + n)(m - 1 + n)}{2(m - 1)(n - 1)}, \\ z_2 &= \frac{m n (m - 2 + n)(m - 1 + n)}{(m - 1)(n - 1)(m + n)^2}, \\ z_3 &= \frac{(m - 1 + n)(m - 2 + n)(m - n)}{(m - 1)(n - 1)(m + n)}, \end{aligned}$$

and

$$\mathbf{u}_0 \mathbf{u}_3 = w_1 (\mathbf{u}_0, \mathbf{u}_1)_2 + w_2 (\mathbf{u}_0, \mathbf{u}_2)_1 + w_3 \mathbf{u}_1 \mathbf{u}_2,$$

where

$$\begin{aligned}
 w_1 &= \frac{(m - 4 + n)(m - 3 + n)}{(m - 2)(n - 2)}, \\
 w_2 &= \frac{(m - 3 + n)(m - 4 + n)(m - n)}{(m - 2)(n - 2)(m - 2 + n)}, \\
 w_3 &= \frac{mn(m - 4 + n)(m - 3 + n)}{(m - 2)(n - 2)(m + n)(m - 1 + n)}.
 \end{aligned}$$

**2.14. A closed form syzygy**

For every  $r \geq 2$ , the space  $\mathbb{K}(m, n; r)$  of quadratic syzygies contains a distinguished syzygy whose coefficients admit a particularly simple form. We deduce it in this section, which gives another proof of Theorem 1.2. We will use the general formalism of [20, §3.2.5] for the symbolic computations.

Let the notation be as in the beginning of §2.7. Consider the map

$$\alpha : S_{2(m+n-r)} \longrightarrow S_2(S_m \otimes S_n)$$

which sends  $f_z^{2(m+n-r)}$  to the form  $Q(\mathbf{p}, \mathbf{u}, \mathbf{q}, \mathbf{v})$ , given by

$$\begin{aligned}
 &(\mathbf{p} \mathbf{u})^r f_{\mathbf{p}}^{m-r} f_{\mathbf{u}}^{n-r} f_{\mathbf{q}}^m f_{\mathbf{v}}^n + (\mathbf{q} \mathbf{v})^r f_{\mathbf{p}}^m f_{\mathbf{u}}^n f_{\mathbf{q}}^{m-r} f_{\mathbf{v}}^{n-r} \\
 &- (\mathbf{q} \mathbf{u})^r f_{\mathbf{p}}^m f_{\mathbf{u}}^{n-r} f_{\mathbf{q}}^{m-r} f_{\mathbf{v}}^n - (\mathbf{p} \mathbf{v})^r f_{\mathbf{p}}^{m-r} f_{\mathbf{u}}^n f_{\mathbf{q}}^m f_{\mathbf{v}}^{n-r}.
 \end{aligned}$$

It is clear that  $Q$  is invariant under the simultaneous exchanges  $\mathbf{p} \leftrightarrow \mathbf{q}$  and  $\mathbf{u} \leftrightarrow \mathbf{v}$ . Notice that it is antisymmetric in each of the pairs  $\mathbf{p}, \mathbf{q}$  and  $\mathbf{u}, \mathbf{v}$ ; i.e., it lies in the image of  $\delta$ . Thus we can deduce a syzygy by calculating  $\eta_4 \circ \dots \circ \eta_1 \circ \alpha$ . Write  $Q = T_1 + T_2 - T_3 - T_4$  (using obvious notation). We should like to assess the effect of  $\eta_3 \circ \eta_2 \circ \eta_1$  on each  $T_k$ .

Now the effect of  $\Omega_{\mathbf{q}\mathbf{v}}$  on (say)  $T_3$  can be seen as follows: we extract one each of the  $\mathbf{q}$  and  $\mathbf{v}$  factors, and contract them against each other. E.g., a contraction of a  $(\mathbf{q}\mathbf{u})$  with an  $f_{\mathbf{v}}$  produces an  $f_{\mathbf{u}}$ . The contraction of  $f_{\mathbf{q}}$  with  $f_{\mathbf{v}}$  leads to  $(f f) = 0$ , hence such a choice contributes nothing. After  $j$  such extractions one sees that  $\Omega_{\mathbf{q}\mathbf{v}}^j \circ T_3$  is a constant multiple of

$$T'_3 = (\mathbf{q} \mathbf{u})^{r-j} f_{\mathbf{p}}^m f_{\mathbf{u}}^{n-r+j} f_{\mathbf{q}}^{m-r} f_{\mathbf{v}}^{n-j}.$$

(This constant, which we will not write down explicitly, is obtained by counting all possible choices of such contractions.) By the same argument,  $\Omega_{\mathbf{p}\mathbf{u}}^j \circ T'_3$  is a constant multiple of

$$T''_3 = (\mathbf{q} \mathbf{u})^{r-i-j} f_{\mathbf{p}}^{m-i} f_{\mathbf{u}}^{n-r+j} f_{\mathbf{q}}^{m-r+i} f_{\mathbf{v}}^{n-j}.$$



After the substitutions  $\mathbf{p}, \mathbf{u} \rightarrow \mathbf{x}$  and  $\mathbf{q}, \mathbf{v} \rightarrow \mathbf{y}$  into  $T_3''$  we get

$$(\mathbf{x} \mathbf{y})^{r-i-j} f_{\mathbf{x}}^{m+n-r-i+j} f_{\mathbf{y}}^{m+n-r+i-j}$$

(up to a sign). A similar analysis applies to  $T_4$ . As to  $T_2$ , notice that if  $j < r$  then at least one bracket factor  $(\mathbf{q} \mathbf{v})$  remains after the extractions, hence the expression goes to zero after  $\mathbf{q}, \mathbf{v} \rightarrow \mathbf{y}$ . Thus  $T_2$  gives a nonzero contribution only for  $i = 0, j = r$ , and  $T_1$  only for  $i = r, j = 0$ .

Now calculating the coefficients is only a matter of keeping track of the multiplying factors. This is straightforward, hence we omit the details. The resulting expression is as follows:

Define  $\delta_{i,j}$  to be 1 if  $i = j$ , and 0 otherwise; and  $\epsilon_{i,j}$  to be 1 if  $i = j$ , and 2 otherwise. Let

$$\beta_{i,j} = \frac{m! n! r! (m+n-2i+1)! (m+n-2j+1)!}{i! j! (n-i)! (m-j)! (r-i-j)! (m+n-i+1)! (m+n-j+1)!},$$

and define

$$(2.11) \quad \vartheta_{i,j} = \epsilon_{i,j} (\delta_{i,0} \delta_{j,r} + \delta_{i,r} \delta_{j,0} - \beta_{i,j} - (-1)^{r+i+j} \beta_{j,i}).$$

Then we have a syzygy in the notation of (1.10). (As before, we have thoroughly checked this formula in MAPLE.)

LEMMA 2.3. — *The coefficient  $\vartheta_{0,r}$  is nonzero (in fact, strictly positive).*

*Proof.* — We are reduced to proving the inequality

$$(2.12) \quad \binom{m+n-r+1}{r} > \binom{m}{r} + \binom{n}{r}.$$

Assume that we have a chest filled with  $(m-r+1)$  Spanish silver coins,  $(r-1)$  Spanish gold coins and  $(n-r+1)$  French gold coins, altogether making a total of  $(m+n-r+1)$  coins. Let  $\mathbf{S}$  be the set of subcollections of  $r$  Spanish coins, and  $\mathbf{G}$  the set of subcollections of  $r$  gold coins. Then  $\mathbf{S} \cap \mathbf{G} = \emptyset$ , but every member of  $\mathbf{S} \cup \mathbf{G}$  gives a subcollection of  $r$  coins from the entire chest. Hence the left hand side of (2.12) is no smaller than the right hand side.

Now consider a subcollection formed out of  $(r-2)$  Spanish gold coins, a single Spanish silver coin, and a single French gold coin. It does not belong either to  $\mathbf{S}$  or  $\mathbf{G}$ , hence the inequality must be strict. □

### 3. The incomplete Segre imbedding

In this section we give a geometric interpretation to the redundancy result.

3.1.

Write  $W = S_{m+n} \oplus S_{m+n-2}$ , and consider the morphism

$$\sigma : \mathbf{P}S_m \times \mathbf{P}S_n \longrightarrow \mathbf{P}W, \quad (A, B) \longrightarrow [u_0, u_1].$$

THEOREM 3.1. — *The morphism  $\sigma$  is an imbedding of algebraic varieties.*

Since  $W$  is a subrepresentation of

$$S_m \otimes S_n \simeq H^0(\mathbf{P}^m \times \mathbf{P}^n, \mathcal{O}_{\mathbf{P}S_m \times \mathbf{P}S_n}(1, 1))$$

(using the self-duality of §1.8), the morphism  $\sigma$  is defined by an incomplete linear subseries of  $|\mathcal{O}_{\mathbf{P}^m \times \mathbf{P}^n}(1, 1)|$ .

*Proof.* — By Theorem 1.2, the  $u_0, u_1$  determine all the higher  $u_r$ . Hence they determine the pair  $(A, B)$  up to an ambiguity of  $(\eta A, \frac{1}{\eta} B)$  for some constant  $\eta \in \mathbb{k}^*$ . This shows that  $\sigma$  is set-theoretically injective.

By [25, Ch. II, Prop. 7.3], it suffices to show that the map  $d\sigma$  on tangent spaces is injective. A tangent vector to  $\mathbf{P}S_m \times \mathbf{P}S_n$  at  $(A, B)$  can be represented by a pair of binary forms  $(M, N)$  of orders  $m, n$ , considered modulo scalar multiples of  $A, B$  respectively (cf. [24, Lecture 16]). Its image via  $d\sigma$  is given by

$$\begin{aligned} \lim_{\delta \rightarrow 0} \frac{1}{\delta} [\sigma(A + \delta M, B + \delta N) - \sigma(A, B)] \\ = (AN + MB, (A, N)_1 + (M, B)_1). \end{aligned}$$

Assume that the image vanishes, then there exists a constant  $c$  such that

$$(3.1) \quad AN + MB = cAB, \quad (A, N)_1 + (M, B)_1 = c(A, B)_1.$$

Let  $N' = N - cB$ , and  $Q = \gcd(A, B)$ . Then we may write  $A = A'Q, B = B'Q$  where  $A', B'$  are coprime. The first equality in (3.1) leads to  $A'N' = -MB'$ , so we must have  $N' = B'R$  for some  $R$ , and then  $M = -A'R$ . Hence

$$(A, N')_1 + (M, B)_1 = (A'Q, B'R)_1 - (A'R, B'Q)_1 = 0.$$

By the next lemma this implies that  $A'B'(Q, R)_1 = 0$ , i.e.,  $(Q, R)_1 = 0$ . This forces  $R = eQ$  for some constant  $e$  (see [21, Lemma 2.2]). But then

$$M = -eA, \quad N = (e + c)B,$$

proving that  $(M, N)$  was the zero vector. This shows that  $d\sigma$  is injective. □

LEMMA 3.2. — *Let  $A, B$  denote binary forms of orders  $m, n$ , and  $Q, R$  of order  $s$ . Then we have an equality*

$$(AQ, BR)_1 - (AR, BQ)_1 = \frac{s(m+n+2s)}{(m+s)(n+s)} AB(Q, R)_1.$$

*Proof.* — Write  $A = a_{\mathbf{x}}^m, B = b_{\mathbf{x}}^n, Q = q_{\mathbf{x}}^s, R = r_{\mathbf{x}}^s$ . A general recipe for calculating transvectants of symbolic products is given in [20, §3.2.5]. It gives the expression

$$(3.2) \quad (AQ, BR)_1 = \frac{1}{(m+s)(n+s)} a_{\mathbf{x}}^{m-1} b_{\mathbf{x}}^{n-1} q_{\mathbf{x}}^{s-1} r_{\mathbf{x}}^{s-1} \times \\ \{mn(ab)q_{\mathbf{x}}r_{\mathbf{x}} + ms(ar)b_{\mathbf{x}}q_{\mathbf{x}} + ns(qb)a_{\mathbf{x}}r_{\mathbf{x}} + s^2(qr)a_{\mathbf{x}}b_{\mathbf{x}}\},$$

and similarly

$$(3.3) \quad (AR, BQ)_1 = \frac{1}{(m+s)(n+s)} a_{\mathbf{x}}^{m-1} b_{\mathbf{x}}^{n-1} q_{\mathbf{x}}^{s-1} r_{\mathbf{x}}^{s-1} \times \\ \{mn(ab)q_{\mathbf{x}}r_{\mathbf{x}} + ms(aq)b_{\mathbf{x}}r_{\mathbf{x}} + ns(rb)a_{\mathbf{x}}q_{\mathbf{x}} + s^2(rq)a_{\mathbf{x}}b_{\mathbf{x}}\}.$$

Use Plücker syzygies to write

$$(ar)b_{\mathbf{x}}q_{\mathbf{x}} = (qr)a_{\mathbf{x}}b_{\mathbf{x}} + (aq)b_{\mathbf{x}}r_{\mathbf{x}}, \quad (qb)a_{\mathbf{x}}r_{\mathbf{x}} = (rb)a_{\mathbf{x}}q_{\mathbf{x}} + (qr)a_{\mathbf{x}}b_{\mathbf{x}}.$$

Substitute these into (3.2), and subtract (3.3) from the result. We are left with

$$(AQ, BR)_1 - (AR, BQ)_1 = \frac{(ms + ns + 2s^2)}{(m+s)(n+s)} (qr)a_{\mathbf{x}}^m b_{\mathbf{x}}^n q_{\mathbf{x}}^{s-1} r_{\mathbf{x}}^{s-1},$$

which completes the calculation, as well as the proof of the theorem.  $\square$

### 3.2.

Theorem 3.1 implies that any expression in the  $\{u_0, u_1, u_2, \dots\}$  admits a ‘formula’ in terms of  $u_0, u_1$ . In order to make this precise, let  $\mathcal{E}$  denote an arbitrary compound transvectant expression which is homogeneous of degree  $e$  and isobaric of weight  $w$ . For instance,

$$(u_1, (u_0, u_3)_3)_2 - 3u_2(u_0, u_5)_2 + 5(u_1, u_0 u_7)_1$$

is of degree 3 (since each term involves three  $u_r$ ), and isobaric of weight 9 (e.g., in the first term  $1 + 0 + 3 + 3 + 2 = 9$ ).

PROPOSITION 3.3. — *With notation as above, there exists an identity of the form*

$$\mathcal{E}(u_0, \dots, u_r) = \frac{\mathcal{Q}(u_0, u_1)}{u_0^N},$$

for some positive integer  $N$ .

*Proof.* — Let  $Y = \text{image } \sigma$ . The expression  $\mathcal{E}$  corresponds to an equivariant morphism

$$\varphi_{\mathcal{E}} : S_{e(m+n)-2w} \longrightarrow H^0(\mathcal{O}_{PS_m \otimes S_n}(e)) \simeq H^0(\mathcal{O}_Y(e)).$$

Consider the exact sequence

$$H^0(\mathcal{O}_{PW}(e + N)) \longrightarrow H^0(\mathcal{O}_Y(e + N)) \longrightarrow H^1(\mathcal{I}_Y(e + N)).$$

Now, to say that  $\mathbf{u}_0^N \mathcal{E}$  can be rewritten as a compound expression  $\mathcal{Q}(\mathbf{u}_0, \mathbf{u}_1)$  is to say that  $\varphi_{\mathbf{u}_0^N} \mathcal{E}$  can be lifted to a morphism

$$S_{(e+N)(m+n)-2w} \longrightarrow S_{e+N} W \simeq H^0(\mathcal{O}_{PW}(e + N)).$$

But this can always be arranged by choosing  $N$  sufficiently large, so that the group  $H^1(\mathcal{I}_Y(e + N)) = 0$ . □

This is analogous to the result on associated forms (see [23, §131]). The smallest such  $N$  is bounded above by the Castelnuovo regularity of  $\mathcal{I}_Y$  (see [34, Lecture 6]).

### 3.3.

It is a natural problem to find a set of  $SL_2$ -invariant defining equations for the variety  $Y = \text{image } (\sigma)$ . The syzygies calculated above can be used to solve this problem; we illustrate this with an example.

*Example 3.4.* — Assume  $m = n = 2$ . In the notation of §2.4, we have  $\mathcal{C} = S_2 \otimes S_2 = S_4 \oplus S_2 \oplus S_0$ . The three summands correspond to the three quadratic syzygies

$$\begin{aligned} \mathbf{u}_0 \mathbf{u}_2 &= \frac{3}{2} \mathbf{u}_1^2 + 3(\mathbf{u}_0, \mathbf{u}_0)_2, & \mathbf{u}_1 \mathbf{u}_2 &= -3(\mathbf{u}_0, \mathbf{u}_1)_2, \\ \mathbf{u}_2^2 &= \frac{3}{2} (\mathbf{u}_0, \mathbf{u}_0)_4 - \frac{3}{2} (\mathbf{u}_1, \mathbf{u}_1)_2. \end{aligned}$$

These are the equations of the usual Segre imbedding  $\mathbf{P}S_2 \times \mathbf{P}S_2 \rightarrow \mathbf{P}(S_4 \oplus S_2 \oplus S_0)$  in disguise. Now isolate  $\mathbf{u}_2$  from the first equation and substitute into the other two, then we get the following defining equations for  $Y$  in degrees 3 and 4 respectively:

$$(3.4) \quad \begin{aligned} \mathbf{u}_1 [\mathbf{u}_1^2 + 2(\mathbf{u}_0, \mathbf{u}_0)_2] + 2\mathbf{u}_0 (\mathbf{u}_0, \mathbf{u}_1)_2 &= 0, \\ [\mathbf{u}_1^2 + 2(\mathbf{u}_0, \mathbf{u}_0)_2]^2 - \frac{2}{3} \mathbf{u}_0^2 [(\mathbf{u}_0, \mathbf{u}_0)_4 - (\mathbf{u}_1, \mathbf{u}_1)_2] &= 0. \end{aligned}$$

However, these equations do not generate the ideal of  $Y$ . We wrote down the map  $\sigma$  in coördinates, and calculated the ideal  $I_Y$  using Macaulay-2. The outcome shows that  $I_Y$  is generated by 20-dimensional space of equations

in degree 3. By construction, the degree 3 part  $(I_Y)_3$  is a subrepresentation of

$$S_3(S_4 \oplus S_2) \simeq S_{12} \oplus S_{10} \oplus (S_8)^{\oplus 2} \oplus (S_6)^{\oplus 5} \oplus (S_4)^{\oplus 4} \oplus (S_2)^{\oplus 4} \oplus S_0.$$

(This was calculated using John Stembridge’s ‘SF’ package for MAPLE.)

Each irreducible summand of  $(I_Y)_3$  corresponds to a cubic syzygy involving only  $u_0, u_1$ . By an exhaustive search we found the syzygies

$$(3.5) \quad \left. \begin{aligned} (u_1^2, u_1)_2 + 2((u_0, u_1)_2, u_0)_2 + 2((u_0, u_0)_2, u_1)_2 \\ ((u_0, u_1)_1, u_1)_2 \end{aligned} \right\} = 0,$$

in order 2, together with

$$(3.6) \quad \left. \begin{aligned} u_1^3 + 9u_0(u_0, u_1)_2 - 7(u_0^2, u_1)_2 \\ 3u_1(u_0, u_1)_1 + 7(u_0^2, u_0)_3 \end{aligned} \right\} = 0,$$

in order 6. This corresponds to the  $SL_2$ -isomorphism

$$(I_Y)_3 \simeq (S_6 \oplus S_2)^{\oplus 2}.$$

To recapitulate, the equations (3.4) define the variety  $Y$  set-theoretically, whereas (3.5) and (3.6) together generate its ideal.

**Problem 1.** Find similar equations for general  $m, n$ .

### 3.4. The minimal equation for $u_1$

Assume  $m = n = 2$ . If  $u_0$  is given, then  $u_1$  may assume  $\binom{4}{2} = 6$  possible values, hence  $u_1$  must satisfy a degree 6 univariate polynomial equation whose coefficients are covariants of  $u_0$ . (The argument leading to this conclusion is very similar to [12, §6.3], hence we will not reproduce it here.) The minimal equation must have the form

$$(3.7) \quad u_1^6 + \varphi_{2,4} u_1^4 + \varphi_{4,8} u_1^2 + \varphi_{6,12} = 0,$$

where  $\phi_{k,2k}$  is a covariant of  $u_0$  of degree  $k$  and order  $2k$ . (Since  $(A, B)_1 = -(B, A)_1$ , only even powers of  $u_1$  appear in the equation.) The actual terms are easily calculated as in [loc. cit.]. Define the following covariants of  $u_0$  (cf. [23, §89]):

$$H = (u_0, u_0)_2, \quad I = (u_0, u_0)_4, \quad T = (u_0, H)_1,$$

and then

$$\varphi_{2,4} = 6H, \quad \varphi_{4,8} = -2Iu_0^2 + 12H^2, \quad \varphi_{6,12} = -16T^2.$$

**Problem 2.** Find the minimal equation of  $u_1$  for any  $m, n$ . It will necessarily be of degree  $\binom{m+n}{m}$ .

### 4. $SL_3$ -representations

It would be of interest to know whether there is an analogue of Theorem 1.2 for  $SL_N$ -representations when  $N \geq 3$ . Specifically, let  $\lambda, \mu$  denote two partitions, and  $\mathbb{S}_\lambda, \mathbb{S}_\mu$  the corresponding irreducible representations of  $SL_N$  (see<sup>(3)</sup> [4] and [18, Ch. 8]). There is a decomposition

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu \simeq \bigoplus_{\nu} (\mathbb{S}_\nu \otimes \mathbb{k}^{\langle \lambda, \mu; \nu \rangle}),$$

quantified over partitions  $\nu$  such that  $|\nu| = |\lambda| + |\mu|$ . The integers  $\langle \lambda, \mu; \nu \rangle$  are usually called Littlewood-Richardson numbers. We have a series of  $SL_N$ -equivariant projection morphisms (described in [4, §IV.2])

$$\pi_\nu^{(w)} : \mathbb{S}_\lambda \otimes \mathbb{S}_\mu \longrightarrow \mathbb{S}_\nu,$$

parametrised by lattice words  $w$  of content  $\mu$  and shape  $\nu - \lambda$ . (Thus there are exactly  $\langle \lambda, \mu; \nu \rangle$  such words.) Let  $A \in \mathbb{S}_\lambda, B \in \mathbb{S}_\mu$  denote generic tensors, and write

$$(4.1) \quad \mathbf{u}_\nu^{(w)} = \pi_\nu^{(w)}(A, B),$$

which are the analogues of transvectants in the  $SL_N$ -case. If  $\langle \lambda, \mu; \nu \rangle = 1$ , then  $w$  may be safely omitted from the notation.

**Problem 3.** Find a subcollection of  $\{\mathbf{u}_\nu^{(w)} : (w, \nu)\}$  which determines the rest.

We will work out one such example for  $SL_3$ ; but first it is necessary to recall some generalities on the (ternary) symbolic method. We will follow the formalism of [32, p. 334 ff].

#### 4.1. The symbolic L-R multiplication

Assume  $N = 3$ . Let  $V$  denote a three-dimensional vector space with basis  $\mathbf{x} = (x_1, x_2, x_3)$ , and  $\mathbf{u} = (u_1, u_2, u_3)$  the dual basis of  $V^*$ . Given  $\lambda = (\lambda_1, \lambda_2)$ , there is a natural split injection (see [19, §15.5])

$$\mathbb{S}_\lambda V \hookrightarrow \text{Sym}^{\lambda_2} V^* \otimes \text{Sym}^{\lambda_1 - \lambda_2} V.$$

Hence an element  $A \in \mathbb{S}_\lambda$  can be represented as a polynomial of degree  $\lambda_2$  in  $\mathbf{u}$ , and of degree  $\lambda_1 - \lambda_2$  in  $\mathbf{x}$ . In classical terminology,  $A$  is of degree  $\lambda_1 - \lambda_2$  and class  $\lambda_2$ .

---

<sup>(3)</sup>Note however that the conventions governing Young diagrams in [4] and [18] are conjugates of each other. We will follow the latter.

Now, for instance, consider the tableau

$$T = \begin{pmatrix} a & a & a & a & a \\ b & b & b & & \end{pmatrix}$$

on the shape  $\lambda = (5, 3)$ . Reading it columnwise, we get the symbolic expression  $\mathcal{E} = (a b \mathbf{u})^3 a_{\mathbf{x}}^2$ . Here

$$(a b \mathbf{u}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ u_1 & u_1 & u_2 \end{vmatrix}, \quad a_{\mathbf{x}} = a_1 x_1 + a_2 x_2 + a_3 x_3,$$

with similar notation to follow.

Given an arbitrary  $A(\mathbf{x}, \mathbf{u}) \in \mathbb{S}_{(5,3)}$ , construct a differential operator  $\tilde{A}$  by replacing each  $x_i$  by  $\frac{\partial}{\partial a_i}$ , and  $u_1, u_2, u_3$  by

$$\frac{\partial^2}{\partial a_2 \partial b_3} - \frac{\partial^2}{\partial b_2 \partial a_3}, \quad \frac{\partial^2}{\partial a_3 \partial b_1} - \frac{\partial^2}{\partial b_3 \partial a_1}, \quad \frac{\partial^2}{\partial a_1 \partial b_2} - \frac{\partial^2}{\partial b_1 \partial a_2}$$

respectively. Then we have an identity

$$A(\mathbf{x}, \mathbf{u}) = \frac{3}{6! 3!} (\tilde{A} \circ \mathcal{E}).$$

In this sense,  $\mathcal{E}$  represents a ‘generic’ form of degree 2 and class 3. The general result is as follows:

LEMMA 4.1. — *Let  $\lambda = (\lambda_1, \lambda_2)$ , and  $\mathcal{E} = (a b \mathbf{u})^{\lambda_2} a_{\mathbf{x}}^{\lambda_1 - \lambda_2}$ . Then for any polynomial  $A(\mathbf{x}, \mathbf{u}) \in \mathbb{S}_{\lambda}$ , we have an identity*

$$A(\mathbf{x}, \mathbf{u}) = \frac{\lambda_1 - \lambda_2 + 1}{(\lambda_1 + 1)! \lambda_2!} (\tilde{A} \circ \mathcal{E}).$$

Hence, every tensor  $A$  can be represented by the corresponding differential operator  $\tilde{A}$ . We will omit the proof of the lemma, since we will make no use of this scaling factor. In general, an element of  $\mathbb{S}_{\lambda}$  may be described by several polynomials  $A(\mathbf{x}, \mathbf{u})$ , because of the identical relation  $x_1 u_1 + x_2 u_2 + x_3 u_3 = 0$ . For instance,  $A = x_1 u_1 + 2 x_2 u_2$  and  $A' = x_2 u_2 - x_3 u_3$  represent the same element of  $\mathbb{S}_{(2,1)}$ . This leads to no complications however, since  $\tilde{A} = \tilde{A}'$ .

### 4.2.

Continuing the example above, let  $\mathcal{E}' = (c d \mathbf{u}) c_{\mathbf{x}}^2$  corresponding to  $T' = \begin{pmatrix} c & c & c \\ d & & \end{pmatrix}$ . Given  $B(\mathbf{x}, \mathbf{u}) \in \mathbb{S}_{(3,1)}$ , define  $\tilde{B}$  by replacing  $x_i$  by  $\frac{\partial}{\partial c_i}$  etc.

The L-R number  $\langle\langle 5, 3 \rangle, (3, 1); (5, 4) \rangle = 2$ , i.e., there are two linearly independent maps

$$\pi_{(5,4)}^{(z_i)} : \mathbb{S}_{(5,3)} \otimes \mathbb{S}_{(3,1)} \longrightarrow \mathbb{S}_{(5,4)}, \quad i = 1, 2.$$

They can be explicitly written down as follows: one can use the L-R procedure to unload the entries of  $T'$  and attach them to  $T$  (see [19, Appendix A]); this leads to two possible tableaux

$$\left( \begin{array}{cccccc} a & a & a & a & a & c \\ b & b & b & c & c & \\ d & & & & & \end{array} \right), \quad \left( \begin{array}{cccccc} a & a & a & a & a & c \\ b & b & b & c & d & \\ c & & & & & \end{array} \right)$$

on the shape  $(6, 5, 1)$ . (Notice that  $\mathbb{S}_{(6,5,1)} \simeq \mathbb{S}_{(5,4)}$  for  $SL_3$ .) If we read the newly added entries from top to bottom and right to left, then we get the corresponding lattice words  $z_1 = ccc d, z_2 = cdcc$ . Form the symbolic expressions

$$Q_1 = (abd) (ab\mathbf{u})^2 (ac\mathbf{u})^2 c_{\mathbf{x}}, \quad Q_2 = (abc) (ab\mathbf{u})^2 (ac\mathbf{u}) (ad\mathbf{u}) c_{\mathbf{x}},$$

and now the required maps are given by

$$\pi_{(5,4)}^{(z_i)} = \tilde{A} \tilde{B} \circ Q_i, \quad i = 1, 2.$$

*Example 4.2.* — Consider the following decomposition<sup>(4)</sup> of representations of  $SL_3$ :

$$\mathbb{S}_{(2,1)} \otimes \mathbb{S}_{(2,1)} \simeq \underbrace{\mathbb{S}_{(4,2)} \oplus \mathbb{S}_{(3)} \oplus \mathbb{S}_{(3,3)} \oplus (\mathbb{S}_{(2,1)} \otimes \mathbb{k}^2)}_{\mathbf{E}} \oplus \mathbb{S}_{(0)},$$

with  $\mathbf{u}_{(4,2)}$  etc. as in (4.1). The point of the example is to show that  $\mathbf{u}_{(0)}$  is redundant, i.e., it can be recovered from the rest of the factors. For instance, the map  $\mathbb{S}_{(2,1)} \otimes \mathbb{S}_{(2,1)} \longrightarrow \mathbb{S}_{(3)}$  takes  $A \otimes B$  to

$$\mathbf{u}_{(3)} = \tilde{A} \tilde{B} \circ (abd) a_{\mathbf{x}} c_{\mathbf{x}}^2.$$

Henceforth the operators  $\tilde{A}, \tilde{B}$  will be understood, and we will avoid writing them explicitly. Thus,

$$\begin{aligned} \mathbf{u}_{(4,2)} &= (ab\mathbf{u}) (ad\mathbf{u}) c_{\mathbf{x}}^2, & \mathbf{u}_{(3,3)} &= (ab\mathbf{u}) (ac\mathbf{u}) (cd\mathbf{u}), \\ \mathbf{u}_{(0)} &= (abc) (acd), \end{aligned}$$

and

$$\mathbf{u}_{(2,1)}^{(w_1)} = (abd) (ac\mathbf{u}) c_{\mathbf{x}}, \quad \mathbf{u}_{(2,1)}^{(w_2)} = (abc) (ad\mathbf{u}) c_{\mathbf{x}}$$

---

<sup>(4)</sup> Throughout this example, all the calculations involving inner and outer plethysms were carried out using the ‘SF’ (Symmetric Functions) package for MAPLE, written by John Stembridge.



corresponding to the words  $w_1 = ccd$  and  $w_2 = cdc$ . As in the binary case, the quadratic syzygies between the  $\mathbf{u}_\nu$  correspond to the summands of

$$\mathcal{C} = \wedge^2 \mathbb{S}_{(2,1)} \otimes \wedge^2 \mathbb{S}_{(2,1)}.$$

Using SF we find that there are 9 copies of the module  $\mathbb{S}_{(4,2)}$  inside  $\mathcal{C}$ , and hence a 9-dimensional space of syzygies of degree 2 and order 2.

Now, in order to build quadratic syzygies, we need to write down all possible maps  $\mathbb{S}_2(\mathbf{E}) \rightarrow \mathbb{S}_{(4,2)}$ ; which is of course done similarly. E.g., there is (up to constant) a unique map  $\mathbb{S}_{(4,2)} \otimes \mathbb{S}_{(3)} \rightarrow \mathbb{S}_{(4,2)}$  given by

$$(ab\mathbf{u})^2 a_{\mathbf{x}}^2 \otimes c_{\mathbf{x}}^3 \rightarrow (abc)(ab\mathbf{u})(ac\mathbf{u}) a_{\mathbf{x}} c_{\mathbf{x}}, \text{ etc.}$$

Using SF again, one sees that the space  $\text{Hom}_{SL_3}(\mathbb{S}_2(\mathbf{E}), \mathbb{S}_{(4,2)})$  is 19-dimensional. We wrote down all the maps explicitly, and found a 9-dimensional subspace of syzygies by solving a system of linear equations. (This was done in MAPLE.) One conveniently chosen syzygy is the following:

$$\begin{aligned} \mathbf{u}_{(4,2)} \mathbf{u}_{(0)} &= \frac{3}{12800} \pi^{(z_1)}(\mathbf{u}_{(4,2)}, \mathbf{u}_{(4,2)}) + \frac{3}{1280} \pi^{(z_2)}(\mathbf{u}_{(4,2)}, \mathbf{u}_{(4,2)}) \\ &- \frac{5}{448} \pi(\mathbf{u}_{(4,2)}, \mathbf{u}_{(3)}) - \frac{1}{1344} \pi(\mathbf{u}_{(4,2)}, \mathbf{u}_{(3,3)}) + \frac{1}{80} \pi^{(z_3)}(\mathbf{u}_{(4,2)}, \mathbf{u}_{(2,1)}^{(w_1)}) \\ &- \frac{11}{400} \pi^{(z_4)}(\mathbf{u}_{(4,2)}, \mathbf{u}_{(2,1)}^{(w_1)}) - \frac{17}{280} \pi^{(z_3)}(\mathbf{u}_{(4,2)}, \mathbf{u}_{(2,1)}^{(w_2)}) - \frac{11}{175} \pi^{(z_4)}(\mathbf{u}_{(4,2)}, \mathbf{u}_{(2,1)}^{(w_2)}) \\ &+ \frac{1}{11520} \pi(\mathbf{u}_{(3,3)}, \mathbf{u}_{(3,3)}) - \frac{1}{96} \pi(\mathbf{u}_{(3,3)}, \mathbf{u}_{(3,2,1)}^{(w_1)}), \end{aligned}$$

where  $z_1 = ccccd$ ,  $z_2 = ccdcdc$ ,  $z_3 = ccd$ ,  $z_4 = cdc$ .

Throughout, we have written  $\pi$  for  $\pi_{(4,2)}$  and omitted the lattice word from the notation whenever it is uniquely determined. This establishes the claim that  $\mathbf{u}_{(0)}$  can be recovered from the rest of the transvectants.

### 5. The standard representation of $\mathfrak{g}_2$

In this section we will give a similar example for the exceptional Lie algebra  $\mathfrak{g}_2$ . A very readable account of its representation theory may be found in [19, Lecture 22] (also see [26]).

#### 5.1.

In conventional notation the two simple roots of  $\mathfrak{g}_2$  can be identified with the vectors

$$\alpha_1 = (1, 0), \quad \alpha_2 = \left( -\frac{3}{2}, \frac{\sqrt{3}}{2} \right) \in \mathbb{R}^2.$$

The two fundamental weights  $\omega_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ ,  $\omega_2 = (0, \sqrt{3})$ , define the closed Weyl chamber

$$\mathcal{W}^+ = \{a\omega_1 + b\omega_2 : a, b \geq 0\}.$$

For integers  $a, b \geq 0$ , let  $\Gamma_{a,b}$  denote the irreducible  $\mathfrak{g}_2$ -representation with highest weight  $a\omega_1 + b\omega_2$ . The 7-dimensional representation  $\Gamma_{1,0}$  is called the standard representation of  $\mathfrak{g}_2$ . We have a decomposition

$$\Gamma_{1,0} \otimes \Gamma_{1,0} \simeq \Gamma_{2,0} \oplus \Gamma_{1,0} \oplus \Gamma_{0,1} \oplus \Gamma_{0,0},$$

with projection maps  $\pi_{i,j} : \Gamma_{1,0} \otimes \Gamma_{1,0} \longrightarrow \Gamma_{i,j}$ . Let  $A, B \in \Gamma_{1,0}$ , and write

$$T_{i,j} = \pi_{i,j}(A \otimes B).$$

By the Weyl character formula (see [19, Prop. 24.48]), there is one copy of  $\Gamma_{2,0}$  inside  $\Gamma_{0,1} \otimes \Gamma_{0,1}$ , and two copies of  $\Gamma_{2,0}$  inside  $\Gamma_{2,0} \otimes \Gamma_{2,0}$ . Let

$$\xi_1 : \Gamma_{0,1} \otimes \Gamma_{0,1} \longrightarrow \Gamma_{2,0},$$

$$\xi_p, \xi_q : \Gamma_{2,0} \otimes \Gamma_{2,0} \longrightarrow \Gamma_{2,0},$$

denote the corresponding projections. (The precise normalisations for these maps will be specified later.)

**THEOREM 5.1.** — *With notation as above, there are identities*

$$-T_{20} T_{00} = \frac{7}{64} \pi_{2,0}(T_{10}, T_{10}) + \frac{1}{4} \xi_p(T_{20}, T_{20}) + \frac{1}{2} \xi_q(T_{20}, T_{20}),$$

and

$$-T_{20} T_{00} = \frac{7}{768} \xi_1(T_{01}, T_{01}) - \frac{1}{16} \xi_p(T_{20}, T_{20}) + \frac{1}{64} \xi_q(T_{20}, T_{20}).$$

Consequently,  $T_{00}$  can be recovered from either of the pairs

$$\{T_{20}, T_{10}\}, \quad \{T_{20}, T_{01}\}.$$

### 5.2.

We will outline the computations which went into deducing these identities. Let  $V$  denote a three-dimensional vector space, and write  $\mathbb{S}_\lambda$  for  $\mathbb{S}_\lambda V$  as in §4.1. Then we can make an identification of  $\mathfrak{g}_2$  with

$$(5.1) \quad \mathbb{S}_{(2,1)} \oplus \mathbb{S}_{(1,0)} \oplus \mathbb{S}_{(1,1)},$$

with  $\mathfrak{sl}_3 \simeq \mathbb{S}_{2,1}$  as a Lie subalgebra. Since every  $\mathfrak{g}_2$ -representation is a *fortiori* an  $\mathfrak{sl}_3$ -representation, the ternary symbolic calculus is available to us. Notice that any  $\mathfrak{g}_2$ -representation  $W$  is naturally  $\mathbb{Z}_3$ -graded: given any  $\mathfrak{sl}_3$ -summand  $S_{(m,n)} \subseteq W$ , the degree of an element in  $\mathbb{S}_{(m,n)}$  is  $m + n \pmod{3}$ .

In symbolic terms, the Lie bracket on  $\mathfrak{g}_2$  can be explicitly written down as follows: let  $X = (A, v, \alpha), Y = (B, w, \beta) \in \mathfrak{g}_2$  in the notation of (5.1), i.e.,  $A \in \mathbb{S}_{(2,1)}$  etc. Then  $[X, Y] = (C, z, \gamma)$ , where

$$\begin{aligned} C &= \tilde{A}\tilde{B} \circ (abc)(ad\mathbf{u})c_{\mathbf{x}} + (\tilde{v}\tilde{\beta} - \tilde{w}\tilde{\alpha}) \circ (ad\mathbf{u})c_{\mathbf{x}}, \\ z &= (\tilde{A}\tilde{w} - \tilde{B}\tilde{v}) \circ (abc)a_{\mathbf{x}} - 2\tilde{\alpha}\tilde{\beta} \circ (abd)c_{\mathbf{x}}, \\ \gamma &= (\tilde{B}\tilde{\alpha} - \tilde{A}\tilde{\beta}) \circ (abd)(ac\mathbf{u}) + \tilde{v}\tilde{w} \circ (ac\mathbf{u}). \end{aligned}$$

In each term, say in  $\tilde{B}\tilde{\alpha} \circ (abd)(ac\mathbf{u})$ , there is a pair of operators acting on a symbolic expression. Our convention is that the operator on the left (i.e.,  $\tilde{B}$ ) is obtained by the substitutions

$$x_i \rightarrow \frac{\partial}{\partial a_i}, \quad u_1 \rightarrow \frac{\partial^2}{\partial a_2 \partial b_3} - \frac{\partial^2}{\partial b_2 \partial a_3}, \text{ etc.}$$

and the one on the right (i.e.,  $\tilde{\alpha}$ ) is obtained by

$$x_i \rightarrow \frac{\partial}{\partial c_i}, \quad u_1 \rightarrow \frac{\partial^2}{\partial c_2 \partial d_3} - \frac{\partial^2}{\partial d_2 \partial c_3}, \text{ etc.}$$

**5.3.**

There are  $\mathbb{Z}_3$ -graded isomorphisms

$$(5.2) \quad \begin{aligned} \Gamma_{1,0} &\simeq \mathbf{Q} \oplus \mathbb{S}_{(1,0)} \oplus \mathbb{S}_{(1,1)}, \\ \Gamma_{2,0} &\simeq (\mathbf{Q} \oplus \mathbb{S}_{(2,1)}) \oplus (\mathbb{S}_{(1,0)} \oplus \mathbb{S}_{(2,2)}) \oplus (\mathbb{S}_{(2,0)} \oplus \mathbb{S}_{(1,1)}), \end{aligned}$$

and  $\Gamma_{0,1} \simeq \mathfrak{g}_2$  is the adjoint representation. We have calculated symbolic descriptions for all the  $\mathfrak{g}_2$ -actions, as well as all the morphisms involved. These descriptions are too laborious to be written down here in their entirety, but an example should suffice to convey the idea. Let

$$X = (A, v, \alpha) \in \mathfrak{g}_2, \quad \text{and} \quad \Psi = (p, B; w, Q; E, \beta) \in \Gamma_{2,0}.$$

The notation follows (5.1) and (5.2); thus  $A \in \mathbb{S}_{(2,1)}$  and  $w \in \mathbb{S}_{(1,0)}$  etc. Let  $\varphi_X(\Psi) = \Psi' = (p', B'; w', Q'; E', \beta')$  denote the image of  $\Psi$  under the action of  $X$ . Then, we have formulae

$$(5.3) \quad \begin{aligned} w' &= \tilde{A}\tilde{w} \circ (abc)a_{\mathbf{x}} + 7pv + \tilde{v}\tilde{B} \circ (acd)c_{\mathbf{x}} + \\ &\quad \frac{1}{2}\tilde{\alpha}\tilde{E} \circ (abc)c_{\mathbf{x}} - 2\tilde{\alpha}\tilde{\beta} \circ (abd)c_{\mathbf{x}}, \\ p' &= \frac{2}{3}\tilde{v}\tilde{\beta} \circ (acd) + \frac{1}{3}\tilde{\alpha}\tilde{w} \circ (abc), \end{aligned}$$

with similar expressions for other factors.

Formulae (5.3) (and others like it) are obtained as follows. The Lie algebra action induces a map of  $\mathfrak{sl}_3$ -representations  $\mathfrak{g}_2 \otimes \Gamma_{2,0} \longrightarrow \Gamma_{2,0}$ . Now,  $\mathfrak{g}_2 \otimes \Gamma_{2,0}$  contains three copies of the trivial representation, coming from the summands  $\mathbb{S}_{(2,1)} \otimes \mathbb{S}_{(2,1)}, \mathbb{S}_{(1,0)} \otimes \mathbb{S}_{(1,1)}$ , and  $\mathbb{S}_{(1,1)} \otimes \mathbb{S}_{(1,0)}$ . This shows the existence of a formula of the type

$$p' = e_1 \tilde{A} \tilde{B} \circ (abc)(acd) + e_2 \tilde{v} \tilde{\beta} \circ (acd) + e_3 \tilde{\alpha} \tilde{w} \circ (abc),$$

for some rational constants  $e_i$ . Now write similar formulae for  $B', w'$  etc. with indeterminate coefficients  $e_i$ . We must have an identity

$$\varphi_X \circ \varphi_Y(\Psi) - \varphi_Y \circ \varphi_X(\Psi) = \varphi_{[X,Y]}(\Psi),$$

which translates into a system of homogeneous quadratic equations in the  $e_i$ . Up to a constant, this system has a unique solution which fixes the action. (Throughout we have used MAPLE for all such computations.) Some of the  $e_i$  may be zero, for instance  $e_1$  is.

The same method was used to deduce symbolic formulae expressing the projections  $\pi_{i,j}$  and  $\xi$ . We have fixed the following normalisations, which determine the projections uniquely:

$$\begin{aligned} \pi_{2,0}(1 \otimes 1) &= 1, & \pi_{1,0}(x_1 \otimes u_1) &= 2, \\ \pi_{0,1}(x_1 \otimes u_1) &= x_1 u_1, & \pi_{0,0}(1 \otimes 1) &= 1, \end{aligned}$$

and

$$\begin{aligned} \xi_p(x_1 u_1 \otimes x_2) &= x_2, & \xi_1(x_1 \otimes u_1) &= -4 + \frac{7}{2} x_1 u_1, \\ \xi_q(x_1 u_1 \otimes x_2) &= u_1 u_2. \end{aligned}$$

Finally, notice that the module

$$(\wedge^2 \Gamma_{1,0}) \otimes (\wedge^2 \Gamma_{1,0}) = (\Gamma_{1,0} \oplus \Gamma_{0,1}) \otimes (\Gamma_{1,0} \oplus \Gamma_{0,1})$$

classifies the quadratic syzygies between the  $T_{ij}$ . It is seen to contain four copies of  $\Gamma_{2,0}$ , and two of the syzygies are those given in Theorem 5.1.

### 6. The standard representation of $\mathfrak{S}_d$

In this section we will give a similar example coming from the standard representation of the permutation group  $\mathfrak{S}_5$ . We conjecture that there is a similar general result to be found for all higher  $\mathfrak{S}_d$ .

Recall that the irreducible representations of  $\mathfrak{S}_d$  are in bijection with the partitions  $\lambda \vdash d$  (see [18, Ch. 7], [19, Lecture 4]). The corresponding

representation  $V_\lambda$  has a basis of standard tableaux on shape  $\lambda$  comprising all the numbers from 1 to  $d$ . For instance, the tableaux

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & \end{bmatrix}, \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & \end{bmatrix}, \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & \end{bmatrix}, \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & \end{bmatrix}, \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & \end{bmatrix}$$

form a basis of  $V_{(3,2)}$ . Usually  $V_{(d-1,1)}$  is called the standard representation of  $\mathfrak{S}_d$ .

### 6.1.

The tensor product  $V_\lambda \otimes V_\mu$  decomposes into a direct sum of irreducibles; let  $\lambda \circ \mu \circ \nu$  denote the multiplicity of  $V_\nu$  in this decomposition. This symbol is invariant under all permutations of the letters, i.e.,

$$\lambda \circ \mu \circ \nu = \mu \circ \lambda \circ \nu = \mu \circ \nu \circ \lambda.$$

If  $\lambda \circ \mu \circ \nu = 1$ , then a matrix  $M$  which describes the projection morphism  $V_\lambda \otimes V_\mu \rightarrow V_\nu$  can be calculated as follows: given an element  $\alpha \in \mathfrak{S}_d$  we have a commutative diagram

$$\begin{array}{ccc} V_\lambda \otimes V_\mu & \xrightarrow{M} & V_\nu \\ Q_\lambda^{(\alpha)} * Q_\mu^{(\alpha)} \downarrow & & \downarrow Q_\nu^{(\alpha)} \\ V_\lambda \otimes V_\mu & \xrightarrow{M} & V_\nu \end{array}$$

where e.g.,  $Q_\nu^{(\alpha)}$  is the matrix describing the action of  $\alpha$  on  $V_\nu$  and  $*$  denotes the Kronecker product of matrices. Once the  $Q$ -matrices are known, the equality  $M Q_\nu^{(\alpha)} = (Q_\lambda^{(\alpha)} \otimes Q_\mu^{(\alpha)}) M$  gives a system of homogeneous linear equations in the entries of the unknown matrix  $M$ . Then  $M$  can be determined (up to a multiplicative scalar) from the combined system of the cycles  $\alpha = (1, 2), (1, 2, 3), \dots, d$ .

For instance, the projection morphism  $V_{(3,1)} \otimes V_{(2,2)} \rightarrow V_{(2,1,1)}$  is given by the matrix

$$M = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ -2 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

This is interpreted as follows: given the tableaux bases

$$\begin{aligned}
 A_1 &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & & \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 4 \\ 3 & & \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 3 & 4 \\ 2 & & \end{bmatrix}, \\
 B_1 &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \\
 C_1 &= \begin{bmatrix} 1 & 2 \\ 3 & \\ 4 & \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 3 \\ 2 & \\ 4 & \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 4 \\ 2 & \\ 3 & \end{bmatrix},
 \end{aligned}$$

the rows of  $M$  sequentially describe the images of

$$A_1 \otimes B_1, A_1 \otimes B_2, A_2 \otimes B_1, A_2 \otimes B_2, A_3 \otimes B_1, A_3 \otimes B_2.$$

E.g.,  $A_2 \otimes B_1 \longrightarrow -2C_1 + C_2 + C_3$ .

**6.2.**

Henceforth assume  $d \geq 5$ . The symmetric square of  $V_{(d-1,1)}$  has the decomposition

$$S_2 V_{(d-1,1)} = V_{(d-1,1)} \oplus V_{(d-2,2)} \oplus V_{(d)},$$

with the associated projection morphisms  $\pi_1, \pi_2, \pi_3$  onto the respective factors. Write  $z_i = \pi_i(u \otimes v)$  for  $u, v \in V_{(d-1,1)}$ . Since  $V_{(d)}$  is the one-dimensional representation with basis  $[1\ 2 \cdots d]$ , one can identify  $z_3$  with a constant. Since  $(d-1, 1) \circ (d-2, 2) \circ (d-1, 1) = 1$ , the projection

$$\eta_1 : V_{(d-1,1)} \otimes V_{(d-2,2)} \longrightarrow V_{(d-1,1)}$$

is defined.

There is an isomorphism  $\wedge^2 V_{(d-1,1)} = V_{(d-2,1,1)}$ , and hence an exact sequence (see §2.1)

$$\begin{aligned}
 0 \rightarrow V_{(d-2,1,1)} \otimes V_{(d-2,1,1)} &\rightarrow S_2 [V_{(d-1,1)} \otimes V_{(d-1,1)}] \\
 &\rightarrow S_2 V_{(d-1,1)} \otimes S_2 V_{(d-1,1)} \rightarrow 0.
 \end{aligned}$$

**6.3.**

Now let  $d = 5$ . A simple calculation with the character table shows that  $(3, 2) \circ (3, 2) \circ (4, 1) = 1$ , let  $\eta_2 : V_{(3,2)} \otimes V_{(3,2)} \rightarrow V_{(4,1)}$  denote the corresponding projection. Moreover, there is precisely one copy of  $V_{(4,1)}$

inside the syzygy module  $V_{(3,1,1)} \otimes V_{(3,1,1)}$ , which must represent a linear relation between the elements

$$\pi_1(z_1 \otimes z_1), \eta_1(z_1 \otimes z_2), \eta_2(z_2 \otimes z_2), z_1 z_3.$$

We calculated the matrices for  $\pi_1, \pi_2, \pi_3, \eta_1, \eta_2$  using the recipe above, and then found the identical relation

$$(6.1) \quad 32 \pi_1(z_1 \otimes z_1) + 100 \eta_1(z_1 \otimes z_2) + 25 \eta_2(z_2 \otimes z_2) - 180 z_1 z_3 = 0,$$

which of course shows that  $z_3$  can be recovered from  $z_1, z_2$ . For the record, the chosen normalisations were as follows:  $\pi_1, \pi_2, \pi_3$  respectively map the tensor

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & & & \end{bmatrix},$$

to the elements

$$-3 \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & & & \end{bmatrix} + \dots, \quad 2 \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & \end{bmatrix} + \dots, \quad 2 [ 1 \ 2 \ 3 \ 4 \ 5 ].$$

Moreover,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & \end{bmatrix} \xrightarrow{\eta_1} -2 \begin{bmatrix} 1 & 2 & 3 & 5 \\ 4 & & & \end{bmatrix} + \dots,$$

and

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & \end{bmatrix} \otimes \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & \end{bmatrix} \xrightarrow{\eta_2} 2 \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & & & \end{bmatrix} + \dots .$$

### 6.4.

We make the following cascading series of conjectures, which would imply that in general  $z_3$  can always be recovered from  $z_1, z_2$ .

*Conjecture 6.1.* — Assume  $d \geq 6$ .

- We have  $(d - 2, 2) \circ (d - 2, 2) \circ (d - 1, 1) = 1$ . This would define the map  $\eta_2 : V_{(d-2,2)} \otimes V_{(d-2,2)} \longrightarrow V_{(d-1,1)}$ .
- We have  $(d - 2, 1, 1) \circ (d - 2, 1, 1) \circ (d - 1, 1) \geq 1$ , which would imply the existence of an identical relation of the form

$$c_1 \pi_1(z_1 \otimes z_1) + c_2 \eta_1(z_1 \otimes z_2) + c_3 \eta_2(z_2 \otimes z_2) + c_4 z_1 z_3 = 0, \quad (c_i \in \mathbf{Q}).$$

Of course, the  $c_i$  would depend on the normalisations chosen for the projections.

- In this relation, the constant  $c_4 \neq 0$ .

We have verified the entire conjecture for  $d = 6, 7$ .

### 7. Wigner symbols

In this section we complete the proof of formula (2.9) from §2.10, which depends on the so-called Ališauskas-Jucys triple sum formula for 9-j symbols.

For the reader's interest we add a short representation-theoretic account of Wigner's 3-j, 6-j and 9-j symbols. A comprehensive discussion of the quantum theory of angular momentum and Wigner symbols may be found<sup>(5)</sup> in [6]. One can find a quick and readable summary of the quantum theory of angular momentum in [8, Appendix A]. We refer the reader to [7, Ch. V] for generalities on Hilbert spaces.

#### 7.1.

Throughout this section, we work over the field of complex numbers  $\mathbf{C}$ . For any  $j \in \frac{1}{2}\mathbf{N}$ , we let  $\mathcal{H}_j = S_{2j}$  which can be seen as the space of homogeneous forms

$$F(\mathbf{z}) = \sum_{k=0}^{2j} \binom{2j}{k} a_k z_1^{2j-k} z_2^k = \sum_{i_1, \dots, i_{2j}=1}^2 f_{i_1, \dots, i_{2j}} z_{i_1} \dots z_{i_{2j}}$$

in the variables  $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ , where the tensor entries  $f_{i_1, \dots, i_{2j}}$  are symmetric in their  $2j$  indices. E.g., a typical element in  $\mathcal{H}_{3/2}$  is of the form

$$F(\mathbf{z}) = f_{111} z_1 z_1 z_1 + f_{112} z_1 z_1 z_2 + f_{121} z_1 z_2 z_1 + \dots \text{(8 terms in all)}.$$

The  $\mathcal{H}_j$  become finite dimensional complex Hilbert spaces when equipped with the natural Hermitian inner product

$$\langle F|G \rangle = \sum_{i_1, \dots, i_{2j}=1}^2 \overline{f_{i_1, \dots, i_{2j}}} g_{i_1, \dots, i_{2j}}.$$

For instance, if

$$F(\mathbf{z}) = \sum_{k=0}^{2j} \binom{2j}{k} a_k z_1^{2j-k} z_2^k \quad \text{and} \quad G(\mathbf{z}) = \sum_{k=0}^{2j} \binom{2j}{k} b_k z_1^{2j-k} z_2^k ;$$

then

$$\langle F|G \rangle = \sum_{k=0}^{2j} \binom{2j}{k} \overline{a_k} b_k .$$

---

<sup>(5)</sup> However, note that errors have crept in some of the formulae in this book; in particular the triple sum formula is not correctly stated on [6, p. 130].



Define the set

$$M_j = \{m : m \in \frac{1}{2} \mathbf{Z}, j - m \in \mathbf{Z}, -j \leq m \leq j\},$$

then the forms

$$e_{jm} = (-1)^{j+m} \sqrt{\binom{2j}{j-m}} z_1^{j-m} z_2^{j+m}, \quad (m \in M_j)$$

constitute an orthonormal basis of the  $(2j + 1)$ -dimensional space  $\mathcal{H}_j$ :

$$\langle e_{jm} | e_{jm'} \rangle = \delta_{mm'}.$$

In the physics literature,  $e_{jm}$  is often written as  $|j m\rangle$ .

### 7.2.

Given  $g \in SL_2\mathbf{C}$ , define  $(g \cdot F)(\mathbf{z}) = F(g^{-1}\mathbf{z})$ . When this action is restricted to  $SU_2$ , the  $\mathcal{H}_j$  turn into unitary representations, i.e.,

$$\langle (g \cdot F) | (g \cdot G) \rangle = \langle F | G \rangle \quad \text{for } g \in SU_2.$$

Let

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

denote the so-called Pauli matrices which generate the Lie algebra  $\mathfrak{su}_2$ . For  $a = 1, 2, 3$ , let  $J_a$  denote the corresponding infinitesimal operators on the representation  $\mathcal{H}_j$ :

$$J_a(F) = -i \frac{d}{d\theta} \left( e^{i\frac{\theta}{2}\sigma_a} \cdot F \right) \Big|_{\theta=0}.$$

They satisfy the so-called angular momentum commutation relations

$$[J_a, J_b] = i \epsilon_{abc} J_c$$

where  $\epsilon_{abc}$  is antisymmetric in  $a, b, c = 1, 2, 3$  with  $\epsilon_{123} = 1$ . If we let  $J_{\pm} = J_1 \pm iJ_2$ , then their actions on  $F(\mathbf{z}) \in \mathcal{H}_j$  can be seen as the following differential operators:

$$\begin{aligned} J_+ &= -z_2 \frac{\partial}{\partial z_1}, \\ J_- &= -z_1 \frac{\partial}{\partial z_2}, \\ J_3 &= \frac{1}{2} \left( z_2 \frac{\partial}{\partial z_2} - z_1 \frac{\partial}{\partial z_1} \right). \end{aligned}$$

In particular,

$$\begin{aligned} J_+ e_{jm} &= \sqrt{j(j+1) - m(m+1)} e_{j,m+1}, \\ J_- e_{jm} &= \sqrt{j(j+1) - m(m-1)} e_{j,m-1}, \\ J_3 e_{jm} &= m e_{jm}, \\ \mathbf{J}^2 e_{jm} &= j(j+1) e_{jm}, \end{aligned}$$

where  $\mathbf{J}^2 = J_1^2 + J_2^2 + J_3^2$ .

**7.3.**

Given two values  $j_1, j_2$  of the angular momentum,  $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$  can be seen as the space of bihomogeneous forms

$$B(\mathbf{x}, \mathbf{y}) = \sum_{p_1, \dots, p_{2j_1}, q_1, \dots, q_{2j_2}=1}^2 b_{p_1, \dots, p_{2j_1}; q_1, \dots, q_{2j_2}} x_{p_1} \cdots x_{p_{2j_1}} y_{q_1} \cdots y_{q_{2j_2}}$$

with complex coefficients, of degree  $2j_1$  in  $\mathbf{x} = (x_1, x_2)$  and of degree  $2j_2$  in  $\mathbf{y} = (y_1, y_2)$ . The tensor entries  $b_{p_1, \dots, p_{2j_1}; q_1, \dots, q_{2j_2}}$  are assumed to be symmetric separately in the  $p$  and  $q$  indices. Once again, we have a Hermitian inner product

$$\langle B|C \rangle = \sum_{p_1, \dots, p_{2j_1}, q_1, \dots, q_{2j_2}=1}^2 \overline{b_{p_1, \dots, p_{2j_1}; q_1, \dots, q_{2j_2}}} c_{p_1, \dots, p_{2j_1}; q_1, \dots, q_{2j_2}}$$

on  $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$ , such that  $\{e_{j_1, m_1} \otimes e_{j_2, m_2} : m_i \in M_{j_i}\}$  is an orthonormal basis. This evidently generalises to tensor products with more than two factors.

We say that  $(j_1, j_2, j)$  is a *triad* if all the three expressions

$$j_1 + j_2 - j, \quad j_2 + j - j_1, \quad j + j_1 - j_2,$$

are nonnegative integers. Moreover, the triad is *stretched* if one of these integers is zero. Then the Clebsch-Gordan decomposition becomes

$$\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} = \bigoplus_{j \in T_{j_1 j_2}} \mathcal{H}_j,$$

where the set  $T_{j_1 j_2}$  consists of those  $j \in \frac{1}{2}\mathbf{N}$  such that  $(j_1, j_2, j)$  is a triad.

An  $SL_2$ -equivariant injection  $\iota_{j_1 j_2 j} : \mathcal{H}_j \rightarrow \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$  is necessarily of the form

$$(\iota_{j_1 j_2 j}(F))(\mathbf{x}, \mathbf{y}) = \frac{c_{j_1 j_2 j}}{(2j)!} (\mathbf{x} \mathbf{y})^{j_1 + j_2 - j} (\mathbf{x} \partial_{\mathbf{z}})^{j + j_1 - j_2} (\mathbf{y} \partial_{\mathbf{z}})^{j + j_2 - j_1} F(\mathbf{z}),$$

where  $c_{j_1 j_2 j}$  is a nonzero constant to be fixed by convention.

Likewise an  $SL_2$ -equivariant projection  $\pi_{j_1 j_2 j} : \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \rightarrow \mathcal{H}_j$  is necessarily of the form

$$(\pi_{j_1 j_2 j}(B))(\mathbf{z}) = d_{j_1 j_2 j} \frac{(j + j_1 - j_2)! (j + j_2 - j_1)!}{(2j_1)! (2j_2)!} [\Omega_{\mathbf{x}\mathbf{y}}^{j_1+j_2-j} B(\mathbf{x}, \mathbf{y})]_{\mathbf{x}, \mathbf{y} \rightarrow \mathbf{z}}$$

for a constant  $d_{j_1 j_2 j}$ .

7.4.

Given the previous natural choices of inner products, one can reduce the arbitrariness by requiring that  $\iota_{j_1 j_2 j}$  be an isometry, i.e.,  $\|F\|^2 = \|\iota_{j_1 j_2 j}(F)\|^2$ . Using the formula on [23, p. 54], this forces

$$|c_{j_1 j_2 j}| = \sqrt{\frac{(2j_1)! (2j_2)! (2j + 1)!}{(j_1 + j_2 + j + 1)! (j_1 + j_2 - j)! (j + j_1 - j_2)! (j + j_2 - j_1)!}}$$

We will also choose  $\pi_{j_1 j_2 j}$  to be the Hermitian transpose of  $\iota_{j_1 j_2 j}$ , i.e.,

$$\langle \iota_{j_1 j_2 j}(F), G \rangle = \langle F, \pi_{j_1 j_2 j}(G) \rangle, \quad \text{for all } F \in \mathcal{H}_j, G \in \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}.$$

This is tantamount to requiring that  $d_{j_1 j_2 j} = \overline{c_{j_1 j_2 j}}$ . At this point the constants are well-determined up to multiplication by a complex number of unit modulus. Several phase conventions are prevalent in physics literature for removing this ambiguity in a consistent manner. Before stating them we need to define the vector coupling coefficients:

$$(7.1) \quad C_{m_1 m_2 m}^{j_1 j_2 j} = \langle e_{j_1 m_1} \otimes e_{j_2 m_2} | \iota_{j_1 j_2 j}(e_{jm}) \rangle,$$

where the inner product is that of  $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$ .

**The Wigner phase convention** requires that

$$C_{j_1, -j_2, j_1 - j_2}^{j_1, j_2, j} > 0,$$

it appears in the 1931 German edition of [43].

**The Condon-Shortley phase convention** requires that with respect to the basis  $\{\iota_{j_1 j_2 j}(e_{jm}) : j \in T_{j_1 j_2}, m \in M_j\}$  of  $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$ , all the matrix elements of  $J_3^{(1)}$  which are nondiagonal with respect to  $j$  must be nonnegative (see [14]). Here  $J_3^{(1)}$  is the infinitesimal generator analogous to  $J_3$ , for the  $SU_2$ -action on  $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$  given by the natural action on the first factor and the trivial one on the second factor.

**The Brussaard phase convention** [9, p. 209] is a less cumbersome re-statement of the Condon-Shortley phase convention. It essentially amounts to the requirement

$$C_{j_1, j-j_1, j}^{j_1, j_2, j} > 0.$$

A similar prescription is used by Racah in [36, §2].

Fortunately we have the following result.

PROPOSITION 7.1. — *All of these conventions are equivalent, and amount to making the most obvious choice:*

$$c_{j_1 j_2 j} = \sqrt{\frac{(2j_1)! (2j_2)! (2j + 1)!}{(j_1 + j_2 + j + 1)! (j_1 + j_2 - j)! (j + j_1 - j_2)! (j + j_2 - j_1)!}}.$$

With this choice, let  $\iota_{j_1 j_2 j}^{\text{PHY}}$  and  $\pi_{j_1 j_2 j}^{\text{PHY}}$  denote the corresponding injection and projection maps respectively; they are the standard ones used in physics literature. To recapitulate,

$$\iota_{\frac{m}{2}, \frac{n}{2}, \frac{m+n-2r}{2}}^{\text{PHY}} = \frac{1}{\sqrt{\mathbf{g}(m, n; r)}} \iota_r, \quad \pi_{\frac{m}{2}, \frac{n}{2}, \frac{m+n-2r}{2}}^{\text{PHY}} = \sqrt{\mathbf{g}(m, n; r)} \pi_r$$

in the notation of §1.6.

### 7.5. The 3-j symbols

Now Wigner’s 3-j symbol is defined to be

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = \frac{(-1)^{j_1-j_2-m}}{\sqrt{2j+1}} C_{m_1, m_2, -m}^{j_1, j_2, j},$$

where  $m_1 \in M_{j_1}$  etc. Its value is given by a terminating  ${}_3F_2$  hypergeometric series. The reader is referred to [2, 3] for more on these symbols and their use, e.g., in proving sharp Castelnuovo-Mumford regularity bounds.

### 7.6. The 6-j symbols

A 6-j symbol is usually represented as an array

$$\mathcal{A} = \left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{matrix} \right\},$$

where  $(j_1, j_2, j_{12}), (j_2, j_3, j_{23}), (j_{12}, j_3, J)$  and  $(j_1, j_{23}, J)$  are assumed to be triads.

Consider the endomorphism  $\phi : \mathcal{H}_J \rightarrow \mathcal{H}_J$  obtained as the composition

$$(7.2) \quad \begin{aligned} \mathcal{H}_J &\longrightarrow \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_{23}} \longrightarrow \mathcal{H}_{j_1} \otimes (\mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}) \\ &\longrightarrow (\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}) \otimes \mathcal{H}_{j_3} \longrightarrow \mathcal{H}_{j_{12}} \otimes \mathcal{H}_{j_3} \longrightarrow \mathcal{H}_J, \end{aligned}$$

using the obvious  $\iota^{\text{PHY}}$  and  $\pi^{\text{PHY}}$  maps. By Schur’s Lemma, this is a multiple  $\alpha \text{Id}_{\mathcal{H}_J}$  of the identity map on  $\mathcal{H}_J$ . Let  $u \in \mathcal{H}_J$  denote an arbitrary vector of unit norm. Then  $\langle u | \phi(u) \rangle$  is independent of  $u$ , and is equal to the multiplying factor  $\alpha$ . Since the maps  $\iota^{\text{PHY}}$  and  $\pi^{\text{PHY}}$  are Hermitian transposes of each other, we also have

$$\alpha = \langle z_L | z_R \rangle,$$

where  $z_L$  and  $z_R$  are respectively the images of  $u$  via the maps

$$(\iota_{j_1, j_2, j_{12}}^{\text{PHY}} \otimes \text{Id}_{\mathcal{H}_{j_3}}) \circ \iota_{j_{12}, j_3, J}^{\text{PHY}}, \quad \text{and} \quad (\text{Id}_{\mathcal{H}_{j_1}} \otimes \iota_{j_2, j_3, j_{23}}^{\text{PHY}}) \circ \iota_{j_1, j_{23}, J}^{\text{PHY}},$$

and the Hermitian inner product is that of  $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}$ .

Now the standard definition of Wigner’s 6-j symbol is (see [16, p. 92])

$$\mathcal{A} = \frac{(-1)^{j_1+j_2+j_3+J}}{\sqrt{(2j_{12}+1)(2j_{23}+1)}} \times \alpha.$$

Appendix B of [8] gives a very good summary of the properties of the 6-j symbols, including Racah’s celebrated single sum formula [36, Appendix B] which expresses it as the value of a terminating  ${}_4F_3$  hypergeometric series.

### 7.7.

The following is essentially the same way of stating the definition. Start with a generic form  $F(\mathbf{z})$  of order  $2J$  and apply the following operators in succession, precisely following the sequence (7.2).

$$\begin{aligned} &(\mathbf{u} \mathbf{y})^{j_1+j_{23}-J} (\mathbf{u} \partial_{\mathbf{z}})^{j_1+J-j_{23}} (\mathbf{y} \partial_{\mathbf{z}})^{j_{23}+J-j_1}, \\ &(\mathbf{v} \mathbf{w})^{j_2+j_3-j_{23}} (\mathbf{v} \partial_{\mathbf{y}})^{j_2+j_{23}-j_3} (\mathbf{w} \partial_{\mathbf{y}})^{j_3+j_{23}-j_2}, \\ &\Omega_{\mathbf{u}\mathbf{v}}^{j_1+j_2-j_{12}}, \quad \{\mathbf{u}, \mathbf{v} \rightarrow \mathbf{x}\}, \quad \Omega_{\mathbf{x}\mathbf{w}}^{j_{12}+j_3-J}, \quad \{\mathbf{x}, \mathbf{w} \rightarrow \mathbf{z}\}. \end{aligned}$$

The result is simply a multiple of the original form, i.e.,  $\tilde{\alpha} F(\mathbf{z})$  for some  $\tilde{\alpha} \in \mathbf{Q}$ . Then

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & J & j_{23} \end{array} \right\} = (-1)^{j_1+j_2+j_3+J} (2J+1) \times \sqrt{\frac{P_1}{P_2 P_3}} \times \tilde{\alpha},$$

where

$$\begin{aligned}
 P_1 &= (j_1 + j_{12} - j_2)! (j_2 + j_{12} - j_1)! (j_{12} + J - j_3)! (j_3 + J - j_{12})!, \\
 P_2 &= (j_1 + j_{23} - J)! (j_1 + J - j_{23})! (j_{23} + J - j_1)! (j_2 + j_3 - j_{23})! \times \\
 &\quad (j_2 + j_{23} - j_3)! (j_3 + j_{23} - j_2)! (j_1 + j_2 - j_{12})! (j_{12} + j_3 - J)!, \\
 P_3 &= (j_1 + j_2 + j_{12} + 1)! (j_2 + j_3 + j_{23} + 1)! (j_1 + j_{23} + J + 1)! \\
 &\quad (j_{12} + j_3 + J + 1)!.
 \end{aligned}$$

### 7.8. The 9-j symbols

A 9-j symbol is usually represented as an array

$$\mathcal{B} = \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & J \end{array} \right\},$$

where all the rows and columns are assumed to be triads.

Consider the endomorphism  $\psi : \mathcal{H}_J \longrightarrow \mathcal{H}_J$  obtained as the composition

$$\begin{aligned}
 \mathcal{H}_J &\longrightarrow \mathcal{H}_{j_{13}} \otimes \mathcal{H}_{j_{24}} \longrightarrow (\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_3}) \otimes (\mathcal{H}_{j_2} \otimes \mathcal{H}_{j_4}) \\
 &\longrightarrow (\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}) \otimes (\mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4}) \longrightarrow \mathcal{H}_{j_{12}} \otimes \mathcal{H}_{j_{34}} \longrightarrow \mathcal{H}_J,
 \end{aligned}$$

of the natural  $\iota^{\text{PHY}}$  and  $\pi^{\text{PHY}}$  maps. By Schur’s lemma,  $\psi = \beta \text{Id}_{\mathcal{H}_J}$ . Now the standard definition of the 9-j symbol (see [27] for instance) is along the same lines as that for 6-j symbols, namely

$$\mathcal{B} = \frac{1}{\sqrt{(2j_{12} + 1)(2j_{34} + 1)(2j_{13} + 1)(2j_{24} + 1)}} \times \beta.$$

One can evaluate  $\beta$  as  $\langle z_L | z_R \rangle$ , where  $z_L$  and  $z_R$  are respectively the images of an arbitrary unit vector  $u$  via the maps

$$(\iota_{j_1, j_2, j_{12}}^{\text{PHY}} \otimes \iota_{j_3, j_4, j_{34}}^{\text{PHY}}) \circ \iota_{j_{12}, j_{34}, J}^{\text{PHY}}$$

and

$$(\text{Id}_{\mathcal{H}_{j_1}} \otimes \tau \otimes \text{Id}_{\mathcal{H}_{j_4}}) \circ (\iota_{j_1, j_3, j_{13}}^{\text{PHY}} \otimes \iota_{j_2, j_4, j_{24}}^{\text{PHY}}) \circ \iota_{j_{13}, j_{24}, J}^{\text{PHY}}$$

with

$$\tau : \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_2} \longrightarrow \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}$$

designating the map that switches the factors.

7.9.

Starting with an arbitrary form  $F(\mathbf{z})$  of order  $2J$ , apply the following operators in succession:

$$\begin{aligned} & (\mathbf{x}\mathbf{y})^{j_{13}+j_{24}-J} (\mathbf{x}\partial_{\mathbf{z}})^{j_{13}+J-j_{24}} (\mathbf{y}\partial_{\mathbf{z}})^{j_{24}+J-j_{13}}, \\ & (\mathbf{p}\mathbf{q})^{j_1+j_3-j_{13}} (\mathbf{p}\partial_{\mathbf{x}})^{j_1+j_{13}-j_3} (\mathbf{q}\partial_{\mathbf{x}})^{j_{13}+j_3-j_1}, \\ & (\mathbf{u}\mathbf{v})^{j_2+j_4-j_{24}} (\mathbf{u}\partial_{\mathbf{y}})^{j_2+j_{24}-j_4} (\mathbf{v}\partial_{\mathbf{y}})^{j_4+j_{24}-j_2}, \\ & \Omega_{\mathbf{p}\mathbf{u}}^{j_1+j_2-j_{12}}, \{\mathbf{p}, \mathbf{u} \rightarrow \mathbf{x}\}, \Omega_{\mathbf{q}\mathbf{v}}^{j_3+j_4-j_{34}}, \{\mathbf{q}, \mathbf{v} \rightarrow \mathbf{y}\}, \Omega_{\mathbf{x}\mathbf{y}}^{j_{12}+j_{34}-J}, \{\mathbf{x}, \mathbf{y} \rightarrow \mathbf{z}\}. \end{aligned}$$

The end result will be of the form  $\tilde{\beta} F(\mathbf{z})$  for some  $\tilde{\beta} \in \mathbf{Q}$ . Then the 9-j symbol is given by

$$(7.3) \quad \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & J \end{array} \right\} = (2J + 1) \sqrt{\frac{Q_1}{Q_2 Q_3}} \times \tilde{\beta},$$

where

$$\begin{aligned} Q_1 &= (j_1 + j_{12} - j_2)! (j_2 + j_{12} - j_1)! (j_3 + j_{34} - j_4)! \times \\ & \quad (j_4 + j_{34} - j_3)! (j_{12} + J - j_{34})! (j_{34} + J - j_{12})!, \\ Q_2 &= (j_1 + j_2 + j_{12} + 1)! (j_3 + j_4 + j_{34} + 1)! (j_{13} + j_{24} + J + 1)! \times \\ & \quad (j_1 + j_3 + j_{13} + 1)! (j_2 + j_4 + j_{24} + 1)! (j_{12} + j_{34} + J + 1)!, \\ Q_3 &= (j_1 + j_2 - j_{12})! (j_3 + j_4 - j_{34})! (j_{13} + j_{24} - J)! (j_{13} + J - j_{24})! \times \\ & \quad (j_{24} + J - j_{13})! (j_1 + j_3 - j_{13})! (j_1 + j_{13} - j_3)! (j_3 + j_{13} - j_1)! \times \\ & \quad (j_2 + j_4 - j_{24})! (j_2 + j_{24} - j_4)! (j_4 + j_{24} - j_2)! (j_{12} + j_{34} - J)!. \end{aligned}$$

*Prima facie*, the multiplicative prefactors entering into the definitions of 3-j, 6-j and 9-j symbols might seem unusual, but their purpose is to ensure maximal symmetry of the symbols.

An important property of the 9-j symbol is embodied in the following proposition (see [27]).

PROPOSITION 7.2. — *The 9-j symbol is invariant with respect to matrix transposition of the array. Any permutation  $\sigma$  of the rows or columns alters the symbol by a sign factor equal to*

$$\epsilon(\sigma) \sum j$$

where  $\epsilon(\sigma)$  is the signature of the permutation, and  $\sum j$  denotes the sum of all the nine entries (which necessarily is an integer).

Except for the Regge symmetries, all known symmetry properties of the 3-j, 6-j and 9-j symbols (such as the one stated in the previous proposition) become trivial if one uses the diagrammatic formalism outlined in [1, 3].

### 7.10. The triple sum formula

The Ališauskas-Jucys formula (see [28, §3]) expresses the 9-j symbol as a triple summation over lattice points. Define

$$\begin{aligned}
 x_1 &= 2j_{34}, & y_1 &= -j_2 + j_4 + j_{24}, & z_1 &= 2j_1, \\
 x_2 &= j_3 + j_4 - j_{34}, & y_2 &= j_{13} + j_{24} - J, & z_2 &= -j_1 + j_2 + j_{12}, \\
 x_3 &= j_{12} - j_{34} + J, & y_3 &= 2j_{24} + 1, & z_3 &= j_1 + j_3 + j_{13} + 1, \\
 x_4 &= -j_3 + j_4 + j_{34}, & y_4 &= j_2 + j_4 - j_{24}, & z_4 &= j_1 + j_3 - j_{13}, \\
 x_5 &= j_{12} + j_{34} - J, & y_5 &= j_{13} - j_{24} + J, & z_5 &= j_1 - j_2 + j_{12}, \\
 p_1 &= j_1 + j_3 - j_{24} + J, & p_2 &= -j_2 + j_3 - j_{34} + j_{24}, & p_3 &= -j_1 + j_2 - j_{34} + J,
 \end{aligned}$$

and

$$[a, b, c] = \sqrt{\frac{(a - b + c)! (a + b - c)! (a + b + c + 1)!}{(-a + b + c)!}}.$$

Let  $\Lambda$  denote the set of integer triples  $(x, y, z)$  satisfying the inequalities

$$\begin{aligned}
 0 &\leq x \leq \min(x_4, x_5), \\
 \max(0, -p_2 - x) &\leq y \leq \min(y_4, y_5), \\
 \max(0, -p_3 - x) &\leq z \leq \min(z_4, z_5, p_1 - y).
 \end{aligned}$$

Then

$$\begin{aligned}
 (7.4) \quad \mathcal{B} &= (-1)^{x_5} \frac{[j_3, j_1, j_{13}] [j_2, j_4, j_{24}] [J, j_{13}, j_{24}]}{[j_3, j_4, j_{34}] [j_2, j_1, j_{12}] [J, j_{12}, j_{34}]} \times \\
 &\sum_{(x, y, z) \in \Lambda} \frac{(-1)^{x+y+z} (x_1 - x)! (x_2 + x)! (x_3 + x)! (y_1 + y)! (y_2 + y)!}{x! y! z! (x_4 - x)! (x_5 - x)! (y_3 + y)! (y_4 - y)! (y_5 - y)!} \times \\
 &\frac{(z_1 - z)! (z_2 + z)! (p_1 - y - z)!}{(z_3 - z)! (z_4 - z)! (z_5 - z)! (p_2 + x + y)! (p_3 + x + z)!}.
 \end{aligned}$$

The triple sum formula was discovered in a rather indirect way in [5]. It is often mistakenly referred to as the Jucys-Bandzaitis formula, perhaps because the first 1965 edition of [29] predates [5]. An elementary yet difficult proof was given in [29], in the style of Racah’s proof of his single-sum formula for 6-j symbols. The simplest method of proof seems to be the one due to Rosengren [37, 38].



7.11.

In general, given  $j_1, j_2, \dots, j_{n+1}$  and  $J$ , one can consider morphisms

$$\mathcal{H}_J \xrightarrow{\psi_1} \bigotimes_{\ell=1}^{n+1} \mathcal{H}_{j_\ell} \xrightarrow{\psi_2} \mathcal{H}_J$$

arising from two different choices of successive transvections; this leads to the general notion of a 3n-j symbol. These are instances of the so-called spin networks which play a prominent role in loop quantum gravity (see [8, 10] and references therein).

7.12. The proof of Formula (2.9)

Recall that by Proposition 2.2, the constant  $\kappa_{(i,j)}^{(a,b)}$  is characterised by the equality

$$\xi = \kappa_{(i,j)}^{(a,b)} \text{Id}_{S_{2(m+n-r)}}.$$

Going through the prescriptions of §2.6-2.7 shows that the action of  $\xi$  on a form  $f_{\mathbf{z}}^{2(m+n-r)}$  amounts to the succession of operators:

$$\begin{aligned} & (\mathbf{x} \mathbf{y})^{r-2a-2b-2} (\mathbf{x} \partial_{\mathbf{z}})^{2m-2a+2b-r} (\mathbf{y} \partial_{\mathbf{z}})^{2n+2a-2b-r}, \\ & (\mathbf{p} \mathbf{q})^{2a+1} (\mathbf{p} \partial_{\mathbf{x}})^{m-2a-1} (\mathbf{q} \partial_{\mathbf{x}})^{m-2a-1}, \\ & (\mathbf{u} \mathbf{v})^{2b+1} (\mathbf{u} \partial_{\mathbf{y}})^{n-2b-1} (\mathbf{v} \partial_{\mathbf{y}})^{n-2b-1}, \\ & \Omega_{\mathbf{p} \mathbf{u}}^i, \quad \{\mathbf{p}, \mathbf{u} \rightarrow \mathbf{x}\}, \quad \Omega_{\mathbf{q} \mathbf{v}}^j, \quad \{\mathbf{q}, \mathbf{v} \rightarrow \mathbf{y}\}, \quad \Omega_{\mathbf{x} \mathbf{y}}^{r-i-j}, \quad \{\mathbf{x}, \mathbf{y} \rightarrow \mathbf{z}\}, \end{aligned}$$

together with multiplication by

$$K = \frac{h(m, n; i) h(m, n; j) h(m+n-2i, m+n-2j; r-i-j)}{(2m+2n-2r)! (2m-4a-2)! (2n-4b-2)!}.$$

Now choose the specific 9-j array

$$\mathcal{B} = \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & J \end{array} \right\} = \left\{ \begin{array}{ccc} \frac{1}{2} m & \frac{1}{2} n & \frac{1}{2} (m+n) - i \\ \frac{1}{2} m & \frac{1}{2} n & \frac{1}{2} (m+n) - j \\ m-2a-1 & n-2b-1 & m+n-r \end{array} \right\},$$

which brings this into perfect agreement with the sequence of operators in §7.9. Hence we get an equality

$$(7.5) \quad \kappa_{(i,j)}^{(a,b)} = \frac{K}{2J+1} \sqrt{\frac{Q_2 Q_3}{Q_1}} \times \mathcal{B}.$$

Now interchange rows 2 and 3 of  $\mathcal{B}$ , then interchange columns 1 and 3, and finally take the transpose. This gives an equivalent array

$$\mathcal{B}' = \left\{ \begin{array}{ccc} \frac{1}{2}(m+n) - i & m+n-r & \frac{1}{2}(m+n) - j \\ \frac{1}{2}n & n-2b-1 & \frac{1}{2}n \\ \frac{1}{2}m & m-2a-1 & \frac{1}{2}m \end{array} \right\}.$$

Now apply the triple sum formula (7.4) to  $\mathcal{B}'$ , and feed the result into (7.5). The outcome exactly boils down to the identity (2.9).  $\square$

The switch  $\mathcal{B} \rightarrow \mathcal{B}'$  is necessary due to the peculiarity that the symmetries given by Proposition 7.2 are not visible from the triple sum formula. Our choice of  $\mathcal{B}'$  ensures that when  $i = a = b = 0$  and  $j = r$ , the array becomes doubly-stretched (i.e., two of its six triads are stretched) according to the pattern

$$\left\{ \begin{array}{ccc} j_3 + j_{13} & j_3 + j_{13} + j_2 & j_2 \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & J \end{array} \right\},$$

which is known to reduce the triple sum to a single term [28, Eq. 18]. This ensures that  $\kappa_{(0,r)}^{(0,0)} \neq 0$ .

**Acknowledgements.** The second author was partly funded by a discovery grant from NSERC. We are thankful to the following authors for the use of their programs: Dan Grayson and Michael Stillman (Macaulay-2), John Stembridge (the ‘SF’ package for Maple), and Anthony Stone (a web-based calculator for 3-j, 6-j and 9-j symbols). The University of Michigan Historical Library (**MiH**) as well as Project Gutenberg (**PG**) have been useful in accessing some classical references.

BIBLIOGRAPHY

[1] A. ABDESSELAM, “The combinatorics of classical invariant theory revisited by modern physics”, Slides of Feb 2007 talk at the Montreal CRM workshop “Combinatorial Problems Raised by Statistical Mechanics”. Available at <http://people.virginia.edu/~aa4cr/MontrealFeb07slides.pdf>.

[2] A. ABDESSELAM & J. CHIPALKATTI, “The bipartite Brill-Gordan locus and angular momentum”, *Transform. Groups* **11** (2006), no. 3, p. 341-370.

[3] ———, “Brill-Gordan Loci, transvectants and an analogue of the Foulkes conjecture”, *Adv. Math* **208** (2007), no. 2, p. 491-520.

- [4] K. AKIN, D. BUCHSBAUM & J. WEYMAN, “Schur functors and Schur complexes”, *Adv. Math.* **44** (1982), no. 3, p. 207-278.
- [5] S. J. ALIŠAUSKAS & A. P. JUCYS, “Weight lowering operators and the multiplicity-free isoscalar factors for the group  $R_5$ ”, *J. Math. Phys.* **12** (1971), no. 4, p. 594-605.
- [6] L. C. BIEDENHARN & J. D. LOUCK, *Angular Momentum in Quantum Physics. Theory and Application*, Encyclopedia of Mathematics and its Applications, vol. 8, Addison-Wesley, 1981.
- [7] N. BOURBAKI, *Topological Vector Spaces*, (translated by H. G. Eggleston and S. Madan), Elements of Mathematics, Springer-Verlag, 1987.
- [8] J. BRUNNEMANN & T. THIEMANN, “Simplification of the spectral analysis of the volume operator in loop quantum gravity”, *Class. Quant. Grav.* **23** (2006), p. 1289-1346.
- [9] P. J. BRUSSAARD, “Clebsch-Gordan- of Wigner coefficienten”, *Ned. Tijdschr. Natuurk.* **33** (1967), p. 202-222.
- [10] J. CARTER, D. FLATH & M. SAITO, *The Classical and Quantum 6j-Symbols*, Mathematical Notes, no. 43, Princeton University Press, 1995.
- [11] A. CAYLEY, *On linear transformations*, Collected Mathematical Works, vol. I, no. 14, Cambridge University Press, 1889.
- [12] J. CHIPALKATTI, “On the invariant theory of the Bezoutiant”, *Beiträge Alg. Geom.* **47** (2006), no. 2, p. 397-417.
- [13] A. CLEBSCH, *Theorie der Binären Algebraischen Formen*, Teubner, Leipzig, 1872, (MiH).
- [14] E. U. CONDON & G. H. SHORTLEY, *The Theory of Atomic Spectra*, Cambridge University Press, 1935.
- [15] I. DOLGACHEV, *Lectures on Invariant Theory*, London Mathematical Society Lecture Notes, no. 296, Cambridge University Press, 2003.
- [16] A. R. EDMONDS, *Angular Momentum in Quantum Mechanics*, Princeton University Press, 1957.
- [17] D. FLATH, “The Clebsch-Gordan formulas”, *Enseign. Math. (2)* **29** (1983), no. 3-4, p. 339-346.
- [18] W. FULTON, *Young Tableaux*, London Mathematical Society Student Texts, no. 35, Cambridge University Press, 1957.
- [19] W. FULTON & J. HARRIS, *Representation Theory, A First Course*, Graduate Texts in Mathematics, Springer-Verlag, 1991.
- [20] O. GLENN, *The Theory of Invariants*, Ginn and Co., Boston, 1915, (PG).
- [21] L. GOLDBERG, “Catalan numbers and branched coverings by the Riemann sphere”, *Adv. Math.* **85** (1991), no. 2, p. 129-144.
- [22] P. GORDAN, “Die Resultante Binärer Formen”, *Rend. Circ. Matem. Palermo* **XXII** (1906), p. 161-196.
- [23] J. H. GRACE & A. YOUNG, *The Algebra of Invariants, 1903*, Reprinted by Chelsea Publishing Co., New York, 1962, (MiH).
- [24] J. HARRIS, *Algebraic Geometry, A First Course*, Graduate Texts in Mathematics, Springer-Verlag, 1992.
- [25] R. HARTSHORNE, *Algebraic Geometry*, Graduate Texts in Mathematics, Springer-Verlag, 1977.
- [26] J.-S. HUANG & C.-B. ZHU, “Weyl’s construction and tensor power decomposition for  $G_2$ ”, *Proc. Amer. Math. Soc.* **127** (1999), no. 3, p. 925-934.
- [27] H. A. JAHN & J. HOPE, “Symmetry properties of the Wigner 9j symbol”, *Phys. Rev.* **93** (1954), no. 2, p. 318-321.
- [28] J. V. JEUGT, N. P. SANGITA & K. SRINIVASA RAO, “Multiple hypergeometric functions and 9-j coefficients”, *J. Phys A: Math. Gen.* **27** (1994), p. 5251-5264.

- [29] A. P. JUCYS & A. A. BANDZAITIS, *Angular Momentum in Quantum Physics*, 2nd Ed., Vilnius: Mokslas, 1977.
- [30] A. A. KIRILLOV, *Elements of the Theory of Representations*, Grundlehren der Mathematischen Wissenschaften, Band 220, Springer-Verlag, 1976.
- [31] J. P. S. KUNG & G.-C. ROTA, "The invariant theory of binary forms", *Bulletin of the A.M.S.* **10** (1984), no. 1, p. 27-85.
- [32] D. E. LITTLEWOOD, "Invariant theory, tensors and group characters", *Philos. Trans. Roy. Soc. London. Ser. A* **239** (1944), no. 807, p. 305-365.
- [33] I. G. MACDONALD, *Symmetric Functions and Hall polynomials*, (2nd ed.), Oxford University Press, 1995.
- [34] D. MUMFORD, *Lectures on Curves on an Algebraic Surface*, Annals of Mathematics Studies, no. 59, Princeton University Press, 1966.
- [35] P. OLVER, *Classical Invariant Theory*, London Mathematical Society Student Texts, Cambridge University Press, 1999.
- [36] G. RACAHA, "Theory of complex spectra II", *Phys. Rev.* **62** (1942), p. 438-462.
- [37] H. ROSENGREN, "On the triple sum formula for Wigner 9j-symbols", *J. Math. Phys.* **39** (1998), no. 12, p. 6730-6744.
- [38] ———, "Another proof of the triple sum formula for Wigner 9j-symbols", *J. Math. Phys.* **40** (1999), no. 12, p. 6689-6691.
- [39] G. SALMON, *Higher Algebra*, Reprinted by Chelsea Publishing Co., New York, 1965.
- [40] T. A. SPRINGER, *Invariant Theory*, Lecture Notes in Mathematics, no. 585, Springer-Verlag, 1977.
- [41] E. STROH, "Entwicklung der Grundsyzyganten der binären Form fünfter Ordnung", *Math. Ann.* **34** (1889), p. 354-370.
- [42] B. STURMFELS, *Algorithms in Invariant Theory*, Texts and Monographs in Symbolic Computation, Springer-Verlag, 1993.
- [43] E. WIGNER, *Group Theory and Its Application to the Quantum Theory of Atomic Spectra*, Academic Press, 1959.

Manuscrit reçu le 2 avril 2008,  
 accepté le 27 mars 2009.

Abdelmalek ABDESSELAM  
 University of Virginia  
 Department of Mathematics  
 Kerchof Hall  
 P. O. Box 400137  
 Charlottesville, VA 22904-4137 (USA)  
 malek@virginia.edu

Jaydeep CHIPALKATTI  
 University of Manitoba  
 Department of Mathematics  
 433 Machray Hall  
 Winnipeg MB R3T 2N2 (Canada)  
 chipalka@cc.umanitoba.ca