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SURPRISING PROPERTIES OF CENTRALISERS IN CLASSICAL LIE ALGEBRAS

by Oksana YAKIMOVA

ABSTRACT. — Let $\mathfrak{g}$ be a classical Lie algebra, *i.e.*, either $\mathfrak{gl}_n$, $\mathfrak{sp}_n$, or $\mathfrak{so}_n$ and let e be a nilpotent element of $\mathfrak{g}$. We study various properties of the centralisers $\mathfrak{g}_e$. The first four sections deal with rather elementary questions, like the centre of $\mathfrak{g}_e$, commuting varieties associated with $\mathfrak{g}_e$, or centralisers of commuting pairs. The second half of the paper addresses problems related to different Poisson structures on $\mathfrak{g}_e^*$ and symmetric invariants of $\mathfrak{g}_e$.

RÉSUMÉ. — Soit $\mathfrak{g}$ une algèbre de Lie classique, *i.e.*, $\mathfrak{gl}_n$, $\mathfrak{sp}_n$, ou $\mathfrak{so}_n$, et soit e un élément nilpotent de $\mathfrak{g}$. Nous étudions dans cet article diverses propriétés du centralisateur $\mathfrak{g}_e$ de e. Les quatre premières sections concernent des problèmes assez élémentaires portant sur le centre de $\mathfrak{g}_e$, la variété commutante de $\mathfrak{g}_e$, ou encore les centralisateurs des paires commutantes. La seconde partie aborde des questions liées aux différentes structures de Poisson sur $\mathfrak{g}_e^*$ et aux invariants symétriques de $\mathfrak{g}_e$.

Introduction

Suppose that G is a connected reductive algebraic group defined over a field $\mathbb{F}$ and $\mathfrak{g} = \operatorname{Lie} G$. For $x \in \mathfrak{g}$ let $\mathfrak{g}_x$ denote the centraliser of x in $\mathfrak{g}$. Due to the existence of the Jordan decomposition many questions about centralisers are readily reduced to nilpotent elements $e \in \mathfrak{g}$. In this paper we restrict ourself to the case of classical $\mathfrak{g}$ and study various properties of centralisers. The first four sections deal with rather elementary questions, like *commuting varieties* associated with $\mathfrak{g}_e$ or centralisers of commuting pairs. The second half of the paper addresses problems related to different Poisson structures on $\mathfrak{g}_e^*$ and symmetric invariants of $\mathfrak{g}_e$. It pursues further an approach and some methods of [15].

Keywords: Nilpotent orbits, centralisers, symmetric invariants.
Math. classification: 17B45.

In Section 1, we introduce a basis of $\mathfrak{g}_e$, which is used throughout the paper. Section 2 is devoted to the description of the centre of $\mathfrak{g}_e$. Let $\mathcal{N}(\mathfrak{g}) \subset \mathfrak{g}$ be the nilpotent cone, *i.e.*, the set of nilpotent elements. Let $\operatorname{rk}\mathfrak{g}$ denote the rank of $\mathfrak{g}$. Answering a question of Hotta and Kashiwara, Sekiguchi wrote a short note [19], where he stated (without a proof) that, for each classical Lie algebra $\mathfrak{g}$ and each $e \in \mathcal{N}(\mathfrak{g})$, there exists $x \in \mathfrak{g}_e$ such that the centraliser $\mathfrak{g}_{(e,x)} = \mathfrak{g}_e \cap \mathfrak{g}_x$ is of dimension $\operatorname{rk}\mathfrak{g}$. He addressed the same problem for the exceptional Lie algebras, but was not able to deal with the E_8 case and overlooked one orbit in type G_2. Recently W. de Graaf [7] calculated (using computer) that in the exceptional Lie algebras there are only three nilpotent orbits Ge such that $\dim\mathfrak{g}_{(e,x)} > \operatorname{rk}\mathfrak{g}$ for all $x \in \mathfrak{g}_e$, one in G_2, one in F_4 and one in E_8. In Section 3, we prove that, for each x in a classical Lie algebra $\mathfrak{g}$, there is a *nilpotent* element $e \in \mathfrak{g}_x$ such that $\dim\mathfrak{g}_{(x,e)} = \operatorname{rk}\mathfrak{g}$.

In Section 4, we study *mixed* commuting varieties, $\mathfrak{C}^*(\mathfrak{g}_e) = \{(x,\alpha) \in \mathfrak{g}_e \times \mathfrak{g}_e^* \mid \alpha([x,\mathfrak{g}_e]) = 0\}$, associated with centralisers. In contrast with the reductive case, these varieties can be reducible. The simplest examples are provided by a minimal nilpotent element in $\mathfrak{sl}_4$ (defined by partition $(2,1,1)$)) and a nilpotent element $e \in \mathfrak{sp}_6$ with Jordan blocks $(4,2)$. On the other hand, we prove that if $e \in \mathcal{N}(\mathfrak{gl}_n)$ has at most two Jordan blocks, then $\mathfrak{C}^*(\mathfrak{g}_e)$ is irreducible.

The last four sections are devoted to the coadjoint representation of $\mathfrak{g}_e$. In those sections we assume that the ground field $\mathbb{F}$ is algebraically closed and of characteristic zero. For a linear action of a Lie algebra $\mathfrak{q}$ on a vector space V, let $\mathfrak{q}_v$ denote the stabiliser of $v \in V$ in $\mathfrak{q}$. Recall that $\operatorname{ind}\mathfrak{q} = \min_{\gamma\in\mathfrak{q}^*}\dim\mathfrak{q}_\gamma$. Set

$$\mathfrak{q}^*_{\text{sing}} := \{\gamma \in \mathfrak{q}^* \mid \dim\mathfrak{q}_\gamma > \operatorname{ind}\mathfrak{q}\}.$$

For a reductive Lie algebra $\mathfrak{g}$ we have $\operatorname{codim}\mathfrak{g}^*_{\text{sing}} \geqslant 3$. In Section 5, the same is shown to be true for the centralisers in type A. In type C there are elements such that $\operatorname{codim}(\mathfrak{g}_e^*)_{\text{sing}} = 2$. In all other simple Lie algebras $\mathfrak{g}$ the codimension of $(\mathfrak{g}_e^*)_{\text{sing}}$ may be 1, see [15, Section 3.9].

The dual space $\mathfrak{q}^*$ of a Lie algebra carries a Poisson structure induced by the Lie-Poisson bracket on $\mathfrak{q}$. Having inequalities like $\operatorname{codim}\mathfrak{q}^*_{\text{sing}} \geqslant 2, 3$ one can construct interesting (maximal) Poisson-commutative subalgebras in $\mathcal{S}(\mathfrak{q})$, see [17].

By the Jacobson-Morozov theorem, e can be included into an $\mathfrak{sl}_2$-triple (e,h,f) in $\mathfrak{g}$. Let us identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of the Killing form on $\mathfrak{g}$. Then $\mathfrak{g}_e^*$ is isomorphic to a so called *Slodowy slice* $\mathbb{S}_e = e + \mathfrak{g}_f \subset \mathfrak{g}^*$ at e to the (co)adjoint orbit Ge. The Slodowy slice $\mathbb{S}_e$ and hence $\mathfrak{g}_e^*$, carries another polynomial Poisson structure, obtained from $\mathfrak{g}^*$ via Weinstein reduction, see e.g. [4] or [5]. This second Poisson bracket is not linear in general and its linear part coincides with the Lie-Poisson bracket on $\mathfrak{g}_e^*$. On the quantum level, one can express the fact by saying that a *finite W-algebra* $W(\mathfrak{g}, e)$ is a deformation of the universal enveloping algebra $\mathbf{U}(\mathfrak{g}_e)$. The centre of $W(\mathfrak{g}, e)$ is a polynomial algebra in $\operatorname{rk}\mathfrak{g}$ variables for all $\mathfrak{g}$ and e. (It can be deduced from the analogous statement on the Poisson level, which is proved e.g. in [15, Remark 2.1].) In [15], the same is shown to be true for the centre of $\mathbf{U}(\mathfrak{g}_e)$, which is isomorphic to $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$, if $\mathfrak{g}$ is of type A or C. In type A another proof is given by Brown and Brundan [2]. In Section 6, we compare construction of [15] and [2] and conclude that they produce the same set of generating symmetric invariants.

In Section 7, we prove that, in types A and C, a generic fibre of the quotient morphism $\mathfrak{g}_e^* \to \mathfrak{g}_e^*/\!\!/G_e$ consists of a single (closed) G_e-orbit. The most interesting fibre of this quotient morphism is the one containing zero, the so called *null-cone* $\mathcal{N}(e)$. In type A it is equidimensional by [15, Section 5]. Contrary to the expectations, see [15, Conjecture 5.1], the null-cone is not reduced (as a scheme). A counterexample is provided by $e \in \mathcal{N}(\mathfrak{gl}_6)$ with partition $(4, 2)$. This implies that the *tangent cone* at e to $\mathcal{N}(\mathfrak{gl}_6)$ is not reduced either. For this nilpotent element there is an irreducible component of $\mathcal{N}(e)$, which contains infinitely many closed G_e-orbits and no regular elements.

If $e \in \mathfrak{gl}_n$ is defined by a rectangular partition d^k, then $\mathfrak{g}_e$ is a truncated current algebra $\mathfrak{gl}_k \otimes \mathbb{F}[t]/(t^d)$ and it is also a so called *Takiff* Lie algebra. As was noticed by Eisenbud and Frenkel [12, Appendix], a deep result of Mustăța [12] implies that $\mathcal{N}(e)$ is irreducible. Apart from that little is known about the number of irreducible components of $\mathcal{N}(e)$. We compute that $\mathcal{N}(e)$ has $m+1$ components for the hook partition $(n, 1^m)$ with $n > 1$, $m > 0$ and $\min(n-m, m)+1$ components for the partition (n, m) with $n \geqslant m$.

Suppose that either $\mathfrak{g}$ is an orthogonal Lie algebra and $e \in \mathfrak{g}$ has only Jordan blocks of odd size or $\mathfrak{g}$ is symplectic and e has only Jordan blocks of even size. Then, as shown in Section 8, all irreducible components of $\mathcal{N}(e)$ are of dimension $\dim \mathfrak{g}_e - \operatorname{rk}\mathfrak{g}$. In type A the same result is proved in [15, Section 5] for all nilpotent elements.

In Sections 1–4, the ground field is supposed to be infinite and whenever dealing with orthogonal or symmetric Lie algebras we assume that $\operatorname{char}\mathbb{F} \neq 2$.

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1. Basis of a centraliser

The main object of this section is to introduce our notation. We construct a certain basis in $\mathfrak{g}_e$, which is used throughout the paper. Let $\mathbb{V}$ be an n-dimensional vector space over $\mathbb{F}$ and let e be a nilpotent element in $\mathfrak{g} = \mathfrak{gl}(\mathbb{V})$. Let k be the number of Jordan blocks of e and $W \subseteq \mathbb{V}$ a (k-dimensional) complement of $\operatorname{Im} e$ in $\mathbb{V}$. Let d_i+1 denote the size of the i-th Jordan block of e. We always assume that the Jordan blocks are ordered such that $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_k$ so that e is represented by the partition $(d_1{+}1, \ldots, d_k{+}1)$ of $n = \dim \mathbb{V}$. Choose a basis $w_1, w_2, \ldots, w_k$ in W such that the vectors $e^j \cdot w_i$ with $1 \leqslant i \leqslant k$, $0 \leqslant j \leqslant d_i$ form a basis for $\mathbb{V}$ and put $\mathbb{V}[i] := \operatorname{span}\{e^j \cdot w_i \mid j \geqslant 0\}$. Note that $e^{d_i+1} \cdot w_i = 0$ for all $i \leqslant k$.

If $\xi \in \mathfrak{g}_e$, then $\xi(e^j \cdot w_i) = e^j \cdot \xi(w_i)$, hence ξ is completely determined by its values on W. Each vector $\xi(w_i)$ can be written as

$$\xi(w_i) = \sum_{j,s} c_i^{j,s} e^s \cdot w_j, \quad c_i^{j,s} \in \mathbb{F}. \tag{1.1}$$

Thus, ξ is completely determined by the coefficients $c_i^{j,s} = c_i^{j,s}(\xi)$. This shows that $\mathfrak{g}_e$ has a basis $\{\xi_i^{j,s}\}$ such that

$$\begin{cases} \xi_i^{j,s}(w_i) = e^s \cdot w_j, \\ \xi_i^{j,s}(w_t) = 0 \ \text{ for } t \neq i, \end{cases} \qquad 1 \leqslant i, j \leqslant k \text{ and } \max\{d_j - d_i, 0\} \leqslant s \leqslant d_j.$$

Note that $\xi \in \mathfrak{g}_e$ preserves each $\mathbb{V}[i]$ if and only if $c_i^{j,s}(\xi) = 0$ for $i \neq j$.

An example of $\xi_i^{j,1}$ with $i > j$ and $d_j = d_i{+}1$ is shown on Figure 1.1. On Figure 1.2, we indicate elements $\xi_i^{j,s}$ using Arnold's description of $\mathfrak{g}_e$ for e with three Jordan blocks. In that interpretation e is given in a standard Jordan form and each $\xi_i^{j,s}$ as a matrix with entries 1 on one of the (above) diagonal lines in one of the nine rectangles.

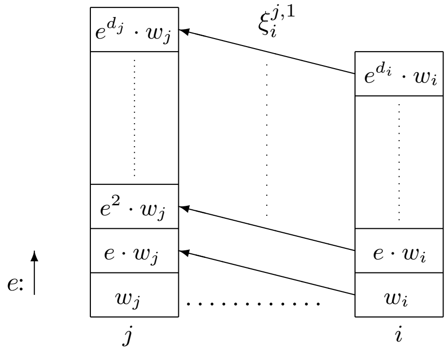

Figure 1.1

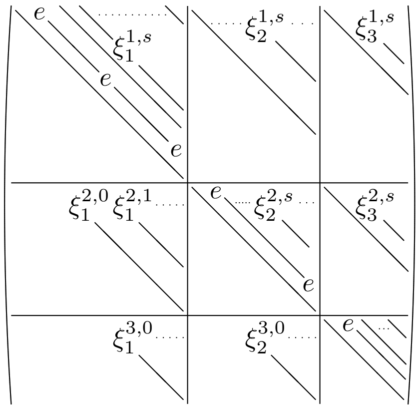

Figure 1.2

A direct computation shows that the following commutator relation holds in $\mathfrak{g}_e$:

$$[\xi_i^{j,s}, \xi] = \sum_{t,\ell} c_t^{i,\ell}(\xi)\xi_t^{j,\ell+s} - \sum_{t,\ell} c_j^{t,\ell}(\xi)\xi_i^{t,\ell+s}, \quad (\forall\, \xi \in \mathfrak{g}_e); \tag{1.2}$$

see [24] for more detail.

Let $(\xi_i^{j,s})^*$ be a linear function on $\mathfrak{g}_e$ such that $(\xi_i^{j,s})^*(\xi) = c_i^{j,s}(\xi)$. Then $\big\langle(\xi_i^{j,s})^*\big\rangle$ form a basis of $\mathfrak{g}_e^*$ dual to the basis $\big\langle\xi_i^{j,s}\big\rangle$ of $\mathfrak{g}_e$.

Let $a\colon \mathbb{F}^\times \to \mathrm{GL}(\mathbb{V})_e$ be the cocharacter such that $a(t)\cdot w_i = t^i w_i$ for all $i \leqslant k$ and $t \in \mathbb{F}^\times$ and define a rational linear action $\rho\colon \mathbb{F}^\times \to \mathrm{GL}(\mathfrak{g}_e^*)$

by the formula

$$\rho(t)\gamma = t\,\mathrm{Ad}^*(a(t)^{-1})\gamma, \quad \left(\forall\,\gamma \in \mathfrak{g}_e^*,\ \ \forall\, t \in \mathbb{F}^{\times}\right). \tag{1.3}$$

Then $\rho(t)(\xi_i^{j,s})^* = t^{i-j+1}(\xi_i^{j,s})^*$ and for the adjoint action, denoted by the same letter, we have $\rho(t)\xi_i^{j,s} = t^{j-i-1}\xi_i^{j,s}$.

Let $(\ ,\ )$ be a nondegenerate symmetric or skew-symmetric bilinear form on $\mathbb{V}$, *i.e.*, $(v,w) = \varepsilon(w,v)$, where $v,w \in \mathbb{V}$ and $\varepsilon = +1$ or -1. Let J be the matrix of $(\ ,\ )$ with respect to a basis B of $\mathbb{V}$. Let X denote the matrix of $x \in \mathfrak{gl}(\mathbb{V})$ relative to B. The linear mapping $x \mapsto \sigma(x)$ sending each $x \in \mathfrak{gl}(\mathbb{V})$ to the linear transformation $\sigma(x)$ whose matrix relative to B equals $-JX^tJ^{-1}$ is an involutory automorphism of $\mathfrak{gl}(\mathbb{V})$ independent of the choice of B. The elements of $\mathfrak{gl}(\mathbb{V})$ preserving $(\ ,\ )$ are exactly the fixed points of σ. We now set $\widetilde{\mathfrak{g}} := \mathfrak{gl}(\mathbb{V})$ and let $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_0 \oplus \widetilde{\mathfrak{g}}_1$ be the symmetric decomposition of $\widetilde{\mathfrak{g}}$ corresponding to the σ-eigenvalues 1 and -1. The elements $x \in \widetilde{\mathfrak{g}}_1$ have the property that $(x\cdot v, w) = (v, x\cdot w)$ for all $v,w \in \mathbb{V}$.

Set $\mathfrak{g} := \widetilde{\mathfrak{g}}_0$ and let e be a nilpotent element of $\mathfrak{g}$. Since $\sigma(e) = e$, the centraliser $\widetilde{\mathfrak{g}}_e$ of e in $\widetilde{\mathfrak{g}}$ is σ-stable and $(\widetilde{\mathfrak{g}}_e)_0 = \widetilde{\mathfrak{g}}_e^{\sigma} = \mathfrak{g}_e$. This yields the $\mathfrak{g}_e$-invariant symmetric decomposition $\widetilde{\mathfrak{g}}_e = \mathfrak{g}_e \oplus (\widetilde{\mathfrak{g}}_e)_1$.

Lemma 1.1. — *In the above setting, suppose that $e \in \widetilde{\mathfrak{g}}_0$ is a nilpotent element. Then the cyclic vectors $\{w_i\}$ and thereby the spaces $\{\mathbb{V}[i]\}$ can be chosen such that there is an involution $i \mapsto i'$ on the set $\{1,\dots,k\}$ satisfying the following conditions:*

- $d_i = d_{i'}$;
- $(\mathbb{V}[i], \mathbb{V}[j]) = 0$ *if* $i \neq j'$;
- $i = i'$ *if and only if* $(-1)^{d_i}\varepsilon = 1$.

Proof. — This is a standard property of the nilpotent orbits in $\mathfrak{sp}(\mathbb{V})$ and $\mathfrak{so}(\mathbb{V})$, see, for example, [3, Sect. 5.1] or [8, Sect. 1]. □

Let $\{w_i\}$ be a set of cyclic vectors chosen according to Lemma 1.1. Consider the restriction of the $\mathfrak{g}$-invariant form $(\ ,\ )$ to $\mathbb{V}[i] + \mathbb{V}[i']$. Since $(w, e^s\cdot v) = (-1)^s(e^s\cdot w, v)$, a vector $e^{d_i}\cdot w_i$ is orthogonal to all vectors $e^s\cdot w_{i'}$ with $s > 0$. Therefore $(w_{i'}, e^{d_i}\cdot w_i) = (-1)^{d_i}(e^{d_i}\cdot w_{i'}, w_i) \neq 0$. There is a (unique up to a scalar) vector $v \in \mathbb{V}[i]$ such that $(v, e^s\cdot w_{i'}) = 0$ for all $s < d_i$. It is not contained in $\mathrm{Im}\, e$, otherwise it would be orthogonal to $e^{d_i}\cdot w_{i'}$ too and hence to $\mathbb{V}[i']$. Therefore there is no harm in replacing w_i by v. Let us always choose the cyclic vectors w_i in such a way that $(w_i, e^s\cdot w_{i'}) = 0$ for $s < d_i$ and normalise them according to:

$$(w_i, e^{d_i}\cdot w_{i'}) = \pm 1 \ \text{ and } \ (w_i, e^{d_i}\cdot w_{i'}) > 0 \ \text{ if } \ i \leqslant i'. \tag{1.4}$$

Then $\mathfrak{g}_e$ is generated (as a vector space) by the vectors $\xi_i^{j,d_j-s} + \varepsilon(i,j,s)\xi_{j'}^{i',d_i-s}$, where $\varepsilon(i,j,s) = \pm 1$ depending on i, j and s in the following way

$$(e^{d_j-s} \cdot w_j, e^s \cdot w_{j'}) = -\varepsilon(i,j,s)(w_i, e^{d_i} \cdot w_{i'}).$$

Elements $\xi_i^{j,d_j-s} - \varepsilon(i,j,s)\xi_{j'}^{i',d_i-s}$ form a basis of $(\widetilde{\mathfrak{g}}_e)_1$. In the following we always normalise w_i as above and enumerate the Jordan blocks such that $i' \in \{i, i+1, i-1\}$ keeping inequalities $d_i \geqslant d_j$ for $i < j$. In this basis $\{e^s \cdot w_i\}$ the matrix of the restriction $(\ ,\ )\big|_{(\mathbb{V}[i]+\mathbb{V}[i'])}$ is anti-diagonal with entries ± 1.

2. The centre of a centraliser

Let $\mathfrak{z}$ be the centre of $\mathfrak{g}_e$. The powers of e (as a matrix) are also elements of $\mathfrak{gl}(V)$. Set $\mathfrak{E} := \mathfrak{g} \cap \langle e^0, e, e^2, \ldots, e^{d_1}\rangle_{\mathbb{F}}$. All higher powers of e are zeros; the first element, e^0, is the identity matrix. Clearly, $\mathfrak{E} \subset \mathfrak{z}$. If $\mathfrak{g}$ is either $\mathfrak{sl}(\mathbb{V})$ or $\mathfrak{sp}(\mathbb{V})$, then this inclusion is in fact the equality and in orthogonal Lie algebras $\mathfrak{z}$ can be larger. For $\mathfrak{g}$ classical, the centre of $\mathfrak{g}_e$ was described by Kurtzke [11] and that description is not quite correct.

The following result is well-known. The proof is easy and illustrates the general scheme of argument very well.

Theorem 2.1. — *If $\mathfrak{g} = \mathfrak{gl}(\mathbb{V})$, then $\mathfrak{z} = \mathfrak{E}$.*

Proof. — We have $e^s = \sum_{i=1}^k \xi_i^{i,s}$ and $e^s \in \mathfrak{g}$ for all $0 \leqslant s \leqslant d_1$. Suppose $\eta \in \mathfrak{z}$. Then η commutes with the maximal torus $\mathfrak{t} := \langle \xi_i^{i,0}\rangle_{\mathbb{F}} \subset \mathfrak{gl}(\mathbb{V})_e$. We have

$$[\xi_i^{i,0}, \xi_j^{t,s}] = \begin{cases} -\xi_i^{t,s} & \text{if } i = j, i \neq t; \\ \xi_j^{i,s} & \text{if } i = t, i \neq j; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\eta \in \langle \xi_i^{i,s}\rangle_{\mathbb{F}}$. Adding an element of $\mathfrak{E}$ we may assume that $c_1^{1,s}(\eta) = 0$ for all s. If $\eta \notin \mathfrak{E}$, then there is some $c_i^{i,s}(\eta)$, which is not zero. Now take $\xi_1^{i,0} \in \mathfrak{g}_e$ and compute that

$$[\eta, \xi_1^{i,0}] = c_i^{i,0}(\eta)\xi_1^{i,0} + c_i^{i,1}(\eta)\xi_1^{i,1} + \cdots + c_i^{i,d_i}(\eta)\xi_1^{i,d_i} \neq 0.$$

A contradiction! Thus $\mathfrak{z} = \mathfrak{E}$. □

Corollary 2.2. — *Suppose that $\mathfrak{g} = \mathfrak{sl}(\mathbb{V})$, then also $\mathfrak{z} = \mathfrak{E}$.*

THEOREM 2.3. — *If* $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ *and* e *is given by a partition* $(d_1+1, d_2+1, d_3+1, \ldots, d_k+1)$ *with* $k \geqslant 2$*, where* $d_2 > d_3$ *and both* d_1 *and* d_2 *are even, then* $\mathfrak{z} = \mathfrak{E} \oplus \mathbb{F}(\xi_1^{2,d_2} - \xi_2^{1,d_1})$*. For all other nilpotent elements of classical simple Lie algebras, we have* $\mathfrak{z} = \mathfrak{E}$*.*

Proof. — First we show that indeed in the special case indicated in the theorem we have an additional central element $x := \xi_1^{2,d_2} - \xi_2^{1,d_1}$. Note that $\xi_1^{2,d_2}, \xi_2^{1,d_1}$ do not commute only with the elements $\xi_1^{1,0}$, $\xi_2^{2,0}$, ξ_2^{1,d_1-d_2} and $\xi_1^{2,0}$. Since $1' = 1, 2' = 2$, the centraliser $\mathfrak{g}_e$ contains no elements of the form $a\xi_1^{1,0} + b\xi_2^{2,0}$ and we have to check only that $[x, \xi_1^{2,0} + \varepsilon(1,2,d_2)\xi_2^{1,d_1-d_2}] = 0$. Here d_1 and d_2 are even, therefore $\varepsilon(1,2,d_2) = -1$. We get

$$\left[x, \xi_1^{2,0} - \xi_2^{1,d_1-d_2}\right] = -\xi_1^{1,d_1} - \xi_2^{2,d_1} + \xi_1^{1,d_1} + \xi_2^{2,d_1} = 0.$$

Let us prove that $\mathfrak{z}$ is not larger than stated in the theorem. The case $\mathfrak{g} = \mathfrak{gl}(\mathbb{V})$ (or $\mathfrak{sl}(\mathbb{V})$) was treated above. Thus assume that $\mathfrak{g}$ is either $\mathfrak{sp}(\mathbb{V})$ or $\mathfrak{so}(\mathbb{V})$. Then $\mathfrak{E}$ is a vector space generated by all odd powers of e.

Suppose that $\eta \in \mathfrak{z}$. If η preserve the cyclic spaces $\mathbb{V}[i]$, then $\eta \in \mathfrak{E}$. It can be shown exactly in the same way as in the $\mathfrak{gl}(V)$ case. Note that whenever $i \neq i'$ there is an $\mathfrak{sl}_2$-triple (subalgebra) $\mathfrak{q}_i = \langle \xi_i^{i,0} - \xi_{i'}^{i',0}, \xi_i^{i',0}, \xi_{i'}^{i,1} \rangle_{\mathbb{F}} \subset \mathfrak{g}_e$. Equality $[\eta, \mathfrak{q}_i] = 0$ forces $c_i^{j,s}(\eta) = 0$ whenever $i \neq i'$ (or $j \neq j'$) and $i \neq j$, also $c_i^{i,s}(\eta) = c_{i'}^{i',s}(\eta)$ for $i \neq i'$.

Assume that $\eta \notin \mathfrak{E}$. Take the minimal i such that there is a non-zero $c_i^{j,s}(\eta)$ with $j \neq i$. (Necessary $i' = i$ and $j' = j$.) Fix this i and take the minimal j and then the minimal s, with this property. Since $c_j^{i,d_j-d_i+s}(\eta) \neq 0$, we have also $j > i$ and therefore $j > 1, 1'$. There is an element $\xi := \xi_j^{1,d_1-s} + \varepsilon(j,1,s)\xi_{1'}^{j,d_j-s} \in \mathfrak{g}_e$. Consider the commutator $[\xi, \eta] = \xi\eta - \eta\xi$. We are interested in the coefficient $a_i := c_i^{1,d_1}([\xi,\eta])$. Since all coefficients $c_i^{1',r}(\eta)$ are zeros and $j \neq i$, we get

$$a_i = c_i^{j,s}(\eta) - \delta_{i,1}\varepsilon(j,1,s)c_j^{1,d_1-d_j+s}(\eta).$$

In particular, if $i \neq 1$, then η is not a central element. Therefore $i = 1$.

In the symplectic case d_1 and d_j are odd, hence $d_j - s$ and s have different parity and $\varepsilon(j,1,s)\varepsilon(1,j,d_j-s) = -1$. Thus $a_i = 2c_i^{j,s} \neq 0$. We get a contradiction.

The orthogonal case is more complicated. If $j > 2$, then also $j > 2'$ and

$$c_1^{2,d_2}\left(\left[\xi_j^{2,d_2-s} + \varepsilon(j,2,s)\xi_{2'}^{j,d_j-s}, \eta\right]\right) = c_1^{j,s}(\eta) \neq 0.$$

Since $\eta \in \mathfrak{z}$, we get $j = 2$. If $d_3 = d_2$, then $3' = 3$ and there is a semisimple element $\xi_2^{3,0} - \xi_3^{2,0} \in \mathfrak{g}_e$, which does not commute with η.

It remains to consider only the special case $d_2 > d_3$. There is no harm in replacing η by $\eta - c_1^{2,d_2}(\eta)(\xi_1^{2,d_2} - \xi_2^{1,d_1})$. In other words, we may assume that $c_1^{2,d_2}(\eta) = 0$ and thereby $s < d_2$. It is not difficult to see that η does not commute either with $\xi_1^{2,1} + \xi_2^{1,d_1-d_2+1}$ or $\xi_1^{2,0} - \xi_2^{1,d_1-d_2}$, depending on the parity of s. Thus if $\eta \notin \mathfrak{C} \oplus \mathbb{F}(\xi_1^{2,d_2} - \xi_2^{1,d_1})$, then η is not a central element. This completes the proof. □

Remark 2.4. — In [11, Proposition 3.5], Kurtzke overlooked nilpotent elements in $\mathfrak{so}(\mathbb{V})$ such that $\mathfrak{C}$ is of codimension 1 in $\mathfrak{z}$ and $k > 2$.

3. Centralisers of commuting pairs

By Vinberg's inequality, $\dim(\mathfrak{g}_e)_\alpha \geqslant \operatorname{rk}\mathfrak{g}$ for any $\alpha \in \mathfrak{g}_e^*$. A famous conjecture of Elashvili states that there is $\alpha \in \mathfrak{g}_e^*$ such that $\dim(\mathfrak{g}_e)_\alpha = \operatorname{rk}\mathfrak{g}$. In the classical case, Elashvili's conjecture is proved in [24] and for the exceptional Lie algebras it is verified (with a computer aid) by W. de Graaf [7]. In [7], de Graaf also showed that in the exceptional Lie algebras there are only three nilpotent orbits Ge such that $\dim(\mathfrak{g}_e)_x > \operatorname{rk}\mathfrak{g}$ for all $x \in \mathfrak{g}_e$. The result was predicted by Elashvili.

By a result of Richardson [18], the commuting variety $\mathfrak{C}(\mathfrak{g}) := \{(x, y) \in \mathfrak{g}\times\mathfrak{g} \mid [x, y] = 0\}$ is irreducible for each reductive Lie algebra $\mathfrak{g}$. It coincides with the closure of a G-saturation $G(\mathfrak{t}, \mathfrak{t})$, where $\mathfrak{t} \subset \mathfrak{g}$ is a maximal torus. Hence $\dim(\mathfrak{g}_e)_x \geqslant \operatorname{rk}\mathfrak{g}$ for all $x \in \mathfrak{g}_e$. A general belief is that in the classical Lie algebras there is always an element $x \in \mathfrak{g}_e$ for which the equality holds. The statement even appeared in the literature without a proof, [19]. Here we prove a slightly stronger statement. Set $\mathfrak{g}_{(e,x)} := (\mathfrak{g}_e)_x = \mathfrak{g}_e \cap \mathfrak{g}_x$.

THEOREM 3.1. — *Suppose that $\mathfrak{g}$ is a classical simple Lie algebra and $e \in \mathcal{N}(\mathfrak{g})$. Then there is a* `nilpotent` *element $x \in \mathfrak{g}_e$ such that* $\dim \mathfrak{g}_{(e,x)} = \operatorname{rk}\mathfrak{g}$.

Proof.

(**1**) If $\mathfrak{g} = \mathfrak{sl}_n$, then e can be included into a so called *principal nilpotent* pair (e, x), where $x := \sum_{i=1}^{k-1} \xi_i^{i+1,0}$ and $\dim \mathfrak{g}_{(e,x)} = n - 1$, see [6].

(**2**) Now assume that $\mathfrak{g} \subset \mathfrak{gl}(\mathbb{V})$ is either symplectic or orthogonal. The required element $x \in \mathfrak{g}_e$ is defined as

$$x := \sum_{i=1}^{k-1} \xi_i^{i+1,0} + \varepsilon(i, i{+}1, 0)\xi_{(i+1)'}^{i',d_i-d_{i+1}}.$$

Set $\widetilde{\mathfrak g} := \mathfrak{gl}(\mathbb V)$. Let $a\colon\ \mathbb F^\times \to \mathrm{GL}(\mathbb V)_e$ be the cocharacter such that $a(t){\cdot}w_i = t^i w_i$ for all $i \leqslant k$ and $t\in\mathbb F^\times$ (the same as in (1.3)). Then $\mathrm{Ad}(a(t))\cdot x = tx_+ + t^{-1}x_-$, where $x = x_+ + x_-$ and $x_+ = \sum_{i=1}^{k-1}\xi_i^{i+1,0}$. As in part (**1**) of the proof, e and x_+ form a principal nilpotent pair in $\widetilde{\mathfrak g}$. Therefore $\dim\widetilde{\mathfrak g}_{(e,x_+)} = n = \dim\mathbb V$. Points $x_+ + t^2x_-$ with $t\in\mathbb F^\times$ form a dense (if $\mathbb F$ is algebraically closed, then open) subset of the line $x_+ + \mathbb F x_-$. Hence, using semi-continuity of dimension, one can show that also $\dim\widetilde{\mathfrak g}_{(e,x)} \leqslant n$.

Consider a product of matrices $e^r x^l$ as an element of $\widetilde{\mathfrak g}_e$. Then $e^r x^l\cdot w_1 = e^r\cdot w_l + v$, where

$$v\in\mathbb V_1\oplus\cdots\oplus\mathbb V_{l-1}\oplus\big\langle w_l, e^1\cdot w_l,\ldots,e^{r-1}\cdot w_l\big\rangle.$$

Hence $\dim\big\langle e^r x^l{\cdot}w_1 \mid r,l\geqslant 0\big\rangle = n$. Clearly each $e^r x^l$ is an element of $\widetilde{\mathfrak g}_{(e,x)}$. Therefore $\dim\widetilde{\mathfrak g}_{(e,x)}\geqslant n$. Taking into account that $\dim\widetilde{\mathfrak g}_{(e,x)}\leqslant n$, we get the equality $\dim\widetilde{\mathfrak g}_{(e,x)} = n$. The centraliser $\widetilde{\mathfrak g}_{(e,x)}$ is the linear span of the vectors $e^r x^l$.

Recall that there is a $\mathfrak g$-invariant bilinear form on $\mathbb V$ such that $(\xi\cdot v, w) = -(v,\xi\cdot w)$ and $(\eta\cdot v,w) = (v,\eta\cdot w)$ for all vectors $v,w\in\mathbb V$, $\xi\in\mathfrak g$, $\eta\in\widetilde{\mathfrak g}_1$. Hence $e^r x^l\in\mathfrak g$ if $r{+}l$ is odd and $e^r x^l\in\widetilde{\mathfrak g}_1$ if $r{+}l$ is even. The centraliser of the pair (e,x) in $\mathfrak g$ is equal to the intersection $\widetilde{\mathfrak g}_{(e,x)}{\cap}\mathfrak g$, which has dimension $[(n{+}1)/2] = \mathrm{rk}\,\mathfrak g$. □

Remark 3.2. — Suppose that $y = y_s + y_n$ is the Jordan decomposition of $y\in\mathfrak g$ and $\mathfrak g$ is classical. Then $\mathfrak g_y = (\mathfrak g_{y_s})_{y_n}$ and $\mathfrak g_{y_s}$ is a direct sum of the centre and simple classical ideals. Therefore Theorem 3.1 is valid for all (not necessary simple) classical Lie algebras $\mathfrak g$ and all (not necessary nilpotent) $y\in\mathfrak g$.

4. Commuting varieties

With a non-reductive Lie algebra $\mathfrak q$ one can associate two different commuting varieties. The usual one $\mathfrak C(\mathfrak q)$, consisting of commuting pairs $(\xi,\eta)\in\mathfrak q{\times}\mathfrak q$, appeared in the previous section. In this section we consider *mixed* commuting varieties

$$\mathfrak C^*(\mathfrak q) := \{(x,\alpha)\in\mathfrak q{\times}\mathfrak q^* \mid \alpha([x,\mathfrak q]) = 0\},$$

associated with centralisers. These varieties are closely related to some questions concerning rings of differential operators. Another way to define $\mathfrak C^*(\mathfrak q)$ is to say that it is the zero fibre of the moment map $\mathfrak q\times\mathfrak q^*\to\mathfrak q^*$.

The usual commuting variety $\mathfrak C(\mathfrak g_e)$ is not always irreducible, see [24]. Here we show that $\mathfrak C^*(\mathfrak g_e)$ can be reducible as well, even if $\mathfrak g$ is of type A.

However, let us start with examples outside of type A. The first of them is related to the following property:

$$\mathfrak{q}_{\rm reg} \cap \left(\bigcup_{\alpha \in \mathfrak{q}^*_{\rm reg}} \mathfrak{q}_\alpha \right) = \varnothing. \tag{4.1}$$

Here $\mathfrak{q}^*_{\rm reg} := \mathfrak{q}^* \setminus \mathfrak{q}^*_{\rm sing}$ and $\xi \in \mathfrak{q}_{\rm reg}$ if and only if the stabiliser $\mathfrak{q}_\xi$ has the minimal possible dimension.

Proposition 4.1. — *Suppose that $\mathfrak{q}$ satisfies (4.1). Then $\mathfrak{C}^*(\mathfrak{q})$ is reducible.*

Proof. — Clearly $U_1 := \mathfrak{C}^*(\mathfrak{q}) \cap (\mathfrak{q}_{\rm reg} \times \mathfrak{q}^*)$ and $U_2 := \mathfrak{C}^*(\mathfrak{q}) \cap (\mathfrak{q} \times \mathfrak{q}^*_{\rm reg})$ are open subsets of $\mathfrak{C}^*(\mathfrak{q})$ and according to (4.1), $U_1 \cap U_2 = \varnothing$. □

Example 4.2. — Let $e \in \mathcal{N}(\mathfrak{sp}_6)$ be defined by the partition $(4,2)$. Then $\mathfrak{g}_e$ has a basis

$$\xi_1^{1,1},\ \xi_2^{2,1},\ \xi_1^{1,3},\ \xi = \xi_1^{2,0} + \xi_2^{1,2},\ \eta = \xi_1^{2,1} - \xi_1^{2,3}$$

with the only non-trivial commutators being $[\xi, \xi_1^{1,1}] = [\xi_2^{2,1}, \xi] = \eta$ and $[\eta, \xi] = 2\xi_1^{1,3}$. Suppose that $\alpha \in (\mathfrak{g}_e^*)_{\rm reg}$ and $x \in (\mathfrak{g}_e)_\alpha$. Since α is regular, it is non-zero on $[\mathfrak{g}_e, \mathfrak{g}_e] = \langle \xi_1^{1,3}, \eta \rangle$. On the other hand $\alpha([x, \mathfrak{g}_e]) = 0$, hence $\dim[x, \mathfrak{g}_e] \leqslant 1$ and $\dim(\mathfrak{g}_e)_x \geqslant 4 > \operatorname{rk}\mathfrak{g}$. Therefore x is not regular and condition (4.1) holds for $\mathfrak{g}_e$.

Remark 4.3. — The simplest example of a Lie algebra satisfying condition (4.1) is a Heisenberg algebra. The centralisers of subregular elements (given by partitions $(2n{-}2, 2)$) in $\mathfrak{sp}_{2n}$ also satisfy (4.1).

The second example is slightly different.

Proposition 4.4. — *Suppose that for each $\alpha \in (\mathfrak{g}_e^*)_{\rm reg}$ the stabiliser $(\mathfrak{g}_e)_\alpha$ consists of nilpotent elements, but $\mathfrak{g}_e$ itself contains semisimple elements. Then $\mathfrak{C}^*(\mathfrak{g}_e)$ is reducible.*

Proof. — Clearly $U_1 := \{(\mathfrak{g}_e)_\alpha \times \{\alpha\} \mid \alpha \in (\mathfrak{g}_e^*)_{\rm reg}\}$ is an open subset of $\mathfrak{C}^*(\mathfrak{g}_e)$. On the other hand, there is an open subset in $\mathfrak{g}_e$ containing no nilpotent elements. Its preimage $U_2 \subset \mathfrak{C}^*(\mathfrak{g}_e)$ is again an open subset. By our assumptions $U_1 \cap U_2 = \varnothing$. □

There are such nilpotent elements in the orthogonal Lie algebra.

Example 4.5. — Let $e \in \mathcal{N}(\mathfrak{so}_7)$ be defined by the partition $(3,2,2)$. Then $x := \xi_2^{2,0} - \xi_3^{3,0} \in \mathfrak{g}_e$ is a semisimple element, which is unique up to conjugation and multiplication by scalars. Suppose that $\alpha \in \mathfrak{g}_e^*$ is such that $(\mathfrak{g}_e)_\alpha$ does not consist of nilpotent elements. Since $(\mathfrak{g}_e)_\alpha$ is the Lie algebra of an algebraic group $(G_e)_\alpha$, it contains a semisimple element, we may assume that x. Then α is zero on $[x,\mathfrak{g}_e]$. Note that the centraliser of x in $\mathfrak{g}_e$ is three dimensional. More precisely, it is generated by x, $\xi_1^{1,1}$ and $\eta:=\xi_2^{2,1}+\xi_3^{3,1}$. Since x is semisimple, $\alpha = a_1((\xi_2^{2,0})^* - (\xi_3^{3,0})^*) + a_2(\xi_1^{1,1})^* + a_3((\xi_2^{2,1})^* + (\xi_3^{3,1})^*)$, where $a_1,a_2,a_3\in\mathbb{F}$. It not difficult to see that $(\mathfrak{g}_e)_\alpha$ contains elements $\xi_1^{2,1}-\xi_3^{1,2}$, $\xi_1^{3,1}+\xi_2^{1,2}$, $\xi_1^{1,1}$ and, by the assumption, x. Hence $\dim(\mathfrak{g}_e)_\alpha \geqslant 4$ and $\alpha\in(\mathfrak{g}_e^*)_{\rm sing}$.

Remark 4.6. — It is possible to show that if $e\in\mathcal{N}(\mathfrak{so}(\mathbb{V}))$ is given by a partition $(d_1+1,\ldots,d_k+1)$ with d_1 being even and all other d_i odd, then $(\mathfrak{g}_e)_\alpha$ consists of nilpotent elements for each $\alpha\in(\mathfrak{g}_e^*)_{\rm reg}$. Note that $\mathfrak{g}_e$ contains semisimple elements, if $k>1$.

Let us say that a point $\gamma\in\mathfrak{g}_e^*$ is *generic* and $(\mathfrak{g}_e)_\gamma$ is a *generic stabiliser* if there is an open subset $U_0\subset\mathfrak{g}_e^*$ such that $(\mathfrak{g}_e)_\delta$ is conjugate to $(\mathfrak{g}_e)_\gamma$ for each $\delta\in U_0$. Suppose that $\mathfrak{g}=\mathfrak{gl}(\mathbb{V})$. Consider a point $\alpha=\sum_{i=1}^k a_i(\xi_i^{i,d_i})^*\in\mathfrak{g}_e^*$, where a_i are pairwise distinct non-zero numbers. Then, as was proved in [24], α is a generic point in $\mathfrak{g}_e^*$ and $\mathfrak{h}:=(\mathfrak{g}_e)_\alpha=\left\langle\xi_i^{i,s}\right\rangle_{\mathbb{F}}$ is a generic stabiliser for the coadjoint action of $\mathfrak{g}_e$. Set $\mathfrak{h}^*:=\left\langle(\xi_i^{i,s})^*\right\rangle_{\mathbb{F}}\subset\mathfrak{g}_e^*$. Then $\{\gamma\in\mathfrak{g}_e^*\mid {\rm ad}^*(\mathfrak{h})\gamma=0\}=\mathfrak{h}^*$ and $\mathfrak{C}_0:=\overline{G_e(\mathfrak{h}\times\mathfrak{h}^*)}$ is an irreducible component of $\mathfrak{C}^*(\mathfrak{g}_e)$. Likewise, if $e\in\mathfrak{sp}(\mathbb{V})$, then $\mathfrak{C}_0\cap\mathfrak{C}^*(\mathfrak{sp}(\mathbb{V})_e)$ is an irreducible component of the mixed commuting variety associated with $\mathfrak{sp}(\mathbb{V})_e$.

Example 4.7. — Let e be a minimal nilpotent element in $\mathfrak{g}=\mathfrak{sl}_{n+2}$ with $n>1$. Then the mixed commuting variety $\mathfrak{C}^*(\mathfrak{g}_e)$ has at least two irreducible components.

Proof. — Let us include e into an $\mathfrak{sl}_2$-triple $\langle e,h,f\rangle$ in $\mathfrak{g}$. Then h defines a $\mathbb{Z}$-grading of $\mathfrak{g}$:

$$\mathfrak{g}(-2)\oplus\mathfrak{g}(-1)\oplus\mathfrak{g}(0)\oplus\mathfrak{g}(1)\oplus\mathfrak{g}(2),$$

where $\mathfrak{g}(-2)=\mathbb{F}f$, $\mathfrak{g}(2)=\mathbb{F}e$, $\mathfrak{g}(-1)\subset\mathfrak{g}_f$ and $\mathfrak{g}(1)\subset\mathfrak{g}_e$. The centraliser $\mathfrak{g}_e$ is a semidirect product of $\mathfrak{gl}_n=\mathfrak{g}(0)_e$ and a (normal) Heisenberg Lie algebra $\mathfrak{n}=V\oplus\mathbb{F}e$, where $V=\mathfrak{g}(1)\cong\mathbb{F}^n\oplus(\mathbb{F}^n)^*$ as a $\mathfrak{gl}_n$-module. Making use of the Killing form, we identify $\mathfrak{g}_e^*$ and $\mathfrak{g}_f$. Let χ_f be the element of $\mathfrak{g}_e^*$ corresponding to f. Fix the h-invariant decomposition $\mathfrak{g}_e^*=\mathfrak{gl}_n^*\oplus V^*\oplus(\mathbb{F}e)^*$.

The theory of $\mathfrak{sl}_2$-actions tells us that $V^*={\rm ad}^*(V)\chi_f$ and that the stabiliser of a point $\gamma+0+\chi_f$, with $\gamma\in\mathfrak{gl}_n^*$, is equal to $(\mathfrak{gl}_n)_\gamma\oplus\mathbb{F}e$. Let $N\subset G_e$ be the unipotent radical. Then ${\rm Lie}\,N=\mathfrak{n}$ and $N(\mathfrak{gl}_n^*+\mathbb{F}\alpha)$ is an open subset of $\mathfrak{g}_e^*$. Taking its preimage in $\mathfrak{C}^*(\mathfrak{g}_e)$, we obtain that the N-saturation

$$Y:=N\Big\{(\mathfrak{gl}_n)_\gamma\oplus\mathbb{F}e)\times(\gamma+0+\mathbb{F}^*\chi_f)\mid\gamma\in\mathfrak{gl}_n^*\Big\}$$

is an open subset of $\mathfrak{C}^*(\mathfrak{g}_e)$. It is irreducible, because the usual commuting variety associated with $\mathfrak{gl}_n$ ($\cong \mathfrak{gl}_n^*$) is irreducible by a result of Richardson [18]. Thus $\overline{Y}$ is an irreducible component of $\mathfrak{C}^*(\mathfrak{g}_e)$. A generic point $\alpha \in \mathfrak{g}_e^*$ can be chosen as $\alpha = \gamma + \chi_f$, where γ is a generic point in $\mathfrak{gl}_n^*$. Therefore $\overline{Y}$ coincides with the irreducible component $\mathfrak{C}_0$ related to generic stabiliser.

Suppose that $((x, y, z) \times (\gamma, \beta, \delta)) \in Y$. Then there is unique $\xi \in V$ such that $\beta = \mathrm{ad}^*(\xi)\delta$. Hence $y = [\xi, x]$ by the construction of Y.

Take a pair $((x, y, z) \times (\gamma, \beta, 0)) \in \mathfrak{g}_e \times \mathfrak{g}_e^*$. It belongs to $\mathfrak{C}^*(\mathfrak{g}_e)$ if and only if $(\gamma + \beta)([x + y, \mathfrak{gl}_n]) = 0$ and $\beta([x, V]) = 0$. Fix $\beta \in V^*$ and $x \in (\mathfrak{gl}_n)_\beta$. Then the second condition is automatically satisfied and the first one can be rewritten as $\mathrm{ad}^*(x)\gamma + \mathrm{ad}^*(y)\beta = 0$. Varying γ we can get any element of $(\mathfrak{gl}_n/(\mathfrak{gl}_n)_x)^*$ on the first place in this sum. Thus, if $\mathrm{ad}^*(y)\beta$ is zero on $(\mathfrak{gl}_n)_x$, *i.e.*, if $\beta([y, (\mathfrak{gl}_n)_x]) = 0$, then there is $\gamma \in \mathfrak{gl}_n^*$ such that $((x, y, z) \times (\gamma, \beta, 0)) \in \mathfrak{C}^*(\mathfrak{g}_e)$.

Suppose that $((x, y, z) \times (\gamma, \beta, 0)) \in \overline{Y}$. Then there are curves $\{\xi(t)\} \subset V$ and $\{x(t)\} \subset \mathfrak{gl}_n$ such that $\lim_{t\to 0} x(t) = x$, $\lim_{t\to 0} \mathrm{ad}^*(\xi(t))t\chi_f = \beta$ and $\lim_{t\to 0}[\xi(t), x(t)] = y$. Clearly this is possible only if either β or x or y is zero.

If $n > 1$, then there are non-zero $x \in \mathfrak{gl}_n$ and $\beta \in V^*$ such that $x \in (\mathfrak{gl}_n)_\beta$. Since $\mathrm{ad}^*(\mathfrak{gl}_n)\beta \neq V^*$, there is also a non-zero $y \in V$ such that $((x, y, z) \times (\gamma, \beta, 0)) \in \mathfrak{C}^*(\mathfrak{g}_e)$. Therefore $\mathfrak{C}^*(\mathfrak{g}_e)$ is reducible. □

Remark 4.8. — It seems that the mixed commuting variety $\mathfrak{C}^*(\mathfrak{g}_e)$ considered in Example 4.7 has exactly two irreducible components. The first one is $\overline{Y}$ and the closure of

$$\Big\{(x, y, z) \times (\gamma, \beta, 0) \mid (\mathfrak{gl}_n)_\beta \cong \mathfrak{gl}_{n-1}, x \in (\mathfrak{gl}_n)_\beta,$$
$$\beta([y, (\mathfrak{gl}_n)_x]) = 0, \mathrm{ad}^*(x)\gamma + \mathrm{ad}^*(y)\beta = 0\Big\}$$

is the second.

If $n = 1$, *i.e.*, the minimal nilpotent element has only two Jordan blocks, then the argument of Example 4.7 does not work. This is not a coincidence. As we will prove below, $\mathfrak{C}^*(\mathfrak{g}_e)$ is irreducible for all nilpotent elements with at most two Jordan blocks. Similar result for $\mathfrak{C}(\mathfrak{g}_e)$ was obtained by Neubauer and Sethuraman in [13].

Theorem 4.9. — *Suppose that $e \in \mathcal{N}(\mathfrak{gl}(\mathbb{V}))$ has at most two Jordan blocks. Then the mixed commuting variety $\mathfrak{C}^*(\mathfrak{g}_e)$ is irreducible.*

Proof. — For regular nilpotent elements the statement is clear. Therefore assume that e is given by a partition (m,n) with $m \geqslant n$. Let $\mathfrak{z}$ be the centre of $\mathfrak{g}_e$ and $\mathrm{Ann}([\mathfrak{g}_e,\mathfrak{g}_e]) \subset \mathfrak{g}_e^*$ the annihilator of the derived algebra $[\mathfrak{g}_e,\mathfrak{g}_e]$. Suppose that $\{(\xi,\alpha)\} \in \mathfrak{C}^*(\mathfrak{g}_e)$. Then also $(\xi+\mathfrak{z}) \times (\alpha + \mathrm{Ann}([\mathfrak{g}_e,\mathfrak{g}_e]) \subset \mathfrak{C}^*(\mathfrak{g}_e)$. The centre $\mathfrak{z}$ is the linear span of vectors $\xi_1^{1,s}+\xi_2^{2,s}$ with $0 \leqslant s < m$. The derived algebra $[\mathfrak{g}_e,\mathfrak{g}_e]$ is spanned by vectors $\xi_i^{j,s}$ with $i \neq j$ and $(\xi_1^{1,s}-\xi_2^{2,s})$. Let us choose complementary subspaces to $\mathfrak{z}$ (in $\mathfrak{g}_e$) and to $\mathrm{Ann}([\mathfrak{g}_e,\mathfrak{g}_e])$ (in $\mathfrak{g}_e^*$) consisting of the elements ξ and α of the following form:

$$\xi = \sum_{i=0}^{n-1} a_{i+1}\xi_1^{2,i} + \sum_{i=0}^{n-1} c_{i+1}\xi_2^{2,i} + \sum_{i=0}^{n-1} b_{i+1}\xi_2^{1,i+m-n}$$

and

$$\alpha = \sum_{i=0}^{n-1} x_{i+1}(\xi_1^{2,i})^* + \sum_{i=0}^{n-1} z_{i+1}((\xi_1^{1,m-n+i})^* - (\xi_2^{2,m-n+i})^*) + \sum_{i=0}^{n-1} y_{i+1}(\xi_2^{1,i+m-n})^*,$$

for some $a_i, b_i, c_i, x_i, z_i, y_i \in \mathbb{F}$. We will prove irreducibility for the set of "commuting" pairs (ξ,α).

Set $X := (x_1,\ldots,x_n)^t$, $Y := (y_1,\ldots,y_n)^t$ and $Z := (z_1,\ldots,z_n)^t$. Consider X, Y and Z as vectors of an n-dimensional vector space W. Let A, B and C be the upper triangular $n\times n$ matrices with entries a_i, b_i and c_i on the ith diagonal line. So the first line of A is $(a_1,a_2,\ldots,a_n)$, the second $(0,a_1,a_2,\ldots,a_{n-1})$ and so on. Note that these matrices lie in the centraliser $\mathfrak{gl}(W)_{\hat{e}}$ of a regular nilpotent element $\hat{e}$. Hence they commute with each other. The mixed commuting variety $\mathfrak{C}^*(\mathfrak{g}_e)$ is defined by equations of three types $\alpha([\xi,\xi_2^{2,s}])=0$, $\alpha([\xi,\xi_1^{2,s}])=0$ and $\alpha([\xi,\xi_2^{1,s}])=0$. Take the first of them with $s=0$. Then we get the following $\sum_{i=1}^n b_iy_i - \sum_{i=1}^n a_ix_i = 0$. The vector $\xi_2^{2,1}$ will give us that $\sum_{i=1}^{n-1} b_iy_{i+1} = \sum_{i=1}^{n-1} a_ix_{i+1}$. In matrix terms this can be expressed as $AX=BY$. Explicitly writing down equations of all three types one can deduce that $\mathfrak{C}^*(\mathfrak{g}_e)$ is defined by the matrix equations

$$AX = BY, \quad CX = BZ, \quad CY = AZ. \tag{4.2}$$

Thus our problem is reduced to a simple exercise in linear algebra. The following lemma solves this exercise and thereby completes the proof. □

Lemma 4.10. — *Suppose that W is an n-dimensional vector space and $\hat{e} \in \mathfrak{gl}(W)$ is a regular nilpotent element. Let P be the set of six-tuples*

$(A,B,C\,;\,X,Y,Z)$, where $A,B,C\in\mathfrak{gl}(W)_{\hat e}$, $X,Y,Z\in W$, satisfying equations (4.2). Then P is irreducible.

Proof. — Suppose that $\hat e$ is written in the normal Jordan form. Keep notation of Theorem 4.9. Let $U\subset P$ be an open subset, where $b_1\neq 0$ or, which is the same, $\operatorname{rk} B=n$. Then $U=\{(A,B,C;X,B^{-1}AX,B^{-1}CX)\mid b_1\neq 0\}$ is a $4n$-dimensional irreducible affine variety. On U the third equation $CY=AZ$ reduces to $CB^{-1}AX=AB^{-1}CX$ and is satisfied automatically because $CB^{-1}A=AB^{-1}C$.

Equations (4.2) are invariant under simultaneous cyclic permutation of (A,B,C) and (Z,Y,X). Therefore we may consider only those solutions, where $\operatorname{rk} B\geqslant\max(\operatorname{rk} A,\operatorname{rk} C)$. Note that $\operatorname{rk} B=n-d$ (with $d>0$) if and only if $b_1=\cdots=b_d=0$ and $b_{d+1}\neq 0$. Set

$$P_d:=\Big\{(A,B,C;X,Y,Z)\in P\mid \operatorname{rk} B=n-d,\operatorname{rk} A\leqslant n-d,\operatorname{rk} C\leqslant n-d\Big\}.$$

Our goal is to show that $P_d\subset\overline{U}$ for each $0<d\leqslant n$.

Let A' be the $(n-d)\times(n-d)$ right upper corner of A and $X':=(x_{d+1},\ldots,x_n)^t$. Define B',C',Y' and Z' in the same way. Then P_d is defined by:

$$b_1=\cdots=b_d=a_1=\cdots=a_d=c_1=\cdots=c_d=0,\quad b_{d+1}\neq 0;$$
$$Y'=(B')^{-1}A'X' \ \text{ and } \ Z'=(B')^{-1}C'X'.$$

Clearly P_d is an irreducible affine variety and it contains an irreducible open subset $(P_d)^\circ$ where $x_ny_nz_n\neq 0$. It suffices to prove that $(P_d)^\circ\subset\overline{U}$. Therefore assume that $x_ny_nz_n\neq 0$. We would like to replace A by $A+\mathcal{E}_A$, where $\mathcal{E}_A\in\mathrm{GL}(W)_{\hat e}$ is non-degenerate and "small", and do the same with B and C. Since $x_ny_nz_n\neq 0$, the vectors X, Y and Z lie in the single open orbit of $\mathrm{GL}(W)_{\hat e}$. In particular $Y=\mathcal{E}_AY$ and $Z=\mathcal{E}_CX$ for some $\mathcal{E}_A,\mathcal{E}_C\in\mathrm{GL}(W)_{\hat e}$. Let $E\in\mathrm{GL}(W)$ be the identity matrix. Then $(A+\lambda\mathcal{E}_A,B+\lambda E,C+\lambda\mathcal{E}_C;X,Y,Z)\in U$ for all $\lambda\in\mathbb{F}^\times$. Taking limit with λ tending to zero, we conclude that $(P_d)^\circ\subset\overline{U}$ and P is irreducible. □

Question 4.11. — Is it true that in the case of two Jordan blocks the defining ideal of $\mathfrak{C}^*(\mathfrak{g}_e)$ is generated by Equations (4.2)? Here the singularities of $\mathfrak{C}^*(\mathfrak{g}_e)$ form a subset of codimension 3 (defined by the equation $a_1=b_1=c_1=0$). Maybe this can help to solve the problem.

Remark 4.12. — Let $x=x_s+x_n$ be the Jordan decomposition of $x\in\mathfrak{gl}_n$. Then $(\mathfrak{gl}_n)_x$ is a sum of centralisers $(\mathfrak{gl}_{n_i})_{e_i}$, where all e_i are nilpotent. Suppose that each e_i has at most two Jordan block. In that case x is said to be *two-regular*, see [13]. The mixed commuting variety associated with $\mathfrak{g}_x$

is a product of mixed commuting varieties associated with $(\mathfrak{gl}_{n_i})_{e_i}$. Hence it is irreducible.

5. Poisson structures on the dual space of a centraliser

From now on, we assume that $\mathbb{F}$ is algebraically closed and of characteristic zero.

By the Jacobson-Morozov theorem, e can be included into an $\mathfrak{sl}_2$-triple (e,h,f) in $\mathfrak{g}$. By means of the Killing form on $\mathfrak{g}$, we identify $\mathfrak{g}$ and $\mathfrak{g}^*$. Consider e as an element of $\mathfrak{g}^*$ and let $\mathbb{S}_e$ denote the *Slodowy slice* $e+\mathfrak{g}_f$ at e to the coadjoint orbit Ge. The Slodowy slice $\mathbb{S}_e$ is a transversal slice to coadjoint G-orbits (symplectic leaves) in a sense of [23] and therefore carries a *transversal* Poisson structure obtained from $\mathfrak{g}^*$ by the Weinstein reduction, see e.g. [4] or [5]. This Poisson structure, which is in general non linear, turns out to be polynomial [4]. For each element $F\in\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ its restriction $F|_{\mathbb{S}_e}$ lies in the centre $\mathcal{Z}\mathbb{F}[\mathbb{S}_e]$ of the Poisson algebra $\mathbb{F}[\mathbb{S}_e]$. Moreover $\mathcal{Z}\mathbb{F}[\mathbb{S}_e]$ is a polynomial algebra in $\operatorname{rk}\mathfrak{g}$ variables generated by the restrictions $F_i|_{\mathbb{S}_e}$ for each generating system of invariants $\{F_1,\ldots,F_{\operatorname{rk}\mathfrak{g}}\}\subset\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$, see e.g. [15, Remark 2.1].

The G-equivariance of the Killing form implies that $\mathfrak{g}_e=[e,\mathfrak{g}]^{\perp}$. On the other hand, $\mathfrak{g}=[e,\mathfrak{g}]\oplus\mathfrak{g}_f$ by the $\mathfrak{sl}_2$-theory. Thereby $\mathbb{S}_e$ is naturally isomorphic to $\mathfrak{g}_e^*$ and $\mathbb{F}[\mathbb{S}_e]\cong\mathbb{F}[\mathfrak{g}_f]\cong\mathcal{S}(\mathfrak{g}_e)$. Remarkably, the linear part of the transversal Poisson structure on $\mathbb{S}_e$ gives us the usual Lie-Poisson bracket on $\mathfrak{g}_e^*$, see e.g. [4]. This leads to a natural construction of symmetric $\mathfrak{g}_e$-invariants.

For a homogeneous $F\in\mathcal{S}(\mathfrak{g})$, let ${}^eF$ be the component of minimal degree of the restriction $F|_{\mathbb{S}_e}$. (The restriction is not necessary homogeneous.) Identifying $\mathbb{F}[\mathbb{S}_e]$ and $\mathbb{F}[\mathfrak{g}_e^*]$, we consider ${}^eF$ as an element of $\mathcal{S}(\mathfrak{g}_e)$.

Lemma 5.1 ([15], Proposition 0.1). — *Keep the above notation. Then ${}^eF\in\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ for each homogeneous $F\in\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$.*

In types A and C it is possible to choose generating sets $\{F_1,\ldots,F_{\operatorname{rk}\mathfrak{g}}\}\subset\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ such that the ${}^eF_i$'s are algebraically independent and generate the whole algebra of symmetric $\mathfrak{g}_e$-invariants, see [15, Theorems 4.2 and 4.4]. The success is partially due to the fact that in those two cases $\operatorname{codim}(\mathfrak{g}_e^*)_{\rm sing}\geqslant 2$. In all other simple Lie algebras there are nilpotent elements, for which the codimension is 1. Here we show that in type A the codimension of $(\mathfrak{g}_e^*)_{\rm sing}$ in $\mathfrak{g}_e^*$ is greater than or equal to 3.

Suppose that $\mathfrak{g} = \mathfrak{gl}(\mathbb{V})$. Then there are certain points $\alpha := \sum_{i=1}^{k} a_i(\xi_i^{i,d_i})^*$, with $a_i \in \mathbb{F}^{\times}$ being pairwise distinct, and $\beta := \sum_{i=1}^{k-1}(\xi_{i+1}^{i,d_i})^*$ in $\mathfrak{g}_e^*$ such that $(\mathbb{F}\alpha \oplus \mathbb{F}\beta) \cap (\mathfrak{g}_e^*)_{\rm sing} = \{0\}$, see [15, Section 3].

To prove that the codimension of $(\mathfrak{g}_e^*)_{\rm sing}$ is greater than 2, we need to find the third, linear independent with α and β, regular point. The following is a slight modification of [15, Proposition 3.2].

Lemma 5.2. — *Suppose that $\mathfrak{g}$ is of type A. Take $\gamma := \sum_{i=1}^{k-1}(\xi_i^{i+1,d_i+1})^*$. Then $\gamma \in (\mathfrak{g}_e^*)_{\rm reg}$.*

Proof. — From (1.2) and the definition of γ it follows that $\gamma([\xi_i^{j,s},\xi]) = c_{j-1}^{i,d_j-s}(\xi) - c_j^{i+1,d_{i+1}-s}(\xi)$ for all $\xi \in \mathfrak{g}_e$. Suppose that ${\rm ad}^*(\xi)\gamma = 0$. Then $\gamma([\xi,\mathfrak{g}_e]) = 0$ forcing $c_{j-1}^{i,d_j-s}(\xi) = c_j^{i+1,d_{i+1}-s}(\xi)$ for all $i,j \in \{1,\dots,k\}$ and all s such that $\max(0,d_j-d_i) \leqslant s \leqslant d_j$.

We claim that $c_j^{i,s}(\xi) = 0$ for $i > j$. Suppose for a contradiction that this is not the case and take the maximal j for which there are $i > j$ and $0 \leqslant t \leqslant d_i$ such that $c_j^{i,t}(\xi) \neq 0$. Recall that, according to our convention, $d_i \leqslant d_j$. Moreover, $d_i \leqslant d_{j+1}$, since $i \geqslant j+1$. Set $s := d_{j+1} - t$. Then $d_{j+1} - d_i \leqslant s \leqslant d_{j+1}$ and $c_j^{i,d_{j+1}-s}(\xi) = c_{j+1}^{i+1,d_{i+1}-s}(\xi)$. As $j+1 > j$ and $i+1 < j+1$, the right hand side of the equality is zero, forcing $c_j^{i,d_{j+1}-s}(\xi) = c_j^{i,t}(\xi)$ to be zero.

Now take $\xi_{i-1}^{i,s} \in \mathfrak{g}_e$ with $0 \leqslant s \leqslant d_i$. Since $\gamma([\xi,\xi_{i-1}^{i,s}]) = 0$, we have $c_i^{i,d_i-s}(\xi) = c_{i-1}^{i-1,d_i-s}(\xi)$. Therefore, $c_i^{i,t}(\xi) = c_{i-1}^{i-1,t}(\xi) = c_1^{1,t}(\xi)$ for $0 \leqslant t \leqslant d_i$. In the same way one can show that $c_{i+\ell}^{i,t}(\xi) = c_{i+\ell-1}^{i-1,t}(\xi) = c_{1+\ell}^{1,t}(\xi)$ for $d_i - d_{i+\ell} \leqslant t \leqslant d_i$. Hence ξ is determined by a pair (ℓ,t), where $0 \leqslant \ell < k$ and $d_1 - d_{\ell+1} \leqslant t \leqslant d_1$ and a scalar $c_{1+\ell}^{1,t}(\xi)$. Thus $\dim(\mathfrak{g}_e)_\gamma \leqslant \dim\mathbb{V}$ and $\gamma \in (\mathfrak{g}_e^*)_{\rm reg}$. $\square$

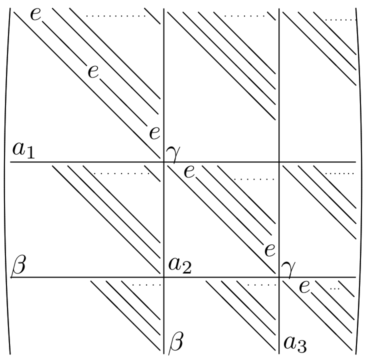

Figure 5.1

Corollary 5.3. — *The stabiliser* $(\mathfrak{g}_e)_\gamma$ *has a basis* $\eta_{i,s}$ *with* $1 \leqslant i \leqslant k$ *and* $d_1 - d_i \leqslant s \leqslant d_1$*, where* $\eta_{i,s} = \xi_i^{1,s} + \xi_{i+1}^{2,s} + \cdots + \xi_k^{k-i+1,s}$.

For a nilpotent element with three Jordan blocks, points α, β and γ are shown on Figure 5.1. Here β and γ are sums of two matrix elements with coefficients 1 and α is the sum with coefficients a_1, a_2, a_3. All of them are considered as elements of $\mathfrak{g}_f$. On the same picture we remind Arnold's description of a generic element of $\mathfrak{g}_e$ (see also Figure 1.2).

Theorem 5.4. — *If* $\mathfrak{g} = \mathfrak{gl}(\mathbb{V})$ *with* $\dim \mathbb{V} \geqslant 3$*, then* $\operatorname{codim}(\mathfrak{g}_e^*)_{\rm sing} \geqslant 3$.

Proof. — If e is a regular element, then $(\mathfrak{g}_e^*)_{\rm sing} = \varnothing$ and the codimension of this subset is equal to $\dim \mathfrak{g}_e = \dim \mathbb{V}$. Suppose that e is not regular and let elements $\alpha = \sum_{i=1}^k a_i(\xi_i^{i,d_i})^*$, $\beta = \sum_{i=1}^{k-1}(\xi_{i+1}^{i,d_i})^*$ and $\gamma = \sum_{i=1}^{k-1}(\xi_i^{i+1,d_{i+1}})^*$ be as above. We claim that $(\mathbb{F}\alpha \oplus \mathbb{F}\beta \oplus \mathbb{F}\gamma) \cap (\mathfrak{g}_e^*)_{\rm sing} = \{0\}$. Indeed each non-zero point $x\alpha + y\beta$ is regular by [15, Proposition 3.3]. In order to prove that $\gamma + x\alpha + y\beta$ is regular for all $x, y \in \mathbb{F}$, we use the action ρ of $\mathbb{F}^*$ defined by Formula (1.3). Direct calculation shows that $\rho(t)(\gamma + x\alpha + y\beta) = \gamma + xt\alpha + yt^2\beta$. Since $\gamma = \lim_{t\to 0} \rho(t)(\gamma + x\alpha + y\beta)$ and it is regular by Lemma 5.2, all points $\rho(t)(\gamma + x\alpha + y\beta)$, including $\gamma + x\alpha + y\beta$, are regular.

The result follows, since the subset $(\mathfrak{g}_e^*)_{\rm sing}$ is conical and Zariski closed. □

Let us say that a subalgebra $\mathcal{A}$ is *Poisson-commutative* if $\{\mathcal{A}, \mathcal{A}\} = 0$. Our main interest in the "codim 3" property is motivated by some application related to Poisson-commutative subalgebras of $\mathcal{S}(\mathfrak{g}_e)$.

Definition 5.5 (Panyushev). — *A Lie algebra* $\mathfrak{q}$ *is said to be* n-wonderful *if*

(i) $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathbb{F}[H_1, \ldots, H_{\operatorname{ind}\mathfrak{q}}]$ *is a polynomial algebra in* $\operatorname{ind}\mathfrak{q}$ *variables;*

(ii) *all* H_i *are homogeneous and* $\sum_{i=1}^{\operatorname{ind}\mathfrak{q}} \deg H_i = \dfrac{\dim\mathfrak{q} + \operatorname{ind}\mathfrak{q}}{2}$*;*

(iii) $\operatorname{codim}(\mathfrak{q}^*_{\rm sing}) \geqslant n$.

The centralisers in types A and C are 2-wonderful by [15]. Now we know that in type A they are 3-wonderful.

For $a \in \mathfrak{q}^*$ let ∂_a be a linear differential operator (partial derivative) on $\mathcal{S}(\mathfrak{q})$ such that $\partial_a \xi = a(\xi)$ on $\xi \in \mathfrak{q}$.

Theorem 5.6 ([17]). — *Suppose that* $\mathfrak{q}$ *is 3-wonderful and* $a \in \mathfrak{q}^*_{\rm reg}$*. Let* $\mathcal{F}_a \subset \mathcal{S}(\mathfrak{q})$ *be a subalgebra generated by the partial derivatives* $\partial_a^m H_i$ *(*$m \geqslant 0$, $1 \leqslant i \leqslant \operatorname{ind}\mathfrak{q}$*). Then* $\mathcal{F}_a$ *is a polynomial algebra in* $(\dim\mathfrak{q} + \operatorname{ind}\mathfrak{q})/2$

variables and it is maximal (with respect to inclusion) Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{q})$.

Theorem 5.6 is applicable to the centralisers $\mathfrak{g}_e$ in type A. Similar results concerning $\mathcal{F}_\alpha$ with $\alpha \in \mathfrak{g}_e^*$ being slightly more general or the same as in Theorem 5.4 are recently obtained by A. Joseph.

In type C the picture is not so nice. There are nilpotent elements such that subalgebras $\mathcal{F}_a$ are never maximal.

Example 5.7. — Let $e \in \mathcal{N}(\mathfrak{sp}_6)$ be defined by the partition $(4,2)$. (It was considered in Example 4.2.) Then $\dim[\mathfrak{g}_e, \mathfrak{g}_e] = 2$, hence $\operatorname{codim}(\mathfrak{g}_e^*)_{\rm sing} = 2$. Let $\mathcal{F}_a$ be as in Theorem 5.6 with $a \in \mathfrak{g}_e^*$. For this centraliser, $\mathcal{F}_a$ is never maximal among Poisson-commutative subalgebras of $\mathcal{S}(\mathfrak{g}_e)$. The general construction of [15] allows us to write down the invariants. They are $H_1 = \xi_1^{1,1} + \xi_2^{2,1}$, $H_2 = \xi_1^{1,3}$ and $H_3 = 4\xi_1^{1,3}e_2 + \eta\eta$, with $\eta = \xi_1^{2,1} - \xi_1^{2,3}$. If a is not regular, *i.e.*, a is zero on $[\mathfrak{g}_e, \mathfrak{g}_e] = \left\langle \xi_1^{1,3}, \eta \right\rangle_{\mathbb{F}}$, then $\partial_a H_3$ is proportional to $\xi_1^{1,3} = H_2$ and $\mathcal{F}_a = \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ is not maximal.

Assume that $a \in (\mathfrak{g}_e^*)_{\rm reg}$. Then $\mathcal{F}_a$ is generated by four elements, the invariants H_i and $x = \partial_a H_3$, which is an element of $(\mathfrak{g}_e)_a$. According to Example 4.2, $\mathfrak{g}_e$ satisfies condition (4.1), hence x is not regular, *i.e.*, $\dim(\mathfrak{g}_e)_x > 3$. Clearly $(\mathfrak{g}_e)_x$ commutes with $\mathcal{F}_a$, but is not contained in it. Therefore $\mathcal{F}_a$ is not maximal.

It is quite possible that there are some wide classes of nilpotent elements in type C for which "codim 3" condition holds. For example, it is satisfied for nilpotent elements given by partitions (d^k) with odd d and even k. By the contrast, it is not satisfied for partitions (d^k) with even d and $k > 1$.

6. Explicit formulas for symmetric invariants of centralisers in type A

In types A and C algebras of symmetric invariants $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ were described in [15]. The outline of that approach is given in Section 5. In type A we have an alternative description of $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ suggested by Brown and Brundan [2]. They reproved that this algebra is a polynomial algebra in $\operatorname{rk}\mathfrak{g}$ variables. Comparing the approaches of [2] and [15] we confirm [15, Conjecture 4.1].

Brown and Brundan used different notation. At first we should reinterpret symbols $e_{i,j;r}$ introduced in [2] in terms of $\xi_i^{j,r}$. According to [2, Formula (1.1)], $e_{i,j;r}$ is a sum of matrix units $e_{h,k}$, where w_h is a basis vector of $\mathbb{V}[i]$ and w_k is a basis vector of $\mathbb{V}[j]$, in the notation of Section 1 of the present paper. Thus $e_{i,j;r} \in \mathrm{Hom}(\mathbb{V}[j], \mathbb{V}[i])$. More precisely, $e_{i,j;r}$ is a sum of the matrix units on the (above) diagonal line in the i,j-rectangular, see Figure 1.2. Hence $e_{i,j;r} = \xi_j^{i,s}$ for some s. In order to calculate s, note that if $r = \lambda_j - 1 = d_j$, then $s = d_i$ and for $r = \lambda_j - \min(\lambda_i, \lambda_j)$ we get $s = d_i - \min(d_i, d_j)$. The final answer is that $e_{i,j;r} = \xi_j^{i,s}$ with $s = r + d_i - d_j$.

The cardinality of a finite set I is denoted by $|I|$. Given a permutation σ of a subset $I = \{i_1, \dots, i_m\} \subset \{1, \dots, k\}$ and a nonnegative function $\bar{s}\colon I \to \mathbb{Z}_{\geqslant 0}$, we associate with the triple $(I, \sigma, \bar{s})$ the monomial

$$\Xi(I, \sigma, \bar{s}) := \xi_{i_1}^{\sigma(i_1),\, \bar{s}(i_1)} \xi_{i_2}^{\sigma(i_2),\, \bar{s}(i_2)} \cdots \xi_{i_m}^{\sigma(i_m),\, \bar{s}(i_m)} \in \mathcal{S}(\mathfrak{g}_e)$$

of degree $m = |I|$. If $\bar{s}(i_j)$ does not satisfies the restriction on s given in Section 1, then we assume that $\xi_{i_j}^{\sigma(i_j), \bar{s}(i_j)} = 0$. For every $\Xi = \Xi(I, \sigma, \bar{s})$ we denote by $\lambda(I, \sigma, \bar{s})$ the weight of Ξ with respect to h, where h is a characteristic of e. Obviously, $\lambda(I, \sigma, \bar{s})$ is the sum of the $\mathrm{ad}\, h$-eigenvalues (h-weights) of the factors $\xi_{i_j}^{\sigma(i_j), \bar{s}(i_j)}$.

Suppose that $\mathfrak{g} = \mathfrak{gl}(\mathbb{V})$. Let $\{\Delta_1, \dots \Delta_{\mathrm{rk}\, \mathfrak{g}}\}$ be a generating set in $\mathbb{F}[\mathfrak{g}]^{\mathfrak{g}}$ such that $\Delta_i(\xi)$ are coefficients of the characteristic polynomial of $\xi \in \mathfrak{g}$. Identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ we identify also $\mathbb{F}[\mathfrak{g}]^{\mathfrak{g}}$ and $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. Let $\{F_i\}$ be the corresponding (to $\{\Delta_i\}$) set of generators of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. By a result of [15], the ${}^eF_i$'s form a generating set of $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$. The following statement was conjectured to be true in [15]. It will be proved in this section.

Theorem 6.1. — *Let $1 \leqslant \ell \leqslant \mathrm{rk}\, \mathfrak{g}$ and set $m := \deg {}^eF_\ell$. Then up to a non-zero constant,*

$${}^eF_\ell = \sum_{|I| = m,\ \lambda(I, \sigma, \bar{s}) = 2(\ell - m)} (\mathrm{sgn}\ \sigma)\, \Xi(I, \sigma, \bar{s}),$$

where the summation is taken over all subsets I, all permutations σ of I and over all functions $\bar{s}$.

Lemma 6.2. — *In the above notation we have $\lambda(I, \sigma, \bar{s}) = 2 \sum_{j \in I} \bar{s}(j)$.*

Proof. — It is not difficult to compute that the weight of $\xi_i^{j,s}$ is equal to $2(d_i - d_j + s)$. Therefore

$$\lambda(I, \sigma, \bar{s}) = 2 \sum_{j \in I} (d_j - d_{\sigma(j)} + \bar{s}(j)) = 2 \sum_{j \in I} \bar{s}(j).$$

The second equality holds because σ is a permutation. □

Set $\tilde{\xi}_i^{j,s} := \xi_i^{j,s} - \delta_{s,0} \delta_{i,j} (i-1)(d_i + 1)$, where $\delta_{i,j} = 1$ for $i = j$ and is zero otherwise. Note that $e_{i,i;0} = \xi_i^{i,0}$ and, as above, for a permutation σ of I we have $\sum_{j \in I} (\bar{s}(j) + d_j - d_{\sigma(j)}) = \sum_{j \in I} \bar{s}(j)$. Taking these two facts into account, we rewrite Formulas (1.2) and (1.3) of [2] in the $\xi_j^{i,s}$-notation. For each set I of indices $1 \leqslant i_1 < i_2 < \cdots < i_m$ and each permutation σ, define

$$\widetilde{\Xi}(I,\sigma,\bar{s}) := \tilde{\xi}_{i_1}^{\sigma(i_1),\,\bar{s}(i_1)}\tilde{\xi}_{i_2}^{\sigma(i_2),\,\bar{s}(i_2)}\ldots\tilde{\xi}_{i_m}^{\sigma(i_m),\,\bar{s}(i_m)} \in \mathbf{U}(\mathfrak{g}_e).$$

Let ℓ be in the range $1 \leqslant \ell \leqslant \operatorname{rk}\mathfrak{g}$ and $m = \deg {}^e\!F_\ell$. In view of Lemma 6.2, we can express elements z_ℓ of [2] as follows

$$z_\ell = \sum_{|I|=m,\ \lambda(I,\sigma,\bar{s})=2(\ell-m)} (\operatorname{sgn}\sigma)\,\widetilde{\Xi}(I,\sigma,\bar{s}), \tag{6.1}$$

where the summation is taken over all subsets I, all permutations σ of I and over all functions $\bar{s}$.

The main theorem of [2] states that the elements z_ℓ generate the centre of $\mathbf{U}(\mathfrak{g}_e)$ and that their symbols, elements of $\mathcal{S}(\mathfrak{g}_e)$, denoted $\overline{z_\ell}$, are algebraically independent.

Proof of Theorem 6.1. — In [15] a slightly weaker statement was proved. More precisely, it was shown that for each $\ell \leqslant \operatorname{rk}\mathfrak{g}$, we have

$${}^e\!F_\ell = \sum_{|I|=m,\ \lambda(I,\sigma,\bar{s})=2(\ell-m)} a(I,\sigma,\bar{s})\,\Xi(I,\sigma,\bar{s})$$

for some $a(I,\sigma,\bar{s}) \in \mathbb{F}$. Here we prove that each ${}^e\!F_\ell$ is a non-zero multiple of the symbol $\overline{z_\ell}$.

Following Brown and Brundan, restrict the invariants to an affine slice $\eta + V \subset \mathfrak{g}_e^*$. In our notation, $\eta = \sum_{i=1}^{k-1}(\xi_{i+1}^{i,d_i})^*$ and V is the subspace generated by $(\xi_1^{i,s})^*$. According to [2], this restriction map $\psi\colon \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e} \to \mathbb{F}[\eta+V]$ is an isomorphism.

Suppose that $\deg {}^e\!F_\ell = m$. Then both $\psi({}^e\!F_\ell)$ and $\psi(\overline{z_\ell})$ are proportional to $\xi_1^{m,s}$ with $s = \ell - (d_1 + \cdots + d_{m-1}) - m$. This completes the proof of Theorem 6.1. □

7. Fibres of the quotient morphism $\mathfrak{g}_e^* \to \mathfrak{g}_e^*/\!\!/G_e$

Suppose that $\mathfrak{g}$ is either of type A or C. Then $\mathcal{S}(\mathfrak{g}_e)^{G_e} = \mathbb{F}[H_1,\ldots,H_{\operatorname{rk}\mathfrak{g}}]$, where $H_i = {}^e\!F_i$ for a certain (good) generating set $\{F_i\} \subset \mathcal{S}(\mathfrak{g})$ of $\mathfrak{g}$-invariants, see [15]. In particular, the algebra of symmetric G_e-invariants is finitely generated and we can consider the quotient morphism $\mathfrak{g}_e^* \to \mathfrak{g}_e^*/\!\!/G_e$, where $\mathfrak{g}_e^*/\!\!/G_e = \operatorname{Spec}\mathcal{S}(\mathfrak{g}_e)^{G_e}$ and each $x \in \mathfrak{g}_e^*$ maps to $(H_1(x),\ldots, H_{\operatorname{rk}\mathfrak{g}}(x))$. In this section we are interested in the fibres of the quotient morphism. By [15, Section 5], in type A all fibres of this morphism are of dimension $\dim\mathfrak{g}_e - \operatorname{rk}\mathfrak{g}$.

Consider a point $\alpha = \sum_{i=1}^{k} a_i(\xi_i^{i,d_i})^* \in \mathfrak{g}_e^*$, where a_i are pairwise distinct non-zero numbers and $\mathfrak{g} = \mathfrak{gl}(\mathbb{V})$. As was already mentioned, it is a generic point and $\mathfrak{h} = (\mathfrak{g}_e)_\alpha$ is a generic stabiliser for the coadjoint action of $\mathfrak{g}_e$. In case $e \in \mathfrak{sp}(\mathbb{V})$, similar statements remain true for the restriction of α to $\mathfrak{sp}(\mathbb{V})_e$ and $\mathfrak{h} \cap \mathfrak{sp}(\mathbb{V})$, see [24]. Set $H := (\mathrm{GL}(\mathbb{V})_e)_\alpha$. Then H is connected and $(\mathrm{GL}(\mathbb{V})_e)_\gamma$ is conjugate to H whenever $(\mathfrak{gl}(\mathbb{V})_e)_\gamma$ is conjugate to $\mathfrak{h}$. In other words, H is a generic stabiliser for the coadjoint action of $\mathrm{GL}(\mathbb{V})_e$. Again, if $e \in \mathfrak{sp}(\mathbb{V})$, then $H \cap \mathrm{Sp}(\mathbb{V})$ is a generic stabiliser for the coadjoint action of $\mathrm{Sp}(\mathbb{V})_e$.

Recall that $\mathfrak{h} = \langle \xi_i^{i,s} \rangle$ and $\mathfrak{h}$ contains a maximal torus $\mathfrak{t} = \langle \xi_i^{i,0} \rangle$ of $\mathfrak{gl}(\mathbb{V})_e$. Thereby $H = T \ltimes U$, where T is a maximal torus of $\mathrm{GL}(\mathbb{V})_e$ and U is contained in the unipotent radical of $\mathrm{GL}(\mathbb{V})_e$. Likewise, for $e \in \mathfrak{sp}(\mathbb{V})$, the generic stabiliser $H \cap \mathrm{Sp}(\mathbb{V})$ contains a maximal torus $T \cap \mathrm{Sp}(\mathbb{V})$ of $\mathrm{Sp}(\mathbb{V})_e$. Applying the following lemma, we get that generic coadjoint orbits of centralisers in types A and C are closed.

Lemma 7.1. — *Suppose that an algebraic group G acts on an affine variety X and a stabiliser G_x of a point $x \in X$ contains a maximal torus T of G. Then the orbit Gx is closed.*

Proof. — Let us choose a Borel subgroup $B \subset G$ containing T. Then the B-orbit Bx is closed, because it coincides with the orbit of a unipotent group, in this case of the unipotent radical of B.

We have a closed subgroup $B \subset G$ such that the quotient G/B is complete and the orbit Bx is closed. It follows that $G \cdot Bx = Gx$ is also closed, see e.g. [21, Lemma 2 in Section 2.13]. □

Lemma 7.1 is a well-known and classical fact. In case of complex reductive group G, similar result was proved by Kostant in 1963, see [10, proof of Lemma 5].

Theorem 7.2. — *If $\mathfrak{g}$ is either $\mathfrak{gl}(\mathbb{V})$ or $\mathfrak{sp}(\mathbb{V})$, then a generic fibre of the quotient morphism $\mathfrak{g}_e^* \to \mathfrak{g}_e^* /\!\!/ G_e$ consists of a single closed G_e-orbit.*

Proof. — In both these cases the coadjoint action of G_e has a generic stabiliser, which contains a maximal torus of G_e, see [24, Section 4] and discussion before Lemma 7.1. By Lemma 7.1, generic orbits are closed. Since $\mathrm{ind}\,\mathfrak{g}_e = \mathrm{rk}\,\mathfrak{g}$, generic coadjoint G_e-orbits and generic fibres of the quotient morphism have the same dimension, $\dim \mathfrak{g}_e - \mathrm{rk}\,\mathfrak{g}$. Hence there is an open subset $U \subset \mathfrak{g}_e^* /\!\!/ G_e$ such that the fibre over each $u \in U$ contains a closed G_e-orbit of maximal dimension and that orbit is an irreducible component of the fibre.

In cases of our interest $\mathcal{S}(\mathfrak{g}_e)^{G_e} = \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$, see [15, Theorems 4.2 and 4.4]. Hence each element of $\mathcal{S}(\mathfrak{g}_e)$, which is algebraic over $\mathrm{Quot}(\mathbb{F}[\mathfrak{g}_e^*]^{G_e})$, is $\mathfrak{g}_e$- and G_e-invariant. This means that $\mathcal{S}(\mathfrak{g}_e)^{G_e}$ is algebraically closed in $\mathcal{S}(\mathfrak{g}_e)$. By Theorem A.1, proved in the appendix, generic fibres of the quotient morphism are connected. Shrinking U if necessary, we may assume that the fibres over elements of U are connected. Then each of them consists of a single closed G_e-orbit of maximal dimension. □

Theorem 7.2 was proved in a discussion with A. Premet during his visit to the Max-Planck Institut für Mathematik (Bonn) in Spring 2007.

Remark 7.3. — The proof of Theorem 7.2 can be completed in a slightly different way. The ring $\mathbb{F}[\mathfrak{g}^*]$ is a unique factorisation domain. If $a \in \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$, then all prime factors of a are also $\mathfrak{g}_e$-invariant. One can show quite elementary that the field $\mathrm{Quot}(\mathbb{F}[\mathfrak{g}_e^*]^{G_e})$ is algebraically closed in $\mathbb{F}(\mathfrak{g}^*)$. Then generic fibres are known to be irreducible, see e.g. [20, Chapter 2, Section 6.1].

Remark 7.4. — If $\mathfrak{g}$ is of type A or C, then, as was mentioned above, the coadjoint action of G_e has a generic stabiliser, which contains a maximal torus of G_e. This means that the ring of semi-invariants $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}_{\mathrm{si}}$ coincides with $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$. Lie algebras $\mathfrak{q}$ with $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}_{\mathrm{si}}$ being a polynomial ring are actively studied, see e.g. [14]. In particular, if $\mathfrak{g}$ is of type A and $\mathfrak{g}_e$ is non-Abelian, then [14, Proposition 1.6] combined with Theorem 5.4, implies that each irreducible component of $(\mathfrak{g}_e^*)_{\mathrm{sing}}$ has dimension $\dim\mathfrak{g}_e - 3$.

In contrast with a generic fibre, the null-cone $\mathcal{N}(e)$ (the fibre containing zero) may have infinitely many closed orbits and there might be no regular elements (and hence no open orbits) in some of its components. Dealing with $\mathcal{N}(e)$, we will freely use the explicit formulas for the generators $H_i = {}^eF_i$, obtained in Section 6.

Example 7.5. — Let $e \in \mathcal{N}(\mathfrak{gl}_6)$ be given by the partition $(4,2)$. Here $\dim\mathfrak{g}_e - \mathrm{rk}\,\mathfrak{g} = 4$, hence all irreducible components of $\mathcal{N}(e)$ are of dimension 4. There are 4 elements in the centre of $\mathfrak{g}_e$, they are linear invariants H_1, H_2, H_3, H_4. The other two invariants H_5 and H_6 are of degree 2. Until the end of the example, we replace $\mathfrak{g}_e^*$ by a subspase $P \subset \mathfrak{g}_e^*$ defined by $H_1 = \cdots = H_4 = 0$ and regard $\mathcal{N}(e) \subset P$ as the zero set of H_5 and H_6.

Then restricted to P, the invariants H_5 and H_6 are expressed by the formulas $H_6 = \xi_1^{2,1}\xi_2^{1,3}$ and $H_5 = \xi_1^{2,1}\xi_2^{1,2} + \xi_1^{2,0}\xi_2^{1,3}$. Both are zero on the linear subspace defined by $\xi_1^{2,1} = \xi_2^{1,3} = 0$. Hence a four-dimensional vector space $X \subset P$ generated by vectors

$$(\xi_1^{1,0})^* - (\xi_2^{2,0})^*, \quad (\xi_1^{1,1})^* - (\xi_2^{2,1})^*, \quad (\xi_1^{2,0})^*, \quad (\xi_2^{1,2})^*$$

is an irreducible component of the null-cone $\mathcal{N}(e)$. The action of G_e on X has a 7-dimensional ineffective kernel. Since coadjoint orbits are even-dimensional, G_e-orbits on X are either trivial or 2-dimensional. Essentially the only non-trivial actions are:

$$\mathrm{ad}^*(\xi_1^{1,0} - \xi_2^{2,0})(\xi_1^{2,0})^* = (\xi_1^{2,0})^*, \quad \mathrm{ad}^*(\xi_1^{1,0} - \xi_2^{2,0})(\xi_2^{1,2})^* = -(\xi_2^{1,2})^*,$$

and

$$-\,\mathrm{ad}^*(\xi_1^{2,0})(\xi_1^{2,0})^* = \mathrm{ad}^*(\xi_2^{1,2})(\xi_2^{1,2})^* = (\xi_1^{1,0})^* - (\xi_2^{2,0})^*.$$

Thus X contains a 2-parameter family of closed 2-dimensional G_e-orbits; two non-closed 2-dimensional orbits; and a 2-parameter family of G_e-invariant points. In particular, X contains no regular elements.

For this nilpotent element the ideal $I = (\mathcal{S}(\mathfrak{g}_e)_\circ^{\mathfrak{g}_e}) \lhd \mathcal{S}(\mathfrak{g}_e)$ generated by the homogeneous invariants of positive degree is not radical. After restriction to P, where I is generated by H_5 and H_6, we have $\xi_1^{2,1}\xi_2^{1,2} \notin I$, but

$$\left(\xi_1^{2,1}\xi_2^{1,2}\right)^2 = \xi_1^{2,1}\xi_2^{1,2}H_5 - \xi_1^{2,0}\xi_2^{1,2}H_6 \in I.$$

A very interesting problem is to describe the irreducible components of $\mathcal{N}(e)$ in type A. Here we compute the number of these components in two particular cases.

Lemma 7.6 ([16], Theorem 1.2). — *Suppose that* $\mathfrak{q}$ *is a Lie algebra such that* $\operatorname{codim}\mathfrak{q}^*_{\mathrm{sing}} \geqslant 2$ *and* $H_1, \ldots, H_{\mathrm{rk}\,\mathfrak{q}}$ *are algebraically independent homogeneous elements of* $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ *with* $\sum_{i=1}^{\mathrm{rk}\,\mathfrak{q}} \deg H_i = (\dim\mathfrak{q} + \mathrm{rk}\,\mathfrak{q})/2$. *Then* $H_1, \ldots, H_{\mathrm{rk}\,\mathfrak{q}}$ *generate the whole algebra* $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ *of symmetric* $\mathfrak{q}$*-invariants.*

Proposition 7.7. — *Suppose that* $e \in \mathfrak{gl}_{m+n}$ *is defined by the partition* $(n, 1^m)$ *with* $n \geqslant 2$. *Then* $\mathcal{N}(e)$ *has* $m+1$ *irreducible components.*

Proof. — Let $P \subset \mathfrak{g}_e^*$ be the zero-set of linear invariants. Then P is isomorphic to the dual space of a Lie algebra $\mathfrak{q} = \mathfrak{gl}_m \ltimes V$, where $V \cong \mathbb{F}^m \oplus (\mathbb{F}^m)^*$ is a commutative ideal. Note that $\mathfrak{q}$ is a quotient of $\mathfrak{g}_e$ and $\mathfrak{g}_e$ acts on P via the coadjoint representation of $\mathfrak{q}$.

Set $L = \mathrm{GL}_m$ and $\mathfrak{l} = \operatorname{Lie} L$. Identifying $\mathfrak{l}^*$ with the annihilator $\operatorname{Ann}(V) \subset \mathfrak{q}^*$ and V^* with $\operatorname{Ann}(\mathfrak{l}) \subset \mathfrak{q}^*$, we consider $\mathfrak{l}^*$ and V^* as subspaces of $\mathfrak{q}^*$ and of $\mathfrak{g}_e^*$. Take $H_i = {}^eF_i$ with $i > n$. Then $\deg H_i = i{-}n{+}1$ and the restriction $H_i|_P$ is a bi-homogeneous polynomial in variables $\mathfrak{l}$ and V of bi-degree $(i{-}n{-}1, 2)$.

The image of the projection $\mathcal{N}(e) \to V^*$ coincides with the zero set $\mathcal{N}(V)$ of $H_{n+1}|_P$. There are four L-orbits in $\mathcal{N}(V)$: the open orbit, zero and two intermediate, in $(\mathbb{F}^m)^*$ and $\mathbb{F}^m$. Note that the subsets $\mathfrak{l}^*\oplus(\mathbb{F}^m)^*$ and $\mathfrak{l}^*\oplus\mathbb{F}^m$ of $\mathfrak{g}_e^*$ are defined by the equations $\xi_1^{1,t}=0$ $(t=0,\dots,m-1)$ and $\xi_1^{i,1}=0$ or $\xi_i^{1,m-1}=0$, respectively (here $i>1$). Explicit formulas exhibited in Section 6 show that both these subspaces are contained in $\mathcal{N}(e)$. Since they are irreducible and of the right dimension, $\dim\mathfrak{g}_e-(m+n)$, they are irreducible components of $\mathcal{N}(e)$.

Let X be an irreducible component of $\mathcal{N}(e)$ distinct from either $\mathfrak{l}^*\oplus(\mathbb{F}^m)^*$ or $\mathfrak{l}^*\oplus\mathbb{F}^m$. Then the image of the projection $X\to V^*$ is either zero or contains an open L-orbit $\mathcal{O}$. The first case is not possible because $\dim\mathfrak{l}^*<\dim\mathcal{N}(e)$. Thus, it remains to deal with the irreducible components of the intersection $\mathcal{N}(e)\cap(\mathfrak{l}^*\times\mathcal{O})$. Since G_e is connected, each irreducible component of $\mathcal{N}(e)$ is G_e-invariant and the problem reduces to the intersection $\mathcal{N}(e)\cap(\mathfrak{l}^*\times\{v\})$, where $v\in\mathcal{O}$. Since V is a commutative ideal of $\mathfrak{q}$, it acts on the fibre $\mathfrak{l}^*\times\{v\}$. This action of V has a slice $S\subset\mathfrak{l}^*\times\{v\}$, isomorphic to $\mathfrak{l}_v^*\times\{v\}$, which meets each V-orbit exactly once, see e.g. [22, Lemma 4]. Since both L_v and V are connected, $\mathcal{N}(e)\cap(\mathfrak{l}^*\times\mathcal{O})$ has exactly the same number of irreducible components as the zero-set of $H_i|_S$.

The restrictions of H_i with $n+2\leqslant i\leqslant n+m$ to S are algebraically independent, otherwise $\mathcal{N}(e)$ would have a component of dimension $(\dim\mathfrak{g}_e-\operatorname{rk}\mathfrak{g})+1$. Identifying S with $\mathfrak{l}_v^*$ we may consider them as $\mathfrak{l}_v$-invariant elements of $\mathcal{S}(\mathfrak{l}_v)$. One readily computes that $\mathfrak{l}_v\cong(\mathfrak{sl}_m)_{\hat e}$, where $\hat e$ is a nilpotent element defined by the partition $(2,1^{m-2})$. Clearly $\deg(H_i|_S)=\deg H_i-2=n-1$ for $i>n$. Therefore we get $m-1=\operatorname{ind}\mathfrak{l}_v$ polynomials of degrees $1,2,\dots,m-1$. The sum of degrees is equal to $(\dim\mathfrak{l}_v+\operatorname{ind}\mathfrak{l}_v)/2$. There is no consequential difference between centralisers in $\mathfrak{gl}_m$ and $\mathfrak{sl}_m$. Therefore, according to Theorem 5.4, the codimension of $(\mathfrak{l}_v^*)_{\rm sing}$ is grater than 2. Thus all conditions of Lemma 7.6 are satisfied. Hence $H_i|_S$ generate $\mathcal{S}(\mathfrak{l}_v)^{\mathfrak{l}_v}$ and $\mathcal{N}(e)\cap S$ is isomorphic to the null-cone $\mathcal{N}(\hat e)$ associated with the nilpotent element $\hat e\in\mathfrak{gl}_m$.

If $m=0$, then $\mathcal{N}(e)$ is irreducible. For $m=1$ there are two irreducible components, since $H_{n+1}|_P=\xi_1^{2,0}\xi_2^{1,n-1}$. Arguing by induction on m, the may assume that $\mathcal{N}(\hat e)$ has $m-1$ components. Then $\mathcal{N}(e)$ has $m-1+2=m+1$ components. □

Proposition 7.8. — *Suppose that $e\in\mathfrak{gl}_{n+m}$ is defined by the partition (n,m) with $n\geqslant m$. Then $\mathcal{N}(e)$ has* $\min(n-m,m)+1$ *irreducible components.*

Proof. — Again we replace $\mathfrak{g}_e^*$ be the zero set $P\subset\mathfrak{g}_e^*$ of the linear $\mathfrak{g}_e$-invariants. Suppose first that $m\leqslant n-m$. Set $x_i:=\xi_1^{2,m-i}$ and $y_i:=\xi_2^{1,n-i}$ for $1\leqslant i\leqslant m$. Then $\mathcal{N}(e)$ is defined by the polynomials $f_q=\sum_{i+j=q}x_iy_j$

with $2 \leqslant q \leqslant m+1$. Each irreducible components is given by a partition $m = a+b$, where $a, b \geqslant 0$. It is a linear subspace defined by $x_1 = \cdots = x_a = 0$, $y_1 = \cdots = y_b = 0$. Hence there are exactly $m+1$ components.

Consider now the second case, there $n-m < m$. Set $k := n-m$. Retain the notation for x_i and y_i. Set in addition $z_i := \xi_2^{2,m-i}$. Then the restrictions of non-linear symmetric invariants H_i to P are given by the polynomials

$$f_q = \sum_{i+j=q} x_i y_j \quad \text{with} \quad 2 \leqslant q \leqslant k+1;$$

and

$$f_p = \sum_{i+j=p} x_i y_j + \sum_{i+j=p-k} z_i z_j \quad \text{with} \quad k+2 \leqslant p \leqslant m+1.$$

For example, here

$$f_{k+2} = x_1 y_{k+1} + \cdots x_{k+1} y_1 + z_1^2$$

and

$$f_{k+3} = x_1 y_{k+2} + \cdots x_{k+2} y_1 + 2 z_1 z_2.$$

Note that variables z_j appear in these equations only for $j \leqslant m-k$. The first equations, f_q, give rise to $k+1$ irreducible components, each of which is a linear subspace. Take one of these components, defined by $x_1 = \cdots = x_a = 0$, $y_1 = \cdots = y_b = 0$ with $a+b = k$ and let $P_{a,b}$ be the intersection of this linear subspace with $\mathcal{N}(e)$. We are going to show that $P_{a,b}$ is irreducible and that these components do not coincide for distinct partitions $k = a+b$.

Let $P^\circ_{a,b}$ be a subset of $P_{a,b}$, where $z_1 \neq 0$. Then $P^\circ_{a,b}$ is irreducible, because it is defined by the equations

$$x_{a+1} y_{b+1} = -z_1^2$$

and

$$z_j = f_{k+1+j}(\overline{x}, \overline{y}, z_2, \ldots, z_{j-1})/z_1 \text{ for } 2 \leqslant j \leqslant m-k.$$

Note that $\dim P^\circ_{a,b} = \dim \mathfrak{g}_e - (m+n) - \dim \mathcal{N}(e)$. On the complement $P_{a,b} \setminus P^\circ_{a,b}$ we have $z_1 = 0$ and equations $f_q = 0$ and $f_p = 0$ reduce to the following

$$x_1 = \cdots = x_a = y_1 = \cdots = y_b = 0, \tag{7.1}$$

$$x_{a+1} y_{b+1} = 0, \quad x_{a+1} y_{b+2} + x_{a+2} y_{b+1} = 0,$$

$$f_p = \sum_{i+j=p} x_i y_j + (z_2 z_{p-k-2} + \cdots + z_{p-k-2} z_2) = 0 \text{ for } \ k+4 \leqslant p \leqslant m+1.$$

Equations (7.1) are very similar to the original f_p's and f_q's. Using induction on $k-m$ and the previous case, where $n-m \geqslant m$, one can say that they define three irreducible components $P_{a+2,b}(\bar{e})$, $P_{a+1,b+1}(\bar{e})$,

$P_{a,b+2}(\bar{e})$ of the null-cone associated with a nilpotent element $\bar{e}$ with Jordan blocks $(n+2,m)$. One thing, which we should keep in mind, is that for $\bar{e}$ variables $z_2,\dots z_{m-k-1}$ are used instead of $z_1,\dots,z_{m-k-2}$. Since $\dim\mathcal{N}(e)=\dim\mathcal{N}(\bar{e})$, the complement $P_{a,b}\setminus P^\circ_{a,b}$ is an irreducible subset of dimension $\dim\mathcal{N}(e)-1$. In particular, it could not be a component of $\mathcal{N}(e)$ and we have proved that $P_{a,b}$ is an irreducible component.

Suppose that $a'+b'=k$ and $a'\neq a$. Then either $a'>a$ or $b'>b$. Anyway, if $P_{a',b'}=P_{a,b}$, then $x_{a+1}y_{b+1}$ is zero on $P_{a,b}$. Hence z_1 is also zero on it. A contradiction, since we know that $z_1\neq 0$ defines a non-empty open subset $P^\circ_{a,b}\subset P_{a,b}$. □

There should be a combinatorial formula for the number of components. Unfortunately, we do not have enough information even to make a conjecture. Apart from two cases considered in this section, little is known. If the partition is rectangular, *i.e.*, all Jordan blocks are of the same size, then $\mathfrak{g}_e$ is a Takiff Lie algebra and the null-cone is irreducible, see [12, Appendix]. A direct calculation shows that the number of irreducible components for the partition $(3,2,1)$ is 4.

8. Further results on the null-cone

Suppose that $\mathfrak{g}\subset\mathfrak{gl}(\mathbb{V})$ is either $\mathfrak{sp}(\mathbb{V})$ or $\mathfrak{so}(\mathbb{V})$ and $e\in\mathfrak{g}$ is such that $i'=i$ for all i (in terms of Lemma 1.1). Here we prove that each irreducible component of $\mathcal{N}(e)$ has dimension $\dim\mathfrak{g}_e-\operatorname{rk}\mathfrak{g}$. Similar result was obtained in [15, Section 5] for all nilpotent elements in $\widetilde{\mathfrak{g}}=\mathfrak{gl}(\mathbb{V})$. Our proof uses the same strategy.

For $m\in\{1,\dots,k\}$, partition the set $\{1,\dots,m\}$ into pairs $(j,m-j+1)$. If m is odd, then there will be a "singular pair" in the middle consisting of the singleton $\{(m+1)/2\}$. Let V_m denote the subspace of $\widetilde{\mathfrak{g}}_e$ spanned by all $\xi_i^{j,s}$ with $i+j=m+1$ and set $V:=\bigoplus_{m\geqslant 1}V_m$. Using the basis $\{(\xi_i^{j,s})^*\}$ of $\widetilde{\mathfrak{g}}_e^*$ dual to the basis $\{\xi_i^{j,s}\}$, we shall regard the dual spaces V_i^* and V^* as subspaces of $\widetilde{\mathfrak{g}}_e^*$.

Since $i'=i$ for all i, the restriction of the $\mathfrak{g}$-invariant form on $\mathbb{V}$ to each $\mathbb{V}_i$ is non-degenerate. Hence the partition into pairs $(j,m-j+1)$ can be pushed down to $\mathfrak{g}_e$. Each V_m is preserved by σ, where σ is an involution of $\widetilde{\mathfrak{g}}$ with $\mathfrak{g}=\widetilde{\mathfrak{g}}^\sigma$. Let $\widetilde{\mathfrak{g}}=\mathfrak{g}\oplus\widetilde{\mathfrak{g}}_1$ be the corresponding symmetric decomposition. Let us identify $\mathfrak{g}_e^*$ with the annihilator of $\widetilde{\mathfrak{g}}_{1,e}$ in $\widetilde{\mathfrak{g}}_e^*$. Then the expressions $V^*_{\mathfrak{g},m}:=V_m^*\cap\mathfrak{g}_e^*$ make sense and $V^*_{\mathfrak{g},m}=(V_m^*)^\sigma$, similarly set $V^*_{\mathfrak{g}}:=V^*\cap\mathfrak{g}^*$. Note also that

$$\bar{\mathfrak{g}}:=\mathfrak{g}\cap\mathfrak{gl}\big(\mathbb{V}[1]\oplus\cdots\oplus\mathbb{V}[k-1]\big)$$

is a semisimple subalgebra of $\mathfrak{g}$, either $\mathfrak{so}\big(\mathbb{V}[1]\oplus\cdots\oplus\mathbb{V}[k-1]\big)$ or $\mathfrak{sp}\big(\mathbb{V}[1]\oplus\cdots\oplus\mathbb{V}[k-1]\big)$, depending on $\mathfrak{g}$. Likewise $\mathfrak{g}_k := \mathfrak{g}\cap\mathfrak{gl}(\mathbb{V}_k)$ is either $\mathfrak{so}(\mathbb{V}_k)$ or $\mathfrak{sp}(\mathbb{V}_k)$.

Set $n := \dim\mathbb{V}$. Let $\Delta_i \in \mathbb{F}[\widetilde{\mathfrak{g}}]^{\widetilde{\mathfrak{g}}}$ (with $1\leqslant i\leqslant n$) be the coefficients of the characteristic polynomial. Unlike Section 6, here we consider Δ_i as elements of $\mathcal{S}(\widetilde{\mathfrak{g}})$. Set $F_i := \Delta_{2i}|_{\mathfrak{g}^*}$ for all i in the range $1\leqslant i\leqslant \operatorname{rk}\mathfrak{g}$. Note that all Δ_i with odd i are zero on $\mathfrak{g}^*$. As was proved in [15, Theorem 4.2 and Lemma 4.5], the polynomials ${}^eF_i$ are algebraically independent and in the symplectic case they generate $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$. Let $\mathcal{N}_F(e)\subset\mathfrak{g}_e^*$ be the zero set of the polynomials ${}^eF_i$.

Theorem 8.1. — *Suppose that $\mathfrak{g}$ and $e\in\mathfrak{g}$ satisfy the assumptions of this section. Then there exists a linear subspace $W_{\mathfrak{g}} = \bigoplus_{m\geqslant 1} W_{\mathfrak{g},m}$ in $V_{\mathfrak{g}}^*$ of dimension $\operatorname{rk}\mathfrak{g}$ such that $W_{\mathfrak{g},m}\subset V_{\mathfrak{g},m}^*$ for all m and $W_{\mathfrak{g}}\cap\mathcal{N}_F(e)=\{0\}$.*

Proof. — We argue by induction on k. If $k=1$, then e is a regular nilpotent element, all ${}^eF_i$ are linear functions and they form a basis of $\mathfrak{g}_e$. Hence $\mathcal{N}_F(e)=\{0\}$ and there is nothing to prove. Assume that $k>1$ and for all $k'<k$ the statement is true.

Regard the dual spaces $\bar{\mathfrak{g}}^*$ and $\mathfrak{g}_k^*$ as subspaces of $\mathfrak{g}^*$. Note that $e = e_k+\bar e$ where e_k and $\bar e$ are the restrictions of e to $\mathbb{V}[k]$ and $\mathbb{V}[1]\oplus\cdots\oplus\mathbb{V}[k-1]$, respectively. Clearly, e_k is a regular nilpotent element in $\mathfrak{g}_k$ and $\bar e\in\bar{\mathfrak{g}}$ is a nilpotent element with Jordan blocks of sizes $d_1+1,\dots,d_{k-1}+1$. Note that $V_{\mathfrak{g},m}^*\subset(\bar{\mathfrak{g}}_{\bar e})^*$ for $m<k$.

The restriction of ${}^eF_i$ (with $1\leqslant 2i\leqslant n-d_k-1$) to $(\bar{\mathfrak{g}}_{\bar e})^*$ can be obtained as follows: first restrict Δ_{2i} to the dual of $\mathfrak{gl}\big(\mathbb{V}[1]\oplus\cdots\oplus\mathbb{V}[k-1]\big)$, getting again a coefficient of the characteristic polynomial, then restrict it further to $\bar{\mathfrak{g}}$ and apply the ${}^{\bar e}F$-construction. Hence by the inductive hypothesis there is a subspace $\overline{W}_{\bar{\mathfrak{g}}} = \bigoplus_{m=1}^{k-1} W_{\mathfrak{g},m}$ with $W_{\mathfrak{g},m}\subset V_{\mathfrak{g},m}^*$ such that $\dim\overline{W}_{\bar{\mathfrak{g}}} = \operatorname{rk}\bar{\mathfrak{g}}$ and $\overline{W}_{\bar{\mathfrak{g}}}\cap\mathcal{N}_F(\bar e)=\{0\}$.

Consider the remaining invariants. For $0\leqslant q\leqslant d_k+1$ set $\hat\varphi_{n-q} := {}^e\Delta_{n-q}|_{V^*}$. By [15, Lemma 5.1], each $\hat\varphi_{n-q}$ is an element of $\mathcal{S}(V_k)$. Let $X\subset V_k^*$ be the zero locus of the $\hat\varphi_\ell$ with $n\geqslant\ell\geqslant n-d_k-1$. Note that

$$\mathcal{N}_F(e)\cap(\overline{W}_{\bar{\mathfrak{g}}}\oplus V_{\mathfrak{g},k}^*) = (\mathcal{N}_F(\bar e)\cap\overline{W}_{\bar{\mathfrak{g}}})\times(X\cap\mathfrak{g}^*) = X\cap\mathfrak{g}^*.$$

Thereby it remains to show that the intersection $X\cap\mathfrak{g}^*$ has no irreducible components of dimension bigger than $\dim V_{\mathfrak{g},k}-\operatorname{rk}\mathfrak{g}+\operatorname{rk}\bar{\mathfrak{g}}$.

The description of X in terms of tuples $\bar s := (s_1,\dots,s_k)$ with $s_i\in\mathbb{Z}_{\geqslant 0}$ is given in [15, Lemma 5.2]. Denote by $X_{\bar s}$ the subspace of V_k^* consisting of all $\gamma\in V_k^*$ such that $\xi_{k-i+1}^{i,d_i-t}(\gamma)=0$ for $0\leqslant t<s_i$. The variety X is a union of linear subspaces $X=\bigcup_{|\bar s|=d_k+1} X_{\bar s}$, where $|\bar s| := s_1+s_2+\cdots+s_k$.

In particular, all irreducible components of X have dimension equal to $\dim V_k - (d_k + 1)$. Then restricted to $\mathfrak{g}^*$ not all of the linear equations $\xi^{i,d_i-t}_{k-i+1} = 0$ stay independent, ξ^{i,d_i-t}_{k-i+1} becomes proportional to ξ^{k-1+1,d_i-t}_{i} and if k is even, then $\xi^{\ell,t}_{\ell}$ with $2\ell = k$ and even t vanishes on $\mathfrak{g}^*$ completely. Summing up, each component of $X\cap\mathfrak{g}^*$ has dimension greater than or equal to $\dim V_{\mathfrak{g},k} - r$, where $r = (d_k+1)/2$ if d_k is odd, $r = d_k/2$ if d_k is even and k is odd and finally if both d_k and k are even, then $r = (d_k + 1)/2$. In any case, $r = \operatorname{rk}\mathfrak{g} - \operatorname{rk}\bar{\mathfrak{g}}$. Therefore we can find a subspace $W_{\mathfrak{g},k} \subset V^*_{\mathfrak{g},k}$ such that $X \cap W_{\mathfrak{g},k} = 0$ and $\dim W_{\mathfrak{g},k} = \operatorname{rk}\mathfrak{g} - \dim \overline{W}_{\bar{\mathfrak{g}}}$. The required subspace $W_{\mathfrak{g}}$ is equal to $\overline{W}_{\bar{\mathfrak{g}}} \oplus W_{\mathfrak{g},k}$. □

Each component of $\mathcal{N}_F(e)$ is a conical Zariski closed subset of $\mathfrak{g}^*_e$ and we found a subspace $W_{\mathfrak{g}} \subset \mathfrak{g}^*_e$ of dimension $\operatorname{rk}\mathfrak{g}$ such that $\mathcal{N}_F(e) \cap W_{\mathfrak{g}} = \{0\}$. Hence

Corollary 8.2. — *All irreducible components of* $\mathcal{N}_F(e)$ *have codimension* $\operatorname{rk}\mathfrak{g}$ *in* $\mathfrak{g}^*_e$ *and* ${}^eF_1, \ldots, {}^eF_{\operatorname{rk}\mathfrak{g}}$ *is a regular sequence in* $\mathcal{S}(\mathfrak{g}_e)$.

Clearly $\mathcal{N}(e)$ is a subset of $\mathcal{N}_F(e)$ and each irreducible component of $\mathcal{N}(e)$ has dimension grater or equal than $\dim\mathfrak{g}_e - \operatorname{rk}\mathfrak{g}$. Therefore we get the following.

Corollary 8.3. — *All irreducible components of the null-cone* $\mathcal{N}(e) \subset \mathfrak{g}^*_e$ *have codimension* $\operatorname{rk}\mathfrak{g}$ *in* $\mathfrak{g}^*_e$.

Let $X \subset \mathbb{A}^d_{\mathbb{F}}$ be a Zariski closed set and let $x = (x_1, \ldots, x_d)$ be a point of X. Let I denote the defining ideal of X in the coordinate algebra $\mathcal{A} = \mathbb{F}[X_1, \ldots, X_d]$ of $\mathbb{A}^d_{\mathbb{F}}$. Each nonzero $f \in \mathcal{A}$ can be expressed as a polynomial in $X_1 - x_1, \ldots, X_d - x_d$, say $f = f_k + f_{k+1} + \cdots$, where f_i is a homogeneous polynomial of degree i in $X_1 - x_1, \ldots, X_d - x_d$ and $f_k \neq 0$. We set $\operatorname{in}_x(f) := f_k$ and denote by $\operatorname{in}_x(I)$ the linear span of all $\operatorname{in}_x(f)$ with $f \in I \smallsetminus \{0\}$. This is an ideal of $\mathcal{A}$ and the affine scheme $TC_x(X) := \operatorname{Spec}\mathcal{A}/\operatorname{in}_x(I)$ is called the *tangent cone* to X at x.

If $\mathfrak{g}$ is of type D, then $n = 2q$ and $F_p = P^2$, where P is the Pfaffian. Set $H_i := {}^eF_i$ for all i in types B, C and for $2i < n$ in type D; and in type D set in addition $H_q := {}^eP$. In exactly the same way as in [15, Subsection 5.4], one can obtain another corollary.

Corollary 8.4. — *Let* $\mathcal{N}$ *be the nilpotent cone of* $\mathfrak{g}$ *and* F_i *as above. Suppose that* $\mathfrak{g}$ *and a nilpotent element* $e \in \mathfrak{g}$ *satisfies the assumptions of this section. Set* $r = \dim\mathfrak{g}_e$. *Then*

$$TC_e(\mathcal{N}(\mathfrak{g})) \cong \mathbb{A}^{\dim\mathfrak{g}-r}_{\mathbb{F}} \times \operatorname{Spec}\mathcal{S}(\mathfrak{g}_e)/(H_1, \ldots, H_{\operatorname{rk}\mathfrak{g}})$$

as affine schemes.

Question 8.5. — Suppose that $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ and $i' = i$ for a nilpotent element $e \in \mathfrak{g}$. Is it true that $H_1, \ldots, H_{\mathrm{rk}\,\mathfrak{g}}$ generate the whole algebra of symmetric $\mathfrak{g}_e$-invariants? The first step is to show that generic fibres of the morphism $\mathfrak{g}_e^* \to \mathrm{Spec}(\mathbb{F}[H_1, \ldots, H_{\mathrm{rk}\,\mathfrak{g}}])$ are connected. Then the subalgebra $\mathbb{F}[H_1, \ldots, H_{\mathrm{rk}\,\mathfrak{g}}]$ will be algebraically closed in $\mathcal{S}(\mathfrak{g}_e)$, see Theorem A.1 in the Appendix. Since it has the right transcendence degree, $\mathrm{ind}\,\mathfrak{g}_e$, it will be shown that at least $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e} \subset \mathrm{Quot}\,\mathbb{F}[H_1, \ldots, H_{\mathrm{rk}\,\mathfrak{g}}]$.

Related, but a slightly different question, is whether $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ is free for the nilpotent elements considered above (in the orthogonal case, the symplectic case is covered by [15]). According to Kac's generalisation of Popov's conjecture, see footnote 1 on page 192 in [9], it should be.

Appendix A. When generic fibres of a morphism are connected

Let $\Bbbk$ be an algebraically closed field of characteristic zero. Suppose that we have a dominant morphism $\varphi\colon X \to Y$ of irreducible affine varieties. Regard $\Bbbk[Y]$ as a subalgebra of $\Bbbk[X]$ and $\Bbbk(Y)$ is a subfield of $\Bbbk(X)$. Let us say that $\Bbbk[Y]$ is algebraically closed in $\Bbbk[X]$, if each element of $\Bbbk[X]$, which is algebraic over $\Bbbk(Y)$, lies in $\Bbbk(Y)$. The following theorem is probably very well known. The proof given below is due to E.B. Vinberg, who explained it to his students at the Moscow University some twenty years ago.

THEOREM A.1. — *Generic fibres of φ are connected if and only if $\Bbbk[Y]$ is algebraically closed in $\Bbbk[X]$.*

Proof. — Suppose first that $\Bbbk[Y]$ is algebraically closed in $\Bbbk[X]$. The algebra $\Bbbk[Y]$ is finitely generated by the assumptions on Y. Let us choose a finite set of generators and let $\mathbb{K} \subset \Bbbk$ be a subfield generated by their coefficients. Then φ is defined over $\mathbb{K}$.

In this proof we say that a point $y \in Y$ is *generic* if the corresponding map $y\colon \mathbb{K}[Y] \to \Bbbk$ is a monomorphism. Informally speaking, being generic means that the coordinates of y are very transcendental elements of $\Bbbk$ with respect to the subfield $\mathbb{K}$. These generic y's form a dense, not necessary open, subset. Since the points $u \in Y$ such that $\varphi^{-1}(u)$ is connected form a closed subset, is suffices to prove that $\varphi^{-1}(y)$ is connected for each generic y.

Suppose y is generic in the above sense. Then

$$\Bbbk[\varphi^{-1}(y)] = \mathbb{K}[X] \otimes_{\mathbb{K}[Y]} \Bbbk = \mathbb{K}[Y]^{-1}\mathbb{K}[X] \otimes_{\mathbb{K}(Y)} \Bbbk,$$

where $\mathbb{K}[Y]$ is embedded into $\Bbbk$ by y and the last equality holds because all elements of $\mathbb{K}[Y]$ are invertible.

Note that a $\mathbb{K}(Y)$-algebra $\mathbb{K}[Y]^{-1}\mathbb{K}[X]$ contains no zero-divisors. (Indeed, if $pq = 0$ in $\mathbb{K}[Y]^{-1}\mathbb{K}[X]$, then multiplying p and q by suitable *invertible* elements of $\mathbb{K}[Y]$, we may assume that $p, q \in \mathbb{K}[X]$. Hence either p or q is zero.) This property might not be preserved by the field extension $\mathbb{K} \subset \Bbbk$. Nevertheless, there are no nilpotent elements in $\Bbbk[\varphi^{-1}(y)]$. In other words, a generic fibre is reduced. If the fibre over y is not connected, then over some Galois extension $\mathbb{K}(Y) \subset L$, the algebra $A := \mathbb{K}[Y]^{-1}\mathbb{K}[X] \otimes_{\mathbb{K}(Y)} L$ decomposes into a direct sum of indecomposable ideals

$$A = A_1 \oplus \cdots \oplus A_m \quad \text{with} \quad m > 1.$$

Let Γ be the Galois group of the extension $\mathbb{K}(Y) \subset L$. Then $\mathbb{K}[Y]^{-1}\mathbb{K}[X] = A^{\Gamma}$. Since this algebra contains no zero-divisors, it could not be a direct sum of two non-trivial ideals. On the other hand, each Γ-orbit in the set of ideals A_i gives rise to an ideal of A^{Γ}. Therefore Γ acts transitively on the set $\{A_i \mid i = 1, \dots, m\}$. Let $\Delta \subset \Gamma$ be the normaliser of A_1. Note that $|\Gamma/\Delta| = m$, hence Δ is a proper subgroup.

Choose a subset $\{\gamma_2, \dots, \gamma_m\} \subset \Gamma$ such that $A_i = \gamma_i \cdot A_1$. If $a \in A^{\Gamma}$, then $a = (a_1, \gamma_2 \cdot a_1, \dots, \gamma_m \cdot a_1)$, where $a_1 \in A_1^{\Delta}$. Thus $\mathbb{K}[Y]^{-1}\mathbb{K}[X] \cong A_1^{\Delta}$. The field L is embedded into A_1 and into any of the other ideals. Threfore L^{Δ} is embedded into A_1^{Δ}. We get a non-trivial extension of $\mathbb{K}(Y)$, which is contained in $\mathbb{K}[Y]^{-1}\mathbb{K}[X]$, *i.e.*, $\mathbb{K}(Y) \subset L^{\Delta} \subset \mathbb{K}[Y]^{-1}\mathbb{K}[X]$. This means that neither $\mathbb{K}[Y]$ nor $\Bbbk[Y]$ is algebraically closed in $\mathbb{K}[X]$ or $\Bbbk[X]$, respectively. A contradiction!

Now suppose that there exists $f \in \Bbbk[X]$, which is algebraic over $\Bbbk(Y)$, but is not an element of $\Bbbk(Y)$. Then there is an open subset $U \subset Y$ such that f takes a finite number of values, more than one, on each fibre $\varphi^{-1}(y)$ with $y \in U$. These values correspond to distinct connected components of $\varphi^{-1}(y)$. $\square$

Remark A.2. — Generic fibres of φ are irreducible if and only if the field $\Bbbk(Y)$ is algebraically closed in the field $\Bbbk(X)$, see e.g. [20, Chapter 2, Section 6.1]. In case X and Y are normal, connectedness of generic fibres implies irreducibility, see [1, Proposition 4]. In general, this is not true.

Here is an example taken form [1] of a dominant morphism with connected but reducible generic fibres.

Example A.3. — Let $X \subset \mathbb{A}^3_{\Bbbk}$ be the irreducible hypersurface defined by the equation $x^2 = y^2 z$. Consider the morphism from X to $Y = \Bbbk$ given by $(x, y, z) \mapsto z$. For any $c \neq 0$, the fibre over $c \in \Bbbk$ consists of two intersecting lines. Hence it is connected and reducible. The set of intersection points $(0, 0, z)$ coincides with the singular locus of X. Evidently, X is not normal.

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