ANNALES

DE

L'INSTITUT FOURIER

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Tome 59, nº 3 (2009), p. 1231-1265.

<http://aif.cedram.org/item?id=AIF_2009__59_3_1231_0>

Ann. Inst. Fourier, Grenoble
59, 3 (2009) 1231-1265

MANIN'S CONJECTURE FOR A QUARTIC DEL PEZZO SURFACE WITH A_4 SINGULARITY

by Tim D. BROWNING & Ulrich DERENTHAL

ABSTRACT. — The Manin conjecture is established for a split singular del Pezzo surface of degree four, with singularity type $\mathbf{A}_4$.

RÉSUMÉ. — Ce papier contient une preuve de la conjecture de Manin pour une surface quartique de del Pezzo, avec singularité $\mathbf{A}_4$.

1. Introduction

The distribution of rational points on del Pezzo surfaces is a challenging topic that has enjoyed a surge of activity in recent years. Guided by the largely unverified conjectures of Manin [11] and his collaborators, the primary aim of this paper is to investigate further the situation for split singular del Pezzo surfaces of degree 4 in $\mathbb{P}^4$, that are defined over $\mathbb{Q}$. Our main achievement will be a proof of the Manin conjecture for the surface

$$x_0x_1 - x_2x_3 = x_0x_4 + x_1x_2 + x_3^2 = 0, \tag{1.1}$$

which we denote by $S \subset \mathbb{P}^4$. This surface contains a unique singularity of type $\mathbf{A}_4$ and exactly three lines, all of which are defined over $\mathbb{Q}$.

Let U be the Zariski open subset formed by deleting the lines from S, and let

$$N_{U,H}(B) := \#\{x \in U(\mathbb{Q}) \mid H(x) \leqslant B\},$$

for any $B \geqslant 1$. Here H is the usual height on $\mathbb{P}^4$, in which the height $H(x)$ is defined as $\max\{|x_0|, \dots, |x_4|\}$ for a point $x = (x_0 : \dots : x_4) \in U(\mathbb{Q})$, provided that $\mathbf{x} = (x_0, \dots, x_4)$ has integral coordinates that are relatively coprime. Bearing this in mind, the following is our principal result.

Keywords: Del Pezzo surfaces, rational points, Manin's conjecture.
Math. classification: 11G35, 14G05, 14G10.

Theorem 1.1. — *We have*

$$N_{U,H}(B) = c_{S,H} B(\log B)^5 + O\big(B(\log B)^{5-2/7}\big),$$

where

$$c_{S,H} = \frac{1}{21600} \cdot \omega_\infty \cdot \prod_p \left(1 - \frac{1}{p}\right)^6 \left(1 + \frac{6}{p} + \frac{1}{p^2}\right)$$

and

$$\omega_\infty = \int_{|t_2|,|t_2 t_6 t_7|,|t_7(t_6^3 t_7 + t_2^2)|,|t_6^2 t_7| \leqslant 1, 0 < t_6 \leqslant 1} \mathrm{d}t_2\, \mathrm{d}t_6\, \mathrm{d}t_7. \tag{1.2}$$

It is easily checked that the surface S is not toric, and we shall see in Lemma 4.1 that it is not an equivariant compactification of $\mathbb{G}_\mathrm{a}^2$. Thus our result does not follow from the work of Tschinkel and his collaborators [1, 5].

As the minimal desingularisation $\widetilde{X_d}$ of a split del Pezzo surface $X_d \subset \mathbb{P}^d$ of degree $d \in \{3, \dots, 7\}$ is the blow-up of $\mathbb{P}^2$ in $9 - d$ points (cf. [6, Introduction]), it has Picard group $\mathrm{Pic}(\widetilde{X_d}) \cong \mathbb{Z}^{10-d}$. In the setting $d = 4$, Manin's conjecture [11] therefore predicts that

$$N_{U,H}(B) \sim \alpha(\widetilde{X_4}) \omega_H(\widetilde{X_4}) B(\log B)^5, \tag{1.3}$$

as $B \to \infty$, where the exponent of $\log B$ is $\operatorname{rank} \mathrm{Pic}(\widetilde{X_4}) - 1$. Moreover, the constants $\alpha(\widetilde{X_4})$ and $\omega_H(\widetilde{X_4})$ are those predicted by Peyre [14]. Note that the exponent of $\log B$ agrees with the statement of the theorem. We shall verify in § 2 that $c_{S,H} = \alpha(\widetilde{S})\omega_H(\widetilde{S})$ in this result.

An overview of progress relating to the Manin conjecture for arbitrary del Pezzo surfaces can be found in the first author's survey [3]. The present paper should be seen as a modest step on the path to its resolution for the singular del Pezzo surfaces of degree 4 that are split over $\mathbb{Q}$. According to the classification of such surfaces found in Coray and Tsfasman [6], it transpires that there are 15 possible singularity types for split singular del Pezzo surfaces of degree 4. It follows from the work of Batyrev and Tschinkel [1], la Bretèche and the first author [2], and the second author's joint work with Tschinkel [10], that the Manin conjecture is already known to hold for 5 explicit surfaces from this catalogue. In view of our theorem, which deals with a surface of singularity type $\mathbf{A}_4$, it remains to deal with the split quartic del Pezzo surfaces that have singularity types

$$\mathbf{A}_n \text{ for } n \in \{1,2,3\}, \quad 3\mathbf{A}_1, \quad \mathbf{A}_1 + \mathbf{A}_n \text{ for } n \in \{1,2,3\}. \tag{1.4}$$

Here one should note that there are two types of surfaces that have singularity type $\mathbf{A}_3$, one containing four lines and one containing five lines. Similarly, there are two types that have $2\mathbf{A}_1$ singularities.

The surface that we have chosen to focus on in the present investigation satisfies the property that the cone of effective divisors associated to the minimal desingularisation $\widetilde{S}$ is not merely generated by the divisors that form a basis for the Picard group $\mathrm{Pic}(\widetilde{S})$, but requires one further divisor to generate it. This leads to some additional considerations in the proof, as we will see shortly.

The proof of the theorem uses a universal torsor. For each split del Pezzo surface of degree d, there is one (essentially unique) universal torsor, which is always an open subset of a $(12-d)$-dimensional affine variety. For toric varieties, universal torsors are open subsets of affine space. Salberger [15] has shown how to establish Manin's conjecture using universal torsors for split toric varieties defined over $\mathbb{Q}$. As a step towards handling non-toric del Pezzo surfaces that still have a relatively simple universal torsor, the second author [7] has determined which del Pezzo surfaces of degree at least 3 have a universal torsor that can be described as a hypersurface in $\mathbb{A}^{13-d}$. Out of the singularity types in (1.4), these include those surfaces of type $\mathbf{A}_1+\mathbf{A}_2$, $\mathbf{A}_1+\mathbf{A}_3$, $3\mathbf{A}_1$ and the $\mathbf{A}_3$ surface with five lines. The surfaces of type $\mathbf{D}_5$ and $\mathbf{D}_4$ considered in [2] and [10] also belong to this class, as does the $\mathbf{A}_4$ surface S considered here. In fact we will see in § 4 that the universal torsor for the present problem is an open subset of the hypersurface

$$\eta_5\alpha_1+\eta_1\alpha_2^2+\eta_3\eta_4^2\eta_6^3\eta_7=0, \tag{1.5}$$

which is embedded in $\mathbb{A}^9 \cong \mathrm{Spec}\,\mathbb{Q}[\eta_1,\ldots,\eta_7,\alpha_1,\alpha_2]$. Note that one of the variables does not explicitly appear in the equation.

Our basic strategy is similar to the one used for the $\mathbf{D}_5$ and $\mathbf{D}_4$ quartic del Pezzo surfaces. The first step is to establish an explicit bijection between the rational points outside the lines on S and certain integral points on the universal torsor. We adopt the approach of Tschinkel and the second author [10] in order to obtain this bijection in an elementary way, motivated by the structure of the minimal desingularisation $\widetilde{S}$ as a blow-up of $\mathbb{P}^2$ in five points. The integral points on the universal torsor are counted in § 5, using the method developed by la Bretèche and the first author [2]. The torsor variables $\eta_1,\ldots,\eta_7,\alpha_1,\alpha_2$ must satisfy (1.5), together with certain coprimality and height conditions. The first step is to fix the variables $\eta_1,\ldots,\eta_7$ and to estimate the relevant number of α_1,α_2 by viewing the equation as a congruence modulo η_5. The resulting estimate is then summed over the remaining variables.

The order in which we handle the remaining variables is crucial and subtle. When it comes to summing over η_6 and η_7 we will run into trouble controlling the overall contribution from the error term each time, because both η_6 and η_7 can be rather big. Summing the number of α_1, α_2 over η_7, for example, leads to an error term that we cannot estimate in a way that is sufficiently small when summed over $\eta_1, \dots, \eta_5$ and large values of η_6. In line with this we shall let the order of summation depend on which of η_6 or η_7 has largest absolute value. When it comes to summing the integral points on the universal torsor that satisfy $|\eta_6| \geqslant |\eta_7|$, we sum first over η_6 and then over η_7. For the alternative contribution we sum first over η_7 and then over η_6. This process leads to two main terms that we put back together to get something of the general shape

$$M(\eta_1, \dots, \eta_5) := \omega_H(\widetilde{S}) \cdot \frac{B}{\eta_1\eta_2\eta_3\eta_4\eta_5}, \tag{1.6}$$

where $\omega_H(\widetilde{S})$ is as in (1.3). The final task is to sum this quantity over the remaining variables $\eta_1, \dots, \eta_5$.

While essentially routine, it is in this final analysis that a further interesting feature of the proof of the theorem is revealed. For $\mathbf{k} \in \mathbb{Z}_{>0}^5$, define the simplex

$$P_{\mathbf{k}} := \big\{(x_1, \dots, x_5) \in \mathbb{R}^5 \mid x_i \geqslant 0, \quad k_1x_1 + \dots + k_5x_5 \leqslant 1\big\}, \tag{1.7}$$

whose volume is easily determined as

$$\mathrm{vol}(P_{\mathbf{k}}) = \frac{1}{5! \cdot k_1 \cdot k_2 \cdot k_3 \cdot k_4 \cdot k_5}.$$

In § 2 we will see that $\alpha(\widetilde{S}) = \mathrm{vol}(P_{(2,4,3,2,3)}) - \mathrm{vol}(P_{(3,6,4,2,5)})$, whence

$$\alpha(\widetilde{S}) = \frac{1}{5! \cdot 2 \cdot 4 \cdot 3 \cdot 2 \cdot 3} - \frac{1}{5! \cdot 3 \cdot 6 \cdot 4 \cdot 2 \cdot 5} = \frac{1}{21600}. \tag{1.8}$$

Returning to the summation of (1.6) over $\eta_1, \dots, \eta_5 \in \mathbb{Z}_{>0}$, which is subject to $\eta_1^2\eta_2^4\eta_3^3\eta_4^2\eta_5^3 \leqslant B$, it will transpire that there is a negligible contribution from those $\eta_1, \dots, \eta_5$ for which $\eta_1^3\eta_2^6\eta_3^4\eta_4^2\eta_5^5 > B$. Summing over the $\eta_1, \dots, \eta_5 \in \mathbb{Z}_{>0}$ that are remaining therefore leads to the final main term

$$\big(\mathrm{vol}(P_{(2,4,3,2,3)}) - \mathrm{vol}(P_{(3,6,4,2,5)})\big) \cdot \omega_H(\widetilde{S})B(\log B)^5,$$

as expected. Thus the main term in the asymptotic formula is really a difference of two main terms that conspire to give the predicted value for $\alpha(\widetilde{S})$. It would be interesting to see whether the same sort of phenomenon occurs for other split del Pezzo surfaces of degree 4, with singularity type among the list (1.4).

Remerciements. The authors are extremely grateful to the anonymous referee for his careful reading of the manuscript and numerous helpful comments. While working on this paper the first author was supported by EPSRC grant number EP/E053262/1. The second author was partially supported by a Feodor Lynen Research Fellowship of the Alexander von Humboldt Foundation.

2. Calculation of Peyre's constant

In this section we wish to show that the value of the constant $c_{S,H}$ obtained in our theorem is in agreement with the prediction (1.3) of Peyre [14]. Beginning with the value of $\omega_H(\widetilde{S})$, whose precise definition we will not include here but which corresponds to a product of local densities, we have

$$\omega_H(\widetilde{S}) = \omega_\infty \prod_p \Big(1-\frac{1}{p}\Big)^6 \omega_p, \tag{2.1}$$

where ω_∞ and ω_p are the real and p-adic densities, respectively. The calculation of ω_p is routine and leads to the conclusion that

$$\omega_p = 1 + \frac{6}{p} + \frac{1}{p^2}.$$

The reader is referred to [2, § 2] for an analogous calculation. We now turn to the calculation of ω_∞, which needs to agree with (1.2).

Recall the equations (1.1) for the surface S, and write $f_1(\mathbf{x}) = x_0x_1 - x_2x_3$ and $f_2(\mathbf{x}) = x_0x_4 + x_1x_2 + x_3^2$. To compute ω_∞, we parametrise the points by writing x_1, x_4 as functions of x_0, x_2, x_3. Thus we have

$$x_1 = \frac{x_2x_3}{x_0}, \quad x_4 = -\frac{x_1x_2 + x_3^2}{x_0} = -\frac{x_2^2x_3 + x_0x_3^2}{x_0^2},$$

and furthermore,

$$\det\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_1}{\partial x_4} & \frac{\partial f_2}{\partial x_4}\end{pmatrix} = \det\begin{pmatrix} x_0 & x_2 \\ 0 & x_0\end{pmatrix} = x_0^2.$$

Since $\mathbf{x}$ and $-\mathbf{x}$ have the same image in $\mathbb{P}^4$, we have

$$\begin{aligned}\omega_\infty &= \frac{1}{2}\int_{|x_0|,\left|\frac{x_2x_3}{x_0}\right|,|x_2|,|x_3|,\left|\frac{x_2^2x_3+x_0x_3^2}{x_0^2}\right|\leqslant 1} x_0^{-2}\,\mathrm{d}x_0\,\mathrm{d}x_2\,\mathrm{d}x_3 \\ &= \frac{1}{2}\int_{|t_6|,|t_2t_6t_7|,|t_2|,|t_6^2t_7|,|t_2^2t_7+t_6^3t_7^2|\leqslant 1} \mathrm{d}t_2\,\mathrm{d}t_6\,\mathrm{d}t_7,\end{aligned}$$

on carrying out the change of variables $x_0 = t_6, x_2 = t_2$ and $x_3 = t_6^2t_7$. But the range of integration is symmetric with respect to the transformation

$(t_2, t_6, t_7) \mapsto (t_2, -t_6, -t_7)$, and so we may restrict to the range $t_6 > 0$. This therefore confirms the equality in (1.2).

It remains to deal with the constant $\alpha(\widetilde{S})$ that appears in (1.3). As we've already commented, the Picard group $\mathrm{Pic}(\widetilde{S})$ of $\widetilde{S}$ has rank 6. Distinguished elements of $\mathrm{Pic}(\widetilde{S})$ are the classes of irreducible curves with negative self intersection number. As described in § 4, these are the classes of four exceptional divisors $E_1, \dots, E_4$ coming from the $\mathbf{A}_4$-singularity of S and the transforms E_5, E_6, E_7 of the three lines on S. By the work of the second author [7, § 7], $E_1, \dots, E_6$ form a basis of $\mathrm{Pic}(\widetilde{S})$. In terms of this basis we have $E_7 = E_1 + 2E_2 + E_3 + 2E_5 - E_6$ and $-K_{\widetilde{S}} = 2E_1 + 4E_2 + 3E_3 + 2E_4 + 3E_5 + E_6$.

The convex cone in $\mathrm{Pic}(\widetilde{S})_{\mathbb{R}} := \mathrm{Pic}(\widetilde{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ generated by classes of effective divisors is generated by $E_1, \dots, E_7$ (see [9, Theorem 3.10]). The intersection of its dual with the hyperplane

$$\{x \in \mathrm{Pic}(\widetilde{S})_{\mathbb{R}} \mid (x, -K_{\widetilde{S}}) = 1\}$$

is a polytope P whose volume is the constant $\alpha(\widetilde{S})$ defined by Peyre [14]. By definition

$$P = \left\{ (x_1, \dots, x_6) \in \mathrm{Pic}(\widetilde{S})_{\mathbb{R}} \,\middle|\, \begin{array}{l} x_i \geqslant 0, \quad x_1 + 2x_2 + x_3 + 2x_5 - x_6 \geqslant 0, \\ 2x_1 + 4x_2 + 3x_3 + 2x_4 + 3x_5 + x_6 = 1 \end{array} \right\}.$$

Eliminating the last coordinate shows that P is isomorphic to

$$P' = \left\{ (x_1, \dots, x_5) \in \mathbb{R}^5 \,\middle|\, \begin{array}{l} x_i \geqslant 0, \quad 2x_1 + 4x_2 + 3x_3 + 2x_4 + 3x_5 \leqslant 1, \\ 3x_1 + 6x_2 + 4x_3 + 2x_4 + 5x_5 \geqslant 1 \end{array} \right\}.$$

Analyzing the volume form with respect to which we must compute the volume of P in order to obtain $\alpha(\widetilde{S})$ (see [9, Section 2], for example), we see that

$$\alpha(\widetilde{S}) = \mathrm{vol}(P') = \mathrm{vol}(P_{(2,4,3,2,3)}) - \mathrm{vol}(P_{(3,6,4,2,5)}),$$

in the notation of (1.7). This therefore establishes (1.8).

An alternative approach to calculating $\alpha(\widetilde{S})$ is available to us through recent work of Joyce, Teitler and the second author [9]. Recall from [8, Table 1] that $\alpha(S_0) = 1/180$ for any non-singular split del Pezzo surface S_0 of degree 4. Since the order of the Weyl group associated to the root system $\mathbf{A}_n$ is $(n+1)!$, as recorded in [9, Table 2], so it follows from [9, Theorem 1.3] that

$$\alpha(\widetilde{S}) = \frac{1}{180} \cdot \frac{1}{5!} = \frac{1}{21600}.$$

This completes the verification that our theorem confirms the Manin conjecture for the split $\mathbf{A}_4$ surface (1.1).

3. Arithmetic functions

In this section we present some elementary facts about certain arithmetic functions and their average order, as required for our argument. Define the multiplicative arithmetic functions

$$\phi^*(n) := \prod_{p\mid n}\Big(1-\frac{1}{p}\Big), \quad \phi^\dagger(n) := \prod_{p\mid n}\Big(1+\frac{1}{p}\Big).$$

Both of these functions have average order $O(1)$, and one has

$$\sum_{n\leqslant x}\frac{\phi^\dagger(n)^j}{n} \ll_j \log x, \tag{3.1}$$

for any $x>1$ and any $j\in\mathbb{Z}_{>0}$. To see this we note that $\phi^\dagger(n)\leqslant\sum_{d\mid n}1/d$, whence

$$\sum_{n\leqslant x}\frac{\phi^\dagger(n)^j}{n} \leqslant \sum_{n\leqslant x}\frac{1}{n}\sum_{d_1,\dots,d_j\mid n}\frac{1}{d_1\dots d_j} \leqslant \sum_{d_1,\dots,d_j=1}^{\infty}\frac{1}{d_1\dots d_j[d_1,\dots,d_j]}\sum_{e\leqslant x}\frac{1}{e},$$

where $[d_1,\dots,d_j]$ denotes the least common multiple of $d_1,\dots,d_j$. The required bound (3.1) then follows from the estimate

$$\sum_{d_1,\dots,d_j=1}^{\infty}\frac{1}{d_1\dots d_j[d_1,\dots,d_j]} \leqslant \sum_{d_1,\dots,d_j=1}^{\infty}\frac{1}{(d_1\dots d_j)^{1+1/j}} \ll_j 1.$$

For given positive integers a,b, our work will lead us to work with the function

$$f_{a,b}(n) := \begin{cases}\phi^*(n)/\phi^*(\gcd(n,a)), & \text{if } \gcd(n,b)=1,\\ 0, & \text{if } \gcd(n,b)>1.\end{cases} \tag{3.2}$$

We begin by establishing the following result.

Lemma 3.1. — *Let $I=[t_1,t_2]$, for $t_1<t_2$. Let $\alpha\in\mathbb{Z}$ such that $\gcd(\alpha,q)=1$. Then we have*

$$\sum_{\substack{n\in I\cap\mathbb{Z}\\ n\equiv\alpha \pmod q}} f_{a,b}(n) = \frac{t_2-t_1}{q}c_0 + O\big(2^{\omega(b)}\log|I|\big),$$

where $|I| := 2+\max\{|t_1|,|t_2|\}$ and

$$c_0 = \frac{\phi^*(b)}{\phi^*(\gcd(b,q))\zeta(2)}\prod_{p\mid abq}\Big(1-\frac{1}{p^2}\Big)^{-1}. \tag{3.3}$$

Proof. — We will follow the convention that $\mu(-n)=\mu(n)$ and $\mu(0)=0$. We begin by calculating the Dirichlet convolution

$$(f_{a,b}*\mu)(n)=\sum_{d\mid n} f_{a,b}(d)\mu(n/d)=\prod_{\substack{p^\nu\| n\\ \nu\geqslant 1}}\big(f_{a,b}(p^\nu)-f_{a,b}(p^{\nu-1})\big).$$

It is clear that $f_{a,b}(1)=1$ and

$$f_{a,b}(p^j)=f_{a,b}(p)=\begin{cases} 1-1/p, & \text{if } p\nmid ab,\\ 1, & \text{if } p\nmid b \text{ and } p\mid a,\\ 0, & \text{if } p\mid b,\end{cases}$$

for any $j\geqslant 1$. Hence it follows that

$$(f_{a,b}*\mu)(n)=\begin{cases} \mu(n)\gcd(b,n)/|n|, & \text{if } \gcd(a,n)\mid b,\\ 0, & \text{otherwise.}\end{cases}$$

In particular

$$\sum_{n\leqslant N}|(f_{a,b}*\mu)(n)|\leqslant \sum_{n\leqslant N}\frac{\gcd(b,n)|\mu(n)|}{|n|}\ll 2^{\omega(b)}\log N,$$

for any $N>1$. Since $f_{a,b}=(f_{a,b}*\mu)*1$, we therefore deduce that

$$\begin{aligned}\sum_{\substack{n\in I\cap\mathbb{Z}\\ n\equiv\alpha \pmod q}} f_{a,b}(n) &= \sum_{\substack{d=1\\ \gcd(d,q)=1}}^{\infty}(f_{a,b}*\mu)(d)\sum_{\substack{m\in d^{-1}I\cap\mathbb{Z}\\ md\equiv\alpha \pmod q}}1\\ &=\frac{t_2-t_1}{q}\sum_{\substack{d=1\\ \gcd(d,q)=1}}^{\infty}\frac{(f_{a,b}*\mu)(d)}{d}+O\big(2^{\omega(b)}\log|I|\big).\end{aligned}$$

Here we have observed that the outer sum in the first line is really a sum over $d\leqslant|I|$, making the previous bound applicable for dealing with the error term. We have then extended the summation over d to infinity, with acceptable error. Finally, it remains to observe that

$$\sum_{\substack{d=1\\ \gcd(d,q)=1}}^{\infty}\frac{(f_{a,b}*\mu)(d)}{d}=\prod_p\Big(1-\frac{1}{p^2}\Big)\prod_{p\mid abq}\Big(1-\frac{1}{p^2}\Big)^{-1}\prod_{\substack{p\mid b\\ p\nmid q}}\Big(1-\frac{1}{p}\Big)=c_0,$$

as required to complete the proof of the lemma. □

Rather than Lemma 3.1, we will actually need a corresponding estimate in which the summand is replaced by $f_{a,b}(n)g(n)$, for suitable real-valued functions g. This is supplied for us by the following result.

Lemma 3.2. — *Let $I = [t_1, t_2]$, for $t_1 < t_2$, and let $g : I \to \mathbb{R}$ be any function such that g has a continuous derivative on I which changes its sign only $R_g(I) < \infty$ times on I. Let $\alpha \in \mathbb{Z}$ such that* $\gcd(\alpha, q) = 1$. *Then we have*

$$\sum_{\substack{n \in I \cap \mathbb{Z} \\ n \equiv \alpha \pmod{q}}} f_{a,b}(n) g(n) = \frac{c_0}{q} \int_I g(t)\,\mathrm{d}t + O\big(2^{\omega(b)} \cdot (\log |I|) \cdot M_I(g)\big),$$

with c_0 given by (3.3) *and* $M_I(g) := (1 + R_g(I)) \cdot \sup_{t \in I} |g(t)|$.

Proof. — We will prove the lemma for $t_1 > 0$, the general case requiring only a trivial modification. Let S denote the sum that is to be estimated, and write

$$M(t) := \sum_{\substack{n \leqslant t \\ n \equiv \alpha \pmod{q}}} f_{a,b}(n),$$

for any $t > 0$. By partial summation,

$$S = M(t_2)g(t_2) - M(t_1)g(t_1) - \int_{t_1}^{t_2} M(t) g'(t)\,\mathrm{d}t.$$

Applying Lemma 3.1 reveals that $M(t) = c_0 t/q + O(2^{\omega(b)} \log(2+t))$. Hence partial integration yields

$$S = \frac{c_0}{q} \int_I g(t)\,\mathrm{d}t + O\Big(2^{\omega(b)} \cdot (\log |I|) \cdot \big(|g(t_2)| + |g(t_1)| + \int_{t_1}^{t_2} |g'(t)|\,\mathrm{d}t\big)\Big).$$

Splitting I into the R_g intervals where g' has constant sign therefore completes the proof of the lemma. □

4. The universal torsor

The purpose of this section is to establish an explicit bijection between the rational points on the open subset U of our $\mathbf{A}_4$ quartic del Pezzo surface S, and the integral points on the universal torsor above $\widetilde{S}$ which are subject to a number of coprimality conditions. In doing so we shall follow the strategy of the second author's joint work with Tschinkel [10].

Along the way we will introduce new variables $\eta_1, \ldots, \eta_7$ and α_1, α_2. It will be convenient to henceforth write

$$\boldsymbol{\eta} = (\eta_1, \ldots, \eta_5), \quad \boldsymbol{\eta}' = (\eta_1, \ldots, \eta_7), \quad \boldsymbol{\alpha} = (\alpha_1, \alpha_2). \tag{4.1}$$

Furthermore, we will make frequent use of the notation

$$\boldsymbol{\eta}^{(k_1,k_2,k_3,k_4,k_5)} := \prod_{i=1}^{5} \eta_i^{k_i}, \tag{4.2}$$

for any $(k_1, \dots, k_5) \in \mathbb{Q}^5$.

In order to derive the bijection alluded to above, we must begin by collecting together some useful information about the geometric structure of S, as defined by equations (1.1). By computing the Segre symbol of S, the definition of which can be found in Hodge and Pedoe [13], we see that S contains exactly one singularity. This has type $\mathbf{A}_4$ and is easily determined as $p = (0 : 0 : 0 : 0 : 1)$. By the classification of singular quartic del Pezzo surfaces found in Coray and Tsfasman [6, Proposition 6.1], S contains exactly three lines. Let us call these lines E_5'', E_6'' and E_7'', where E_5'' and E_6'' intersect in the singularity p, and E_7'' intersects E_6'' outside p. We easily determine these lines as $E_5'' = \{x_0 = x_2 = x_3 = 0\}$, $E_6'' = \{x_0 = x_1 = x_3 = 0\}$ and $E_7'' = \{x_1 = x_3 = x_4 = 0\}$.

The projection $\mathbf{x} \mapsto (x_1 : x_3 : x_4)$ from E_7'' is a birational map $\phi : S \dashrightarrow \mathbb{P}^2$, which maps

$$U := S \setminus (E_5'' \cup E_6'' \cup E_7'') = \{(x_0 : \ldots : x_4) \in S \mid x_3 \neq 0\}$$

isomorphically to

$$\{(\alpha_2 : \eta_5 : \alpha_1) \in \mathbb{P}^2 \mid \eta_5 \neq 0, \alpha_1\eta_5 + \alpha_2^2 \neq 0\} \subset \mathbb{P}^2.$$

The inverse map is $\psi : \mathbb{P}^2 \dashrightarrow S$ given by

$$\psi : (\alpha_2 : \eta_5 : \alpha_1) \mapsto (\eta_5^3 : \alpha_2\eta_7 : \alpha_2\eta_5^2 : \eta_5\eta_7 : \alpha_1\eta_7), \tag{4.3}$$

where $\eta_7 = -(\alpha_1\eta_5 + \alpha_2^2)$.

By [6, Proposition 6.1, Diagram 12], blowing up the singularity p leads to a minimal desingularisation $\pi_0 : \widetilde{S} \to S$ containing four (-2)-curves $E_1, \dots, E_4$ (the four exceptional divisors obtained by blowing up p) and three (-1)-curves E_5, E_6, E_7 (the strict transforms of the lines E_5'', E_6'', E_7'' on S). The configuration of these (-1)- and (-2)-curves on $\widetilde{S}$ is described by Figure 4.1, where the number of edges between two curves is the intersection number, and self intersection numbers are given as upper indices. The divisors A_1, A_2 will be introduced momentarily.

The surface $\widetilde{S}$ is a blow-up $\pi : \widetilde{S} \to \mathbb{P}^2$ in five points. While there are several ways to construct $\widetilde{S}$ as such a blow-up of $\mathbb{P}^2$, we describe a map π that is compatible with the map $\phi : S \dashrightarrow \mathbb{P}^2$ in the sense that $\phi \circ \pi_0 : \widetilde{S} \to S \dashrightarrow \mathbb{P}^2$ coincides with π where it is defined. Such a map $\pi : \widetilde{S} \to \mathbb{P}^2$ is obtained by contracting E_6, E_4, E_3, E_2, E_1 on $\widetilde{S}$ in this order. We choose the same coordinates $(\alpha_2 : \eta_5 : \alpha_1)$ on $\mathbb{P}^2$ as before. Then π maps E_1, E_2, E_3, E_4, E_6 to $(0 : 0 : 1)$. Furthermore, E_7 is the strict transform of $E_7' = \{\eta_7 = -(\alpha_1\eta_5 + \alpha_2^2) = 0\} \subset \mathbb{P}^2$ and E_5 is the strict transform of $E_5' = \{\eta_5 = 0\} \subset \mathbb{P}^2$ under π.

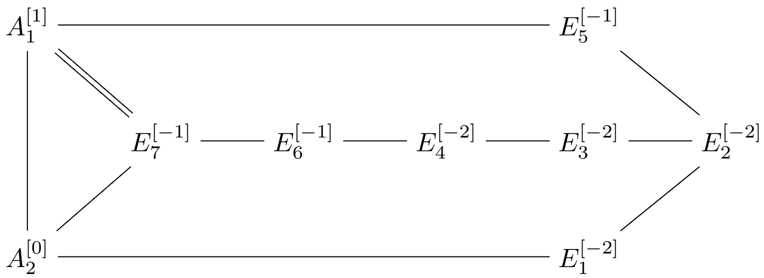

Figure 4.1. Configuration of curves on $\widetilde{S}$.

To describe which points on $\mathbb{P}^2$ we must blow up in order to recover $\widetilde{S}$, we introduce $A_1' = \{\alpha_1 = 0\} \subset \mathbb{P}^2$ and $A_2' = \{\alpha_2 = 0\} \subset \mathbb{P}^2$. We note that its strict transforms A_1, A_2 under π on $\widetilde{S}$ intersect $E_1, \dots, E_7$ as described by Figure 4.1, where A_1, A_2, E_7 meet in one point which maps under π_0 to $(1:0:0:0:0) \in S$. Given $E_5', E_7', A_1', A_2' \subset \mathbb{P}^2$ as above, we may now perform the following sequence of five blow-ups to obtain $\widetilde{S}$:

- blow up the intersection of E_5, E_7, A_2 to obtain E_1;
- blow up the intersection of E_1, E_5, E_7 to obtain E_2;
- blow up the intersection of E_2, E_7 to obtain E_3;
- blow up the intersection of E_3, E_7 to obtain E_4;
- blow up the intersection of E_4, E_7 to obtain E_6.

Here we have renamed E_i' to E_i and A_j' to A_j, and we have used the same names for a divisor and its strict transform in each blow-up in the sequence. We proceed to establish the claim made in § 1.

LEMMA 4.1. — *The surface S is not an equivariant compactification of $\mathbb{G}_\mathrm{a}^2$.*

Proof. — To establish the lemma we assume for a contradiction that S is of this type and apply the work of Hassett and Tschinkel [12]. If S is an equivariant compactification of $\mathbb{G}_\mathrm{a}^2$ then the map $\phi : S \dashrightarrow \mathbb{P}^2$ has to be $\mathbb{G}_\mathrm{a}^2$-equivariant, resulting in an action of $\mathbb{G}_\mathrm{a}^2$ on $\mathbb{P}^2$ which leaves $E_7' = \{\eta_7 = -(\alpha_1\eta_5 + \alpha_2^2) = 0\}$ invariant. However, we can check that the two distinct $\mathbb{G}_\mathrm{a}^2$-structures on $\mathbb{P}^2$ (see [12, Proposition 3.2]) do not leave any irreducible quadric curve invariant. □

We are now ready to derive the promised bijection between $U(\mathbb{Q})$ and integral points on the universal torsor lying above $\widetilde{S}$. The map ψ given by

(4.3) induces a bijection

$$\psi_0 : (\alpha_1, \alpha_2, \eta_5, \eta_7) \mapsto (\eta_5^3, \alpha_2\eta_7, \alpha_2\eta_5^2, \eta_5\eta_7, \alpha_1\eta_7)$$

between

$$\{(\boldsymbol{\alpha}, \eta_5, \eta_7) \in \mathbb{Z}^2 \times \mathbb{Z}_{>0} \times \mathbb{Z}_{\neq 0} \mid \alpha_1\eta_5 + \alpha_2^2 + \eta_7 = 0, \gcd(\alpha_1, \alpha_2, \eta_5) = 1\}$$

and

$$U(\mathbb{Q}) = \{(x_0 : \ldots : x_4) \in S(\mathbb{Q}) \mid x_3 \neq 0\} \subset S(\mathbb{Q}).$$

Note that

$$H(\psi_0(\alpha_1, \alpha_2, \eta_5, \eta_7)) = \frac{\max_{0 \leqslant i \leqslant 4} |\psi_0(\alpha_1, \alpha_2, \eta_5, \eta_7)_i|}{\gcd(\{\psi_0(\alpha_1, \alpha_2, \eta_5, \eta_7)_i \mid 0 \leqslant i \leqslant 4\})}.$$

Motivated by the sequence of blow-ups above, we introduce new variables

$$\begin{aligned} &\eta_1 := \gcd(\alpha_2, \eta_5, \eta_7), \quad &&\eta_2 := \gcd(\eta_1, \eta_5, \eta_7), \quad &&\eta_3 := \gcd(\eta_2, \eta_7),\\ &\eta_4 := \gcd(\eta_3, \eta_7), &&\eta_6 := \gcd(\eta_4, \eta_7), \end{aligned}$$

and in each step transform and rename the previous variables accordingly.

Observe that this gives a bijection

$$(\boldsymbol{\eta}', \boldsymbol{\alpha}) \mapsto (\boldsymbol{\eta}^{(2,4,3,2,3)}\eta_6, \boldsymbol{\eta}^{(1,1,1,1,0)}\eta_6\eta_7\alpha_2, \boldsymbol{\eta}^{(2,3,2,1,2)}\alpha_2, \boldsymbol{\eta}^{(1,2,2,2,1)}\eta_6^2\eta_7, \eta_7\alpha_1),$$

which we call Ψ, between

$$\mathcal{T} := \left\{ (\boldsymbol{\eta}', \boldsymbol{\alpha}) \in \mathbb{Z}_{>0}^6 \times \mathbb{Z}_{\neq 0} \times \mathbb{Z}^2 \,\middle|\, \begin{array}{l} \eta_5\alpha_1 + \eta_1\alpha_2^2 + \eta_3\eta_4^2\eta_6^3\eta_7 = 0 \\ \text{coprimality conditions hold} \end{array} \right\}$$

and $U(\mathbb{Q})$. The coprimality conditions are described by the extended Dynkin diagram of $E_1, \ldots, E_7, A_1, A_2$ in Figure 4.1, following the rule that any of the variables η_i, α_j are coprime if and only if there is no line connecting the divisors E_i, A_j in the Dynkin diagram. Once taken in conjunction with the equation

$$T(\boldsymbol{\eta}', \boldsymbol{\alpha}) = \eta_5\alpha_1 + \eta_1\alpha_2^2 + \eta_3\eta_4^2\eta_6^3\eta_7 = 0,$$

that is satisfied by the elements of $\mathcal{T}$, it is easily checked that the coprimality conditions can be rewritten as

(4.4) $\qquad \gcd(\alpha_1, \eta_2\eta_6) = 1,$

(4.5) $\qquad \gcd(\alpha_2, \eta_2\eta_3\eta_4) = 1,$

(4.6) $\qquad \gcd(\eta_6, \eta_1\eta_2\eta_3\eta_5) = 1,$

(4.7) $\qquad \gcd(\eta_7, \eta_1\eta_2\eta_3\eta_4\eta_5) = 1,$

(4.8) $\qquad \gcd(\eta_1, \eta_3\eta_4\eta_5) = 1, \ \gcd(\eta_2, \eta_4) = 1, \ \gcd(\eta_5, \eta_3\eta_4) = 1.$

In particular it follows that $H(\Psi(\boldsymbol{\eta}',\boldsymbol{\alpha})) = \max_{0\leqslant i\leqslant 4}|\Psi(\boldsymbol{\eta}',\boldsymbol{\alpha})_i|$, since the five coordinates of $\Psi(\boldsymbol{\eta}',\boldsymbol{\alpha})$ are necessarily coprime for $(\boldsymbol{\eta}',\boldsymbol{\alpha}) \in \mathcal{T}$. The height conditions may therefore be written as

$$\max\left\{\begin{array}{l} |\boldsymbol{\eta}^{(2,4,3,2,3)}\eta_6|, |\boldsymbol{\eta}^{(1,1,1,1,0)}\eta_6\eta_7\alpha_2|, \\ |\boldsymbol{\eta}^{(2,3,2,1,2)}\alpha_2|, |\boldsymbol{\eta}^{(1,2,2,2,1)}\eta_6^2\eta_7|, |\eta_7\alpha_1| \end{array}\right\} \leqslant B. \tag{4.9}$$

The equation $T(\boldsymbol{\eta}',\boldsymbol{\alpha}) = 0$ is an embedding of the universal torsor over $\widetilde{S}$ in $\mathbb{A}^9$. Our argument so far has given us a parametrisation of rational points of bounded height in the complement U of the lines in S. This will play a pivotal role in our proof of the theorem.

5. The main argument

In this section we give an overview of the proof of the theorem, and make our final preparations for its proof. Recall the notation introduced in (4.1) and (4.2) for $\boldsymbol{\eta}, \boldsymbol{\alpha}$ and $\boldsymbol{\eta}^{(k_1,k_2,k_3,k_4,k_5)}$. We define the quantities

$$\begin{aligned} Y_0 &:= \left(\frac{\boldsymbol{\eta}^{(2,4,3,2,3)}}{B}\right)^{1/5}, \\ Y_2 &:= \left(\frac{B}{\boldsymbol{\eta}^{(2,-1,-2,-3,-2)}}\right)^{1/5}, \\ Y_6 &:= Y_0^{-1}, \\ Y_7 &:= \left(\frac{B}{\boldsymbol{\eta}^{(-3,-6,-2,2,-7)}}\right)^{1/5}, \end{aligned}$$

which clearly depend only on $\boldsymbol{\eta}$ and B. Using the equation $T(\boldsymbol{\eta}',\boldsymbol{\alpha}) = 0$, a little thought reveals that we may write the height condition (4.9) as

$$|Y_0^4(\eta_6/Y_6)| \leqslant 1, \tag{5.1}$$

$$|Y_0^2(\eta_6/Y_6)(\eta_7/Y_7)(\alpha_2/Y_2)| \leqslant 1, \tag{5.2}$$

$$|Y_0^4(\alpha_2/Y_2)| \leqslant 1, \tag{5.3}$$

$$|Y_0^2(\eta_6/Y_6)^2(\eta_7/Y_7)| \leqslant 1, \tag{5.4}$$

$$|(\eta_7/Y_7)((\eta_6/Y_6)^3(\eta_7/Y_7) + Y_0^2(\alpha_2/Y_2)^2)| \leqslant 1, \tag{5.5}$$

with $\eta_1, \ldots, \eta_6 > 0$. For example, eliminating α_1 from $|\eta_7\alpha_1| \leqslant B$ using $T(\boldsymbol{\eta}',\boldsymbol{\alpha}) = 0$ gives (5.5). It follows from the contents of § 4 that $N_{U,H}(B)$ is equal to the number of $(\boldsymbol{\eta}',\boldsymbol{\alpha}) \in \mathbb{Z}_{>0}^6 \times \mathbb{Z}_{\neq 0} \times \mathbb{Z}^2$ such that (1.5) holds, with (4.4)–(4.8) and (5.1)–(5.5) all holding. As indicated in the introduction it will be necessary to follow different arguments according to which of η_6 or

$|\eta_7|$ is biggest in the summation over the variables $\boldsymbol{\eta}'$. Accordingly, we write $N_a(B)$ for the overall contribution to $N_{U,H}(B)$ from $(\boldsymbol{\eta}', \boldsymbol{\alpha})$ such that

$$\eta_6 \geqslant |\eta_7|, \tag{5.6}$$

and $N_b(B)$ for the remaining contribution from $(\boldsymbol{\eta}', \boldsymbol{\alpha})$ such that

$$\eta_6 < |\eta_7|. \tag{5.7}$$

These quantities will be estimated in § 5.3 and § 5.4, respectively.

Let us now recall the broad outlines of our approach to estimating $N_a(B)$ and $N_b(B)$, as discussed in § 1. Thus the idea is to view the torsor equation (1.5) as a congruence modulo η_5, in order to take care of the summation over the variable α_1. In § 5.2 we shall use this strategy to count the total number of permissible $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$. This will lead to a preliminary estimate for both $N_a(B)$ and $N_b(B)$, since it will make no difference whether (5.6) or (5.7) holds. It will then remain to sum this estimate over all of the remaining variables $\boldsymbol{\eta}'$. We will estimate the overall contribution from the error term in § 5.2. For the treatment of the main term, however, we will need to treat the cases in which (5.6) or (5.7) holds differently. In estimating $N_a(B)$, we will sum the main term over η_6 and then over η_7. This will be undertaken § 5.3. Alternatively, to estimate $N_b(B)$, we will sum the main term over η_7 and then over η_6. This will be the object of § 5.4. Finally, in § 5.5 we will recombine our estimates and sum over the remaining variables $\boldsymbol{\eta} = (\eta_1, \dots, \eta_5)$.

5.1. Real-valued functions

In estimating $N_a(B)$ and $N_b(B)$ we will meet a number of real-valued functions, whose basic properties it will be crucial to understand. Let

$$h(t_0, t_2, t_6, t_7) := \max\{|t_0^4 t_6|, |t_0^2 t_2 t_6 t_7|, |t_0^4 t_2|, |t_0^2 t_6^2 t_7|, |t_7(t_6^3 t_7 + t_0^2 t_2^2)|\}. \tag{5.8}$$

Bearing this notation in mind, one notes that the height conditions in (5.1)–(5.5) are equivalent to $h(Y_0, \alpha_2/Y_2, \eta_6/Y_6, \eta_7/Y_7) \leqslant 1$. Finally, it is easy to see that

$$\omega_\infty = \int_{h(1,t_2,t_6,t_7)\leqslant 1, t_6>0} \mathrm{d}t_2 \, \mathrm{d}t_6 \, \mathrm{d}t_7,$$

where ω_∞ is given by (1.2).

We define the real-valued functions

$$g_0(t_0,t_6,t_7) := \int_{h(t_0,t_2,t_6,t_7)\leqslant 1} 1 \,\mathrm{d}t_2, \tag{5.9}$$

$$g_1^a(t_0,t_7;\boldsymbol{\eta};B) := \int_{Y_6t_6\geqslant |Y_7t_7|,t_6>0} g_0(t_0,t_6,t_7)\,\mathrm{d}t_6, \tag{5.10}$$

$$g_1^b(t_0,t_6;\boldsymbol{\eta};B) := \int_{|Y_7t_7|>\max\{Y_6t_6,1\}} g_0(t_0,t_6,t_7)\,\mathrm{d}t_7, \tag{5.11}$$

$$\begin{aligned} g_2^a(t_0;\boldsymbol{\eta};B) &:= \int_{h(t_0,t_2,t_6,t_7)\leqslant 1, Y_6t_6\geqslant|Y_7t_7|>1} \mathrm{d}t_2\,\mathrm{d}t_6\,\mathrm{d}t_7 \\ &= \int_{|t_7|>1/Y_7} g_1^a(t_0,t_7;\boldsymbol{\eta};B)\,\mathrm{d}t_7, \end{aligned} \tag{5.12}$$

$$\begin{aligned} g_2^b(t_0;\boldsymbol{\eta};B) &:= \int_{h(t_0,t_2,t_6,t_7)\leqslant 1,|Y_7t_7|>\max\{Y_6t_6,1\},t_6>0} \mathrm{d}t_2\,\mathrm{d}t_6\,\mathrm{d}t_7 \\ &= \int_0^\infty g_1^b(t_0,t_6;\boldsymbol{\eta};B)\,\mathrm{d}t_6. \end{aligned} \tag{5.13}$$

We clearly have

$$\begin{aligned} g_2(t_0;\boldsymbol{\eta};B) &:= g_2^a(t_0;\boldsymbol{\eta};B) + g_2^b(t_0;\boldsymbol{\eta};B) \\ &= \int_{h(t_0,t_2,t_6,t_7)\leqslant 1,|Y_7t_7|>1,t_6>0} \mathrm{d}t_2\,\mathrm{d}t_6\,\mathrm{d}t_7. \end{aligned} \tag{5.14}$$

Finally, we define

$$G_2(t_0) := \int_{h(t_0,t_2,t_6,t_7)\leqslant 1,t_6>0} \mathrm{d}t_2\,\mathrm{d}t_6\,\mathrm{d}t_7.$$

The function $G_2 : \mathbb{R}_{>0} \to \mathbb{R}$ is intimately related to the real density ω_∞, as the following result shows.

Lemma 5.1. — *We have*

$$G_2(t_0) = \frac{\omega_\infty}{t_0^2}.$$

Proof. — This result follows on making the change of variables

$$t_2 = T_2t_0^{-4}, \quad t_6 = T_6t_0^{-4}, \quad t_7 = T_7t_0^6.$$

Under this transformation one therefore obtains

$$G_2(t_0) = \frac{1}{t_0^2}\int_{h(t_0,T_2t_0^{-4},T_6t_0^{-4},T_7t_0^6)\leqslant 1,T_6>0} \mathrm{d}T_2\,\mathrm{d}T_6\,\mathrm{d}T_7,$$

where $h(t_0,T_2t_0^{-4},T_6t_0^{-4},T_7t_0^6) = h(1,T_2,T_6,T_7)$ is independent of t_0. □

During the course of our main argument it will be absolutely critical to control the size of the functions (5.9)–(5.11), as t_0, t_6, t_7 vary. We may and shall assume that $t_0, t_6, |t_7|$ take only positive values.

Lemma 5.2. — *Let $\boldsymbol{\eta} \in \mathbb{Z}_{>0}^5$ be given. Then the following hold:*

(1) $g_0(t_0, t_6, t_7) \ll \frac{1}{t_0 |t_7|^{1/2}}$.

(2) $g_1^a(t_0, t_7; \boldsymbol{\eta}; B) \leqslant \int_0^\infty g_0(t_0, t_6, t_7)\, \mathrm{d}t_6 \ll \min\left\{\frac{1}{t_0|t_7|^{7/6}}, \frac{1}{t_0^8}\right\}$.

(3) $g_1^b(t_0, t_6; \boldsymbol{\eta}; B) \leqslant \int_{-\infty}^\infty g_0(t_0, t_6, t_7)\, \mathrm{d}t_7 \ll \frac{1}{t_0 t_6^{3/4}}$.

Proof. — Recall the definition (5.8) of h. The upper bound $O(t_0^{-8})$ that appears in (2) is easy. Indeed, it follows from $h(t_0, t_2, t_6, t_7) \leqslant 1$ that $|t_2| \leqslant 1/t_0^4$ and $|t_6| \leqslant 1/t_0^4$.

For the remaining statements, we distinguish the case $|t_6^3 t_7^2| \leqslant 2$ and its opposite. Note that the inequality $h(t_0, t_2, t_6, t_7) \leqslant 1$ implies

$$|t_6^3 t_7^2 + t_0^2 t_2^2 t_7| \leqslant 1. \tag{5.15}$$

Let us begin with the first case, in which case $|t_0^2 t_2^2 t_7| \leqslant 3$. We therefore obtain

$$t_2 \ll \frac{1}{t_0 |t_7|^{1/2}}, \quad t_6 \ll \frac{1}{|t_7|^{2/3}}, \quad t_7 \ll \frac{1}{t_6^{3/2}}.$$

The first of these inequalities implies statement (1), the first and second imply the first bound in statement (2), and finally, integrating the bound for $g_0(t_0, t_6, t_7)$ from statement (1) over $t_7 \ll 1/t_6^{3/2}$ gives statement (3).

In the second case $|t_6^3 t_7^2| > 2$, the inequality (5.15) implies $t_7 < 0$ and

$$\frac{t_6^3 t_7^2 - 1}{t_0^2 |t_7|} \leqslant t_2^2 \leqslant \frac{t_6^3 t_7^2 + 1}{t_0^2 |t_7|}.$$

Note that the condition $\sqrt{x} \leqslant t_2 \leqslant \sqrt{x+y}$ describes an interval for t_2 of length $O(y/x^{1/2})$. Here, $x = (t_6^3 t_7^2 - 1)/(t_0^2 |t_7|) \geqslant t_6^3 |t_7|/(2t_0^2)$ and $y = 2/(t_0^2 |t_7|)$, whence

$$g_0(t_0, t_6, t_7) \ll \frac{1}{t_0 t_6^{3/2} |t_7|^{3/2}}.$$

The inequality $t_6 > 2^{1/3}/|t_7|^{2/3}$ implies statement (1) and integrating over $t_6 > 2^{1/3}/|t_7|^{2/3}$ results in the first bound in statement (2). Finally, integrating over $|t_7| > 2^{1/2}/t_6^{3/2}$ gives statement (3). □

5.2. Estimating $N_a(B)$ and $N_b(B)$ — first step

We are now ready to begin our estimation of $N_a(B)$ and $N_b(B)$ in earnest. In what follows, we always have $\eta_1, \ldots, \eta_6 \in \mathbb{Z}_{>0}$ and $\eta_7 \in \mathbb{Z}_{\neq 0}$.

For fixed $\boldsymbol{\eta}' = (\boldsymbol{\eta}, \eta_6, \eta_7)$ subject to the coprimality conditions (4.6), (4.7) and (4.8), we let $N_0 := N_0(\boldsymbol{\eta}'; B)$ be the total number of $\alpha_1, \alpha_2 \in \mathbb{Z}$ which satisfy the equation (1.5), subject to $h(Y_0, \alpha_2/Y_2, \eta_6/Y_6, \eta_7/Y_7) \leqslant 1$ and the coprimality conditions (4.4) and (4.5). Employing a Möbius inversion for (4.4), we obtain

$$N_0 = \sum_{k_1 \mid \eta_2\eta_6} \mu(k_1) \# \left\{ \alpha_2 \;\middle|\; \begin{array}{l} \alpha_2^2\eta_1 \equiv -\eta_3\eta_4^2\eta_6^3\eta_7 \pmod{k_1\eta_5}, \\ h(Y_0, \alpha_2/Y_2, \eta_6/Y_6, \eta_7/Y_7) \leqslant 1, \\ \text{(4.5) holds} \end{array} \right\}.$$

It is easy to see that the summand vanishes unless $\gcd(k_1, \eta_1\eta_3\eta_4) = 1$. Indeed, if $p \mid k_1, \eta_1$ then $p \mid \eta_1, \eta_3\eta_4\eta_6\eta_7$, which is forbidden, and furthermore, if $p \mid k_1, \eta_3\eta_4$ then $p \mid \eta_3\eta_4, \alpha_2\eta_1$, which is also forbidden.

Let k_1 be a squarefree divisor of $\eta_2\eta_6$. Since $\gcd(\eta_2, \eta_6) = 1$, we can write $k_1 = k_{12}k_{16}$ with $k_{12} \mid \eta_2$ and $k_{16} \mid \eta_6$. Furthermore such a representation is unique. Writing $\eta_6 = k_{16}\eta_6'$ we therefore obtain

$$N_0 = \sum_{\substack{k_{16} \mid \eta_6, k_{12} \mid \eta_2 \\ \gcd(k_{12}k_{16}, \eta_1\eta_3\eta_4) = 1}} \mu(k_{12})\mu(k_{16}) N_0(k_{12}, k_{16})$$

where

$$N_0(k_{12}, k_{16}) := \# \left\{ \alpha_2 \;\middle|\; \begin{array}{l} \alpha_2^2\eta_1 \equiv -\eta_3\eta_4^2 k_{16}^3 \eta_6'^3\eta_7 \pmod{k_{12}k_{16}\eta_5}, \\ h(Y_0, \alpha_2/Y_2, \eta_6/Y_6, \eta_7/Y_7) \leqslant 1, \\ \text{(4.5) holds} \end{array} \right\}.$$

In view of the congruence we have $k_{16} \mid \alpha_2^2\eta_1$, whence $k_{16} \mid \alpha_2$ since $\gcd(k_{16}, \eta_1) = 1$ and k_{16} is squarefree. Writing $\alpha_2 = k_{16}\alpha_2'$, we divide through the congruence by k_{16} to obtain

$$\alpha_2'^2 k_{16}\eta_1 \equiv -\eta_3\eta_4^2 k_{16}^2 \eta_6'^3 \eta_7 \pmod{k_{12}\eta_5}.$$

Using the relation $\gcd(\eta_6, \eta_2\eta_5) = 1$, we see that $\gcd(k_{16}, k_{12}\eta_5) = 1$, whence we can remove a further factor of k_{16} in this congruence. It therefore follows that

$$N_0(k_{12}, k_{16}) = \# \left\{ \alpha_2' \;\middle|\; \begin{array}{l} \alpha_2'^2\eta_1 \equiv -\eta_3\eta_4^2 k_{16} \eta_6'^3\eta_7 \pmod{k_{12}\eta_5}, \\ h(Y_0, \alpha_2' k_{16}/Y_2, \eta_6/Y_6, \eta_7/Y_7) \leqslant 1, \\ \gcd(\alpha_2', \eta_2\eta_3\eta_4) = 1 \end{array} \right\},$$

since $\gcd(k_{16}, \eta_2\eta_3\eta_4) = 1$.

Note that $\gcd(k_{12}\eta_5, \eta_1) = 1$ and $\gcd(k_{12}\eta_5, \eta_3\eta_4^2k_{16}\eta_6'^3\eta_7) = 1$. It therefore follows that for each α_2' satisfying the congruence, there is a unique $1 \leqslant \varrho \leqslant k_{12}\eta_5$, with

$$\gcd(\varrho, k_{12}\eta_5) = 1, \quad \varrho^2\eta_1 \equiv -\eta_3\eta_6\eta_7 \pmod{k_{12}\eta_5}, \tag{5.16}$$

such that

$$\alpha_2' \equiv \varrho\eta_4\eta_6' \pmod{k_{12}\eta_5}.$$

Thus we obtain

$$N_0(k_{12},k_{16}) = \sum_{\substack{1\leqslant\varrho\leqslant k_{12}\eta_5\\ (5.16)\text{ holds}}} \#\left\{\alpha_2' \;\middle|\; \begin{array}{l} \alpha_2' \equiv \varrho\eta_4\eta_6' \pmod{k_{12}\eta_5},\\ h(Y_0,\alpha_2'k_{16}/Y_2,\eta_6/Y_6,\eta_7/Y_7)\leqslant 1,\\ \gcd(\alpha_2',\eta_2\eta_3\eta_4)=1 \end{array}\right\}.$$

We remove $\gcd(\alpha_2',\eta_2\eta_3\eta_4)=1$ by a further application of Möbius inversion. Writing $\alpha_2' = k_2\alpha_2''$, we see that $N_0(k_{12},k_{16})$ is equal to

$$\sum_{\substack{1\leqslant\varrho\leqslant k_{12}\eta_5\\ (5.16)\text{ holds}}} \sum_{k_2\mid\eta_2\eta_3\eta_4} \mu(k_2)\#\left\{\alpha_2'' \;\middle|\; \begin{array}{l} k_2\alpha_2'' \equiv \varrho\eta_4\eta_6' \pmod{k_{12}\eta_5},\\ h(Y_0,\alpha_2''k_{16}k_2/Y_2,\eta_6/Y_6,\eta_7/Y_7)\leqslant 1 \end{array}\right\}.$$

The summand vanished unless $\gcd(k_2,k_{12}\eta_5)=1$, since $p\mid k_2, k_{12}\eta_5$ implies $p\mid k_{12}\eta_5, \varrho\eta_4\eta_6'$, which is forbidden. Thus we may restrict our summation over k_2 to $\gcd(k_2,k_{12}\eta_5)=1$, and it therefore follows that the number of available α_2'' is

$$\frac{Y_2}{k_{12}k_{16}k_2\eta_5} g_0(Y_0,\eta_6/Y_6,\eta_7/Y_7) + O(1),$$

where g_0 is given by (5.9). Recall the definition of the function ϕ^* from § 3. We are now ready to establish the following result.

Lemma 5.3. — *We have*

$$N_0 = \frac{Y_2}{\eta_5} g_0(Y_0,\eta_6/Y_6,\eta_7/Y_7)\vartheta_0(\boldsymbol{\eta},\eta_6,\eta_7) + O(R_0(\boldsymbol{\eta},\eta_6,\eta_7;B))$$

with

$$\vartheta_0(\boldsymbol{\eta},\eta_6,\eta_7) := \frac{\phi^*(\eta_6)\phi^*(\eta_2\eta_3\eta_4)}{\phi^*(\gcd(\eta_6,\eta_4))} \sum_{\substack{k_{12}\mid\eta_2\\ \gcd(k_{12},\eta_1\eta_3\eta_4)=1}} \frac{\mu(k_{12})}{k_{12}\phi^*(\gcd(\eta_2,k_{12}\eta_5))} \sum_{\substack{1\leqslant\varrho\leqslant k_{12}\eta_5\\ (5.16)\ \textit{holds}}} 1,$$

and

$$\sum_{\boldsymbol{\eta},\eta_6,\eta_7} R_0(\boldsymbol{\eta},\eta_6,\eta_7;B) \ll B(\log B)^3.$$

The final statement in Lemma 5.3 should be taken to mean that the overall contribution from the error term in the asymptotic formula for N_0, once summed over all of the available $\boldsymbol{\eta},\eta_6,\eta_7$, is $O(B(\log B)^3)$. What is crucial here is that the exponent of $\log B$ is strictly smaller than 5, so that this truly is an acceptable error term from the point of view of the main theorem. In the case of Lemma 5.3 we need to sum $R_0(\boldsymbol{\eta},\eta_6,\eta_7;B)$ over all $\boldsymbol{\eta},\eta_6,\eta_7$ which satisfy the height conditions (5.1)–(5.5), and the coprimality conditions (4.6)–(4.8). In the arguments to follow there will be several points at which the overall contribution from various error terms needs to be estimated. In each case we will not stress the precise conditions on the variables to be summed over, these being invariably self-evident.

Proof of Lemma 5.3. — Tracing through our argument above, it follows that

$$N_0 = \frac{Y_2}{\eta_5} g_0(Y_0, \eta_6/Y_6, \eta_7/Y_7)\vartheta_0(\boldsymbol{\eta}, \eta_6, \eta_7) + O(R_0(\boldsymbol{\eta}, \eta_6, \eta_7; B)),$$

with

$$\vartheta_0 = \sum_{\substack{k_{16}|\eta_6, k_{12}|\eta_2 \\ \gcd(k_{12}k_{16}, \eta_1\eta_3\eta_4)=1}} \frac{\mu(k_{12})\mu(k_{16})}{k_{12}k_{16}} \sum_{\substack{1\leqslant \varrho \leqslant k_{12}\eta_5 \\ (5.16) \text{ holds}}} \sum_{\substack{k_2|\eta_2\eta_3\eta_4 \\ \gcd(k_2, k_{12}\eta_5)=1}} \frac{\mu(k_2)}{k_2},$$

and

$$\begin{aligned} R_0(\boldsymbol{\eta}, \eta_6, \eta_7; B) &\ll 2^{\omega(\eta_2\eta_3\eta_4)+\omega(\eta_6)} \sum_{k_{12}|\eta_2} |\mu(k_{12})| \sum_{\substack{1\leqslant \varrho \leqslant k_{12}\eta_5 \\ (5.16) \text{ holds}}} 1 \\ &\ll 2^{\omega(\eta_2)+\omega(\eta_2\eta_5)} 2^{\omega(\eta_2\eta_3\eta_4)+\omega(\eta_6)} \\ &\leqslant 8^{\omega(\eta_2)} 2^{\omega(\eta_3)+\omega(\eta_4)+\omega(\eta_5)+\omega(\eta_6)}. \end{aligned}$$

We have used here the fact that the congruence in (5.16) has at most $2^{\omega(k_{12}\eta_5)} \leqslant 2^{\omega(\eta_2\eta_5)}$ solutions ϱ modulo $k_{12}\eta_5$.

On noting that $\gcd(\eta_6, \eta_1\eta_3) = 1$ and $\gcd(\eta_3\eta_4, k_{12}\eta_5) = 1$, we deduce that

$$\vartheta_0 = \sum_{\substack{k_{12}|\eta_2 \\ \gcd(k_{12}, \eta_1\eta_3\eta_4)=1}} \frac{\mu(k_{12})}{k_{12}} \frac{\phi^*(\eta_6)}{\phi^*(\gcd(\eta_6, \eta_4))} \frac{\phi^*(\eta_2\eta_3\eta_4)}{\phi^*(\gcd(\eta_2, k_{12}\eta_5))} \sum_{\substack{1\leqslant \varrho \leqslant k_{12}\eta_5 \\ (5.16) \text{ holds}}} 1.$$

This completes the proof of the main term in the lemma.

To show that $R_0(\boldsymbol{\eta}, \eta_6, \eta_7; B)$ makes a satisfactory contribution once it is summed over all $\boldsymbol{\eta}, \eta_6, \eta_7$ satisfying the height conditions in (4.9), we begin by summing over η_7. Thus it follows that

$$\begin{aligned} \sum_{\boldsymbol{\eta}, \eta_6, \eta_7} R_0(\boldsymbol{\eta}, \eta_6, \eta_7; B) &\ll \sum_{\boldsymbol{\eta}, \eta_6} \frac{8^{\omega(\eta_2)} 2^{\omega(\eta_3)+\omega(\eta_4)+\omega(\eta_5)+\omega(\eta_6)} B}{\boldsymbol{\eta}^{(1,2,2,2,1)} \eta_6^2} \\ &\ll B(\log B)^3, \end{aligned}$$

as required to complete the proof of the lemma. ☐

5.3. Estimating $N_a(B)$ — second step

In this section our task is to sum the main term in Lemma 5.3 over all of the relevant η_6 and η_7, such that (5.6) holds. As we've already indicated, we will begin by summing over the η_6. For fixed $\boldsymbol{\eta}, \eta_7$ satisfying the coprimality conditions (4.7) and (4.8), define $N_1^a := N_1^a(\boldsymbol{\eta}, \eta_7; B)$ to be the sum of the main term in Lemma 5.3 over all $\eta_6 \in \mathbb{Z}_{>0}$ such that the coprimality condition (4.6) holds, and furthermore, $\eta_6 \geqslant |\eta_7|$.

We begin by noting that it is possible to remove η_5 from (4.6), replacing this coprimality condition by $\gcd(\eta_6, \eta_1\eta_2\eta_3) = 1$. Indeed, if $p \mid \eta_6, \eta_5$ then (5.16) implies that we must have $p \mid \varrho^2\eta_1$, which is forbidden. Since $\gcd(\eta_3\eta_7, k_{12}\eta_5) = 1$, so there exists a unique integer $\beta \in [1, k_{12}\eta_5]$ such that

$$\varrho^2\eta_1 \equiv -\eta_3\eta_7\beta \pmod{k_{12}\eta_5}.$$

It therefore follows that

$$N_1^a = \frac{Y_2}{\eta_5}\phi^*(\eta_2\eta_3\eta_4) \sum_{\substack{k_{12}\mid\eta_2 \\ \gcd(k_{12},\eta_1\eta_3\eta_4)=1}} \frac{\mu(k_{12})}{k_{12}\phi^*(\gcd(\eta_2, k_{12}\eta_5))} \sum_{\substack{1\leqslant\varrho\leqslant k_{12}\eta_5 \\ \gcd(\varrho,k_{12}\eta_5)=1}} A,$$

where

$$A = \sum_{\substack{\eta_6\in\mathbb{Z}_{>0} \\ \eta_6\geqslant|\eta_7| \\ \eta_6\equiv\beta \pmod{k_{12}\eta_5}}} f_{\eta_4,\eta_1\eta_2\eta_3}(\eta_6) g_0(Y_0, \eta_6/Y_6, \eta_7/Y_7).$$

Here $f_{\eta_4,\eta_1\eta_2\eta_3}$ is given by (3.2). Since $g_0(Y_0, \eta_6/Y_6, \eta_7/Y_7) = 0$ for $\eta_6 > B$, we may restrict the summation to η_6 in the range $|\eta_7| \leqslant \eta_6 \leqslant B$.

We will estimate A using Lemma 3.2. This produces a main term and an error term, the latter having size

$$\ll 2^{\omega(\eta_1\eta_2\eta_3)}(\log B) \sup_{t_6} g_0(Y_0, t_6, \eta_7/Y_7),$$

where the supremum is over all $t_6 \in \mathbb{R}$ such that $Y_6 t_6 \geqslant |\eta_7|$. This therefore gives an overall contribution

$$\ll Y_2 2^{\omega(\eta_2)+\omega(\eta_1\eta_2\eta_3)}(\log B) \sup_{t_6} g_0(Y_0, t_6, \eta_7/Y_7), \tag{5.17}$$

to N_1^a, since

$$\frac{1}{k_{12}\eta_5} \sum_{\substack{1\leqslant\varrho\leqslant k_{12}\eta_5 \\ \gcd(\varrho,k_{12}\eta_5)=1}} 1 = \phi^*(k_{12}\eta_5).$$

The main term in our application of Lemma 3.2 to A is simply

$$\Theta(\boldsymbol{\eta}, k_{12}) \frac{Y_6}{k_{12}\eta_5} \int_{Y_6 t_6\geqslant|\eta_7|, t_6>0} g_0(Y_0, t_6, \eta_7/Y_7)\, \mathrm{d}t_6,$$

with

$$\begin{aligned}\Theta(\boldsymbol{\eta}, k_{12}) &= \frac{\phi^*(\eta_1\eta_2\eta_3)}{\zeta(2)\phi^*(\gcd(\eta_1\eta_2\eta_3, k_{12}\eta_5))} \prod_{p\mid\eta_1\eta_2\eta_3\eta_4\eta_5} \Big(1-\frac{1}{p^2}\Big)^{-1} \\ &= \frac{\phi^*(\eta_1\eta_2\eta_3)}{\zeta(2)\phi^*(\gcd(\eta_2, k_{12}\eta_5))} \prod_{p\mid\eta_1\eta_2\eta_3\eta_4\eta_5} \Big(1-\frac{1}{p^2}\Big)^{-1}.\end{aligned}$$

Here we have used the fact that $\gcd(\eta_1\eta_3, k_{12}\eta_5) = 1$. Note for future reference that $\Theta(\boldsymbol{\eta}, k_{12}) \ll 1$. We are now ready to establish the following result.

Lemma 5.4. — *We have*

$$N_1^a = \frac{Y_2Y_6}{\eta_5} g_1^a(Y_0, \eta_7/Y_7; \boldsymbol{\eta}; B)\vartheta_1^a(\boldsymbol{\eta}) + O(R_1^a(\boldsymbol{\eta}, \eta_7; B))$$

with

$$\vartheta_1^a(\boldsymbol{\eta}) := \sum_{\substack{k_{12}\mid\eta_2 \\ \gcd(k_{12},\eta_1\eta_3\eta_4)=1}} \frac{\mu(k_{12})}{k_{12}} \Theta(\boldsymbol{\eta}, k_{12})\phi^*(\eta_2\eta_3\eta_4\eta_5),$$

and

$$\sum_{\boldsymbol{\eta},\eta_7} R_1^a(\boldsymbol{\eta}, \eta_7; B) \ll B(\log B)^3.$$

Proof. — It is clear from our calculations above that the main term in our estimate for N_1^a is equal to $Y_2Y_6 g_1^a(Y_0, \eta_7/Y_7; \boldsymbol{\eta}; B)\vartheta_1^a(\boldsymbol{\eta})/\eta_5$, with

$$\begin{aligned}\vartheta_1^a(\boldsymbol{\eta}) &= \sum_{\substack{k_{12}\mid\eta_2 \\ \gcd(k_{12},\eta_1\eta_3\eta_4)=1}} \frac{\mu(k_{12})\phi^*(\eta_2\eta_3\eta_4)\phi^*(k_{12}\eta_5)}{k_{12}\phi^*(\gcd(\eta_2\eta_3\eta_4, k_{12}\eta_5))} \Theta(\boldsymbol{\eta}, k_{12}) \\ &= \phi^*(\eta_2\eta_3\eta_4\eta_5) \sum_{\substack{k_{12}\mid\eta_2 \\ \gcd(k_{12},\eta_1\eta_3\eta_4)=1}} \frac{\mu(k_{12})}{k_{12}} \Theta(\boldsymbol{\eta}, k_{12}),\end{aligned}$$

since $k_{12}\eta_5$ is coprime to $\eta_3\eta_4$ and every divisor of k_{12} divides η_2. This completes the proof of the main term in the lemma.

Turning to the overall contribution from the error term $R_1^a(\boldsymbol{\eta}, \eta_7; B)$, which we have already seen has size (5.17), we conclude from (4.9) and (5.6) that

$$|\boldsymbol{\eta}^{(2,4,3,2,3)}\eta_7| \leqslant B,$$

for the $\boldsymbol{\eta}, \eta_7$ that we need to sum over. We therefore deduce from Lemma 5.2(1) that

$$\begin{aligned}\sum_{\boldsymbol{\eta},\eta_7} R_1^a(\boldsymbol{\eta},\eta_7;B) &\ll \log B \sum_{\boldsymbol{\eta},\eta_7} Y_2 4^{\omega(\eta_2)} 2^{\omega(\eta_1)+\omega(\eta_3)} \cdot \frac{Y_7^{1/2}}{Y_0|\eta_7|^{1/2}} \\ &= \log B \sum_{\boldsymbol{\eta},\eta_7} \frac{4^{\omega(\eta_2)} 2^{\omega(\eta_1)+\omega(\eta_3)} B^{1/2}}{\boldsymbol{\eta}^{(1/2,0,0,0,-1/2)}|\eta_7|^{1/2}} \\ &\ll \log B \sum_{\boldsymbol{\eta}} \frac{4^{\omega(\eta_2)} 2^{\omega(\eta_1)+\omega(\eta_3)} B}{\boldsymbol{\eta}^{(3/2,2,3/2,1,1)}} \\ &\ll B(\log B)^3,\end{aligned}$$

as required to complete the proof of the lemma. □

Lemma 5.4 takes care of the summation of the main term in Lemma 5.3 over all of the relevant η_6. We proceed to sum the resulting main term over the η_7. Thus we let

$$N_2^a := N_2^a(\boldsymbol{\eta};B) = \sum_{\substack{\eta_7 \in \mathbb{Z}_{\neq 0} \\ (4.7) \text{ holds}}} \frac{Y_2Y_6}{\eta_5} g_1^a(Y_0, \eta_7/Y_7; \boldsymbol{\eta}; B)\vartheta_1^a(\boldsymbol{\eta})$$

We begin with an application of Möbius inversion to remove the coprimality condition (4.7). This gives

$$N_2^a = \frac{Y_2Y_6}{\eta_5}\vartheta_1^a(\boldsymbol{\eta}) \sum_{k_7|\eta_1\eta_2\eta_3\eta_4\eta_5} \mu(k_7) \sum_{|\eta_7'|\geqslant 1} g_1^a(Y_0, k_7\eta_7'/Y_7; \boldsymbol{\eta}; B),$$

where we have written $\eta_7 = k_7\eta_7'$. Partial summation now yields

$$\begin{aligned}N_2^a =& \frac{Y_2Y_6Y_7}{\eta_5}\vartheta_1^a(\boldsymbol{\eta}) \sum_{k_7|\eta_1\eta_2\eta_3\eta_4\eta_5} \frac{\mu(k_7)}{k_7} \int_{|t_7|\geqslant k_7/Y_7} g_1^a(Y_0, t_7; \boldsymbol{\eta}; B)\,\mathrm{d}t_7 \\ &+ O\Big(\frac{Y_2Y_6}{\eta_5}|\vartheta_1^a(\boldsymbol{\eta})| \sum_{k_7|\eta_1\eta_2\eta_3\eta_4\eta_5} |\mu(k_7)| \sup_{|t_7|\geqslant k_7/Y_7} g_1^a(Y_0, t_7; \boldsymbol{\eta}; B)\Big).\end{aligned}$$

The following result constitutes the final outcome of our summation over η_7.

Lemma 5.5. — *We have*

$$N_2^a = \frac{Y_2Y_6Y_7}{\eta_5} g_2^a(Y_0; \boldsymbol{\eta}; B)\vartheta_2^a(\boldsymbol{\eta}) + O(R_2^a(\boldsymbol{\eta}; B))$$

with

$$\vartheta_2^a(\boldsymbol{\eta}) := \phi^*(\eta_1\eta_2\eta_3\eta_4\eta_5)\vartheta_1^a(\boldsymbol{\eta}),$$

and

$$\sum_{\boldsymbol{\eta}} R_2^a(\boldsymbol{\eta}; B) \ll B(\log B)^{5-2/7}.$$

Proof. — The effect of replacing $\int_{|t_7|\geqslant k_7/Y_7} g_1^a(Y_0, t_7; \boldsymbol{\eta}; B)\,\mathrm{d}t_7$ by the expression $g_2^a(Y_0;\ \boldsymbol{\eta}; B)$ in our estimate for N_2^a, is to create an additional term

$$\frac{Y_2Y_6Y_7}{\eta_5}|\vartheta_1^a(\boldsymbol{\eta})| \sum_{k_7|\eta_1\eta_2\eta_3\eta_4\eta_5} \frac{|\mu(k_7)|}{k_7} \int_{1/Y_7<|t_7|<k_7/Y_7} g_1^a(Y_0, t_7; \boldsymbol{\eta}; B)\,\mathrm{d}t_7,$$

that must become part of $R_2^a(\boldsymbol{\eta}; B)$. Let us think of this as the first term in $R_2^a(\boldsymbol{\eta}; B)$. The second term that appears in $R_2^a(\boldsymbol{\eta}; B)$ is the error appearing in the asymptotic formula for N_2^a that directly precedes the statement of the lemma.

We will need to estimate the overall contribution from both of these terms separately. It will be convenient to note that $\vartheta_1^a(\boldsymbol{\eta}) = O(\phi^\dagger(\eta_2))$, in the notation of § 3. Let $\lambda > 0$ be a parameter to be selected in due course. Our argument will depend upon whether or not $\boldsymbol{\eta}^{(3,6,4,2,5)} < \lambda B$ in the summation over the $\boldsymbol{\eta}$. Accordingly let $E_1(\lambda)$ denote the overall contribution from the two errors terms once summed over $\boldsymbol{\eta}$ such that

$$\boldsymbol{\eta}^{(3,6,4,2,5)} < \lambda B, \tag{5.18}$$

and let $E_2(\lambda)$ denote the remaining contribution from $\boldsymbol{\eta}$ such that

$$\boldsymbol{\eta}^{(3,6,4,2,5)} \geqslant \lambda B. \tag{5.19}$$

Beginning with the estimation of $E_1(\lambda)$, we employ Lemma 5.2(2) to conclude that

$$\int_{1/Y_7}^{k_7/Y_7} g_1^a(Y_0, t_7; \boldsymbol{\eta}; B)\,\mathrm{d}t_7 \ll \int_{1/Y_7}^{k_7/Y_7} \frac{1}{Y_0|t_7|^{7/6}}\,\mathrm{d}t_7 \ll \frac{Y_7^{1/6}}{Y_0}.$$

Once summed over all $\boldsymbol{\eta}$ such that (5.18) holds, we use (3.1) to estimate the overall contribution from the first term in $R_2^a(\boldsymbol{\eta}; B)$ as

$$\begin{aligned}
&\ll \sum_{\boldsymbol{\eta}} \sum_{k_7|\eta_1\eta_2\eta_3\eta_4\eta_5} \frac{|\mu(k_7)|}{k_7} \frac{\phi^\dagger(\eta_2)Y_2Y_6Y_7^{7/6}}{\eta_5 Y_0}\\
&\ll \sum_{\boldsymbol{\eta}} \frac{\phi^\dagger(\eta_1\eta_2\eta_3\eta_4\eta_5)\phi^\dagger(\eta_2)B^{5/6}}{\boldsymbol{\eta}^{(1/2,0,1/3,2/3,1/6)}}\\
&\ll \sum_{\eta_1,\eta_2,\eta_3,\eta_4} \frac{\phi^\dagger(\eta_1)\phi^\dagger(\eta_3)\phi^\dagger(\eta_4)\phi^\dagger(\eta_2)^2\lambda^{1/6}B}{\boldsymbol{\eta}^{(1,1,1,1,0)}}\\
&\ll \lambda^{1/6}B(\log B)^4.
\end{aligned}$$

Turning to the overall contribution from the second term in $R_2^a(\boldsymbol{\eta}; B)$, we again deduce from Lemma 5.2(2) that

$$\sup_{|t_7|\geqslant k_7/Y_7} g_1^a(Y_0, t_7; \boldsymbol{\eta}; B) \ll \sup_{|t_7|\geqslant k_7/Y_7} \frac{1}{Y_0|t_7|^{7/6}} \ll \frac{Y_7^{7/6}}{Y_0 k_7^{7/6}}.$$

Hence, in this case too, we obtain the overall contribution

$$\ll \sum_{\boldsymbol{\eta}} \frac{\phi^\dagger(\eta_2) Y_2 Y_6 Y_7^{7/6}}{\eta_5 Y_0} \ll \sum_{\eta_1,\eta_2,\eta_3,\eta_4} \frac{\phi^\dagger(\eta_2)\lambda^{1/6} B}{\boldsymbol{\eta}^{(1,1,1,1,0)}} \ll \lambda^{1/6} B (\log B)^4.$$

Thus far we have shown that $E_1(\lambda) \ll \lambda^{1/6} B (\log B)^4$.

It remains to produce a suitable upper bound for $E_2(\lambda)$. It will be convenient to record the estimates

$$\sum_{n\leqslant x} \frac{2^{\omega(n)}\phi^\dagger(n)}{n} \ll (\log x)^2, \quad \sum_{n>x} \frac{h^{\omega(n)}}{n^a} \ll x^{1-a}(\log x)^{h-1}.$$

The second inequality is valid for any $h \in \mathbb{Z}_{>0}$ and any $a > 1$, and follows on combining partial summation with the bound

$$\sum_{n\leqslant x} h^{\omega(n)} \leqslant \sum_{n\leqslant x} \sum_{n=d_1\dots d_h} 1 \ll \sum_{d_1,\dots,d_{h-1}\leqslant x} \frac{x}{d_1 \dots d_{h-1}} \ll x(\log x)^{h-1}.$$

Beginning with the first term in $R_2^a(\boldsymbol{\eta}; B)$, we deduce from Lemma 5.2(2) that

$$\int_{1/Y_7}^{k_7/Y_7} g_1^a(Y_0, t_7; \boldsymbol{\eta}; B)\,\mathrm{d}t_7 \ll \int_{1/Y_7}^{k_7/Y_7} \frac{1}{Y_0^8}\,\mathrm{d}t_7 \ll \frac{k_7}{Y_7 Y_0^8}.$$

Summing over $\boldsymbol{\eta}$ such that (5.19) holds, we therefore obtain the overall contribution

$$\begin{aligned}
&\ll \sum_{\boldsymbol{\eta}} \sum_{k_7|\eta_1\eta_2\eta_3\eta_4\eta_5} \frac{|\mu(k_7)|}{k_7} \frac{\phi^\dagger(\eta_2) k_7 Y_2 Y_6}{\eta_5 Y_0^8} \\
&\ll \sum_{\boldsymbol{\eta}} \frac{2^{\omega(\eta_1\eta_2\eta_3\eta_4\eta_5)}\phi^\dagger(\eta_2) B^2}{\boldsymbol{\eta}^{(4,7,5,3,6)}} \\
&\ll \sum_{\eta_1,\dots,\eta_4} \frac{2^{\omega(\eta_1\eta_2\eta_3\eta_4)}\phi^\dagger(\eta_2) B \log B}{\lambda \boldsymbol{\eta}^{(1,1,1,1,0)}} \\
&\ll \lambda^{-1} B (\log B)^9,
\end{aligned}$$

by (3.1). Similarly, for the contribution from the second term in $R_2^a(\boldsymbol{\eta}; B)$, we may use Lemma 5.2(2) to deduce the overall contribution

$$\ll \sum_{\boldsymbol{\eta}} \frac{2^{\omega(\eta_1\eta_2\eta_3\eta_4\eta_5)}\phi^\dagger(\eta_2) Y_2 Y_6}{\eta_5 Y_0^8} \ll \lambda^{-1} B (\log B)^9.$$

Taken together this shows that $E_2(\lambda) \ll \lambda^{-1}B(\log B)^9$. We choose $\lambda = (\log B)^{30/7}$, which therefore gives the overall contribution

$$\sum_{\boldsymbol{\eta}} R_2^a(\boldsymbol{\eta}; B) \ll B(\log B)^{33/7},$$

as required. □

5.4. Estimating $N_b(B)$ — second step

We must now return to the main term in Lemma 5.3, but this time reverse the order of summation for η_6 and η_7. This will allow us to make use of the inequality (5.7) in our treatment of the error terms. We begin with the summation over η_7. For fixed $\boldsymbol{\eta}, \eta_6$ satisfying the coprimality conditions (4.6) and (4.8), define $N_1^b := N_1^b(\boldsymbol{\eta}, \eta_6; B)$ to be the sum of the main term in Lemma 5.3 over all $\eta_7 \in \mathbb{Z}_{\neq 0}$ such that the coprimality condition (4.7) holds, and furthermore, $|\eta_7| > \eta_6 = \max\{\eta_6, 1\}$.

Our argument is very similar in spirit to the preceding section. Removing (4.7) with an application of Möbius inversion, we find that

$$\begin{aligned} N_1^b = &\frac{Y_2}{\eta_5} \frac{\phi^*(\eta_6)\phi^*(\eta_2\eta_3\eta_4)}{\phi^*(\gcd(\eta_6, \eta_4))} \sum_{\substack{k_{12} \mid \eta_2 \\ \gcd(k_{12}, \eta_1\eta_3\eta_4)=1}} \frac{\mu(k_{12})}{k_{12}\phi^*(\gcd(\eta_2, k_{12}\eta_5))} \\ &\times \sum_{\substack{1 \leqslant \varrho \leqslant k_{12}\eta_5 \\ \gcd(\varrho, k_{12}\eta_5)=1}} \sum_{\substack{k_7 \mid \eta_1\eta_2\eta_3\eta_4\eta_5 \\ \gcd(k_7, k_{12}\eta_5)=1}} \mu(k_7) A, \end{aligned}$$

where

$$A = \sum_{\substack{\eta_7' \in \mathbb{Z}_{\neq 0} \\ \varrho^2\eta_1 \equiv -\eta_3\eta_6 k_7 \eta_7' \pmod{k_{12}\eta_5} \\ k_7|\eta_7'| > \eta_6}} g_0(Y_0, \eta_6/Y_6, k_7\eta_7'/Y_7),$$

and we have written $\eta_7 = k_7\eta_7'$. Note that we have been able to add the constraint $\gcd(k_7, k_{12}\eta_5) = 1$ in the sum over k_7, since $A = 0$ otherwise.

Since $\gcd(\eta_3\eta_6k_7, k_{12}\eta_5) = 1$, it follows from an easy application of partial summation that

$$A = \frac{Y_7}{k_{12}k_7\eta_5} g_1^b(Y_0, \eta_6/Y_6; \boldsymbol{\eta}; B) + O\Big(\sup_{t_7} g_0(Y_0, \eta_6/Y_6, t_7)\Big),$$

where the supremum is over $t_7 \in \mathbb{R}$ such that $|t_7| > \eta_6/Y_7$. We may now establish the following result.

Lemma 5.6. — *We have*

$$N_1^b = \frac{Y_2Y_7}{\eta_5} g_1^b(Y_0, \eta_6/Y_6; \boldsymbol{\eta}; B)\vartheta_1^b(\boldsymbol{\eta})\frac{\phi^*(\eta_6)}{\phi^*(\gcd(\eta_6,\eta_4))} + O(R_1^b(\boldsymbol{\eta},\eta_6;B))$$

with

$$\vartheta_1^b(\boldsymbol{\eta}) := \phi^*(\eta_2\eta_3\eta_4\eta_5)\phi^*(\eta_1\eta_2\eta_3\eta_4) \sum_{\substack{k_{12}\mid\eta_2\\ \gcd(k_{12},\eta_1\eta_3\eta_4)=1}} \frac{\mu(k_{12})}{k_{12}\phi^*(\gcd(\eta_2,k_{12}\eta_5))},$$

and

$$\sum_{\boldsymbol{\eta},\eta_6} R_1^b(\boldsymbol{\eta},\eta_6;B) \ll B(\log B)^3.$$

Proof. — It is clear that the main term in the lemma is valid with

$$\begin{aligned}
\vartheta_1^b &= \sum_{\substack{k_{12}\mid\eta_2\\ \gcd(k_{12},\eta_1\eta_3\eta_4)=1}} \frac{\mu(k_{12})\phi^*(\eta_2\eta_3\eta_4)\phi^*(k_{12}\eta_5)}{k_{12}\phi^*(\gcd(\eta_2,k_{12}\eta_5))}\frac{\phi^*(\eta_1\eta_2\eta_3\eta_4)}{\phi^*(\gcd(\eta_1\eta_2\eta_3\eta_4,k_{12}\eta_5))}\\
&= \sum_{\substack{k_{12}\mid\eta_2\\ \gcd(k_{12},\eta_1\eta_3\eta_4)=1}} \frac{\mu(k_{12})}{k_{12}}\phi^*(k_{12}\eta_2\eta_3\eta_4\eta_5)\frac{\phi^*(\eta_1\eta_2\eta_3\eta_4)}{\phi^*(\gcd(\eta_2,k_{12}\eta_5))}\\
&= \phi^*(\eta_2\eta_3\eta_4\eta_5)\phi^*(\eta_1\eta_2\eta_3\eta_4) \sum_{\substack{k_{12}\mid\eta_2\\ \gcd(k_{12},\eta_1\eta_3\eta_4)=1}} \frac{\mu(k_{12})}{k_{12}\phi^*(\gcd(\eta_2,k_{12}\eta_5))},
\end{aligned}$$

as claimed. We have used here the fact that $\gcd(k_{12}\eta_5,\eta_1\eta_3\eta_4)=1$.

For the error term, we deduce from (4.9) that $\boldsymbol{\eta}^{(2,4,3,2,3)}\eta_6 \leqslant B$, for the $\boldsymbol{\eta},\eta_6$ that we need to sum $R_1^b(\boldsymbol{\eta},\eta_6;B)$ over. Using Lemma 5.2(1) to bound g_0, we easily deduce that

$$\begin{aligned}
\sum_{\boldsymbol{\eta},\eta_6} R_1^b(\boldsymbol{\eta},\eta_6;B) &\ll \sum_{\boldsymbol{\eta},\eta_6} Y_2 2^{\omega(\eta_2)+\omega(\eta_1\eta_2\eta_3\eta_4)} \sup_{|t_7|>\eta_6/Y_7} g_0(Y_0,\eta_6/Y_6,t_7)\\
&\ll \sum_{\boldsymbol{\eta},\eta_6} \frac{2^{\omega(\eta_2)+\omega(\eta_1\eta_2\eta_3\eta_4)}Y_2Y_7^{1/2}}{Y_0\eta_6^{1/2}}\\
&= \sum_{\boldsymbol{\eta},\eta_6} \frac{2^{\omega(\eta_2)+\omega(\eta_1\eta_2\eta_3\eta_4)}B^{1/2}}{\boldsymbol{\eta}^{(1/2,0,0,0,-1/2)}\eta_6^{1/2}}\\
&\ll \sum_{\boldsymbol{\eta}} \frac{2^{\omega(\eta_2)+\omega(\eta_1\eta_2\eta_3\eta_4)}B}{\boldsymbol{\eta}^{(3/2,2,3/2,1,1)}}\\
&\ll B\sum_{\eta_4,\eta_5} \frac{2^{\omega(\eta_4)}}{\eta_4\eta_5}.
\end{aligned}$$

But this is $O(B(\log B)^3)$, as required. This completes the proof of the lemma. $\square$

We must now sum the main term in Lemma 5.6 over all of the relevant η_6, and then over $\eta_1,\dots,\eta_5$. In doing so it will be convenient distinguish between values of $\boldsymbol{\eta},\eta_6$ such that

$$\boldsymbol{\eta}^{(2,4,3,2,3)} \leqslant \frac{B}{(\log B)^A}, \tag{5.20}$$

for some $A>0$, and those for which this inequality does not hold. We write $N_{b_1}(B;A)$ and $N_{b_2}(B;A)$ for the corresponding contributions. The following result shows that $N_{b_2}(B;A)$ makes a negligible contribution to $N_{U,H}(B)$.

Lemma 5.7. — *We have $N_{b_2}(B;A) \ll_A B(\log B)^4(\log\log B)$.*

Proof. — Once taken in conjunction with the inequalities for $\boldsymbol{\eta},\eta_6$ in (4.9), the failure of (5.20) clearly implies that we must sum over $\boldsymbol{\eta},\eta_6$ for which

$$\eta_1^2\eta_2^4\eta_3^3\eta_4^2\eta_5^3\eta_6 \leqslant B, \quad \eta_6 < (\log B)^A. \tag{5.21}$$

Recalling the definition of the main term from Lemma 5.6 and using $\vartheta_1^b(\boldsymbol{\eta}) \ll \phi^\dagger(\eta_2)$, we see that

$$\begin{aligned} N_{b_2}(B;A) &\ll \sum_{\substack{\boldsymbol{\eta},\eta_6\\ (5.21)\text{ holds}}} \frac{Y_2Y_7}{\eta_5} g_1^b(Y_0,\eta_6/Y_6;\boldsymbol{\eta};B)\vartheta_1^b(\boldsymbol{\eta})\frac{\phi^*(\eta_6)}{\phi^*(\gcd(\eta_6,\eta_4))}\\ &\ll \sum_{\substack{\boldsymbol{\eta},\eta_6\\ (5.21)\text{ holds}}} \frac{Y_2Y_6^{3/4}Y_7\phi^\dagger(\eta_2)}{Y_0\eta_5\eta_6^{3/4}}, \end{aligned}$$

using Lemma 5.2(3). In view of the definitions of the Y_i we conclude that

$$\begin{aligned} N_{b_2}(B;A) &\ll B^{3/4} \sum_{\substack{\boldsymbol{\eta},\eta_6\\ (5.21)\text{ holds}}} \frac{\phi^\dagger(\eta_2)}{\boldsymbol{\eta}^{(1/2,0,1/4,1/2,1/4)}\eta_6^{3/4}}\\ &\ll B \sum_{\substack{\eta_2,\dots,\eta_6\\ (5.21)\text{ holds}}} \frac{\phi^\dagger(\eta_2)}{\boldsymbol{\eta}^{(0,1,1,1,1)}\eta_6}. \end{aligned}$$

This last expression is clearly satisfactory for the lemma by (3.1) with $j=1$ and the fact that the η_6 summation is over $\eta_6<(\log B)^A$. $\square$

Our focus now shifts to estimating $N_{b_1}(B;A)$, deemed to be the overall contribution from the main term in Lemma 5.6 that arises from $\boldsymbol{\eta},\eta_6$ for which (5.20) holds. For the moment let $N_2^b := N_2^b(\boldsymbol{\eta}; B)$ be the quantity obtained by summing the main term in Lemma 5.6's estimate for $N_1^b(\boldsymbol{\eta}, \eta_6; B)$, over all $\eta_6 \in \mathbb{Z}_{>0}$ such that (4.6) holds. An application of Lemma 3.2 with $\alpha = 0$ and $q = 1$ therefore reveals that

$$\begin{aligned}
N_2^b =& \frac{Y_2Y_7}{\eta_5}\vartheta_1^b(\boldsymbol{\eta}) \sum_{\eta_6 \geqslant 1} f_{\eta_4,\eta_1\eta_2\eta_3\eta_5}(\eta_6) g_1^b(Y_0, \eta_6/Y_6; \boldsymbol{\eta}; B)\\
=& \frac{Y_2Y_6Y_7}{\eta_5} g_2^b(Y_0; \boldsymbol{\eta}; B)\vartheta_1^b(\boldsymbol{\eta}) \frac{\phi^*(\eta_1\eta_2\eta_3\eta_5)}{\zeta(2)} \prod_{p \mid \eta_1\eta_2\eta_3\eta_4\eta_5} \left(1 - \frac{1}{p^2}\right)^{-1}\\
&+ O\left(\frac{Y_2Y_7}{\eta_5} |\vartheta_1^b(\boldsymbol{\eta})| (\log B) 2^{\omega(\eta_1\eta_2\eta_3\eta_5)} \sup_{t_6} g_1^b(Y_0, t_6; \boldsymbol{\eta}; B)\right)\\
&+ O\left(\frac{Y_2Y_6Y_7}{\eta_5} |\vartheta_1^b(\boldsymbol{\eta})| \int_{0<t_6<1/Y_6} g_1^b(Y_0, t_6; \boldsymbol{\eta}; B)\, \mathrm{d}t_6\right),
\end{aligned}$$

where the supremum is over all $t_6 \geqslant 1/Y_6$. The following result is now straightforward.

Lemma 5.8. — *We have*

$$N_2^b = \frac{Y_2Y_6Y_7}{\eta_5} g_2^b(Y_0; \boldsymbol{\eta}; B)\vartheta_2^b(\boldsymbol{\eta}) + O(R_2^b(\boldsymbol{\eta}; B))$$

with

$$\vartheta_2^b(\boldsymbol{\eta}) := \vartheta_1^b(\boldsymbol{\eta}) \frac{\phi^*(\eta_1\eta_2\eta_3\eta_5)}{\zeta(2)} \prod_{p \mid \eta_1\eta_2\eta_3\eta_4\eta_5} \left(1 - \frac{1}{p^2}\right)^{-1},$$

and

$$\sum_{\substack{\boldsymbol{\eta} \\ (5.20)\ \text{holds}}} R_2^b(\boldsymbol{\eta}; B) \ll B(\log B)^{9-A/4}.$$

Proof. — The value of $\vartheta_2^b(\boldsymbol{\eta})$ in the main term for N_2^b is a direct consequence of our manipulations above. In considering the overall contribution from the error term it will be convenient to note that $\vartheta_2^b(\boldsymbol{\eta}) \ll \vartheta_1^b(\boldsymbol{\eta}) \ll \phi^\dagger(\eta_2)$.

Once again the error $R_2^b(\boldsymbol{\eta}; B)$ is comprised of two basic terms, the first one involving a supremum of g_1^b over t_6 in an appropriate range, and the second involving an integration of g_1^b. We begin with dealing with the first term. It is here that we will make critical use of the inequality (5.20), that underpins our definition of $N_{b_1}(B; A)$. The first term in $R_2^b(\boldsymbol{\eta}; B)$ clearly makes an overall contribution of

$$\ll \sum_{\boldsymbol{\eta}} \frac{2^{\omega(\eta_1\eta_2\eta_3\eta_5)}\phi^\dagger(\eta_2) Y_2Y_7 \log B}{\eta_5} \sup_{t_6 \geqslant 1/Y_6} g_1^b(Y_0, t_6; \boldsymbol{\eta}; B),$$

where the summation is restricted to $\boldsymbol{\eta}$ for which (5.20) holds. Using Lemma 5.2(3) to estimate g_1^b, we may bound this as

$$\begin{aligned}
&\ll \sum_{\boldsymbol{\eta}} \frac{2^{\omega(\eta_1\eta_2\eta_3\eta_5)}\phi^\dagger(\eta_2)Y_2Y_7Y_6^{3/4}\log B}{\eta_5 Y_0} \\
&= \sum_{\boldsymbol{\eta}} \frac{2^{\omega(\eta_1\eta_2\eta_3\eta_5)}\phi^\dagger(\eta_2)B^{3/4}\log B}{\boldsymbol{\eta}^{(1/2,0,1/4,1/2,1/4)}} \\
&\ll (\log B)^{1-A/4} \sum_{\eta_1,\eta_2,\eta_3,\eta_5} \frac{2^{\omega(\eta_1\eta_2\eta_3\eta_5)}\phi^\dagger(\eta_2)B}{\boldsymbol{\eta}^{(1,1,1,0,1)}} \\
&\ll B(\log B)^{9-A/4}.
\end{aligned}$$

Turning to the contribution from the second term in $R_2^b(\boldsymbol{\eta};B)$, we employ Lemma 5.2(3) and (5.20) to derive the overall contribution

$$\begin{aligned}
\ll \sum_{\boldsymbol{\eta}} \frac{\phi^\dagger(\eta_2)Y_2Y_6Y_7}{\eta_5}\int_0^{1/Y_6}\frac{1}{Y_0t_6^{3/4}}\,\mathrm{d}t_6 &\ll \sum_{\boldsymbol{\eta}} \frac{\phi^\dagger(\eta_2)Y_2Y_6^{3/4}Y_7}{\eta_5Y_0} \\
&= \sum_{\boldsymbol{\eta}} \frac{\phi^\dagger(\eta_2)B^{3/4}}{\boldsymbol{\eta}^{(1/2,0,1/4,1/2,1/4)}} \\
&\ll (\log B)^{-A/4} \sum_{\eta_1,\eta_2,\eta_3,\eta_5} \frac{\phi^\dagger(\eta_2)B}{\boldsymbol{\eta}^{(1,1,1,0,1)}} \\
&\ll B(\log B)^{4-A/4}.
\end{aligned}$$

Together these two upper bounds complete the proof of the lemma. □

5.5. The final step

Let us take a moment to compile our work so far. We saw at the start of § 5 that

$$N_{U,H}(B) = N_a(B) + N_b(B).$$

It will be convenient to set $B_0 = B/(\log B)^{36}$ in what follows.

The union of Lemmas 5.3, 5.4 and 5.5 shows that

$$N_a(B) = \sum_{\boldsymbol{\eta}\in\mathcal{E}(B)} \frac{Y_2Y_6Y_7}{\eta_5}\vartheta_2^a(\boldsymbol{\eta})g_2^a(Y_0;\boldsymbol{\eta};B) + O\big(B(\log B)^{5-2/7}\big),$$

where $\vartheta_2^a(\boldsymbol{\eta})$ is as in the statement of Lemma 5.5, and

$$\mathcal{E}(B) := \big\{\boldsymbol{\eta}\in\mathbb{Z}_{>0}^5 : (4.8) \text{ holds and } \boldsymbol{\eta}^{(2,4,3,2,3)} \leqslant B\big\}.$$

Similarly, we can combine Lemmas 5.3, 5.6, 5.7 and 5.8, taking $A = 36$ in the latter two results, to deduce that

$$N_b(B) = \sum_{\substack{\boldsymbol{\eta}\in\mathcal{E}(B)\\ \boldsymbol{\eta}^{(2,4,3,2,3)}\leqslant B_0}} \frac{Y_2Y_6Y_7}{\eta_5}\vartheta_2^b(\boldsymbol{\eta})g_2^b(Y_0;\boldsymbol{\eta};B) + O\big(B(\log B)^4(\log\log B)\big),$$

where $\vartheta_2^b(\boldsymbol{\eta})$ is as in the statement of Lemma 5.8.

We would now like to remove the constraint that $\boldsymbol{\eta}^{(2,4,3,2,3)} \leqslant B_0$ in our estimate for $N_b(B)$. In view of the fact that $\vartheta_2^b(\boldsymbol{\eta}) \ll \phi^\dagger(\eta_2)$, it easily follows from (5.13) and Lemma 5.2(3) that

$$\begin{aligned}\sum_{\substack{\boldsymbol{\eta}\in\mathcal{E}(B)\\ B_0<\boldsymbol{\eta}^{(2,4,3,2,3)}\leqslant B}} \frac{Y_2Y_6Y_7}{\eta_5}\vartheta_2^b(\boldsymbol{\eta})g_2^b(Y_0;\boldsymbol{\eta};B) &\ll \sum_{\boldsymbol{\eta}} \frac{Y_2Y_6Y_7\phi^\dagger(\eta_2)}{\eta_5}\int_0^{1/Y_0^4}\frac{\mathrm{d}t_6}{Y_0t_6^{3/4}}\\ &\ll \sum_{\boldsymbol{\eta}}\frac{Y_2Y_6^3Y_7\phi^\dagger(\eta_2)}{\eta_5}\\ &= \sum_{\boldsymbol{\eta}}\frac{B\phi^\dagger(\eta_2)}{\boldsymbol{\eta}^{(1,1,1,1,1)}}\\ &\ll B(\log B)^4(\log\log B).\end{aligned}$$

In deducing the first bound we have used the fact that $g_1^b(t_0,t_6;\boldsymbol{\eta};B) = 0$ unless $0 < t_6 \leqslant 1/t_0^4$, which follows from the definition of (5.8). Thus we may replace the above formula for $N_b(B)$ by

$$\sum_{\boldsymbol{\eta}\in\mathcal{E}(B)} \frac{Y_2Y_6Y_7}{\eta_5}\vartheta_2^b(\boldsymbol{\eta})g_2^b(Y_0;\boldsymbol{\eta};B) + O\big(B(\log B)^4(\log\log B)\big).$$

For given $\boldsymbol{\eta}\in\mathcal{E}(B)$, define

$$\begin{aligned}\vartheta(\boldsymbol{\eta}) :=& \frac{\phi^*(\eta_1\eta_2\eta_3)\phi^*(\eta_1\eta_2\eta_3\eta_4\eta_5)\phi^*(\eta_2\eta_3\eta_4\eta_5)}{\zeta(2)\prod_{p\mid\eta_1\eta_2\eta_3\eta_4\eta_5}(1-1/p^2)}\\ &\times \sum_{\substack{k_{12}\mid\eta_2\\ \gcd(k_{12},\eta_1\eta_3\eta_4)=1}} \frac{\mu(k_{12})}{k_{12}\phi^*(\gcd(\eta_2,k_{12}\eta_5))}.\end{aligned}$$

It is easily seen that $\vartheta(\boldsymbol{\eta}) = \vartheta_2^a(\boldsymbol{\eta})$, in the notation of Lemmas 5.4 and 5.5. Furthermore, on noting that

$$\phi^*(\eta_1\eta_2\eta_3)\phi^*(\eta_1\eta_2\eta_3\eta_4\eta_5) = \phi^*(\eta_1\eta_2\eta_3\eta_4)\phi^*(\eta_1\eta_2\eta_3\eta_5),$$

since $\gcd(\eta_4, \eta_5) = 1$, we see that $\vartheta(\boldsymbol{\eta}) = \vartheta_2^b(\boldsymbol{\eta})$ also. Thus we may draw together our argument so far to conclude that

$$N_{U,H}(B) = \sum_{\boldsymbol{\eta}\in\mathcal{E}(B)} \frac{Y_2Y_6Y_7}{\eta_5}\vartheta(\boldsymbol{\eta})g_2(Y_0;\boldsymbol{\eta};B) + O\big(B(\log B)^{5-2/7}\big),$$

where $g_2(t_0;\boldsymbol{\eta};B)$ is given by (5.14). It turns out that there is a negligible contribution to $N_{U,H}(B)$ from summing $Y_2Y_6Y_7\vartheta(\boldsymbol{\eta})g_2(t_0;\boldsymbol{\eta};B)/\eta_5$ over small values of $\boldsymbol{\eta} \in \mathcal{E}(B)$. The $\boldsymbol{\eta}$ that give the dominant contribution belong to the set

$$\mathcal{E}^*(B) := \big\{\boldsymbol{\eta} \in \mathbb{Z}_{>0}^5 : (4.8) \text{ holds}, \ \boldsymbol{\eta}^{(2,4,3,2,3)} \leqslant B \text{ and } \boldsymbol{\eta}^{(3,6,4,2,5)} > B\big\}.$$

We also wish to remove the dependence on $\boldsymbol{\eta}$ and B from the real-valued function $g_2(Y_0;\boldsymbol{\eta};B)$. All of this will be achieved in the following result.

Lemma 5.9. — *We have*

$$N_{U,H}(B) = \omega_\infty B \sum_{\boldsymbol{\eta}\in\mathcal{E}^*(B)} \frac{\vartheta(\boldsymbol{\eta})}{\boldsymbol{\eta}^{(1,1,1,1,1)}} + O\big(B(\log B)^{5-2/7}\big),$$

where ω_∞ is given by (1.2).

Proof. — We begin by showing that

$$M_1(B) := \sum_{\substack{\boldsymbol{\eta}\in\mathcal{E}(B)\\ \boldsymbol{\eta}^{(3,6,4,2,5)}\leqslant B}} \frac{Y_2Y_6Y_7}{\eta_5}\vartheta(\boldsymbol{\eta})g_2(Y_0;\boldsymbol{\eta};B) \ll B(\log B)^4.$$

Now it follows from (5.9) and (5.14) that

$$\begin{aligned} g_2(t_0;\boldsymbol{\eta};B) &= \int_{h(t_0,t_2,t_6,t_7)\leqslant 1, |Y_7t_7|>1, t_6>0} \mathrm{d}t_2\,\mathrm{d}t_6\,\mathrm{d}t_7 \\ &\leqslant \int_{|t_7|>1/Y_7}\int_0^\infty g_0(t_0,t_6,t_7)\,\mathrm{d}t_6\,\mathrm{d}t_7. \end{aligned}$$

Hence Lemma 5.2(2) yields

$$g_2(Y_0;\boldsymbol{\eta};B) \ll \int_{|t_7|>1/Y_7} \frac{1}{Y_0|t_7|^{7/6}}\,\mathrm{d}t_7 \ll \frac{Y_7^{1/6}}{Y_0}.$$

Applying this we deduce that

$$\begin{aligned} M_1(B) \ll \sum_{\boldsymbol{\eta}^{(3,6,4,2,5)}\leqslant B} \frac{\phi^\dagger(\eta_2)Y_2Y_6Y_7^{7/6}}{\eta_5Y_0} &\ll \sum_{\boldsymbol{\eta}^{(3,6,4,2,5)}\leqslant B} \frac{\phi^\dagger(\eta_2)B^{5/6}}{\boldsymbol{\eta}^{(1/2,0,1/3,2/3,1/6)}} \\ &\ll \sum_{\eta_1,\eta_2,\eta_3,\eta_4} \frac{\phi^\dagger(\eta_2)B}{\boldsymbol{\eta}^{(1,1,1,1,0)}} \\ &\ll B(\log B)^4, \end{aligned}$$

by (3.1), which therefore shows that

$$N_{U,H}(B) = \sum_{\boldsymbol{\eta}\in\mathcal{E}^*(B)} \frac{Y_2Y_6Y_7}{\eta_5}\vartheta(\boldsymbol{\eta})g_2(Y_0;\boldsymbol{\eta};B) + O\big(B(\log B)^{5-2/7}\big).$$

It remains to deal with the real-valued function $g_2(Y_0;\boldsymbol{\eta};B)$.

We will show that

$$M_2(B) := \sum_{\boldsymbol{\eta}\in\mathcal{E}^*(B)} \frac{Y_2Y_6Y_7}{\eta_5}\vartheta(\boldsymbol{\eta}) \int_{\substack{h(Y_0,t_2,t_6,t_7)\leqslant 1\\ |Y_7t_7|\leqslant 1, t_6>0}} \mathrm{d}t_2\,\mathrm{d}t_6\,\mathrm{d}t_7 \ll B(\log B)^4.$$

Once achieved, this will suffice to complete the proof of the lemma, since an application of Lemma 5.1 reveals that

$$\int_{h(Y_0,t_2,t_6,t_7)\leqslant 1, t_6>0} \mathrm{d}t_2\,\mathrm{d}t_6\,\mathrm{d}t_7 = \frac{\omega_\infty}{Y_0^2},$$

and clearly

$$\frac{Y_2Y_6Y_7}{Y_0^2\eta_5} = \frac{B}{\boldsymbol{\eta}^{(1,1,1,1,1)}}.$$

To establish the bound for $M_2(B)$ we appeal to Lemma 5.2(2), which in a similar manner to our treatment of $M_1(B)$, implies that

$$\begin{aligned} M_2(B) \ll \sum_{\boldsymbol{\eta}\in\mathcal{E}^*(B)} \frac{Y_2Y_6Y_7\phi^\dagger(\eta_2)}{\eta_5}\int_{|t_7|\leqslant 1/Y_7}\frac{\mathrm{d}t_7}{Y_0^8} &\ll \sum_{\boldsymbol{\eta}\in\mathcal{E}^*(B)} \frac{Y_2Y_6\phi^\dagger(\eta_2)}{Y_0^8\eta_5}\\ &= \sum_{\boldsymbol{\eta}\in\mathcal{E}^*(B)} \frac{\phi^\dagger(\eta_2)B^2}{\boldsymbol{\eta}^{(4,7,5,3,6)}}\\ &\ll \sum_{\eta_1,\eta_2,\eta_3,\eta_4} \frac{\phi^\dagger(\eta_2)B}{\boldsymbol{\eta}^{(1,1,1,1,0)}}\\ &\ll B(\log B)^4. \end{aligned}$$

This completes the proof of the lemma. ◻

Let us redefine the function $\vartheta(\boldsymbol{\eta})$ so that it is equal to zero if $\boldsymbol{\eta}$ fails to satisfy the coprimality relations in (4.8). For $\mathbf{k} = (k_1,\dots,k_5) \in \mathbb{Z}_{>0}^5$, let

$$\Delta_{\mathbf{k}}(n) := \sum_{\substack{\boldsymbol{\eta}\in\mathbb{Z}_{>0}^5\\ \boldsymbol{\eta}^{(k_1,k_2,k_3,k_4,k_5)}=n}} \frac{\vartheta(\boldsymbol{\eta})}{\boldsymbol{\eta}^{(1,1,1,1,1)}}.$$

Then Lemma 5.9 implies that

(5.22)

$$N_{U,H}(B) = \omega_\infty B\sum_{n\leqslant B}\big(\Delta_{(2,4,3,2,3)}(n) - \Delta_{(3,6,4,2,5)}(n)\big) + O\big(B(\log B)^{5-2/7}\big).$$

We will want to establish an asymptotic formula for

$$M_{\mathbf{k}}(t) := \sum_{n \leqslant t} \Delta_{\mathbf{k}}(n),$$

as $t \to \infty$. We shall do so by studying the corresponding Dirichlet series

$$F_{\mathbf{k}}(s) = \sum_{n=1}^{\infty} \frac{\Delta_{\mathbf{k}}(n)}{n^s} = \sum_{\boldsymbol{\eta} \in \mathbb{Z}_{>0}^5} \frac{\vartheta(\boldsymbol{\eta})}{\boldsymbol{\eta}^{(k_1 s+1, k_2 s+1, k_3 s+1, k_4 s+1, k_5 s+1)}},$$

which is absolutely convergent for $\Re e(s) > 0$.

By multiplicativity we clearly have an Euler product $F_{\mathbf{k}}(s) = \prod_p F_{\mathbf{k},p}(s)$, and a cumbersome computation reveals that the local factors $F_{\mathbf{k},p}(s)$ are equal to

$$\begin{aligned}
&(1-1/p) \cdot \Bigg((1+1/p) + \frac{1-1/p}{p^{k_1 s+1}-1} \\
&+ \frac{1-1/p}{p^{k_2 s+1}-1} \left((1-2/p) + \frac{1-1/p}{p^{k_1 s+1}-1} + \frac{1-1/p}{p^{k_3 s+1}-1} + \frac{1-1/p}{p^{k_5 s+1}-1} \right) \\
&+ \frac{(1-1/p)^2}{p^{k_3 s+1}-1} \left(1 + \frac{1}{p^{k_4 s+1}-1} \right) + \frac{1-1/p}{p^{k_4 s+1}-1} + \frac{1-1/p}{p^{k_5 s+1}-1} \Bigg).
\end{aligned}$$

Let $\varepsilon > 0$ and assume that $\mathbf{k} \in \{(2,4,3,2,3), (3,6,4,2,5)\}$. Then it follows that for all $s \in \mathbb{C}$ belonging to the half-plane $\Re e(s) \geqslant -1/12 + \varepsilon$, we have

$$F_{\mathbf{k},p}(s) \prod_{j=1}^{5} \left(1 - \frac{1}{p^{k_j s+1}} \right) = 1 + O_\varepsilon(p^{-1-\varepsilon}).$$

Thus, on defining

$$E_{\mathbf{k}}(s) := \prod_{j=1}^{5} \zeta(k_j s + 1), \quad G_{\mathbf{k}}(s) := \frac{F_{\mathbf{k}}(s)}{E_{\mathbf{k}}(s)},$$

we may conclude that $F_{\mathbf{k}}(s)$ has a meromorphic continuation to the half-plane $\Re e(s) \geqslant -1/12 + \varepsilon$, with a pole of order 5 at $s = 0$. It will be useful to note that

$$G_{\mathbf{k}}(0) = \prod_p \left(1 - \frac{1}{p} \right)^6 \left(1 + \frac{6}{p} + \frac{1}{p^2} \right). \tag{5.23}$$

To estimate $M_{\mathbf{k}}(t)$ we now have everything in place to apply the following standard Tauberian theorem, which is recorded in work of Chambert-Loir and Tschinkel [4, Appendice A].

Lemma 5.10. — *Let $\{c_n\}_{n \in \mathbb{Z}_{>0}}$ be a sequence of positive real numbers, and let $f(s) = \sum_{n=1}^{\infty} c_n n^{-s}$. Assume that:*

(1) *the series defining $f(s)$ converges for $\Re e(s) > 0$;*

(2) *it admits a meromorphic continuation to* $\Re e(s) > -\delta$ *for some* $\delta > 0$*, with a unique pole at* $s = 0$ *of order* $b \in \mathbb{Z}_{>0}$*;*
(3) *there exists* $\kappa > 0$ *such that*

$$\left|\frac{f(s)s^b}{(s+2\delta)^b}\right| \ll (1 + \Im m(s))^\kappa,$$

for $\Re e(s) > -\delta$*.*

Then there exists a monic polynomial P *of degree* b*, and a constant* $\delta' > 0$ *such that*

$$\sum_{n \leqslant t} c_n = \frac{\Theta}{b!} P(\log t) + O(t^{-\delta'}),$$

as $t \to \infty$*, where* $\Theta = \lim_{s \to 0} s^b f(s)$*.*

In fact [4, Appendice A] deals only with Dirichlet series possessing a unique pole at $s = a > 0$, but the extension to a pole at $s = 0$ is straightforward. We apply Lemma 5.10 to estimate $M_{\mathbf{k}}(t)$, for

$$\mathbf{k} \in \{(2,4,3,2,3), (3,6,4,2,5)\}.$$

We have already seen that the corresponding Dirichlet series $F_{\mathbf{k}}(s)$ satisfies parts (1) and (2) of the lemma, with $b = 5$. The third part follows from the boundedness of $G_{\mathbf{k}}(s)$ on the half-plane $\Re e(s) \geqslant -1/12 + \varepsilon$, and standard upper bounds for the size of the Riemann zeta function in the critical strip. In view of the fact that

$$\lim_{s \to 0} s^b F(s) = \frac{G_{\mathbf{k}}(0)}{\prod_{j=1}^5 k_j},$$

we therefore conclude that

$$M_{\mathbf{k}}(t) = \frac{G_{\mathbf{k}}(0) P(\log t)}{5! \cdot \prod_{j=1}^5 k_j} + O(t^{-\delta}), \tag{5.24}$$

for some $\delta > 0$ and some monic polynomial P of degree 5.

We are now ready to complete the proof of our theorem. Recall the definition (2.1) of $\omega_H(\widetilde{S})$. It therefore follows on combining (1.8), (5.22), (5.23) and (5.24) that

$$N_{U,H}(B) = \alpha(\widetilde{S})\omega_H(\widetilde{S}) B(\log B)^5 + O\big(B(\log B)^{5-2/7}\big),$$

as required.

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Manuscrit reçu le 21 septembre 2007,
accepté le 24 octobre 2008.

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