ANNALES

DE

L'INSTITUT FOURIER

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Tome 59, nº 2 (2009), p. 595-619.

<http://aif.cedram.org/item?id=AIF_2009__59_2_595_0>

Ann. Inst. Fourier, Grenoble
59, 2 (2009) 595-619

A NEW PROOF OF OKAJI'S THEOREM FOR A CLASS OF SUM OF SQUARES OPERATORS

by Paulo D. CORDARO & Nicholas HANGES (*)

ABSTRACT. — Let P be a linear partial differential operator with analytic coefficients. We assume that P is of the form "sum of squares", satisfying Hörmander's bracket condition. Let q be a characteristic point for P. We assume that q lies on a symplectic Poisson stratum of codimension two. General results of Okaji show that P is analytic hypoelliptic at q. Hence Okaji has established the validity of Treves' conjecture in the codimension two case. Our goal here is to give a simple, self-contained proof of this fact.

RÉSUMÉ. — Soit P un opérateur différentiel analytique, de la forme "somme de carrés", avec la condition d'Hörmander réalisée. Soit q un point caractéristique de P. On suppose que q est un point d'un "symplectic Poisson stratum" de codimension deux (au sens de Treves). D'après le théorème d'Okaji, P est hypoelliptique analytique en q. Autrement dit, la conjecture de Treves est vraie en codimension deux. On donne dans ce travail une preuve élémentaire de ce fait.

1. Introduction

Let $\mathcal{M}$ be a real analytic manifold and let $X_0, \dots, X_\nu$ be real valued, real analytic vector fields on $\mathcal{M}$. We study an operator P of the form "sum of squares". That is P has the form

$$P = {X_0}^2 + \cdots + {X_\nu}^2. \tag{1.1}$$

DEFINITION 1.1. — *We say that P is analytic hypoelliptic (in the strong sense) on $\mathcal{M}$ if for every open $O \subset \mathcal{M}$ we have the following: Pu analytic on O implies that u is analytic on O. Here u is a distribution on O.*

Keywords: Analytic hypoelliptic, sum of squares.
Math. classification: 35H10, 35H20, 35A17, 35A20, 35A27.
(*) Both authors were partially suported by CNPq (Brazil). The first author was also partially suported by Fapesp, (Brazil) and the second author was also partially suported by NSF grant INT–0227100.

We always assume that the X_j satisfy Hörmander's bracket ("finite type") condition. That is, at each point of $\mathcal{M}$, the Lie algebra generated by the X_j (under the commutation bracket) has dimension equal to $\dim \mathcal{M}$.

Under these conditions a classical result of Hörmander [7] guarantees the hypoellipticity of P. However analytic hypoellipticity will not hold unless further assumptions are made.

If we assume that Σ, the characteristic set of P, is a symplectic manifold and that the principal symbol of P vanishes precisely to second order on Σ, then P is analytic hypoelliptic. This follows from results of Treves [16] and Tartakoff [15]. Further work in this direction was done by Métivier [10]. All this work assumes that the characteristic set is a symplectic manifold and that the principal symbol vanishes uniformly on the characteristic set.

Very general results on analytic hypoellipticity were obtained by Okaji [11] in the symplectic case, allowing non-uniform vanishing of the principal symbol.

However, there are examples when the characteristic set is symplectic, and analytic hypoellipticity fails. See Oleinik [12], Hanges – Himonas [6] and Cordaro – Hanges [3]. These examples motivated Treves to introduce the Poisson stratification of the characteristic set.

1.1. The Poisson Stratification of Σ

For simplification we assume that P is defined in an open subset Ω of $\mathbb{R}^N$ and let $iF_j, j = 0, \dots, \nu$ ($i = \sqrt{-1}$) denote the symbols of the X_j. The characteristic set of P is defined as

$$\Sigma = \{p \in T^*\Omega \setminus 0 : F_j(p) = 0, j = 0, \dots, \nu\}.$$

It is a theorem of Treves [17] that Σ can be decomposed in the following way:

(1) There exist connected, pairwise disjoint analytic submanifolds $\Sigma_j \subset \Sigma$ such that

$$\Sigma = \cup \Sigma_j.$$

Furthermore, the union is locally finite.

(2) For each j we have

$$T_p\Sigma_j \cap T_p\Sigma_j{}^{\perp}$$

has constant dimension at each $p \in \Sigma_j$ (here $T_p\Sigma_j{}^{\perp}$ denotes the orthogonal space with respect to the natural symplectic form on $T^*\Omega$).

(3) There exists, for each j, an integer n_j such that F_I vanishes on Σ_j for all $|I| < n_j$, but for each $p \in \Sigma_j$, there exists I with $|I| = n_j$ such that $F_I(p) \neq 0$. Note that if $I = (i_1, \dots, i_q)$, then $F_I = \{F_{i_1}, \dots, \{F_{i_{q-1}}, F_{i_q}\}, \dots\}$.

(4) Each Σ_j is maximal with respect to properties (2) and (3).

Each Σ_j is called a *Poisson stratum* of Σ.

1.2. Treves' Conjecture

Treves' conjecture [17] asserts that the following statement is true:
(S) *For P to be analytic hypoelliptic on Ω it is necessary and sufficient that every Poisson stratum of Σ be symplectic.*

Treves' conjecture is consistent with all known results. However, the analog of the conjecture is not true in the global sense or in the sense of germs. See Cordaro–Himonas [4] and Hanges [5]. Also, the contribution of Bove, Derridj, Tartakoff [1] is a very interesting generalization of [5]. Indeed, these papers have motivated Treves to give a more generalized conjecture, see [18]. For this we need to introduce the concept of bicharacteristic leaves.

On each stratum Σ_j the association $p \mapsto T_p\Sigma_j \cap T_p\Sigma_j^{\perp}$ defines a vector subbundle of $T\Sigma_j$ which is closed under Lie brackets. By the Frobenius theorem we can foliate Σ_j by leaves $\mathcal{L}$ such that $T\mathcal{L} = (T\Sigma_j \cap T\Sigma_j^{\perp})|_{\mathcal{L}}$. Such leaves $\mathcal{L}$ are called *bicharacteristic leaves of the operator P*.

The generalized Treves conjecture can be stated as follows:
(C) *For P to be analytic hypoelliptic on Ω it is necessary and sufficient that every bicharacteristic leaf is vertical and relatively compact in $T^*\Omega$.*

Note that here *vertical* means that the set is either empty or projects (under the canonical projection $\pi : T^*\Omega \to \Omega$) to a point.

1.3. The main result

The following theorem is our main result, which follows from the work of Okaji [11]. In particular this establishes Treves' conjecture (in the positive direction) in the codimension two case. Our goal here is to give a simple, self-contained proof of this.

THEOREM 1.2. — *Let $\Omega \subset \mathbb{R}^N$ be open and let $X_0, \dots, X_\nu$ be real valued, real analytic vector fields on Ω which satisfy Hörmander's condition. Let P have the form (1.1) with Σ the characteristic set of P. Let $p \in \Sigma$. We assume that near p, Σ is a symplectic Poisson stratum of codimension 2. Then P is analytic hypoelliptic at p.*

This means that whenever $u \in \mathcal{D}'(\Omega)$, with $p \notin WF_A(Pu)$, it follows that $p \notin WF_A(u)$. Note that $\mathcal{D}'(\Omega)$ denotes the space of distributions on Ω and $WF_A(u)$ denotes the analytic wave front set of u.

The proof of Theorem 1.2 is given in the next three sections. In section 2 the geometric assumption is discussed. We make a detailed study of symplectic Poisson strata of codimension two. In particular we choose local coordinates which are convenient for the analysis. In sections 3 and 4 we prove the regularity result. We work in special coordinates given in section 3. We use a version of the FBI transform specially suited to our needs. A key tool is the Green's function for an associated ordinary differential operator. Finally we employ a simple symbolic calculus based on the work of Sjöstrand [14] to estimate error terms.

Finally we remark that once the operator P of Theorem 1.2 is written in the special coordinates of section 3, then the regularity result proved in section 4, also follows from the main result of [11]. Our methods are much different from those of [11] and it is our belief that our technique will extend to strata of higher codimension. We can also obtain optimal Gevrey regularity results in the codimension two case with our techniques. This result will appear elsewhere.

2. The Geometric Hypothesis

2.1. Preliminaries

Let U be an open set of $\mathbb{R}^{2N}$, equiped with the standard symplectic structure. We assume given real-analytic functions $F_0, \ldots, F_\mu$ on U and we let

$$\Sigma \doteq \{p \in U : F_j(p) = 0,\, j = 0, \ldots, \mu\}.$$

In other words, Σ is the locus of the ideal I spanned by $F_0, \ldots, F_\mu$.

Given $p \in U$ we shall denote by $\mathbf{I}_p$ the ideal of germs of real-analytic functions at p spanned by the germs $\mathbf{F}_0 \ldots, \mathbf{F}_\mu$ of the functions $F_0, \ldots, F_\mu$ at p.

Given $k \geqslant 1$ we shall consider the ideals $\mathbf{I}_p^k$ spanned by the germs of Poisson brackets of length $\ell \leqslant k$ at p

$$\mathbf{F}_J \doteq \{\mathbf{F}_{j_\ell}, \ldots \{\mathbf{F}_{j_1}, \mathbf{F}_{j_0}\} \ldots\}, \quad J = (j_0, \ldots, j_\ell).$$

Thus we have an ascending chain of ideals $\mathbf{I}_p = \mathbf{I}_p^0 \subset \mathbf{I}_p^1 \subset \ldots \subset \mathbf{I}_p^k \subset \mathbf{I}_p^{k+1} \subset \ldots$

Needless to say that the ideals $\mathbf{I}_p^k$ are generated by global sections.

Our main hypothesis, which will be assumed throughout, is the validity of the *microlocal version* of the Hörmander finite type condition, that is, for every $p \in U$ there is $k \in \mathbb{N}$ such that $\mathbf{I}_p^k$ equals the space of germs of all real-analytic functions at p. The smallest k for which this property holds is then denoted by $\rho(p)$.

Lemma 2.1. — *ρ is upper semicontinuous.*

Proof. — We must show that each point p_0 has an open neighborhood V_0 such that $\rho(p) \leqslant \rho(p_0)$ for $p \in V_0$. Since the germ of the constant function 1 at p_0 belongs to $\mathbf{I}_{p_0}^{\rho(p_0)}$ there are a_I, real-analytic in an open neighborhood V_0 of p_0, such that

$$\sum a_I F_I = 1 \quad \text{in } V_0,$$

where F_I are multiple Poisson brackets of the functions $F_0, \dots, F_\mu$ of length $\leqslant \rho(p_0)$. But then, for any $p \in V_0$, the germ of the constant function 1 at p belongs to the ideal spanned by the germs of the F_I at p. Hence $\rho(p) \leqslant \rho(p_0)$. $\square$

We shall then say that Σ *satisfies property* $(\star)$ *with respect to* $\mathbf{I}$ *if* ρ *is constant on* Σ.

2.2. A model

We now take $\Omega \subset \mathbb{R}^{m+1}$, an open neighborhood of the origin, where the coordinates are written as $(x,t) = (x_1, \dots, x_m, t)$ and let $U = \Omega \times (\mathbb{R}^{m+1} \setminus \{(0,0)\})$, where now the coordinates are written as $(x,t,\xi,\tau) = (x_1, \dots, x_m, t, \xi_1, \dots, \xi_m, \tau)$. We shall take

$$(2.1) \quad F_0 = \tau, \qquad F_j(x,t,\xi) \doteq f_j(x,t,\xi) = \sum_{\ell=1}^{m} a_{j\ell}(x,t)\xi_\ell, \quad j = 1, \dots, \mu.$$

We assume that, in an open and conic neighborhood V_0 of $p_0 = (0,0;\xi_0,0)$, Σ is a symplectic manifold of codimension 2. Thus we can assume that Σ is defined, in V_0, by the equations $\tau = 0$, $t - \lambda(x,\xi) = 0$, where λ is real-analytic in a conic neighborhood W_0 of $(0,\xi_0)$ and positively homogeneous of degree zero. In other words we are assuming

$$\Sigma \cap V_0 = \{(x, \lambda(x,\xi); \xi, 0) : (x,\xi) \in W_0\}.$$

We shall also assume that Σ satisfies property $(\star)$ with respect to the ideal generated by $f_0, \dots, f_\mu$ in V_0.

We start by proving the following:

Lemma 2.2. — *Let g be a real-analytic function in V_0 vanishing in $\Sigma \cap V_0$. If $p \in \Sigma \cap V_0$ and if $(\partial_t g)(p) = 0$ then the Hamilton vector field H_g vanishes at p.*

Proof. — Indeed we have $g(x, \lambda(x,\xi), \xi) = 0$ for all $(x,\xi) \in W_0$. Differentiation of this expression with respect to x_ℓ and ξ_q proves the result. □

Lemma 2.3. — *Let $p \in \Sigma \cap V_0$ and suppose that, for some $r \geqslant 1$, we have $(\partial_t^s f_j) = 0$ in $\Sigma \cap V_0$ for all $j = 1 \ldots, \mu$ and all $s \in \{0, \ldots, r-1\}$ and that $(\partial_t^r f_j)(p) = 0$ for all $j = 1 \ldots, \mu$. Then every Poisson bracket of length $\leqslant r$ vanishes at p.*

Proof. — Lemma 2.1 implies that $H_{\partial_t^s f_j}$ vanishes at p for all $j = 1, \ldots, \mu$ and all $s = 0, \ldots, r-1$. Consider then a general Poisson bracket of length $\sigma \leqslant r$

$$F = H_{f_{i_\sigma}} H_{f_{i_{\sigma-1}}} \ldots H_{f_{i_1}} f_{i_0}.$$

If $i_\sigma = \ldots = i_1 = 0$ then $F = (\partial_t^\sigma f_{i_0})$ vanishes at p by hypothesis. Otherwise let i_q be such that $i_q \in \{1, \ldots, \mu\}$ and $i_\sigma = i_{\sigma-1} = \ldots = i_{q+1} = 0$. We then have

$$F = (\partial_t)^{\sigma-q} H_{f_{i_q}} G$$

for some G. Now given any g we have

$$\partial_t H_g = H_g \partial_t + [\partial_t, H_g] = H_g \partial_t + H_{\partial_t g}.$$

Applying this remark to the expression of F we see that F can be expressed as a sum of terms, each of them starting as $H_{\partial_t^j f_{i_q}} \ldots$, with $j = 0, \ldots, \sigma - q$. Since $q \geqslant 1$ it follows that $\sigma - q \leqslant \sigma - 1 \leqslant r - 1$ and hence F vanishes at p. □

We then have

Proposition 2.4. — *Let $\kappa \doteq \rho(p)$ for $p \in \Sigma \cap V_0$. Then there are j_0 and V_0', an open neighborhood of $p_0 = (0, 0; \xi_0, 0)$ in $\Sigma \cap V_0$, such that $\partial_t^\kappa f_{j_0} \neq 0$ in V_0' and $\partial_t^p f_j = 0$ on V_0' for all $0 \leqslant p < \kappa$ and $j = 1 \ldots, \mu$.*

Proof. — There is $\beta \in \mathbb{N}$ such that $\partial_t^p f_j = 0$ on $\Sigma \cap V_0$ for every $p = 0, 1, \ldots, \beta$ and $j = 1, \ldots, \mu$ and such furthemore that $(\partial_t^{\beta+1} f_q)(A) \neq 0$ for some q and some $p \in \Sigma \cap V_0$. From Lemma 2.3 we obtain $\beta + 1 = \kappa$. We now claim the existence of $j_0 \in \{1, \ldots, \mu\}$ such that $(\partial_t^\kappa f_{j_0})(p_0) \neq 0$. Indeed if this were not true Lemma 2.3 would again imply that all Poisson brackets of length $\leqslant \beta$ vanish at p_0 and consequently $\rho(p_0) > \beta = \rho(p)$, which is a contradiction. □

Application of Taylor's formula then gives

Corollary 2.5. — *Same hypotheses of Proposition 2.4. We can write, for* (x,t) *near* $(0,0)$ *and* ξ *in a conic neighborhood of* ξ_0,

$$f_j(x,t,\xi) = (t-\lambda(x,\xi))^{\kappa}\ E_j(x,\xi) + \mathrm{O}\left((t-\lambda(x,\xi))^{\kappa+1}\,|\xi|\right)$$

where E_j *is positively homogeneous of degree one and*

$$E(x,\xi) = \sum_{j=1}^{m} E_j(x,\xi)^2$$

is an elliptic symbol of degree 2 defined in a conic neighborhood of $(0,\xi_0)$ *in* $\mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\})$.

Remark 2.6. — Still under the hypotheses of Proposition 1, since every Poisson bracket of length $\leqslant \kappa - 1$ vanishes identically on $\Sigma \cap V_0$, we in particular have

$$\partial_t^p\{f_j, f_k\} = 0 \text{ in } \Sigma \cap V_0 \text{ for all } j,k \text{ and all } 0 \leqslant p \leqslant \kappa-2.$$

Thus

$$\{f_j, f_k\}(x,t,\xi) = \mathrm{O}\left((t-\lambda(x,\xi))^{\kappa-1}|\xi|\right)$$

for (x,t) near $(0,0)$ and ξ in a conic neighborhood of ξ_0.

3. Proof of Theorem 1.2 (beginning)

3.1. First reduction

We recall that we are dealing with a "sum of squares operator" of the form (1.1) which is defined in an open subset Ω of $\mathbb{R}^N$. Here $X_0, X_1, \dots, X_\mu$ are real-analytic, real vector fields defined in Ω. Our main hypotheses are the following:

[H1] The vector fields $X_0, X_1, \dots, X_\mu$ satisfy Hörmander condition.

[H2] Near a characteristic point $p_0 \in T^*\Omega \setminus 0$ the characteristic set is a two-codimensional symplectic manifold which satisfies property $(\star)$ with respect to ideal generated by the symbols of the vector fields $X_0, \dots, X_\mu$.

Assume, without loss of generality, that the base projection of p_0 is the origin. [H1] implies in particular that one of the vector fields does not vanish at the origin. We choose local coordinates $(x_1, \dots, x_m, t)$, where $N = m+1$, in such a way that $X_0 = \partial/\partial t$. Let

$$X_j = b_j(x,t)\partial_t + \sum_{k=1}^{m} a_{j,k}(x,t)\partial_{x_k}, \qquad j = 1, \dots, \mu.$$

According to our previous notation we can write

$$X_j = b_j(x,t)\frac{\partial}{\partial t} + f_j(x,t,\partial_x).$$

If we denote the dual coordinates as $(\xi_1,\dots,\xi_m,\tau)$ it follows that the characteristic set of P is defined by the vanishing of the functions (2.1).

Define a positive definite matrix $D_{jk}(x,t)$ by

$$\frac{1}{A(x,t)}\sum_{j,k=1}^{\mu} D_{jk}(x,t)v_j v_k = |v|^2 - \left(\sum_{j=1}^{\mu}\frac{b_j(x,t)}{A(x,t)^{1/2}}\,v_j\right)^2,$$

where $v=(v_1,\dots,v_\mu)\in\mathbb{R}^\mu$ and $A = 1+\sum b_j^2$.

By a direct computation it can be seem that

$$P = Q + a^\bullet(x,t)\partial_t + \sum_{j=1}^{\mu}\left[b_j^\bullet(x,t)f_j(x,t,\partial_x) + c_j^\bullet(x,t)(\partial_t f_j)(x,t,\partial_x)\right],$$

where $a^\bullet$, $b_j^\bullet$ and $c_j^\bullet$ are real-analytic and real-valued and Q is the operator

$$Q \doteq A(x,t)\left[\left(\partial_t + \frac{1}{A(x,t)}\sum_{j=1}^{\mu} b_j(x,t)f_j(x,t,\partial_x)\right)^2 + \sum_{j,k=1}^{\mu} D_{jk}(x,t)f_j(x,t,\partial_x)f_k(x,t,\partial_x)\right].$$

We shall now perform a diffeomorphism near the origin in $\mathbb{R}^{m+1}$ of the form

$$x' = x'(x,t), \qquad t' = t \tag{3.1}$$

such that the vector field $\partial/\partial t + \sum b_j f_j(x,t,\partial_x)/A$ becomes $\partial/\partial t'$. We consider the local (symplectic) diffeomorphism of $T^*\Omega$ defined near $A = (0,0,\xi_0,0)$ and associated to (3.1): it can be written as

$$(x,t,\xi,\tau)\mapsto(x',t',\xi',\tau') = \chi(x,t,\xi,\tau)$$

where

$$x' = x'(x,t),\quad t' = t,\quad \xi' = M(x,t)\xi,\quad \tau' = \tau + \sum_{j=1}^{\mu}\frac{b_j(x,t)}{A(x,t)}f_j(x,t,\xi) \tag{3.2}$$

and $M(x,t)$ is the transpose of the matrix $(\partial x'/\partial x)^{-1}$ at (x,t).

We write $A^\sharp = A \circ \chi^{-1}$, $f_j^\sharp = f_j \circ \chi^{-1}$, etc... Since χ is a symplectic diffeomorphism, property $(\star)$ remains true for the ideal $\mathbf{I}$ generated by $f_j^\sharp$, $j = 0, 1, \ldots, \mu$, where f_j are given by (2.1). We have

$$f_0^\sharp(x', t', \xi', \tau') = \tau' - \sum_{j=1}^{\mu} \frac{b_j^\sharp(x', t')}{A^\sharp(x', t')} f_j^\sharp(x', t', \xi')$$

and then it is easily seem that the ideals $\mathbf{I}_X^q$ $(q = 0, 1, \ldots,)$ are also spanned by the germs at X of the Poisson brackets of length $\leqslant q$ of the functions $\tau', f_1^\sharp, \ldots, f_\mu^\sharp$. Consequently, the conclusions of Corollary 2.5 hold for the functions $f_j^\sharp$ substituted for f_j and the variables (x', t', ξ', t') substituted for (x, t, ξ, τ).

As a further remark we observe that in the new variables P can be written as

$$\begin{aligned} P = A^\sharp(x', t') &\left[\partial_{t'}^2 + \sum_{j,k=1}^{\mu} D_{jk}^\sharp(x', t') f_j^\sharp(x', t', \partial_{x'}) f_k^\sharp(x', t', \partial_{x'}) \right] \\ &+ a^{\bullet\sharp}(x', t') \left(\partial_{t'} + \sum_{j=1}^{\mu} \frac{b_j^\sharp(x', t')}{A^\sharp(x', t')} f_j^\sharp(x', t', \partial_{x'}) \right) \\ &+ \sum_{j=1}^{\mu} \left[b_j^{\bullet\sharp}(x', t') f_j^\sharp(x', t', \partial_{x'}) + c_j^{\bullet\sharp}(x', t') (\partial_t f_j)^\sharp(x', t', \partial_{x'}) \right]. \end{aligned}$$

Since finally we have

$$(\partial_t f_j)^\sharp = (H_{f_0} f_j)^\sharp = H_{f_0^\sharp} f_j^\sharp = \partial_{t'} f_j^\sharp - \sum_k \left\{ \frac{b_k^\sharp}{A^\sharp} f_k^\sharp, f_j^\sharp \right\}$$

and

$$\left\{ \frac{b_k^\sharp}{A^\sharp} f_k^\sharp, f_j^\sharp \right\} = f_k^\sharp \left\{ \frac{b_k^\sharp}{A^\sharp}, f_j^\sharp \right\} + \frac{b_k^\sharp}{A^\sharp} \left\{ f_k^\sharp, f_j^\sharp \right\}$$

we can apply Corollary 2.5 and Remark 2.6 to reach the conclusion that we can assume P written as

$$P = \partial_t^2 + \sum_{j,k=1}^{\mu} A_{jk}(x, t) X_j X_k + b(x, t) \partial_t + Y \tag{3.3}$$

where:

(1) X_j and Y are real-analytic, real vector fields in $\partial/\partial x_1, \ldots, \partial/\partial x_m$ defined in Ω;

(2) There exists $C > 0$ such that

$$\langle A\theta, \theta\rangle = \sum_{j,k=1}^{\mu} A_{jk}(x,t)\theta_j\theta_k \geqslant C|\theta|^2,$$

for all $\theta \in \mathbb{R}^\mu$ and $(x,t) \in \Omega$;

(3) b is a real-valued, real-analytic function;

(4) The characteristic set is defined by the equations $\tau = 0$, $f_j(x,t,\xi) = 0$, where f_j are such that $X_j = f_j(x,t,\partial_x)$.

(5) The conclusions of Corollary 2.5 hold for f_j.

(6) If $Y = g(x,t,\partial_x)$ then

$$|g(x,t,\xi)| = \mathrm{O}\left((t-\lambda(x,\xi))^{\kappa-1}|\xi|\right)$$

for (x,t) near $(0,0)$ and ξ in a conic neighborhood of ξ_0.

3.2. Final Simplification

By Corollary 2.5 we may write

$$X_l = f_l(x,t,\partial_x) = \sum_{j=1}^{m} b_j^l(x,t)\partial_x\,,$$

where

$$f_l(x,t,\xi) = (t-\lambda(x,\xi))^\kappa E_l(x,\xi) + \mathrm{O}((t-\lambda(x,\xi))^{\kappa+1}|\xi|), \tag{3.4}$$

for (x,t) near $(0,0)$ and ξ in a conic neighborhood of ξ_0.

Next we study the growth of each individual $b_j^l(x,t)$. We will show that for all $l = 1,\dots,\mu$ and $j = 1,\dots,m$ we have

$$b_j^l(x,t) = \mathrm{O}\left((t-\lambda(x,\xi))^\kappa\right), \tag{3.5}$$

for (x,t) near $(0,0)$ and ξ in a conic neighborhood of ξ_0.

Indeed, differentiating with respect to ξ_j yields

$$b_j^l(x,t) = (t-\lambda(x,\xi))^{\kappa-1}\frac{\partial\lambda}{\partial\xi_j}(x,\xi)E_\ell(x,\xi) + \mathrm{O}\left((t-\lambda(x,\xi))^\kappa\right).$$

Differentiating again with respect to ξ_j shows that $(\partial\lambda/\partial\xi_j)E_l$ vanishes when $t = \lambda(x,\xi)$, and (3.5) follows.

We now define $\lambda_0(x) = \lambda(x,\xi_0)$. It follows from (3.5) that

$$b_j^l(x,t) = \mathrm{O}\left((t-\lambda_0(x))^\kappa\right), \tag{3.6}$$

for (x,t) near $(0,0)$ and all l,j. A similar statement (with $\kappa-1$ in place of κ) holds for the coefficients of the vector field Y. More precisely, we have

$$f_j(x,t,\xi) = (t-\lambda_0(x))^\kappa E_j^\flat(x,\xi) + \mathrm{O}\left((t-\lambda_0(x))^{\kappa+1}|\xi|\right)$$

where, by Corollary 2.5,

$$\sum_{j=1}^{\mu} E_j^\flat(x,\xi_0)^2 = \sum_{j=1}^{\mu} E(x,\xi_0)^2 > 0 \tag{3.7}$$

for all x near the origin.

We now make a change of variable near the origin in (x,t) space. We define new coordinates (y,s) as follows:

$$y = x, \quad s = t-\lambda_0(x).$$

Notice then that the corresponding covector changes as

$$(\eta,\sigma) \mapsto (\xi+\tau(\nabla\lambda_0)(x),\tau).$$

In particular $(\xi_0,0)$ is fixed under such transformation.

We start with P in the form (3.3). In the new variables (y,s) we can write

$$X_j = d_j^*(y,s)\frac{\partial}{\partial s} + f_j^*(y,s,\partial_y),\ Y = c^*(y,s)\frac{\partial}{\partial s} + g^*(y,s,\partial_y)$$

where $d^*(y,s)=(d_1^*(y,s),\dots,d_\mu^*(y,s))=\mathrm{O}(s^\kappa)$, and $g^*(y,s,\eta)=\mathrm{O}(s^{\kappa-1}|\eta|)$ for s near 0. Notice moreover that

$$f_j^*(y,s,\eta) = s^\kappa E_j^\flat(y,\eta) + \mathrm{O}(s^{\kappa+1}|\eta|).$$

Hence, we can write [*cf.* (3.3)]

$$\begin{aligned} P = &\ \partial_s^2 + \sum_{j,k=1}^{\mu} A_{jk}^*(y,s)\left(d_j^*\partial_s + f_j^*(y,s,\partial_y)\right)\left(d_k^*\partial_s + f_k^*(y,s,\partial_y)\right) \\ &\ + b^*(y,s)\partial_s + g^*(y,s,\partial_y) \\ = &\ G^*\left\{\partial_s + h^*(y,s,\partial_y)\right\}^2 + \sum_{j,k=1}^{\mu} A_{jk}^{**}(y,s) f_j^*(y,s,\partial_y) f_k^*(y,s,\partial_y) \\ &\ + b^{**}(y,s)\partial_s + g^{**}(y,s,\partial_y). \end{aligned}$$

Here $G^* = 1+\langle A^*d^*,d^*\rangle$ and $\{A_{j,k}^{**}(y,s)\}$ denotes the quadratic form

$$\mathbb{R}^\mu \ni \theta \mapsto \langle A^*\theta,\theta\rangle - \left(\frac{\langle A^*d^*,\theta\rangle + \langle\, {}^tA^*d^*,\theta\rangle}{2G^*}\right)^2,$$

which is still positive definite in a small neighborhood of the origin in the (y,s)-space since $d^*(y,s)=\mathrm{O}(s^\kappa)$. Furthermore $g^{**}(y,s,\eta)=\mathrm{O}(s^{\kappa-1}|\eta|)$.

It follows then, that after a new local diffeomorphim

$$(y,s) \mapsto (x,t) = (x(y,s),s)$$

which changes the vector field $\partial_s + h^*(y,s,\partial_y)$ into ∂_t, we obtain the following form for P/G^* near the origin:

(3.8)

$$P/G^* = \partial_t^2 + t^{2\kappa} \sum_{j,k=1}^{\mu} A^{\diamond}_{jk}(x,t) Z_j(x,t,\partial_x) Z_k(x,t,\partial_x) + t^{\kappa-1} W(x,t,\partial_x) \\ + b(x,t)\partial_t$$

where $A^{\diamond}_{jk}$ is positive definite for (x,t) near $(0,0)$,

$$Z_j(x,t,\xi) = f_j^*(y(x,t),t,B(x,t)\xi)/t^{\kappa},$$
$$W(x,t,\xi) = g^*(y(x,t),t,B(x,t)\xi)/t^{\kappa-1}$$

and $B(x,t)$ denotes the transpose of the matrix $(\partial x/\partial y)$ at $(y(x,t),t)$. Since $x(y,t)$ can be chosen in such a way that $x(0,0)=0$ and that $\partial x/\partial y$ is the identity at the origin we have

$$Z_j(x,t,\xi_0) = E_j(y(x,t),\xi_0) + \mathrm{O}(t), \quad j = 1,\dots,\nu$$

when (x,t) is near the origin. Hence (3.7) implies the existence of a constant $C>0$ such that

$$\sum_{j,k=1}^{m} A^{\diamond}_{jk}(x,t) Z_j(x,t,\xi) Z_k(x,t,\xi) \geqslant C|\xi|^2 \tag{3.9}$$

for (x,t) near the origin and ξ conically close to ξ_0.

We summarize what we have reached in the following key result:

Proposition 3.1. — *Same hypotheses as in Theorem 1.2. We can choose local coordinates $(x,t) \in \mathbb{R}^m \times \mathbb{R}$ $(m+1=N)$ near the origin in such a way that $p = (0,0;\xi_0,0)$ and, up to a non vanishing analytic factor, the operator P can be written as*

$$P = \partial_t^2 + t^{2\kappa} \sum_{j,k=1}^{m} a_{jk}(x,t)\partial_{x_j}\partial_{x_k} + t^{\kappa-1}\sum_{j=1}^{m} b_j(x,t)\partial_{x_j} + b(x,t)\partial_t, \tag{3.10}$$

where $\kappa \geqslant 1$ is an integer, a_{jk}, b_j, b are real-valued, real-analytic functions defined near the origin and, for some constant $C>0$,

$$\sum_{j,k=1}^{m} a_{jk}(x,t)\xi_j\xi_k \geqslant C|\xi|^2 \tag{3.11}$$

for (x,t) near $(0,0)$ and ξ in a conic neighborhood of ξ_0.

4. Proof of Theorem 1.2 (conclusion)

Thanks to Proposition 3.1, the proof of Theorem 1.2 will follow if the following slightly more general statement can be proved:

PROPOSITION 4.1. — *Let P have the following form*

(4.1)
$$P = \partial_t^2 + t^{2\kappa} \sum_{j,k=1}^{m} a_{jk}(x,t)\partial_{x_j}\partial_{x_k} + t^{\kappa-1}\sum_{j=1}^{m} b_j(x,t)\partial_{x_j} + a(x,t)\partial_t + b(x,t)\,,$$

where $\kappa \geqslant 1$ is an integer and a_{jk}, b_j, a, b are real-analytic functions defined near the origin, with a_{jk}, b_j real valued (a, b are allowed to be complex valued). Assume also the validity of (3.11) for (x,t) near the origin and ξ conically close to ξ_0. Then if $u(x,t)$ is a distribution near the origin such that $(0,0;\xi_0,0) \notin WF_A(Pu)$ it follows that $(0,0;\xi_0,0) \notin WF_A(u)$.

From now on we shall work under this set-up.

4.1. A preliminary reduction

Let u be a distribution near $(0,0) \in \mathbb{R}^{m+1}$ such that

$$(0,0;\xi_0,0) \notin WF_A(Pu). \tag{4.2}$$

Our goal is to show that $(0,0;\xi_0,0) \notin WF_A(u)$. We now make some preliminary reductions involving u.

After multiplication by a smooth cutoff function we may assume that u has compact support, with (4.2) still satisfied. We now apply Corollary 8.4.13 of [8]. Let $\Gamma \subset \mathbb{R}^{m+1}$ be a small (so that Γ contains no points of the form $(0,\tau)$), closed, convex, proper cone containing $(\xi_0,0)$ as an interior point. Then there exist $u_1, u_2 \in \mathcal{S}'(\mathbb{R}^{m+1})$, the space of tempered distributions, such that $u = u_1 + u_2$ and

$$WF_A(u_1) \subset \mathbb{R}^{m+1} \times \Gamma \tag{4.3}$$

and

$$(0,0;\xi_0,0) \notin WF_A(u_2). \tag{4.4}$$

It follows then that

$$(0,0;\xi_0,0) \notin WF_A(Pu_1). \tag{4.5}$$

Furthermore we see that $(0,0;\xi_0,0) \notin WF_A(u)$ if and only if $(0,0;\xi_0,0) \notin WF_A(u_1)$. It is important to note that u_1 has well defined traces on all hyperplanes $t = t_0$, since Γ avoids the normals to these hyperplanes; this follows from Theorem 8.2.4 in [8], whose proof also implies that u_1 is a smooth function in t valued in the space of distributions in x.

Next let $\chi(x) = \chi \in C_0^\infty(\mathbb{R}^m)$ with $\chi \equiv 1$ near $x = 0$. Note that $(0,0;\xi_0,0) \notin WF_A(P(\chi u_1))$ and that $(0,0;\xi_0,0) \notin WF_A(u)$ if and only if $(0,0;\xi_0,0) \notin WF_A(\chi u_1)$. Next we will cut in the t variable. Let $\varphi(t)$ be a smooth function with compact support, depending only on t. Assume that φ is identically equal to 1 near the origin. We have

$$P(\varphi\chi u_1) = \varphi P(\chi u_1) + \varphi''\chi u_1 + 2\varphi'(\chi u_1)_t + a\varphi'\chi u_1\,. \tag{4.6}$$

Because of (4.6), we see that $(0,t;\xi_0,0) \notin WF_A(\chi u_1)$ for all $t \neq 0$ and small. Indeed, the characteristic set of P is defined, near $(0,0;\xi_0,0)$ by $t = \tau = 0$. Hence it follows from (4.6) that

$$(0,t;\xi_0,0) \notin WF_A(P(\varphi\chi u_1)) \tag{4.7}$$

for all $t \in \mathbb{R}$, as long as the support of φ is small enough.

Summing up, we have the right to assume, from the beginning, that $u \in \mathcal{E}'(\mathbb{R}^{m+1}) \cap C^\infty(\mathbb{R}_t, \mathcal{D}'(\mathbb{R}^m_x))$ has support contained in an arbitrarily small neighborhood of the origin in $\mathbb{R}^{m+1}$ and that

$$(0,t;\xi_0,0) \notin WF_A(Pu) \tag{4.8}$$

for all $t \in \mathbb{R}$.

Notice moreover that, if $(Pu)_t \in \mathcal{D}'(\mathbb{R}^m)$ denotes the trace of Pu at t then

$$(0;\xi_0) \notin WF_A((Pu)_t) \tag{4.9}$$

for all $t \in \mathbb{R}$ (*cf.* Theorem 8.2.4 in [8]).

Remark 4.2. — Our assumptions show that $\Gamma = \mathcal{C}^0$, where $\mathcal{C}$ is an open convex cone, and $\mathcal{C}^0$ denotes the dual cone. We now apply Theorem 8.4.15 of [8]. It follows that if U is a small neighborhood of the origin in $\mathbb{R}^{m+1}$ and if V is an open convex cone with closure contained in $\mathcal{C} \cup \{0\}$, then there exist a function f holomorphic on $U \times iV_\delta$, of slow growth, such that $u = \lim_{(y,s)\to(0,0)} f(\cdot + iy, \cdot + is)$. Here the limit is of course taken when $(y,s) \in V_\delta$ and exists in the sense of distributions on U. [1]

[1] Here, for $\delta > 0$, V_δ denotes the truncated cone $\{(y,s) \in V : |(y,s)| < \delta\}$.

4.2. The FBI transform

We introduce the FBI transform of u, defined by

$$I[u](z,t,\xi)=\int e^{-iy\cdot\xi-|\xi|(z-y)^2/2}u(y,t)dy. \tag{4.10}$$

Note that, for each $\xi\in(\mathbb{R}^m\backslash 0)$ fixed, $I[u]$ is a smooth function of t valued in the space of entire functions of z in $\mathbb{C}^m$. By the standard characterization of the analytic wave front set via the FBI transform it follows from (4.9) that there exist $C>0$ and an $\epsilon>0$ such that if $|z|<\epsilon$, then

$$|I[Pu](z,t,\xi)|\leqslant Ce^{-\epsilon|\xi|}, \tag{4.11}$$

for all $t\in\mathbb{R}$ and for all ξ in a conic neighborhood of ξ_0.

Our goal is, of course, to derive a similar estimate for $I[u]$.

We have the following formulas for $I[u]$:

$$I[\partial_{x_j}u]=(\partial_{z_j}+i\xi_j)I[u], \tag{4.12}$$

and

$$I[y_ju]=\left(\frac{1}{|\xi|}\partial_{z_j}+z_j\right)I[u]. \tag{4.13}$$

Note that (4.13) follows from

$$(y_j-z_j)e^{-iy\cdot\xi-|\xi|(z-y)^2/2}=\frac{1}{|\xi|}\partial_{z_j}\left(e^{-iy\cdot\xi-|\xi|(z-y)^2/2}\right). \tag{4.14}$$

It follows that we have, for all α,

$$(y-z)^\alpha e^{-iy\cdot\xi-|\xi|(z-y)^2/2}=F_\alpha(e^{-iy\cdot\xi-|\xi|(z-y)^2/2}), \tag{4.15}$$

where we define the partial differential operator $F_\alpha=F_\alpha\left(\frac{1}{|\xi|},\frac{1}{|\xi|}\partial_z\right)$ as follows:

$$F_\alpha=\frac{\alpha!}{(-2)^{|\alpha|}}\sum_{2\beta\leqslant\alpha}\frac{1}{\beta!(\alpha-2\beta)!}\left(\frac{2}{|\xi|}\right)^{|\alpha|-|\beta|}\partial_z^{\alpha-2\beta}. \tag{4.16}$$

Observe that, given $\alpha=(\alpha_1,\dots,\alpha_m)$, the sum is taken over all $\beta=(\beta_1,\dots,\beta_m)$ such that $0\leqslant 2\beta_j\leqslant\alpha_j$.

Formula (4.15) follows from the following one–variable version. If $t\in\mathbb{R}$ and m is a positive integer we have

$$t^me^{-t^2}=\frac{m!}{(-2)^m}\sum_{0\leqslant 2j\leqslant m}\frac{1}{j!(m-2j)!}\left(\frac{d}{dt}\right)^{m-2j}(e^{-t^2}). \tag{4.17}$$

This formula follows by induction. It is clearly true for $m = 1, 2$. Now we assume true for m and $m-1$. Since

$$t^{m+1}e^{-t^2} = \frac{m}{2}t^{m-1}e^{-t^2} - \frac{1}{2}\frac{d}{dt}(t^m e^{-t^2})$$

(4.17) follows by induction hypothesis.

Next let $a(z,t)$ be smooth function of $t \in \mathbb{R}$ valued in the space of holomorphic functions of $z \in U$. Here U is a neighborhood of the origin in $\mathbb{C}^m$. We also assume that there is $h > 0$ such that

$$\sup\{|\partial_z^\alpha a(z,t)| : (z,t) \in U \times \mathbb{R}\} \leqslant h^{|\alpha|+1}\,\alpha!, \quad \alpha \in \mathbb{Z}^m\,.$$

We wish to compare $I[au]$ with $aI[u]$. If, say, $|x| \leqslant 1/(2h)$ on the support of u we can write

$$\begin{aligned} &I[au](z,t,\xi) - a(z,t)I[u](z,t,\xi) \\ &\qquad\qquad = \int e^{-iy\cdot\xi - |\xi|(z-y)^2/2} \sum_{\alpha\neq 0} \frac{\partial_z^\alpha a(z,t)}{\alpha!}(y-z)^\alpha u(y,t)dy. \end{aligned}$$

for z in a neighborhood of the origin in $\mathbb{C}^m$. Note that given $\delta > 0$, there exist $C > 0$ and $\epsilon > 0$, such that

$$\left| \int e^{-iy\cdot\xi - |\xi|(z-y)^2/2} \sum_{|\alpha|>\delta|\xi|} \frac{\partial_z^\alpha a(z,t)}{\alpha!}(y-z)^\alpha u(y,t)dy \right| \leqslant Ce^{-\epsilon|\xi|} \tag{4.18}$$

for z near 0, $t \in \mathbb{R}$ and all ξ large. So we see that we may truncate such sums as above, modulo an exponentially decreasing error, which leads us to introduce the operators

$$R_{a,\delta} = \sum_{1\leqslant j\leqslant \delta|\xi|} \frac{A_j}{|\xi|^j}, \tag{4.19}$$

where A_j is the differential operator of order j given by:

$$A_j = \sum_{|\alpha-\beta|=j,\, 2\beta\leqslant\alpha} \frac{\partial_z^\alpha a(z,t)}{2^{|\beta|}}\frac{(-1)^{|\alpha|}}{\beta!(\alpha-2\beta)!}\partial_z^{\alpha-2\beta}. \tag{4.20}$$

Formula (4.18) can then be written as

$$I[au] \sim (a + R_{a,\delta})I[u], \tag{4.21}$$

where $\sim$ means that the difference is exponentially decreasing in $|\xi|$ uniformly for z near 0 and $t \in \mathbb{R}$.

4.3. A Formula for $I[Pu]$

We wish to write $I[Pu]$ in terms of $I[u]$. Since u is compactly supported we can assume that the coefficients of P in (4.1) have been extended for all $t \in \mathbb{R}$ as a compactly supported function of t, real-analytic near the origin in $\mathbb{R}^{m+1}$. Using our formulas, and assuming that the support of u is contained in an appropriately small neighborhood of the origin, we see that there exists a partial differential operator $Q = Q(z,t,\xi,\partial_z,\partial_t)$ such that

$$Q\, I[u](z,t,\xi) \sim I[Pu](z,t,\xi) \tag{4.22}$$

for (z,t) near $(0,0)$ and $\xi \in \mathbb{R}^m$. We see that Q can be written as follows :

(4.23)

$$\begin{aligned} Q(z,t,\xi,\partial_z,\partial_t) = \partial_t^2 + t^{2\kappa} & \sum_{j,k=1}^{m} (a_{jk}(z,t) + R_{a_{jk},\delta})(i\xi_j + \partial_{z_j})(i\xi_k + \partial_{z_k}) \\ & + t^{\kappa-1}\sum_{j=1}^{m}(b_j(z,t) + R_{b_j,\delta})(i\xi_j + \partial_{z_j}) \\ & + (a(z,t) + R_{a,\delta})\partial_t + (b(z,t) + R_{b,\delta}). \end{aligned}$$

Next we introduce the change of variable

$$s = t|\xi|^{1/(\kappa+1)}.$$

We denote by $Q^{\#}(z,s,\xi,\partial_z,\partial_s)$ the transformed operator. We have the following

(4.24)

$$\begin{aligned} |\xi|^{-2/(\kappa+1)}Q^{\#} = \partial_s^2 + s^{2\kappa}|\xi|^{-2} & \sum_{j,k=1}^{m} (a_{jk} + R_{a_{jk},\delta})(i\xi_j + \partial_{z_j})(i\xi_k + \partial_{z_k}) \\ & + s^{\kappa-1}|\xi|^{-1}\sum_{j=1}^{m}(b_j + R_{b_j,\delta})(i\xi_j + \partial_{z_j}) \\ & + |\xi|^{-1/(\kappa+1)}(a + R_{a,\delta})\partial_s + |\xi|^{-2/(\kappa+1)}(b + R_{b,\delta}), \end{aligned}$$

where, by abuse of notation, we denote the transformed coefficients and operators (e.g. $a, R_{a,\delta}$) by the same letters.

Note that all terms are of order zero in ξ, or of lower order. We first examine the term of order precisely zero. We will write, for example, $a(z,t) = a^0(z) + t\tilde{a}(z,t)$ near the origin. We see that the term of order zero is

(4.25)
$$\partial_s^2 - s^{2\kappa} \sum_{j,k=1}^m a^0_{jk}(z)\frac{\xi_j\xi_k}{|\xi|^2} + is^{\kappa-1}\sum_{j=1}^m b^0_j(z)\frac{\xi_j}{|\xi|} = \partial_s^2 - s^{2\kappa}\mathcal{E}(z,\xi) - is^{\kappa-1}\varphi(z,\xi),$$

where $\mathcal{E}$ and φ are defined by left hand side. Note that $\mathcal{E}$ and φ are real for real z, and positively homogeneous of degree 0 in ξ, with $\mathcal{E} > 0$ in a conic neighborhood of $(0,\xi_0)$ in $\mathbb{R}^m \times (\mathbb{R}^m \setminus \{0\})$ [*cf.* (3.9)].

Next we discuss the terms of negative order. We define operators T_j, $j = 1,2,3,4$, as follows

$$T_1 = -s^{2\kappa+1}|\xi|^{-1/(\kappa+1)} \sum_{j,k=1}^m \tilde{a}_{jk}\frac{\xi_j\xi_k}{|\xi|^2}\,; \tag{4.26}$$

$$T_2 = s^{\kappa}|\xi|^{-1/(\kappa+1)}\sum_{j=1}^m \tilde{b}_j\left(i\frac{\xi_j}{|\xi|} + \frac{1}{|\xi|}\partial_{z_j}\right) + |\xi|^{-2/(\kappa+1)}(b + R_{b,\delta})\,; \tag{4.27}$$

$$T_3 = |\xi|^{-1/(\kappa+1)}(a + R_{a,\delta})\partial_s\,; \tag{4.28}$$

$$\begin{aligned} T_4 = s^{2\kappa}\sum_{j,k=1}^m a_{jk}&\left(i\frac{\xi_j}{|\xi|^2}\partial_{z_k} + i\frac{\xi_k}{|\xi|^2}\partial_{z_j} + \frac{1}{|\xi|^2}\partial_{z_j}\partial_{z_k}\right)\\ &+ s^{2\kappa}\sum_{j,k=1}^m R_{a_{jk},\delta}\left(i\frac{\xi_j}{|\xi|} + \frac{1}{|\xi|}\partial_{z_j}\right)\left(i\frac{\xi_k}{|\xi|} + \frac{1}{|\xi|}\partial_{z_k}\right)\\ &+ s^{\kappa-1}\sum_{j=1}^m R_{b_j,\delta}\left(i\frac{\xi_j}{|\xi|} + \frac{1}{|\xi|}\partial_{z_j}\right). \end{aligned} \tag{4.29}$$

If we finally define $T^{\#} = -(T_1 + T_2 + T_3 + T_4)$, we see that

$$|\xi|^{-2/(\kappa+1)}Q^{\#} = q^{\#} - T^{\#}\,, \tag{4.30}$$

where we have written

$$q^{\#} = q^{\#}(s,\partial_s;z,\xi) = \partial_s^2 - s^{2\kappa}\mathcal{E}(z,\xi) - is^{\kappa-1}\varphi(z,\xi). \tag{4.31}$$

4.4. The Green function for the operator $q^{\#}$

We now study the Green's function for $q^{\#}$. Note that $q^{\#}$ can be transformed into the operator q given by

$$q(s,\partial_s;z,\xi) = \partial_s^2 - s^{2\kappa} - i\mathcal{E}(z,\xi)^{-1/2}\varphi(z,\xi)s^{\kappa-1}. \tag{4.32}$$

The operators q and $q^{\#}$ are related as follows: $q^{\#}f^{\#}=0$ if and only if $qf=0$, where f and $f^{\#}$ are related by the formula

$$f^{\#}(s;z,\xi)=f(s\mathcal{E}(z,\xi)^{1/(2\kappa+2)};z,\xi) \tag{4.33}$$

(recall that automaticaly f and $f^{\#}$ are entire functions of s).

We first analyse the Green's function of q. First observe that this operator is injective on the Schwartz space for each (z,ξ) fixed, because φ and $\mathcal{E}$ are real valued for real z. Also, the coefficient of $s^{\kappa-1}$ is positively homogeneous of degree 0 in ξ and hence is uniformly bounded for z near 0 in $\mathbb{C}^m$ for ξ in $\mathbb{R}^m\setminus\{0\}$.

Following the arguments in [2], almost without change, we have, when κ is *even*, two linearly independent functions f,g, both in the kernel of q such that

$$f(s;z,\xi)=(1/2)s^{-(\kappa-i\psi(z,\xi))/2}e^{|s|^{\kappa+1}/(\kappa+1)}\left[1+\mathrm{O}\left(1/s\right)\right],\quad s\to+\infty; \tag{4.34}$$

$$f(s;z,\xi)=-(1/2)(-s)^{-(\kappa-i\psi(z,\xi))/2}e^{-|s|^{\kappa+1}/(\kappa+1)}\left[1+\mathrm{O}\left(1/s\right)\right],\quad s\to-\infty; \tag{4.35}$$

$$g(s;z,\xi)=-(1/2)s^{-(\kappa+i\psi(z,\xi))/2}e^{-|s|^{\kappa+1}/(\kappa+1)}\left[1+\mathrm{O}\left(1/s\right)\right],\quad s\to+\infty; \tag{4.36}$$

$$g(s;z,\xi)=(1/2)(-s)^{-(\kappa+i\psi(z,\xi))/2}e^{|s|^{\kappa+1}/(\kappa+1)}\left[1+\mathrm{O}\left(1/s\right)\right],\quad s\to-\infty, \tag{4.37}$$

where we have written $\psi(z,\xi)=\varphi(z,\xi)/\mathcal{E}(z,\xi)^{1/2}$.

Similarly, when κ is *odd*, we have two linearly independent functions f,g, both in the kernel of q such that

$$f(s;z,\xi)=(1/2)s^{-(\kappa-i\psi(z,\xi))/2}e^{|s|^{\kappa+1}/(\kappa+1)}\left[1+\mathrm{O}\left(1/s\right)\right],\quad s\to+\infty; \tag{4.38}$$

$$f(s;z,\xi)=-(1/2)(-s)^{-(\kappa+i\psi(z,\xi))/2}e^{-|s|^{\kappa+1}/(\kappa+1)}\left[1+\mathrm{O}\left(1/s\right)\right],\quad s\to-\infty; \tag{4.39}$$

$$g(s;z,\xi)=-(1/2)s^{-(\kappa+i\psi(z,\xi))/2}e^{-|s|^{\kappa+1}/(\kappa+1)}\left[1+\mathrm{O}\left(1/s\right)\right],\quad s\to+\infty; \tag{4.40}$$

$$g(s;z,\xi)=(1/2)(-s)^{-(\kappa-i\psi(z,\xi))/2}e^{|s|^{\kappa+1}/(\kappa+1)}\left[1+\mathrm{O}\left(1/s\right)\right],\quad s\to-\infty. \tag{4.41}$$

The terms $\mathrm{O}(1/s)$ are uniform for z near 0 in $\mathbb{C}^m$ for ξ in $\mathbb{R}^m\setminus\{0\}$: this follows from Theorem 6.1 of Sibuya [13], since the same is true for the coefficient of $s^{\kappa-1}$ in (4.32). Similar uniform asymptotics are valid for the derivatives of f and g. These series can be obtained by formal differentiation of the given series.

We introduce G, the Green's function for q. We have

$$G(s,s';z,\xi) = W(z,\xi)^{-1}\left[g(s;z,\xi)f(s';z,\xi)H(s-s') + g(s';z,\xi)\right.\\ \left. f(s;z,\xi)\; H(s'-s)\right],$$

where H denotes the Heaviside function

$$H(s) = 1, \quad s \geqslant 0; \quad H(s) = 0, \quad s < 0,$$

and W denotes the Wronskian of f and g.

The asymptotics for f and g show that there exists a constant $C > 0$ such that

$$C^{-1} \leqslant |W(z,\xi)| \leqslant C$$

for for z near 0 in $\mathbb{C}^m$ for ξ in $\mathbb{R}^m \setminus \{0\}$.

Using the aymptotics above, (or following arguments of Menikoff [9]), we have $C > 0$ such that

$$\int_{-\infty}^{+\infty} |s'|^j |G(s,s';z,\xi)| ds' \leqslant C, \quad 0 \leqslant j \leqslant 2\kappa, \tag{4.42}$$

for all $s \in \mathbb{R}$, z near 0 in $\mathbb{C}^m$ and ξ in $\mathbb{R}^m \setminus \{0\}$.

We denote by $G^{\#}(s,s';z,\xi)$ the Green's function for $q^{\#}$. We see that we have

$$G^{\#}(s,s';z,\xi) = W^{\#}(z,\xi)^{-1}\left[g^{\#}(s;z,\xi)f^{\#}(s';z,\xi)H(s-s')\right.\\ \left. + g^{\#}(s';z,\xi)f^{\#}(s;z,\xi)H(s'-s)\right], \tag{4.43}$$

where $W^{\#}(z,\xi)$ denotes the Wronskian of $f^{\#}, g^{\#}$. Clearly we have $C > 0$ such that

$$C^{-1} \leqslant |W^{\#}(z,\xi)| \leqslant C$$

for z near 0 in $\mathbb{C}^m$ and ξ in $\mathbb{R}^m \setminus \{0\}$.

We now obtain the analog of (4.42). There exists $C > 0$ such that

$$\int_{-\infty}^{+\infty} |s'|^j |G^{\#}(s,s';z,\xi)| ds' \leqslant C, \quad 0 \leqslant j \leqslant 2\kappa, \tag{4.44}$$

for all $s \in \mathbb{R}$, z near 0 in $\mathbb{C}^m$ and ξ in $\mathbb{R}^m \setminus \{0\}$.

4.5. Final Estimates

We begin by defining

$$U^{\#}(z,s,\xi) = I[u](z,s|\xi|^{-1/(\kappa+1)},\xi),$$
$$V^{\#}(z,s,\xi) = |\xi|^{-2/(\kappa+1)} I[Pu](z,s|\xi|^{-1/(\kappa+1)},\xi).$$

We now make use of formula (4.22). We see that we have

$$Q^{\#}U^{\#} \sim |\xi|^{2/(\kappa+1)}V^{\#}. \tag{4.45}$$

Making use of (4.30) and (4.31) we obtain

$$(q^{\#} - T^{\#})U^{\#} \sim V^{\#} \tag{4.46}$$

and hence

$$(I - G^{\#}T^{\#})U^{\#} \sim G^{\#}V^{\#}. \tag{4.47}$$

Here $G^{\#}$ will denote both the operator and the Green's function for $q^{\#}$.

Recall that $I[u]$ has compact support in t. We choose $\chi \in C_0^{\infty}(\mathbb{R})$ such that

$$I[u](z,t,\xi) = \chi(t)I[u](z,t,\xi)$$

for all (z,t,ξ). Define $\chi^{\#}(s,\xi) = \chi(s|\xi|^{-1/(\kappa+1)})$. Hence it follows from (4.47) that we have

$$(I - \chi^{\#}G^{\#}T^{\#})U^{\#} \sim \chi^{\#}G^{\#}V^{\#}. \tag{4.48}$$

It follows by induction that we have

$$U^{\#} \sim (\chi^{\#}G^{\#}T^{\#})^N U^{\#} + \left(\sum_{j=0}^{N-1} (\chi^{\#}G^{\#}T^{\#})^j\right) \chi^{\#}G^{\#}V^{\#}, \tag{4.49}$$

for all $N \geqslant 1$.

Now we introduce the spaces where we shall work. First choose $\rho > 0$ small, such that χ is supported in the interval $\{|t| \leqslant \rho\}$. Let $H(\tau)$ denote the Banach space of all continuous functions on the closure of $B_\tau \times \mathbb{R}$, which are holomorphic on

$$B_\tau = \{z \in \mathbb{C}^m : |z_j| < \tau, j = 1,\dots,m\} \tag{4.50}$$

for each $s \in \mathbb{R}$ and vanish for $|s| \geqslant \rho|\xi|^{1/(\kappa+1)}$. We must keep in mind that the spaces $H(\tau)$ depend on ξ.

If $v(z,s) \in H(\tau)$ we set

$$\|v\|_\tau = \sup\{|v(z,s)| : z \in B_\tau, s \in \mathbb{R}\}.$$

We wish to estimate $\|(\chi^{\#}G^{\#}T^{\#})^M\|_{\tau,\tau'}$ for $0<\tau'<\tau$, where $\|\cdot\|_{\tau,\tau'}$ denotes the norm in the space of bounded linear operators between $H(\tau)$ and $H(\tau')$ and M is an arbitrary positive integer.

We begin by studying $\chi^{\#}G^{\#}T_1$. We have, for $v\in H(\tau)$ and τ small enough,

$$\begin{aligned}\left|(\chi^\sharp G^\sharp T^\sharp)(v)(z,s;\xi)\right| &= \left|\chi\left(s|\xi|^{-1/(\kappa+1)}\right)\int G^\sharp(s,s')(T_1)(v)(z,s',\xi)\,\mathrm{d}s'\right|\\ &\leqslant C\rho\left(\int |s'|^{2\kappa}\,|G^\sharp(s,s';z,\xi)|\,\mathrm{d}s'\right)\|v\|_\tau,\end{aligned}$$

by (4.26) and (4.44), since $|s'||\xi|^{-1/(\kappa+1)}\leqslant\rho$ on the support of v. Hence there exist $C>0$ and $\tau_0>0$, depending only on the operator P, such that

$$\|(\chi^{\#}G^{\#}T_1)^M\|_{\tau,\tau}\leqslant(\rho C)^M, \tag{4.51}$$

for all M and $0<\tau<\tau_0$.

Using similar arguments, we may also estimate $\chi^{\#}G^{\#}S$, where S is the first term in T_2. We see that there exist $C>0$ and $0<\tau'<\tau$, such that for all $M\geqslant 1$ we have

$$\|(\chi^{\#}G^{\#}S)^M\|_{\tau,\tau'}\leqslant|\xi|^{-M/(\kappa+1)}C^M. \tag{4.52}$$

The second term in T_2 is $|\xi|^{-2/(\kappa+1)}(b+R_{b,\delta})$ and we see that $\chi^{\#}G^{\#}(b+R_{b,\delta})$ can be written as

$$\chi^{\#}G^{\#}(b+R_{b,\delta})=\sum_{0\leqslant j\leqslant\delta|\xi|}\frac{B_j^{\#}}{|\xi|^j}, \tag{4.53}$$

where $B_j^{\#}$ satisfies the following : there exist $D>0$ and $\tau_0>0$ such that we have

$$\|B_j^{\#}\|_{\tau,\tau'}\leqslant\frac{D^j j^j}{(\tau-\tau')^j}, \tag{4.54}$$

for all $j\geqslant 0$ and all $0<\tau'<\tau<\tau_0$. Since

$$(\chi^{\#}G^{\#}(b+R_{b,\delta}))^M=\sum_{0\leqslant j_\ell\leqslant\delta|\xi|}\frac{B_{j_1}^{\#}\cdots B_{j_M}^{\#}}{|\xi|^{j_1+\cdots+j_M}} \tag{4.55}$$

we have

$$\|(\chi^{\#}G^{\#}(b+R_{b,\delta}))^M\|_{\tau,\tau'}\leqslant\sum_{j_\ell\geqslant 0}\left(\frac{D\delta}{\tau-\tau'}\right)^{j_1+\cdots+j_M}. \tag{4.56}$$

We choose $\delta > 0$ and $0 < \tau' < \tau$ such that $D\delta/(\tau - \tau') < 1$ and then we obtain

$$\|(\chi^{\#} G^{\#}(b + R_{b,\delta}))^M\|_{\tau,\tau'} \leqslant C^M \tag{4.57}$$

for some constant $C > 0$.

Thus, from (4.52) and (4.57), there are $0 < \tau' < \tau$ and $C > 0$ such that, if $\delta > 0$ is small enough,

$$\|(\chi^{\#} G^{\#} T_2)^M\|_{\tau,\tau'} \leqslant |\xi|^{-M/(\kappa+1)} C^M. \tag{4.58}$$

Next we estimate $(\chi^{\#} G^{\#} T_3)^N$ using the same methods, combined with the asymptotics (4.34) - (4.41) for the derivatives of f and g and the analogous estimate of (4.44) for $\partial G^{\#}/\partial s'$. We see that there exists $C > 0$ and $0 < \tau' < \tau$, such that for all $M \geqslant 1$ we have

$$\|(\chi^{\#} G^{\#} T_3)^M\|_{\tau,\tau'} \leqslant |\xi|^{\frac{-M}{\kappa+1}} C^M. \tag{4.59}$$

Finally we estimate $(\chi^{\#} G^{\#} T_4)^N$. We see, using the estimates (4.44), that $\chi^{\#} G^{\#} T_4$ can be written as

$$\chi^{\#} G^{\#} T_4 = \sum_{j \geqslant 1} \frac{B_j^{\#\#}}{|\xi|^j}, \tag{4.60}$$

where $B_j^{\#\#}$ satisfies the following : there exist $\tilde{D} > 0$ and $\tau_0 > 0$ such that we have

$$\|B_j^{\#\#}\|_{\tau,\tau'} \leqslant \frac{\tilde{D}^j j^j}{(\tau - \tau')^j}, \tag{4.61}$$

for all $j \geqslant 1$ and all $0 < \tau' < \tau < \tau_0$. It then follows, as before, that for all M,

$$\|(\chi^{\#} G^{\#} T_4)^M\|_{\tau,\tau'} \leqslant \sum_{j_\ell \geqslant 1} \left(\frac{D\delta}{\tau - \tau'}\right)^{j_1 + \cdots + j_M}. \tag{4.62}$$

Since $j \geqslant 1$ in the summation we see that given $\epsilon > 0$ we can choose $\delta > 0$ and $0 < \tau' < \tau$ such

$$\|(\chi^{\#} G^{\#} T_4)^M\|_{\tau,\tau'} \leqslant \epsilon^M. \tag{4.63}$$

If we combine estimates (4.51), (4.58), (4.59) and (4.63) we obtain the following: there exist $0 < \tau' < \tau$, $\delta > 0$, $C > 0$ such that

$$\|(\chi^{\#} G^{\#} T^{\#})^M\|_{\tau,\tau'} \leqslant 2(C\rho)^M \tag{4.64}$$

for all $M \geqslant 1$ and all large ξ in a conic neighborhood of ξ_0.

From this the proof of Proposition 4.1 can be easily concluded. It follows from (4.11) and (4.49) that there exist $C, \eta > 0$ such that

(4.65)
$$\|U^{\#}\|_{\tau'} \leqslant \|(\chi^{\#}G^{\#}T^{\#})^N U^{\#}\|_{\tau'} + Ce^{-\eta|\xi|} \leqslant 2(C\rho)^N \|U^{\#}\|_{\tau} + Ce^{-\eta|\xi|}.$$

We know that we have, for some $C > 0$ and $R > 0$,

$$\|U^{\#}\|_{\tau} \leqslant Ce^{R|\xi|}.$$

Hence, if $\rho > 0$ is small and $N \geqslant |\xi|$ we see that there exists a new constant $C > 0$ such that

$$\|U^{\#}\|_{\tau'} \leqslant C\left(e^{-|\xi|} + e^{-\eta|\xi|}\right) \tag{4.66}$$

for ξ in a conic neighborhood of ξ_0. Then $(0, 0; \xi_0, 0) \notin WF_A(u)$ and the proof is complete.

BIBLIOGRAPHY

[1] A. Bove, M. Derridj & D. Tartakoff, "Analytic hypoellipticity in the presence of non–symplectic characteristic points", *J. Funct. Anal.* **234** (2006), no. 2, p. 464-472.

[2] P. D. Cordaro & N. Hanges, "Impact of lower order terms on a model PDE in two variables", in *Geometric analysis of PDE and several complex variables* (Providence, RI) (A. M. Soc., ed.), vol. 368, Contemp. Math., 2005, p. 157-176.

[3] ———, "Symplectic strata and analytic hypoellipticity", in *Progresss in Nonlinear Differential Equations and Their Applications* (Birkhäuser, ed.), Phase Space Analysis of Partial Differential Equations, vol. 69, 2006, p. 83-94.

[4] P. D. Cordaro & A. A. Himonas, "Global analytic regularity for sums of squares of vector fields", *Trans. Amer. Math. Soc.* **350** (1998), no. 12, p. 4993-5001.

[5] N. Hanges, "Analytic regularity for an operator with Treves curves", *J. Funct. Anal.* **210** (2004), no. 2, p. 295-320.

[6] N. Hanges & A. A. Himonas, "Non–analytic hypoellipticity in the presence of symplecticity", *Proceedings of the AMS* **126** (1998), no. 2, p. 405-409.

[7] L. Hörmander, "Hypoelliptic second order differential equations", *Acta Math.* **119** (1967), p. 147-171.

[8] ———, *The analysis of linear partial differential operators I*, Springer–Verlag, 1983.

[9] A. Menikoff, "Some examples of hypoelliptic partial differential equations", *Math. Ann.* **221** (1976), p. 167-181.

[10] G. Métivier, "Analytic hypoellipticity for operators with multiple characteristics", *Commun. Partial Differ. Equations* **6** (1981), no. 1, p. 1-90.

[11] T. Okaji, "Analytic hypoellipticity for operators with symplectic characteristics", *J. Math. Kyoto Univ.* **25** (1985), no. 3, p. 489-514.

[12] O. A. Oleinik, "On the analyticity of solutions of partial differential equations and systems", *Astérisque* **2/3** (1973), p. 272-285.

[13] Y. Sibuya, *Global Theory of a Second Order Linear Ordinary Differential Equation with a Polynomial Coefficient*, North–Holland, 1975.

[14] J. Sjöstrand, "Singularités analytiques microlocales", *Astérisque* **95** (1982), p. 1-166.

[15] D. Tartakoff, "On the local real analyticity of solutions to $\Box_b$ and the $\bar{\partial}-$Neumann problem", *Acta Math.* **145** (1980), p. 117-204.

[16] F. Treves, "Analytic hypoellipticity of a class of pseudodifferential operators with double characteristics and applications to the $\bar{\partial}$–Neumann problem", *Commun. Partial Differ. Equations* **3** (1978), p. 475-642.

[17] ———, "Symplectic geometry and analytic hypo-ellipticity", *Proceedings of Symposia in Pure Mathematics* **65** (1999), p. 201-219.

[18] ———, "On the analyticity of solutions of sums of squares of vector fields", in *Progress in Nonlinear Differential Equations and Their Applications* (Birkhäuser, ed.), Phase space analysis of partial differential equations, vol. 69, 2006, p. 315-329.

Manuscrit reçu le 7 février 2008,
révisé le 10 juillet 2008,
accepté le 17 juillet 2008.

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