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POISSON BOUNDARY OF TRIANGULAR MATRICES IN A NUMBER FIELD

by Bruno SCHAPIRA

ABSTRACT. — The aim of this note is to describe the Poisson boundary of the group of invertible triangular matrices with coefficients in a number field. It generalizes to any dimension and to any number field a result of Brofferio concerning the Poisson boundary of random rational affinities.

RÉSUMÉ. — L'objet de cette note est de décrire la frontière de Poisson du groupe des matrices triangulaires supérieures inversibles à coefficients dans un corps de nombre. C'est une généralisation en dimension supérieure d'un résultat de Brofferio concernant la frontière de Poisson du groupe des applications affines rationnelles.

1. Introduction

The Poisson boundary is a measure space which describes the asymptotic behavior of random walks on groups. In the same time it gives information on the geometry of the group and provides a representation of bounded harmonic functions (we refer for instance to [6] [7] [8] or [10] for more details). Our aim in this paper is to explore the case of a group of matrices with coefficients in a number field. More precisely we study the group of upper triangular matrices with non zero diagonal coefficients.

This example was treated previously by Brofferio [2] for matrices of size 2 and rational coefficients, which corresponds to the case of rational affinities. She proved that the Poisson boundary is the product over all prime numbers p (including $p = \infty$) of "local" boundaries C_p, which are either a p-adic line, or a point. This can be determined explicitly in function of the random walk. In each case one can see C_p as a subspace of the p-adic projective

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line, which is the Poisson boundary of the group of p-adic affinities (*cf.* [5] [3]). The goal of this paper is to generalize this result in higher dimension $d \geqslant 2$. In other words we will prove that the Poisson boundary is a product of local factors C_p, where for every p, C_p is a subspace of the Furstenberg boundary, which is also the space of flags on $\mathbb{Q}_p^d$. It is known that this space, or a quotient, is the Poisson boundary of a large class of random walks on groups of real matrices (see *e.g.*[7] [12] or [13]). There is a well known decomposition of the Furstenberg boundary called Bruhat decomposition, which coincides for $d = 2$ with the decomposition of the projective line into a line and a point. So we will prove that each C_p is a component, also called a Bruhat cell, of this decomposition, that we determine in function of the random walk.

Our proof follows very closely the general strategy of Brofferio [2] in dimension 2. However, as the technical details are a bit different, we will repeat all arguments here. So this paper can be read independently of [2]. The main tools are the law of large numbers (for contraction) and Kaimanovich's entropy criterion (for maximality).

Such factor decompositions of the Poisson boundary were already observed so far. For instance Bader and Shalom [1] proved recently a general factor theorem in an adelic setting. It is in fact rather likely that our result should extend to a more general class of groups, such as $SL_d(\mathbb{Q})$. For this our proof "with hands" should probably be replaced by more powerful tools, such as the Oseledec' theorem (see for instance its use by Ledrappier [12] for the study of discrete subgroups of semisimple groups), or a geometric argument (using for instance Kaimanovich's strip approximation criterion).

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2. Statement of results

Let $\mathbb{K}$ be some number field. The reader non familiar with this may think to the particular case of the rational field $\mathbb{K} = \mathbb{Q}$ (see also section 7 for more details). Let $\mathcal{P}$ be the set of prime ideals (the set of prime numbers in the case $\mathbb{K} = \mathbb{Q}$). For $\mathfrak{p} \in \mathcal{P}$, we denote by $|\cdot|_{\mathfrak{p}}$ the associated norm on $\mathbb{K}$ and by $\mathbb{K}_{\mathfrak{p}}$ the associated completion of $\mathbb{K}$. Let μ be some measure on $A(\mathbb{K})$, the space of upper triangular matrices with coefficients in $\mathbb{K}$ and non zero diagonal coefficients. For $\mathfrak{p} \in \mathcal{P}$ and $i \in [1, \dots, d]$, we set

$$\phi_{\mathfrak{p}}(i) := \int_{A(\mathbb{K})} \ln |a_{i,i}|_{\mathfrak{p}} \; d\mu(a). \tag{2.1}$$

We denote by $B_{\mathfrak{p}}$ the space of flags in $\mathbb{K}_{\mathfrak{p}}^d$. Let W be the group of permutations of the set $\{1, \dots, d\}$. Let w be the unique element in W such that $w(i) > w(j)$ if $i < j$ and $\phi_{\mathfrak{p}}(i) \geqslant \phi_{\mathfrak{p}}(j)$. Let $C_{\mathfrak{p}}(\mu)$ be the Bruhat cell associated to w in the Bruhat decomposition of $B_{\mathfrak{p}}$ (see next section for a definition). If V_∞ is the set of archimedean norms on $\mathbb{K}$ (reduced to the usual absolute value if $\mathbb{K} = \mathbb{Q}$), then for every $v \in V_\infty$, $|\cdot|_v$, $\mathbb{K}_v$,..., are defined analogously.

The main result of this paper is the

Theorem 2.1. — *Let*

$$\mathbf{B} := \prod_{\mathfrak{p} \in \mathcal{P}} B_{\mathfrak{p}} \times \prod_{v \in V_\infty} B_v,$$

be the product of all flag manifolds. For every μ on $A(\mathbb{K})$ satisfying

$$\int_{A(\mathbb{K})} \sum_{i \leqslant j} \left(\sum_{\mathfrak{p} \in \mathcal{P}} |\ln |a_{i,j}|_p| + \sum_{v \in V_\infty} |\ln |a_{i,j}|_v| \right) d\mu(a) < +\infty, \tag{2.2}$$

there exists a measure ν on $\mathbf{B}$ such that $(\mathbf{B}, \nu)$ is the Poisson boundary of $(A(\mathbb{K}), \mu)$. Furthermore, ν is supported on the product of the $C_{\mathfrak{p}}(\mu)$'s and the $C_v(\mu)$'s.

We will prove this theorem in three steps, corresponding to the three following propositions:

Proposition 2.2. — *There exists a measure ν on $\mathbf{B}$, such that the measure space $(\mathbf{B}, \nu)$ is a μ-boundary.*

Proposition 2.3. — *If μ satisfies* (2.2), *then μ has finite entropy.*

Proposition 2.4. — *For ν-almost all $z \in \mathbf{B}$, the asymptotic entropy h^z of the conditional measure $\mathbb{P}^z$ vanishes.*

We will recall all necessary definitions about entropy in the next section. Assuming these propositions, Theorem 2.1 is then a consequence of Kaimanovich's criterion:

Theorem 2.5 (Kaimanovich [10] Theorem 4.6). — *Let G be a countable group and μ a probability measure on G with finite entropy. Then a μ-boundary $(\mathbf{B}, \nu)$ is the Poisson boundary if, and only if, for ν-almost all $z \in \mathbf{B}$, the asymptotic entropy h^z of the conditional measure $\mathbb{P}^z$ vanishes.*

Let us describe now the organization of the paper. In the next section we detail our notations and recall some preliminary background on μ-boundaries, Poisson boundary, entropy and Bruhat decomposition. Then we prove the three propositions above in the particular case of the rational field, which is easier in a first reading. Section 4 is devoted to the proof of Proposition 2.2, section 5 to Proposition 2.3, and section 6 to Proposition 2.4. The last section is devoted to the case of number fields. There are some adjustments to make in the proof that we explain. Finally the appendix is devoted to the proof of a technical result.

3. Preliminaries

Let $d \geqslant 1$ be an integer. Recall that $A(\mathbb{K})$ is the set of upper triangular matrices of size d with coefficients in $\mathbb{K}$ and non zero diagonal coefficients. So if $a = (a_{i,j})_{i,j} \in A(\mathbb{K})$, we have $a_{i,j} = 0$, if $i > j$, and $a_{i,i} \neq 0$ for $1 \leqslant i \leqslant d$. For $n \geqslant 1$ and $z = (z_1, \cdots, z_n) \in \mathbb{K}_{\mathfrak{p}}^n$, we set $|z|_{\mathfrak{p}} = \max_{i=1,\cdots,n} |z_i|_{\mathfrak{p}}$.

Random walk, μ-boundaries, and Poisson boundary

Let μ be a measure on $A(\mathbb{K})$. We consider a sequence $(g_n)_{n\geqslant 1}$ of i.i.d. random variables of law μ on $A(\mathbb{K})$. The random walk $(x_n)_{n\geqslant 0}$ of law μ on $A(\mathbb{K})$ is defined by

$$x_n := g_1 \cdots g_n.$$

We denote by $\mathbb{P}$ the law of $(x_n)_{n\geqslant 1}$ on the path space $A(\mathbb{K})^{\mathbb{N}}$.

Assume that B is a locally compact space, endowed with a measure ν and an action of $A(\mathbb{Q})$. We say that ν is μ-invariant (also known as μ-stationary or μ-harmonic), if

$$\int_{A(\mathbb{K})} (g\nu) d\mu(g) = \nu,$$

where for all $g \in A(\mathbb{K})$, $g\nu$ is defined by

$$g\nu(f) = \int_{\mathbf{B}} f(gz) d\nu(z),$$

for all continuous functions f. In this case, according to Furstenberg [7, 8], we say that (B,ν) is a μ-boundary if, $\mathbb{P}$-almost surely $x_n\nu$ converges vaguely to a Dirac measure. Then the Poisson boundary $(\mathbf{B},\nu)$ is defined as the maximal μ-boundary, *i.e.*, it is a μ-boundary such that any other μ-boundary is one of its measurable G-equinvariant quotients. For any element $\mathbf{x} = (x_n)_{n\geqslant 1} \in A(\mathbb{K})^{\mathbb{N}}$, we define $\mathbf{bnd}\ \mathbf{x} \in \mathbf{B}$ by

$$\lim_{n\to+\infty} x_n\nu = \delta_{\mathbf{bnd}\ \mathbf{x}}.$$

If $z \in \mathbf{B}$, it is possible to define (in the sense of Doob, see [10] for details), the law $\mathbb{P}^z$ of $(x_n)_{n\geqslant 1}$ conditioned by $\mathbf{bnd}\ \mathbf{x} = z$. Then for $n \geqslant 0$, $\mathbb{P}^z_n$ denotes the projection of $\mathbb{P}^z$ on the n^{th} coordinate. In fact if (B,ν) is any μ-boundary and $z \in B$, $\mathbb{P}^z$ and $\mathbb{P}^z_n$ are defined similarly.

Let $\mathrm{sgr}(\mu)$ be the semi-group generated by the support of μ, *i.e.*, $\mathrm{sgr}(\mu) = \cup_n \mathrm{supp}(\mu^{*n})$. We say that a function f on $\mathrm{sgr}(\mu)$ is μ-harmonic if

$$\int_{A(\mathbb{K})} f(gg')d\mu(g') = f(g),$$

for all $g \in \mathrm{sgr}(\mu)$. Furstenberg [7] proved that there is an isometry between the space $\mathcal{H}^\infty(A(\mathbb{K}),\mu)$ of bounded μ-harmonic functions f on $\mathrm{sgr}(\mu)$ and the space $L^\infty(\mathbf{B},\nu)$ of bounded functions F on $\mathbf{B}$. The isometry is given by the formula

$$F(\mathbf{bnd}\ \mathbf{x}) = \lim_{n\to\infty} f(x_n), \qquad f(g) = \int_{\mathbf{B}} F(gz)\ d\nu(z).$$

The second formula is the so-called Poisson integral representation formula of bounded harmonic functions.

Entropy and asymptotic entropy

The entropy of a measure μ on a countable group G is given by the formula:

$$H(\mu) = -\sum_{g\in G} \mu(g) \ln \mu(G).$$

If (B,ν) is a μ-boundary and $z \in B$ the conditional asymptotic entropy h^z is defined by:

$$h^z := -\lim_{n\to+\infty} \frac{\log \mathbb{P}^z_n(x_n)}{n}.$$

Some structure and the Bruhat decomposition

Let $G = GL_d$. We denote by Δ the set of invertible diagonal matrices. We denote by $\delta = \mathrm{diag}(\delta_1, \ldots, \delta_d)$ the diagonal matrix with entries $\delta_{i,i} = \delta_i$, $i = 1, \ldots, d$. Let U be the group of upper triangular matrices with one's in the diagonal (unipotent matrices). The notation $U(\mathcal{R})$, where $\mathcal{R}$ is some ring, means that the coefficients strictly upper the diagonal are in $\mathcal{R}$. Let $\overline{U}$ be the group of lower triangular matrices with one's in the diagonal. We set $A = \Delta U$ and $\overline{A} = \Delta\overline{U}$. We denote by W the Weyl group, identified with the subgroup of permutation matrices. Its action by conjugation on Δ permutes the coordinates of the diagonal. In this way W can also be identified with the group of permutations of the set $\{1, \ldots, d\}$. For $w \in W$, we set $U_w = w\overline{U}w^{-1} \cap U$ and $U^w = wUw^{-1} \cap U$. We have $U = U^w U_w$ and $U^w \cap U_w = \{\mathrm{Id}\}$. An element $u \in U$ lies in U^w if, and only if, $u_{i,j} = 0$ whenever $i < j$ and $w(i) > w(j)$.

The *Bruhat decomposition* (see *e.g.*[9] or [16]) says that G can be decomposed in the following disjoint union:

$$G = \coprod_{w \in W} Aw\overline{A} = \coprod_{w \in W} U^w w\overline{A}.$$

The components $Aw\overline{A}$ are called the Bruhat cells. In the quotient space $G/\overline{A}$ they are identified with the groups U^w by the map

$$U^w \to G/\overline{A}, \quad u \mapsto uw\overline{A}.$$

For $\mathfrak{p} \in \mathcal{P}$, resp. $v \in V_\infty$, we recall that $C_\mathfrak{p}(\mu)$, resp. $C_v(\mu)$, denotes the cell $U^w(\mathbb{K}_\mathfrak{p})$, resp. $U^w(\mathbb{K}_v)$, associated to the $w \in W$ such that $w(i) > w(j)$ if $i < j$ and $\phi_\mathfrak{p}(i) \geqslant \phi_\mathfrak{p}(j)$, resp. $\phi_v(i) \geqslant \phi_v(j)$. If $u \in U(\mathbb{K}_\mathfrak{p})$, resp. $u \in U(\mathbb{K}_v)$, we denote by $\overline{u}$ its component in this $U^w(\mathbb{K}_\mathfrak{p})$, resp. $U^w(\mathbb{K}_v)$, according to the product decomposition $U = U^w U_w$.

The action of $a \in A(\mathbb{K})$ on $b \in C_p(\mu)$ is defined as follows. Assume that $a = u\delta$ with $u \in U$ and $\delta \in \Delta$. Then

$$a \cdot b := \overline{ab\delta^{-1}}.$$

In other words $a \cdot b$ is the unique element of U^w representing ab in $G/\overline{A}$.

4. Proof of Proposition 2.2

To simplify a little the notations and the arguments, we assume here and in the next three sections, that $\mathbb{K} = \mathbb{Q}$. In this case we denote by $\mathcal{P}^*$ the union of $\mathcal{P}$, the set of prime numbers, and $\{\infty\}$, which corresponds to the usual absolute value. So the notation $\mathbb{Q}_\infty$ denotes the field of reals $\mathbb{R}$, and $|\cdot|_\infty$ the absolute value on $\mathbb{R}$. Let $p \in \mathcal{P}^*$. Let $(e_1, \ldots, e_d)$ be the canonical basis of $\mathbb{Q}_p^d$. Let

$$J_d = \{i \leqslant d \mid \phi_p(i) \geqslant \phi_p(d)\},$$

and let r be the cardinality of J_d. Assume that $j_1 < \cdots < j_r = d$ are the elements of J_d. We denote by $\Lambda^r_{\mathrm{sub}}\mathbb{Q}_p^d$ the subspace of $\Lambda^r\mathbb{Q}_p^d$ generated by the elements $e_{i_1} \wedge \ldots \wedge e_{i_r}$ such that $i_s \leqslant j_s$ for all $s \in [1, \ldots, r]$. We denote by $\mathcal{B}_r$ the basis of $\Lambda^r_{\mathrm{sub}}\mathbb{Q}_p^d$ made up of these elements ranged in lexicographical order. Let also m be the dimension of this subspace. Each $a \in A(\mathbb{Q})$ defines an endomorphism of $\Lambda^r\mathbb{Q}_p^d$, by setting $a(v_1 \wedge \ldots \wedge v_r) := av_1 \wedge \ldots \wedge av_r$. We denote by $a^{(r)}$ the restriction of this endomorphism to $\Lambda^r_{\mathrm{sub}}\mathbb{Q}_p^d$. Observe that it has a triangular matrix representation in the basis $\mathcal{B}_r$. This provides a representation of $A(\mathbb{Q})$ on the subspace $P\mathbb{Q}_p^m$ of $\mathbb{Q}_p^m$ whose vectors have last coordinate equal to 1. Indeed for $u \in P\mathbb{Q}_p^m$ and $a \in A(\mathbb{Q})$, we define the (projective) action $a \cdot u$ of a on u by

$$a \cdot u := \frac{1}{\prod_{j \in J} a_{j,j}} a^{(r)} u.$$

LEMMA 4.1. — *For any $u \in P\mathbb{Q}_p^m$, the sequence $(x_n \cdot u)_{n \geqslant 1}$ converges $\mathbb{P}$-a.s. Moreover the limit, that we denote by $(Z_k^p(d))_{k \leqslant m}$, does not depend on the choice of u.*

Proof. — We assume that $m \geqslant 2$, otherwise there is nothing to prove. Let $a \in A(\mathbb{Q})$. We have observed that $a^{(r)}$ has a triangular matrix representation in the basis $\mathcal{B}_r$. We put $a' = \frac{1}{\prod_{j \in J} a_{j,j}} a^{(r)}$. Then $a'_{m,m} = 1$, and for $k < m$, there exists a subset $K \subset \{1, \ldots, d\}$ of cardinality r, such that $a'_{k,k} = \frac{\prod_{j \in K} a_{j,j}}{\prod_{j \in J} a_{j,j}}$. Therefore

$$\begin{aligned} \phi'_p(k) &:= \mathbb{E}[\ln |a'_{k,k}|_p] = \sum_{j \in K} \mathbb{E}[\ln |a_{j,j}|_p] - \sum_{j \in J} \mathbb{E}[\ln |a_{j,j}|_p] \\ &= \sum_{j \in K \cap J^c} \underbrace{(\phi_p(j) - \phi_p(d))}_{<0} - \sum_{j \in J \cap K^c} \underbrace{(\phi_p(j) - \phi_p(d))}_{\geqslant 0} < 0. \end{aligned} \tag{4.1}$$

For $n \geqslant 1$, let

$$x'_n = g'_1 \cdots g'_n.$$

If $k < m$, by the law of large number, a.s.

$$\frac{\ln |(x'_n)_{k,k}|_p}{n} = \frac{1}{n} \sum_{l=1}^{n} \ln |(g'_l)_{k,k}|_p \to \phi'_p(k), \text{ when } n \to \infty. \tag{4.2}$$

First step. Let k be fixed. Let us prove by induction on $k' \in [k, \dots, m-1]$, that for all $k \leqslant k' \leqslant m-1$, there exists a.s. $\alpha > 0$ such that $\frac{1}{n} \ln |(x'_n)_{k,k'}|_p \leqslant -\alpha$ for n large enough. If $k' = k$ the result is immediate from (4.2). We assume now the result for $l \leqslant k'-1$ (with $k' > k$), and we prove it for k'. By the induction hypothesis there exists a.s. $\alpha > 0$ and $N_1 \geqslant 1$ such that for all $l \in [k, \dots, k'-1]$,

$$n \geqslant N_1 \Rightarrow |(x'_n)_{k,l}|_p \leqslant e^{-n\alpha}.$$

Let $\epsilon > 0$ be such that $\alpha - 4\epsilon > 0$, and $\phi'_p(k') + \epsilon < 0$. By (4.2) there exists a.s. $N_2 \geqslant N_1$ such that

$$n \geqslant N_2 \Rightarrow e^{n(\phi'_p(k)-\epsilon)} \leqslant |(x'_n)_{k',k'}|_p \leqslant e^{n(\phi'_p(k)+\epsilon)}. \tag{4.3}$$

We set $c_n = \max_{i,j} |(g'_{n+1})_{i,j}|_p$. We have a.s. $\frac{\ln c_n}{n} \to 0$. Thus there is a.s. some integer $N_3 \geqslant N_2$ such that $n \geqslant N_3 \Rightarrow c_n \leqslant e^{n\epsilon}$. Finally we set $u_n := \frac{(x'_n)_{k,k'}}{(x'_n)_{k',k'}}$. We have for all $n \geqslant 1$,

$$u_{n+1} = u_n + \underbrace{\frac{1}{(x'_{n+1})_{k',k'}} \sum_{l=k}^{k'-1} (x'_n)_{k,l} (g'_{n+1})_{l,k'}}_{:=r_n}. \tag{4.4}$$

With the previous notations we have a.s. for $n \geqslant N_3$,

$$|r_n|_p \leqslant d e^{-n\phi'_p(k') - n(\alpha - 3\epsilon)}. \tag{4.5}$$

We set $C = \max_{n \leqslant N_3} |r_n|_p$. Hence by (4.3), (4.4) and (4.5), we have a.s. for any $N \geqslant N_3$,

$$|(x'_N)_{k,k'}|_p \leqslant |(x'_N)_{k',k'}|_p \sum_{n=1}^{N} |r_n|_p \leqslant N d (e^{-N(\alpha-4\epsilon)} + C e^{N(\phi'_p(k')+\epsilon)}),$$

and the result for k' follows.

Second step. We have for $n \geqslant 1$, and any $k < m$,

$$(x'_{n+1})_{k,m} = (x'_n)_{k,m} + \sum_{l=k}^{m-1} (x'_n)_{k,l} (g'_{n+1})_{l,m}.$$

As a consequence $(x'_n)_{k,m}$ is the partial sum of a series whose general term converges a.s. to 0 exponentially fast (by the first step). Thus it is almost surely convergent. Now take $u \in P\mathbb{Q}_p^m$. By definition $x_n \cdot u = x'_n u$ for all n. So we see that $x_n \cdot u$ converges a.s. to some $(Z_k^p(d))_{k \leqslant m} \in P\mathbb{Q}_p^m$, where for every $k < m$, $Z_k^p(d)$ is the limit of $(x'_n)_{k,m}$, which is independent of u. This finishes the proof of the lemma. □

This lemma says that $P\mathbb{Q}_p^m$ equipped with the law of $(Z_k^p(d))_{k\leqslant m}$ is a μ-boundary. But it implies in fact the

Corollary 4.2. — *For every $p \in \mathcal{P}^*$, there exists a measure ν_p on $C_p(\mu)$ such that the measure space $(C_p(\mu), \nu_p)$ is a μ-boundary.*

Proof. — For all $d' \leqslant d$, we define the minor of size d' of an element $a \in A(\mathbb{Q})$ as the matrix $d' \times d'$ in the upper left corner of a. These matrices act in the same way on $\Lambda_{\text{sub}}^{r(d')}\mathbb{Q}_p^{d'}$, where $r(d')$ is the cardinality of $J_{d'}$. Hence Lemma 4.1 holds as well in this setting. This provides new μ-boundaries and new vectors $(Z_k^p(d'))_{k\leqslant m(d')}$, where $m(d')$ is the dimension of $\Lambda_{\text{sub}}^{r(d')}\mathbb{Q}_p^{d'}$. We claim that the set of vectors $(Z^p(2), \cdots, Z^p(d))$ is associated to an element $(Z_{i,j}^p)_{1\leqslant i,j\leqslant d}$ of $C_p(\mu)$. More precisely we claim that we can define the columns Z_j^p of Z^p, for $j = 1, \cdots, d$, recursively by

$$Z_{i_1}^p \wedge \cdots \wedge Z_j^p = Z^p(j),$$

where $i_1, \cdots, j$ are the elements of J_j. Indeed the set, let say S, of vectors $(V(2), \cdots, V(d))$ which are associated to an element of $C_p(\mu)$ by this way is stable under the action of $A(\mathbb{Q})$. But since the limit $(Z^p(2), \cdots, Z^p(d))$ is independent of the starting point, which can be chosen in S, it must be also in S. Thus if ν_p is the law of the associated Z^p, we get that $(C_p(\mu), \nu_p)$ is a μ-boundary. □

Proof of Proposition 2.2. — It suffices to observe the elementary fact that a product of μ-boundaries is a μ-boundary. So if we define ν on $\mathbf{B}$ to be the law of $(Z^p)_{p\in\mathcal{P}^*}$, we get from Corollary 4.2 that $(\mathbf{B}, \nu)$ is a μ-boundary. □

5. Gauges on $A(\mathbb{Q})$ and proof of Proposition 2.3

We denote by $\mathbb{A}$ the adele ring of $\mathbb{Q}$, *i.e.*, the restricted product $\Pi'_{p\in\mathcal{P}^*}\mathbb{Q}_p$ (see *e.g.*[17]). The notation Π' means that if $(z^p)_{p\in\mathcal{P}} \in \mathbb{A}$, then for all p but a finite number, $|z^p|_p \leqslant 1$. Let H be the group of upper triangular matrices with non zero rational diagonal coefficients and strictly upper diagonal coefficients in $\mathbb{A}$. In other words

$$H := U(\mathbb{A})\ \Delta(\mathbb{Q}).$$

We have a natural injection $i_{\mathbb{A}}$ from $\mathbb{Q}$ into $\mathbb{A}$ and therefore also an injection i_H from $A(\mathbb{Q})$ into H. Via i_H we will sometimes identify elements in $A(\mathbb{Q})$ with their image in H. For $q \in \mathbb{Q}^*$, we set

$$\langle q\rangle := \sum_{p\in\mathcal{P}} |\ln |q|_p|.$$

In particular for every irreducible fraction $q = \pm r/s$ of integers, one has $\langle q\rangle = \ln r + \ln s$. If $\delta = \mathrm{diag}(\delta_1, \dots, \delta_d) \in \Delta(\mathbb{Q})$, we set

$$\langle\delta\rangle := \sum_{i=1}^{d} \langle\delta_i\rangle.$$

For $b = (b^p)_{p\in\mathcal{P}^*} \in \mathbb{A}$ and $p \in \mathcal{P}^*$, we set

$$\langle b\rangle_p^+ := \ln^+ |b^p|_p,$$

where $\ln^+$ denotes the positive part of the function ln and

$$\langle b\rangle^+ := \sum_{p\in\mathcal{P}^*} \langle b\rangle_p^+$$

If $u \in U(\mathbb{A})$ and $p \in \mathcal{P}^*$, we set

$$\langle u\rangle_p^+ := \sum_{i<j} \langle u_{i,j}\rangle_p^+ \quad \text{and} \quad \langle u\rangle^+ := \sum_{i<j} \langle u_{i,j}\rangle^+.$$

Let $h \in H$ and let $h = u\delta$ be its decomposition in $U(\mathbb{A})\Delta(\mathbb{Q})$. We define the adelic length of h by

$$||h|| := \langle u\rangle^+ + \langle\delta\rangle.$$

The adelic length is not sub-additive but it is almost the case. Indeed for any $q, q' \in \mathbb{Q}^*$,

$$\langle qq'\rangle \leqslant \langle q\rangle + \langle q'\rangle,$$

and for any $b, b', b'' \in \mathbb{A}$, and $q \in \mathbb{Q}^*$,

$$\langle b + qb'b''\rangle^+ \leqslant \ln 2 + \langle b\rangle^+ + \langle b'\rangle^+ + \langle b''\rangle^+ + \langle q\rangle.$$

Using these relations we can find constants $K > 0$ and $K' > 0$ such that for all $h, h' \in H$,

$$||hh'|| \leqslant K + K'(||h|| + ||h'||).$$

Now we consider the family of gauges $(\mathcal{G}_k^h)_{k\in\mathbb{N}}$ on $A(\mathbb{Q})$ defined for $k \geqslant 0$ and $h \in H$, by

$$\mathcal{G}_k^h := \{a \in A(\mathbb{Q}) \mid ||a^{-1}h|| \leqslant k\}. \tag{5.1}$$

We have

Lemma 5.1. — *The family of gauges $\{\mathcal{G}^h\}_{h\in H}$ has uniform exponential growth, i.e., there exists $C' > 0$ such that* $\mathrm{Card}\{\mathcal{G}_k^h\} \leqslant e^{C'k}$ *for all $h \in H$ and all $k \in \mathbb{N} - \{0\}$.*

Proof. — First, since the inverse map is a bijection of $A(\mathbb{Q})$, we can always replace a^{-1} by a in the definition of the gauges. Now let h_0 be the unit element of H, and let $a \in A(\mathbb{Q})$ be such that $||ah_0|| \leqslant k$. In this case $\langle a_{i,i} \rangle \leqslant k$ for any $i \leqslant d$, and $\langle a_{i,j}/a_{j,j} \rangle^+ \leqslant k$ for any $i < j$. But the number of rational $q \neq 0$ such that $\langle q \rangle \leqslant k$ is lower than $2e^{2k}$. Moreover, for any rational q, $\langle q \rangle^+ \geqslant \langle q \rangle / 2$. Thus

$$\Big\langle \frac{a_{i,j}}{a_{j,j}} \Big\rangle^+ \geqslant \frac{1}{2} \Big\langle \frac{a_{i,j}}{a_{j,j}} \Big\rangle \geqslant \frac{1}{2}(\langle a_{i,j} \rangle - \langle a_{j,j} \rangle),$$

which implies

$$\mathrm{Card}(\mathcal{G}_k^{h_0}) \leqslant (2e^{6k})^{d^2}.$$

Now let $h \in H$. Since the multiplication by any element is a bijection of $A(\mathbb{Q})$, we do not change the cardinality of the $\mathcal{G}_k^h$ if we multiply to the left h by an element in $A(\mathbb{Q})$. Hence, multiplying them if necessary by $\mathrm{diag}(h_1^{-1}, \dots, h_d^{-1})$ we can always suppose that $h_1 = \cdots = h_d = 1$. Then it is elementary to find $b \in A(\mathbb{Q})$ such that $||h^{-1}b|| = 0$. Hence for any $a \in A(\mathbb{Q})$,

$$||ab|| = ||ahh^{-1}b|| \leqslant K + K'(||ah|| + ||h^{-1}b||).$$

Thus $\mathcal{G}_k^h \subseteq \mathcal{G}_{K+K'k}^b$, which has the same cardinality as $\mathcal{G}_{K+K'k}^{h_0}$, since $b \in A(\mathbb{Q})$. This concludes the proof of the lemma. $\square$

If $\mathcal{G} = (\mathcal{G}_n)_{n \geqslant 1}$ is a gauge on a countable group G, and if $g \in G$, we set

$$|g|_{\mathcal{G}} := \inf\{n \mid g \in \mathcal{G}_n\}.$$

Then if μ is a measure on G, the first moment $|\mu|_{\mathcal{G}}$ of μ with respect to $\mathcal{G}$ is defined by:

$$|\mu|_{\mathcal{G}} := \sum_{g \in G} |g|_{\mathcal{G}} \mu(g).$$

The proof of Proposition 2.3 follows now from Derriennic's criterion:

Theorem 5.2 (Derriennic [4]). — *Let μ be a probability measure on a countable group G. If μ has finite first moment with respect to some exponentially growing gauge, then μ has finite entropy.*

Indeed Hypothesis (2.2) says exactly that μ has finite first moment with respect to the gauge $(\mathcal{G}_n^{h_0})_{n \geqslant 1}$, where h_0 is the identity matrix, and Lemma 5.1 assures in particular that $\mathcal{G}^{h_0}$ has exponential growth.

6. Proof of Proposition 2.4

We start by some preliminary estimates. Remember that if $q \in \mathbb{Q}^*$, then

$$\langle q\rangle = \sum_{p\in\mathcal{P}} |\ln|q|_p|.$$

Remember also the definition of ϕ_p from (2.1). We have

Lemma 6.1. — *For $i \leqslant d$, and $n \geqslant 1$, let $q_n^i = \prod_{p\in\mathcal{P}} p^{-\left[n\frac{\phi_p(i)}{\ln p}\right]}$, where for $x \in \mathbb{R}$, $[x]$ denotes the integer part of x if $x \geqslant 0$, and the opposite of the integer part of $-x$ otherwise. Then*

$$\frac{\langle (x_n)_{i,i}^{-1} q_n^i\rangle}{n} \to 0, \ \textit{in } L^1.$$

Proof. — For $p \in \mathcal{P}$, the ergodic theorem implies that

$$\frac{\ln|(x_n)_{i,i}^{-1}q_n^i|_p}{n} = \frac{-\sum_{k=1}^n \ln|(g_k)_{i,i}|_p + \ln p\left[n\frac{\phi_p}{\ln p}\right]}{n} \to 0$$

in L^1. Thus by the dominated convergence theorem, the sequence

$$\mathbb{E}\left[\frac{\langle (x_n)_{i,i}^{-1} q_n^i\rangle}{n}\right] = \sum_{p\in\mathcal{P}} \frac{\mathbb{E}\left[|\ln|(x_n)_{i,i}^{-1}q_n^i|_p|\right]}{n}$$

converges to zero because each term of the infinite sum converges to zero and is dominated by $\mathbb{E}\left[|\ln|a_{i,i}|_p|\right] + |\phi_p|$ whose series is convergent by (2.2). □

Let now P be some finite subset of $\mathcal{P}^*$ and let $q_n = \mathrm{diag}(q_n^1, \dots, q_n^d)$. We set

$$\begin{array}{rccc} \pi_n^P : & \prod_{p\in P^*} B_p & \longrightarrow & H \\ & (z^p)_{p\in P^*} & \mapsto & \mathbf{z}\ q_n, \end{array}$$

where for $i<j$, $\mathbf{z_{i,j}} \in \mathbb{A}$ is defined by

$$\mathbf{z^P_{i,j}} = \begin{cases} z^p_{i,j} & \text{if } p \in P, \\ 0 & \text{otherwise.}\end{cases}$$

We set also $Z^P := (Z^p)_{p\in P} \in \prod_{p\in P} B_p$. The main result of this section is the following proposition.

Lemma 6.2. — *Let P be some finite subset of $\mathcal{P}^*$ containing ∞. For $p \in \mathcal{P}^*$, let $K_p = \sum_{r\leqslant s}\int_{A(\mathbb{Q})} |\ln|a_{r,s}|_p|\ d\mu(a)$. Let $\epsilon > 0$ be some constant. Then there exists a constant $C > 0$, such that*

$$\mathbb{P}\left[\frac{||x_n^{-1}\pi_n^P(Z^P)||}{n} \leqslant \epsilon + C\sum_{p\notin P} K_p\right] \to 1.$$

Proof. — Assume that

$$x_n = u_n \delta_n \quad \forall n \geqslant 1,$$

with $u_n \in U$ and $\delta_n \in \Delta$. We have

$$||x_n^{-1}\pi_n^P(Z^P)|| = \langle \delta_n^{-1} q_n \rangle + \langle x_n^{-1}\pi_n^P(Z^P) q_n^{-1}\delta_n\rangle^+.$$

First we know by Lemma 6.1 that $\langle \delta_n^{-1} q_n\rangle/n$ converges to 0 in L^1. So it converges also to 0 in probability. Next

$$\langle x_n^{-1}\pi_n^P(Z^P) q_n^{-1}\delta_n\rangle^+ = \sum_{p\in P} \langle x_n^{-1} Z^p \delta_n\rangle_p^+ + \sum_{p\notin P}\langle x_n^{-1}\delta_n\rangle_p^+.$$

First step: the sum over $p \notin P$.

For $i \leqslant j$ and $N \geqslant 1$ we have

$$\frac{(x_{N+1}^{-1})_{i,j}}{(x_{N+1}^{-1})_{i,i}} = \frac{(x_N^{-1})_{i,j}}{(x_N^{-1})_{i,i}} + \underbrace{\sum_{k=i+1}^{j} \frac{(x_N^{-1})_{k,j}}{(x_{N+1}^{-1})_{i,i}} (g_{N+1}^{-1})_{i,k}}_{:=r_N}.$$

By the ultra-metric property we get

$$\ln^+ |(x_{N+1}^{-1})_{i,j}|_p \leqslant |\ln|(x_{N+1}^{-1})_{i,i}|_p| + \max_{1\leqslant n\leqslant N} \ln^+ |r_n|_p. \tag{6.1}$$

For $n \geqslant 1$, we set $c_n = \max_{r,s} |(g_{n+1}^{-1})_{r,s}|_p$. Observe that for some constant $C_1 > 0$, $E[|\ln c_n|] \leqslant C_1 K_p$. By (6.1), we have

$$\begin{aligned}\max_{1\leqslant n\leqslant N+1} \ln^+ |(x_n^{-1})_{i,j}|_p \quad &\leqslant \quad 2\sum_{n=1}^{N} (|\ln|(g_{n+1}^{-1})_{i,i}|_p| + |\ln c_n|) \\ &\quad + \quad \max_{i+1\leqslant k\leqslant j}\ \max_{1\leqslant n\leqslant N} \ln^+ |(x_n^{-1})_{k,j}|_p.\end{aligned} \tag{6.2}$$

Now by an elementary induction on $(j-k) \in [0,\dots,j-i]$ (with j fixed), we get from (6.2)

$$\forall N \geqslant 1 \quad \frac{1}{N}\mathbb{E}\Big[\max_{1\leqslant n\leqslant N+1} \ln^+ |(x_n^{-1})_{i,j}|_p\Big] \leqslant CK_p, \tag{6.3}$$

for some constant $C > 0$. Let now

$$\alpha_n^p := \frac{1}{n} \ln^+ |(x_n^{-1})_{i,j}|_p.$$

Again from (6.2) we get by induction on $j-k$ that a.s. for all $p \notin P$, $(\alpha_n^p - CK_p)^+$ tends to 0, when $n \to +\infty$. By (6.3) and Lebesgue theorem, we have even that $\sum_p (\alpha_n^p - CK_p)^+$ converges to 0 in L^1. So for some constant $C' > 0$,

$$\mathbb{P}\Big[\frac{1}{n}\sum_{p\notin P} \langle x_n^{-1}\delta_n\rangle_p^+ \leqslant C' \sum_{p\notin P} K_p\Big] \to 1. \tag{6.4}$$

Second step: the sum over $p \in P$.

We will show now that for all $i \leqslant j$ and all $p \in P$,

$$\left|\frac{(x_n^{-1}Z^p)_{i,j}}{(x_n^{-1})_{j,j}}\right|_p \leqslant e^{n\epsilon}$$

for n large enough. Together with (6.4) this will conclude the proof of the lemma. Without loss of generality we can always suppose that $i=1$ and $j=d$. For $n \geqslant 1$, and $l \geqslant 1$, we have

$$\frac{(x_n^{-1}Z^p)_{l,d}}{(x_n^{-1})_{d,d}} = \sum_{k=l}^{d} \frac{(x_n^{-1})_{l,k}}{(x_n^{-1})_{d,d}} Z^p_{k,d} := u_n^l$$

Using that $x_n x_n^{-1} = \mathrm{Id}$, we get by an elementary induction on l that, for $l \geqslant 1$,

$$u_n^l = \frac{(x_n)_{d,d}}{(x_n)_{l,l}} Z^p_{l,d} - \sum_{k=l+1}^{d} \frac{(x_n)_{l,k}}{(x_n)_{l,l}} u_n^k. \tag{6.5}$$

Now for $l < k$ let $A(l,k) := ((x_n)_{i,j})_{l \leqslant i < k, l < j \leqslant k}$. Next we will need the elementary formula

$$\frac{\det A(l,k)}{\prod_{l'=l+1}^{k}(x_n)_{l',l'}} = \sum_{l'=l+1}^{k} (-1)^{l'-l+1} \frac{(x_n)_{l,l'}}{(x_n)_{l',l'}} \frac{\det A(l',k)}{\prod_{l''=l'+1}^{k}(x_n)_{l'',l''}}, \tag{6.6}$$

where by convention $A(k,k) = (1)$. We denote also for $l < k_1 < k$, by $A(l,\widehat{k_1},k)$ the matrix $A(l,k)$, where the k_1^{th} line and the $(k_1-1)^{th}$ column are omitted. With evident notation we define analogously $A(l,\widehat{k_1},\ldots,\widehat{k_r},k)$ for $l < k_1 < \cdots k_r < k$. For any $l < d$ we set $I_d^l := \{l\} \cup (J - \{d\})$, $J_d^l := \{j \in J /\ j \neq l\}$ and

$$S_n^l := \frac{\epsilon_l^p}{\prod_{j\in J}(x_n)_{j,j}} \det\Big(((x_n)_{i,j})_{(i,j)\in I_d^l \times J_d^l}\Big),$$

where $\epsilon_l^p = (-1)^{\mathrm{Card}\{l<i<d \mid \phi_p(i) \geqslant \phi_p(d)\}}$. By convention we set also $S_n^d = -1$. We will need the

Lemma 6.3. — *When $l \notin J$,*

$$Z^p_{l,d} = \lim_{n\to\infty} S_n^l.$$

We postpone the proof of this lemma to the appendix. Let $\{i_1,\ldots,i_s\} = J^c$ and for $l \geqslant 1$, let $k_l = \min\{k \leqslant s \mid i_k \geqslant l\}$. First we prove by induction on $d-l \geqslant 0$, that

$$u_n^l = -\,\mathbf{1}_{(l\in J)}\frac{(x_n)_{d,d}}{(x_n)_{l,l}}S_n^l + \sum_{k=k_l}^{s}(-1)^{i_k-l}\frac{(x_n)_{d,d}^{i_k-l+1}\det A(l,i_k)}{(x_n)_{l,l}\cdots(x_n)_{i_k,i_k}}(Z^p_{i_k,d}-S_n^{i_k}), \tag{6.7}$$

where we recall our convention $A(l,i_{k_l})=(1)$ if $i_{k_l}=l$ (i.e., if $l\notin J$). In fact the result is trivial for $l=d$. Now we suppose that it is true for l strictly greater than some l_0. Then Formula (6.7) for l_0 is a direct consequence of (6.5) and (6.6), which proves the induction step. Next we prove also by induction on $d-l\geqslant 0$, that

(6.8)

$$u_n^l = \frac{(x_n)_{d,d}}{(x_n)_{l,l}}(Z^p_{l,d}-S_n^l) + \sum_{k=k_l}^{s}\frac{(-1)^{i_k-l}\det A(l,\widehat{i_{k_l}},\dots,\widehat{i_{k-1}},i_k)}{(x_n)_{l,l}\cdots(\widehat{x_n})_{i_{k_l},i_{k_l}}\cdots(\widehat{x_n})_{i_{k-1},i_{k-1}}\cdots(x_n)_{i_k-1,i_k-1}}u_n^{i_k},$$

where the notation $\hat{x}$ means that x is omitted in the list. Formula (6.8) is true for $l=d$. So we suppose that it is true for l strictly greater than some l_0. Then observe that for any $l<k'<k$,

$$\det A(l,k)=\det A(l,k')\det A(k',k)+(-1)^{k'-l}(x_n)_{k',k'}\det A(l,\widehat{k'},k).$$

Injecting this in (6.7) and using the induction hypothesis we get (6.8) for l_0, and we can conclude by the induction principle. Eventually we prove again by induction on $d-l\geqslant 0$, that $|u_n^l|_p\leqslant e^{n\epsilon}$ for n large enough. We suppose that it is true for l strictly greater than some l_0. For any l and any $k>k_l$, $\det A(l,\dots,\widehat{i_{k-1}},i_k)$ is equal to the component on $e_l\cdots\wedge\widehat{e_{i_{k-1}}}\cdots\wedge e_{i_k-1}$ of $(ae_2\cdots\wedge\widehat{ae_{i_{k-1}}}\cdots\wedge ae_{i_k})$. Therefore as in the proof of Lemma 4.1, we see that

$$\Big|\frac{\det A(l,\widehat{i_{k_l}},\dots,\widehat{i_{k-1}},i_k)}{(x_n)_{l,l}\cdots(\widehat{x_n})_{i_{k_l},i_{k_l}}\cdots(\widehat{x_n})_{i_{k-1},i_{k-1}}\cdots(x_n)_{i_k-1,i_k-1}}\Big|_p\leqslant e^{n\epsilon},$$

for n large enough. Moreover if $l_0\in J$, in which case $Z^p_{l_0,d}=0$, we have also by the same argument $|\frac{(x_n)_{d,d}}{(x_n)_{l_0,l_0}}S_n^{l_0}|_p\leqslant e^{n\epsilon}$ for n large enough. Then we immediately prove the result for l_0, by using the induction hypothesis and Formula (6.8). This finishes the proof of the lemma. □

We are now ready for the

Proof of Proposition 2.4. — Let P be some finite subset of $\mathcal{P}^*$ containing ∞. For $z\in\prod_{p\in\mathcal{P}^*}B_p$, let z^P be its natural projection on $\prod_{p\in P}B_p$. Let

$K = \epsilon + C\sum_{p\notin P} K_p$, where ϵ, C and K_p are as in Lemma 6.2. Then by Lemma 6.2 (remember also (5.1))

$$\mathbb{P}\left[x_n \in \mathcal{G}_{nK}^{\pi_n^P(Z_\infty^P)}\right] = \int_{\mathbf{B}} \mathbb{P}_n^z\left[\mathcal{G}_{nK}^{\pi_n^P(z^P)}\right] d\nu(z) \to 1. \tag{6.9}$$

Remember that h^z denotes the $\mathbb{P}^z$-almost sure limit of $-\ln \mathbb{P}_n^z(x_n)/n$. Consider the set

$$A_n = \{g \in A(\mathbb{Q}) \mid -h^z - \epsilon < \ln \mathbb{P}_n^z(g)/n < -h^z + \epsilon\}.$$

Then

$$\mathbb{P}_n^z(A_n \cap \mathcal{G}_{nK}^{\pi_n^P(z^P)}) \leqslant e^{n(\epsilon - h)}\,\mathrm{Card}\left(\mathcal{G}_{nK}^{\pi_n(z^P)}\right) \leqslant e^{n(\epsilon-h^z)}e^{C'nK},$$

where C' is the parameter of the exponential growth of the gauges $(\mathcal{G}^g)_{g\in H}$. Thus we must have $C'K - h^z + \epsilon \geqslant 0$ for ν-almost all $z \in \mathbf{B}$. Otherwise this would contradict (6.9). Since ϵ was arbitrarily chosen, we get

$$h^z \leqslant C'C\sum_{p\notin P} K_p.$$

Letting now P grow to $\mathcal{P}$, we obtain $h^z = 0$, which concludes the proof of the proposition. $\square$

7. The case of a number field

In this section $\mathbb{K}$ denotes a number field, *i.e.*, a finite extension of $\mathbb{Q}$. We refer to [14] [15] [17] for the general theory. Let $\mathcal{O}$ be the ring of integers of $\mathbb{K}$. The main difference with the rational case is that except for $\mathbb{Q}$ or imaginary quadratic extensions of $\mathbb{Q}$, the set $\mathcal{O}^*$ of units (the invertible elements of $\mathcal{O}$) of $\mathbb{K}$ is infinite. So we have to be careful when defining the gauges, to keep them with uniform exponential growth. Namely we have to define $\langle k\rangle$ for $k \in \mathbb{K}$, in a suitable way. More precisely, let $\mathcal{P}$ be the set of prime ideals of $\mathcal{O}$, and for $\mathfrak{p} \in \mathcal{P}$ let $v_\mathfrak{p}$ be the associated discrete valuation. Let $N_\mathfrak{p} = \mathrm{Card}(\mathcal{O}/\mathfrak{p})$. Following a usual convention (see *e.g.*[11]), we define the norm associated to $\mathfrak{p}$ by

$$|k|_\mathfrak{p} := N_\mathfrak{p}^{-v_\mathfrak{p}(k)},$$

for all $k \in \mathbb{K}^*$. Let N be the norm function on $\mathcal{O}$. If $k = x^{-1}y$ with $x, y \in \mathcal{O}$ such that $(x) \wedge (y) = 1$, then

$$\sum_{\mathfrak{p}\in\mathcal{P}} |\ln|k|_\mathfrak{p}| = \ln|N(x)| + \ln|N(y)|.$$

Remember that the norm of any unit is equal to ± 1. Thus with the previous notation we can not define $\langle k \rangle$ as the sum $\ln |N(x)| + \ln |N(y)|$ like in the rational case. Otherwise the associated gauges would have an infinite cardinality. So we have to add a term corresponding to archimedean norms. Remember that V_∞ denotes the set of norms extending the usual absolute value $|\cdot|$ on $\mathbb{Q}$. If $v \in V_\infty$ we will write (with a slight abuse of notation) $v" = "|\cdot|_v$ and we define the norm $|\cdot|_v := |\cdot|_v^{\epsilon_v}$ on $\mathbb{K}$, where $\epsilon_v = 1$ if $\mathbb{K}_v = \mathbb{R}$ whereas $\epsilon_v = 2$ if $\mathbb{K}_v = \mathbb{C}$. Then we have the product formula (see [11])

$$\prod_{\mathfrak{p} \in \mathcal{P}} |k|_{\mathfrak{p}} \times \prod_{v \in V_\infty} |k|_v = 1,$$

for all $k \in K^*$, which implies by the way the identity

$$\sum_{\mathfrak{p} \in \mathcal{P}} \phi_{\mathfrak{p}} + \sum_{v \in V_\infty} \phi_v = 0. \tag{7.1}$$

Now we fix some archimedean norm $|\cdot|_{v_0}$ and we define

$$\langle k \rangle := \sum_{\mathfrak{p} \in \mathcal{P}} |\ln |k|_{\mathfrak{p}}| + \sum_{v \neq v_0} |\ln |k|_v|.$$

In this way the set of $k \in \mathbb{K}^*$ such that $\langle k \rangle \leqslant C$ has a cardinality bounded by $\text{const} \cdot e^{\text{const} \cdot C}$, where the constants are independent of C. Then we can define the height function on the adele ring and the associated gauges, in the same way as in the rational case. The only other change in the proof is the definition of the q_n^i (see section 6). Remember that the set of units is isomorphic to $\mathbb{Z}^{r_1+r_2-1} \times G$, where G is cyclic, r_1 is the number of embedding of $\mathbb{K}$ in $\mathbb{R}$ and $2r_2$ the number of embedding in $\mathbb{C}$. We set

$$q_n^i = \prod_{\mathfrak{p} \in \mathcal{P}} p^{-[\frac{n\phi_{\mathfrak{p}}(i)}{\ln p}]} \prod_{v \neq v_0} u_v^{-[n\alpha_v]},$$

where in the first product, for each $\mathfrak{p}$ the prime number p is such that $v_{\mathfrak{p}}$ extends v_p on $\mathbb{Q}^*$, and in the second product the $u_v \in \mathcal{O}^*$ and the $\alpha_v \in \mathbb{R}$ are chosen as follows. For $(u_v)_{v \neq v_0}$ take any basis of $\mathbb{Z}^{r_1+r_2-1}$ (seen as a subset of $\mathcal{O}^*$). Then the matrix $(\ln ||u_v||_w)_{v,w \neq v_0}$ is invertible (see the proof of Theorem 1 p.72 in [14]). So one can choose $(\alpha_v)_{v \neq v_0}$ such that

$$\sum_{v \neq v_0} \alpha_v \ln ||u_v||_w = \sum_{v \neq w} \phi_v(i),$$

for all $w \neq v_0$. Thus with (7.1) one can check that the analogue of Lemma 6.1 holds. The other parts of the proof are unchanged. We leave the details to the reader.

8. Appendix

Proof of Lemma 6.3. — Let $l \notin J$. Let $i_1 < \cdots < i_r$ be the elements of I_d^l. Assume that $e_{i_1} \wedge \cdots e_{i_r}$ is the k^{th} element of the basis $\mathcal{B}_r$ (with the notation of section 4). First by definition of Z^p, we can see that $Z_{l,d}^p = \epsilon_l^p Z_k^p(d)$. Next we have seen in the proof of Lemma 4.1 that $Z_k^p(d) = \lim_{n\to\infty}(x'_n)_{k,m}$. We will show in fact directly that for any $a \in A(\mathbb{Q})$,

$$a_{k,m}^{(r)} = \det\Big(\underbrace{(a_{i,j})_{(i,j)\in I_d^l\times J}}_{:=M(l)}\Big).$$

Naturally it will imply the lemma. We prove the result by induction on $h = d-l$. If $h = 1$, *i.e.*, $l = d-1$, then $I_d^l = \{j_1, \ldots, j_{r-1}, d-1\}$, and $a_{k,m}^{(r)} = a_{j_1,j_1} \ldots a_{j_{r-1},j_{r-1}} a_{d-1,d}$. Then the result is immediate. We prove now the induction step from h to $h+1$. We suppose that $j_{s-1} < l < j_s$ for some s (if $s = 1$ we have just $l < j_s$). The coefficient $a_{k,m}^{(r)}$ is equal to the component on $e_{j_1} \cdots \wedge e_l \cdots \wedge e_{j_{r-1}}$ of $(ae_{j_1} \wedge \cdots \wedge ae_{j_r})$. This component is equal to the sum over $k \in [s, \ldots, r]$, of the components of $(\prod_{j<l} a_{j,j})(e_{j_1} \cdots \wedge ae_{j_s} \cdots \wedge a_{l,j_k} e_l \cdots \wedge \widehat{ae_{j_k}} \cdots \wedge ae_{j_r})$ on $e_{j_1} \cdots \wedge e_{j_k} \cdots \wedge e_{j_{r-1}}$. But by the induction hypothesis, for each $k \in [s, \ldots, r]$, the corresponding component is equal to a_{l,j_k} times the cofactor of a_{l,j_k} in the matrix $M(l)$. This gives exactly the formula of the determinant of $M(l)$. Therefore the proof of the lemma is finished. □

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