ANNALES

DE

L'INSTITUT FOURIER

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Tome 59, nº 2 (2009), p. 543-562.

<http://aif.cedram.org/item?id=AIF_2009__59_2_543_0>

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Ann. Inst. Fourier, Grenoble
59, 2 (2009) 543-562

AN O-MINIMAL STRUCTURE WHICH DOES NOT ADMIT C^∞ CELLULAR DECOMPOSITION

by Olivier LE GAL & Jean-Philippe ROLIN

Abstract. — We present an example of an o-minimal structure which does not admit C^∞ cellular decomposition. To this end, we construct a function H whose germ at the origin admits a C^k representative for each integer k, but no C^∞ representative. A number theoretic condition on the coefficients of the Taylor series of H then insures the quasianalyticity of some differential algebras $\mathcal{A}_n(H)$ induced by H. The o-minimality of the structure generated by H is deduced from this quasianalyticity property.

Résumé. — Nous présentons un exemple de structure o-minimale n'admettant pas la propriété de décomposition cellulaire C^∞. Pour ce faire, nous construisons une fonction H dont le germe en 0 admet un représentant C^k pour tout entier k, mais n'admet aucun représentant C^∞. Une condition de transcendance sur les coefficients de la série de Taylor de H assure alors la quasi-analyticité de certaines algèbres différentielles $\mathcal{A}_n(H)$ engendrées par H. La o-minimalité de la structure engendrée par H est enfin déduite de cette quasi-analyticité.

Consider a family $\mathcal{F}$ of functions $f\colon \mathbb{R}^m \to \mathbb{R}$ for various $m \in \mathbb{N}$. The model theoretic notion of **structure generated by** $\mathcal{F}$, denoted by $\mathbb{R}_{\mathcal{F}}$ or $\langle \mathbb{R}, <, 0, 1, +, -, \cdot, \mathcal{F}\rangle$, provides useful information about the real geometry generated by functions in $\mathcal{F}$. Recall that a subset of $\mathbb{R}^m$, is said to be **definable** in the structure $\mathbb{R}_{\mathcal{F}}$ (also called the **expansion** of the ordered field of real numbers by $\mathcal{F}$) if it belongs to the smallest collection of subsets of $\mathbb{R}^n$, $n \in \mathbb{N}$, which

(i) contains the graphs of addition and multiplication, and all the graphs of functions in $\mathcal{F}$, and of constant maps;
(ii) contains the graph of the order relation $<$, and of the equality;
(iii) is closed under taking cartesian products, finite unions or intersections, complements, and images under linear projection maps.

A map $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is also said to be definable if its graph is definable. The two following properties express the *tameness* of definable sets. If in the

Keywords: o-minimal, smooth cell decomposition.
Math. classification: 03C64 57-99 26A27 57R45.

previous definition, the operation of taking complement is superfluous, the structure is **model complete**. This property is analogous to the classical *Gabrielov's complement Theorem*. On another hand, if each definable set has finitely many connected components, the structure $\mathbb{R}_{\mathcal{F}}$ is **o-minimal**. The o-minimality of a structure provides many nice geometric properties for definable sets, such as Whitney stratification.

Let us mention classical examples of o-minimal structures. The family of so-called **restricted analytic functions** generates the structure $\mathbb{R}_{\mathrm{an}}$, which is also model complete (see [2, 6]). The structure $\mathbb{R}_{\mathrm{Pfaff}}$, where $\mathcal{F}$ is the family of so-called **pfaffian** functions is o-minimal as well (see [10]). The preceding implies the o-minimality of $\mathbb{R}_{\exp}$, generated by the exponential function.

An important consequence of o-minimality is the property of $\mathcal{C}^k$**-cell decomposition** [3]: given a positive integer k, for any definable subset $A \subset \mathbb{R}^n$, there exists a finite cellular decomposition of $\mathbb{R}^n$ into $\mathcal{C}^k$ cells adapted to A (*i.e.* A and its complement $\mathbb{R}^n \setminus A$ are finite unions of cells). It turns out that most of the known o-minimal expansions of the real field admit **analytic** cell decomposition. It has been proved in [9] that it is not always the case: the o-minimal structures generated by convenient **quasianalytic Denjoy-Carleman** classes admit $\mathcal{C}^\infty$ cell decomposition but no analytic cell decomposition.

The present work is devoted to the proof of the following result:

Theorem. — *There exists an o-minimal expansion of the real field which doesn't admit $\mathcal{C}^\infty$ cell decomposition.*

We actually give an explicit construction of such a structure. More precisely, we build a function $H\colon \mathbb{R} \to \mathbb{R}$ such that the structure $\mathbb{R}_H$ is o-minimal and whose germ at the origin admits a $\mathcal{C}^k$ representative for any $k \geqslant 0$ but no $\mathcal{C}^\infty$ representative. The o-minimality of $\mathbb{R}_H$ is, as in [9], a consequence of the quasianalyticity of algebras of germs at $0 \in \mathbb{R}^n$, for $n \geqslant 0$, containing the germ of H and closed under classical operations, such as composition, partial derivation and solution of implicit equations. The major part of our work shows how this quasianalyticity property is guaranteed by a convenient choice of the formal Taylor expansion $\widehat{H}$ of H at 0. We prove that it is sufficient to impose a transcendence condition on the coefficients of $\widehat{H}$ (see section 2).

Let us conclude by the following questions regarding the methods developed in this paper. Could they be used to produce an o-minimal structure $\mathbb{R}_{\mathcal{F}}$ which doesn't admit $\mathcal{C}^\infty$ cell decomposition but defines restricted analytic functions? On another hand, could they lead to another construction of an o-minimal structure which does admit smooth but not analytic cell decomposition? The interest of such an approach would be to circumvent the use of the deep (and difficult) Mandelbrojt's theorem needed in [9], which states that any $\mathcal{C}^\infty$ function defined on a compact interval is the sum of two functions belonging to different Denjoy-Carleman classes [8].

1. Main results

Throughout this paper, the letters i, j, k, m, n, p and s range over $\mathbb{N}$. We let $x = (x', x_n)$ with $x' = (x_1, \dots, x_{n-1})$, and $y = (y', y_m)$ with $y' = (y_1, \dots, y_{m-1})$ be tuples of indeterminates. A real valued germ is called **weakly** $\mathcal{C}^\infty$ if it admits, for any non negative integer k, a $\mathcal{C}^k$ representative. Likewise, a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called **weakly** $\mathcal{C}^\infty$ if its germ at 0 is weakly $\mathcal{C}^\infty$. We denote by $\mathcal{W}_n$ the algebra of weakly $\mathcal{C}^\infty$ germs at the origin of $\mathbb{R}^n$. For any germ $g \in \mathcal{W}_n$, $\hat{g} \in \mathbb{R}[[x_1, \dots, x_n]]$ denotes its (infinite) Taylor expansion. Finally, we put $I = [-1, 1]$, and for $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$, $I_r := [-r_1, r_1] \times \cdots \times [-r_n, r_n]$.

Our proof of o-minimality is inspired by the method of [9], which involves the **quasianalyticity** of algebras closed under some classical operations. Recall that a subalgebra $\mathcal{A}_n \subset \mathcal{W}_n$ is **quasianalytic** if the only element of $\mathcal{A}_n$ with zero Taylor expansion is the zero germ. The algebras considered here are defined in the following way. Consider a weakly $\mathcal{C}^\infty$ function $H\colon \mathbb{R} \to \mathbb{R}$. We set $\mathcal{A}(H)$ to be the smallest collection of algebras $\mathcal{A}_n(H) \subset \mathcal{W}_n$, with $n \in \mathbb{N}$, satisfying the following conditions:

1) The germ of H belongs to $\mathcal{A}_1(H)$, and polynomial germs in n variables belong to $\mathcal{A}_n(H)$;
2) If $f \in \mathcal{A}_n(H)$, and if f_i denotes the restriction of f to the hyperplane $x_i = 0$, for $i = 1, \dots, n$, then the germ which continuously extends $(f - f_i)/x_i$ at $0 \in \mathbb{R}^m$ belongs to $\mathcal{A}_n(H)$;
3) If $g_1, \dots, g_m \in \mathcal{A}_n(H)$ and $f \in \mathcal{A}_m(H)$, then:
$$f(g_1 - g_1(0), \dots, g_m - g_m(0)) \in \mathcal{A}_n(H);$$
4) If $n > 1$ and $f \in \mathcal{A}_n(H)$, let
$$g(x) = f(x) - f(0) - x_n \partial f/\partial x_n(0) + x_n,$$
so that $\partial g/\partial x_n(0) = 1$. Then the germ $\varphi \in \mathcal{W}_{n-1}$ defined by $g(x', \varphi(x')) = 0$ belongs to $\mathcal{A}_{n-1}(H)$.

Remark 1.1.

1) A straightforward computation shows that these hypotheses imply the closure of the algebras $\mathcal{A}_n(H)$ under partial differentiation.
2) The definition of the algebras $\mathcal{A}_n(H)$ involves a slightly modified version of the usual composition of functions and implicit equations. These modified operations may be applied to any family of weakly $\mathcal{C}^\infty$ germs, without checking the classical assumptions. Of course, if a germ $f \in \mathcal{A}_n(H)$ satisfies $f(0) = 0$ and $\partial f / \partial x_n (0) \neq 0$, the germ g associated with $F(x) = \dfrac{1}{\partial f / \partial x_n (0)} f(x)$ as in the point 4) of the foregoing definition is actually equal to F. Hence the algebra $\mathcal{A}_{n-1}(H)$ contains the implicit function defined by $f(x) = 0$.

The main result is an immediate consequence of the following theorems:

Theorem A. — *There exists a weakly $\mathcal{C}^\infty$ function $H\colon \mathbb{R} \to \mathbb{R}$ satisfying:*

1) *the germ of H at the origin of $\mathbb{R}$ is not $\mathcal{C}^\infty$;*
2) *the restriction of H to the complement of any neighborhood of $0 \in \mathbb{R}$ is piecewise given by finitely many polynomials (or* piecewise polynomial *for short);*
3) *the algebras $\mathcal{A}_n(H)$ are quasianalytic.*

Theorem B. — *Consider a weakly $\mathcal{C}^\infty$ function $H\colon \mathbb{R} \to \mathbb{R}$ satisfying properties 2) and 3) of Theorem A. Then the structure $\mathbb{R}_H$ is o-minimal. Moreover, if $\mathcal{H}$ denotes the collection of all derivatives $H^{(i)}\colon I_{r_i} \to \mathbb{R}$ of H restricted to a neighborhood I_{r_i} of 0 where $H \in \mathcal{C}^i(I_{r_i})$, the structure $\mathbb{R}_{\mathcal{H}}$ is model complete.*

The main result follows from these theorems in the following way: if H is a function satisfying properties of Theorem A, there cannot exist in the o-minimal structure $\mathbb{R}_H$ any $\mathcal{C}^\infty$ cell decomposition of the plane $\mathbb{R}^2$ adapted to the graph of H.

We give in section 2 the proof of Theorem A, and in section 3 the proof of Theorem B.

2. Proof of Theorem A

In order to build a function which satisfies the three properties of Theorem A, we proceed in two steps. We first show, following Borel's methods, how any formal power series in one variable can be realized as the Taylor expansion at the origin of a weakly $\mathcal{C}^\infty$ function H satisfying the two first properties. Then we give a number theoretic condition on the coefficients of the power series which insures, for this function H, the quasianalyticity of the algebras $\mathcal{A}_n(H)$.

2.1. A weakly $\mathcal{C}^\infty$ version of Borel's Theorem

Notation. — For any compact polydisk $B \subset \mathbb{R}^n$ and any integer $m \geqslant 0$, we denote by $\|\cdot\|_{B,m}$ (or simply $\|\cdot\|_m$ if B is clear from context) the norm defined on $\mathcal{C}^m(B)$ by:

$$\|f\|_{B,m} = \max_{|\alpha| \leqslant m} \left(\|\partial^\alpha f / \partial x_\alpha\|_{B,\infty} \right)$$

where $\|\cdot\|_{B,\infty}$ denotes the supremum norm on B, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Likewise, if $B \subset \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_s}$ is the cartesian product of compact polydisks $B_i \subset \mathbb{R}^{n_i}$, $i = 1, \ldots, s$, we denote by $\|\cdot\|_{B,m}$ (or $\|\cdot\|_m$) the norm defined on $\mathcal{C}^m(B_1) \times \cdots \times \mathcal{C}^m(B_s)$ by

$$\|(f_1, \ldots, f_s)\|_{B,m} = \max_{i=1,\ldots,s} \|f_i\|_{B_i,m}\,.$$

From now on, we consider each space $\mathcal{C}^m(B)$ with its topology of Banach space induced by the norm $\|\cdot\|_{B,m}$.

LEMMA 2.1. — *Let $\hat{f}(x_1) \in \mathbb{R}[[x_1]]$. There exists a weakly $\mathcal{C}^\infty$ function whose germ at $0 \in \mathbb{R}$ is not $\mathcal{C}^\infty$, which admits $\hat{f}$ as a Taylor expansion, and which is piecewise polynomial in restriction to the complement of any neighborhood of the origin.*

Proof. — We adapt the classical proof of Borel's Theorem, as it can be found for example in [7].

For each integer i, let P_i be the polynomial $P_i(x_1) = (1 - x_1)^i (1 + x_1)^i$. We define, for $\varepsilon \in (0, 1)$, a function v_i^ε by:

$$v_i^\varepsilon(x_1) = \begin{cases} x_1^i P_i\left(\dfrac{x_1}{\varepsilon}\right) & \text{if } x_1 \in (-\varepsilon, \varepsilon) \\ 0 & \text{otherwise.} \end{cases}$$

A straightforward computation shows that, for $i \geqslant 1$, the functions v_i^ε and their derivatives satisfy the following properties:

i) For $0 \leqslant m < i$, we have $v_i^\varepsilon \in \mathcal{C}^m(I)$, $(v_i^\varepsilon)^{(m)}(0) = 0$ and $\|v_i^\varepsilon\|_m \to 0$ when $\varepsilon \to 0$.

ii) $(v_i^{\varepsilon})^{(i)}(0) = i!$. This value does in particular not depend on ε.

Consider now a formal power series $\hat{f}(x_1) = \sum a_i x_1^i$. For each $i \in \mathbb{N}$, we define recursively the values ε_i, b_i. We fix $b_0 = a_0$ and $\varepsilon_0 = 1$. For $i > 0$, suppose $b_0, b_1, \ldots, b_{i-1}$ and $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{i-1}$ defined, and let $h_{i-1} = \sum_{k=0}^{i-1} b_k v_k^{\varepsilon_k}$. We fix $b_i = a_i - \dfrac{h_{i-1}^{(i)}(0)}{i!}$. We then have

$$(h_{i-1} + b_i v_i^{\varepsilon})^{(i)}(0) = i!\ a_i.$$

By i), there exists a value ε_i in $(0, \inf(1/i, \varepsilon_{i-1}))$ such that $\|b_i v_i^{\varepsilon_i}\|_{I,i-1} < 1/2^i$. Therefore, for any $m \geqslant 0$, the series $\sum b_i v_i^{\varepsilon_i}$ is normally convergent in $\mathcal{C}^m(I_{\varepsilon_m})$. It defines a weakly $\mathcal{C}^\infty$ function on $(-1,1)$, which is not $\mathcal{C}^\infty$ on the sequence $\varepsilon_i \to 0$. Hence its germ at 0 is not $\mathcal{C}^\infty$. Finally, since $(v_i^{\varepsilon_i})^{(m)}(0) = 0$ for $i > m$, its Taylor expansion at 0 is exactly the series $\hat{f}$. □

Notation. — Given a power series $\hat{f}(x_1) \in \mathbb{R}[[x_1]]$, the function built in the preceding lemma is called the **special realization** of $\hat{f}$.

2.2. Definition of operators

Given a weakly $\mathcal{C}^\infty$ function H, the algebras $\mathcal{A}_n(H)$, $n \in \mathbb{N}$, are the smallest algebras containing the germ of H at the origin, closed under some classical operations (see section 1). In order to study their properties, we establish a formalism of *operators*, such that every element of these algebras is the image of the germ of H under such an operator. Moreover, having in mind a proof of quasianalyticity, we also need to describe the action of these operators on formal power series.

Operators acting on weakly $\mathcal{C}^\infty$ germs. An **elementary operator** is one of the following, where n, m denote any non negative integers:

1) the sum and the product acting on $\mathcal{W}_n \times \mathcal{W}_n$;
2) the natural embedding $\mathcal{W}_n \to \mathcal{W}_{n+1}$;
3) for any $c \in \mathbb{R}$, the constant operator $\mathcal{W}_1 \to \mathcal{W}_0$ defined by $f \mapsto c$;
4) for $1 \leqslant i \leqslant n$, the coordinates operators $\mathcal{W}_1 \to \mathcal{W}_n$ defined by $f \mapsto x_i$;
5) the monomial division operators $\mathcal{W}_n \to \mathcal{W}_n$ defined for $i = 1, \ldots, n$ by $f \mapsto D_i(f)$, where $D_i(f)$ is the germ at $0 \in \mathbb{R}^n$ of the continuous extension of $(f - f_i)/x_i$, if f_i denotes the restriction of f to the hyperplane $\{x_i = 0\}$;
6) the composition operators $\mathcal{W}_m \times \mathcal{W}_n^m \to \mathcal{W}_n$ defined by
$$(f, g_1, \ldots, g_m) \mapsto f(g_1 - g_1(0), \ldots, g_m - g_m(0));$$

7) the implicit function operator $\mathcal{W}_n \to \mathcal{W}_{n-1}$ defined for $n > 1$ by $f \mapsto \varphi$, where $\varphi \in \mathcal{W}_{n-1}$ is the germ characterized by $\varphi(0) = 0$ and $g(x', \varphi(x')) = 0$, with $x' = (x_1, \ldots, x_{n-1})$ and

$$g = f - f(0) - x_n \partial f / \partial x_n (0) + x_n$$

(note that g satisfies the hypotheses of the Implicit Function Theorem).

We call **operator** a finite composition of elementary operators. According to 1.1, the *partial differentiation with respect to any coordinate* is an operator. Moreover, given a germ $H \in \mathcal{W}_1$ and a positive n , for every element $g \in \mathcal{A}_n(H)$ there exists (at least) one operator $\mathcal{L}$ such that $\mathcal{L}(H) = g$.

Operators acting on formal power series. We remark that, for any $(f_1, \ldots, f_s) \in \mathcal{W}_{n_1} \times \cdots \times \mathcal{W}_{n_s}$ and any operator $\mathcal{L}$ acting on $\mathcal{W}_{n_1} \times \cdots \times \mathcal{W}_{n_s}$, the Taylor expansion of $\mathcal{L}(f_1, \ldots, f_s)$ only depends on the Taylor expansion of $f_1, \ldots, f_s$. Therefore $\mathcal{L}$ has a "formal counterpart" $\widehat{\mathcal{L}}$ acting on $\mathbb{R}[[x_1, \ldots, x_{n_1}]] \times \cdots \times \mathbb{R}[[x_1, \ldots, x_{n_s}]]$, such that

$$\mathcal{L}\,(\widehat{f_1, \ldots, f_s}) = \widehat{\mathcal{L}}\left(\hat{f}_1 \ldots, \hat{f}_s\right), \quad (f_1, \ldots, f_s) \in \mathcal{W}_{n_1} \times \cdots \times \mathcal{W}_{n_s}.$$

By 2.1, since each formal series can be realized by a weakly $\mathcal{C}^\infty$ germ, it would be equivalent to define formal operators as in the definition of germ's operators, replacing $\mathcal{W}_n$ by $\mathbb{R}[[x_1, \ldots, x_n]]$. In particular, each formal operator is a finite composition of elementary formal operators, and if $\mathcal{L} = \mathcal{L}_1 \circ \cdots \circ \mathcal{L}_k$, where the $\mathcal{L}_i$'s are elementary, then we have $\widehat{\mathcal{L}} = \widehat{\mathcal{L}}_1 \circ \cdots \circ \widehat{\mathcal{L}}_k$.

2.3. Continuity of operators

In order to prove a quasianalyticity property for operators, we need in the next section the following classical property, which states how operators act on functions. Actually, for a map f and an operator $\mathcal{L}$, since the definition domain of $\mathcal{L}(f)$ depends *a priori* on f, operators are not well defined on maps spaces. But we have the following:

Proposition 2.2. — *Let $\mathcal{L}\colon \mathcal{W}_{n_1} \times \cdots \times \mathcal{W}_{n_s} \to \mathcal{W}_m$ be an operator. Then $\mathcal{L}$ admits an* order of derivation *d, in the following meaning. Set $r = (r_1, \ldots, r_s) \in \mathbb{R}_{>0}^{n_1} \times \cdots \times \mathbb{R}_{>0}^{n_s}$ and $U = I_{r_1} \times \cdots \times I_{r_s}$. Then, for $l \geqslant d$ and any s-tuple of functions $f = (f_1, \ldots, f_s)$ in $\mathcal{C}^l(U)$, shrinking r if necessary, there exist a neighborhood W of $(f_1, \ldots, f_s)$ in $\mathcal{C}^l(I_{r_1}) \times \cdots \times \mathcal{C}^l(I_{r_s})$ and $r' \in \mathbb{R}_{>0}^m$ such that $\mathcal{L}$ is well defined and continuous from W to $\mathcal{C}^{l-d}(I_{r'})$.*

Let us give a sketch of proof for this proposition.

Proof. — An operator $\mathcal{L}$ being a finite composition of elementary operators, we proceed by induction on the *length* of $\mathcal{L}$. In a first step we prove the proposition for elementary operators, and more precisely for the monomial division, composition and implicit function operators (the result for other elementary operators being obvious).

Proof of 2.2 for the monomial division operator. Consider the monomial division operator $\mathcal{L}$ defined on $\mathcal{W}_n$ by:

$$\mathcal{L}(f)(x) = \frac{f(x', x_n) - f(x', 0)}{x_n}.$$

We claim that the *order of derivation* of $\mathcal{L}$ is 1. Consider indeed an integer $l \geqslant 1$, $r \in \mathbb{R}^n_{>0}$ and a function $f \in \mathcal{C}^l(I_r)$. Then we have

$$\mathcal{L}(f)(x) = \int_0^1 \frac{\partial f}{\partial x_n}(x', tx_n)\,\mathrm{d}t, \text{ for } x \in I_r.$$

This equality shows that the linear operator $\mathcal{L}$ is a bounded operator from $\mathcal{C}^l(I_r)$ to $\mathcal{C}^{l-1}(I_r)$.

Proof of 2.2 for the implicit functions. We actually prove the result for a system of implicit equations. Consider an integer $l \geqslant 1$, a polyradius $r \in \mathbb{R}^{n+m}_{>0}$, and a tuple $F = (f_1, \ldots, f_m)$ of elements of $\mathcal{C}^l(I_r)$. We denote by x the n-tuple $(x_1, \ldots, x_n)$, and by y the m-tuple $(y_1, \ldots, y_m)$. For any $\overline{F} \in \mathcal{C}^l(I_r)$, let $\overline{G}$ be the $\mathcal{C}^l$ map defined on I_r by:

$$\overline{G}(x,y) = \overline{F}(x,y) - \overline{F}(0) + \left(\mathrm{Id} - \partial_y \overline{F}(0)\right)(y), \tag{2.1}$$

where Id is the identity of $\mathbb{R}^m$. Notice that $\partial_y \overline{G}(0) = \mathrm{Id}$. Consider now the map $\Psi\colon \mathcal{C}^l(I_r) \times I_r \to \mathbb{R}^m$ defined by

$$\Psi\left(\overline{F}, x, y\right) = \overline{G}(x,y).$$

We see that Ψ is continuously differentiable and satisfies the implicit function hypothesis with respect to y. It follows from the Implicit Function Theorem applied to the Banach space $\mathcal{C}^l(I_r) \times \mathbb{R}^{n+m}$ that there exist a $\mathcal{C}^l$-neighborhood U of F, a polyradius $(r', r'') \in \mathbb{R}^n_{>0} \times \mathbb{R}^m_{>0}$, and a continuously differentiable map $\Phi\colon U \times I_{r'} \to I_{r''}$ such that, for $\left(\overline{F}, x, y\right) \in U \times I_{r'} \times I_{r''}$,

$$\Psi\left(\overline{F}, x, y\right) = 0 \Leftrightarrow y = \Phi\left(\overline{F}, x\right).$$

Since x ranges over the compact set $I_{r'}$ the operator $\mathcal{L}\colon \overline{F} \mapsto \Phi\left(\overline{F}, \cdot\right)$ is continuous from U to $\mathcal{C}^l(I_{r'})$. Therefore, the implicit function operator admits 1 as an order of derivation.

Proof of 2.2 for the composition operator. We claim that the order of derivation of the composition operator is at most one. Actually, this operator may be defined via a system of implicit equations, in the following way. Consider a polydisk $I_r \times (I_{r'})^m$, with $r \in \mathbb{R}^m_{>0}$ and $r' \in \mathbb{R}^n_{>0}$. Let $l \geqslant 1$ and $(f, g_1, \ldots, g_m) \in \mathcal{C}^l(I_r) \times \left(\mathcal{C}^l(I_{r'})\right)^m$. After shrinking r' if necessary, there exists a $\mathcal{C}^l$-neighborhood U of $(f, g_1, \ldots, g_m)$ such that the composition $\bar{f}(\bar{g}_1 - \bar{g}_1(0), \ldots, \bar{g}_m - \bar{g}_m(0))$ is well defined for $\left(\bar{f}, \bar{g}_1, \ldots, \bar{g}_m\right) \in U$. Moreover the equation

$$z = \bar{f}\left(\bar{g}_1(x) - \bar{g}_1(0), \ldots, \bar{g}_m(x) - \bar{g}_m(0)\right),$$

for $x \in I_{r'}$, is equivalent to the system

$$\exists y = (y_1, \ldots, y_m) \in I_r \begin{cases} \bar{g}_1(x) - \bar{g}_1(0) - y_1 & = & 0 \\ \quad\vdots & \vdots & \vdots \\ \bar{g}_m(x) - \bar{g}_m(0) - y_m & = & 0 \\ \bar{f}(y_1, \ldots, y_m) - z & = & 0 \end{cases}$$

which satisfies the implicit function hypothesis with respect to the variables $(y_1, \ldots, y_m, z)$. Consequently, if we let

$$\overline{F}(x, y, z) = \left(\bar{g}_1(x) - \bar{g}_1(0) - y_1, \ldots, \bar{g}_m(x) - \bar{g}_m(0) - y_m, \bar{f}(y) - z\right)$$

we simply apply the foregoing continuity result for the implicit system

$$\left(\partial_{y,z}\overline{F}(0)\right)^{-1} \cdot \overline{F}(x, y, z) = 0$$

which continuously depends on $\overline{F}$. The order of derivation of the composition operator is therefore at most 1.

Proof of 2.2 for any operator. Suppose that the proposition holds for any operator of length less than or equal to $\ell - 1 \geqslant 1$, and let $\mathcal{L}\colon \mathcal{W}_{n_1} \times \cdots \times \mathcal{W}_{n_s} \to \mathcal{W}_m$ be an operator of length ℓ. We may write $\mathcal{L} = \mathcal{L}_0(\mathcal{L}_1, \ldots, \mathcal{L}_k)$ where $\mathcal{L}_0$ is elementary and each $\mathcal{L}_i$, $i = 1, \ldots, k$, has length less than ℓ. We easily conclude by applying the induction hypothesis to the $\mathcal{L}_i$'s and the previous continuity results to $\mathcal{L}_0$. □

Remark 2.3. — Let $H\colon \mathbb{R} \to \mathbb{R}$ be a weakly $\mathcal{C}^\infty$ function, with germ $\widetilde{H}$ at 0, and $f = \mathcal{L}(\widetilde{H})$ be a germ in $\mathcal{A}_n(H)$. As a direct consequence of the proposition, for each i, there exist a restriction H_i of H, and a polyradius $r_i \in \mathbb{R}^n_{>0}$, such that $\mathcal{L}(H_i)$ be well defined and $\mathcal{C}^i$ on I_{r_i}. Each germ f in $\mathcal{A}_n(H)$ then admits a **natural representative** defined on an appropriate I_{r_0}, and $\mathcal{C}^i$ on a small enough polydisk I_{r_i}.

2.4. Quasianalyticity for operators

This section is devoted to the proof of the following *quasianalyticity property for operators*, which is a key step in the proof of Theorem A. It shows that no "flat" operator exists, which would map each germ on a "flat" germ, excepted for the null one.

Lemma 2.4. — *Let $\mathcal{L}$ be an operator acting on $\mathcal{W}_1$. If $\widehat{\mathcal{L}} = 0$ then $\mathcal{L} = 0$.*

Proof. — Consider an operator $\mathcal{L}\colon \mathcal{W}_1 \to \mathcal{W}_n$ with order of derivation d, such that $\widehat{\mathcal{L}} = 0$. We fix a germ $f \in \mathcal{W}_1$. It follows from Proposition 2.2 that there exist $r > 0$, and a representative f_0 of f in $\mathcal{C}^d\left(I_r\right)$, a $\mathcal{C}^d$-neighborhood U of f and a compact neighborhood V of $0 \in \mathbb{R}^n$ such that the operator $\mathcal{L}$ is well defined and continuous from U to $\mathcal{C}^0\left(V\right)$.

Consider now a sequence (P_k) of polynomials which converges to f_0 in $\mathcal{C}^d\left(I_r\right)$. The sequence $\mathcal{L}\left(P_k\right)$ tends to $\mathcal{L}\left(f_0\right)$ in $\mathcal{C}^0\left(V\right)$. Since $\widehat{\mathcal{L}} = 0$, we have $\widehat{\mathcal{L}\left(P_k\right)} = \widehat{\mathcal{L}}\left(\widehat{P}_k\right) = 0$. This implies, since the functions $\mathcal{L}\left(P_k\right)$ are analytic, that $\mathcal{L}\left(P_k\right) = 0$. By continuity, $\mathcal{L}\left(f_0\right) = 0$, and then $\mathcal{L}\left(f\right) = 0$. Since $\mathcal{L}(f) = 0$ for arbitrary f, we deduce that $\mathcal{L} = 0$. $\square$

2.5. Algebraic expression for operators

In order to obtain a condition for the quasianalyticity of the algebras $\mathcal{A}_n(H)$, we need a precise description of the action of operators on formal power series. This classical result of commutative algebra is explained in the following lemma, for which we give a short proof:

Lemma 2.5. — *Let $\widehat{\mathcal{L}}$ be a formal operator acting on $\mathbb{R}\left[\left[x_1, \ldots, x_{n_1}\right]\right] \times \cdots \times \mathbb{R}\left[\left[x_1, \ldots, x_{n_s}\right]\right]$ with values in $\mathbb{R}\left[\left[x_1, \ldots, x_n\right]\right]$. If $\hat{g}_i \in \mathbb{R}\left[\left[x_1, \ldots, x_{n_i}\right]\right]$, $i = 1, \ldots, s$, are given by $\hat{g}_i\left(x_1, \ldots, x_{n_i}\right) = \sum g_{i,\beta} x^\beta$, then there exist a tuple $a \in \mathbb{R}^q$ and, for any multi-index $\alpha \in \mathbb{N}^n$, an integer N_α and a polynomial*

$$P_\alpha \in \mathbb{Z}\left[X_1, \ldots, X_q, Y_1, \ldots, Y_{N_\alpha}\right],$$

such that:

$$\widehat{\mathcal{L}}\left(\hat{g}_1, \ldots, \hat{g}_s\right) = \sum_{\alpha \in \mathbb{N}^n} P_\alpha\left(a, \tilde{g}_\alpha\right) x^\alpha \tag{2.2}$$

where $\tilde{g}_\alpha$ denotes a N_α-tuple of coefficients of $\hat{g}_1, \ldots, \hat{g}_s$.

Proof. — The operator $\widehat{\mathcal{L}}$ is a finite composition of elementary operators. The proof is an induction on the length ℓ of this composition.

1) If $\ell = 1$, the operator $\widehat{\mathcal{L}}$ is elementary. If $\widehat{\mathcal{L}}$ is an operator of type 1), 2), 3) , 4) or 5), the result is clear. If $\widehat{\mathcal{L}}$ is a composition operator $\mathbb{R}[[x_1,\dots,x_m]] \times (\mathbb{R}[[x_1,\dots,x_n]])^m \to \mathbb{R}[[x_1,\dots,x_n]]$, we have:

$$\widehat{\mathcal{L}}\,(f,g_1,\dots,g_m) = \sum_{\alpha\in\mathbb{N}^m} f_\alpha \left(\sum_{\beta\in\mathbb{N}^n\smallsetminus\{0\}} g_{1,\beta}x^\beta\right)^{\alpha_1}\cdots\left(\sum_{\beta\in\mathbb{N}^n\smallsetminus\{0\}} g_{m,\beta}x^\beta\right)^{\alpha_m}$$

The result is obtained by expanding the above expression and rearranging the powers of x.

Finally, if $\widehat{\mathcal{L}}$ is an implicit function operator, and $\hat{f} \in \mathbb{R}\,[[x_1,\dots,x_n]]$, we expand the equation solved by $\widehat{\mathcal{L}}\,(f)$. We obtain a triangular system of polynomial equations. Notice that we actually solve an implicit equation $g\,(x', x_n) = 0$ with $\partial g/\partial x_n\,(0) = 1$ (see section 2.2 for the definition of elementary operators). Therefore, solutions of the above triangular systems may be expressed as polynomials (and not rational fractions) in the coefficients of $\hat{f}$.

2) If $\ell > 1$, the operator $\widehat{\mathcal{L}}$ is the composition of an elementary operator $\widehat{\mathcal{B}}$ with operators whose length are strictly smaller than ℓ:

$$\widehat{\mathcal{L}}\,(\hat{g}_1,\dots,\hat{g}_s) = \widehat{\mathcal{B}}\left(\widehat{\mathcal{L}}_1\,(\hat{g}_1,\dots,\hat{g}_s)\,,\dots,\widehat{\mathcal{L}}_k\,(\hat{g}_1,\dots,\hat{g}_s)\right),$$

with each $\hat{g}_i \in \mathbb{R}\,[[x_1,\dots,x_{n_i}]]$. According to the induction hypothesis, for each operator $\widehat{\mathcal{L}}_i$ there exist $a_i \in \mathbb{R}^{q_i}$ and polynomials $P_{\alpha,i}$ such that

$$\widehat{\mathcal{L}}_i\,(\hat{g}_1,\dots,\hat{g}_s) = \sum_{\alpha\in\mathbb{N}^{m_i}} P_{\alpha,i}\,(a_i,\tilde{g}_{\alpha,i})\,x^\alpha$$

where $x = (x_1,\dots,x_{m_i})$ and each $\tilde{g}_{\alpha,i}$ is a finite family of coefficients of $\hat{g}_1,\dots,\hat{g}_s$. On another hand, to the operator $\widehat{\mathcal{B}}$ are associated a tuple $a \in \mathbb{R}^q$ and polynomials Q_β such that for any $\hat{h}_1,\dots,\hat{h}_k$, with $\hat{h}_i \in \mathbb{R}\,[[x_1,\dots,x_{m_i}]]$:

$$\widehat{\mathcal{B}}\left(\hat{h}_1,\dots,\hat{h}_k\right) = \sum_{\beta\in\mathbb{N}^n} Q_\beta\left(a,\tilde{h}_\beta\right)x^\beta$$

where $x = (x_1,\dots,x_n)$ and each $\tilde{h}_\beta$ is a finite family of coefficients of $\hat{h}_1,\dots,\hat{h}_k$. Therefore,

$$\widehat{\mathcal{B}}\left(\widehat{\mathcal{L}}_1\,(\hat{g}_1,\dots,\hat{g}_s)\,,\dots,\widehat{\mathcal{L}}_k\,(\hat{g}_1,\dots,\hat{g}_s)\right) = \sum_{\beta\in\mathbb{N}^n} Q_\beta\left(a,(P_{\alpha,i}\,\widetilde{(a_i,\tilde{g}_{\alpha,i})})_\beta\right)x^\beta.$$

Hence the expansion of the formal power series $\widehat{\mathcal{B}}\left(\widehat{\mathcal{L}}_1,\dots,\widehat{\mathcal{L}}_k\right)(\hat{g}_1,\dots,\hat{g}_s)$ has the required shape. $\square$

2.6. Transcendence and quasianalyticity

The last step in the proof of Theorem A consists in building a formal power series $\widehat{H} \in \mathbb{R}[[x_1]]$ such that, if H is a weakly $\mathcal{C}^\infty$ function with Taylor expansion $\widehat{H}$, the algebras $\mathcal{A}_n(H)$ are quasianalytic. This means, in the language of operators, that for every operator $\mathcal{L}$:

$$\widehat{\mathcal{L}(H)} = 0 \text{ implies } \mathcal{L}(H) = 0.$$

The next lemma gives an answer to this question:

Lemma 2.6. — *Consider a sequence h_i of real numbers such that*

$$\operatorname{trdeg} \mathbb{Q}(h_1, \dots, h_i) = i, \text{ for } i \in \mathbb{N}.$$

If $H\colon \mathbb{R} \to \mathbb{R}$ is a weakly $\mathcal{C}^\infty$ function whose Taylor expansion at the origin is $\widehat{H}(x_1) = \sum h_i x_1^i \in \mathbb{R}[[x_1]]$, then the algebras $\mathcal{A}_n(H)$, $n \in \mathbb{N}$, are quasianalytic.

Remark 2.7. — Let us provide an explicit example of such a sequence (h_i). It is well known that the reals $\sqrt{p_i}$, where p_i denotes the i-th prime number, are $\mathbb{Q}$-linearly independent. Hence, according to Lindemann's Theorem, the sequence $h_i = \exp\left(\sqrt{p_i}\right)$, $i \in \mathbb{N}$, satisfies the hypothesis of the lemma.

Proof of Lemma 2.6. — In a first step, we notice that the transcendence hypothesis satisfied by the coefficients h_i implies that, for any $a \in \mathbb{R}^q$, there exists an integer $N \geqslant 0$ such that every finite family of coefficients h_i disjoint from $\{h_1, \dots, h_N\}$ is algebraically independent on $\mathbb{Q}(a, h_1, \dots, h_N)$. Indeed, since $d_i = \operatorname{trdeg} \mathbb{Q}(a, h_1, \dots, h_i) \geqslant i$ for any integer i, the numbers of indexes i such that $d_{i+1} = d_i$ is finite. If i_0 denotes the greatest of these indexes, the required number N is $i_0 + 1$.

Next we claim that the power series $\widehat{H}(x_1) = \sum h_i x_1^i$ satisfies the following property: for any operator $\widehat{\mathcal{L}}$ acting on $\mathbb{R}[[x_1]]$, if $\widehat{\mathcal{L}}\left(\widehat{H}\right) = 0$, there exists an integer $N \geqslant 0$ such that

$$\text{for all } \hat{g} \in \mathbb{R}[[x_1]], \ \widehat{\mathcal{L}}\left(h_1 x_1 + \dots + h_N x_1^N + x_1^{N+1}\hat{g}(x_1)\right) = 0. \tag{2.3}$$

This property is a consequence of the expansion 2.2. Consider an operator $\mathcal{L}$ such that $\widehat{\mathcal{L}}(H) = 0$. There exists $a \in \mathbb{R}^q$ and polynomials P_α such that, for each $\hat{f}(x_1) = \sum f_i x_1^i \in \mathbb{R}[[x_1]]$, we have

$$\widehat{\mathcal{L}}\left(\hat{f}\right)(x) = \sum_{\alpha \in \mathbb{N}^n} P_\alpha\left(a, \tilde{f}_\alpha\right) x^\alpha$$

where $\tilde{f}_\alpha$ denotes a finite family of coefficients of $\hat{f}$. Let N be the integer produced for this tuple a in the first step of the proof. If $\hat{f}$ has initial part $h_1x_1 + \cdots + h_N x_1^N$, i.e. $f_i = h_i$ for $0 \leqslant i \leqslant N$, this equality becomes

$$\widehat{\mathcal{L}}\left(\hat{f}\right)(x) = \sum_{\alpha\in\mathbb{N}^n} P_\alpha\left(a, h_1, \ldots, h_N, \bar{f}_\alpha\right) x^\alpha,$$

where $\bar{f}_\alpha$ denotes a finite family of coefficients f_i disjoint from $\{f_1, \ldots, f_N\}$. Setting $Q_\alpha\left(\bar{f}_\alpha\right) = P_\alpha\left(a, h_1, \ldots, h_N, \bar{f}_\alpha\right)$, it comes

$$\widehat{\mathcal{L}}\left(\hat{f}\right)(x) = \sum_{\alpha\in\mathbb{N}^n} Q_\alpha\left(\bar{f}_\alpha\right) x^\alpha, \tag{2.4}$$

where the coefficients of Q_α belong to $\mathbb{Q}(a, h_1, \ldots, h_N)$. If we apply these notations to $\hat{f} = \widehat{H}$, since $\widehat{\mathcal{L}}\left(\widehat{H}\right) = 0$, we have $Q_\alpha\left(\bar{h}_\alpha\right) = 0$ for each $\alpha \in \mathbb{N}^n$. But the family $\bar{h}_\alpha$ is algebraically independent on $\mathbb{Q}(a, h_1, \ldots, h_N)$. Hence the polynomials Q_α vanish identically, which implies, by the expression 2.4, that any formal power series with initial part $h_1x_1 + \cdots + h_N x_1^N$ belongs to the kernel of $\widehat{\mathcal{L}}$.

Consider finally a weakly $\mathcal{C}^\infty$ germ $g \in \mathcal{A}_n(H)$ such that $\hat{g} = 0$. There exists an operator $\mathcal{L}$ acting on $\mathcal{W}_1$ such that $g = \mathcal{L}(H)$. Hence $\widehat{\mathcal{L}}\left(\widehat{H}\right) = 0$. According to the previous claim, there exists an integer $N \geqslant 0$ such that, for any power series $\hat{f} \in \mathbb{R}[[x_1]]$, $\widehat{\mathcal{L}}\left(h_1x_1 + \cdots + h_N x_1^N + x_1^{N+1}\hat{f}(x_1)\right) = 0$. Hence, the operator $\mathcal{M}$ defined on $\mathcal{W}_1$ by

$$\mathcal{M}(f)(x_1) = \mathcal{L}\left(h_1x_1 + \cdots + h_N x_1^N + x_1^{N+1} f(x_1)\right),$$

verify $\widehat{\mathcal{M}} = 0$. It follows from the "quasianalyticity property for operators" (see Lemma 2.4) that $\mathcal{M} = 0$. If $f_0 \in \mathcal{W}_1$ is the germ defined by

$$H(x_1) = h_1x_1 + \cdots + h_N x_1^N + x_1^{N+1} f_0(x_1),$$

we have $\mathcal{M}(f_0) = \mathcal{L}(H) = g$. By the nullity of $\mathcal{M}$, we conclude that $g = 0$, which proves the quasianalyticity of the algebra $\mathcal{A}_n(H)$. □

Proof of Theorem A. We simply sum up the different steps. Let $\widehat{H}$ be a formal series satisfying the transcendence hypothesis of Lemma 2.6, and H be its special realization obtained by the Borel's process. The construction of H prove that H is weakly $\mathcal{C}^\infty$ with no $\mathcal{C}^\infty$ germ at 0, and that H is piecewise polynomial on the complement of each neighborhood of 0 (see Lemma 2.1). Moreover, Lemma 2.6 shows that the algebras $\mathcal{A}_n(H)$ are quasianalytic. Points 1), 2) and 3) then hold, and Theorem A is proved.

3. Proof of Theorem B

The proof of Theorem B is merely an adaptation of Gabrielov's method, which has been already used in several contexts ([1] for analytic functions, [4] for generalized power series, [5] for multisummable series). The introduction of algebras similar to our $\mathcal{A}_n(H)$ is done in [9]: the o-minimality of the generated structure is shown to be a consequence of a **normalization process** applied to the elements of these algebras, followed by the proof of a convenient **Λ-Gabrielov property**. We follow the same scheme for the structure $\mathbb{R}_H$. We won't give all over again the details of these well known proofs, but we will recall their main steps and insist on some precise points specific to our framework.

3.1. Basic definitions

We fix H a weakly $\mathcal{C}^\infty$ function which satisfies hypothesis of Theorem B. We noticed in section 2.3 that each element of $\mathcal{A}_n(H)$ admits a **natural representative** defined on an appropriate polydisk. We do not distinguish between a germ in $\mathcal{A}_n(H)$ and its natural representative, when there is no confusion. If $f, g_1, \ldots, g_p \in \mathcal{A}_n(H)$ admit natural representatives on a common polydisk $I_r \subset \mathbb{R}^n$, they define the H-**basic set**

$$A = \Big\{x \in I_r \,;\, f(x) = 0, g_1(x) > 0, \ldots, g_p(x) > 0\Big\}.$$

A finite union of H-basic sets is called an H-**set**. A subset $A \subset \mathbb{R}^n$ is H-**semianalytic at** $a \in \mathbb{R}^n$ if there exists $r > 0$ such that $(A - a) \cap I_r$ is an H-set; the set A is H-**semianalytic** if it is H-semianalytic at each point of $\mathbb{R}^n$. Consider an integer $d \in \mathbb{N}$. If a d-dimensional manifold $A \subset \mathbb{R}^n$ is H-semianalytic at $a \in \mathbb{R}^n$, and if there exist $f_1, \ldots, f_{n-d}$ in $\mathcal{A}_n(H)$ vanishing on $A - a$ with linearly independent gradients at 0, then A is an H-**semianalytic manifold at** a. If A is an H-semianalytic manifold at each point of $\mathbb{R}^n$, it is an H-**semianalytic manifold**.

Remark 3.1. — The graph $\mathrm{gr}(f)$ of a natural representative of $f \in \mathcal{A}_n(H)$ is H-semianalytic at $(0, f(0)) \in \mathbb{R}^{n+1}$, but is not, *a priori*, H-semianalytic at any other point. This is a typical difference with the previous frameworks. Nevertheless, being semialgebraic on the complement of each neighborhood of 0, the graph of H is H-semianalytic.

Given $m \leqslant n$, we let $\Pi^n_m\colon \mathbb{R}^n \to \mathbb{R}^m$ be the projection map $\big(x_1, \ldots, x_n\big) \mapsto (x_1, \ldots, x_m)$. More generally, given an injective $\lambda\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$, we let $\Pi_\lambda^n\colon \mathbb{R}^n \to \mathbb{R}^m$ be the linear map $\Pi_\lambda^n(x) = \left(x_{\lambda(1)}, \dots, x_{\lambda(m)}\right)$. We simply write Π_m and Π_λ if n is clear from context. The following manifolds play a crucial role in the proof of Theorem B.

Definition 3.2. — *Let $r \in \mathbb{R}^n_{>0}$. A subset $B \subset I_r$ is H-**trivial** if one of the following holds:*

1) *There exist sign conditions $\sigma_1, \dots, \sigma_n \in \{<, =, >\}$ such that*

$$B = \{x \in I_r \colon x_1\sigma_1 0, \dots, x_n\sigma_n 0\}.$$

2) *There exist a permutation λ of $\{1, \dots, n\}$, an H-trivial set $C \subset I_s$ where $s = \left(r_{\lambda(1)}, \dots, r_{\lambda(n-1)}\right)$ and an element g of $\mathcal{A}_{n-1}(H)$ with C^1 natural representative defined on I_s, such that $g(I_s) \subset \left(-r_{\lambda(n)}, r_{\lambda(n)}\right)$ and $\Pi_\lambda(B) = \mathrm{gr}\left(g_{|C}\right)$.*

The image of an H-trivial set under a translation is said to be H-trivial again.

Remark.

1) Consider a germ $g = \mathcal{L}(H)$ in $\mathcal{A}_n(H)$, where $\mathcal{L}$ is an operator. Since $\mathcal{L}$ is a composition of elementary operators, the graph of a natural representative of g is the projection of the solution of a semialgebraic system involving H and its derivatives. Hence every H-trivial set is the projection of a bounded H-semianalytic set.

We call **Λ-set** an H-semianalytic subset of I^n, for $n \in \mathbb{N}$. A subset $B \subset I^m$ is a **sub-Λ-set** if there exist $n \geqslant m$ and a Λ-set $A \subset I^n$ such that $B = \Pi_m(A)$. If in addition B is a manifold, it is a **sub-Λ-manifold**. In order to state the property which implies Theorem B, let us give a final definition. A subset $A \subset I^n$ satisfies the **Λ-Gabrielov property** if for each $m \leqslant n$ there are connected sub-Λ-manifolds $B_i \subset I^{n+q_i}$, with $i = 1, \dots, k$ and $q_1, \dots, q_k \geqslant 0$, such that

$$\Pi_m(A) = \Pi_m(B_1) \cup \dots \cup \Pi_m(B_k),$$

and for each $i \in \{1, \dots, k\}$ we have

(G1) $\mathrm{fr}B_i$ is contained in a closed sub-Λ-set $D_i \subset I^{n+q_i}$ such that D_i has dimension and $\dim(D_i) < \dim(B_i)$;

(G2) $d := \dim(B_i) \leqslant m$ and there is a strictly increasing $\lambda\colon \{1, \dots, d\} \to \{1, \dots, m\}$ such that $\Pi_\lambda|B_i\colon B_i \to \mathbb{R}^n$ is an immersion.

Remark 3.3. — Notice that an H-trivial set $B \subset I_n$ is a sub-Λ-manifold which satisfies condition (G1).

Since the restriction to I^2 of the graph of H is a Λ-set, the structure generated by all Λ-sets is exactly $\mathbb{R}_H$. It is well known (see [5]), that model completeness and o-minimality — and hence Theorem B — are consequence of the following:

Proposition 3.4. — *Every Λ-set has the Λ-Gabrielov property.*

We prove this statement in the next section. To this end, we establish several properties of H-semianalytic sets.

3.2. Towards the Λ-Gabrielov property

We follow the main steps of section 4 of [9], and sometimes adapt it to a "local" frame. We first apply the *normalization process* of [9]; it provides a local description of H-semianalytic sets in terms of finite union of diffeomorphic projections of H-trivial manifolds. We then prove a *fiber cutting lemma*. Run over geometric arguments then conclude.

Proposition 3.5 (compare to 3.8 in [9]). — *Let $A \subset \mathbb{R}^n$ be H-semianalytic at $a \in \mathbb{R}^n$, and W be a neighborhood of a. Then, there is a polyradius $r \subset \mathbb{R}^n_{>0}$, with $I_r \subset W$, and for $i = 1, \dots, s$, there are $n_i \geqslant n$ and H-trivial manifolds $B_i \subset \mathbb{R}^{n_i}$ such that*

$$(a + I_r) \cap A = \Pi_n (B_1) \cup \cdots \cup \Pi_n (B_s)$$

and for each i and each $b \in (\Pi_n^{n_i})^{-1}(a)$, B_i is H-semianalytic at b, $\Pi_n (B_i)$ is a manifold and $\Pi_n|B_i \colon B_i \to \Pi_n (B_i)$ is a diffeomorphism. In particular, $(a + I_r) \cap A$ has dimension.

Proof. — As the proof follows faithfully the one given in [9], we recall only its main ideas. The statement is a consequence of a **normalization process** of the elements of the algebras $\mathcal{A}_n (H)$. Let us recall that a germ $f \in \mathcal{A}_n$ is called **normal** if $f(x) = x^r u(x)$, with $r \in \mathbb{N}^n$ and u a unit of $\mathcal{A}_n$. A set defined by sign condition on a normal germ is a trivial manifold. Remark also that, by the quasianalyticity of the algebras $\mathcal{A}_n(H)$, a germ f in $\mathcal{A}_n(H)$ is normal as soon as $\hat{f}$ is normal. The normalization then deals with Taylor expansions. Theorem 2.5 in [9] states that there exists a finite sequence of *admissible substitutions* $\tau_1, \dots, \tau_m$ such that $\tau_1\tau_2 \cdots \tau_m(\hat{f})$ is normal. An admissible substitution may be a linear substitution, a translation by formal series $\hat{g}$ obtained from $\hat{f}$ by the action of an operator, a power substitution or a blow-up substitution. To each substitution corresponds a geometric counterpart. Applying these geometric transformations allows one to describe locally a set defined by a sign condition on an element of $\mathcal{A}_n(H)$ as the union of finitely many projections of trivial sets. All the closure properties of the algebras $\mathcal{A}_n(H)$ are required all along this process. □

Remark 3.6.

1) The previous proposition joint with the compactness of I^n implies corollary 4.4 of [9].
2) Although trivial manifolds are not H-semianalytic in general, the manifolds B_i of the above property are H-semianalytic at each point of $(\Pi_n^{n_i})^{-1}(a)$. It simply reflects the fact that if f belongs to $\mathcal{A}(H)$, the germ of the image of f under a blowing up transformation at any point of the exceptional divisor belongs to $\mathcal{A}(H)$.

The next statement is a version of the classical fiber cutting lemma adapted to our framework.

Lemma 3.7 (compare with 4.5 in [9]). — *Let $0 \leqslant \ell < m \leqslant n$, and $k \leqslant n$. Consider $M \subset \mathbb{R}^n$ an m-dimensional H-semianalytic manifold at $a \in \mathbb{R}^n$. Suppose that $\Pi_k|M$ has constant rank ℓ. Then there exist a polyradius $r \in \mathbb{R}^n$, and a set $A \subset M$ such that:*

a) *A is H-semianalytic at a;*
b) *$\Pi_k(A) = \Pi_k(M \cap (a + I_r))$;*
c) *$\dim(A) < \dim(M)$.*

Proof. — We adapt the traditional proof (see [1]) to a local frame. We denote $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ by $x = (y, z)$, where $y = (x_1, \ldots, x_k)$, $z = (x_{k+1}, \ldots, x_n)$, and reduce to the case where $a = 0$ and M is an H-basic set, with non empty germ at a. Let $r \in \mathbb{R}^n_{>0}$ be a polyradius such that

$$M = \Big\{x \in I_r;\ f_1(x) = 0, \ldots, f_{n-m}(x) = 0,\ g_1(x) > 0, \ldots, g_q(x) > 0\Big\},$$

where $f_1, \ldots, f_{n-m}, g_1, \ldots, g_q$ belong to $\mathcal{A}_n(H)$, and $\nabla f_1, \ldots, \nabla f_{n-m}$ are linearly independent. We denote by F the map $F = (f_1, \ldots, f_{n-m})$. Each fiber $M_y = \{z \in \mathbb{R}^{n-k};\ (y, z) \in M\}$, for $y \in \Pi_k(I_r)$, is empty or has dimension $m - \ell$. We put

$$\varphi_0(x) = \prod_{i=1}^{q} g_i(x) \cdot \prod_{i=1}^{n} (r_i - x_i)\,,$$

$$A^0 = \Big\{x = (y, z) \in M;\ \varphi_0|M_y \text{ is critical at } z\Big\}.$$

Since φ_0 has positive values on M and vanishes on its boundary, φ_0 admits critical points on each non empty fiber M_y. Then $\Pi_k(A^0) = \Pi_k(M)$. Since the equations which define A^0 depends algebraically on the functions which define M, A^0 is H-semianalytic at 0. If $\dim(A^0) < \dim(M)$ in a neighborhood of 0, the proof is done.

Suppose then that the restrictions of A^0 to any small neighborhood of 0 have dimension m. We claim that there exists a polyradius r' such that φ_0 is constant on every connected component of each fiber M_y restricted to $I_{r'}$. Indeed, A^0 contains a set B, open in $M' = F^{-1}(0)$, with $0 \in \overline{B}$. The tangent plane of M'_y admits a base whose coordinates depend algebraically on x and the coordinates of $\nabla f_1, \dots, \nabla f_{n-m}$. Any derivative of φ_0 with respect to a vector of this base then belongs to $\mathcal{A}_n(H)$, and vanishes on B. Since by Proposition 3.5, M' is parametrized by functions whose germs belong to $\mathcal{A}_m(H)$, the quasianalyticity allows one to conclude that there exists a neighborhood of 0 in M' where all theses derivatives vanish. Hence, on this neighborhood, φ_0 is constant along each connected component of each fiber M_y. In particular, there exists a polyradius r' such that: $A^0 \cap I_{r'} = M \cap I_{r'} = M' \cap I_{r'}$.

Therefore, up to replacing M by $M \cap I_{r'}$, each fiber M_y is a compact manifold whose boundary is the intersection of M_y with the frontier of $I_{r'}$. For $i = k+1, \dots, n$, let us define $\varphi_i \colon x \mapsto |x|^2 + x_i^2$, and

$$A^i = \Big\{(y,z) \in M;\ \varphi_i|M_y \text{ is critical at } z\Big\}.$$

The level sets $\{x \in \mathbb{R}^n;\ \varphi_i(x) = \varepsilon\}$ don't intersect the frontier of I_r for small ε. Hence, by the same arguments as above, shrinking r' if necessary, we have:

a) A^i is H-semianalytic at 0;
b) $\Pi_k(A^i \cap I_{r'}) = \Pi_k(M \cap I_{r'})$;
c) If $\dim(A^i \cap I_{r'}) = m$, then $\varphi_i|M_y$ has constant value on any connected component of each restricted fiber $M_y \cap I_{r'}$.

At least one A^i has dimension less than m and then satisfies the conclusion of the lemma. Otherwise, all the functions φ_i are constant on the restricted fibers $M_y \cap I_{r'}$. Now, on each quadrant, the forms $d\varphi_{k+1}, \dots, d\varphi_n$ generate the space $< dx_{k+1}, \dots, dx_n >$. Hence, for $i = k+1 \cdots n$, the coordinate x_i is constant on the connected components of the restricted fibers $M_y \cap I_{r'}$, which is impossible since $M \cap I_{r'}$ has dimension m. □

The normalization process, joint with the fiber cutting lemma imply the following.

PROPOSITION 3.8 (compare to 4.7 in [9]). — *Let $A \subset \mathbb{R}^n$ be H-semianalytic at a, and $k \leqslant n$. Then, there exist a polyradius r, and trivial manifolds $N_i \subset \mathbb{R}^{n_i}$, $i = 1, \dots, s$ such that*

$$\Pi_k(A \cap (a + I_r)) = \Pi_k(N_1) \cup \cdots \cup \Pi_k(N_s).$$

Moreover, for each i we have $d = \dim(N_i) \leqslant k$, and there is a strictly increasing $\iota\colon [1, \dots, d] \to [1, \dots, k]$ such that $\Pi_\iota | N_i\colon N_i \to \mathbb{R}^d$ is an immersion.

Proof (see [9], 4.6 and 4.7 for details). — We proceed by induction on $\dim(A)$. If $\dim(A) = 0$, there exists a polyradius r such that $A \cap I_r$ is empty or is a single point, and the proposition hold. Otherwise, we apply Proposition 3.5. There exists a polyradius r, and finitely many trivial manifolds N_i, $i = 1, \dots, \ell$ with dimension less than or equal to $\dim(A)$ such that $\bigcup_{1 \leqslant i \leqslant \ell} \Pi_k(N_i) = \Pi_k(A \cap (a + I_r))$. Notice that these manifolds are H-semianalytic at each point of the exceptional divisor (see remark 2 in 3.6). If for all $1 \leqslant i \leqslant \ell$, $\dim(N_i) = \operatorname{rank}(\Pi_k | N_i)$, the statement is obtained by the use of Proposition 3.5 applied to each submanifold P of the N_i's such that a canonical projection restricted to P is an immersion, and by the use of the inductive hypothesis for the rest. Otherwise, by a decomposition of N_i into manifolds where projections have constant rank, we may suppose that N_i satisfies hypothesis of Lemma 3.7. The fiber cutting lemma then provides, for all $1 \leqslant i \leqslant \ell$, and all $b \in \Pi_k^{-1}(a)$, a set $A_{i,b}$ which is H-semianaytic at b, such that $\Pi_k(A_{i,b}) = \Pi_k(N_i \cap (b + I_{r_{i,b}}))$, with $\dim(A_{i,b}) < \dim(A)$. By induction, the proposition holds for each $A_{i,b}$. The compactness of $\Pi_k^{-1}(a) \cap \overline{N_i}$ then allows one to consider only a finite number of trivial manifolds, which achieve the proof. □

Proof of Theorem B. Recall that we only need to prove Proposition 3.4. Let $A \subset \mathbb{R}^n$ be a Λ-set, and $a \in I^n$. We apply Proposition 3.8 to the set A, H-semianalytic at a. There exists a polyradius r_a such that

$$\Pi_m(A \cap (a + I_{r_a})) = \Pi_m(B_{a,1}) \cup \cdots \cup \Pi_m(B_{a,s_a}),$$

where all the manifolds $B_{a,i}$ satisfy condition (G2). We may suppose that these manifolds $B_{a,i}$ are included in I^{n_i}. According to remark 3.3, they satisfy also condition (G1). By the compactness of I^n, there exists a finite family of points $b_1, \dots, b_k$ in I^n such that $A = \bigcup_{1 \leqslant i \leqslant k} A \cap (b_i + I_{r_{b_i}})$. We then have:

$$\Pi_m(A) = \bigcup_{1 \leqslant i \leqslant k} \Pi_m(B_{b_i,1}) \cup \cdots \cup \Pi_m\left(B_{b_i,k_{b_i}}\right),$$

where each $B_{b_i,j}$ satisfies (G1) and (G2). Hence, any Λ-set A has the Λ-Gabrielov property, and the proof is done.

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Manuscrit reçu le 1er février 2008,
accepté le 25 avril 2008.

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