Hajer BAHOURI, Jean-Yves CHEMIN & Chao-Jiang XU

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TRACE THEOREM ON THE HEISENBERG GROUP

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1. Introduction

In this work, we continue the study of the problem of restriction of functions that belong to Sobolev spaces associated to left invariant vector fields for the Heisenberg group $\mathbb{H}^d$ initiated in [2]. As observed in [2], the case when $d = 1$ is not very different from the case when $d \geq 2$, but the statement in this particular case are less pleasant. Thus, for the sake of simplicity, we shall assume that $d \geq 2$. Let us recall that the Heisenberg group is the space $\mathbb{R}^{2d+1}$ of the (non commutative) law of product

$$w \cdot w' = (x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' + (y| x' ) - (y' | x)).$$

The left invariant vector fields are

$$X_j = \partial_{x_j} + y_j \partial_s, \quad Y_j = \partial_{y_j} - x_j \partial_s, \quad j = 1, \ldots, d \quad \text{and} \quad S = \partial_s = \frac{1}{2} [Y_j, X_j].$$

In all that follows, we shall denote by $Z = (Z_\ell)_{1 \leq \ell \leq 2d}$ defined by $Z_j = X_j$ and $Z_{j+d} = Y_j$ for $j$ in $\{1, \ldots, d\}$ and by $\hat{Z}$ the modulus of smooth vector fields generated by $Z$. Moreover, for any $C^1$ function $f$, we shall state

$$\nabla_H f \overset{\text{def}}{=} (Z_1 \cdot f, \ldots, Z_{2d} \cdot f).$$

Keywords: Trace and trace lifting, Heisenberg group, Hörmander condition, Hardy’s inequality.

The key point is that $Z$ satisfies Hörmander’s condition at order 2, which means that the family $(Z_1, \ldots, Z_{2d}, [Z_1, Z_{d+1}])$ spans the whole tangent space $T\mathbb{R}^{2d+1}$.

For $k \in \mathbb{N}$ and $V$ an open subset of $\mathbb{R}^d$, we define the associated Sobolev space as following

$$H^k(\mathbb{H}^d, V) = \left\{ f \in L^2(\mathbb{R}^{2d+1}) | \text{Supp } f \subset V \right\},$$

where $\alpha \in \{1, \ldots, 2d\}^{k'}$, $|\alpha| \overset{\text{def}}{=} k'$ and $Z^\alpha \overset{\text{def}}{=} Z_{\alpha_1} \cdots Z_{\alpha_{k'}}$. As in the classical case, when $s$ is any real number, we can define the function space $H^s(\mathbb{H}^d)$ through duality and complex interpolation, Littlewood-Paley theory on the Heisenberg group (see [4]), or Weyl-Hörmander calculus (see [7], [8] and [9]).

It turns out that these spaces have properties which look very much like the ones of usual Sobolev spaces, see [2] and their references.

The purpose of this paper is the study of the problems of trace and trace lifting on a smooth hypersurface of $\mathbb{H}^d$ in the frame of Sobolev spaces. Let us point out that the problem of existence of trace appears only when $s$ is less than or equal to 1. Indeed, under the sub-ellipticity of system $Z$, the space $H^s(\mathbb{H}^d)$ is included locally in $H^{\frac{s}{2}}(\mathbb{R}^{2d+1})$. So if $s$ is strictly larger than 1, this implies that the trace on any smooth hypersurface exists and belongs locally to the usual Sobolev space $H^{s-\frac{1}{2}}$ of the hypersurface. Thus the case when $s = 1$ appears as the critical one. It is the case we study here.

1.1. Statement of the results

Two very different cases then appear: the one when the hypersurface is non characteristic, which means that any point $w_0$ of the hypersurface $\Sigma$ is such that $Z|_{w_0} \not\subset T_{w_0} \Sigma$, and the one when some point $w_0$ of the hypersurface $\Sigma$ is characteristic, which means that $Z|_{w_0} \subset T_{w_0} \Sigma$.

The non characteristic case is now well understood. In [2], we give a full account of trace and trace lifting results on smooth non characteristic hypersurfaces for $s > \frac{1}{2}$. This result generalize various previous results (see among others [6], [10] and [12]).

Let us recall this theorem in the case of $H^1$ (see [2] for the details). If $w_0$ is any non characteristic point of $\Sigma$, then there exists at least one of the vector fields $Z_1, \ldots, Z_{2d}$ which is transverse to $\Sigma$ at $w_0$. We define $\hat{Z}_\Sigma$ by
\( \hat{Z}_{\Sigma|_w} = T_w \Sigma \cap \hat{Z}_{|_w} \) (for \( w \) in \( \Sigma \)). It is easily checked that, if \( g \) is a local defining function of \( \Sigma \), the family \( R_{\Sigma} \)

\[
R_{j,k} \overset{\text{def}}{=} (Z_j \cdot g)Z_k - (Z_k \cdot g)Z_j
\]
generates \( \hat{Z}_{\Sigma} \) and that it satisfies the Hörmander condition at order 2 (see for instance Lemma 4.1 of [2]). We define, for an open subset \( W \) of \( \Sigma \),

\[
H^k(R_{\Sigma}, W) = \{ f \in L^2(W) \mid \forall (j, k), \ R_{j,k}u \in L^2(W) \}.
\]

We have proved the following trace and trace lifting theorem in [2]:

**Theorem 1.1.** — *Let us suppose that \( \Sigma \) is non characteristic on an open subset \( V \) of \( \mathbb{H}^d \), then the trace operator on \( \Sigma \) denoted by \( \gamma_{\Sigma} \) is an onto continuous map from \( H^1(\mathbb{H}^d, V) \) onto\]*

\[
\left[ H^1(R_{\Sigma}, V \cap \Sigma), L^2(V \cap \Sigma) \right]_1 \overset{\text{def}}{=} H^{\frac{1}{2}}(R_{\Sigma}, V \cap \Sigma).
\]

**Remark.** — As the system \( R_{\Sigma} \) satisfies the Hörmander’s condition at order 2, Theorem 1.1 implies in particular that \( \gamma_{\Sigma} \) maps \( H^1(\mathbb{H}^d, V) \) into the classical Sobolev space \( H^{\frac{1}{2}}(V \cap \Sigma) \).

We shall now consider the characteristic case. The set of characteristic points of \( \Sigma \)

\[
\Sigma_c = \left\{ w \in \Sigma \mid \hat{Z}_{|w} \subset T_w \Sigma \right\},
\]

may have a complicated structure. Let us introduce the following definition.

**Definition 1.1.** — *A characteristic point \( w_0 \) of a hypersurface \( \Sigma \) is a regular point of order \( r \) if and only if*

i) for any 1-form \( \theta \in T^*\mathbb{R}^{2d+1} \) that vanishes on \( T\Sigma \) and such that \( \theta(w_0) \neq 0 \), the system \( (L_{Z_j}(\theta|_{T_{w_0} \Sigma}))_{1 \leq j \leq 2d} \) is of rank \( r \);

ii) near \( w_0 \), the characteristic set \( \Sigma_c \) is a submanifold of \( \Sigma \) of codimension \( r \) in \( \Sigma \).

Let us make some comments about this definition. A regular characteristic point of order \( 2d \) is exactly the familiar notion of non degenerate characteristic point. This notion of non degenerate characteristic point have been used in our preceding work [2] to study this problem of trace.

As we shall prove in forthcoming Proposition 2.2, if \( g \) is a local defining function of \( \Sigma \), the condition i) means exactly that the matrix \( (Z_i \cdot Z_j \cdot g)_{1 \leq i, j \leq 2d} \) is of rank \( r \) at \( w_0 \). Let us notice that, because, if \( i \in \{1, \ldots, d\} \) and \( j \neq i + d \),

\[
(Z_i \cdot Z_{i+d} \cdot g)(w_0) - (Z_{i+d} \cdot Z_i \cdot g)(w_0) = -2\partial_s g(w_0) \neq 0
\]
and
\[ (Z_i \cdot Z_j \cdot g) = (Z_j \cdot Z_i \cdot g), \]
the rank of the matrix \((Z_i \cdot Z_j \cdot g)_{1 \leq i,j \leq 2d}\) is at least \(d\) at \(w_0\).

Let us give some examples. First let us consider the case when the hypersurface \(\Sigma\) is given by an equation of the type \(s - P(x, y)\) where \(P\) is a homogeneous polynomial of degree 2 on \(\mathbb{R}^{2d}\). Let us observe that this equation is homogeneous of order 2 with respect to the dilations of Heisenberg group \(d_{\lambda}(x, y, s) \overset{\text{def}}{=} (\lambda x, \lambda y, \lambda^2 s)\). In this case \(w_0 = (0, 0, 0)\) is always a regular characteristic point. Indeed the family \((Z_j \cdot g)_{1 \leq j \leq 2d}\) is a family of linear forms on \(\mathbb{R}^{2d}\). As \(X_j|w_0 = \partial_{x_j}\) and \(Y_j|w_0 = \partial_{y_j}\), the rank of the family is exactly the rank of the matrix \((Z_i \cdot Z_j \cdot g)_{1 \leq i,j \leq 2d}\) at the point \(w_0\). Thus \(\Sigma_c\) is obviously a linear submanifold of codimension \(r\) of \(\Sigma\). This particular case is treated in \([3]\).

Now let us exhibit an example of non regular characteristic point. In the case when \(d = 2\), let us define, for \(a\) in \(\mathbb{R}\),
\[ \Sigma_a = \left\{ (x_1, y_1, x_2, y_2, s) \in \mathbb{R}^5 \mid s = x_1 y_1 + a(x_1^3 + y_1^3) \right\}. \]
If \(a = 0\), as observed above, the origin is a regular characteristic point. A very easy computation shows that the rank of the matrix \((Z_i \cdot Z_j \cdot g)_{1 \leq i,j \leq 4}\) is three. But the characteristic set \(\Sigma_{a,c}\) is the set of points of \(\Sigma_a\) such that
\[ 3ax_1^2 = -2x_1 + 3ay_1^2 = y_2 = x_2 = 0. \]
If \(a \neq 0\), the characteristic set \(\Sigma_{a,c}\) reduces to the origin.

Let us introduce some rings of functions adapted to our situation.

**Definition 1.2.** — Let \(W\) be any open subset of \(\Sigma\) and \(F\) a closed subset of \(W\). Let us denote by \(C^\infty_F(W)\) the set of smooth functions \(a\) on \(W \setminus F\) such that for any multi-index \(\alpha\), a constant \(C_\alpha\) exists such that
\[ \forall \alpha \in \mathbb{N}^d \mid \partial^\alpha a(z) \mid \leq C_\alpha d(z, F)^{-|\alpha|}, \]
where \(d\) denotes the distance on \(\Sigma\) induced by the euclian distance on \(\mathbb{R}^{2d+1}\).

Now let us define the vector fields on \(\Sigma\) which will describe the regularity on \(\Sigma\).

**Definition 1.3.** — Let \(w_0\) a characteristic point of a hypersurface \(\Sigma\). Let \(W\) be a neighbourhood of \(w_0\). We denote by \(Z_\Sigma\) the \(C^\infty_{\Sigma_c}(W)\) modulus spanned by the set vector fields of \(Z \cap T\Sigma|_W\) that vanish on \(\Sigma_c\).

As we shall see in Proposition 3.1, the modulus \(Z_\Sigma\) is of finite type (of course as a \(C^\infty_{\Sigma_c}(W)\) modulus) if \(w_0\) is a regular characteristic point and \(W\).
is chosen sufficiently small. If \( g \) is a local defining function of \( \Sigma \), a generating system is given by
\[
R_{j,k} \overset{\text{def}}{=} (Z_j \cdot g)Z_k - (Z_k \cdot g)Z_j \quad \text{for} \quad 1 \leq j < k \leq 2d.
\]
Now we are ready to introduce the space of traces.

**Definition 1.4.** — Let \( w_0 \) a regular characteristic point of a hypersurface \( \Sigma \). Let \( W \) be a sufficiently small neighborhood of \( w_0 \). We denote by \( H^1(Z\Sigma, W) \) the space of functions \( v \) of \( L^2(W) \) such that
\[
\|v\|_{H^1(Z\Sigma, W)} \overset{\text{def}}{=} \|v\|_{L^2(W)}^2 + \sum_{1 \leq j,k \leq 2d} \|R_{j,k}v\|_{L^2(W)}^2 < \infty,
\]
where the family \( (R_{j,k})_{1 \leq j,k \leq 2d} \) is given by (1.1). If \( s \in [0, 1] \), we define \( H^s(Z\Sigma, V) \) by complex interpolation.

Our theorem is the following.

**Theorem 1.2.** — Let \( w_0 \) a regular characteristic point of a hypersurface \( \Sigma \). Let \( V \) be a sufficiently small neighborhood of \( w_0 \). Then the restriction map \( \gamma_{\Sigma} \) is an onto continuous map from \( H^1(\mathbb{H}^d, V) \) onto \( H^{\frac{s}{2}}(Z\Sigma, V \cap \Sigma) \).

Let us remark that, if \( w_0 \) is a non degenerate characteristic point (i.e. a regular characteristic point of order \( 2d \)) this theorem is Theorem 1.8 of [2].

### 1.2. Structure of the proof

In our paper [2], we use a blow up of the point \( w_0 \) (which is \( \Sigma_c \) in the case when the characteristic point \( w_0 \) is of order \( 2d \)). Here we shall blow up the submanifold \( \Sigma_c \). In order to do it, let us introduce a function \( \varphi \in D(\mathbb{R}_+ \setminus \{0\}) \) such that
\[
\forall t \in [-1, 1] \setminus \{0\}, \quad \sum_{p=0}^{\infty} \varphi(2^p t) = 1.
\]
Let us define the function \( \rho_c \) by \( \rho_c \overset{\text{def}}{=} (g^2 + |\nabla g|^4)^{\frac{1}{2}} \). Now writing that for any function \( u \) in \( L^2(\rho_c \leq 1) \),
\[
u = \sum_{p=0}^{\infty} \varphi_p u \quad \text{with} \quad \varphi_p(w) \overset{\text{def}}{=} \varphi(2^p \rho_c(w)),
\]
we apply Theorem 1.1 of trace and trace lifting to each piece \( \varphi_p u \) which is supported in a domain where \( \Sigma \) is non characteristic because \( \rho_c \sim 2^{-p} \) in this domain. This decomposition leads immediately to the problem of
estimating the norm $H^1(\mathbb{H}^d)$ of each piece $\varphi_p u$. Leibnitz formula and the chain rule tell us that
\[
\nabla_{\mathbb{H}}(\varphi_p u) = \varphi_p \nabla_{\mathbb{H}} u + 2^p \varphi'(2^p \rho_c) u \nabla_{\mathbb{H}} \rho_c.
\]
Let us observe that, as
\[
Z_j \rho_c^4 = 2g Z_j \cdot g + 4|\nabla_{\mathbb{H}} g|^2 \sum_{k=1}^{2d}(Z_k \cdot g) Z_j \cdot (Z_k \cdot g),
\]
we have, for any real number $s$, $|\nabla_{\mathbb{H}} \rho_c^s| \leq C_s \rho_c^{s-1}$. As the support of $\varphi'(2^p \rho_c)$ included in $\rho_c \sim 2^{-p}$, the supports of $\varphi'(2^p \rho_c)$ and $\varphi'(2^p \rho_c)$ are disjoint if $|p - p'| \geq N_0$ for some $N_0$. Thus, we get that
\[
\sum_{p=0}^{\infty} 2^{2p} \| \varphi'(2^p \rho_c) u \nabla_{\mathbb{H}} \rho_c \|_{L^2}^2 \leq C \left( \frac{u}{\rho_c} \right)_{L^2}^2.
\]
This leads to the proof of the following Hardy type inequality.

**Theorem 1.3.** — If $w_0$ is a regular characteristic point of $\Sigma$, a neighbourhood $V$ of $w_0$ exists such that, for any $u$ in the space $H^1(\mathbb{H}^d, V)$ of $H^1(\mathbb{H}^d)$ functions supported in $V$,
\[
\int_{\mathbb{H}^d} \frac{u^2}{\rho_c^2} dw \leq C \| \nabla_{\mathbb{H}} u \|_{L^2}^2 \text{ with } \rho_c = \left( g^2 + |\nabla_{\mathbb{H}} g|^4 \right)^{\frac{1}{4}}.
\]
This theorem implies that, for any $u$ in $H^1(\mathbb{H}^d, V)$,
\[
\sum_{p=0}^{\infty} \| \nabla_{\mathbb{H}} (\varphi_p u) \|_{L^2}^2 \leq C \| \nabla_{\mathbb{H}} u \|_{L^2}^2. \tag{1.4}
\]

The proof of this theorem, which is the core of this work, is the purpose of the second section.

In the third section, we first straighten the submanifolds $\Sigma$ and $\Sigma_c$, and after dilation, we apply Theorem 1.1. This gives a rather unpleasant description of the trace space. Then, we prove an interpolation result which allows to conclude the proof of Theorem 1.2.

### 2. A Hardy type inequality

#### 2.1. The classical Hardy inequality

As a warm up, let us recall briefly the usual proof of the classical Hardy inequality
\[(1)\]

\[\text{(1)}\] For a different approach based on Fourier analysis, see [1].
(2.1) \( \int_{\mathbb{H}_d^+} \frac{u^2}{\rho^2} dw \leq C \| \nabla_{\mathbb{H}} u \|_{L^2}^2 \) with \( \rho(w) = (s^2 + (|x|^2 + |y|^2)^2)^{\frac{1}{4}} \).

As \( D(\mathbb{H}_d^+ \setminus \{0\}) \) is dense in \( H^1(\mathbb{H}_d) \), we restrict ourselves to functions \( u \) in \( D(\mathbb{H}_d^+ \setminus \{0\}) \). Then the proof mainly consists in an integration by parts with respect to the radial vector field \( R_{\mathbb{H}} \) adapted to the structure of \( \mathbb{H}_d^+ \), namely

\[
R_{\mathbb{H}} \overset{\text{def}}{=} 2s \partial_s + \sum_{j=1}^d (x_j \partial_{x_j} + y_j \partial_{y_j}) = s[Y_1, X_1] + \sum_{j=1}^d (x_j X_j + y_j Y_j),
\]

divided by \( s^2 \) and \( \div R_{\mathbb{H}} = 2d + 2 \). More precisely, this gives

\[
- d \int \frac{u^2}{\rho^2} dw = \int \sum_{j=1}^d \frac{u}{\rho} (\frac{x_j}{\rho} X_j + \frac{y_j}{s} Y_j) udw
- d \int (\frac{s}{\rho^2}) u(X_1 u) dw + \int (\frac{s}{\rho^2}) u(Y_1 u) dw.
\]

As we have \( |Z_j \left( \frac{s}{\rho^2} \right)| \leq C \rho^{-1} \), Cauchy-Schwarz inequality gives (2.1).

2.2. Reduction to the construction of substitute of \( \rho \) and \( R_{\mathbb{H}} \)

The proof of Theorem 1.3 is based on the following proposition that we admit for the time being.

**Proposition 2.1.** — There is a couple of vector fields \((Z_0, \ol Z_0)\) in \((T\Sigma \cap \hat{\Sigma}) \times \mathcal{Z}\) such that

\[
D(\ol Z_0 \cdot g)(w_0) \neq 0 \text{ and } [Z_0, \ol Z_0] - 2\partial_s \in \hat{\Sigma}.
\]

Moreover, a smooth function \( \beta \) exists such that, if

\[
R_1 \overset{\text{def}}{=} 2g \partial_s + \beta(\ol Z_0 \cdot g)Z_0 \text{ and } \rho_0 \overset{\text{def}}{=} (g^2 + (\ol Z_0 \cdot g)^4)^{\frac{1}{2}},
\]

then, we have

\[
R_1 \rho_0^4 = 4 \rho_0^4 \text{ and } (\div R_1)(w_0) = 3.
\]

With this proposition, we shall prove that a neighbourhood \( V \) of \( w_0 \) exists such that, for any \( u \) in \( H^1(\mathbb{H}_d^+, V) \),

\[
(2.2) \quad \int \frac{u^2}{\rho_0^2} dw \leq C \| \nabla_{\mathbb{H}} u \|_{L^2}^2.
\]
It is obvious that this inequality implies Theorem 1.3. Surprisingly, we are not able to prove directly the inequality of Theorem 1.3.

In order to prove the inequality (2.2), let us first reduce to the case of smooth functions compactly supported outside \( \rho_0^{-1}(0) \). Indeed, Proposition 2.1 implies that, near \( w_0 \), the set \( \rho_0^{-1}(0) \) is a submanifold of \( \mathbb{H}^d \) of codimension 2. The following lemma will allow us to assume all along the proof that \( u \) belongs to \( \mathcal{D}(V \setminus \rho_0^{-1}(0)) \).

**Lemma 2.1.** — Let \( V \) be a bounded domain of \( \mathbb{H}^d \) and \( \Gamma \) a submanifold of codimension \( \geq 2 \). Then \( \mathcal{D}(V \setminus \Gamma) \) is dense in the space \( H_0^1(\mathbb{H}^d, V) \) of functions of \( H_0^1(\mathbb{H}^d) \) supported in \( V \) equipped with the norm

\[
\left( \| u \|_{L^2}^2 + \| \nabla_H u \|_{L^2}^2 \right)^{\frac{1}{2}}.
\]

**Proof of Lemma 2.1.** — As \( H_0^1(\mathbb{H}^d, V) \) is a Hilbert space, it is enough to prove that the orthogonal of \( \mathcal{D}(V \setminus \Gamma) \) is \{0\}. Let \( u \) be in this space. For any \( v \in \mathcal{D}(V \setminus \Gamma) \), we have

\[
(u|v)_{L^2} + (\nabla_H u|\nabla_H v)_{L^2} = 0.
\]

By integration by parts, this implies that

\[
\forall v \in \mathcal{D}(V \setminus \Gamma), \quad (u - \Delta_H u, v) = 0.
\]

Thus the support of \( u - \Delta_H u \) is included in \( \Gamma \). As \( Z_j u \) belongs to \( L^2 \), then \( Z_j^2 u \) belongs to \( H^{-1}(\mathbb{R}^{2d+1}) \) (the classical Sobolev space). And except 0, no distribution of \( H^{-1}(\mathbb{R}^{2d+1}) \) can be supported in a submanifold of codimension greater than 1. Thus \( u - \Delta_H u = 0 \). Taking the \( L^2 \) scalar product with \( u \) implies that \( u \equiv 0 \).

Thanks to Proposition 2.1, we have

\[
\rho_0^{-2} = -\frac{1}{2} R_1 \cdot \rho_0^{-2}.
\]

Thus by integration by parts, we have, using Proposition 2.1,

\[
\int \frac{u^2}{\rho_0^2} dw = \frac{3}{2} \int \frac{u^2}{\rho_0^2} dw + \int \frac{\theta u^2}{\rho_0^2} dw + I
\]

with

\[
I \overset{\text{def}}{=} \int \frac{u}{\rho_0^2} (R_1 \cdot u) dw \quad \text{and} \quad \theta \overset{\text{def}}{=} \text{div } R_1 - 3.
\]

As the function \( \theta \) vanishes at the point \( w_0 \), we can assume that \( V \) is sufficiently small such that \( \| \theta \|_{L^\infty(V)} \leq \frac{1}{4} \). This gives

\[
\int \frac{u^2}{\rho_0^2} dw \leq 4|I|.
\]
In order to estimate $I$, which contains terms of the type $g\partial_s u$, we have to compute the vector field $R_1$ in term of elements of $Z$. Proposition 2.1 claims that there are two families $(\beta_k)_{1 \leq k \leq 2d}$ and $(\gamma_k)_{1 \leq k \leq 2d}$ of smooth functions such that

$$R_1 = 2g[Z_0, Z_0] + \sum_{k=1}^{2d} (\beta_k g + \gamma_k (Z_0 \cdot g)) Z_k.$$  

We deduce that

$$I = J_1 + J_2$$

with

$$J_1 \overset{\text{def}}{=} \sum_{k=1}^{2d} \int \frac{u}{\rho_0} \beta_k g + \gamma_k (Z_0 \cdot g) \frac{(Z_k \cdot u)}{\rho_0} dw$$

and

$$J_2 \overset{\text{def}}{=} \int \frac{u}{\rho_0^2} g[Z_0, Z_0] \cdot udw.$$

As $V$ is supposed bounded, we have that the functions

$$\frac{\beta_k g + \gamma_k (Z_0 \cdot g)}{\rho_0}$$

are bounded on $V$. Cauchy-Schwarz inequality yields

$$|J_1| \leq C \left\| \frac{u}{\rho_0} \right\|_{L^2} \left\| \nabla_H u \right\|_{L^2}.$$  

The estimate about $J_2$ is a little bit more difficult to obtain. Let us write that

$$K_1 \overset{\text{def}}{=} \int \frac{g}{\rho_0} Z_0 \cdot (Z_0 \cdot u) dw \quad \text{and} \quad K_2 \overset{\text{def}}{=} \int \frac{g}{\rho_0} Z_0 \cdot (Z_0 \cdot u) dw.$$

By integration by parts, we have $K_1 = -K_{11} - K_{12}$ with

$$K_{11} \overset{\text{def}}{=} \int \frac{g}{\rho_0} (Z_0 \cdot u) \frac{(Z_0 \cdot u)}{\rho_0} dw \quad \text{and} \quad K_{12} \overset{\text{def}}{=} \int f \frac{u}{\rho_0} (Z_0 \cdot u) dw$$

with $f \overset{\text{def}}{=} \left( \text{div} Z_0 \right) \frac{g}{\rho_0} + \rho_0 \left( Z_0 \cdot \frac{g}{\rho_0^2} \right)$.  

By definition of $\rho_0$, it is obvious that

$$|K_{11}| \leq C \left\| \nabla_{\Xi} u \right\|_{L^2}^2.$$  

As we can assume that $V$ is included in $\rho_0^{-1}(0,1)$, we have that $\rho_0^{-1} g |\text{div} Z_0| \leq C$ on $V$. Moreover using that $Z_0 \cdot g = 0$, we get

$$\left| Z_0 \cdot \frac{g}{\rho_0^2} \right| = 2g \frac{2}{\rho_0^2} \left| Z_0 \cdot (Z_0 \cdot g) \right| \left| Z_0 \cdot g \right| ^3 \leq C \frac{g}{\rho_0^2} \leq \frac{C}{\rho_0}.$$  

This ensures that $f$ is bounded on $V$ and thus by Cauchy-Schwarz inequality,

$$\left\| K_{12} \right\| \leq C \left\| \frac{u}{\rho_0} \right\|_{L^2} \left\| \nabla_{\Xi} u \right\|_{L^2}.$$

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Together with (2.7), this proves that

\[ |K_1| \leq C \left( \left\| \frac{u}{\rho_0} \right\|_{L^2} + \| \nabla H u \|_{L^2} \right) \| \nabla H u \|_{L^2}. \]

In order to estimate \( K_2 \), let us write that, by integration by parts,

\[ K_2 = -\int \frac{g}{\rho_0^4}(Z_0 \cdot u)(Z_0 \cdot u)dw - \int \rho_0 \left( \frac{Z_0 \cdot g}{\rho_0^4} \right) u \frac{Z_0 \cdot u}{\rho_0} dw. \]

Using that

\[ Z_0 \cdot \rho_0^4 = 2g(Z_0 \cdot g) + 4 \left( Z_0 \cdot (Z_0 \cdot g) \right) (Z_0 \cdot g)^3, \]

we immediately get that the function \( \rho_0 \left( \frac{Z_0 \cdot g}{\rho_0^4} \right) u \frac{Z_0 \cdot u}{\rho_0} \) is bounded on \( V \) and we deduce that

\[ |K_2| \leq C \left( \left\| \frac{u}{\rho_0} \right\|_{L^2} + \| \nabla H u \|_{L^2} \right) \| \nabla H u \|_{L^2}. \]

Together with (2.4), (2.6) and (2.8), we infer that

\[ \left\| \frac{u}{\rho_0} \right\|_{L^2}^2 \leq C \left( \left\| \frac{u}{\rho_0} \right\|_{L^2} + \| \nabla H u \|_{L^2} \right) \| \nabla H u \|_{L^2} \]

which concludes the proof of the inequality 2.2 (and thus Theorem 1.3) provided of course we prove Proposition 2.1

### 2.3. Construction of substitute of \( \rho \) and \( R_H \)

Let us start with some remarks about the relations between \( \Sigma_c \) and the vector fields \( Z_j \) in the case when \( w_0 \) is a regular characteristic point.

**Proposition 2.2.** — The condition i) of Definition 1.1 is equivalent to the fact that, for any defining function \( g \) of \( \Sigma \), the rank of the matrix

\[ (Z_i \cdot Z_j \cdot g(w_0))_{1 \leq i,j \leq 2d} \]

is \( r \).

**Proof of Proposition 2.2.** — Let \( g \) be a local defining function of \( \Sigma \). Of course, \( Dg \) vanishes on \( T\Sigma \). As \( Z_j(w_0) \) belongs to \( T_{w_0} \Sigma \), we have

\[ L_{Z_j}(Dg)(w_0) = D(Z_j \cdot g)(w_0). \]

By definition of \( Z \), we infer that

\[ D(Z_j \cdot g)(w_0) = \sum_{i=1}^{2d} (Z_i \cdot Z_j \cdot g)(w_0)dz_i. \]

Thus the rank of matrix \( (Z_i \cdot Z_j \cdot g)_{1 \leq i,j \leq 2d} \) is the rank of \( L_{Z_j}(Dg)(w_0) \).

Conversely, let \( \theta \) be a 1-form that vanishes on \( T\Sigma \) and such that \( \theta(w_0) \neq 0 \) and \( g \) a local defining function of \( \Sigma \). A function \( a \) that does not vanish at \( w_0 \) exists such that \( \theta = a Dg \). Thanks to Leibnitz formula, \( L_{Z_j}(\theta)(w_0)\vert_{T_{w_0}\Sigma} = \)
a(w_0)D(Z_j \cdot g)(w_0)|_{\tau_{w_0}\Sigma}. The fact that the function a does not vanish at point \( w_0 \) implies the proposition. 

In all that follows, \( g \) will denote a defining function of \( \Sigma \) of the form \( g(x, y, s) = s + f(x, y) \) (this is allowed by the implicit function theorem) near \( w_0 \), assumed to be the origin of \( H^d \) which is assumed to be a characteristic regular point of order \( r < 2d \).

As the matrix \( (Z_i \cdot Z_j \cdot g)_{1 \leq i, j \leq 2d} \) is of rank \( r \) in \( w_0 \), and as \( Z_i|_{w_0} = \partial_{z_i} \), a family \( (j_\ell)_{1 \leq \ell \leq r} \) exists in \( \{1, \ldots, 2d\}^r \) such that the linear forms \( (D(Z_{j_\ell} \cdot g))_{1 \leq \ell \leq r} \) are linearly independent near \( w_0 \). Moreover, the function \( Z_i \cdot g \) are independent of \( s \) and \( Dg(w_0) = (ds, 0, 0) \). Thus the family of functions

\[
(2.9) \quad (g, (Z_{j_1} \cdot g), \ldots, (Z_{j_r} \cdot g))
\]

is a family of \( r + 1 \) independent functions. They vanish on the submanifold \( \Sigma_c \) which is by hypothesis a submanifold of \( H^d \) of codimension \( r + 1 \). This implies that, near \( w_0 \),

\[
(2.10) \quad \Sigma_c = \{ w \mid g(w) = (Z_{j_1} \cdot g)(w) = \cdots = (Z_{j_r} \cdot g)(w) = 0 \}.
\]

We shall keep these notations all along this text.

The definition of substitute to \( \rho \) and \( R_H \) relies on the following two lemmas.

**Lemma 2.2.** — A couple of vector fields \( (\tilde{Z}_0, Z_0) \) exists in \( (\mathcal{Z} \setminus \{Z_{j_1}, \ldots, Z_{j_r}\}) \times (\pm \mathcal{Z}) \) such that

\[
[\tilde{Z}_0, Z_0] = 2\partial_s \text{ and } D(\tilde{Z}_0 \cdot g)(w_0) \neq 0.
\]

**Proof of Lemma 2.2.** — Let us consider \( \tilde{Z}_0 \in \mathcal{Z} \setminus \{Z_{j_1}, \ldots, Z_{j_r}\} \) and \( Z_0 \) in \( \pm \mathcal{Z} \) such that \([\tilde{Z}_0, Z_0] = 2\partial_s \). If \( \pm Z_0 \) belongs to \( \{Z_{j_1}, \ldots, Z_{j_r}\} \), then (2.9) implies that \( D(Z_0 \cdot g)(w_0) \) is different from 0 and then \( Z_0 = \tilde{Z}_0 \) fits. If \( \pm Z_0 \) is not in \( \{Z_{j_1}, \ldots, Z_{j_r}\} \), as

\[
(\tilde{Z}_0 \cdot (Z_0 \cdot g))(w_0) - (Z_0 \cdot (\tilde{Z}_0 \cdot g))(w_0) = 2
\]

either \( D(\tilde{Z}_0 \cdot g)(w_0) \) or \( D(Z_0 \cdot g)(w_0) \) is different from 0. Thus if

\[
D(Z_0 \cdot g)(w_0) = 0,
\]

we get the lemma interchanging the role of \( Z_0 \) and \( \tilde{Z}_0 \).

For the reader’s convenience, we recall the proof given in [3] in the case when \( g = s - P(x, y) \) where \( P \) is a homogeneous polynomial of degree 2. In
this case, the functions $Z_j \cdot g$ are linear forms on $\mathbb{R}^{2d}$. Thus, by hypothesis, there is a family of real numbers $(\alpha_\ell)_{1 \leq \ell \leq r}$ such that

$$\tilde{Z}_0 \cdot g = \sum_{\ell=1}^r \alpha_\ell Z_{j_\ell} \cdot g.$$ 

Now let us define

$$Z_0 \overset{\text{def}}{=} \tilde{Z}_0 - \sum_{\ell=1}^r \alpha_\ell Z_{j_\ell}, \quad R_1 \overset{\text{def}}{=} 2g \partial_s + \frac{1}{2}(Z_0 \cdot g)Z_0$$

and $\rho_0 \overset{\text{def}}{=} (g^2 + (Z_0 \cdot g)^4)^{\frac{1}{4}}$.

As $Z_0$ and the $Z_{j_\ell}$ commute, we have

$$[Z_0, Z_0] = 2\partial_s, \quad Z_0 \cdot g = 0 \quad \text{and} \quad Z_0 \cdot Z_0 \cdot g = 2.$$

This implies that

$$R_1 \cdot g = 2, \quad R_1 \cdot Z_0 \cdot g = 1 \quad \text{and} \quad \text{div} R_1 = 2\partial_s + \frac{1}{2}Z_0 \cdot Z_0 \cdot g = 3.$$

We immediately infer that $R_1 \cdot \rho_0^4 = 4\rho_0^4$. Proposition 2.1 is proved in that case.

Let us treat now the general case. Noticing that the functions $Z_j \cdot g$ do not depend on $s$, the set of the zeros of the functions $Z_{j_\ell} \cdot g$ is a submanifold of $\mathbb{R}^{2d}$, the standard division theorem implies the existence (on a sufficiently small neighbourhood $V$ of $w_0$) of a family $(\alpha_\ell)_{1 \leq \ell \leq r}$ of functions of $C^\infty(V)$ such that

$$\tilde{Z}_0 \cdot g = \sum_{\ell=1}^r \alpha_\ell (Z_{j_\ell} \cdot g).$$

As in (2.11), let us define

$$Z_0 \overset{\text{def}}{=} \tilde{Z}_0 - \sum_{\ell=1}^r \alpha_\ell Z_{j_\ell}, \quad R_1 \overset{\text{def}}{=} 2g \partial_s + \beta(Z_0 \cdot g)Z_0$$

and

$$\rho_0 \overset{\text{def}}{=} (g^2 + (Z_0 \cdot g)^4)^{\frac{1}{4}}$$

where $\beta$ is a function determined later on. By definition of the function $\rho_0$, we have

$$R_1 \cdot \rho_0^4 = 2g(R_1 \cdot g) + 4(Z_0 \cdot g)^3 \left( R_1 \cdot (Z_0 \cdot g) \right).$$

By definition of $Z_0$, it is tangent to $\Sigma$. Using that $\partial_s g \equiv 1$, this implies that $R_1 \cdot g = 2g$. Let us compute $R_1 \cdot (Z_0 \cdot g)$. As $\partial_s(Z_0 \cdot g) = 0$, we have

$$R_1 \cdot (Z_0 \cdot g) = \beta(Z_0 \cdot g) (Z_0 \cdot (Z_0 \cdot g)).$$
Let us notice that $Z_0$ does not belong to the family $(Z_{j\ell})_{1 \leq \ell \leq r}$. Thus $Z_0$ commutes with the vector fields $Z_{j\ell}$. By definition of $Z_0$, we infer

$$[Z_0, Z_0] = [Z_0, Z_0] - \sum_{\ell=1}^{r} [\alpha_\ell Z_{j\ell}, Z_0]$$

(2.12)

$$= 2\partial_s + \sum_{\ell=1}^{r} (Z_0 \cdot \alpha_\ell) Z_{j\ell}.$$  

By definition of $Z_0$, we have $Z_0 \cdot g = 0$. Thus we get

$$Z_0 \cdot (Z_0 \cdot g) = (Z_0 \cdot g) + \sum_{\ell=1}^{r} (Z_0 \cdot \alpha_\ell)(Z_{j\ell} \cdot g)$$

(2.13)

$$= 2 + \tilde{\theta} \quad \text{with} \quad \tilde{\theta} \overset{\text{def}}{=} \sum_{\ell=1}^{r} (Z_0 \cdot \alpha_\ell)(Z_{j\ell} \cdot g).$$

It turns out that $R_1 \cdot \rho_1^4 = 4g^2 + 4(Z_0 \cdot g)^4 \beta(2 + \tilde{\theta})$. Choosing $\beta \overset{\text{def}}{=} (2 + \tilde{\theta})^{-1}$ ensures that $R_1 \cdot \rho_0^4 = \rho_0^4$. Now, let us compute $\text{div} R_1$. We have

$$\text{div} R_1 = 2\partial_s g + \beta Z_0 \cdot (Z_0 \cdot g) + (Z_0 \cdot g)(\text{div} Z_0 + Z_0 \cdot \beta).$$

Using that $\partial_s g \equiv 1$ and (2.13), we get

$$\text{div} R_1 = 2 + \beta(2 + \tilde{\theta}) + (Z_0 \cdot g)(\text{div} Z_0 + Z_0 \cdot \beta)$$

$$= 3 + (Z_0 \cdot g)(\text{div} Z_0 + Z_0 \cdot \beta).$$

This proves the lemma with $\theta \overset{\text{def}}{=} (Z_0 \cdot g)(\text{div} Z_0 + Z_0 \cdot \beta)$. \qed

3. The proof of the trace and trace lifting theorem

3.1. Some preliminary properties

Proposition 3.1. — A neighbourhood $W$ of $w_0$ exists such that the $C_{\Sigma_c}(W)$ modulus $Z_{\Sigma}$ spanned by the vector fields of $Z \cap T\Sigma |_W$ which vanish on the characteristic submanifold $\Sigma_c$ is of finite type and generated by

$$R_{j,k} \overset{\text{def}}{=} (Z_j \cdot g)Z_k - (Z_k \cdot g)Z_j.$$  

Proof of Proposition 3.1. — It is enough to prove that any element $L$ of $Z \cap T\Sigma$ which vanish on $\Sigma_c$ is a combinaison (with coefficients in $C_{\Sigma_c}^\infty(W)$) of the $R_{j,k}$. By definition

$$L = \sum_{j=1}^{2d} \alpha_j Z_j \quad \text{with} \quad \alpha_j|_{\Sigma_c} = 0 \quad \text{and} \quad \sum_{j=1}^{2d} \alpha_j(Z_j \cdot g) = 0.$$
Let us introduce a partition of unity \((\tilde{\psi}_j)_{1 \leq j \leq 2d}\) of the sphere \(S^{2d-1}\) such that the support of \(\tilde{\psi}_j\) is included in the set of \(\zeta\) of \(S^{2d-1}\) such that \(|\zeta| \geq (4d)^{-1}\). Let us state

\[
\psi_j \overset{\text{def}}{=} \tilde{\psi}_j \left( \frac{\nabla H g}{|\nabla H g|} \right).
\]

It is an exercise left to the reader to check that \(\psi_j\) belongs to \(C^\infty_{\Sigma_c}(W)\). On \(\Sigma \setminus \Sigma_c\), we have, for any \(j\) in \(\{1, \ldots, 2d\}\),

\[
\psi_j (L \cdot g) = \sum_{k=1}^{2d} \psi_j \alpha_k (Z_k \cdot g) = 0.
\]

By definition of \(\psi_j\), \((Z_j \cdot g)\) does not vanish on the support of \(\psi_j\). Thus we have

\[
\alpha_j \psi_j = -\frac{1}{(Z_j \cdot g)} \sum_{k \neq j} \psi_j \alpha_k (Z_k \cdot g).
\]

From this, we deduce that

\[
\psi_j L = \sum_{k \neq j} \psi_j \alpha_k \left( Z_k - \frac{(Z_k \cdot g)}{(Z_j \cdot g)} Z_j \right)
\]

\[
= \sum_{k \neq j} \frac{\psi_j \alpha_k}{(Z_j \cdot g)} ((Z_j \cdot g) Z_k - (Z_k \cdot g) Z_j).
\]

Now the facts that \(\alpha_k \in C^\infty_{\Sigma_c}\) and that \((Z_j \cdot g)\) does not vanish on the support of \(\psi_j\) ensure that

\[
\alpha_{j,k} = \frac{\psi_j \alpha_k}{(Z_j \cdot g)} \in C^\infty_{\Sigma_c}.
\]

So we have

\[
L = \sum_{1 \leq j < k \leq 2d} \alpha_{j,k} ((Z_j \cdot g) Z_k - (Z_k \cdot g) Z_j)
\]

and the proposition is proved. \(\square\)

The blow up procedure requires to straighten the submanifolds \(\Sigma\) and \(\Sigma_c\).

**Lemma 3.1.** — A neighbourhood \(V\) of \(w_0\) and a diffeomorphism \(\chi\) from \(V\) onto \(\chi(V)\) exist which satisfy the following properties.

- It straighten the submanifolds \(\Sigma\) and \(\Sigma_c\), namely

  \[
  \chi(\Sigma \cap V) = \{ w \mid s = 0 \} \cap \chi(V)
  \]

  and

  \[
  \chi(\Sigma_c \cap V) = \{ w \mid s = z_1 = \cdots z_r = 0 \} \cap \chi(V).
  \]
• The transported vector fields are of the form
\[ \chi^*(\partial_s) = \partial_s \]
and
\[ Z_j^D \overset{\text{def}}{=} \chi^*(Z_j) = \frac{\partial}{\partial e_j} + \left( \sum_{\ell=1}^{r} \alpha_{j\ell}(z)z_\ell \right) \partial_s + h_j(z, \partial_z) \]
where \((e_j)_{1 \leq j \leq 2d}\) is a basis of \(\mathbb{R}^{2d}\), the \((\alpha_{k})\) are smooth bounded functions on \(V\) such that, for \(j \in \{1, \ldots, r\}\), \(\alpha_{j}^\ell \equiv \delta_{j}^\ell\) and \((h_{j})_{1 \leq j \leq 2d}\) is a family of smooth vector fields which vanish at \(z = 0\).

Proof of Lemma 3.1. — Let \((L_k)_{r+1 \leq k \leq 2d}\) be a family of linear forms on \(\mathbb{R}^{2d}\) such that
\[ (D(Z_{j\ell} \cdot g)(w_0))_{1 \leq \ell \leq r}, (L_k)_{r+1 \leq k \leq 2d} \]
is a basis of the dual space of \(\mathbb{R}^{2d}\). It is easily checked that the map defined by
\[ \chi(x, y, s) = \left( \begin{array}{c} s + f(x, y) \\ (Z_{j\ell} \cdot g)(x, y) \quad \text{if} \quad k \leq r \\ (L_k, (x, y)) \quad \text{if} \quad k > r \end{array} \right) \]
is a local diffeomorphism which satisfies the property of the lemma. \(\square\)

From now on, we shall work only in the straighten situation and to avoid excessive heaviness of notations, we shall still denote \(Z_j^D\) by \(Z_j\).

3.2. The blow up procedure

Let us write that, for any function \(u\), we can write (at least in \(L^2\)) that
\[ u = \sum_{p=0}^{\infty} \varphi_p u \quad \text{with} \quad \varphi_p(z, s) \overset{\text{def}}{=} \varphi(2^p(s^2 + |z'|^4)^{\frac{1}{4}}) \]
where \(\varphi\) is the function introduced in (1.2) and if \(z = (z_j)_{1 \leq j \leq 2d}, z' \overset{\text{def}}{=} (z_1, \ldots, z_r, 0, \ldots, 0)\). Let us define the space of trace in this straightened situation.

Definition 3.1. — Let us denote by \([A, B]_{\theta}\) the complex interpolation between \(A\) and \(B\) and by \(H^{1}(\mathbb{R}, W)\) the space of functions of \(H^{1}(\mathbb{R})\) supported in \(W\). The space \(T_{\frac{1}{2}}^1\) is the space of functions \(v \in L^2(|z'| \leq 1)\) such
that
\[ \|v\|_{T^1/2}^2 \overset{\text{def}}{=} \sum_{p=0}^{\infty} \| \varphi_p^s v \|_{H^{1/2}(R_p)}^2 < \infty \]
with \( H^s(R_p) \overset{\text{def}}{=} [L^2(2^{-p}C_\Sigma), H^{1}(R, 2^{-p}C_\Sigma)]_s \)
where \( C_\Sigma \overset{\text{def}}{=} \{ c \leq |z'| \leq C \} \) and \( \varphi_p^s(z) \overset{\text{def}}{=} \varphi_p(z, 0) = \varphi(2^p|z'|) \).

We shall prove the following theorem.

**Theorem 3.1.** — The restriction map to the hypersurface \( (s = 0) \) can be extended in a continuous onto map from \( H^1(Z; \{ \rho_c \leq 1 \}) \) onto \( T^{1/2} \).

**Proof of Theorem 3.1.** — Once noticed that the Hardy inequality given by Theorem 1.3 becomes
\[
\int \frac{u^2(z, s)}{(s^2 + |z'|^4)^{1/2}} dz ds \leq C \sum_{j=1}^{2d} \| Z_j u \|_{L^2}^2,
\]
we get, by computations very similar to the ones done at the beginning of Subsection 1.2, an analogous of (1.4), namely
\[
\sum_{p=0}^{\infty} \sum_{j=1}^{2d} \| Z_j(\varphi_p u) \|_{L^2}^2 \leq C \sum_{j=1}^{2d} \| Z_j u \|_{L^2}^2.
\]
Let us notice that outside \( \Sigma_c = \{(z, s) | s = 0, z' = 0\} \), thus in particular in the support of \( \varphi_p \), the hypersurface \( \Sigma \) is non characteristic for \( Z \). Thus locally we can apply Theorem 1.1 to each piece \( \varphi_p u \). The key point is the control of the constant when \( p \) tends to \( \infty \). In order to do so, it is convenient to use the quasi-homogeneous dilations \( \delta_p(z, s) \overset{\text{def}}{=} (2^p z, 2^{2p} s) \). Let us define
\[
u_p(z, s) \overset{\text{def}}{=} \varphi_0(z, s) u(2^p z, 2^{2p} s)
\]
and
\[ Z_{j,p} \overset{\text{def}}{=} \frac{\partial}{\partial e_j} + \sum_{\ell=1}^{r} \alpha_{j,\ell}^p (2^{-p} z) z_\ell \partial_s + h_j(2^{-p} z, \partial z). \]
It is obvious that a one to one map \( \sigma \) of \( \{1, \ldots, 2d\} \) exists such that
\[
[Z_{j,p}, Z_{k,p}] = 2 \delta_{k,\sigma(j)} \partial_s.
\]
Moreover, as \( \| u_p \|_{L^2}^2 = 2^{2p(d+1)} \| \varphi_p u \|_{L^2}^2 \), we have, thanks to Hardy’s inequality (3.1),
\[
\sum_{p=0}^{\infty} 2^{-2pd} \| u_p \|_{L^2}^2 \leq C \sum_{j=1}^{2d} \| Z_j u \|_{L^2}^2.
\]
Applying (3.2), we infer
\[
\sum_{p=0}^{\infty} 2^{-2pd} \left( \|u_p\|_{L^2}^2 + \sum_{j=1}^{2d} \|Z_{j,p} u_p\|_{L^2}^2 \right) \leq C \sum_{j=1}^{2d} \|Z_j u\|_{L^2}^2.
\]

On the support of $\varphi_0$, the hypersurface ($s = 0$) is non characteristic with respect to the family $(Z_{j,p})_{1 \leq j \leq 2d}$ because, for $j$ between 1 and $r$,
\[ Z_{j,p} = \frac{\partial}{\partial e_j} + h_j(2^{-p}z, \partial z) + z_j \partial_s. \]

Let us notice that the transverse component of $Z_{j,p}$ does not depend on $p$. Thus we can apply Theorem 4.6 of [2] together with a result of interpolation between Sobolev spaces (see Remark 4.2 page 89 in [7]) to each $u_p$. Using that $\|u_p\|_{H^1(R^p)} = 2^{2pd} \|\varphi_p u\|_{H^1(R^s)}$, this gives in particular that a constant $C$ exists (independent of $p$) such that
\[
\|\gamma(u_p)\|_{L^2(R^{2d}), H^1(\tilde{R}_p, R^d)} \leq C 2^{2pd} \|\varphi_p u\|_{H^1(R^s)}
\]
with $\tilde{R}_p$ is the union of
\[
\tilde{R}_p = (\varphi_0^\Sigma(Z_{j,p} \cdot s) Z_{k,p} - \varphi_0^\Sigma(Z_{k,p} \cdot s) Z_{j,p})_{1 \leq j,k \leq 2d} \quad \text{and} \quad (1 - \varphi_0^\Sigma)_{1 \leq j \leq 2d}
\]
where $\varphi_0^\Sigma$ is a smooth function supported in $C^\Sigma$ such that $\varphi_0^\Sigma \equiv 1$ near the support of $\varphi_0^\Sigma$.

At this point, let us recall the definition of complex interpolation. For details of this theory, we refer in particular to [5] and [11].

**Definition 3.2.** — Let $(\mathcal{H}_j, \|\cdot\|_{j})_{j \in \{0,1\}}$ be two Hilbert spaces such that $\mathcal{H}_1$ is densely included in $\mathcal{H}_0$. Let $F(\mathcal{H}_0, \mathcal{H}_1)$ be the space of holomorphic functions $f$ from the strip $0 < \Re \zeta < 1$ into $\mathcal{H}_0$ such that $f(j + it)$ is continuous and vanishes at infinity in $\mathcal{H}_j$. Then, for $\theta \in ]0,1[$, the space $[\mathcal{H}_0, \mathcal{H}_1]_\theta$ is
\[ [\mathcal{H}_0, \mathcal{H}_1]_\theta \overset{\text{def}}{=} \left\{ v \in L^2 \mid \exists f \in F(\mathcal{H}_0, \mathcal{H}_1) \mid f(\theta) = v \right\} \]
equipped with the norm
\[ \|v\|_{[\mathcal{H}_0, \mathcal{H}_1]_\theta} \overset{\text{def}}{=} \inf_{f \in F(\mathcal{H}_0, \mathcal{H}_1) \atop j \in \{0,1\}} \max_{t \in \mathbb{R}} \|f(j + it)\|_{\mathcal{H}_j}. \]

As the support of $\gamma(u_p)$ is included in the support of $\varphi_0^\Sigma$, let us consider a smooth function $\varphi_1^\Sigma$ supported in the set where $\varphi_0^\Sigma$ has value 1 and such that $\varphi_1^\Sigma$ has value 1 near the support of $\varphi_0^\Sigma$. If $f$ is a function in $F(L^2(R^{2d}), H^1(\tilde{R}_p, R^{2d}))$ such that $f(1/2) = v$, then the function
\( \zeta \mapsto \varphi_1 f(\zeta) \) belongs to \( \mathcal{F}(L^2(C_\Sigma), H^1(\widetilde{\mathcal{R}}_p, C_\Sigma)) \) and \( \varphi_1 f(1/2) = v \). As we obviously have that \( H^1(\widetilde{\mathcal{R}}_p, C_\Sigma) = H^1(\mathcal{R}_p, C_\Sigma) \), inequality (3.5) becomes

\[
(3.6) \quad \| \gamma(u_p) \|_{[L^2(C_\Sigma), H^1(\mathcal{R}_p, C_\Sigma)]_{\frac{1}{2}}} \leq C 2^{pd} \| \varphi_p u \|_{H^1(\Sigma)}.
\]

Moreover, dilations on \( \mathbb{R}^{2d} \) of ratio \( 2^{-p} \) maps \( L^2(C_\Sigma) \) (resp. \( H^1(\mathcal{R}_p, C_\Sigma) \)) into \( L^2(2^{-p}C_\Sigma) \) (resp. \( H^1(\mathcal{R}, 2^{-p}C_\Sigma) \)) with norm equal to \( 2^{-pd} \). Thus by the functorial property of complex interpolation, inequality (3.6) becomes

\[
(3.7) \quad \| \gamma(\varphi_p u) \|_{H^\frac{1}{2}(\mathcal{R}_p)} \leq C \| \varphi_p u \|_{H^1(\Sigma)}.
\]

Inequality (3.2) implies that \( \gamma \) can be extended to a continuous linear map from \( H^1(\Sigma) \) into \( T^\frac{1}{2} \).

In order to prove that \( \gamma \) is onto, let us consider \( v \in T^\frac{1}{2} \). By definition of \( T^\frac{1}{2} \), and after dilation, we infer that

\[
(3.8) \quad \| \varphi_0^\Sigma v(2^{-p} \cdot) \|_{[L^2(C_\Sigma), H^1(\mathcal{R}_p, C_\Sigma)]_{\frac{1}{2}}} \leq C 2^{pd} c_p \| v \|_{T^\frac{1}{2}} \quad \text{with} \quad \sum_p c_p^2 = 1.
\]

As \( L^2(\mathcal{C}_\Sigma) \) (resp. \( H^1(\mathcal{R}_p, C_\Sigma) \)) is a subspace of \( L^2(\mathbb{R}^{2d}) \) (resp. \( H^1(\mathcal{R}_p; \mathbb{R}^{2d}) \)), using Theorem 4.6 of [2] together with Remarque 4.2 of [7] and (3.8), we claim the existence of a function \( \tilde{u}_p \) in the space \( H^1(\tilde{Z}_p) \) such that a constant \( C \) (independent of \( p \)) exists which satisfies, for any \( p \),

\[
(3.9) \quad \| \tilde{u}_p \|_{H^1(\tilde{Z}_p)} \leq C 2^{pd} c_p \| v \|_{T^\frac{1}{2}} \quad \text{with} \quad \sum_p c_p^2 = 1.
\]

where \( \tilde{Z}_p \) is the union of the families

\[
(\varphi_0 Z_{j,p})_{1 \leq j, k \leq 2d}, ((1 - \varphi_0) \partial_j)_{1 \leq j \leq 2d} \text{ and } (1 - \varphi_0) \partial_s.
\]

Let us consider a smooth function \( \varphi_1 \) supported in the domain where \( \tilde{\varphi} \) has value 1 and such that \( \varphi_1 \equiv 1 \) near the support of \( \varphi_0 \). Defining \( u_p \equiv \varphi_1 \tilde{u}_p \), we have, by definition of \( \mathcal{R}_p \) and by (3.9)

\[
u_p \in H^1(\tilde{Z}_p, C_\Sigma) = H^1(\mathcal{Z}_p, C_\Sigma) \quad \text{and} \quad \| u_p \|_{H^1(\mathcal{R}_p)} \leq C \| \tilde{u}_p \|_{H^1(\mathcal{R}_p, C_\Sigma)}.
\]

After dilation, this gives

\[
\sum_p \| u_p(2^p \cdot) \|_{H^1(\Sigma)}^2 \leq C \| v \|_{T^\frac{1}{2}}^2.
\]

As an integer \( N_0 \) exists such that

\[
|p - p'| \geq N_0 \implies u_p(2^p \cdot) \perp u_{p'}(2^{p'} \cdot) \quad \text{in} \quad H^1(\Sigma),
\]

the series \((u_p(2^p \cdot))_p\) converge in \( H^1(\Sigma) \) to a function \( u \) whose trace is obviously \( v \). This concludes the proof of Theorem 3.1. \( \square \)
Remark. — The trace lifting theorem provides functions in $H^1(\mathbb{H}^d)$ supported in a set of the form $s^2 \leq C(|x|^2 + |y|^2)^2$. It obviously prevents, using this method, to prove trace lifting theorem for very regular (for instance continuous) functions.

### 3.3. The space of trace as an interpolation space

The description given by Theorem 3.1 is not totally satisfactory. We want to describe this space of trace as an interpolation space to get Theorem 1.2. In order to do so, let us define, for $s \in [0,1]$, the space

$$T^s \overset{\text{def}}{=} \left\{ v \in L^2 \mid \|v\|_T^2 = \sum_p \|\varphi^\Sigma_p v\|_{H^s(\mathcal{R}_p)}^2 < \infty \right\}.$$ 

Let us start with the proof of the following lemma.

**Lemma 3.2.** — The space $T^1$ is equal to $H^1(\mathcal{R})$ and the norm are equivalent.

**Proof of Lemma 3.2.** — By definition of the norm on $H^1(\mathcal{R}_p)$, we have

$$\|((\varphi^\Sigma_p v)(2^{-p} \cdot))\|_{H^1(\mathcal{R}_p)}^2 = \|((\varphi^\Sigma_p v)(2^{-p} \cdot))\|_{L^2}^2 + \sum_{j,k} \|\mathcal{R}_{j,k,p}((\varphi^\Sigma_p v)(2^{-p} \cdot))\|_{L^2}^2.$$ 

By definition of $\mathcal{R}_{j,k,p}$, we have

$$2^{-2pd}\|\mathcal{R}_{j,k,p}((\varphi^\Sigma_p v)(2^{-p} \cdot))\|_{L^2}^2 = \|\mathcal{R}_{j,k}(\varphi^\Sigma_p v)\|_{L^2}^2.$$ 

By Leibnitz formula and by definition of $\varphi^\Sigma_p$, we have

$$\mathcal{R}_{j,k}(\varphi^\Sigma_p v)(z) = \varphi^\Sigma_p \mathcal{R}_{j,k} \cdot v(z) + (\mathcal{R}_{j,k} \cdot \varphi^\Sigma_p) v(z) = \varphi^\Sigma_p (z)(\mathcal{R}_{j,k} \cdot v)(z) + 2^p (\mathcal{R}_{j,k} \cdot |z'|)\varphi'(2^p|z'|)v(z).$$ 

As the vector fields $\mathcal{R}_{j,k}$ vanishes at 0, we have

$$\sup_{p,j,k} \|\mathcal{R}_{j,k} \varphi^\Sigma_p\|_{L^\infty} < \infty.$$ 

This gives that

$$|\mathcal{R}_{j,k}(\varphi^\Sigma_p v)(z) - \varphi^\Sigma_p \mathcal{R}_{j,k} v(z)| \leq C\varphi'(2^p|z'|)\varphi'(2^p|z'|)v(z).$$ 

As, for some positive integer $N_0$, the support of the two functions $\varphi(2^p|z'|)$ and $\varphi(2^p|z'|)$ are disjoint when $|p - p'| \geq N_0$, this gives the lemma. □

Now Theorem 1.2 will be an easy consequence of the following abstract interpolation lemma.
Lemma 3.3. — Let us consider \((H_j, \|\cdot\|_j)_{j \in \{0,1\}}\) two Hilbert spaces such that \(H_1\) is densely included in \(H_0\) and a family \((H_{j,p})_{(j,p) \in \{0,1\} \times \mathbb{N}}\) such that, for any \(p\), \(H_{j,p}\) is a closed subset of \(H_j\).

Let us assume that a family of \((\Lambda_p)_{p \in \mathbb{N}}\) of (unbounded) selfadjoints operators on \(H_{0,p}\) exists such that \(H_{1,p}\) equals to the domain of \(\Lambda_p\) and

\[
\forall u \in H_{1,p}, \|u\|_{H_1} \sim \|\Lambda_p u\|_{H_0}.
\]

Let us assume in addition that a family of operators \((A_p)_{p \in \mathbb{N}}\) exists such that, for any \((j,p)\) in \(\{0,1\} \times \mathbb{N}\), the operator \(A_p\) is continuous from \(H_j\) into \(H_{j,p}\) and

\[
\forall v \in H_j, \lim_{p \to \infty} \left\| v - \sum_{p=0}^{N} v_p \right\| = 0 \text{ and } \|v\|_{H_j}^2 \sim \sum_{p} \|A_p v\|_{H_{s,p}}^2.
\]

Then,

\[
[H_0, H_1]_s = \left\{ v \in H_0 \mid \|v\|_{H_j}^2 \overset{\text{def}}{=} \sum_{p=0}^{\infty} \|A_p v\|_{H_{s,p}}^2 \right\}
\]

with \(H_{s,p} \overset{\text{def}}{=} [H_{0,p}, H_{1,p}]_s\).

Proof of Lemma 3.3. — It is enough to prove that the two norms are equivalent on the dense space of \(v\) such that

\[
v = \sum_{p=0}^{N} v_p \text{ with } v_p \in H_{1,p}.
\]

Let us first estimate \(\|v\|_{[H_0, H_1]_s}\). By definition of the norm on \(H_{s,p}\), a function \(f_p\) exists in \(F(H_{0,p}, H_{1,p})\) such that

\[
f_p(s) = A_p v \text{ and } \max_{j \in \{0,1\}} \sup_{t \in \mathbb{R}} \|f_p(j + it)\|_{H_j} \leq 2 \|A_p v\|_{H_{s,p}}.
\]

Now let us define

\[
F_N(\zeta) \overset{\text{def}}{=} e^{\zeta^2} \sum_{p=0}^{N} f_p(\zeta).
\]

As the sum is finite, it is obvious that \(F_N\) belongs to \(F(H_0, H_1)\). Because of (3.11), we have, for \(j \in \{0,1\}\),

\[
\|F_N(j + it)\|_{H_j}^2 \leq C e^{-t^2} \sum_{p=0}^{N} \|f_p(j + it)\|_{H_j}^2 \leq C e^{-t^2} \sum_{p=0}^{N} \|A_p v\|_{H_{s,p}}^2 \leq C \|v\|_{T_s}^2.
\]
Thus by definition of the complex interpolation norm, we deduce that
\[ \|v\|_{[\mathcal{H}_0, \mathcal{H}_1]_s} \leq C\|v\|_{T^s}. \]

Now let us estimate \( \|v\|_{T^s} \). In order to do so, let us consider \( F \) in \( \mathcal{F}(\mathcal{H}_0, \mathcal{H}_1) \) such that
\[ F(s) = v \text{ and } \max_{j \in \{0, 1\}} \sup_t \|F(j + it)\|_{\mathcal{H}_j} \leq 2\|v\|_{[\mathcal{H}_0, \mathcal{H}_1]_s}. \]

For \( a \) greater than 1, let us introduce
\[ N_a(\zeta) \stackrel{\text{def}}{=} e^{C^2 - s^2} \sum_{p=0}^{N} \int_1^{a} \lambda^{2\zeta} d\mu_p(A_p F(\zeta), A_p F(\zeta)) \]
where \( \mu_p \) is the spectral measure of \( \Lambda_p \). Then, by using (3.10) and (3.11),
\[ |N_a(j + it)| \leq C e^{-t^2} \left| \int_1^{a} \lambda^{2it} \lambda^{2j} d\mu_p \left( A_p F(j + it), A_p F(j + it) \right) \right| \]
\[ \leq C e^{-t^2} \sum_{p=0}^{N} \int_1^{a} \lambda^{2j} d\mu_p \left( A_p F(j + it), A_p F(j + it) \right) \]
\[ \leq C e^{-t^2} \sum_{p=0}^{N} \|A_p F(j + it)\|_{\mathcal{H}_j}^2 \]
\[ \leq C e^{-t^2} \|F_j(j + it)\|_{\mathcal{H}_j}^2 \]
\[ \leq C \|v\|_{[\mathcal{H}_0, \mathcal{H}_1]_s}^2. \]

Then using the Phragmen-Lindelöf principle, we get that
\[ N_a(s) \leq \sup_t |N_a(it)|^{1-s} |N_a(1 + it)|^s \leq C \|v\|_{[\mathcal{H}_0, \mathcal{H}_1]_s}^2. \]

Thus a constant \( C \) exists such that, for any \( a \),
\[ \sum_{p=0}^{N} \int_1^{a} \lambda^{2s} d\mu_p(A_p v, A_p v) \leq C \|v\|_{[\mathcal{H}_0, \mathcal{H}_1]_s}^2. \]

By definition of \( \mathcal{H}_{s,p} \) and using that
\[ \|w\|_{\mathcal{H}_{s,p}}^2 = \int_1^{\infty} \lambda^{2s} d\mu_p(w, w), \]
we infer by passing to the limit when \( a \) tends to infinity in (3.12) that
\[ \sum_{p=0}^{N} \|A_p v\|_{\mathcal{H}_{s,p}}^2 \leq C \|v\|_{[\mathcal{H}_0, \mathcal{H}_1]_s}^2. \]

This conclude the proof of Lemma 3.3. \( \square \)
3.4. Conclusion of the proof of Theorem 1.2

Theorem 1.2 follows, observing that the hypothesis of Lemma 3.3 are satisfied with $\mathcal{H}_0 = L^2$, $\mathcal{H}_1 = H^1(\mathbb{R})$, $\mathcal{H}_{j,p}$ is the set of $v$ in $\mathcal{H}_j$ supported in $2^{-p}\mathbb{C}$ and $\Lambda_p$ is the square root of Dirichlet realization on $2^{-p}\mathbb{C}$ of the operator

$$\text{Id} + \Delta_{\Sigma} \text{ with } \Delta_{\Sigma} \overset{\text{def}}{=} \sum_{j,k} R_{j,k}^* R_{j,k}.$$

To be able to apply Lemma 3.3, and then to conclude the proof of Theorem 1.2, it is enough to prove the following proposition.

**Proposition 3.2.** — A neighbourhood $V$ of $w_0$ exists such that the operator $\Delta_{\Sigma}$ is selfadjoint on $L^2(V)$ with domain

$$\left\{ v \in L^2(V) \mid \forall (j, k, j', k') \in \{1, \ldots, 2d\}^4, R_{j,k}v \in L^2(V) \text{ and } R_{j,k}R_{j',k'}v \in L^2(V) \right\}.$$

**Proof of Proposition 3.2.** — Up to an omitted regularization process, it is enough to prove that, for any $v \in D(V)$,

$$\sum_{j,k} \|R_{j,k}v\|_{L^2}^2 + \sum_{j,k,j',k'} \|R_{j,k}R_{j',k'}v\|_{L^2}^2 \leq C \left( \|v\|_{L^2}^2 + \|\Delta_{\Sigma}v\|_{L^2}^2 \right). \quad (3.13)$$

Let us start with the observation that

$$\sum_{j,k} \|R_{j,k}v\|_{L^2}^2 \leq C \sum_{j,k} (R_{j,k}v| R_{j,k}v)_{L^2}$$

$$\leq C \sum_{j,k} (R_{j,k}^* R_{j,k}v|v)_{L^2}$$

$$\leq C \sum_{j,k} (R_{j,k}^* R_{j,k}v|v)_{L^2} \leq C (\Delta_{\Sigma}v|v)_{L^2}$$

$$\leq C \|\Delta_{\Sigma}v\|_{L^2} \|v\|_{L^2}. \quad (3.14)$$

In order to estimate $\|R_{j,k}R_{j',k'}v\|_{L^2}$, we are going to proceed as in the proof of Lemma 3.2. Let us write that

$$R_{j,k}R_{j',k'}(\varphi_p v) - \varphi_p R_{j,k}R_{j',k'}v$$

$$= (R_{j,k} \varphi_p)(R_{j',k'}v) + (R_{j',k'} \varphi_p)(R_{j,k}v) + \varphi_p (R_{j,k}R_{j',k'}v). \quad (3.15)$$

As the coefficients of the vector fields $R_{j,k}$ vanish on $\Sigma_c$, we have

$$\sup_{p,j,k,j',k'} \|R_{j,k} \varphi_p\|_{L^\infty} + \|R_{j,k}R_{j',k'} \varphi_p\|_{L^\infty} < \infty.$$
Thus, using (3.14), we have

\[
\| \varphi_p R_{j,k} R_{j',k'} v - R_{j,k} R_{j',k'} (\varphi_p v) \|_{L^2} \leq C c_p \| \Delta^\Sigma v \|_{L^2} \frac{1}{2} \| v \|_{L^2} \tag{3.16}
\]

with \( \sum_{p=0}^{\infty} c_p^2 = 1. \)

We have

\[
R_{j,k} R_{j',k'} (\varphi_p v) = R_{j,k,p} R_{j',k',p} (\varphi_0 v (2^p \cdot)).
\]

Lemma 4.1 of [2] tells us that the systems \((R_{j,k,p})_{j,k}\) satisfy the Hörmander condition at order 2 uniformly with respect to \( p \) on \( C \). Thus, the classical maximal estimate tells us that

\[
\| R_{j,k,p} R_{j',k',p} w \|_{L^2} \leq C \left( \sum_{j,k} R_{j,k,p}^* R_{j,k,p} w \|_{L^2} + \| w \|_{L^2} \right). \tag{3.17}
\]

Applied with \( w = \varphi_0 v (2^p \cdot) \), this gives

\[
\| R_{j,k} R_{j',k'} (\varphi_p v) \|_{L^2} \leq C 2^{pd} \left( \sum_{j,k} R_{j,k,p}^* R_{j,k,p} \varphi_0 v (2^p \cdot) \|_{L^2} + \| \varphi_0 v (2^p \cdot) \|_{L^2}^2 \right) \leq C \left( \| \Delta^\Sigma (\varphi_p v) \|_{L^2} + \| \varphi_p v \|_{L^2} \right). \tag{3.17}
\]

Then (3.16) implies that

\[
\| \Delta^\Sigma (\varphi_p v) - \varphi_p \Delta^\Sigma v \|_{L^2} \leq C c_p \| \Delta^\Sigma v \|_{L^2} \frac{1}{2} \| v \|_{L^2} \| \frac{1}{2} \| v \|_{L^2} \text{ with } \sum_{p=0}^{\infty} c_p^2 = 1.
\]

Thus, by using (3.15) and (3.17) we infer that

\[
\| \varphi_p R_{j,k} R_{j',k'} v \|_{L^2} \leq C c_p (\| \Delta^\Sigma v \|_{L^2} + \| v \|_{L^2}) \text{ with } \sum_{p=0}^{\infty} c_p^2 = 1.
\]

This proves (3.13) and thus Proposition 3.2. \( \square \)

BIBLIOGRAPHY


