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Formal geometric quantization


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FORMAL GEOMETRIC QUANTIZATION

by Paul-Émile PARADAN

Abstract. — Let $K$ be a compact Lie group acting in a Hamiltonian way on a symplectic manifold $(M, \Omega)$ which is pre-quantized by a Kostant-Souriau line bundle. We suppose here that the moment map $\Phi$ is proper so that the reduced space $M_\mu := \Phi^{-1}(K \cdot \mu)/K$ is compact for all $\mu$. Then, we can define the “formal geometric quantization” of $M$ as

$$Q^-\infty_K(M) := \sum_{\mu \in \hat{K}} Q(M_\mu)V^K_\mu.$$ 

The aim of this article is to study the functorial properties of the assignment $(M, K) \mapsto Q^-\infty_K(M)$.

Résumé. — Considérons l’action hamiltonienne d’un groupe de Lie compact $K$ sur une variété symplectique $(M, \Omega)$ préquantifiée par un fibré en droites de Kostant-Souriau. On suppose que l’application moment $\Phi$ est propre, ainsi les réductions symplectiques $M_\mu := \Phi^{-1}(K \cdot \mu)/K$ sont compactes pour tout $\mu$. On peut alors définir la quantification formelle de $M$ comme

$$Q^-\infty_K(M) := \sum_{\mu \in \hat{K}} Q(M_\mu)V^K_\mu.$$ 

Le but de ce travail est l’étude de certaines propriétés fonctorielles de l’application $(M, K) \mapsto Q^-\infty_K(M)$.

The aim of this article is to study the functorial properties of the “formal geometric quantization” process, which is defined on non-compact Hamiltonian manifolds when the moment map is proper. For this purpose, we introduce a technique of symplectic cutting that uses the (wonderful) compactifications of de Concini-Procesi [14, 15] and Brion [11], and we prove an extension of the “quantization commutes with reduction” theorem to the singular setting (here the singular manifolds that we consider are those obtained by symplectic reduction).

Keywords: Geometric quantization, moment map, symplectic reduction, index, transversally elliptic.

1. Introduction and statement of results

Let \((M, \Omega)\) be a symplectic manifold which is equipped with a Hamiltonian action of a compact connected Lie group \(K\). Let us denote by \(\mathfrak{t}^*\) the dual of the Lie algebra of \(K\). Let \(\Phi : M \rightarrow \mathfrak{t}^*\) be the moment map. We assume the existence of a \(K\)-equivariant line bundle \(L\) on \(M\) having a connection with curvature equal to \(-i\Omega\). In other words \(M\) is pre-quantizable in the sense of [21] and we call \(L\) a Kostant-Souriau line bundle.

In the process of quantization one tries to associate a unitary representation of \(K\) to the data \((M, \Omega, \Phi, L)\). When \(M\) is compact one associates to this data a virtual representation \(Q_K(M) \in R(K)\) of \(K\) defined as the equivariant index of a Dolbeault-Dirac operator: \(Q_K(M)\) is the geometric quantization of \(M\).

This quantization process satisfies the following functorial properties:

\[ P1 \] When \(N\) and \(M\) are respectively pre-quantized compact Hamiltonian \(K_1\) and \(K_2\)-manifolds, the product \(M \times N\) is a pre-quantized compact Hamiltonian \(K_1 \times K_2\)-manifold, and we have

\[
Q_{K_1 \times K_2}(M \times N) = Q_{K_1}(M) \otimes Q_{K_2}(N)
\]

in \(R(K_1 \times K_2) \simeq R(K_1) \otimes R(K_2)\).

\[ P2 \] If \(H \subset K\) is a closed and connected Lie subgroup, then the restriction of \(Q_K(M)\) to \(H\) is equal to \(Q_H(M)\).

Note that \([P1]\) and \([P2]\) give the following functorial property:

\[ P3 \] When \(N\) and \(M\) are pre-quantized compact Hamiltonian \(K\)-manifolds, the product \(M \times N\) is a pre-quantized compact Hamiltonian \(K\)-manifold, and we have \(Q_K(M \times N) = Q_K(M) \cdot Q_K(N)\), where \(\cdot\) denotes the product in \(R(K)\).

Another fundamental property is the behaviour of the \(K\)-multiplicities of \(Q_K(M)\) that is known as “quantization commutes with reduction”.

Let \(T\) be a maximal torus of \(K\). Let \(\mathfrak{t}^*\) be the dual of the Lie algebra of \(T\) containing the weight lattice \(\wedge^*: \alpha \in \wedge^*\) if \(i\alpha : t \rightarrow i\mathbb{R}\) is the differential of a character of \(T\). Let \(C_K \subset \mathfrak{t}^*\) be a Weyl chamber, and let \(\hat{K} := \wedge^* \cap C_K\) be the set of dominant weights. The ring of characters \(R(K)\) has a \(\mathbb{Z}\)-basis \(V^K_{\mu}, \mu \in \hat{K}\): \(V^K_{\mu}\) is the irreducible representation of \(K\) with highest weight \(\mu\).

For any \(\mu \in \hat{K}\) which is a regular value of \(\Phi\), the reduced space (or symplectic quotient) \(M_\mu := \Phi^{-1}(K \cdot \mu)/K\) is an orbifold equipped with a symplectic structure \(\Omega_{\mu}\). Moreover \(L_{\mu} := (L|_{\Phi^{-1}(\mu)} \otimes \mathbb{C}_{-\mu})/K_{\mu}\) is a
Kostant-Souriau line orbibundle over \((M_\mu, \Omega_\mu)\). The definition of the index of the Dolbeault-Dirac operator carries over to the orbifold case, hence \(Q(M_\mu) \in \mathbb{Z}\) is defined. In [26], this is extended further to the case of singular symplectic quotients, using partial (or shift) desingularization. So the integer \(Q(M_\mu) \in \mathbb{Z}\) is well defined for every \(\mu \in \hat{K}\): in particular \(Q(M_\mu) = 0\) if \(\mu \notin \Phi(M)\).

The following theorem was conjectured by Guillemin-Sternberg [17] and is known as “quantization commutes with reduction” [25, 26, 31, 29]. For complete references on the subject the reader should consult [30, 33].

**Theorem 1.1 (Meinrenken, Meinrenken-Sjamaar).** — We have the following equality in \(R(K)\):

\[
Q_K(M) = \sum_{\mu \in \hat{K}} Q(M_\mu) V^K_\mu.
\]

Suppose now that \(M\) is **non-compact** but that the moment map \(\Phi: M \rightarrow \mathfrak{k}^*\) is assumed to be **proper** (we will simply say “\(M\) is proper”). In this situation the geometric quantization of \(M\) as an index of an elliptic operator is not well defined. Nevertheless the integers \(Q(M_\mu), \mu \in \hat{K}\) are well defined since the symplectic quotients \(M_\mu\) are **compact**.

Following Weitsman [34], we introduce the following definition.

**Definition 1.2.** — The **formal quantization** of \((M, \Omega, \Phi)\) is the element of \(R^{-\infty}(K) := \text{hom}_{\mathbb{Z}}(R(K), \mathbb{Z})\) defined by

\[
Q^{-\infty}_K(M) = \sum_{\mu \in \hat{K}} Q(M_\mu) V^K_\mu.
\]

A representation \(E\) of \(K\) is admissible if it has finite \(K\)-multiplicities: \(\dim(\text{hom}_K(V^K_\mu, E)) < \infty\) for every \(\mu \in \hat{K}\). Here \(R^{-\infty}(K)\) is the Grothendieck group associated to the \(K\)-admissible representations. We have a canonical inclusion \(i: R(K) \hookrightarrow R^{-\infty}(K)\): to \(V \in R(K)\) we associate the map \(i(V): R(K) \rightarrow \mathbb{Z}\) defined by \(W \mapsto \dim(\text{hom}_K(V, W))\). In order to simplify notation, \(i(V)\) will be written \(V\). Moreover the tensor product induces an \(R(K)\)-module structure on \(R^{-\infty}(K)\) since \(E \otimes V\) is an admissible representation when \(V\) and \(E\) are, respectively, a finite dimensional and an admissible representation of \(K\).

It is an easy matter to see that [P1] holds for the formal quantization process \(Q^{-\infty}\). Let \(N\) and \(M\) be respectively pre-quantized proper Hamiltonian \(K_1\) and \(K_2\)-manifolds: the product \(M \times N\) is then a pre-quantized proper Hamiltonian \(K_1 \times K_2\)-manifold. For the reduced spaces we have
(M \times N)_{(\mu_1, \mu_2)} \simeq M_{\mu_1} \times N_{\mu_2}, \text{ for all } \mu_1 \in \hat{K}_1, \mu_2 \in \hat{K}_2. \text{ It follows then that} \\
(1.2) \quad Q^{-\infty}_{K_1 \times K_2}(M \times N) = Q^{-\infty}_{K_1}(M) \otimes Q^{-\infty}_{K_2}(N) \\
in R^{-\infty}(K_1 \times K_2) \simeq R^{-\infty}(K_1) \otimes R^{-\infty}(K_2).

The purpose of this article is to show that the functorial property [P2] still holds for the formal quantization process $Q^{-\infty}$.

**Theorem 1.3.** — Let $M$ be a pre-quantized Hamiltonian $K$-manifold which is proper. Let $H \subset K$ be a closed connected Lie subgroup such that $M$ is still proper as a Hamiltonian $H$-manifold. Then $Q^{-\infty}_K(M)$ is $H$-admissible and we have the following equality in $R^{-\infty}(H)$:

$$
Q^{-\infty}_K(M)|_H = Q^{-\infty}_H(M).
$$

For $\mu \in \hat{K}$ and $\nu \in \hat{H}$ we denote $N^\mu_\nu = \dim(\text{hom}_H(V^H_\nu, V^K_\mu|_H))$ the multiplicity of $V^H_\nu$ in the restriction $V^K_\mu|_H$. In the situation of Theorem 1.3, the moment maps relative to the $K$ and $H$-actions are $\Phi_K$ and $\Phi_H = p \circ \Phi_K$, where $p : \mathfrak{t}^* \to \mathfrak{h}^*$ is the canonical projection.

**Corollary 1.4.** — In the situation of Theorem 1.3, we have for every $\nu \in \hat{H}$:

$$
Q(M_{\nu,H}) = \sum_{\mu \in \hat{K}} N^\mu_\nu Q(M_{\mu,K}).
$$

Here $M_{\nu,H} = \Phi^{-1}_H(H \cdot \nu)/H$ and $M_{\mu,K} = \Phi^{-1}_K(K \cdot \mu)/K$ are respectively the symplectic reductions relative to the $H$ and $K$-actions.

Since $V^K_\mu$ is equal to the $K$-quantization of $K \cdot \mu$, the “quantization commutes with reduction” theorem tells us that $N^\mu_\nu \neq 0$ implies that $\nu \in p(K \cdot \mu) \iff \mu \in K \cdot p^{-1}(\nu)$. Finally

$$
N^\mu_\nu Q(M_{\mu,K}) \neq 0 \implies \mu \in K \cdot p^{-1}(\lambda) \quad \text{and} \quad \Phi^{-1}_K(\mu) \neq \emptyset.
$$

These two conditions imply that we can restrict the sum of RHS of (1.4) to

$$
\mu \in \hat{K} \cap \Phi_K(K \cdot \Phi^{-1}_H(\nu))
$$

which is finite since $\Phi_H$ is proper.

Theorem 1.3 and (1.2) give the following extended version of [P3].

**Theorem 1.5.** — Let $N$ and $M$ be two pre-quantized Hamiltonian $K$-manifolds where $N$ is compact and $M$ is proper. The product $M \times N$ is then proper and we have the following equality in $R^{-\infty}(K)$:

$$
Q^{-\infty}_K(M \times N) = Q^{-\infty}_K(M) \cdot Q_K(N).
$$
For \( \mu, \lambda, \theta \in \widehat{K} \) we denote \( C_{\lambda, \theta}^\mu = \dim(\text{hom}_K(V^K_\mu, V^K_\lambda \otimes V^K_\theta)) \) the multiplicity of \( V^K_\mu \) in the tensor product \( V^K_\lambda \otimes V^K_\theta \). Since \( V^K_\lambda \otimes V^K_\theta \) is equal to the quantization of the product \( K \cdot (K \cdot \lambda \times K \cdot \theta) \), the “quantization commutes with reduction” theorem tells us that \( C_{\lambda, \theta}^\mu = Q((K \cdot (K \cdot \lambda \times K \cdot \theta))_\mu) \): in particular \( C_{\lambda, \theta}^\mu \neq 0 \) implies that (*) \( \| \lambda \| \leq \| \theta \| + \| \mu \| \).

**Corollary 1.6.** — In the situation of Theorem 1.5, we have for every \( \mu \in \widehat{K} \):

\[
(1.7) \quad Q((M \times N)_\mu) = \sum_{\lambda, \theta \in \widehat{K}} C_{\lambda, \theta}^\mu Q(M_\lambda) Q(N_\theta).
\]

Since \( N \) is compact, \( Q(N_\theta) \neq 0 \) for \( \theta \in \{ \text{finite set} \} \). Then (*) and (**) show that the sum in the RHS of (1.7) is finite.

Weitsman proved in [34] the validity of (1.6) in a particular case. The natural strategy to obtain Theorem 1.3 can be summarized as follows:

1. **Cut the non-compact manifold** \( M \) at different levels “\( n \)” to obtain Hamiltonian \( K \)-manifolds \( M^{(n)} \), possibly singular, but which are compact and pre-quantized. We require that the manifold \( M \) is the limit of the sequence \( M^{(n)} \) in the following sense. Each \( M^{(n)} \) contains an invariant and dense open subset \( U^n \) which is symplectomorphic to an invariant open subset \( \widetilde{U^n} \) of \( M \). The sequence \( \widetilde{U^n} \) is increasing and we have \( M = \bigcup_n \widetilde{U^n} \).

2. **Compute** \( Q_K(M^{(n)}) \).

We then expect to have another definition of \( Q^{-\infty}_K(M) \) as the limit of \( Q_K(M^{(n)}) \) when “\( n \)” goes to infinity. Then we can prove that “\( Q^{-\infty} \)” satisfies [P2].

Weitsman worked out point (1) in the case where \( K \) is the unitary group \( U(r) \). He defines the cut spaces \( M^{(n)} \) via symplectic reductions of \( M \times \text{Mat}_r(\mathbb{C}) \), where \( \text{Mat}_r(\mathbb{C}) \) is the vector space of complex \( r \times r \) matrices, viewed as a Hamiltonian \( U(r) \times U(r) \)-manifold. He could handle point (2) **under the hypothesis that all the cut spaces** \( M^{(n)} \) **are smooth**. Under this strong smoothness hypothesis, Weitsman was then able to show Theorem 1.5.

A natural way to carry out point (1) for any compact connected Lie group is by using another version of symplectic cutting due to C. Woodward [36] (see also [26]): each non-abelian cut space \( M^{(n)}_{CW} \) is defined by patching together abelian cut spaces (made on each symplectic slice of \( M \)). But the cut spaces \( M^{(n)}_{CW} \) are either singular or not pre-quantized, hence the main difficulty is point (2).
Let $K_{\mathbb{C}}$ be the complexification of the Lie group $K$. In this article, a smooth projective compactification of $K_{\mathbb{C}}$ is a smooth projective complex variety $\mathcal{X}$ embedded in $\mathbb{P}(E)$ where

i) $E$ is a $K_{\mathbb{C}} \times K_{\mathbb{C}}$-module,

ii) $\mathcal{X}$ is $K_{\mathbb{C}} \times K_{\mathbb{C}}$-stable,

iii) $\mathcal{X}$ contains an open and dense $K_{\mathbb{C}} \times K_{\mathbb{C}}$-orbit $\mathcal{O}$ isomorphic to $K_{\mathbb{C}}$.

In this paper, we work out point (1) for any compact connected Lie group $K$ by introducing another method of symplectic cutting which uses projective compactifications of $K_{\mathbb{C}}$. Each cut space $M_{\text{PEP}}^{(n)}$ is defined as the symplectic reduction of a Hamiltonian $K \times K$-manifold of the type $M \times \mathcal{X}$: here $\mathcal{X}$ is a smooth projective compactification of $K_{\mathbb{C}}$ viewed as a Hamiltonian $K \times K$-manifold. We make the reduction relatively to one copy of $K$, so that the reduced space $M_{\text{PEP}}^{(n)}$ is a Hamiltonian $K$-manifold. These cut spaces are in general singular, but each of them contains an open and dense subset of smooth points which is symplectomorphic to an invariant open subset of $M$.

Originally, projective compactifications of $K_{\mathbb{C}}$ were defined by de Concini-Procesi in the case of an adjoint group: these compactifications were wonderful [14, 15]. This construction was extended by Brion [11] to the case of a connected reductive group. In Section 3.1, we recall the construction of these compactifications and we study them from the Hamiltonian point of view. We show in particular that the open $K_{\mathbb{C}} \times K_{\mathbb{C}}$-orbit in $\mathcal{X}$ is $K \times K$-equivariantly symplectomorphic to an open subset of the cotangent bundle $T^*K$.

In order to work out point (2), we have to handle the non-smoothness of the cut spaces. For this purpose, we prove an extension of Theorem 1.1 to the singular setting.

Let $N$ be a smooth pre-quantized Hamiltonian $K \times H$-manifold. Let us denote by $N_{\mathcal{O}}$ the symplectic reduction of $N$ at 0 relatively to the $H$-action: we assume that the moment map relatively to $H$ is proper so that $N_{\mathcal{O}}$ is a compact Hamiltonian $K$-manifold. Even if $N_{\mathcal{O}}$ is singular, one can still define its geometric quantization $Q_{K}(N_{\mathcal{O}}) \in R(K)$. In Section 2, we prove the following

**Theorem 2.4 (Quantization commutes with reduction in the singular setting).** — We have the following equality in $R(K)$:

$$Q_{K}(N_{\mathcal{O}}) = \sum_{\mu \in \hat{K}} Q\left((N_{\mathcal{O}})_{\mu}\right) V_{\mu}^{K},$$

where the reduced space $(N_{\mathcal{O}})_{\mu}$ is equal to $(N \times K \cdot \mu)_{\mathcal{O}} K \times H$. 

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Note that Theorem 2.4 applies naturally to the cut spaces $M_{\text{PEP}}^{(n)}$, but a priori not to the cut spaces $M_{\text{CW}}^{(n)}$.

In a forthcoming paper, we will exploit these results to compute the multiplicities of the holomorphic discrete series representations of a real semi-simple Lie group $S$ relatively to a compact subgroup $H \subset S$.

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2. Quantization commutes with reduction

In this section we give the precise definition of the geometric quantization of a smooth and compact Hamiltonian manifold. We extend the definition to the case of a singular Hamiltonian manifold and we prove a “quantization commutes with reduction” theorem in the singular setting.

Let $K$ be a compact connected Lie group, with Lie algebra $\mathfrak{k}$. In the Kostant-Souriau framework, a Hamiltonian $K$-manifold $(M, \Omega, \Phi)$ is pre-quantized if there is an equivariant Hermitian line bundle $L$ with an invariant Hermitian connection $\nabla$ such that

\begin{equation}
\mathcal{L}(X) - \nabla_{X_M} = i\langle \Phi, X \rangle \quad \text{and} \quad \nabla^2 = -i\Omega,
\end{equation}

for every $X \in \mathfrak{k}$. Here $X_M$ is the vector field on $M$ defined by $X_M(m) = \frac{d}{dt} e^{-tX} m|_0$.

$(L, \nabla)$ is also called a Kostant-Souriau line bundle. Remark that conditions (2.1) imply via the equivariant Bianchi formula the relation

\begin{equation}
\iota(X_M)\Omega = -d\langle \Phi, X \rangle, \quad X \in \mathfrak{k}.
\end{equation}

We will now recall the notion of geometric quantization.

2.1. Geometric quantization: the compact and smooth case

We suppose here that $(M, \Omega, \Phi)$ is compact and is pre-quantized by a Hermitian line bundle $L$. Choose a $K$-invariant almost complex structure $J$ on $M$ which is compatible with $\Omega$ in the sense that the symmetric bilinear form $\Omega(\cdot, J\cdot)$ is a Riemannian metric. Let $\overline{\partial}_L$ be the Dolbeault operator with coefficients in $L$, and let $\overline{\partial}_{L*}$ be its (formal) adjoint. The Dolbeault-Dirac operator on $M$ with coefficients in $L$ is $D_L = \overline{\partial}_L + \overline{\partial}_{L*}$, considered as an operator from $\mathcal{A}^{0,\text{even}}(M, L)$ to $\mathcal{A}^{0,\text{odd}}(M, L)$. 

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Definition 2.1. — The geometric quantization of $(M, \Omega, \Phi)$ is the element $Q_K(M) \in R(K)$ which is defined as the equivariant index of the Dolbeault-Dirac operator $D_L$.

Remark 2.2.
- We can define the Dolbeault-Dirac operator $D^J_L$ for any invariant almost complex structure $J$. If $J_0$ and $J_1$ are equivariantly homotopic the indices of $D^J_{L_0}$ and $D^J_{L_1}$ coincide (see [29]).
- Since the set of compatible invariant almost complex structures on $M$ is path-connected, the element $Q_K(M) \in R(K)$ does not depend of the choice of $J$.

2.2. Geometric quantization: the compact and singular case

We are interested in defining the geometric quantization of singular compact Hamiltonian manifolds: here “singular” means that the manifold is obtained by symplectic reduction.

Let $(N, \Omega)$ be a smooth symplectic manifold equipped with a Hamiltonian action of $K \times H$; we denote $(\Phi_K, \Phi_H) : N \rightarrow \mathfrak{t}^* \times \mathfrak{h}^*$ the corresponding moment map. We assume that $N$ is pre-quantized by a $K \times H$-equivariant line bundle $L$ and we suppose that the map $\Phi_H$ is proper. One wants to define the geometric quantization of the (compact) symplectic quotient $N/\!/0_H := \Phi_H^{-1}(0)/H$.

When $0$ is a regular value of $\Phi_H$, $N/\!/0_H$ is a compact symplectic orbifold equipped with a Hamiltonian action of $K$; the corresponding moment map is induced by the restriction of $\Phi_K$ to $\Phi_H^{-1}(0)$. The symplectic quotient $N/\!/0_H$ is pre-quantized by the line orbibundle

$$L_0 := (L|_{\Phi_H^{-1}(0)}) / H.$$ 

Definition 2.1 extends to the orbifold case, so one can still define the geometric quantization of $N/\!/0_H$ as an element $Q_K(N/\!/0_H) \in R(K)$.

Suppose now that $0$ is not a regular value of $\Phi_H$. Let $T_H$ be a maximal torus of $H$, and let $C_H \subset t_H^*$ be a Weyl chamber. Since $\Phi_H$ is proper, the convexity theorem says that the image of $\Phi_H$ intersects $C_H$ in a closed locally polyhedral convex set, that we denote $\Delta_H(N)$, [23].

We consider an element $a \in \Delta_H(N)$ which is generic and sufficiently close to $0 \in \Delta_H(N)$; we denote $H_a$ the subgroup of $H$ which stabilizes $a$. When $a \in \Delta_H(N)$ is generic, one can show (see [26]) that

$$N/\!/a_H := \Phi_H^{-1}(a)/H_a$$
is a compact Hamiltonian $K$-orbifold, and that
$$L_a := \left( L|_{\Phi_H^{-1}(a)} \right) / H_a.$$ 

is a $K$-equivariant line orbibundle over $N//_a H$: we can then define, like in Definition 2.1, the element $Q_K(N//_a H) \in R(K)$ as the equivariant index of the Dolbeault-Dirac operator on $N//_a H$ (with coefficients in $L_a$).

**Proposition-Definition 2.3.** — The elements $Q_K(N//_a H) \in R(K)$ do not depend on the choice of the generic element $a \in \Delta_H(N)$, when $a$ is sufficiently close to 0. Their common value will be taken as the geometric quantization of $N//_0 H$, and still be denoted by $Q_K(N//_0 H)$.

**Proof.** — When $N$ is compact and $K = \{e\}$, the proof can be found in [26] and in [29]. The $K$-theoretic proof of [29] extends naturally to our case. \hfill $\square$

### 2.3. Quantization commutes with reduction: the singular case

In Section 2.2, we have defined the geometric quantization $Q_K(N//_0 H) \in R(K)$ of a compact symplectic reduced space $N//_0 H$. We will compute its $K$-multiplicities like in Theorem 1.1.

For every $\mu \in \hat{K}$, we consider the co-adjoint orbit $K \cdot \mu \simeq K/K_{\mu}$, which is pre-quantized by the line bundle $\mathbb{C}[\mu] \simeq K \times K_{\mu} \mathbb{C}_{\mu}$. We consider the product\(^{(1)}\) $N \times K^{-\mu}$ which is a Hamiltonian $K \times H$-manifold pre-quantized by the $K \times H$-equivariant line bundle $L \otimes \mathbb{C}_{[\mu]}^{-1}$. The moment map $N \times K^{-\mu} \rightarrow \mathfrak{k}^* \times \mathfrak{h}^*$, $(n, \xi) \mapsto (\Phi_K(n) - \xi, \Phi_H(n))$ is proper, so that the reduced space

$$\left( N//_0 H \right)_\mu := \left( N \times K^{-\mu} \right)/(0, 0) K \times H$$

is compact. Following Proposition 2.3, we can then define its quantization $Q((N//_0 H)_\mu) \in \mathbb{Z}$. The main result of this section is the

**Theorem 2.4.** — We have the following equality in $R(K)$:

$$Q_K(N//_0 H) = \sum_{\mu \in \hat{K}} Q \left( (N//_0 H)_\mu \right) V^K_{\mu}.$$ 

**Proof.** — The proof will occupy the remainder of this section. The starting point is to state another definition of the geometric quantization of a symplectic reduced space which uses the Atiyah-Singer theory of transversally elliptic operators. \hfill $\square$

\(^{(1)}\) $K^{-\mu}$ denotes the co-adjoint orbit with the opposite symplectic form.
2.3.1. Transversally elliptic symbols

Here we give the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah-Singer in [6]. For an axiomatic treatment of the index morphism see Berline-Vergne [8, 9] and for a short introduction see [29].

Let \( \mathcal{X} \) be a compact \( K_1 \times K_2 \)-manifold. Let \( p : T\mathcal{X} \to \mathcal{X} \) be the projection, and let \( (\cdot, \cdot)_{\mathcal{X}} \) be a \( K_1 \times K_2 \)-invariant Riemannian metric. If \( E^0, E^1 \) are \( K_1 \times K_2 \)-equivariant complex vector bundles over \( \mathcal{X} \), a \( K_1 \times K_2 \)-equivariant morphism \( \sigma \in \Gamma(T\mathcal{X}, \text{hom}(p^*E^0, p^*E^1)) \) is called a symbol. The subset of all \( (x, v) \in T\mathcal{X} \) where \( \sigma(x, v) : E^0_x \to E^1_x \) is not invertible is called the characteristic set of \( \sigma \), and is denoted by \( \text{Char}(\sigma) \).

Let \( T_{K_2}\mathcal{X} \) be the following subset of \( T\mathcal{X} \):

\[
T_{K_2}\mathcal{X} = \{ (x, v) \in T\mathcal{X}, \ (v, X_{\mathcal{X}}(x))_\chi = 0 \quad \text{for all } X \in \mathfrak{k}_2 \}.
\]

A symbol \( \sigma \) is elliptic if \( \sigma \) is invertible outside a compact subset of \( T\mathcal{X} \) (\( \text{i.e.} \ \text{Char}(\sigma) \) is compact), and is \( K_2 \)-transversally elliptic if the restriction of \( \sigma \) to \( T_{K_2}\mathcal{X} \) is invertible outside a compact subset of \( T_{K_2}\mathcal{X} \) (\( \text{i.e.} \ \text{Char}(\sigma) \cap T_{K_2}\mathcal{X} \) is compact). An elliptic symbol \( \sigma \) defines an element in the equivariant \( \mathbf{k} \)-theory of \( T\mathcal{X} \) with compact support, which is denoted by \( \mathbf{k}_{K_1 \times K_2}(T\mathcal{X}) \), and the index of \( \sigma \) is a virtual finite dimensional representation of \( K_1 \times K_2 \) [2, 3, 4, 5].

A \( K_2 \)-transversally elliptic symbol \( \sigma \) defines an element of \( \mathbf{k}_{K_1 \times K_2}(T_{K_2}\mathcal{X}) \), and the index of \( \sigma \) is defined as a trace class virtual representation of \( K_1 \times K_2 \) (see [6] for the analytic index and [8, 9] for the cohomological one): in fact Index\(^\mathcal{X}\)(\( \sigma \)) belongs to the tensor product \( R(K_1) \hat{\otimes} R^{-\infty}(K_2) \).

Remark that any elliptic symbol of \( T\mathcal{X} \) is \( K_2 \)-transversally elliptic, hence we have a restriction map \( \mathbf{k}_{K_1 \times K_2}(T\mathcal{X}) \to \mathbf{k}_{K_1 \times K_2}(T_{K_2}\mathcal{X}) \), and a commutative diagram

\[
\begin{array}{ccc}
\mathbf{k}_{K_1 \times K_2}(T\mathcal{X}) & \longrightarrow & \mathbf{k}_{K_1 \times K_2}(T_{K_2}\mathcal{X}) \\
\text{Index}^{\mathcal{X}} & & \text{Index}^{\mathcal{X}} \\
R(K_1) \otimes R(K_2) & \longrightarrow & R(K_1) \hat{\otimes} R^{-\infty}(K_2).
\end{array}
\]  

Using the excision property, one can easily show that the index map \( \text{Index}^{\mathcal{U}} : \mathbf{k}_{K_1 \times K_2}(T_{K_2}\mathcal{U}) \to R(K_1) \hat{\otimes} R^{-\infty}(K_2) \) is still defined when \( \mathcal{U} \) is a \( K_1 \times K_2 \)-invariant relatively compact open subset of a \( K_1 \times K_2 \)-manifold (see [29], Section 3.1).
2.3.2. Quantization of singular spaces: second definition

Let \((\mathcal{X}, \Omega)\) be a Hamiltonian \(K_1 \times K_2\)-manifold pre-quantized by a \(K_1 \times K_2\)-equivariant line bundle \(L\). The moment map \(\Phi_2 : \mathcal{X} \to \mathfrak{k}_2^*\) relative to the \(K_2\)-action is supposed to be proper. Take a compatible \(K_1 \times K_2\)-invariant almost complex structure on \(\mathcal{X}\). We choose a \(K_1 \times K_2\)-invariant Hermitian metric \(\|v\|^2\) on the tangent bundle \(T\mathcal{X}\), and we identify the cotangent bundle \(T^*\mathcal{X}\) with \(T\mathcal{X}\). For \((x, v) \in T\mathcal{X}\), the principal symbol of the Dolbeault-Dirac operator \(\overline{\partial}_L + \overline{\partial}_L^*\) is the Clifford multiplication \(c_\mathcal{X}(v)\) on the complex vector bundle \(\Lambda^*T\mathcal{X} \otimes L_x\). It is invertible for \(v \neq 0\), since \(c_\mathcal{X}(v)^2 = -\|v\|^2\).

When \(\mathcal{X}\) is compact, the symbol \(c_\mathcal{X}\) is elliptic and then defines an element of the equivariant \(K\)-group of \(T\mathcal{X}\). The topological index of \(c_\mathcal{X} \in K_{K_1 \times K_2}(T\mathcal{X})\) is equal to the analytical index of the Dolbeault-Dirac operator \(\overline{\partial}_L + \overline{\partial}_L^*\):

\[
Q_{K_1 \times K_2}(\mathcal{X}) = \text{Index}^\mathcal{X}(c_\mathcal{X}) \quad \text{in} \quad R(K_1) \otimes R(K_2).
\]

When \(\mathcal{X}\) is not compact the topological index of \(c_\mathcal{X}\) is not defined. In order to give a topological definition of \(Q_{K_1}(\mathcal{X} \parallel_0 K_2)\), we will deform the symbol \(c_\mathcal{X}\) as follows. Consider the identification \(\mathfrak{k}_2^* \simeq \mathfrak{k}_2\) defined by a \(K_2\)-invariant scalar product on the Lie algebra \(\mathfrak{k}_2\). From now on the moment map \(\Phi_2\) will take values in \(\mathfrak{k}_2\), and we define the vector field on \(\mathcal{X}\)

\[
\kappa_x = (\Phi_2(x))|_\mathcal{X}(x), \quad x \in \mathcal{X}.
\]

We consider now the symbol

\[
c_\mathcal{X}^\kappa(v) = c(v - \kappa_x), \quad v \in T_x\mathcal{X}.
\]

Note that \(c_\mathcal{X}^\kappa(v)\) is invertible except if \(v = \kappa_x\). If furthermore \(v\) belongs to the subset \(T_{K_2}\mathcal{X}\) of tangent vectors orthogonal to the \(K_2\)-orbits, then \(v = 0\) and \(\kappa_x = 0\). Indeed \(\kappa_x\) is tangent to \(K_2 \cdot x\) while \(v\) is orthogonal.

Since \(\kappa\) is the Hamiltonian vector field of the function \(-\frac{1}{2}\|\Phi_2\|^2\), the set of zeros of \(\kappa\) coincides with the set \(\text{Cr}(\|\Phi_2\|^2)\) of critical points of \(\|\Phi_2\|^2\).

Let \(\mathcal{U} \subset \mathcal{X}\) be a \(K_1 \times K_2\)-invariant open subset which is relatively compact. If the boundary \(\partial \mathcal{U}\) does not intersect \(\text{Cr}(\|\Phi_2\|^2)\), then the restriction \(c_\mathcal{X}^\kappa|_\mathcal{U}\) defines a class in \(K_{K_1 \times K_2}(T_{K_2}\mathcal{U})\) since

\[
\text{Char}(c_\mathcal{X}^\kappa|_\mathcal{U}) \cap T_{K_2}\mathcal{U} \simeq \text{Cr}(\|\Phi_2\|^2) \cap \mathcal{U}
\]

is compact. In this situation the index of \(c_\mathcal{X}^\kappa|_\mathcal{U}\) is defined as an element \(\text{Index}^\mathcal{U}(c_\mathcal{X}^\kappa|_\mathcal{U}) \in R(K_1) \hat{\otimes} R^{-\infty}(K_2)\).

**Theorem 2.5.** — The \(K_2\)-invariant part of \(\text{Index}^\mathcal{U}(c_\mathcal{X}^\kappa|_\mathcal{U})\) is equal to:
• $Q_{K_1}(X\sslash_0 K_2)$ when $\Phi_2^{-1}(0) \subset \mathcal{U}$,
• 0 in the other case.

Proof. — When $K_1 = \{e\}$, the proof is done in [29] (see Section 7). This proof works equally well in the general case. \qed

Remark 2.6. — If $X$ is compact we can take $\mathcal{U} = X$ in the last theorem. In this case the symbols $c^\kappa_X$ and $c_X$ define the same class in $K_{K_1 \times K_2}(TX)$ so they have the same index. Theorem 2.5 corresponds then to the traditional “quantization commutes with reduction” phenomenon: $[Q_{K_1 \times K_2}(X)]^{K_2} = Q_{K_1}(X\sslash_0 K_2)$.

>From now one we will work with this (topological) definition of the geometric quantization of the reduced $K_1$-Hamiltonian manifold $X\sslash_0 K_2$ (which is possibly singular):

$$(2.7) \quad Q_{K_1}(X\sslash_0 K_2) = [\text{Index}^\mathcal{U}(c_X^\kappa|_\mathcal{U})]^{K_2},$$

where $\mathcal{U}$ is any relatively compact neighborhood of $\Phi_2^{-1}(0)$ such that $\partial \mathcal{U} \cap \text{Cr}(\|\Phi_2\|^2) = \emptyset$.

Remark 2.7. — In this topological definition of $Q_{K_1}(X\sslash_0 K_2)$ one has to check that such open subset $\mathcal{U}$ exists. Take $\mathcal{U} = \{\|\Phi_2\|^2 < \epsilon\}$ for $\epsilon > 0$: one can check that $\partial \mathcal{U} = \{\|\Phi_2\|^2 = \epsilon\}$ does not intersect $\text{Cr}(\|\Phi_2\|^2)$ for $\epsilon$ small enough.

The functorial properties still hold in this singular setting. In particular:

[P2] If $H \subset K_1$ is a closed and connected Lie subgroup, then the restriction of $Q_{K_1}(X\sslash_0 K_2)$ to $H$ is equal to $Q_H(X\sslash_0 K_2)$.

2.3.3. Proof of Theorem 2.4

We go back to the situation of Sections 2.2 and 2.3.

First we apply Theorem 2.5 to $X = N$, $K_1 = K$ and $K_2 = H$. (2.3) is trivially true when $0 \notin \text{Image}(\Phi_H)$. So we suppose now that $0 \in \text{Image}(\Phi_H)$, and we consider a $K \times H$-invariant open subset $\mathcal{U} \subset N$ which is relatively compact and such that

$$\Phi_H^{-1}(0) \subset \mathcal{U} \quad \text{and} \quad \partial \mathcal{U} \cap \text{Cr}(\|\Phi_H\|^2) = \emptyset.$$ 

We have $Q_K(N\sslash_0 H) = [\text{Index}^\mathcal{U}(c_N^\kappa|_\mathcal{U})]^H$ and one want to compute its $K$- multiplicities $m_\mu, \mu \in \hat{K}$. Here $\kappa^H$ is the vector field on $N$ associated to the moment map $\Phi_H$ (see (2.6)).
Take $\mu \in \hat{K}$. We denote $c_{-\mu}$ the principal symbol of the Dolbeault-Dirac operator on $\overline{K} \cdot \mu$ with values in the line bundle $C_{[-\mu]}$: we have $\text{Index}^{K\cdot\mu}(c_{-\mu}) = (V^\mu_K)_*$. We know then that the multiplicity of $[\text{Index}^U(c^\mu_N|_U \odot c_{-\mu})]^{K\times H}$ relatively to $V^\mu_K$ is equal to

\[ (2.8) \quad m_\mu := \left[ \text{Index}^V (c^\mu_N|_U \odot c_{-\mu}) \right]^{K\times H}, \]

with $V = U \times K\cdot \mu$. This identity is due to the fact that we have a “multiplication”

\[ K_{K\times H}(T_H U) \times K_K(T(K\cdot\mu)) \longrightarrow K_{K\times H}(T_{K\times H}(U \times K\cdot\mu)) \]

\[ (\sigma_1, \sigma_2) \longmapsto \sigma_1 \odot \sigma_2, \]

so that $\text{Index}^{U \times K\cdot\mu}(\sigma_1 \odot \sigma_2) = \text{Index}^U(\sigma_1) \cdot \text{Index}^{K\cdot\mu}(\sigma_2)$ in $R^{-\infty}(K \times H)$. See [6].

Consider now the case where $X = N \times \overline{K} \cdot \mu$, $K_1 = \{ e \}$ and $K_2 = K \times H$. By Theorem 2.5, we know that

\[ (2.9) \quad Q((N//_0 H)_\mu) = \left[ \text{Index}^V (c^\mu_N|_V) \right]^{K\times H}, \]

where $\kappa$ is the vector field on $N \times \overline{K} \cdot \mu$ associated to the moment map

\[ \Phi : N \times \overline{K} \cdot \mu \longrightarrow \mathfrak{k}^* \times \mathfrak{h}^* \]

\[ (x,\xi) \longmapsto (\Phi_K(x) - \xi, \Phi_H(n)). \]

Note that $V = U \times K\cdot \mu$ is a neighborhood of $\Phi^{-1}(0) \subset (\Phi_H)^{-1}(0)$.

Our aim now is to prove that the quantities (2.8) and (2.9) are equal.

Since the definition of $\kappa$ requires the choice of an invariant scalar product on the Lie algebra $\mathfrak{k} \times \mathfrak{h}$, we give a precise definition of it. Let $\| \cdot \|_K$ and $\| \cdot \|_H$ be two invariant Euclidean norms respectively on $\mathfrak{k}$ and $\mathfrak{h}$. For any $r > 0$, we consider on $\mathfrak{k} \times \mathfrak{h}$ the invariant Euclidean norm $\|(X,Y)\|^2 = r^2\|X\|_K^2 + \|Y\|_H^2$.

Let $\kappa^K$ be the vector field on $N \times \overline{K} \cdot \mu$ associated to the map $N \times \overline{K} \cdot \mu \rightarrow \mathfrak{k}^*$, $(x,\xi) \mapsto \Phi_K(x) - \xi$, and where the identification $\mathfrak{k} \simeq \mathfrak{k}^*$ is made via the Euclidean norm $\| \cdot \|_K$ (see (2.6)). For $(x,\xi) \in N \times \overline{K} \cdot \mu$, we have the decomposition

\[ \kappa^K(x,\xi) = (\kappa_1(x,\xi), \kappa_2(x,\xi)) \in T_x N \times T_{\xi}(K\cdot\mu). \]

Let $\kappa^H$ be the vector field on $N \times \overline{K} \cdot \mu$ associated to the map $N \times \overline{K} \cdot \mu \rightarrow \mathfrak{h}^*$, $(x,\xi) \mapsto \Phi_H(x)$, and where the identification $\mathfrak{h} \simeq \mathfrak{h}^*$ is made via the
Euclidean norm $\| \cdot \|_H$. For $(x, \xi) \in N \times K \cdot \mu$, we have the decomposition
\[ \kappa^H(x, \xi) = (\kappa^H(x), 0) \in T_x N \times T_\xi (K \cdot \mu). \]
For any $r > 0$, we denote by $\kappa_r$ the vector field on $N \times K \cdot \mu$ associated to the map (2.10), and where the identification $\mathfrak{k} \times \mathfrak{h} \simeq \mathfrak{t}^* \times \mathfrak{h}^*$ is made via the Euclidean norm $\| \cdot \|_r$. We have then
\[ \kappa_r = \kappa^H + r \kappa^K = (\kappa^H + r \kappa_1, r \kappa_2). \]

Now we can specify (2.9). Take an invariant relatively compact neighborhood $U$ of $\Phi^{-1}_H(0)$ such that $\partial U \cap \{ \text{zeros of } \kappa^H \} = \emptyset$. With the help of an invariant Riemannian metric on $X$ we define
\[ \varepsilon_H = \inf_{x \in \partial U} \| \kappa^H(x) \| > 0 \quad \text{and} \quad \varepsilon_K = \sup_{(x, \xi) \in \partial U \times K \cdot \mu} \| \kappa_1(x, \xi) \|. \]
Note that for any $0 \leq r < \frac{\varepsilon_H}{\varepsilon_K}$, we have $\partial U \times K \cdot \mu \cap \{ \text{zeros of } \kappa^H + r \kappa_1 \} = \emptyset$, and then $\partial V \cap \{ \text{zeros of } \kappa_r \} = \emptyset$ for the neighborhood $V := U \times K \cdot \mu$ of $\Phi^{-1}(0)$. We can then use Theorem 2.5: for $0 < r < \frac{\varepsilon_H}{\varepsilon_K}$ we have
\[ Q((N \parallel_0 H)_\mu) = \left[ \text{Index}^V (c_{\kappa_r}^{K \times H}) \right]^{K \times H}. \]

We are now close to the end of the proof. Let us compare the symbols $c_{\kappa_r}^{K \times H}|_V$ and $c_{\kappa^H}^{K \times H}|_U \odot c_{-\mu}$ in $K_{K \times H}(T_{K \times H}(U \times K \cdot \mu))$. First one sees that the symbol $c_{\kappa_r}$ is equal to the product $c_N \odot c_{-\mu}$ hence the symbol $c_{\kappa^H}^{K \times H}|_U \odot c_{-\mu}$ is equal to $c_{\kappa^H}^{K \times H}|_V$ when $r = 0$. Since for $r < \frac{\varepsilon_H}{\varepsilon_K}$ the path $s \in [0, r] \to c_{\kappa_r}^{K \times H}|_V$ defines a homotopy of $K \times H$-transversally elliptic symbols on $V$, we get
\[ \text{Index}^V (c_{\kappa_r}^{K \times H}|_V) = \text{Index}^V (c_{\kappa^H}^{K \times H}|_U \odot c_{-\mu}) \]
and then $m_\mu = Q((N \parallel_0 H)_\mu)$. \qed

3. Wonderful compactifications and symplectic cutting

In this section we use projective compactifications of $K_{\mathbb{C}}$ “à la de Concini-Procesi” [14, 15] to perform symplectic cutting. These compactifications are special cases of Spherical varieties, see [10].
3.1. Wonderful compactifications: definitions

Here we study the projective compactifications of $K_C$ defined by Brion [11] from the Hamiltonian point of view. This construction generalizes previous work of de Concini-Procesi [14, 15], where wonderful compactifications of an adjoint group were defined.

We consider a compact connected Lie group $K$ and its complexification $K_C$. Let $T$ be a maximal torus of $K$, and let $W := N(T)/T$ be the Weyl group. Let $t^\ast$ be the dual of the Lie algebra of $T$ containing the lattice of weights $\wedge^\ast$. Let $C_K \subset t^\ast$ be a Weyl chamber and let $\hat{K} := \wedge^\ast \cap C_K$ be the set of dominant weights. An element $\xi \in t^\ast$ is called regular if its stabilizer subgroup $K_\xi$ is equal to $T$.

We recall the notion of Delzant polytope [28]. Let $P$ be a convex polytope in $t^\ast$.

**Definition 3.1.** — $P$ is a Delzant polytope (relatively to $\wedge^\ast$) if:

i) the vertices of $P$ belong to $\wedge^\ast$,

ii) $P$ is simple: there are exactly $\dim(t^\ast)$ edges through each vertex,

iii) at each vertex $\xi$, the tangent cone to $P$ at $\{\xi\}$ is generated by a $\mathbb{Z}$-basis of the lattice $\wedge^\ast$.

We need the following refinement of the notion of Delzant polytope.

**Definition 3.2.** — A convex polytope $P$ in $t^\ast$ is $K$-adapted if:

i) $P$ is a Delzant polytope (relatively to $\wedge^\ast$),

ii) the vertices of $P$ are regular elements of $t^\ast$,

iii) $P$ is $W$-invariant.

**Example 1.** — When $K = T$ is a torus, a $T$-adapted polytope is just a Delzant polytope.

**Example 2.** — We consider the Lie groups $SU(3)$ and $PSU(3) := SU(3)/Z$, where $Z \simeq \mathbb{Z}/3\mathbb{Z}$ is the center of $SU(3)$. Note that $PSU(3)$ has a trivial center. In Figures 3.1 and 3.2, the lattice $\wedge^\ast_{PSU}$ of weights for $PSU(3)$ is formed by the black dots and the lattice $\wedge^\ast_{SU}$ of weights for $SU(3)$ is formed by all the dots (grey and black). In Figure 3.1, the polytope is a Delzant polytope relatively to $\wedge^\ast_{PSU}$, but it is not a Delzant polytope relatively to $\wedge^\ast_{SU}$: hence the polytope is $PSU(3)$-adapted but not $SU(3)$-adapted.

**Example 3.** — When $K$ has trivial center, the convex hull of $W \cdot \mu$ is a $K$-adapted polytope for any regular dominant weight $\mu$. Figure 3.1 is an example of this case for the Lie group $PSU(3)$. 
Proposition 3.3. — For any compact connected Lie group $K$, there exist $K$-adapted polytopes in $t^*$.

Proof. — Let us use the dictionary between polytopes and projective fans [28]. Conditions $i)$ and $iii)$ of Definition 3.2 means that we are looking after a smooth projective $W$-invariant fan $\mathcal{F}$ in $t$. Condition $ii)$ means that each cone of $\mathcal{F}$ of maximal dimension should not be fixed by any element
of $W \setminus \{\text{Id}\}$. For a proof of the existence of such a fan, see [12, 13]. In particular condition $(*)$ in Proposition 2 of [13] implies ii).

In the rest of this section, we consider a $K$-adapted polytope $P$. Let

\begin{equation}
\{\lambda_1, \ldots, \lambda_N\}
\end{equation}

be the set of regular dominant weights which are on the edges of $P$ (i.e. on the 1-dimensional faces of $P$). Note that some of the $\lambda_i$ are the vertices of $P$ which belong to the (interior) of the Weyl chamber.

Let $V_{\lambda_i}$ be an irreducible representation of $K$ with highest weight $\lambda_i$: this representation extends canonically to the complexification $K_C$. We denote $\rho: K_C \to \Pi_i^N GL(V_{\lambda_i})$ the representation of $K_C$ on $V := \oplus_{i=1}^N V_{\lambda_i}$.

Let $T_C \subset K_C$ be the complexification of the (compact) torus $T \subset K$. Let $\Delta(T_C, V)$ be the set of weights relative to the action of $T_C$ on $V$. Let us sum up the basic but essential properties concerning the set $\Delta(T_C, V)$.

**Lemma 3.4.**

- We have $W \cdot \{\lambda_1, \ldots, \lambda_N\} \subset \Delta(T_C, V) \subset P$.
- $P$ is equal to the convex hull of $W \cdot \{\lambda_1, \ldots, \lambda_N\}$.
- For any vertex $\lambda$ of $P$, the $\mathbb{Z}$-basis of the lattice $\wedge^*$ which generates the tangent cone to $P$ at $\{\lambda\}$ is of the form: $\alpha_1 - \lambda, \ldots, \alpha_r - \lambda$ where $\alpha_k \in \Delta(T_C, V)$.

**Proof.** — Since each $\lambda_i$ is a weight for the action of $T_C$ on $V_{\lambda_i}$, we have $\lambda_i \in \Delta(T_C, V)$. Using the $W$-invariance of $\Delta(T_C, V)$, we get one inclusion of the first point. The other inclusion follows from the fact that the set of weights relative to the action of $T_C$ on $V_{\lambda_i}$ is contained in the convex hull of $W \cdot \lambda_i$. The second point is due to the fact that all the vertices of $P$ belong to $W \cdot \{\lambda_1, \ldots, \lambda_N\}$.

Let us prove the last point for a vertex $\lambda$ which is dominant. Since $P$ is a Delzant polytope, the tangent cone to $P$ at $\{\lambda\}$ is generated by a $\mathbb{Z}$-basis of the lattice $\wedge^*$ that we denote $\alpha_1 - \lambda, \ldots, \alpha_r - \lambda$. Let us show that all the $\alpha_k$ belong to $\Delta(T_C, V)$. We consider the segment $[\lambda, \alpha_k] \subset \mathfrak{t}^*$ which is part of an edge of $P$. If $[\lambda, \alpha_k]$ is included in the interior of the Weyl chamber, we have then $\alpha_k \in \{\lambda_1, \ldots, \lambda_N\} \subset \Delta(T_C, V)$. Suppose now that the segment $[\lambda, \alpha_k]$ intersects the wall $\Pi_\alpha$ of the Weyl chamber defined by a simple root $\alpha$. Let $s_\alpha \in W$ be the symmetry relative to the wall $\Pi_\alpha$. Since $P$ is $W$-invariant, the segment

$$[s_\alpha(\lambda), s_\alpha(\alpha_k)] = s_\alpha([\lambda, \alpha_k])$$

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is also part of an edge of $P$, and it intersects $[\lambda, \alpha_k]$. Since two distinct edges can intersect only at the vertices, the line $(\lambda, \alpha_k)$ must be invariant under $s_\alpha$.

Let us sum up the properties of the weight $\alpha_k$: the segment $[\lambda, \alpha_k]$ intersects the wall $\Pi_\alpha$ orthogonally and $\lambda - \alpha_k$ is part of a $\mathbb{Z}$-basis of $\wedge^*$. There are only two possibilities: either $\alpha_k \in \Pi_\alpha$ or $\alpha_k = s_\alpha(\lambda)$. Both of them implies that

$$\alpha_k = \lambda - \alpha.$$ 

Finally, it is a standard fact of representation theory that, for any simple root $\alpha$ and any regular dominant weight $\lambda$, $\lambda - \alpha$ is a weight relative to the action of $T_C$ on $V_\lambda$. We have proved that $\alpha_k = \lambda - \alpha \in \Delta(T_C, V)$. $\square$

We consider now the vector space

$$E = \bigoplus_{i=1}^N \text{End}(V_{\lambda_i})$$

equipped with the action of $K_C \times K_C$ given by: $(g_1, g_2) \cdot f = \rho(g_1) \circ f \circ \rho(g_2)^{-1}$. Let $\mathbb{P}(E)$ be the projective space associated to $E$: it comes equipped with an algebraic action of the reductive group $K_C \times K_C$. We consider the map $g \mapsto [\rho(g)]$ from $K_C$ into $\mathbb{P}(E)$, and we denote it $\bar{\rho}$.

**Lemma 3.5.** — The map $\bar{\rho}: K_C \to \mathbb{P}(E)$ is an embedding.

**Proof.** — Let $g \in K_C$ such that $\bar{\rho}(g) = [\text{Id}]$: there exists $a \in \mathbb{C}^*$ such that $\rho(g) = a \text{Id}$. The Cartan decomposition gives

$$\rho(k) = \frac{a}{|a|} \text{Id} \quad \text{and} \quad \rho(e^{iX}) = |a| \text{Id}$$

for $g = ke^{iX}$ with $k \in K$ and $X \in \mathfrak{k}$. Since there exist $Y, Y' \in \mathfrak{k}$ and $u, u' \in K$ such that $k = ue^Y u^{-1}$ and $X = u' \cdot Y'$, (3.2) gives

$$\rho(e^Y) = \frac{a}{|a|} \text{Id} \quad \text{and} \quad \rho(e^{iY'}) = |a| \text{Id}.$$ 

and then

$$e^{i(\alpha - \alpha', Y)} = 1 \quad \text{and} \quad e^{i(\alpha - \alpha', Y')} = 1,$$

for every $\alpha, \alpha' \in \Delta(T_C, V)$. Using now the last point of Lemma 3.4, we see that (3.3) implies $Y' = 0$ and $Y \in \ker(Z \in \mathfrak{k} \to e^Z)$. We have proved that $g = e$. $\square$

We can now define the projective compactification $\mathcal{X}_P$ of $K_C$.

**Definition 3.6.** — Let $P$ be a $K$-adapted polytope in $\mathfrak{k}^*$. Let $\{\lambda_1, \ldots, \lambda_N\}$ be the set of regular dominant weights which are on the edges of $P$. Let $E := \bigoplus_{i=1}^N \text{End}(V_{\lambda_i})$. We define the varieties:

- $\mathcal{X}_P$ which is the Zariski closure of $\bar{\rho}(K_C)$ in $\mathbb{P}(E)$,
• $\mathcal{Y}_P \subset \mathcal{X}_P$ which is the Zariski closure of $\bar{\rho}(T_C)$ in $\mathbb{P}(E)$.

Since $\bar{\rho}(K_C) = K_C \times K_C \cdot [\text{Id}]$ and $\bar{\rho}(T_C) = T_C \times T_C \cdot [\text{Id}]$ are orbits of algebraic group actions their Zariski closures coincide with their closures for the Euclidean topology.

**Theorem 3.7. —** The varieties $\mathcal{X}_P$ and $\mathcal{Y}_P$ are smooth.

The proof will be given in the next section.

**Remark 3.8. —** In the definition of $\mathcal{X}_P$, we work with the representation $V = \bigoplus_{i=1}^N V_{\lambda_i}$, where the $\lambda_i$ run over the set of regular dominant weights that belong to the edges of $P$. We can be interested to work with a subset $\Delta \subset \{\lambda_1, \ldots, \lambda_N\}$. We consider then the representations $V(\Delta) := \bigoplus_{\lambda \in \Delta} V_{\lambda}$ and $E(\Delta) := \bigoplus_{\lambda \in \Delta} \text{End}(V_{\lambda})$. We define the variety $\mathcal{X}(\Delta)$ as the Zariski closure of $\bar{\rho}(K_C)$ in $\mathbb{P}(E(\Delta))$.

Suppose now that $\Delta$ contains all the vertices of $P$ which are in the Weyl chamber: the first two points of Lemma 3.4 apply to $\Delta(T_C, V(\Delta))$. One can show by the method described in Section 3.2 that $\mathcal{X}(\Delta)$ is smooth if $\Delta(T_C, V(\Delta))$ satisfies the third point of Lemma 3.4. In other words we have the following

*Smoothness criterion for $\mathcal{X}(\Delta)$:* for any vertex $\lambda$ of $P$, the $\mathbb{Z}$-basis of the lattice $\wedge^* \mathfrak{a}$ which generates the tangent cone to $P$ at $\{\lambda\}$ is of the form: $\alpha_1 - \lambda, \ldots, \alpha_r - \lambda$ where $\alpha_k \in \Delta(T_C, V(\Delta))$.

When $K$ has trivial center (see Figure 3.1) one can work with the polytope equal to the convex hull of $W \cdot \mu$, with $\mu$ a regular dominant weight. In this case one can take $\Delta := \{\mu\}$: the variety $\mathcal{X}(\Delta) \subset \mathbb{P}(\text{End}(V_{\mu}))$ is a smooth compactification of $K_C$. This was the situation studied originally by de Concini-Procesi [14].

In the example of Figure 3.1, if one takes $\Delta := \{\lambda_2, \lambda_3\}$, the variety $\mathcal{X}(\Delta)$ is a smooth compactification of $\text{SL}(3, \mathbb{C})$.

### 3.2. Smoothness of $\mathcal{X}_P$ and $\mathcal{Y}_P$

Let $E$ be a complex vector space equipped with a linear action of a reductive group $G$. Let $Z \subset \mathbb{P}(E)$ be a projective variety which is $G$-stable. We have the classical fact

**Lemma 3.9.**

- $Z$ has closed $G$-orbits.
- $Z$ is smooth if $Z$ is smooth near its closed $G$-orbits.
\( Z \) is smooth near an orbit \( G \cdot z \) if \( Z \) is smooth near \( z \).

We are interested here respectively in
\( \bullet \) the \( K_C \times K_C \)-variety \( \mathcal{X}_P \subset \mathbb{P}(E) \subset \mathbb{P}(\text{End}(V)) \),
\( \bullet \) the \( T_C \times T_C \)-variety \( \mathcal{Y}_P \subset \mathbb{P}(E) \).

Since the diagonal \( Z_C = \{(t, t) | t \in T_C \} \) stabilizes \([\text{Id}]\), its action on \( \mathcal{Y}_P \) is trivial. Hence we will restrict ourselves to the action of \( T_C \times T_C / Z_C \simeq T_C \) on \( \mathcal{Y}_P \): for \( t \in T_C \) and \([y] \in \mathcal{Y}_P \) we take \( t \cdot [y] = [\rho(t) \circ y] \).

3.2.1. The case of \( \mathcal{Y}_P \)

We apply Lemma 3.9 to the \( T_C \)-variety \( \mathcal{Y}_P = T_C / [\text{Id}] \) in \( \mathbb{P}(E) \). Let \( \{\alpha_j, j \in J\} \) be the \( T_C \)-weights on \( V = \oplus_{i=1}^N V_{\lambda_i} \), counted with their multiplicities. We suppose that a \( K \)-invariant Hermitian metric is fixed on each representation \( V_{\lambda_i} \).

Their exists an orthonormal basis \( \{v_j, j \in J\} \) of \( V = \oplus_{i=1}^N V_{\lambda_i} \) such that \( \text{Id} = \sum_{j \in J} v_j \otimes v_j^* \) and
\[
\rho(e^Z) = \sum_{j \in J} e^{i(\alpha_j, Z)} v_j \otimes v_j^*, \quad Z \in t_C.
\]

So the action of \( e^Z \in T_C \) on \([\text{Id}] \in \mathbb{P}(E)\) is \( e^Z \cdot [\text{Id}] = \left[ \sum_{j \in J} e^{i(\alpha_j, Z)} v_j \otimes v_j^* \right] \).

We introduce a subset \( J' \) of \( J \) such that for every \( j \in J \) there exists a unique \( j' \in J' \) such that \( \alpha_j = \alpha_{j'} \). So the variety \( \mathcal{Y}_P \) lives into \( \mathbb{P}(E') \) where \( E' = \oplus_{j' \in J'} \mathbb{C}m_{j'} \) with \( m_{j'} = \sum_{j \in J, \alpha_j = \alpha_{j'}} v_j \otimes v_j^* \). The closed \( T_C \)-orbits in \( \mathbb{P}(E') \) are the fixed points \([m_{j'}], j' \in J' \).

**Lemma 3.10.** — \( [m_{j_o}] \in \mathcal{Y}_P \) if and only if \( \alpha_{j_o} \) is a vertex of the polytope \( P \).

**Proof.** — If \( \alpha_{j_o} \) is a vertex of \( P \), there exists \( X \in t \) such that \( \langle \alpha_{j_o}, X \rangle > \langle \alpha_j, X \rangle \) whenever \( \alpha_{j_o} \neq \alpha_j \). Hence \( e^{-isX} \cdot [\text{Id}] \) tends to \([m_{j_o}]\) when \( s \to +\infty \). If \( \alpha_{j_o} \) is not a vertex of \( P \), we can find \( L \subset J' \setminus \{j_o\} \) such that \( \alpha_{j_o} = \sum_{l \in L} a_l \alpha_l \) with \( 0 < a_l < 1 \) and \( \sum_{l} a_l = 1 \). So \( \mathcal{Y}_P \) is included into the closed subset defined by
\[
\left\{ \left[ \sum_{j' \in J'} \delta_{j'} m_{j'} \right] \in \mathbb{P}(E') : \prod_{l \in L} |\delta_l|^{a_l} = |\delta_{j_o}| \right\}.
\]
Hence \([m_{j_o}] \notin \mathcal{Y}_P \). \( \square \)

**Remark 3.11.** — When \( \alpha_j \) is a vertex of the polytope \( P \), the multiplicity of \( \alpha_j \) in \( \oplus_{i=1}^N V_{\lambda_i} \) is one, so \( m_j = v_j \otimes v_j^* \).
Consider now a vertex \( \alpha_{j_o} \) of \( P \) (for \( j_o \in J' \)). We consider the open neighborhood \( \mathcal{V} \subset \mathbb{P}(E') \) of \( [m_{j_o}] \) defined by \( [\sum_{j' \in J'} \delta_{j'} m_{j'}] \in \mathcal{V} \Leftrightarrow \delta_{j_o} \neq 0 \), and the diffeomorphism \( \psi : \mathcal{V} \rightarrow \mathbb{C}' \setminus \{ j_o \} \), \( \sum_{j' \in J'} \delta_{j'} m_{j'} \mapsto (\delta_{j'})_{j' \neq j_o} \). The map \( \psi \) realizes a diffeomorphism between \( \mathcal{V}_P \cap \mathcal{V} \) and the affine subvariety

\[
\mathcal{Z} := \{(t^{\alpha_{j'} - \alpha_{j_o}})_{j' \neq j_o} \mid t \in T'_C \} \subset \mathbb{C}' \setminus \{ j_o \}.
\]

The set of weights \( \{ \alpha_{j}, j \in J \} \) contains all the lattice points that belong to the edges of \( P \). Since the polytope \( P \) is \( K \)-adapted, there exists a subset \( L_{j_o} \subset J' \) such that \( \alpha_l - \alpha_{j_o}, l \in L_{j_o} \) is a \( \mathbb{Z} \)-basis of the group of weights \( \Lambda^* \). And for every \( j' \neq j_o \) we have

\[
\alpha_{j'} - \alpha_{j_o} = \sum_{l \in L_{j_o}} n_{j'}(\alpha_l - \alpha_{j_o}) \quad \text{with} \quad n_{j'} \in \mathbb{N}.
\]

We define on \( \mathbb{C}^{L_{j_o}} \) the monomials \( P_{j'}(Z) = \Pi_{l \in L_{j_o}} (Z_l)^{n_{j'}} \). Note that \( P_{j'}(Z) = Z_l \) when \( j' = l \in L_{j_o} \). Now it is not difficult to see that the map

\[
\mathbb{C}^{L_{j_o}} \longrightarrow \mathbb{C}' \setminus \{ j_o \}
\]

\[
Z \mapsto (P_{j'}(Z))_{j' \neq j_o}
\]

realizes a diffeomorphism between \( \mathbb{C}^{L_{j_o}} \) and \( \mathcal{Z} \).

We have shown that \( \mathcal{Y}_P \) is smooth near \( [m_{j_o}] \); hence \( \mathcal{Y}_P \) is a smooth subvariety of \( \mathbb{P}(E) \). Since \( T'_C \) acts on \( \mathcal{Y}_P \) with a dense orbit, \( \mathcal{Y}_P \) is a smooth projective toric variety.

### 3.2.2. The case of \( X_P \)

Recall that \( E := \bigoplus_{i=1}^N \text{End}(V_{\lambda_i}) \). The closed \( K_C \times K_C \)-orbits in \( \mathbb{P}(E) \) are those passing through \( [v_{\lambda_i} \otimes v_{\lambda_i}^*] \) where \( v_{\lambda_i} \in V_{\lambda_i} \) is a highest weight vector. Recall that all the \( \lambda_i \) are regular elements of \( t^* \).

**Lemma 3.12.** — \( [v_{\lambda_i} \otimes v_{\lambda_i}^*] \in X_P \) if and only if \( \lambda_i \) is a vertex of the polytope \( P \).

**Proof.** — If \( \lambda_i \) is a vertex of \( P \), we have proved in Lemma 3.10 that \( [v_{\lambda_i} \otimes v_{\lambda_i}^*] \) belongs to \( \mathcal{Y}_P \) and so belongs to \( X_P \). We shall prove the converse in Corollary 3.17. \( \square \)

For the remainder of this section we consider a vertex \( \lambda_{i_o} \in \hat{R} \) of the polytope \( P \). Let \( B^+, B^- \) be the subgroups fixing respectively the elements \( [v_{\lambda_{i_o}}] \in \mathbb{P}(V_{\lambda_i}) \) and \( [v_{\lambda_{i_o}}^*] \in \mathbb{P}(V_{\lambda_i}^*) \): since \( \lambda_{i_o} \) is regular, \( B^+ \) and \( B^- \) are
We consider the open subset \( V_{\text{End}} \subset \mathbb{P}(E) \) of elements \([f]\) such that \( \langle v_{\lambda_{io}}^*, f(v_{\lambda_{io}}) \rangle \neq 0\): \( V_{\text{End}} \) is a \( B^- \times B^+ \)-stable neighborhood of \([v_{\lambda_{io}} \otimes v_{\lambda_{io}}^*]\). Consider the open subsets \( V \subset \mathbb{P}(V_{\lambda_{io}}) \) and \( V^* \subset \mathbb{P}(V_{\lambda_{io}}^*) \) defined by:

- \([v] \in V \iff \langle v_{\lambda_{io}}^*, v \rangle \neq 0\): \( V \) is \( B^- \) stable,
- \([\xi] \in V^* \iff \langle \xi, v_{\lambda_{io}} \rangle \neq 0\): \( V^* \) is \( B^+ \) stable.

We define now the rational maps \( l : \mathbb{P}(E) \rightarrow \mathbb{P}(V_{\lambda_{io}}), [f] \mapsto [f(v_{\lambda_{io}})] \) and \( r : \mathbb{P}(E) \rightarrow \mathbb{P}(V_{\lambda_{io}}^*), [f] \mapsto [v_{\lambda_{io}}^* \circ f] \). The maps \( l \) and \( r \) are defined on \( V_{\text{End}} \): they define respectively a \( B^- \)-equivariant map from \( V_{\text{End}} \) to \( V \), and a \( B^+ \)-equivariant map from \( V_{\text{End}} \) to \( V^* \).

The orbits \( K_C \cdot [v_{\lambda_{io}}] \subset \mathbb{P}(V_{\lambda_{io}}) \) and \( K_C \cdot [v_{\lambda_{io}}^*] \subset \mathbb{P}(V_{\lambda_{io}}^*) \) are closed and we have

\[
K_C \cdot [v_{\lambda_{io}}] \cap V = N^- \cdot [v_{\lambda_{io}}] \simeq N^- \\
K_C \cdot [v_{\lambda_{io}}^*] \cap V^* = N^+ \cdot [v_{\lambda_{io}}^*] \simeq N^+.
\]

The rational map \( (l, r) : \mathbb{P}(E) \rightarrow \mathbb{P}(V_{\lambda_{io}}) \times \mathbb{P}(V_{\lambda_{io}}^*) \) then induces a map

\[
g : V_{\text{End}} \cap \chi_P \rightarrow N^- \times N^+
\]

which is \( N^- \times N^+ \)-equivariant: \( q((n^-, n^+) \cdot x) = (n^-, n^+) \cdot q(x) \) for \( x \in V_{\text{End}} \cap \chi_P \), and \( n^\pm \in N^\pm \).

We can now finish the proof. The set \( N^- T_C N^+ \subset K_C \) is dense in \( K_C \), so it is now easy to see that the map

\[
N^- \times N^+ \times (\mathcal{Y}_P \cap V_{\text{End}}) \longrightarrow \chi_P \cap V_{\text{End}} \\
(n^-, n^+, y) \longmapsto (n^-, n^+) \cdot y
\]

is a diffeomorphism. We proved above that \( \mathcal{Y}_P \cap V_{\text{End}} \) is a smooth affine variety, hence \( \chi_P \) is smooth near \([v_{\lambda_{io}} \otimes v_{\lambda_{io}}^*] \in \chi_P \cap V_{\text{End}} \). Lemma 3.9 then tells us that \( \chi_P \) is a smooth variety.

### 3.3. Hamiltonian actions

First consider a Hermitian vector space \( V \). The Hermitian structure on \( \text{End}(V) \) is \((A, B) := \text{Tr}(AB^*)\), hence the associated symplectic structure on \( \text{End}(V) \) is defined by the relation \( \Omega_{\text{End}}(A, B) := -\text{Im}(\text{Tr}(AB^*)) \).

Let \( U(V) \) be the unitary group and \( u(V) \) its Lie algebra. We will use the identification \( \epsilon : u(V) \simeq u(V)^*, X \mapsto \epsilon_X \) where \( \epsilon_X(Y) = -\text{Tr}(XY) \). The
action of $U(V) \times U(V)$ on $\mathrm{End}(V)$ is $(g, h) \cdot A = gAh^{-1}$. The moment map relative to this action is

$$\begin{array}{c}
\mathrm{End}(V) \rightarrow u(V)^* \times u(V)^* \\
A \mapsto - \frac{1}{2} (iAA^*, -iA^*A) .
\end{array}$$

We now consider the projective space $\mathbb{P}(\mathrm{End}(V))$ equipped with the Fubini-Study symplectic form $\Omega_{FS}$. Here the action of $U(V) \times U(V)$ on $\mathbb{P}(\mathrm{End}(V))$ is Hamiltonian with moment map

$$\begin{array}{c}
\mathbb{P}(\mathrm{End}(V)) \rightarrow u(V)^* \times u(V)^* \\
[A] \mapsto - \frac{1}{\|A\|^2} (\pi_K(iAA^*), -\pi_K(iA^*A)).
\end{array}$$

where $\|A\|^2 = \mathrm{Tr}(AA^*)$ (see [27], Section 7). If $\rho : K \hookrightarrow U(V)$ is a closed connected Lie subgroup, we can consider the action of $K \times K$ on $\mathbb{P}(\mathrm{End}(V))$. Let $\pi_K : u(V)^* \rightarrow \mathfrak{t}^*$ be the projection which is dual to the inclusion $\rho : \mathfrak{t} \hookrightarrow u(V)$. The moment map for the action of $K \times K$ on $(\mathbb{P}(\mathrm{End}(V)), \Omega_{FS})$ is then

$$(3.6) \quad \mathbb{P}(\mathrm{End}(V)) \rightarrow \mathfrak{t}^* \times \mathfrak{t}^*$$

$$[A] \mapsto \frac{1}{\|A\|^2} (\pi_K(iAA^*), -\pi_K(iA^*A)).$$

Here we are interested in

- the projective variety $\mathcal{X}_P \subset \mathbb{P}(\mathrm{End}(V))$ with the action of $K \times K$,
- the projective variety $\mathcal{Y}_P \subset \mathbb{P}(\mathrm{End}(V))$ with the action of $T \times T$,

where $V = \bigoplus_{i=1}^N V_{\lambda_i}$. The Fubini-Study two-form restricts to symplectic forms on $\mathcal{X}_P$ and $\mathcal{Y}_P$. The action of $K \times K$ on $\mathcal{X}_P$ is Hamiltonian with moment map

$$(3.7) \quad \Phi_{K \times K} : \mathcal{X}_P \rightarrow \mathfrak{t}^* \times \mathfrak{t}^*$$

$$[x] \mapsto \frac{1}{\|x\|^2} (\pi_K(ixx^*), -\pi_K(ix^*x)).$$

Since the diagonal $Z = \{(t, t) | t \in T\}$ acts trivially on $\mathcal{Y}_P$ we restrict ourselves to the action of $T \times T / Z \simeq T$ on $\mathcal{Y}_P$. Let us compute the moment map $\Phi_T : \mathcal{Y}_P \rightarrow \mathfrak{t}^*$ associated to this action. First we have

$$(3.8) \quad \Phi_T([y]) = \frac{\pi_T(iy^*y)}{\|y\|^2} = \frac{\pi_T(iyy^*)}{\|y\|^2}$$

where $\pi_T : u(V)^* \rightarrow \mathfrak{t}^*$ is the projection which is dual to $\rho : \mathfrak{t} \rightarrow u(V)$. Since $\rho(X) = i \sum_{j \in J} \alpha_j(X)v_j \otimes v_j^*$, a small computation shows that for
B ∈ u(V) ∼= u(V)∗ we have πT(B) = −i∑j∈J(Bv,j,v)aj. Finally for any [y] ∈ YP we get

$$\Phi_T([y]) = \sum_{j \in J} \frac{\|y v_j\|^2}{\|y\|^2} \alpha_j.$$ 

Together with the action of T, we also have an action of the Weyl group

$$W = N(T)/T$$
on YP: for \(\bar{w}\) \in W we take

$$(3.9) \quad \bar{w} \cdot [y] = [\rho(w) \circ y \circ \rho(w)^{-1}], \quad [y] \in Y_P.$$ 

This action is well defined since the diagonal \(Z \subset T \times T\) acts trivially on \(Y_P\). The set of weights \(\{\alpha_j, j \in J\}\) is stable under the action of \(W\), hence it is easy to verify that the map \(\Phi_T\) is \(W\)-equivariant.

A dense part of \(Y_P\) is formed by the elements \(e^Z \cdot [\text{Id}] = [\rho(e^Z)]\) with \(Z = X + iY \in \mathfrak{t}_C\). We have \(\Phi_T(e^Z \cdot [\text{Id}]) = \psi_T(Y) \in \mathfrak{t}^*\) with

$$(3.10) \quad \psi_T(Y) = \frac{1}{\sum_{j \in J} e^{-2(\alpha_j,Y)}} \sum_{j \in J} e^{-2(\alpha_j,Y)} \alpha_j.$$ 

Hence the image of the moment map \(\Phi_T : Y_P \rightarrow \mathfrak{t}^*\) is equal to the closure of the image of the map \(\psi_T : \mathfrak{t} \rightarrow \mathfrak{t}^*\).

**Proposition 3.13.** — The map \(\psi_T\) realizes a diffeomorphism between \(\mathfrak{t}\) and the interior of the polytope \(P \subset \mathfrak{t}^*\).

**Proof.** — Consider the function \(F_T : \mathfrak{t} \rightarrow \mathbb{R}, F_T(Y) = \ln \left(\sum_j e^{\langle \alpha_j, Y \rangle}\right)\), and let \(L_T : \mathfrak{t} \rightarrow \mathfrak{t}^*\) be its Legendre transform: \(L_T(X) = dF_T|_X\). Note that we have \(L_T(-2Y) = \psi_T(Y)\).

We see that \(F_T\) is strictly convex. So, it is a classical fact that \(L_T\) realizes a diffeomorphism of \(\mathfrak{t}\) onto its image, and for \(\xi \in \mathfrak{t}^*\) we have

$$\xi \in \text{Image}(L_T) \iff \lim_{Y \rightarrow \infty} F_T(Y) - \langle \xi, Y \rangle = \infty$$

$$\iff \lim_{Y \rightarrow \infty} \sum_{j \in J} e^{\langle \alpha_j - \xi, Y \rangle} = \infty.$$ 

In order to conclude we need the following

**Lemma 3.14.** — Let \(\{\beta_j, j \in J\}\) be a sequence of elements of \(\mathfrak{t}^*\), and let \(Q\) be its convex hull. We have

$$\lim_{Y \rightarrow \infty} \sum_{j \in J} e^{\langle \beta_j, Y \rangle} = \infty \iff 0 \in \text{Interior}(Q).$$
Proof. — First we see that $0 \notin \text{Interior}(Q)$ if and only there exists $v \in t - \{0\}$ such that $\langle \beta_j, v \rangle \leq 0$ for all $j$: for such a vector $v$, the map $t \to \sum_{j \in J} e^{t \langle \beta_j, v \rangle}$ is bounded for $t \geq 0$. Suppose now that $\lim_{Y \to \infty} \sum_{j \in J} e^{t \langle \beta_j, Y \rangle} \neq \infty$. Then there exists a sequence $(X_k)_k \in t$ such that $\lim_k |X_k| = \infty$ and for all $j$ the sequence $(\langle \beta_j, X_k \rangle)_k$ remains bounded from above. If $v$ is a limit of a sub-sequence of $(\frac{X_k}{|X_k|})_k$ we have then $\langle \beta_j, v \rangle \leq 0$ for all $j$. 

**Lemma 3.15.** — For $[y] \in \mathcal{Y}_P$ we have $\Phi_{K \times K}([y]) = (\Phi_T([y]), -\Phi_T([y]))$.

**Proof.** — It is sufficient to consider the case

$$y = \rho(e^Z) = \sum_{j \in J} e^{i\langle \alpha_j, Z \rangle} v_j \otimes v_j^*, \text{ for } Z = X + iY \in t_C.$$  

Then $yy^* = y^*y = \sum_{j \in J} e^{-2\langle \alpha_j, Y \rangle} v_j \otimes v_j^* = \rho(e^{2iY})$. So what remains to prove is that $\pi_K(iyy^*) = \pi_T(iyy^*)$. We have to check that $\langle \pi_K(iyy^*), [U, V] \rangle = 0$ for $U \in t$ and $V \in t$

$$\langle \pi_K(iyy^*), [U, V] \rangle = -i \text{ Tr} \left( yy^* \rho([U, V]) \right) = -i \text{ Tr} \left( \rho(e^{2iY}) \rho(U) \rho(V) \right) = -i \text{ Tr} \left( [\rho(e^{2iY}), \rho(U)] \rho(V) \right) = 0.$$

**Theorem 3.16.** — We have

- $\text{Image}(\Phi_T) = P$,
- $\text{Image}(\Phi_{K \times K}) = \{ (k_1 \cdot \xi, -k_2 \cdot \xi) \mid \xi \in P \text{ and } k_1, k_2 \in K \}$,
- $\mathcal{Y}_P \subset \Phi_{K \times K}^{-1}(t^* \times t^*)$,
- $\Phi_{K \times K}^{-1}(\text{interior}(C)) \subset \mathcal{Y}_P$, where $C = C_K \times -C_K$.

**Proof.** — The first point follows from Proposition 3.13. Since the map $(k_1, t_2, k_2) \mapsto k_1tk_2$ from $K \times T_C \times K$ to $K_C$ is onto, we have

$$\mathcal{X}_P = (K \times K) \cdot \mathcal{Y}_P.$$  

So if $[x] \in \mathcal{X}_P$, there exist $[y] \in \mathcal{Y}$ and $k_1, k_2 \in K$ such that $[x] = (k_1, k_2) \cdot [y]$, hence

$$\Phi_{K \times K}([x]) = (k_1, k_2) \cdot \Phi_{K \times K}([y]) = (k_1 \cdot \Phi_T([y]), -k_2 \cdot \Phi_T([y])).$$  

The second point is proved. The third point follows also from the identity (3.12) when $k_1 = k_2 = e$. Consider now $[x] = (k_1, k_2) \cdot [y]$ such that $\Phi_{K \times K}([x])$ belongs to the interior of the cone $C_K \times -C_K$. Then $k_1 \cdot \Phi_T([y])$
and \(k_2 \cdot \Phi_T([y])\) are regular points of \(C_K\). This implies that \(k_1, k_2 \in N(T)\) and \(k_2 k_1^{-1} \in T\). So

\[
[x] = (k_1, k_2) \cdot [y] = (e, k_2 k_1^{-1}) \cdot ((k_1, k_1) \cdot [y]) \in \mathcal{Y}_P
\]

since \(\mathcal{Y}_P\) is stable under the actions of \(T \times T\) and \(W\).

Let \(O_i\) be the closed \(K_C \times K_C\)-orbit in \(\mathbb{P}(E)\) passing through \([v_{\lambda_i} \otimes v_{\lambda_i}^*]\), where \(v_{\lambda_i} \in V_{\lambda_i}\) is a highest weight vector and \(\lambda_i\) is regular dominant weight.

**Corollary 3.17.** — If \(O_i \subset \mathcal{X}_P\) then \(\lambda_i\) is a vertex of the polytope \(P\).

**Proof.** — Let \(x = v_{\lambda_i} \otimes v_{\lambda_i}^*\), and suppose that \([x]\) belongs to \(\mathcal{X}_P\). In order to show that \([x]\) belongs to \(\mathcal{Y}_P\), we compute \(\Phi_{K \times K}([x])\). We see that \(xx^* = x^* x = x\) and \(\|x\| = 1\), so \(\Phi_{K \times K}([x]) = (\pi_K(ix), -\pi_K(ix))\). For \(X \in \mathfrak{k}\) we have

\[
\langle \pi_K(ix), X \rangle = -i \ \text{Tr} \left( v_{\lambda_i} \otimes v_{\lambda_i}^* \rho(X) \right) = -i \ (\rho(X)v_{\lambda_i}, v_{\lambda_i}) = \langle \lambda_i, X \rangle.
\]

We then have \(\Phi_{K \times K}([x]) = (\lambda_i, -\lambda_i)\) with \(\lambda_i\) being a regular point of \(C_K\): then the last point of Theorem 3.16 shows that \([x] \in \mathcal{Y}_P\). Now we can conclude with the help of Lemma 3.10. Since \([v_{\lambda_i} \otimes v_{\lambda_i}^*]\) belongs to \(\mathcal{Y}_P\), the weight \(\lambda_i\) is a vertex of the polytope \(P\).

**Remark 3.18.** — In this section, Theorem 3.16 was obtained without using the fact that the varieties \(\mathcal{X}_P\) and \(\mathcal{Y}_P\) are smooth. Hence Corollary 3.17 can be used to prove the smoothness of \(\mathcal{X}_P\) (cf. Lemma 3.12).

### 3.4. Symplectic cutting

Let \((M, \Omega_M, \Phi_M)\) be a Hamiltonian \(K\)-manifold. At this stage the moment map \(\Phi_M\) is not assumed to be proper. We also consider the Hamiltonian \(K \times K\)-manifold \(\mathcal{X}_P\) associated to a \(K\)-adapted polytope \(P\).

The purpose of this section is to define a symplectic cutting of \(M\) which uses \(\mathcal{X}_P\). The notion of symplectic cutting was introduced by Lerman in [22] in the case of a torus action. Later Woodward [36] extended this procedure to the case of a non-abelian group action (see also [25, 26]). The method of symplectic cutting that we define in this section is different from that of Woodward.
We have two actions of $K$ on $\mathcal{X}_P$: the action from the left (resp. right), denoted $\cdot_l$ (resp. $\cdot_r$), with moment map $\Phi_l : \mathcal{X}_P \to \mathfrak{k}^*$ (resp. $\Phi_r$). We consider now the product $M \times \mathcal{X}_P$ with
- the action $k \cdot_1 (m, x) = (k \cdot m, k \cdot_r x)$: the corresponding moment map is $\Phi_1(m, x) = \Phi_M(m) + \Phi_r(x)$,
- the action $k \cdot_2 (m, x) = (m, k \cdot_l x)$: the corresponding moment map is $\Phi_2(m, x) = \Phi_l(x)$.

**Definition 3.19.** — We denote $M_P$ the symplectic reduction at 0 of $M \times \mathcal{X}_P$ for the action $\cdot_1$: $M_P := (\Phi_1)^{-1}(0)/K$.

Note that $M_P$ is compact when $\Phi_M$ is proper. The action $\cdot_2$ on $M \times \mathcal{X}_P$ induces an action of $K$ on $M_P$. The moment map $\Phi_2$ induces an equivariant map $\Phi_{M_P} : M_P \to \mathfrak{k}^*$. Let $Z \subset (\Phi_1)^{-1}(0)$ be the set of points where $(K, \cdot_1)$ has a trivial stabilizer.

**Definition 3.20.** — We denote $M'_P$ the quotient $Z/K \subset M_P$.

$M'_P$ is an open subset of smooth points of $M_P$ which is invariant under the $K$-action. The symplectic structure of $M \times \mathcal{X}_P$ induces a canonical symplectic structure on $M'_P$ that we denote $\Omega_{M'_P}$. The action of $K$ on $(M'_P, \Omega_{M'_P})$ is Hamiltonian with moment map equal to the restriction of $\Phi_{M_P} : M_P \to \mathfrak{k}^*$ to $M'_P$.

We start with the easy

**Lemma 3.21.** — The image of $\Phi_{M_P} : M_P \to \mathfrak{k}^*$ is equal to the intersection of the image of $\Phi_M : M \to \mathfrak{k}^*$ with $K \cdot P$.

Let $\mathcal{U}_P = K \cdot \text{Interior}(P) \subset K \cdot P$. We will show now that the open and dense subset $\Phi_{M_P}^{-1}(\mathcal{U}_P)$ of $M_P$ is contained in $M'_P$. Afterwards we will prove that $\Phi_{M_P}^{-1}(\mathcal{U}_P)$ is quasi-symplectomorphic to the open subset $\Phi_{M_P}^{-1}(\mathcal{U}_P)$ of $M$.

We consider the open and dense subset of $\mathcal{X}_P$ which is equal to the open orbit $\bar{\rho}(K_C)$. From Lemma 3.5, we know that

$$\Theta : K \times \mathfrak{k} \longrightarrow \bar{\rho}(K_C)$$

$$(k, X) \longmapsto [\rho(k e^{iX})]$$

is a diffeomorphism. Via $\Theta$, the action of $K \times K$ on $K \times \mathfrak{k}$ is $k \cdot_1 (a, X) = (ka, X)$ for the action “from the left” and $k \cdot_r (a, X) = (ak^{-1}, k \cdot X)$ for the action “from the right”.

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We now consider the map $\psi_K : \mathfrak{k} \rightarrow \mathfrak{k}^*$ defined by $\psi_K(X) = \Phi_l(\rho(e^{iX}))$. In other words,

$$\psi_K(X) = \frac{\pi_K(i\rho(e^{i2X}))}{\text{Tr}(\rho(e^{i2X}))}.$$ 

Consider the function $F_K : \mathfrak{k} \rightarrow \mathbb{R}$, $F_K(X) = \ln(\text{Tr}(\rho(e^{-iX})))$. Let $L_K : \mathfrak{k} \rightarrow \mathfrak{k}^*$ be its Legendre transform.

**Proposition 3.22.**

- We have $\psi_K(X) = L_K(-2X)$, for $X \in \mathfrak{k}$.
- The function $F_K$ is strictly convex.
- The map $\psi_K$ realizes an equivariant diffeomorphism between $\mathfrak{k}$ and $U_P$.
- The image of $\Phi_l : X_P \rightarrow \mathfrak{k}^*$ is equal to the closure of $U_P$.
- $\Phi_l^{-1}(U_P) = \overline{\rho(K_C)}$.

**Proof.** — For $X, Y \in \mathfrak{k}$ we consider the function $\tau(s) = F_K(X + sY)$. Since $F_K$ is $K$-invariant we can restrict our computation to the case where $X \in \mathfrak{t}$. We will use the decomposition of $Y \in \mathfrak{k}$ relatively to the $T$-weights on $\mathfrak{k}_C$: $Y = \sum \alpha Y_\alpha$ where $\text{ad}(Z)Y_\alpha = i\alpha(Z)Y_\alpha$ for any $Z \in \mathfrak{t}$, and $Y_0 \in \mathfrak{t}$. We have

$$\tau'(s) = \frac{-i}{\text{Tr}(\rho(e^{-iX_s}))} \text{Tr} \left( \rho(e^{-iX_s}) \rho \left( \frac{e^{i\text{ad}(X_s)}}{i\text{ad}(X_s)} - 1 \right) Y \right)$$

$$= \frac{-i}{\text{Tr}(\rho(e^{-iX_s}))} \text{Tr} \left( \rho(e^{-iX_s}) \rho(Y) \right)$$

$$= \frac{1}{\text{Tr}(\rho(e^{-iX_s}))} \langle \pi_K(i\rho(e^{-iX_s})), Y \rangle$$

where $X_s = X + sY$. Since by definition $\tau'(0) = \langle L_K(X), Y \rangle$, the first point is proved. For the second derivative we have

$$\tau''(0) = -\left( \frac{\text{Tr}(\rho(e^{-iX})\rho(iY))}{\text{Tr}(\rho(e^{-iX}))} \right)^2 + \frac{\text{Tr} \left( \rho(e^{-iX}) \rho \left( \frac{e^{i\text{ad}(X)}}{i\text{ad}(X)} - 1 \right) Y \rho(iY) \right)}{\text{Tr}(\rho(e^{-iX}))}$$

$$= R_1 + R_2$$

where

$$R_1 = \frac{\text{Tr} \left( \rho(e^{-iX}) \rho(iY_0) \rho(iY_0) \right)}{\text{Tr}(\rho(e^{-iX}))} - \left( \frac{\text{Tr}(\rho(e^{-iX}) \rho(iY_0))}{\text{Tr}(\rho(e^{-iX}))} \right)^2$$

$$= \frac{\sum_j e^{-\langle \alpha_j, X \rangle} \langle \alpha_j, Y_0 \rangle^2}{\sum_j e^{-\langle \alpha_j, X \rangle}} - \left( \frac{\sum_j e^{-\langle \alpha_j, X \rangle} \langle \alpha_j, Y_0 \rangle}{\sum_j e^{-\langle \alpha_j, X \rangle}} \right)^2$$
and
\[
R_2 = \frac{1}{\text{Tr}(\rho(e^{-iX}))} \sum_{\alpha \neq 0, \beta \neq 0} \frac{e^{-\langle \alpha, X \rangle} - 1}{-\langle \alpha, X \rangle} \text{Tr} \left( \rho(e^{-iX})\rho(iY_\alpha)\rho(iY_\beta) \right)
\]
\[
= \frac{1}{\text{Tr}(\rho(e^{-iX}))} \sum_{\alpha \neq 0, j} \frac{e^{-\langle \alpha, X \rangle} - 1}{-\langle \alpha, X \rangle} e^{-\langle \alpha_j, X \rangle} \|\rho(Y_\alpha)v_j\|^2.
\]

It is now easy to see that \(R_1 \) and \(R_2 \) are nonnegative and that \(R_1 + R_2 > 0\) if \(Y \neq 0\). We have proved that \(F_K\) is strictly convex. So, its Legendre transform \(L_K\) realizes a diffeomorphism of \(\mathfrak{k}\) onto its image. Using the first point we know that \(\psi_K\) realizes a diffeomorphism of \(\mathfrak{k}\) onto its image. The map \(\psi_K\) is equivariant and coincides with \(\psi_T\) on \(\mathfrak{t}\). We have proved in Proposition 3.13 that the image of \(\psi_T\) is equal to the interior of \(P\), hence the image of \(\psi_K\) is \(U_P\).

For the last two points we first remark that
\[
(3.14) \quad \Phi_I(\rho(ke^{iX})) = k \cdot \psi_K(X)
\]

hence the image of \(\Phi_I\) is the closure of \(U_P\). If we use the fact that \(\psi_K\) is a diffeomorphism from \(\mathfrak{k}\) onto \(U_P\), (3.14) shows that \(\Phi_I^{-1}(K \cdot \xi) \cap \tilde{\rho}(K_C)\) is a non empty and closed subset of \(\Phi_I^{-1}(K \cdot \xi)\) for any \(\xi \in U_P\) (in fact it is a \(K \times K\)-orbit). On the other hand \(\Phi_I^{-1}(K \cdot \xi) \cap (\mathcal{X}_P \setminus \tilde{\rho}(K_C))\) is also a closed subset of \(\Phi_I^{-1}(K \cdot \xi)\) since \(\tilde{\rho}(K_C)\) is open in \(\mathcal{X}_P\). Since \(\Phi_I^{-1}(K \cdot \xi)\) is connected the second subset is empty: in other words \(\Phi_I^{-1}(K \cdot \xi) \subset \tilde{\rho}(K_C)\).

We introduce now the equivariant diffeomorphism
\[
(3.15) \quad \Upsilon : K \times U_P \longrightarrow \tilde{\rho}(K_C)
\]
\[
(k, \xi) \longmapsto \Theta(k, \psi_K^{-1}(\xi)).
\]

We now consider \(K \times U_P\) equipped with the symplectic structure \(\Upsilon^*(\Omega_{\mathcal{X}_P})\), and the Hamiltonian action of \(K \times K\): the moment maps satisfy
\[
(3.16) \quad \Upsilon^*(\Phi_I)(k, \xi) = k \cdot \xi \text{ and } \Upsilon^*(\Phi_T)(k, \xi) = -\xi.
\]

**Proposition 3.23.** — We have
\[
\Upsilon^*(\Omega_{\mathcal{X}_P}) = d\lambda + d\eta
\]
where \(\lambda\) is the Liouville 1-form on \(K \times \mathfrak{k}^* \simeq T^*K\) and \(\eta\) is an invariant 1-form on \(U_P \subset \mathfrak{k}^*\) which is killed by the vectors tangent to the \(K\)-orbits.

**Proof.** — Let \(E_1, \ldots, E_r\) be a basis of \(\mathfrak{k}\), with dual basis \(\xi^1, \ldots, \xi^r\). Let \(\omega^i\) the 1-form on \(K\), invariant by left translation and equal to \(\xi^i\) at the identity. The Liouville 1-form is \(\lambda = -\sum_{i} \omega^i \otimes E_i\). For \(X \in \mathfrak{k}\) we denote \(X_l(k, \xi) = \frac{d}{dt}|_{t=0} e^{-tX} \cdot l(k, \xi)\) and \(X_r(k, \xi) = \frac{d}{dt}|_{t=0} e^{-tX} \cdot r(k, \xi)\) the vector
fields generated by the action of $K \times K$. Since $\iota(X) d\lambda = -d(\Phi_t, X)$ and $\iota(X_r) d\lambda = -d(\Phi_r, X)$, the closed invariant 2-form $\beta = \Upsilon^* (\Omega_{\Phi_r}) - d\lambda$ is $K \times K$ invariant and is killed by the vectors tangent to the orbits: (*) $
abla^*(\Phi_r)$.

We have $\iota(X) \beta = \iota(X_r) \beta = 0$ for all $X \in \mathfrak{k}$. We have $\beta = \beta_2 + \beta_1 + \beta_0$ where $\beta_2 = \sum_{i,j} a_{ij} (\xi) \omega^i \wedge \omega^j$, $\beta_1 = \sum_{i,j} b_{ij} (\xi) \omega^i \wedge dE_j$, and $\beta_0$ is an invariant 2-form on $U_P$. The equalities (*) gives $\iota(X) \beta_2 = \iota(X) \beta_1 = 0$ which imply that $\beta_2 = \beta_1 = 0$. So $\beta = \beta_0$ is a closed invariant 2-form on $U_P$ which is killed by the vectors tangent to the $K$-orbits. Since $U_P$ admits a retraction to $\{0\}$, $\beta = d\eta$ where $\eta$ is an invariant 1-form on $U_P$ which is killed by the vectors tangent to the $K$-orbits.

If $(m, x) \in M \times X_P$ belongs to $\Phi_1^{-1}(0)$, we denote $[m, x]$ the corresponding element in $M_P$. By definition we have $\Phi_{M_P}([m, x]) = \Phi_t(x)$ for $[m, x] \in M_P$, hence the image of $\Phi_{M_P}$ is included in the closure of $U_P$. We see also that $[m, x] \in \Phi_{M_P}^{-1}(U_P)$ if and only if $x \in \Phi_{M_P}^{-1}(U_P) = \rho(K_C)$. Since $(K, \cdot, r)$ acts freely on $\rho(K_C)$, we see that $(K, \cdot, 1)$ acts freely on $\Phi^{-1}_{M_P}(U_P)$: the open and dense set $\Phi_{M_P}^{-1}(U_P) \subset M_P$ is then contained in $M_P$.

Now, we can state our main result which compares the open invariant subsets $\Phi_M^{-1}(U_P) \subset M$ and $\Phi_{M_P}^{-1}(U_P) \subset M_P$ equipped respectively with the symplectic structures $\Omega_M$ and $\Omega_{M_P}$.

**Theorem 3.24.** — $\Phi_{M_P}^{-1}(U_P)$ is an open and dense subset of smooth points in $M_P$. There exists an equivariant diffeomorphism $\Psi : \Phi_{M_P}^{-1}(U_P) \to \Phi_{M_P}^{-1}(U_P)$ such that

$$\Psi^* (\Omega_{M_P}) = \Omega_M + d\Phi_M^* \eta.$$ 

Here $\eta$ is an invariant 1-form on $U_P$ which is killed by the vectors tangent to the $K$-orbits. Moreover the path $\Omega^t = \Omega_M + t d\Phi_M^* \eta$, defines a homotopy of symplectic 2-forms between $\Omega_M$ and $\Psi^* (\Omega_{M_P}).$

**Remark 3.25.** — The map $\Psi$ will be called a quasi-symplectomorphism.

**Proof.** — Consider the immersion

$$\psi : \Phi_{M}^{-1}(U_P) \longrightarrow M \times X_P,$$

$$m \longmapsto (m, \Upsilon(e, \Phi_M(m))).$$

We have $\Phi_1(\psi(m)) = \Phi_M(m) + \Upsilon^* \Phi_r(e, \Phi_M(m)) = 0$, and $\Phi_2(\psi(m)) = \Upsilon^* \Phi_t(e, \Phi_M(m)) = \Phi_M(m) \in U_P$ (see (3.16)). Hence for all $m \in \Phi_{M}^{-1}(U_P)$, we have $\psi(m) \in \Phi_{M}^{-1}(0)$, and its class $[\psi(m)] \in M_P$ belongs to $\Phi_{M_P}^{-1}(U_P)$.

We denote $\Psi : \Phi_{M}^{-1}(U_P) \to \Phi_{M_P}^{-1}(U_P)$ the map $m \mapsto [\psi(m)]$. Let us show that it defines a diffeomorphism. If $\Psi(m) = \Psi(m')$, there exists $k \in K$ such
that
\[ (m, \Upsilon(e, \Phi_M(m))) = k \cdot 1 (m', \Upsilon(e, \Phi_M(m'))) \]
\[ = (k \cdot m', k \cdot \Upsilon(e, \Phi_M(m'))) \]
\[ = (k \cdot m', \Upsilon(k^{-1}, k \cdot \Phi_M(m'))) \].

Since \( \Upsilon \) is a diffeomorphism, we must have \( k = e \) and \( m = m' \): the map \( \Psi \) is one to one. Consider now \((m, x) \in \Phi^{-1}_1(0)\) such that \( \Phi_{M_p}([m, x]) = \Phi_{t}(x) \in \mathcal{U}_P \): then \( x \in \Phi^{-1}_1(\mathcal{U}_P) = \tilde{\rho}(K_C) = \text{Image}(\Upsilon) \). We have \( x = \Upsilon(k, \xi) \) where \( \xi = -\Phi_r(x) = \Phi_M(m) \). Finally
\[ (m, x) = (m, \Upsilon(k, \Phi_M(m))) \]
\[ = k^{-1} \cdot 1 (k \cdot m, \Upsilon(e, k \cdot \Phi_M(m))) \]
\[ = k^{-1} \cdot 1 \psi(k \cdot m). \]

We have proved that \( \Psi \) is onto.

In order to show that \( \Psi \) is a submersion we must show that for \( m \in \Phi_{M}^{-1}(\mathcal{U}_P) \)
\[ \text{Image}(T_m\psi) \oplus T_{\psi(m)}(K \cdot 1 \psi(m)) = T_{\psi(m)}\Phi^{-1}_1(0). \]

Here \( T_m\psi : T_mM \rightarrow T_{\psi(m)}(M \times \mathcal{X}_P) \) is the tangent map, and \( T_{\psi(m)}(K \cdot 1 \psi(m)) \) denotes the tangent space at \( \psi(m) \) of the \((K, \cdot 1)\)-orbit. We have \( \dim(\text{Image}(T_m\psi)) + \dim(T_{\psi(m)}(K \cdot 1 \psi(m))) = \dim(T_{\psi(m)}\Phi^{-1}_1(0)) \) so it is sufficient to prove that
\[ \text{Image}(T_m\psi) \cap T_{\psi(m)}(K \cdot 1 \psi(m)) = \{0\}. \]

Consider \((v, w) \in \text{Image}(T_m\psi) \cap T_{\psi(m)}(K \cdot 1 \psi(m)) \). There exists \( X \in \mathfrak{X} \) such
\[ (v, w) = \frac{\partial}{\partial t} \Upsilon(e, \Phi_M(e^{tX} \cdot m)) \]
\[ = \frac{\partial}{\partial t} e^{tX} \cdot m \]
\[ = \frac{\partial}{\partial t} e^{tX} \cdot \Upsilon(e, \Phi_M(m)). \]

On the other hand, since \((v, w) \in \text{Image}(T_m\psi), \) we have
\[ w = \frac{\partial}{\partial t} \Upsilon(e, \Phi_M(e^{tX} \cdot m)). \]

Since \( e^{tX} \cdot \Upsilon(e, \Phi_M(m)) = \Upsilon(e^{-tX}, \Phi_M(e^{tX} \cdot m)) \) we obtain that
\[ \frac{\partial}{\partial t} \Upsilon(e^{-tX}, \Phi_M(e^{tX} \cdot m)) = \frac{\partial}{\partial t} \Upsilon(e, \Phi_M(e^{tX} \cdot m)), \]
or in other words \( \frac{\partial}{\partial t} \Upsilon(e^{-tX}, \Phi_M(m)) = 0 \). Since \( \Upsilon \) is a diffeomorphism we have \( X = 0 \), and then \((v, w) = 0\).
We can now compute the pull-back by $\Psi$ of the symplectic form $\Omega_{M'}$. We have
\[\Psi^*(\Omega_{M'}) = \psi^*(\Omega_M + \Omega_{X_P}) = \Omega_M + \Phi_M^* \Upsilon^*(\Omega_{X_P}) = \Omega_M + d\Phi_M^* \eta.\]

It remains to prove that for every $t \in [0, 1]$, the 2-form $\Omega^t = \Omega_M + td\Phi_M^* \eta$ is non-degenerate. Take $t \neq 0$, $m \in \Phi_M^{-1}(U_P)$ and suppose that the contraction of $\Omega^t|_m$ by $v \in T_m M$ is equal to 0. For every $X \in \mathfrak{t}$ we have
\[0 = \Omega^t(X_M(m), v) = -\iota(v)d(\Phi_M, X)|_m + t\iota(v)d(X_M)d\Phi_M^* \eta|_m = -\iota(v)d(\Phi_M, X)|_m,
\]
since $\iota(X_M)d\Phi_M^* \eta = d\Phi_M^*(\iota(X_P) \eta) = 0$. Thus we have $T_m \Phi_M(v) = 0$, and then $\iota(v)d\Phi_M^* \eta = 0$. Finally we have that $0 = \iota(v)\Omega^t|_m = \iota(v)\Omega_M|_m$. But $\Omega_M$ is non-degenerate, so $v = 0$. \hfill \Box

### 3.5. Formal quantization: second definition

We suppose here that the Hamiltonian $K$-manifold $(M, \Omega_M, \Phi_M)$ is proper and admits a Kostant-Souriau line bundle $L$. Now we consider the complex $K \times K$-submanifold $X_P$ of $\mathbb{P}(E)$. Since $\mathcal{O}(-1)$ is a $K \times K$-equivariant Kostant-Souriau line bundle on the projective space $\mathbb{P}(E)$ the restriction
\[(3.17) \quad L_P = \mathcal{O}(-1)|_{X_P},\]
is a Kostant-Souriau line bundle on $X_P$. Hence $L \boxtimes L_P$ is a Kostant-Souriau line bundle on the product $M \times X_P$. In Section 2.2 we have defined the quantization $\mathcal{Q}_K(M_P)$ of the (singular) reduced space $M_P := (M \times X_P)/\{0(K, \cdot_1)\}$.

**Notation.** — $O_K(r)$ will be any element $\sum_{\mu \in \mathbb{R}} m_\mu V^K_\mu$ of $R^{-\infty}(K)$ where $m_\mu = 0$ if $\|\mu\| < r$. The limit $\lim_{r \to +\infty} O_K(r) = 0$ defines the notion of convergence in $R^{-\infty}(K)$.

**Proposition 3.26.** — Let $\varepsilon_P > 0$ be the radius of the biggest ball center at $0 \in t^*$ which is contained in the polytope $P$. We have
\[(3.18) \quad \mathcal{Q}_K(M_P) = \sum_{\|\mu\| < \varepsilon_P} \mathcal{Q}(M_\mu)V^K_\mu + O_K(\varepsilon_P).\]
Proof. — Theorem 2.4—“Quantization commutes with reduction in the singular setting”—tells us that $Q_K(M_P) = \sum_{\mu \in \hat{K}} Q((M_P)_{\mu}) V^K_{\mu}$ where $(M_P)_{\mu}$ is the symplectic reduction $(M_P \times K_{\mu})/\gamma_{0}K$.

Since the image of $\Phi_M$ is equal to the intersection of $K \cdot P = \mathcal{U}_P$ with the image of $\Phi_M$, we have $Q((M_P)_{\mu}) = 0$ if $\mu \not\in P \cap \text{Image}(\Phi_M)$. We will now exploit Theorem 3.24 to show that $Q((M_P)_{\mu}) = Q(M_{\mu})$ if $\mu$ belongs to the interior of $P$.

There exists a quasi-symplectomorphism $\Psi$ between the open subset $\Phi^{-1}_M(U_P)$ of $M$ and the open and dense subset $\Phi^{-1}_M(U_P)$ of $M_P$. Moreover one can see easily that the restriction of the Kostant-Souriau line bundle $L_P \to \chi_P$ to the open subset $\bar{\rho}(K_C)$ is trivial. If $L_M$ is the Kostant-Souriau line bundle on $M_P$ induced by $L \otimes L_P$, then the pull-back of the restriction $L_M|_{\Phi^{-1}_M(U_P)}$ by $\Psi$ is equivariantly diffeomorphic to the restriction of $L$ to $\Phi^{-1}_M(U_P)$.

Take now $\mu \in \hat{K}$ that belongs to the interior of the polytope $P$. The element $Q((M_P)_{\mu}) \in \mathbb{Z}$ is given by the index of a transversally elliptic symbol defined in a (small) neighborhood of $\Phi^{-1}_M(\mu) \subset M_P$. This symbol is defined through two auxiliary data: the Kostant-Souriau line bundle $L_M$ and a compatible almost complex structure $J$ which is defined in a neighborhood of $\Phi^{-1}_M(\mu)$. If we pull back everything by $\Psi$, we get a transversally elliptic symbol living in a (small) neighborhood of $\Phi^{-1}_M(\mu) \subset M$ which is defined by the Kostant-Souriau line bundle $L$ and an almost complex structure $J_1$ compatible with the symplectic structure $\Omega_1 := \Omega_M + d\Phi^*_M\eta$. But since $\Omega_t = \Omega_M + t\Phi^*_M\eta$ defines a homotopy of symplectic structures, any almost complex structure compatible with $\Omega_M$ is homotopic to $J_1$. We have then shown that $Q(M_{\mu}) = Q((M_P)_{\mu})$ for any $\mu$ belonging to the interior of $P$. So we have

$$Q_K(M_P) = \sum_{\mu \in \text{Interior}(P)} Q(M_{\mu}) V^K_{\mu} + \sum_{\nu \in \partial P} Q((M_P)_{\nu}) V^K_{\nu}.$$ 

Since for $\nu \in \partial P$ we have $\|\nu\| \geq \varepsilon_P$, the last equality proves (3.18). 

We work now with the dilated polytope $nP$, for any integer $n \geq 1$. The polytope $nP$ is still $K$-adapted, so one can consider the reduced space (2) $M_{nP}$ and Proposition 3.26 gives that

$$Q_K(M_{nP}) = \sum_{\|\mu\| < n\varepsilon_P} Q(M_{\mu}) V^K_{\mu} + O_K(n\varepsilon_P).$$

(2) These are the cut spaces denoted $M_{nP}$ in the introduction.
for any integer \( n \geq 1 \). We can summarize the result of this section in the following

**Proposition 3.27.** — Let \((M, \Omega_M)\) be a pre-quantized Hamiltonian \( K \)-manifold, with a proper moment map \( \Phi_M \).

- For any integer \( n \geq 1 \), the (singular) compact Hamiltonian manifold \( M_{nP} \) contains as an open and dense subset, the open subset \( \Phi_M^{-1}(nU_P) \) of \( M \).
- We have \( Q_K^{-\infty}(M) = \lim_{n \to \infty} Q_K(M_{nP}) \).

4. Functorial properties: Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. We will use in a crucial way the characterisation of \( Q^{-\infty}_K \) given in Proposition 3.27.

Let \( H \subset K \) be a closed and connected Lie subgroup. Here we consider a pre-quantized Hamiltonian \( K \)-manifold \( M \) which is proper as a Hamiltonian \( H \)-manifold. We want to compare \( Q_K^{-\infty}(M) \) and \( Q_H^{-\infty}(M) \).

For \( \mu \in \hat{K} \) and \( \nu \in \hat{H} \) we denote \( N^\mu_\nu \) the multiplicity of \( V^H_\nu \) in the restriction \( V^K_\mu|_H \). We have seen in the introduction that \( N^\mu_\nu Q(M_\mu,K) \neq 0 \) only for the \( \mu \) belonging to finite subset \( \hat{K} \cap \Phi_K(K \cdot \Phi_H^{-1}(\nu)) \). Then \( Q_K^{-\infty}(M) \) is \( H \)-admissible and we have the following equality in \( R^{-\infty}(H) \):

\[
Q_K^{-\infty}(M)|_H = \sum_{\nu \in \hat{H}} m_\nu V^K_\nu
\]

with \( m_\nu = \sum_\mu N^\mu_\nu Q(M_\mu,K) \). We will now prove that

\[
Q_K^{-\infty}(M)|_H = Q_H^{-\infty}(M).
\]

**Lemma 4.1.** — The restriction \( Q_K^{-\infty}(M)|_H \) is equal to \( \lim_{n \to \infty} Q_K(M_{nP})|_H \).

**Proof.** — Let us denote by \( P^o \) and \( \partial P \) respectively the interior and the boundary of the \( K \)-adapted polytope \( P \). We write

\[
Q_K^{-\infty}(M) = \sum_{\mu \in nP^o} Q(M_{\mu,K})V^K_\mu + \sum_{\mu \notin nP^o} Q(M_{\mu,K})V^K_\mu.
\]

On the other side

\[
Q_K(M_{nP}) = \sum_{\mu \in nP^o} Q(M_{\mu,K})V^K_\mu + \sum_{\mu \in \partial P} Q((M_{nP})_{\mu,K})V^K_\mu.
\]

So the difference \( D(n) = Q_K^{-\infty}(M) - Q_K(M_{nP}) \) is equal to

\[
D(n) = \sum_{\mu' \in \partial P} Q((M_{nP})_{\mu',K})V^K_{\mu'} + \sum_{\mu \notin nP^o} Q(M_{\mu,K})V^K_\mu.
\]
We show now that the restriction \( D(n)|_H \) tends to 0 in \( R^{-\infty}(H) \) as \( n \) goes to infinity. For this purpose, we will prove that for any \( c > 0 \) there exists \( n_c \in \mathbb{N} \) such that \( D(n)|_H = O_H(c) \) for any \( n \geq n_c \).

For \( c > 0 \) we consider the compact subset of \( \mathfrak{P}^* \) defined by

\[
K_c = \Phi_K \left( K \cdot \Phi_H^{-1}(\xi \in \mathfrak{h}^*, \|\xi\| \leq c) \right).
\]

Let \( n_c \in \mathbb{N} \) such that \( K_c \) is included in \( K \cdot (n_c P^o) \); hence \( K_c \subset K \cdot (nP^o) \) for any \( n \geq n_c \). We know that for \( \mu \in \hat{K} \), we have \( N^\mu_{\nu} Q(M_{\mu, K}) \neq 0 \) only for \( \mu \in \Phi_K \left( K \cdot \Phi_H^{-1}(\nu) \right) \), and for \( \mu' \in \hat{K} \), we have \( N^\mu_{\nu'} Q((M_{nP})_{\mu', K}) \neq 0 \) only for \( \mu' \in nP \cap \Phi_K \left( K \cdot \Phi_H^{-1}(\nu) \right) \).

Then if \( n \geq n_c \), we have

\[
N^\mu_{\nu} Q(M_{\mu, K}) = N^\mu_{\nu'} Q((M_{nP})_{\mu', K}) = 0
\]

for any \( \nu \in \hat{H} \cap \{ \xi \in \mathfrak{h}^*, \|\xi\| \leq c \} \), \( \mu \notin nP^o \) and \( \mu' \in nP \). This means that \( D(n)|_H = O_H(c) \) for any \( n \geq n_c \). \( \square \)

Since \( Q_K(M_{nP})|_H = Q_H(M_{nP}) \), we are led to the

**Lemma 4.2.** — The limit \( \lim_{n \to \infty} Q_H(M_{nP}) \) is equal to \( Q_H^{-\infty}(M) \).

**Proof.** — Theorem 2.4 – “Quantization commutes with reduction in the singular setting” – tells us that \( Q_H(M_{nP}) = \sum_{\nu \in \hat{H}} Q((M_{nP})_{\nu, H})V^\nu_H \) where \( (M_{nP})_{\nu, H} \) is the symplectic reduction

\[
(M_{nP} \times \overline{H} \cdot \nu)|_0 H \cong (M \times X_{nP} \times \overline{H} \cdot \mu)|_{(0, 0)} H \times K.
\]

For \( c > 0 \) we consider the compact subset of \( \mathcal{K}_c \) defined in (4.2). Let \( n_c \in \mathbb{N} \) such that \( \mathcal{K}_c \subset K \cdot (nP^o) \) for any \( n \geq n_c \). This implies that

\[
\Phi_H^{-1}(\xi \in \mathfrak{h}^*, \|\xi\| \leq c) \subset \Phi_K^{-1}(K \cdot (nP^o))
\]

for \( n \geq n_c \). Since \( M_{nP} \) “contains” the open subset \( \Phi_K^{-1}(K \cdot (nP^o)) \), arguments similar to those used in the proof of Proposition 3.26 show that \( Q((M_{nP})_{\nu, H}) = Q(M_{\nu, H}) \) for \( \|\nu\| \leq c \) and \( n \geq n_c \). This means that

\[
Q_H(M_{nP}) = \sum_{\|\nu\| \leq c} Q(M_{\nu, H})V^\nu_H + O_H(c) \quad \text{when} \quad n \geq n_c.
\]

It follows that \( \lim_{n \to \infty} Q_H(M_{nP}) = \sum_{\nu \in \hat{H}} Q(M_{\nu, H})V^\nu_H = Q_H^{-\infty}(M). \) \( \square \)

### 5. The case of a Hermitian vector space

Let \( (E, h) \) be a Hermitian vector space of dimension \( n \).
5.1. The quantization of $E$

Let $U := U(E)$ be the unitary group with Lie algebra $\mathfrak{u}$. We use the isomorphism $\epsilon : \mathfrak{u} \to \mathfrak{u}^*$ defined by $\langle \epsilon(X), Y \rangle = -\text{Tr}(XY) \in \mathbb{R}$. For $v, w \in E$, let $v \otimes w^* : E \to E$ be the linear map $x \mapsto h(x, w)v$.

Let $E_\mathbb{R}$ be the space $E$ view as a real vector space. Let $\Omega$ be the imaginary part of $-h$, and let $J$ the complex structure on $E_\mathbb{R}$. Then on $E_\mathbb{R}$, $\Omega$ is a (constant) symplectic structure and $\Omega(-, J-)$ defines a scalar product. The action of $U$ on $(E_\mathbb{R}, \Omega)$ is Hamiltonian with moment map $\Phi : E \to \mathfrak{u}^*$ defined by $\langle \Phi(v), X \rangle = \frac{1}{2}\Omega(Xv, v)$. Via $\epsilon$, the moment map $\Phi$ is defined by

$$\Phi(v) = \frac{1}{2i}v \otimes v^*.$$ (5.1)

The pre-quantization data $(L, \langle -, -, \nabla \rangle)$ on the Hamiltonian U-manifold $(E_\mathbb{R}, \Omega, \Phi)$ is a trivial line bundle $L$ with a trivial action of $U$ equipped with the Hermitian structure $\langle s, s' \rangle_v = e^{-\frac{h(v, v)}{2}}ss^\top$ and the Hermitian connexion $\nabla = d - i\theta$ where $\theta$ is the 1-form on $E$ defined by $\theta = \frac{1}{2}\Omega(v, dv)$.

The traditional quantization of the Hamiltonian U-manifold $(E_\mathbb{R}, \Omega, \Phi)$, that we denote $Q_U^{L^2}(E)$, is the Bargman space of entire holomorphic functions on $E$ which are $L^2$ integrable with respect to the Gaussian measure $e^{-\frac{h(v, v)}{2}}\Omega^n$. The representation $Q_U^{L^2}(E)$ of $U$ is admissible. The irreducible representations of $U$ that occur in $Q_U^{L^2}(E)$ are the vector subspaces $S^j(E^*)$ formed by the homogeneous polynomials on $E$ of degree $j \geq 0$.

On the other hand, the moment map $\Phi$ is proper (see (5.1)). Hence we can consider the formal quantization $Q_U^{-\infty}(E) \in R^{-\infty}(U)$ of the $U$-action on $E$.

**Lemma 5.1.** — The two quantizations of $(E, \Omega, \Phi)$, $Q_U^{L^2}(E)$ and $Q_U^{-\infty}(E)$ coincide in $R^{-\infty}(U)$. In other words, we have

$$Q_U^{-\infty}(E) = S^*(E^*) := \sum_{j \geq 0} S^j(E^*) \quad \text{in} \quad R^{-\infty}(U).$$ (5.2)

**Proof.** — Let $T \subset U$ be a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{u}$. There exist an orthonormal basis $(e_k)_{k=1,\ldots,n}$ of $E$ and characters $(\chi_k)_{k=1,\ldots,n}$ of $T$ such that $t \cdot e_k = \chi_k(t)e_k$ for all $k$. The family $(ie_k \otimes e_k^*)_{k=1,\ldots,n}$ is then a basis of $\mathfrak{t}$ such that $\frac{i}{2} d\chi_l(ie_k \otimes e_k^*) = \delta_{l,k}$. The set $\tilde{U} \subset \mathfrak{t}^* \subset \mathfrak{u}^*$ of dominant weights is composed, via $\epsilon$, by the elements

$$\lambda = i \sum_{k=1}^n \lambda_k e_k \otimes e_k^*,$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a decreasing sequence of integers.
The formal quantization \( Q_U^{-\infty}(E) \in R^{-\infty}(U) \) is defined by
\[
Q_U^{-\infty}(E) = \sum_{\lambda_1 \geq \ldots \geq \lambda_n} Q(E_{\lambda}) V_{\lambda}
\]
where \( E_{\lambda} = \Phi^{-1}(U \cdot \lambda)/U \) is the reduced space and \( V_{\lambda} \) is the irreducible representation of \( U \) with highest weight \( \lambda \).

It is now easy to check that \( E_{\lambda} = \begin{cases} \{ \text{pt} \} & \text{if } \lambda = (0, \ldots, 0, -j) \text{ with } j \geq 0, \\ \emptyset & \text{in the other cases,} \end{cases} \)
and then
\[
Q(E_{\lambda}) = \begin{cases} 1 & \text{if } \lambda = (0, \ldots, 0, -j) \text{ with } j \geq 0, \\ 0 & \text{in the other cases.} \end{cases}
\]
Finally (5.2) follows from the fact that \( V_{(0, \ldots, 0, -j)} = S^j(E^*) \). \( \square \)

### 5.2. The quantization of \( E \) restricted to a subgroup of \( U \)

Let \( K \subset U \) be a closed connected Lie subgroup with Lie algebra \( \mathfrak{k} \). Let \( K_C \subset \text{GL}(E) \) be its complexification. The moment map relative to the \( K \)-action on \((E_{\mathbb{R}}, \Omega)\) is the map
\[
\Phi_K : E \to \mathfrak{k}^* \equiv \text{the composition of } \Phi \text{ with the projection } u^* \to \mathfrak{k}^*.
\]
equal to the composition of \( \Phi \) with the projection \( u^* \to \mathfrak{k}^* \).

**Lemma 5.2.** — The following conditions are equivalent:

(a) the map \( \Phi_K \) is proper,
(b) \( \Phi^{-1}_K(0) = \{0\} \),
(c) \( \{0\} \) is the only closed \( K_C \)-orbit in \( E \),
(d) for every \( v \in E \) we have \( 0 \in \overline{K_C \cdot v} \),
(e) \( S^{\ast}(E^*) \) is an admissible representation of \( K \),
(f) the \( K \)-invariant polynomials on \( E \) are the constant polynomials.

**Proof.** — The equivalence \((a) \iff (b)\) is due to the fact that \( \Phi_K \) is quadratic.

Let \( \mathcal{O} \) be a \( K_C \)-orbit in \( E \). Classical results of Geometric Invariant Theory \([27, 19]\) assert that \( \overline{\mathcal{O}} \cap \Phi^{-1}_K(0) \neq \emptyset \) and that \( \mathcal{O} \) is closed if and only if \( \mathcal{O} \cap \Phi^{-1}_K(0) \neq \emptyset \). Hence \((b) \iff (c) \iff (d)\).

>From Lemma 5.1 we know that \( Q^{-\infty}_U(E) = S^{\ast}(E^*) \). Since \( Q^{-\infty}_U(E) \) is \( K \)-admissible when \( \Phi_K \) is proper (see Section 4), we have \((a) \implies (e)\).
For every $\mu \in \hat{K}$, the $\mu$-isotopic component $[S^\bullet(E^\ast)] \mu$ is a module over $[S^\bullet(E^\ast)] \mu = [S^\bullet(E^\ast)]^K$. Hence $\dim [S^\bullet(E^\ast)] \mu < \infty$ implies that $[S^\bullet(E^\ast)]^K = \mathbb{C}$. We have $(e) \implies (f)$.

Finally $(f) \implies (d)$ follows from the following fundamental fact. For any $v, w \in E$ we have $K \cdot v \cap K \cdot w \neq \emptyset$ if and only if $P(v) = P(w)$ for all $P \in [S^\bullet(E^\ast)]^K$. □

Theorem 1.3 implies the following

**Proposition 5.3.** — Let $K \subset U(E)$ be a closed connected subgroup such that $S^\bullet(E^\ast)$ is an admissible representation of $K$. For every $\mu \in \hat{K}$, we have

$$\dim ([S^\bullet(E^\ast)] \mu) = Q(E_{\mu,K})$$

where $[S^\bullet(E^\ast)] \mu$ is the $\mu$-isotopic component of $S^\bullet(E^\ast)$ and $E_{\mu,K}$ is the reduced space $\Phi_{K}^{-1}(K \cdot \mu)/K$.

In the following examples the condition $\Phi_{K}^{-1}(0) = \{0\}$ is easy to check.

1) the subgroup $K \subset U(E)$ contains the center of $U(E)$,
2) $E = \wedge^2 \mathbb{C}^n$ or $E = S^2(\mathbb{C}^n)$ and $K = U(n) \subset U(E)$,
3) $E = M_{n,k}$ is the vector space of $n \times k$-matrices and $K = U(n) \times U(k) \subset U(E)$.

**BIBLIOGRAPHY**


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