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KAC-MOODY GROUPS, HOVELS AND LITTELMANN PATHS

by Stéphane GAUSSENT & Guy ROUSSEAU

Abstract. — We give the definition of a kind of building $\mathcal{I}$ for a symmetrizable Kac-Moody group over a field K endowed with a discrete valuation and with a residue field containing $\mathbb{C}$. Due to the lack of some important property of buildings, we call it a hovel. Nevertheless, some good ones remain, for example, the existence of retractions with center a sector-germ. This enables us to generalize many results proved in the semisimple case by S. Gaussent and P. Littelmann. In particular, if $K = \mathbb{C}((t))$, the geodesic segments in $\mathcal{I}$, ending in a special vertex and retracting onto a given path π, are parametrized by a Zariski open subset P of $\mathbb{C}^N$. This dimension N is maximal when π is a LS path and then P is closely related to some Mirković-Vilonen cycle.

Résumé. — Nous définissons une sorte d'immeuble $\mathcal{I}$ associé à un groupe de Kac-Moody symétrisable sur un corps K muni d'une valuation discrète avec un corps résiduel contenant $\mathbb{C}$. Nous l'appelons masure (hovel) à cause de l'absence d'une propriété importante des immeubles. Cependant, de bonnes propriétés restent, par exemple l'existence de retractions de centre un germe de quartier. Cela nous permet de généraliser plusieurs résultats prouvés par S. Gaussent et P. Littelmann dans le cas semi-simple. En particulier, si $K = \mathbb{C}((t))$, les segments géodésiques dans $\mathcal{I}$, d'extrémité un sommet spécial et se rétractant sur un chemin donné π, sont paramétrés par un ouvert de Zariski P de $\mathbb{C}^N$. Cette dimension N est maximale quand π est un chemin LS et alors P est fortement associé à un cycle de Mirković-Vilonen.

1. Introduction

Let $\mathfrak{g}^\vee$ be a complex symmetrizable Kac-Moody algebra. To capture the combinatorial essence of the representation theory of $\mathfrak{g}^\vee$, P. Littelmann [10, 11] introduced the path model. Particularly, this model gives a method to compute the multiplicity of a weight μ in an irreducible representation of highest weight λ (a dominant weight), by counting some

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"Lakshmibai-Seshadri" (or LS) paths of shape λ starting from 0 and ending in μ. When $\mathfrak{g}^\vee$ is semi-simple and G is an algebraic group with Lie algebra the Langlands dual $\mathfrak{g}$ of $\mathfrak{g}^\vee$, I. Mirković and K. Vilonen [13] gave a new interpretation of this multiplicity: it is the number of irreducible components (the MV cycles) in some subvariety X_λ^μ of the affine grassmannian $\mathcal{G} = G(\mathbb{C}(\!(t)\!))/G(\mathbb{C}[[t]])$.

S. Gaussent and P. Littelmann [5] gave a link between these two theories (when G is semi-simple). Actually, the LS paths are drawn in a vector space V which is an apartment A of the Bruhat-Tits building of G (over any non archimedean valuated field K, in particular $K = \mathbb{C}(\!(t)\!)$). They replaced the LS paths of shape λ from 0 to μ by "LS galleries" of type λ from 0 to μ. This gives a new "gallery model" for the representations of $\mathfrak{g}^\vee$. Moreover, let ρ be the retraction of the Bruhat-Tits building $\mathcal{I}$ of G over $K = \mathbb{C}(\!(t)\!)$ onto A with center some sector-germ $\mathfrak{S}_{-\infty}$ in A. Then, the image under ρ of a minimal gallery Γ of type λ starting from 0 in $\mathcal{I}$ is a gallery γ in A of type λ which looks much like a LS gallery: it is "positively folded". Conversely, any positively folded gallery γ in A of type λ from 0 to μ is the image under ρ of many minimal galleries Γ in $\mathcal{I}$. These galleries are parametrized by a complex variety X_γ combinatorially defined from γ. Moreover, γ is a LS gallery if, and only if, $\dim(X_\gamma)$ is maximal, and then X_γ is isomorphic to an open subset of a MV cycle in X_λ^μ.

It was natural (and suggested to us by P. Littelmann) to try to generalize this when G is a Kac-Moody group. Actually, G. Rousseau [17] had constructed some building for a Kac-Moody group over a discretely valuated field K. But, the apartments of this microaffine building are not appropriate to define LS paths. So, we construct a new set $\mathcal{I}$ (see 3.15) associated to the Kac-Moody group G over $K = \mathbb{C}(\!(t)\!)$ (or, more generally, over any discretely valuated field K with residue field containing $\mathbb{C}$). By definition, the group $G(K)$ acts on $\mathcal{I}$, LS paths can be drawn on its apartments and the action of $G(K)$ on them is transitive. Unfortunately, any two points in $\mathcal{I}$ are not always in a same apartment: this was already noticed (in a different language and in the affine case) by H. Garland [4], who remarked that Cartan decomposition is true only after some twist (cf. Remark 6.10). Because of this pathological behaviour, $\mathcal{I}$ is called a *hovel*. Moreover, the system of walls in an apartment of $\mathcal{I}$ is not discrete, so the notion of chamber in $\mathcal{I}$ is unusual (but follows an idea of F. Bruhat and J. Tits [3]) and $\mathcal{I}$ is not gallery-connected (see Section 2.2). Therefore, we have to come back to the path model.

Nevertheless, we get good generalizations of Gaussent-Littelmann's results. First, any sector-germ and any point in $\mathcal{I}$ are always in a same apartment (this is equivalent to Iwasawa decomposition, Proposition 3.6). Next, we fix a maximal torus in G and a system of positive roots, this gives us an apartment A in $\mathcal{I}$ and a sector-germ $\mathfrak{S}_{-\infty}$ in A. So, we get a retraction ρ of $\mathcal{I}$ onto A with center $\mathfrak{S}_{-\infty}$ (4.4). Following a definition of M. Kapovich and J. Millson [8], we say that a Hecke path is a piecewise linear path π_1 in A which is positively folded along true walls (see Definition 5.2). Now, an analogue of some results due, in the semisimple case, to Kapovich-Millson or Gaussent-Littelmann may be proven:

Theorem 1.1 (see 6.2). — *If π is a geodesic segment (of shape λ) in $\mathcal{I}$, then $\rho\pi$ is a Hecke path (of shape λ) in A.*

Conversely, any Hecke path π_1 in A is the image under ρ of a geodesic segment π in $\mathcal{I}$. But, if we want a finite dimensional variety of parameters, we can no longer look at segments with a given starting point but rather at segments with a given end. We get (when $K = \mathbb{C}((t))$):

Theorem 1.2 (see 6.3). — *Let π_1 be a Hecke path of shape λ in A with endpoint a special vertex y. Then, there exist geodesic segments π in $\mathcal{I}$ with endpoint y such that $\rho\pi = \pi_1$ and they are parametrized by a Zariski open subset $P(\pi_1, y)$ of $\mathbb{C}^N$, stable under the natural action of $(\mathbb{C}^*)^N$.*

Here, N is the so-called *dual dimension* of π_1 (5.7) and it is maximal (among Hecke paths of shape λ with the same starting and ending points) if and only if π_1 is a LS path.

This result enables us to state that $P(\pi_1, y)$ is isomorphic to a dense open subset of some Mirković-Vilonen cycle. In the semi-simple case, this MV cycle is, up to isomorphism, the classical one associated to the reverse path of π_1.

The paper is organized as follows. In Section 2, we recall some results on Kac-Moody groups and their affine apartments. Actually, in the literature one finds many kinds of Kac-Moody groups. We choose the minimal one, the most algebraic. But (in Section 3.3), we will also have to use the maximal one which appears to be a formal completion of the minimal one and has better commutation relations.

The construction of the hovel $\mathcal{I}$ is explained completely in Section 3 and the first properties are developed in Section 4. The proofs are rather involved but we get all what is needed: in particular, the Iwasawa decomposition (3.6), the retraction with respect to a sector-germ (4.4), the twin building structure of the residue of $\mathcal{I}$ at some point x (4.5) and, for some "good" subsets Ω in apartments, the structure of their fixator (*i.e.* pointwise stabilizer) G_Ω and the transitivity of the action of G_Ω on apartments containing Ω (4.1 to 4.3).

In Section 5, we give the definitions of LS paths, Hecke paths, dual dimension and codimension. We prove the characterization of LS paths as Hecke paths with maximal dual dimension (resp. minimal codimension).

We get in Section 6 the results on paths explained above, in particular Theorems 1.1 and 1.2. The last theorem (Theorem 6.9) asserts that there is, on $\mathcal{I}$, a preorder relation which induces, on each apartment, the preorder given by the Tits cone.

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2. Kac-Moody groups and the apartment

We recall here the main results on Kac-Moody groups (in 2.1). A good reference is [15], see also [17]. We introduce the model apartment of our hovel and call it, in analogy with the classical case, the affine apartment (see 2.2). Because the set of walls is not locally finite anymore, the definition of faces needs the notion of filter (see Sections 2.2.2 to 2.2.4).

2.1. Kac-Moody groups

2.1.1. Kac-Moody algebras

A *Kac-Moody matrix* (or generalized Cartan matrix) is a square matrix $A = (a_{i,j})_{i,j\in I}$, with integer coefficients, indexed by a finite set I and satisfying:

(i) $a_{i,i} = 2 \quad \forall i \in I$,

(ii) $a_{i,j} \leqslant 0 \quad \forall i \neq j$,

(iii) $a_{i,j} = 0 \iff a_{j,i} = 0$.

A *root generating system* [1] is a 5-tuple $\mathcal{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ made of a Kac-Moody matrix A indexed by I, of two dual free $\mathbb{Z}-$modules X (of characters) and Y (of cocharacters) of finite rank $\mathrm{rk}(X)$, a family $(\alpha_i)_{i\in I}$ (of simple roots) in X and a family $(\alpha_i^\vee)_{i\in I}$ (of simple coroots) in Y. They have to satisfy the following compatibility condition: $a_{i,j} = \alpha_j(\alpha_i^\vee)$.

The *Langlands dual* of $\mathcal{S}$ is $\mathcal{S}^\vee = ({}^tA, Y, X, (\alpha_i^\vee)_{i\in I}, (\alpha_i)_{i\in I})$, where ${}^tA$ is the transposed matrix of A.

The *Kac-Moody algebra* $\mathfrak{g} = \mathfrak{g}_\mathcal{S}$ is a complex Lie algebra generated by the standard Cartan subalgebra $\mathfrak{h} = Y \otimes \mathbb{C}$ and the Chevalley generators $(e_i)_{i\in I}$, $(f_i)_{i\in I}$; we shall not explain here the relations, see for instance [6].

The adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ gives a grading on $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta}\ \mathfrak{g}_\alpha)$, where $\Delta \subset X \setminus \{0\} \subset \mathfrak{h}^*$ is the set of *roots* of $\mathfrak{g}$ (with respect to $\mathfrak{h}$). For all $i \in I$, $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$ and $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$. If $Q^+ = \sum_i \ \mathbb{N}\alpha_i$, $\Delta^+ = \Delta \cap Q^+$ and $\Delta^- = -\Delta^+$, one has $\Delta = \Delta^+ \bigsqcup \Delta^-$.

N.B. — For simplicity we shall assume throughout the paper the following condition:

$F(\mathcal{S})$ The family $(\alpha_i)_{i\in I}$ is free in X and the family $(\alpha_i^\vee)_{i\in I}$ is free in Y.

See Section 2.1.5.

Starting from Section 3.7, we shall also assume that the Lie algebra $\mathfrak{g}$ is symmetrizable, that is, endowed with a nondegenerate invariant $\mathbb{C}-$valued symmetric bilinear form.

2.1.2. Weyl group and real roots

Let $V = Y \otimes \mathbb{R} \subset \mathfrak{h}$; every element in X defines a linear form on this $\mathbb{R}-$vector space. For $i \in I$, the formula $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$ defines an involution in V (or $\mathfrak{h}$), more precisely a reflection of hyperplane $\mathrm{Ker}(\alpha_i)$.

The (vectorial) Weyl group W^v is the subgroup of $GL(V)$ generated by the set $\{r_i\}_{i\in I}$. One knows that W^v is a Coxeter group; it stabilizes the lattice Y of V. It also acts on X and stabilizes Δ.

One denotes by $\Phi = \Delta_{\mathrm{re}}$ the set of real roots, those which can be written as $\alpha = w(\alpha_i)$ with $w \in W^v$ and $i \in I$. This set Φ is infinite except in the classical case, where A is a Cartan matrix and $\mathfrak{g}$ is finite-dimensional (reductive). If $\alpha \in \Phi$, then $r_\alpha = w.r_i.w^{-1}$ is well determined by α, independently of the choice of w and of i such that $\alpha = w(\alpha_i)$. For $v \in V$ one has $r_\alpha(v) = v - \alpha(v)\alpha^\vee$, where the coroot $\alpha^\vee \in Y$ associated to α satisfies $\alpha(\alpha^\vee) = 2$. Hence r_α is the reflection with respect to the hyperplane $M(\alpha) = \mathrm{Ker}(\alpha)$ which is called the *wall* of α. The *half-apartment* associated to α is $D(\alpha) = \{v \in V \mid \alpha(v) \geqslant 0\}$.

The set Φ is a system of (real) roots in the sense of [14]. The set Δ is a system of roots in the sense of [1]. The imaginary roots (those in $\Delta_{\mathrm{im}} = \Delta \setminus \Phi$) will not be very much used here. We define $\Phi^\pm = \Phi \cap \Delta^\pm$.

A subset Ψ of Φ (or Δ) is said to be *closed* in Φ (or Δ) if: $\alpha, \beta \in \Psi$, $\alpha+\beta \in \Phi$ (or Δ) $\Rightarrow \alpha+\beta \in \Psi$. The subset Ψ is said to be *prenilpotent* if there exist $w, w' \in W^v$ such that $w\Psi \subset \Delta^+$ and $w'\Psi \subset \Delta^-$. Then Ψ is finite and contained in the subset $w^{-1}(\Phi^+) \cap (w')^{-1}(\Phi^-)$ which is *nilpotent* (*i.e.* prenilpotent and closed).

One denotes by $Q^\vee$ (resp. $P^\vee, Q$) the *"coroot-lattice"* (resp. *"coweight-lattice"*, *"root-lattice"*), *i.e.* the subgroup of Y generated by the $\alpha_i^\vee$ (resp. $P^\vee = \{y \in Y \otimes \mathbb{Q} \mid \alpha_i(y) \in \mathbb{Z}, \forall i \in I\}$, $Q = \sum_i \mathbb{Z}\alpha_i$); one has $Q^\vee \subset Y \subset P^\vee$. Actually, $Q^\vee$, $P^\vee$ or Q is a lattice in V or V^* if and only if the $\alpha_i^\vee$ generate V *i.e.* the α_i generate V^* *i.e.* $|I| = \mathrm{rk}(X) = \dim(V)$. We define the set of *dominant weights* $X^+ = \{\chi \in X \mid \chi(\alpha_i^\vee) \geqslant 0, \forall i \in I\}$ and $X^- = -X^+$. Dually, the set of *dominant coweights* is $Y^+ = \{\lambda \in Y \mid \lambda(\alpha_i) \geqslant 0, \forall i \in I\}$ and $Y^- = -Y^+$.

2.1.3. The Tits cone

The *positive fundamental chamber* $C_f^v = \{u \in V \mid \alpha_i(u) > 0 \quad \forall i \in I\}$ is a nonempty open convex cone. Its closure $\overline{C_f^v}$ is the disjoint union of the *faces* $F^v(J) = \{\, u \in V \mid \alpha_i(u) = 0\ \forall i \in J\,;\ \alpha_i(u) > 0\ \forall i \notin J\}$ for $J \subset I$; one has $C_f^v = F^v(\emptyset)$. We define $V_0 = F^v(I)$, it is a vector subspace. These faces are called vectorial because they are convex cones with base point 0. One says that the face $F^v(J)$ or the set J is *spherical* (or *of finite type*) if the matrix $A(J) = (a_{i,j})_{i,j \in J}$ is a Cartan matrix (in the classical sense), i.e. if $W^v(J) = \langle r_i \mid i \in J\rangle$ is finite. This holds for the chamber C_f^v or its *panels* $F^v(\{i\})$, $\forall i \in I$.

The *Tits cone* is the union $\mathcal{T}$ of the positive *closed-chambers* $w.\overline{C_f^v}$ for $w \in W^v$. Its interior is the *open Tits cone* $\mathcal{T}^o$, disjoint union of the (positive) *spherical faces* $w.F^v(J)$ for w in W^v and J spherical. Both $\mathcal{T}$, $\mathcal{T}^o$ and their closure $\overline{\mathcal{T}}$ are convex cones, stable under W^v. They may be defined as:

$$\mathcal{T} = \{v \in V \mid \alpha(v) < 0 \text{ only for a finite number of } \alpha \in \Delta^+ (\text{or } \Phi^+)\ \},$$

$$\mathcal{T}^o = \{v \in V \mid \alpha(v) \leqslant 0 \text{ only for a finite number of } \alpha \in \Delta^+ (\text{or } \Phi^+)\ \},$$

$$\overline{\mathcal{T}} = \{v \in V \mid \alpha(v) \geqslant 0 \quad \forall \alpha \in \Delta^+_{\mathrm{im}}\ \}.$$

The action of W^v on the positive chambers is simply transitive. The fixator (pointwise stabilizer) or the stabilizer of $F^v(J)$ is $W^v(J)$.

We shall also consider the negative Tits cones $-\mathcal{T}$, $-\mathcal{T}^o$, $-\overline{\mathcal{T}}$ and all negative faces, chambers... which are obtained by change of sign.

Actually, $\mathcal{T}^o \cap -\mathcal{T}^o = \emptyset$ except in the classical case (where $\mathcal{T}^o = -\mathcal{T}^o = V$) and $\mathcal{T} \cap -\mathcal{T} = \{v \in V \mid \alpha(v) = 0$ for almost all $\alpha \in \Phi$ (or Δ)$\}$ is reduced to $V_0 = \bigcap_{\alpha \in \Delta} \mathrm{Ker}(\alpha)$ if no connected component of I is spherical.

2.1.4. The Kac-Moody groups

One considers the (split, complex) Kac-Moody group $G = G_{\mathcal{S}}$ associated to the above root generating system as defined by Tits [22], see also [15, Chapitre 8]. It is actually an affine ind-algebraic-group [9, 7.4.14].

For any field K containing $\mathbb{C}$, the group $G(K)$ of the points of G in K is generated by the following subgroups:

- the fundamental torus $T(K)$ where $T = \mathrm{Spec}(\mathbb{Z}[X])$, hence $T(K)$ is isomorphic to the group $(K^*)^n = (K^*) \otimes_{\mathbb{Z}} Y$ and the character (resp. cocharacter) group of T is X (resp. Y).
- the root subgroups $U_\alpha(K)$ for $\alpha \in \Phi$, each isomorphic to the additive group $(K, +)$ by an isomorphism (of algebraic groups) x_α.

Actually, we consider an isomorphism $x_\alpha : K \simeq \mathfrak{g}_\alpha \otimes K \to U_\alpha(K)$ where the additive group $\mathfrak{g}_\alpha \otimes K$ is identified with K by the choice of a Chevalley generator e_α of the $1-$dimensional complex space $\mathfrak{g}_\alpha$.

Let M be an $\mathfrak{h}-$diagonalizable $\mathfrak{g}-$module with weights in X, and where the action of each $\mathfrak{g}_\alpha$ is locally nilpotent (*e.g.* $\mathfrak{g}$ itself). Then $G(K)$ acts on $M \otimes K$: the torus $T(K)$ acts via the character λ on $M_\lambda \otimes K$ and the action of $x_\alpha(a)$ for $a \in \mathfrak{g}_\alpha \otimes K$ is the exponential of the action of a.

2.1.5. About the freedom condition $F(\mathcal{S})$

This condition is used in 2.1.3 to show that C_f^v is nonempty, and in 5.3 below for the existence of ρ_{Φ^+}. If it fails, then W^v as defined in 2.1.2 could be smaller than wanted (finite), and the roots of $\mathfrak{g}_{\mathcal{S}}$ could not be defined by the adjoint action of $\mathfrak{h}$. But, actually, $F(\mathcal{S})$ is not necessary to define $\mathfrak{g}_{\mathcal{S}}$ or $G_{\mathcal{S}}$ [15].

In [6], [9] and [10], the condition $F(\mathcal{S})$ and a minimality condition for the rank of X: $\mathrm{rk}(X) = |I| + \mathrm{corank}(A)$ are assumed. Kumar requires moreover a simple-connectedness condition:

$$(SC) \qquad \sum_{i \in I} \mathbb{Z}\alpha_i^\vee \text{ is cotorsion-free in } Y.$$

For a root generating system $\mathcal{S}$, define $\mathcal{S}_{sc} = (A, X_{sc}, Y_{sc}, (\alpha_i)_{i\in I}, (\alpha_i^*)_{i\in I})$ by $X_{sc} = X \oplus \mathbb{Z}^I$, Y_{sc} is the dual of X_{sc} and $\alpha_i^*(x + (n_j)_{j\in I}) = \alpha_i^\vee(x) + n_i$. The group $G_{\mathcal{S}}$ is a quotient of $G_{\mathcal{S}_{sc}}$ by a subtorus of the torus $T_{\mathcal{S}_{sc}}$, central in $G_{\mathcal{S}_{sc}}$.

Starting from $\mathcal{S}$ satisfying $F(\mathcal{S})$, then $\mathcal{S}_{sc}$ satisfies $F(\mathcal{S}_{sc})$ and (SC). The group $G_{\mathcal{S}_{sc}}$ is the direct product of a torus and a group with the properties assumed by Kumar. So, there is no trouble in using the results of [9] for the groups we define.

2.1.6. Some commuting relations

We present here some relations in the group G, for more details, see [17, 1.5 and 1.6].

If $\{\alpha,\beta\}$ is a prenilpotent pair of roots in Φ (hence $\alpha\neq-\beta$), one denotes by $]\alpha,\beta[$ the finite set of the roots $\gamma=p\alpha+q\beta\in\Phi$ with p and q strictly positive integers and $[\alpha,\beta]=]\alpha,\beta[\cup\{\alpha,\beta\}$; we choose any total order on this set. Then, the product map: $\prod_{\gamma\in[\alpha,\beta]}U_\gamma(K)\to G(K)$ is a bijection onto the group $U_{[\alpha,\beta]}(K)$ generated by these $U_\gamma(K)$; it is actually an isomorphism of algebraic varieties. The commutator group $[U_\alpha(K),U_\beta(K)]$ is contained in $U_{]\alpha,\beta[}(K)$. More precisely, for $u,v\in K$, one has: $[x_\alpha(u),x_\beta(v)]=\prod x_\gamma(C_{p,q}u^pv^q)$ where the product runs over the $\gamma=p\alpha+q\beta\in]\alpha,\beta[$ (in the fixed order) and the $C_{p,q}$ are integers.

The group $T(K)$ normalizes $U_\alpha(K)$: if $t\in T(K)$ and $u\in K$ one has $tx_\alpha(u)t^{-1}=x_\alpha(\alpha(t)u)$. The subgroup $G^{(\alpha)}$ of G generated by U_α, $U_{-\alpha}$ and T is, up to its center, isomorphic to PGL_2. In particular, for $\alpha\in\Phi$ and $u\in U_\alpha(K)$, the set $U_{-\alpha}(K)uU_{-\alpha}(K)$ contains a unique element $m(u)$ conjugating $U_\beta(K)$ to $U_{r_\alpha(\beta)}(K)$ for each $\beta\in\Phi$. Moreover, for $v,v'\in K^*$:

$$\begin{aligned}
&m(x_\alpha(v))=m(x_{-\alpha}(v^{-1}))=x_{-\alpha}(-v^{-1})x_\alpha(v)x_{-\alpha}(-v^{-1})=\cdots\\
&\qquad\qquad\qquad\cdots=\alpha^\vee(v)m(x_\alpha(1))=m(x_\alpha(1))\alpha^\vee(v^{-1})\\
&m(x_\alpha(v))x_\alpha(v')m(x_\alpha(v))^{-1}=x_{-\alpha}(v^{-2}v')\\
&m(x_\alpha(v))^2=\alpha^\vee(-1).
\end{aligned}$$

If N is the normalizer of T in G, then $N(K)$ is the group generated by $T(K)$ and the $m(u)$ for all $\alpha\in\Phi$ and all $u\in U_\alpha(K)$.

Lemma 2.1. — *([17] or [15]) There exists an algebraic homomorphism ν^v from N onto W^v such that $\nu^v(m(u))=r_i$ for $u\in U_{\pm\alpha_i}(K)$ and* $\mathrm{Ker}(\nu^v)=T$. *As K is infinite, $N(K)$ is the normalizer of $T(K)$ in $G(K)$ and all maximal split subtori of $G(K)$ are conjugate of $T(K)$.*

The conjugacy action of N on T is given by ν^v where W^v acts on T through its action on X or Y.

2.1.7. Borel subgroups

The subgroup $U^+(K)$ of $G(K)$ is generated by the groups $U_\alpha(K)$ for $\alpha\in\Phi^+$; it is normalized by $T(K)$. We define the same way $U^-(K)$ and $U(\Psi)(K)$ for any subset Ψ of Φ^+ or Φ^-.

The groups $B(K)=B^+(K)=T(K).U^+(K)$, $B^-(K)=T(K).U^-(K)$ are the *standard (positive, negative) Borel subgroups* of $G(K)$.

One has $U^+(K)\cap B^-(K) = U^-(K)\cap B^+(K) = \{1\}$; more generally, one has the following decompositions.

Bruhat decompositions:

$$G(K) = U^+(K)N(K)U^+(K) = U^-(K)N(K)U^-(K)$$

Moreover, the maps from $N(K)$ onto $U^\pm(K)\backslash G(K)/U^\pm(K)$ are one to one.

Birkhoff decompositions:

$$G(K) = U^+(K)N(K)U^-(K) = U^-(K)N(K)U^+(K)$$

Moreover, the maps from $N(K)$ onto $U^\pm(K)\backslash G(K)/U^\mp(K)$ are one to one.

2.2. The affine apartment

The affine apartment $\mathbb{A}$ is V considered as an affine space.

2.2.1. Affine Weyl group and preorder relation

The group W^v acts $\mathbb{Z}-$linearly on Y, hence it acts $\mathbb{R}-$linearly on $\mathbb{A} = V$. One has also an action of V by translations. Finally, one obtains an affine action on $\mathbb{A}$ of the semi-direct product $W_\mathbb{R} = W^v \ltimes V$.

For $\alpha \in \Phi$ and $k \in \mathbb{Z}$, $M(\alpha, k) = \{v \in \mathbb{A} \mid \alpha(v) + k = 0\}$ is the *wall* associated to (α, k), it is closed in $\mathbb{A}$. One has $M(\alpha, k) = M(-\alpha, -k)$.

For $\alpha \in X\backslash\{0\}$ and $k \in \mathbb{Z}$, we define $D(\alpha, k) = \{v \in \mathbb{A} \mid \alpha(v)+k \geqslant 0\}$, it is closed in $\mathbb{A}$. When $\alpha \in \Phi$, we call $D(\alpha, k)$ the *half-apartment* associated to (α, k) and the set $D^\circ(\alpha, k) = D(\alpha, k) \setminus M(\alpha, k) = V \setminus D(-\alpha, -k)$ is the *open-half-apartment* associated to (α, k).

The *reflection* associated to the wall $M(\alpha, k)$ is $r_{\alpha,k} : \mathbb{A} \to \mathbb{A}$ given by the formula:

$$r_{\alpha,k}(y) = r_\alpha(y) - k\alpha^\vee.$$

The group generated by the $r_{\alpha,k}$ is $W = W^v \ltimes Q^\vee \subset W_\mathbb{R}$.

The subgroup of $W_\mathbb{R}$ of all elements stabilizing the set of walls is $W_P = W^v \ltimes P^\vee$. One defines also $W_Y = W^v \ltimes Y$ and one has:

$$W \subset W_Y \subset W_P \subset W_\mathbb{R} = W^v \ltimes V.$$

Definition 2.2. — *The affine space $\mathbb{A}$ together with its walls and its Tits cone is the affine apartment of G associated to T, its (affine) Weyl group $W = W(\mathbb{A})$ is generated by the reflections with respect to the walls.*

As the Tits cone $\mathcal{T}$ is convex, we can define a *preorder-relation* on $\mathbb{A}$ given by $x \leqslant y \ \Leftrightarrow \ y - x \in \mathcal{T}$. This is a genuine (= antisymmetric) order relation only when Φ generates V and the Kac-Moody matrix A has no factor of finite type.

For $x \in \mathbb{A}$, the set Δ_x of all roots α such that $\alpha(x) \in \mathbb{Z}$ is a closed subsystem of roots of Δ in the sense of [1] Section 5.1. The associated Weyl group $W_x^{\min}$ is the subgroup of W generated by all the reflections associated to the walls containing x. It is isomorphic to its image W_x^v in W^v and is a Coxeter group, as shown in [loc. cit.; 5.1.12]. The canonical generators of W_x^v are the r_α for α simple in $\Phi_x^+ = \Delta_x \cap \Phi^+$; their number may be infinite.

The point x is *special* when $\Phi_x = \Delta_x \cap \Phi$ is equal to Φ, *i.e.* when $W_x^v = W^v$.

2.2.2. Faces

The faces in $\mathbb{A}$ are associated to the above systems of walls and half-apartments. As in [3], they are no longer subsets of $\mathbb{A}$, but filters of subsets of $\mathbb{A}$.

Definition 2.3 ([3], [18] or [17]). — *A filter in a set E is a nonempty set F of nonempty subsets of E, such that, if $S, S' \in F$ then $S \cap S' \in F$ and, if $S' \supset S \in F$ then $S' \in F$. If Z is a nonempty subset of E, the set $F(Z)$ of subsets of E containing Z is a filter (usually identified with Z). If $E \subset E'$, to any filter F in E is associated the filter $F_{E'}$ in E' consisting of all subsets of E' containing some S in F; one usually makes no difference between F and $F_{E'}$.*

A filter F is said to be contained in another filter F': $F \subset F'$ (resp. in a subset Z in E: $F \subset Z$) if and only if any set in F' (resp. if Z) is in F. The union of a family of filters in E is the filter consisting of subsets which are in all the filters. Note that these definitions are opposite the natural ones.

A group Γ acting on E fixes pointwise (resp. stabilizes) a filter F, if and only if every γ in Γ fixes pointwise some $S \in F$ (resp. for all γ in Γ and all S in F, $\gamma S \in F$).

If E is a topological space, the closure of a filter F in E is the filter $\overline{F}$ consisting of all subsets of E containing the closure of a set in F.

If F is a subset of E containing an element x in its closure, the *germ* of F in x is the filter $\mathrm{germ}_x(F)$ consisting of all subsets of E which are intersections of F and neighbourhoods of x.

If E is a real affine space and $x \neq y \in E$, then the *segment-germ* $[x, y)$ is the germ of the segment $[x, y]$ in x.

All the above definitions for filters are compatible with the corresponding definitions for subsets and the identification of a subset Z with the filter $F(Z)$.

We say that a family $\mathcal{F}$ of filters *generates* a filter Ω if: a set S is in Ω if and only if it is in some filter $F \in \mathcal{F}$. If $\mathcal{B}$ is a basis of the filter Ω, then the (filters canonically associated to the) sets in $\mathcal{B}$ generate Ω.

The *enclosure* $cl(F)$ of a filter F of subsets of $\mathbb{A}$ is the filter made of the subsets of $\mathbb{A}$ containing any intersection of half-spaces $D(\alpha, k)$ (for $\alpha \in \Delta$ and $k \in \mathbb{Z}$), which is in F. With this definition, the enclosure of a subset Ω is the closed subset intersection of all $D(\alpha, k)$ (for $\alpha \in \Delta$ and $k \in \mathbb{Z}$) containing Ω. For $\mathcal{P}$ a non empty subset of $X \setminus \{0\}$, we define also the $\mathcal{P}-$*enclosure* $cl_{\mathcal{P}}(F)$ by the same definition, just replacing Δ by $\mathcal{P}$.

Definition 2.4. — *A face F in the apartment $\mathbb{A}$ is associated to a point $x \in \mathbb{A}$ and a vectorial face F^v in V; it is called spherical according to the nature of F^v. More precisely, a subset S of $\mathbb{A}$ is an element of the face $F(x, F^v)$ if and only if it contains an intersection of half-spaces $D(\alpha, k)$ or $D^\circ(\alpha, k)$ (for $\alpha \in \Delta$ and $k \in \mathbb{Z}$) which contains $\Omega \cap (x + F^v)$, where Ω is an open neighborhood of x in $\mathbb{A}$. The enclosure of a face $F = F(x, F^v)$ is its closure: the closed-face $\overline{F}$; it is the enclosure of the local-face in x,* $\mathrm{germ}_x(x + F^v)$.

Actually, in the classical case where Φ is finite, this definition is still valid: $F(x, F^v)$ is a subset Z of $\mathbb{A}$ (more precisely: is the filter of subsets containing a subset Z of $\mathbb{A}$), and this subset Z is a face in the sense of [3, §1] or [2, 6.1].

Note that the union of the faces $F(x, F^v)$ is not always the filter of neighborhoods of x; it is contained in $(x + \mathcal{T}) \cup (x - \mathcal{T})$ if x is special.

2.2.3. Chambers, panels...

There is an order on the faces: the assertions "F is a face of F' ", "F' covers F " and "$F \leqslant F'$ " are by definition equivalent to $F \subset \overline{F'}$.

Any point $x \in \mathbb{A}$ is contained in a unique face $F(x, V_0)$ which is minimal (but seldom spherical); x is a *vertex* if and only if $F(x, V_0) = \{x\}$. When Φ generates V (*i.e.* $\mathrm{rk}(X) = |I|$), a special point is a vertex, but the converse is not true.

The *dimension* of a face F is the smallest dimension of an affine space generated by some $S \in F$. The (unique) such affine space E of minimal dimension is the *support* of F. Any $S \in F$ contains a non empty open subset of E.

A *chamber* (or alcove) is a maximal face, or, equivalently, a face such that all its elements contain a nonempty open subset of $\mathbb{A}$ or a face of dimension $\mathrm{rk}(X) = \dim(\mathbb{A})$.

A *panel* is a spherical face maximal among faces which are not chambers, or, equivalently, a spherical face of dimension $n-1$. Its support is a wall.

So, the set of spherical faces of $\mathbb{A}$ completely determines the set $\mathcal{H}$ of walls.

A *wall of a chamber* C is the support M of a panel F covered by C. Two chambers are called *adjacent* (along F or M) if they cover a common panel (F of support M). But there may exist a chamber covering no panel, and hence having no wall. So, $\mathbb{A}$ is far from being "gallery-connected".

2.2.4. Sectors

A *sector* in $\mathbb{A}$ is a $V-$translate $\mathfrak{s} = x + C^v$ of a vectorial chamber $C^v = \pm w.C^v_f$ ($w \in W^v$), x is its *base point* and C^v its *direction*. Two sectors have the same direction if and only if they are conjugate by $V-$translation, and if and only if their intersection contains another sector.

The *sector-germ* of a sector $\mathfrak{s} = x + C^v$ in $\mathbb{A}$ is the filter $\mathfrak{S}$ of subsets of $\mathbb{A}$ consisting of the sets containing a $V-$translate of $\mathfrak{s}$, it is well determined by the direction C^v. So the set of translation classes of sectors in $\mathbb{A}$, the set of vectorial chambers in V and the set of sector-germs in $\mathbb{A}$ are in canonical bijection.

The sector-germ associated to the positive (resp. negative) fundamental chamber C^v_f (resp. $-C^v_f$) is called the positive (resp. negative) fundamental sector-germ and is denoted by $\mathfrak{S}_{+\infty}$ (resp. $\mathfrak{S}_{-\infty}$).

A *sector-face* in $\mathbb{A}$ is a $V-$translate $\mathfrak{f} = x + F^v$ of a vectorial face $F^v = \pm wF^v(J)$. The *sector-face-germ* of $\mathfrak{f}$ is the filter $\mathfrak{F}$ of subsets containing a translate $\mathfrak{f}'$ of $\mathfrak{f}$ by an element of F^v (*i.e.* $\mathfrak{f}' \subset \mathfrak{f}$). If F^v is spherical, then $\mathfrak{f}$ and $\mathfrak{F}$ are also called spherical. The sign of $\mathfrak{f}$ and $\mathfrak{F}$ is the sign of F^v.

3. The hovel, definition

To define something like an affine building associated to the Kac-Moody group G and the apartment $\mathbb{A}$ of $T(K)$, we have to define the action of $N(K)$ on $\mathbb{A}$ and the fixator $\hat{P}_x$ in $G(K)$ of a point x in $\mathbb{A}$, *i.e.* the associated parahoric subgroup. This fixator $\hat{P}_x$ should contain the fixator $\hat{N}_x$ of x in $N(K)$ and the groups $U_{\alpha,k}$ for $x \in D(\alpha, k)$. When x is 0 ("origin" of $\mathbb{A}$), the group $\hat{P}_x$ should be $G(\mathcal{O})$, so that the orbit of 0 in the "building" is the affine grassmannian $\mathcal{G} = G(K)/G(\mathcal{O})$ (see Example 3.14 below).

But, we have also to define and study parahoric subgroups associated to more general points or faces in $\mathbb{A}$ and this will lead to difficulties. Moreover, the expected Bruhat decomposition for parahoric subgroups is actually false in our case (see Remark 6.10). So, the "building" we can construct has bad properties, therefore, we call it an *hovel* (in french "masure").

Here, we give an overview of the present section. First, we describe the action of $N(K)$ on the apartment $\mathbb{A}$ (3.1). Then, given a filter of subsets Ω in $\mathbb{A}$, we define a subgroup $P_\Omega^{\min}$ of $G(K)$ (3.2), in the same way as in [3]. But, due to some bad commutation relations in $G(K)$, we have to work in larger groups, the formal completions of G. There are two ways of doing it which lead to two groups P_Ω^{pm} and P_Ω^{nm} (still in $G(K)$) both containing $P_\Omega^{\min}$ (see 3.4). All these groups are defined by generators. The ideal situation is when they coincide, but in general, we need another group to compare them. So, we define a fourth group $\widetilde{P}_\Omega$ as the stabilizer of some subalgebra and some submodule for the action of $G(K)$ on highest weight representations (in 3.6); it contains all the previous ones. However, $\widetilde{P}_\Omega$ is a bit too big to be the fixator of Ω for the action of $G(K)$ on the hovel. We get the "parahoric group" P_Ω by assuming that $\mathfrak{g}$ is symmetrizable and by taking the fixator of some subalgebra for the action of $\widetilde{P}_\Omega$. Finally, the "right candidate", as a fixator of Ω, is the group $\hat{P}_\Omega$ obtained by adding to P_Ω the fixator $\hat{N}_\Omega$ of Ω in $N(K)$ (for its action on $\mathbb{A}$), see 3.7.

3.1. Action of $N(K)$ on $\mathbb{A}$

We suppose now the field K endowed with a discrete valuation ω, assumed normalized: $\omega(K^*) = \mathbb{Z}$. The ring of integers is $\mathcal{O}$; we choose a uniformizing parameter ϖ, so $\omega(\varpi) = 1$, $\mathcal{O}^* = \mathcal{O} \setminus \varpi\mathcal{O}$ and the residue field is $\kappa = \mathcal{O}/\varpi\mathcal{O}$. Moreover, we assume that κ contains $\mathbb{C}$ (so, if K is complete for ω, then $K = \kappa(\!(\varpi)\!)$ and $\mathcal{O} = \kappa[[\varpi]]$). For the definition of the hovel for a Kac-Moody group G over any valuated field, one needs more knowledge about G, it should appear in [19].

For $\alpha \in \Phi, u \in U_\alpha(K)$, $u \neq 1$, we define: $\varphi_\alpha(u) = \omega(t)$, if $u = x_\alpha(t)$ with $t \in K$. For all $k \in \mathbb{R} \cup \{+\infty\}$, the set $U_{\alpha,k} = \varphi_\alpha^{-1}([k, +\infty])$ is a subgroup of $U_\alpha(K)$ and $U_{\alpha,\infty} = \{1\}$. See also [17, 2.2].

The group $T(K)$ acts on $\mathbb{A}$ by translations: if $t \in T(K)$, $\nu(t)$ is the element in V such that $\chi(\nu(t)) = -\omega(\chi(t))$, $\forall \chi \in X$. This action is W^v-equivariant.

The following lemma is a trivial consequence of the corresponding result 2.9.2 in [17].

Lemma 3.1. — *There exists an action ν of $N(K)$ on $\mathbb{A}$ which induces the preceding one on $T(K)$ and such that for $n \in N(K)$, $\nu(n)$ is an affine map with associated linear map $\nu^v(n)$.*

Remarks 3.2. —
1) The image of $N(K)$ in $Aut(\mathbb{A})$ is $\nu(N) = W_Y$. The kernel $H = \mathrm{Ker}(\nu) \subset T(K)$ is $H = \mathcal{O}^* \otimes Y = T(\mathcal{O})$.
2) By construction $\nu(N(\mathbb{C}))$ fixes 0, the origin of $\mathbb{A}$, so, $\nu(m(x_\alpha(1)))$ is the reflection $r_\alpha = r_{\alpha,0}$ with respect to the wall $M(\alpha,0)$. Moreover, $m(x_\alpha(u)) = \alpha^\vee(u)m(x_\alpha(1))$, hence the image $\nu(m(x_\alpha(u)))$ is the reflection $r_{\alpha,\omega(u)}$ with respect to the wall $M(\alpha,\omega(u))$, as by definition one has: $\alpha(\nu(\alpha^\vee(u))) = -\omega(\alpha(\alpha^\vee(u))) = -\omega(u^2) = -2\omega(u)$.

3.2. First objects associated to Ω and the group $P_\Omega^{\min}$

Let Ω be a filter of subsets in $\mathbb{A}$. For $\alpha \in \Delta$, let $f_\Omega(\alpha) = \inf\{k \in \mathbb{Z} \mid \Omega \subset D(\alpha,k)\} = \inf\{k \in \mathbb{Z} \mid \alpha(\Omega) + k \subset [0,+\infty)\} \in \mathbb{Z} \cup \{+\infty\}$; by this second equality, f_Ω is defined on X. The function f_Ω is concave[3]: $\forall \alpha,\beta \in X$, $f_\Omega(\alpha+\beta) \leqslant f_\Omega(\alpha) + f_\Omega(\beta)$ and $f_\Omega(0) = 0$; in particular $f_\Omega(\alpha) + f_\Omega(-\alpha) \geqslant 0$. We say that Ω is *narrow* (resp. *almost open*) if and only if $f_\Omega(\alpha) + f_\Omega(-\alpha) \in \{0,1\}$ (resp. $\neq 0$), $\forall \alpha \in \Phi$. The filter Ω is almost open if and only if it is not contained in any wall, this is true for a chamber. A point or a face is narrow. Actually, in the classical case, Ω is narrow if and only if it is included in the closure of a chamber.

We define U_Ω as the subgroup of $G(K)$ generated by the groups $U_{\alpha,\Omega} = U_{\alpha,f_\Omega(\alpha)}$ for $\alpha \in \Phi$, and $U_\Omega^\pm = U_\Omega \cap U^\pm(K)$. For $\alpha \in \Phi$, $U_\Omega^{(\alpha)}$ $(\subset U_\Omega)$ is generated by $U_{\alpha,\Omega}$ and $U_{-\alpha,\Omega}$; $N_\Omega^{(\alpha)} = N(K) \cap U_\Omega^{(\alpha)}$. The group N_Ω^u $(\subset N(K) \cap U_\Omega)$ is generated by all $N_\Omega^{(\alpha)}$ for $\alpha \in \Phi$.

All these groups are normalized by H. In particular, one can define the groups $N_\Omega^{\min} = H.N_\Omega^u$ and $P_\Omega^{\min} = H.U_\Omega$. These groups depend only on the enclosure of Ω (not on Ω itself).

Lemma 3.3. — *Let Ω be a filter of subsets in $\mathbb{A}$ and $\alpha \in \Phi$ a root.*

1) $U_\Omega^{(\alpha)} = U_{\alpha,\Omega}.U_{-\alpha,\Omega}.N_\Omega^{(\alpha)} = U_{-\alpha,\Omega}.U_{\alpha,\Omega}.N_\Omega^{(\alpha)}$.

2) *If* $f_\Omega(\alpha)+f_\Omega(-\alpha)>0$, *then* $N_\Omega^{(\alpha)}\subset H$. *If* $f_\Omega(\alpha)=-f_\Omega(-\alpha)=k$, *then* $\nu(N_\Omega^{(\alpha)})=r_{\alpha,k}$.
3) $N_\Omega^{(\alpha)}$ *fixes* Ω *i.e.* $\forall n\in N_\Omega^{(\alpha)}$, $\exists S\in\Omega$ *pointwise fixed by* $\nu(n)$.

Consequence. — The group $W_\Omega^{\min}=N_\Omega^{\min}/H$ is isomorphic to its image W_Ω^v in W^v, it is generated by the reflections $r_{\alpha,k}$ for which $\Omega\subset M(\alpha,k)$ ($\alpha\in\Phi$, $k\in\mathbb{Z}$). The group $N_\Omega^{\min}$ is included in the group $\hat{N}_\Omega$, fixator in $N(K)$ of Ω which normalizes H, U_Ω and $P_\Omega^{\min}$. The group $\hat{W}_\Omega=\hat{N}_\Omega/H$ is also isomorphic to a subgroup of W^v.

Proof. — Parts 1) and 2) are proved by an easy computation in SL_2 or PGL_2; one can also refer to [3, 6.4.7] where a more complicated result (non split case) is proved. Clearly, 3) is a consequence of 2). □

We define $\mathfrak{g}_\Omega=\mathfrak{h}_{\mathcal{O}}\bigoplus(\oplus_{\alpha\in\Delta}\,\mathfrak{g}_{\alpha,\Omega})$, where $\mathfrak{h}_{\mathcal{O}}=\mathfrak{h}\otimes_{\mathbb{C}}\mathcal{O}$, $\mathfrak{g}_{\alpha,\Omega}=\mathfrak{g}_{\alpha,f_\Omega(\alpha)}$ and (in general) $\mathfrak{g}_{\alpha,k}=\mathfrak{g}_\alpha\otimes_{\mathbb{C}}\{t\in K\mid\omega(t)\geqslant k\}$. This is a sub-$\mathcal{O}$-Lie-algebra of $\mathfrak{g}_K=\mathfrak{g}\otimes_{\mathbb{C}}K$.

The Lie algebra $\mathfrak{g}_\Omega$ depends only on the enclosure of Ω (not on Ω itself). This is also true for the algebras and groups defined above in the Consequence of Lemma 3.3 (except for $\hat{N}_\Omega$ and $\hat{W}_\Omega$) and below in Sections 3.3 and 3.4.

If Ω is bounded, then $\mathfrak{g}_\Omega$ is a lattice in $\mathfrak{g}_K$.

Let M be a $\mathfrak{g}$-module of highest weight (resp. lowest weight) $\Lambda\in X$, then M is the sum of its weight spaces: $M=\oplus_{\lambda\in X}\,M_\lambda$. We define $M_\Omega=\oplus_{\lambda\in X}\,M_{\lambda,\Omega}$, where $M_{\lambda,\Omega}=M_{\lambda,f_\Omega(\lambda)}$ and (in general) $M_{\alpha,k}=M_\alpha\otimes_{\mathbb{C}}\{t\in K\mid\omega(t)\geqslant k\}$. This is a sub$-\mathfrak{g}_\Omega-$module of $M\otimes K$, and a lattice when Ω is bounded.

If the module is integrable, then $\Lambda\in X^+$ (resp. $\Lambda\in X^-$) and $G(K)$ acts on $M\otimes K$. As we are in equal characteristic 0, it is clear that U_Ω stabilizes M_Ω.

3.3. Maximal Kac-Moody groups

1) The *positively-maximal Kac-Moody algebra* associated to $\mathfrak{g}$ is the Lie algebra $\hat{\mathfrak{g}}^p=(\oplus_{\alpha\in\Delta^-}\;\mathfrak{g}_\alpha)\oplus\mathfrak{h}\oplus\hat{\mathfrak{n}}^+$ where $\hat{\mathfrak{n}}^+=\prod_{\alpha\in\Delta^+}\;\mathfrak{g}_\alpha$ is the completion of $\mathfrak{n}^+=\oplus_{\alpha\in\Delta^+}\;\mathfrak{g}_\alpha$ [9].
2) The *positively-maximal Kac-Moody group* G^{pmax} is defined in [9] (under the name $\mathcal{G}$); it contains G as a subgroup (G is denoted by $\mathcal{G}^{\min}$ by Kumar). For any closed subset Ψ of Δ^+, G^{pmax} contains the pro-unipotent pro-group $U^{\max}(\Psi)$ with Lie algebra $\hat{\mathfrak{n}}(\Psi)=$

$\prod_{\alpha\in\Psi}\ \mathfrak{g}_\alpha$; *i.e.* $U^{\max}(\Psi)(K)=\prod_{\alpha\in\Psi}\ U_\alpha(K)$ where $U_\alpha(K)$ is isomorphic, via an isomorphism x_α, to $\mathfrak{g}_\alpha\otimes K$ (already defined when α is real).

One has the *Bruhat decomposition*:

$$G^{pmax}(K)=\coprod_{n\in N(K)} U^{\max}(\Delta^+)(K)nU^{\max}(\Delta^+)(K),$$

and the *Birkhoff decomposition*:

$$G^{pmax}(K)=\coprod_{n\in N(K)} U^-(K)nU^{\max}(\Delta^+)(K).$$

Moreover,

$$U^-(K)\cap N(K)U^{\max}(\Delta^+)(K)=N(K)\cap U^{\max}(\Delta^+)(K)=\{1\}.$$

N.B. — In all the preceeding or following notations, a sign $^+$ may replace (Ψ) when $\Psi=\Delta^+$.

3) The following subalgebras or subgroups associated to a filter Ω are also defined:
   - $\hat{\mathfrak{g}}^p_\Omega=\mathfrak{n}^-_\Omega\oplus\mathfrak{h}_\mathcal{O}\oplus\hat{\mathfrak{n}}^+_\Omega$, where $\mathfrak{n}^-_\Omega=\oplus_{\alpha\in\Delta^-}\ \mathfrak{g}_{\alpha,\Omega}$ and $\hat{\mathfrak{n}}_\Omega(\Psi)=\cup_{S\in\Omega}\left(\prod_{\alpha\in\Psi}\ \mathfrak{g}_{\alpha,S}\right)$;
   - $U^{\max}_\Omega(\Psi)=\cup_{S\in\Omega}\ \left(\prod_{\alpha\in\Psi}\ U_{\alpha,S}\right)$, where $U_{\alpha,S}=U_{\alpha,f_S(\alpha)}$ is $x_\alpha(\mathfrak{g}_{\alpha,S})$; as we are in equal characteristic zero, the Campbell-Hausdorf formula proves that this is a subgroup of $U^{\max}(\Psi)(K)$;
   - $U^{pm}_\Omega(\Psi)=G(K)\cap U^{\max}_\Omega(\Psi)$, actually $U^{pm}_\Omega(\Psi)=U^+(K)\cap U^{\max}_\Omega(\Psi)$ because by [9, 7.4.3], $U^+(K)=G(K)\cap U^{\max}(\Delta^+)(K)$. We have $U^{pm}_\Omega(\Psi)=\cup_{S\in\Omega}\ U^{pm}_S(\Psi)$ and $U^{pm}_{\Omega\cup\Omega'}(\Psi)=U^{pm}_\Omega(\Psi)\cap U^{pm}_{\Omega'}(\Psi)$.

4) Let α be a simple root, then by [9, 6.1.2, 6.1.3], $U^{\max +}(K)=U_\alpha(K)\ltimes U^{\max}(\Delta^+\setminus\{\alpha\})(K)$. Using the same proof, one can show that $U^{\max +}_\Omega=U_{\alpha,\Omega}\ltimes U^{\max}_\Omega(\Delta^+\setminus\{\alpha\})$ and, intersecting with $G(K)$, one gets $U^{pm+}_\Omega=U_{\alpha,\Omega}\ltimes U^{pm}_\Omega(\Delta^+\setminus\{\alpha\})$.

   The groups $U^{\max}_\Omega(\Delta^+\setminus\{\alpha\})$ and $U^{pm}_\Omega(\Delta^+\setminus\{\alpha\})$ above are normalized by $H.U^{(\alpha)}_\Omega$ and $U^{\max}(\Delta^+\setminus\{\alpha\})(K)$ is normalized by

$$G^{(\alpha)}(K)=\langle\, T(K),U_\alpha(K),U_{-\alpha}(K)\,\rangle.$$

5) One has also to consider the *negatively-maximal Kac-Moody algebra* associated to $\mathfrak{g}$, $\hat{\mathfrak{g}}^n=(\oplus_{\alpha\in\Delta^+}\ \ \mathfrak{g}_\alpha)\oplus\mathfrak{h}\oplus(\prod_{\alpha\in\Delta^-}\ \ \mathfrak{g}_\alpha)$ and the associated *negatively-maximal Kac-Moody group* G^{nmax}. More generally, one can change p to n and $\pm$ to $\mp$ in 1),2),3), and 4) above in order to obtain similar groups (with similar properties) in the negative case.

3.4. The groups P_Ω^{pm} and P_Ω^{nm}

Proposition 3.4. — *Let Ω be a filter of subsets in $\mathbb{A}$. We have 3 subgroups of $G(K)$ associated to Ω and independent of the choice of a set of positive roots in its W^v-conjugacy class:*

1) *The group U_Ω (generated by all $U_{\alpha,\Omega}$) is equal to $U_\Omega = U_\Omega^-.U_\Omega^+.N_\Omega^u = U_\Omega^+.U_\Omega^-.N_\Omega^u$.*
2) *The group U_Ω^{pm} generated by the groups U_Ω and U_Ω^{pm+} is equal to $U_\Omega^{pm} = U_\Omega^{pm+}.U_\Omega^-.N_\Omega^u$.*
3) *Symmetrically, the group U_Ω^{nm} generated by U_Ω and U_Ω^{nm-} is equal to $U_\Omega^{nm} = U_\Omega^{nm-}.U_\Omega^+.N_\Omega^u$.*
4) *One has:*
   i) $U_\Omega \cap N(K) = N_\Omega^u$
   ii) $U_\Omega^{pm} \cap N(K) = N_\Omega^u$
   iii) $U_\Omega \cap (N(K).U^\pm(K)) = N_\Omega^u.U_\Omega^\pm$
   iv) $U_\Omega^{pm} \cap (N(K).U^+(K)) = N_\Omega^u.U_\Omega^{pm+}$
   v) $U_\Omega \cap U^\pm(K) = U_\Omega^\pm$
   vi) $U_\Omega^{pm} \cap U^+(K) = U_\Omega^{pm+}$

   and symmetrically for U_Ω^{nm}.

Remarks 3.5. — The group $H = T(\mathcal{O})$ normalizes also U_Ω^{pm} and U_Ω^{nm}, moreover, $P_\Omega^{\min}$ is contained in $P_\Omega^{pm} = H.U_\Omega^{pm}$ and in $P_\Omega^{nm} = H.U_\Omega^{nm}$. The group U_Ω^{++} generated by the $U_{\alpha,\Omega}$ for $\alpha \in \Phi^+$ is included in U_Ω^+, itself included in U_Ω^{pm+}. The first inclusion may be strict even for Ω reduced to a special point and A of affine type. The equality $U_\Omega^+ = U_\Omega^{pm+}$ is equivalent to $U_\Omega = U_\Omega^{pm}$, it may be false for Ω large (*e.g.* a negative sector). The situation should be better for Ω narrow. Actually, we shall prove that $P_\Omega^{pm} = P_\Omega^{nm}$ when Ω is reduced to a special point or is a spherical face (3.7). The problem is then to know if this group is generated by its intersections with the torus and the (real) root groups.

In the classical case of reductive groups, one has $G = G^{pmax} = G^{nmax}$ and

$$U_\Omega^{++} = U_\Omega^+ = U_\Omega^{\max +} = U_\Omega^{pm+},$$
$$U_\Omega^{--} = U_\Omega^- = U_\Omega^{\max -} = U_\Omega^{nm-};$$

moreover U_Ω ($= U_\Omega^{pm} = U_\Omega^{nm}$) is the same as the group defined in [3, 6.4.2, 6.4.9]. The group $P_\Omega^{\min}$ is called P_Ω by Bruhat and Tits.

Proof. — after [3, 6.4.9]

a) Let $\mathcal{U} = U_\Omega^{pm}(\Delta^+).U_\Omega^{nm}(\Delta^-).N_\Omega^u \subset G(K)$. By 3.3.4) and Lemma 3.3, for α simple, one has:

$$\begin{aligned}\mathcal{U} &= U_\Omega^{pm}(\Delta^+ \setminus \{\alpha\}).U_\Omega^{nm}(\Delta^- \setminus \{-\alpha\}).U_{\alpha,\Omega}.U_{-\alpha,\Omega}.N_\Omega^u \\ &= U_\Omega^{pm}(\Delta^+ \setminus \{\alpha\}).U_\Omega^{nm}(\Delta^- \setminus \{-\alpha\}).U_{-\alpha,\Omega}.U_{\alpha,\Omega}.N_\Omega^u \\ &= U_\Omega^{pm}(\Delta^+ \setminus \{\alpha\}).U_{-\alpha,\Omega}.U_\Omega^{nm}(\Delta^- \setminus \{-\alpha\}).U_{\alpha,\Omega}.N_\Omega^u \\ &= U_\Omega^{pm}(r_\alpha(\Delta^+)).U_\Omega^{nm}(r_\alpha(\Delta^-)).N_\Omega^u.\end{aligned}$$

So $\mathcal{U}$ does not change when Δ^+ is changed by the Weyl group W^v.

b) Hence $\mathcal{U}$ is stable by left multiplication by U_Ω^{pm+} and all $U_{\alpha,\Omega}$ for $\alpha \in \Phi$. Moreover, it contains these subgroups, so $\mathcal{U} \supset U_\Omega^{pm} \supset U_\Omega$.

c) In $G^{pmax}(K)$, let us prove that $\mathcal{U} \cap U^{\max +}(K) = U_\Omega^{pm+}$: if $xyz \in U^{\max +}(K)$ with $x \in U_\Omega^{pm+}$, $y \in U_\Omega^{nm-}$ and $z \in N_\Omega^u$, then $yz \in U^{\max +}(K)$ and by the Birkhoff decomposition (3.3.2) one has $y = z = 1$.

d) So $\quad U_\Omega^{pm} \cap U^+(K) = U_\Omega^{pm} \cap U^{max+}(K) = U_\Omega^{pm+}$.

The group $U_\Omega(\Delta^+ \setminus \{\alpha\}) := U_\Omega \cap U_\Omega^{pm}(\Delta^+ \setminus \{\alpha\}) = U_\Omega^+ \cap U_\Omega^{pm}(\Delta^+ \setminus \{\alpha\})$ is normalized by $U_{\alpha,\Omega}$ and $U_{-\alpha,\Omega}$. By 3.3.4), $U_\Omega^+ = U_{\alpha,\Omega} \ltimes U_\Omega(\Delta^+ \setminus \{\alpha\})$ and symmetrically for U_Ω^- .

e) Now we are able to argue as in a), b) above with a new $\mathcal{U}$, where $U_\Omega^{pm}(\Delta^+)$ is changed to U_Ω^+ and/or $U_\Omega^{nm}(\Delta^-)$ to U_Ω^-. This proves 1), 2) and 3).

f) Concerning 4), v) holds by definition, and vi) was proved in d). We prove now iv) and ii); iii) and i) are similar. Let $n \in N(K)$ and $v \in U^+(K)$ be such that $nv \in U_\Omega^{pm}$. There exist $n' \in N_\Omega^u$, $u' \in U_\Omega^-$ and $v' \in U_\Omega^{pm+}$ such that $nv = n'u'v'$. Now $n'^{-1}n = u'v'v^{-1}$ and by the Birkhoff decomposition $n = n' \in N_\Omega^u$, $v = u'v'$, so, $u' = 1$ and $v = v' \in U_\Omega^{pm+}$.

□

3.5. Iwasawa decomposition

Proposition 3.6. — *Suppose Ω narrow, then*

$$G(K) = U^+(K).N(K).U_\Omega.$$

Suppose, moreover, Ω almost open. Then the natural map from $W_Y = N(K)/H$ onto $U^+(K)\backslash N(K)/U_\Omega$ is one to one.

Remarks 3.7. — We also have $G(K) = U^-(K).N(K).U_\Omega$ and similarly with the maximal groups $G^{pmax}(K) = U^{pmax+}(K).N(K).U_\Omega$ and $G^{nmax}(K) = U^{nmax-}(K).N(K).U_\Omega$.

As a consequence, when Ω is narrow, every subgroup P of $G(K)$ containing U_Ω may be written $P = (P \cap (U^+(K).N(K))).U_\Omega$. If, moreover, $P \cap (U^+(K).N(K)) = U_P^+.N_P$ with $U_P^+ = P \cap U^+(K)$ and $N_P = P \cap N(K)$ normalizing U_Ω, then $P = U_P^+.N_P.U_\Omega^-$. We shall use this to (almost) identify U_Ω^{pm} and U_Ω^{nm} (see Section 3.7).

The idea of the proof of the Iwasawa decomposition goes back to Steinberg. We follow [3, 7.3.1], see also [7, 3.7] and [4, 1.6]. We first need a lemma.

Lemma 3.8. — *Let α be in Φ, then $Z_\alpha := U_\alpha(K).\{1, r_\alpha\}.T(K).U_\Omega^{(\alpha)}$ contains $G^{(\alpha)}(K)$.*

Proof. — By the Bruhat decomposition,

$$G^{(\alpha)}(K) \subset U_\alpha(K).\{1, r_\alpha\}.T(K).U_\alpha(K).$$

So it suffices to prove that, for $m_\alpha \in N(K)$ such that $\nu^v(m_\alpha) = r_\alpha$ and $u \in U_\alpha(K)$, $m_\alpha u \in Z_\alpha$. If $\varphi_\alpha(u) \geqslant f_\Omega(\alpha)$, it's clear: $u \in U_{\alpha,\Omega}$. Otherwise $\varphi_\alpha(u) \leqslant f_\Omega(\alpha) - 1 \leqslant -f_\Omega(-\alpha)$ and $u = v'mv$" with $\nu^v(m) = r_\alpha$, $v', v'' \in U_{-\alpha,-\varphi_\alpha(u)} \subset U_{-\alpha,\Omega}$. So $m_\alpha u = m_\alpha v'mv$" $\in U_\alpha(K).T(K).U_{-\alpha,\Omega} \subset Z_\alpha$, and the lemma is proved.

Proof of Proposition 3.6. — The set $Z = U^+(K).N(K).U_\Omega$ is stable by left multiplication by $U^+(K)$ and $T(K)$. It remains to prove that it is stable by left multiplication by $U_{-\alpha}(K)$ for α a simple root. Let $U(\Phi^+ \setminus \{\alpha\})(K) = G(K) \cap U^{\max}(\Delta^+ \setminus \{\alpha\})(K) \subset U^+(K)$, using the Lemma 3.3 and discussion in Section 3.3.4), one gets:

$$\begin{aligned}
U_{-\alpha}(K)Z &= U_{-\alpha}(K).U(\Phi^+ \setminus \{\alpha\})(K).U_\alpha(K).N(K).U_\Omega \\
&\subset U(\Phi^+ \setminus \{\alpha\})(K).G^{(\alpha)}(K).N(K).U_\Omega \\
&\subset U(\Phi^+ \setminus \{\alpha\})(K).U_\alpha(K).\{1, r_\alpha\}.T(K).U_\Omega^{(\alpha)}.N(K).U_\Omega \\
&\subset U^+(K).T(K).U_{-\alpha}(K).N(K).U_\Omega \\
&\qquad \cup \;\; U^+(K).T(K).r_\alpha.U_{-\alpha}(K).U_\alpha(K).N(K).U_\Omega \\
&\subset U^+(K).T(K).U_{-\alpha}(K).N(K).U_\Omega.
\end{aligned}$$

It remains to show that $U_{-\alpha}(K).N(K) \subset Z$. But $un = n.n^{-1}un \in nU_\beta(K) \subset nU_{-\beta}(K).\{1, r_\beta\}.T(K).U_\Omega^{(\beta)}$ with $\beta = -\nu^v(n^{-1})\alpha$. So $un \in U_\alpha(K).n.\{1, r_\beta\}.T(K).U_\Omega^{(\beta)} \subset U_\alpha(K).N(K).U_\Omega$.

With obvious notation, suppose $n' \in U^+(K)nU_\Omega$. Then, by Lemma 3.3 and Proposition 3.4 one has: $n'^{-1}n \in U_\Omega n^{-1}U^+(K)n$. But, $n^{-1}U^+(K)n = U(n^{-1}\Phi^+)$. Further, $U_\Omega.U(n^{-1}\Phi^+)(K) \subset H.U_\Omega(n^{-1}\Phi^-).U(n^{-1}\Phi^+)(K) \subset U(n^{-1}\Phi^-)(K).H.U(n^{-1}\Phi^+)(K)$.

Finally, by the Birkhoff decomposition, $n'^{-1}n \in H$.

□

3.6. The group $\widetilde{P}_\Omega$

In this section Ω is asked to be a nonempty set.

Clearly, $U_\Omega^{\max +}$ stabilizes $\hat{\mathfrak{g}}_\Omega^p$ and $G(K)$ stabilizes $\mathfrak{g}_K$; so $U_\Omega^{pm+} = G(K) \cap U_\Omega^{\max +}$ stabilizes $\mathfrak{g}_\Omega = \hat{\mathfrak{g}}_\Omega^p \cap \mathfrak{g}_K$. Finally, U_Ω^{pm} and also U_Ω^{nm} (or H) stabilize $\mathfrak{g}_\Omega$. If M is a highest weight integrable $\mathfrak{g}-$module, then $U_\Omega^{\max +}$ stabilizes M_Ω. The group $U_\Omega^{\max -}$ stabilizes $\hat{M}_\Omega = \prod_{\lambda \in X} M_{\lambda,\Omega}$ and $G(K)$ stabilizes $M \otimes K$. Finally, U_Ω^{pm} and also U_Ω^{nm} (or H) stabilize M_Ω for every highest (or lowest) weight integrable $\mathfrak{g}-$module M.

Definition 3.9. — *The group $\widetilde{P}_\Omega$ is the subgroup of all elements in $G(K)$ stabilizing $\mathfrak{g}_\Omega$ and M_Ω for every highest (or lowest) weight integrable $\mathfrak{g}-$module M.*

Hence, $\widetilde{P}_\Omega$ contains U_Ω^{pm}, U_Ω^{nm}, U_Ω and H. When Ω is narrow, we have $\widetilde{P}_\Omega = (\widetilde{P}_\Omega \cap U^+(K).N(K)).U_\Omega = (\widetilde{P}_\Omega \cap U^-(K).N(K)).U_\Omega$.

Lemma 3.10. — *Let $\widetilde{N}_\Omega = \widetilde{P}_\Omega \cap N(K)$, then $\widetilde{P}_\Omega \cap (U^+(K).N(K)) = U_\Omega^{pm+}.\widetilde{N}_\Omega$ and $\widetilde{P}_\Omega \cap (U^-(K).N(K)) = U_\Omega^{nm-}.\widetilde{N}_\Omega$. Moreover, $\widetilde{N}_\Omega$ normalizes U_Ω and is the stabilizer (in $N(K)$ for the action ν on $\mathbb{A}$) of the $\mathcal{P}-$enclosure $cl_{\mathcal{P}}(\Omega)$ of Ω, where $\mathcal{P} \subset X$ is the union of Δ and the set of all weights of $\mathfrak{h}$ in all the modules M above.*

Proof. —

a) Let $n \in N(K)$ and $u \in U^+(K)$ be such that $un \in \widetilde{P}_\Omega$ and $w = \nu^v(n)$. For $\mathcal{M} = M_\Omega$ or $\mathfrak{g}_\Omega$, $g \in \widetilde{P}_\Omega$ and $\mu, \mu' \in \mathfrak{h}^*$, we define ${}_{\mu'}|g|_\mu$ as the restriction of g to $\mathcal{M}_\mu$ followed by the projection onto $\mathcal{M}_{\mu'}$ (with kernel $\oplus_{\mu'' \neq \mu'} \mathcal{M}_{\mu''}$). Now for all μ, ${}_{w\mu}|un|_\mu = {}_{w\mu}|n|_\mu$ and $n = \oplus_\mu \, {}_{w\mu}|n|_\mu$ (in an obvious sense); so $n \in \widetilde{N}_\Omega$. We have $\widetilde{P}_\Omega \cap (U^+(K).N(K)) = (\widetilde{P}_\Omega \cap U^+(K)).\widetilde{N}_\Omega$ and it remains to determine $\widetilde{N}_\Omega$ and $\widetilde{P}_\Omega \cap U^+(K)$ (or $\widetilde{P}_\Omega \cap U^-(K)$).

b) $\widetilde{P}_\Omega \cap U^+(K) = U_\Omega^{pm+}$: the inclusion $\supset$ is already proved in the discussion before Definition 3.9. So, consider $u = \prod_{\alpha \in \Delta^+} u_\alpha \in$

$U^{\max +}(K)$ such that u stabilizes $\mathfrak{g}_\Omega$ (the order on the $u_\alpha \in U_\alpha(K)$ is chosen such that the height of α is increasing from right to left). We shall prove by induction that each u_α is in $U_{\alpha,\Omega}$. We may suppose $u_{\alpha'} \in U_{\alpha',\Omega}$ for $u_{\alpha'}$ on the right of u_α; moreover, as $U_{\alpha',\Omega}$ stabilizes $\mathfrak{g}_\Omega$, we may suppose these $u_{\alpha'}$ equal to 1. So $u = (\prod_{\beta\neq\alpha} u_\beta).u_\alpha$ where the u_β are in $U_\beta(K)$ and $ht(\beta) \geqslant ht(\alpha)$. But ${}_\alpha|u|_0 = {}_\alpha|u_\alpha|_0$ sends $\mathfrak{h}_\mathcal{O}$ into $\mathfrak{g}_{\alpha,\Omega}$, so $u_\alpha \in U_{\alpha,\Omega}$. Now if $u \in \widetilde{P}_\Omega \cap U^+(K)$, it is in $U^{\max +}(K) \cap G(K)$ and stabilizes $\mathfrak{g}_\Omega$; by the above argument, $u \in U_\Omega^{\max +} \cap G(K) = U_\Omega^{pm+}$.

c) Let $n = n_0 t$, $n_0 \in N(\mathbb{C})$, $\nu^v(n) = w$ and $t \in T(K)$, then $nM_{\lambda,k} = M_{w\lambda,k+\omega(\lambda(t))}$. Consider now the action on $\mathbb{A}$: $nD(\lambda,k) = n_0 D(\lambda, k+\omega(\lambda(t))) = D(w\lambda, k+\omega(\lambda(t)))$. But, $\mathfrak{g}_\Omega$ is generated by $\mathfrak{h}_\mathcal{O}$ and the $\mathfrak{g}_{\alpha,\Omega}$ for $\alpha \in \Delta$, so n is in $\widetilde{P}_\Omega$ if and only if, for all $\lambda \in \mathcal{P}$ $f_\Omega(\lambda) + \omega(\lambda(t)) = f_\Omega(w\lambda)$ if and only if, for all $\lambda \in \mathcal{P}$, $nD(\lambda, f_\Omega(\lambda)) = D(w\lambda, f_\Omega(w\lambda))$. This is equivalent to the fact that n stabilizes the set $cl_\mathcal{P}(\Omega)$. Moreover, as $\widetilde{N}_\Omega$ stabilizes $\mathfrak{g}_\Omega$, it normalizes U_Ω.

□

We know that $N_\Omega^{\min} = H.N_\Omega^u \subset \widetilde{N}_\Omega$. So, to determine $\widetilde{N}_\Omega$, we only have to determine the subgroup $\widetilde{W}_\Omega = \widetilde{N}_\Omega/H = \nu(\widetilde{N}_\Omega)$ of $\nu(N(K))$; it contains $W_\Omega^{\min} = \nu(N_\Omega^{\min})$.

Examples 3.11. — 1) *Let us now assume that* Ω *is bounded.* As $\mathcal{P} \supset \Delta \cup X^+ \cup X^-$, it is easy to prove that each $\chi \in X$ is a positive linear combination of some $\lambda \in \mathcal{P}$. Hence, the intersection $cl_\mathcal{P}(\Omega)$ of all $D(\lambda, f_\Omega(\lambda))$'s (for $\lambda \in \mathcal{P}$) is a nonempty convex compact set. But $\widetilde{N}_\Omega$ stabilizes $cl_\mathcal{P}(\Omega)$ and, as it acts affinely, it fixes a point x_Ω in $cl_\mathcal{P}(\Omega)$.

2) *Suppose now* Ω *narrow*, then

$$\widetilde{P}_\Omega = U_\Omega^{pm+}.\widetilde{N}_\Omega.U_\Omega = U_\Omega^{pm+}.U_\Omega.\widetilde{N}_\Omega = U_\Omega^{pm+}.U_\Omega^-.\widetilde{N}_\Omega$$

and

$$\widetilde{P}_\Omega = U_\Omega^{nm-}.\widetilde{N}_\Omega.U_\Omega = U_\Omega^{nm-}.U_\Omega.\widetilde{N}_\Omega = U_\Omega^{nm-}.U_\Omega^+.\widetilde{N}_\Omega.$$

In particular $\widetilde{P}_\Omega$, which contains always P_Ω^{pm} and P_Ω^{nm}, is not much greater than them in this case.

3.7. The (parahoric) group P_Ω and the "fixator" $\hat{P}_\Omega$

From now on, we suppose $\mathfrak{g}$ *symmetrizable.*

As $\mathfrak{g}_\Omega$ is generated by $\mathfrak{h}_\mathcal{O}$ and the $\mathfrak{g}_{\alpha,\Omega}$ for $\alpha \in \Delta$, the derived algebra of $\mathfrak{g}_\Omega$ is $\mathfrak{g}'_\Omega = (\sum_{\alpha\in\Delta} [\mathfrak{g}_{\alpha,\Omega}, \mathfrak{g}_{-\alpha,\Omega}]) \oplus (\oplus_{\alpha\in\Delta} \mathfrak{g}_{\alpha,\Omega})$. Consider the quotient algebra $\overline{\mathfrak{g}}_\Omega = \mathfrak{g}_\Omega/\varpi\mathfrak{g}_\Omega = (\mathfrak{h}\otimes\kappa) \oplus (\oplus_{\alpha\in\Delta} \mathfrak{g}_{\alpha,\Omega}/\varpi\mathfrak{g}_{\alpha,\Omega})$. As $\mathfrak{g}$ is symmetrizable, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}\alpha^\vee$ for all $\alpha \in \Delta$, so the derived algebra of $\overline{\mathfrak{g}}_\Omega$ is $\overline{\mathfrak{g}}'_\Omega = (\sum_{\alpha\in\Delta_\Omega} \kappa\alpha^\vee)\oplus(\oplus_{\alpha\in\Delta} \mathfrak{g}_\alpha\otimes\kappa)$, where $\Delta_\Omega = \{\alpha \in \Delta \mid f_\Omega(\alpha)+f_\Omega(-\alpha) = 0\}$ is the set of $\alpha \in \Delta$ such that $\alpha(\Omega)$ is reduced to a point in $\mathbb{Z}$.

As $\mathfrak{g}$ is symmetrizable, the orthogonal $(\overline{\mathfrak{g}}'_\Omega)^\perp$ of $\overline{\mathfrak{g}}'_\Omega$ in $\overline{\mathfrak{g}}_\Omega$ is $\{x \in \mathfrak{h}\otimes\kappa \mid \alpha(x) = 0, \forall\alpha \in \Delta_\Omega\}$. If Ω is a set, the action of $\widetilde{P}_\Omega$ (by inner automorphisms) is compatible with the invariant bilinear form; so $\widetilde{P}_\Omega$ stabilizes $\mathfrak{g}_\Omega$, $\overline{\mathfrak{g}}_\Omega$, $\overline{\mathfrak{g}}'_\Omega$ and $(\overline{\mathfrak{g}}'_\Omega)^\perp$. Let $\widetilde{P}'_\Omega$ be the fixator of $(\overline{\mathfrak{g}}'_\Omega)^\perp$ for this action of $\widetilde{P}_\Omega$.

Definition 3.12. — *For a filter Ω, $P_\Omega = \cup_{S\in\Omega} (\cap_{S'\subset S} \widetilde{P}'_{S'})$ is a "parahoric" group associated to Ω.*

An element g of U_Ω (or U_Ω^{pm+}, U_Ω^{nm-}) is in some U_S (or U_S^{pm+}, U_S^{nm-}) for $S \in \Omega$, hence in $U_{S'}$ (or $U_{S'}^{pm+}$, $U_{S'}^{nm-}$) for any $S' \subset S$. But $U_{S'}^{max+}$ and H induce the identity on $\mathfrak{h}_\mathcal{O}$, so $\widetilde{P}'_{S'}$ contains H, $U_{S'}^{pm+}$ and also $U_{S'}^{nm-}$. Moreover $U_{S'}$ is generated by elements in $U_{S'}^{pm+}$ or $U_{S'}^{nm-}$. Finally P_Ω contains U_Ω, U_Ω^{pm+}, U_Ω^{nm-} and H.

The group $N_\Omega = P_\Omega\cap N(K)$ contains $N_\Omega^{\min}$ (and is often equal to it, as we shall see). The quotient group $W_\Omega = N_\Omega/H$ contains $W_\Omega^{\min}$ and is included in $\widetilde{W}_\Omega := \cup_{S\in\Omega} \widetilde{W}_S$. Actually, $W_\Omega = \cup_{S\in\Omega} (\cap_{S'\subset S} \widetilde{W}'_{S'})$ with $\widetilde{W}'_{S'} = (N(K)\cap\widetilde{P}'_{S'})/H$. If S' is a non empty bounded set , then, by Example 3.11, $\widetilde{W}_{S'}$ fixes a point $x_{S'}$ in $cl_\mathcal{P}(S') \subset \{x \in \mathbb{A} \mid \alpha(x) = \alpha(S') \quad \forall\alpha \in \Delta_{S'}\} \subset \{x \in \mathbb{A} \mid \alpha(x) = \alpha(S) \quad \forall\alpha \in \Delta_S\}$ (if $S' \subset S$); in particular, it is isomorphic to its image in W^v. But, by definition, the image in W^v of $\widetilde{W}'_{S'}$ is the fixator (in the image of $\widetilde{W}_{S'}$) of the direction of the affine space $\{x \in \mathbb{A} \mid \alpha(x) = \alpha(S') \quad \forall\alpha \in \Delta_{S'}\} \supset cl_\mathcal{P}(S')$. Hence, by Lemma 3.10, $\widetilde{W}'_{S'}$ is the fixator in W_Y of $\{x \in \mathbb{A} \mid \alpha(x) = \alpha(S') \quad \forall\alpha \in \Delta_{S'}\}$. It follows that W_Ω is always the fixator in W_Y of $\{x \in \mathbb{A} \mid \alpha(x) = \alpha(\Omega) \quad \forall\alpha \in \Delta_\Omega\}$. In particular N_Ω normalizes U_Ω.

When there exists $x, y \in \{x \in \mathbb{A} \mid \alpha(x) = \alpha(\Omega) \quad \forall\alpha \in \Delta_\Omega\}$ such that $y - x$ is in the open-Tits-cone (in particular when Ω is a spherical face), it is known that $W_\Omega = W_\Omega^{\min}$.

When Ω is narrow, $P_\Omega = U_\Omega^{pm+}.U_\Omega^-.N_\Omega = U_\Omega^{nm-}.U_\Omega^+.N_\Omega = U_\Omega^{pm}.N_\Omega = U_\Omega^{nm}.N_\Omega$.

In particular, when Ω is a spherical face (or a special point), $N_\Omega = N_\Omega^{\min}$ and $P_\Omega = P_\Omega^{pm} = P_\Omega^{nm}$ is called the parahoric subgroup *associated to Ω.*

Definition 3.13. — *The "fixator" $\hat{P}_\Omega$ associated to Ω is the group generated by P_Ω and the fixator $\hat{N}_\Omega$ (in $N(K)$ for the action ν) of Ω.*

Actually, $\hat{N}_\Omega$ is also the fixator of the *support* of Ω: the smallest affine subspace of $\mathbb{A}$ generated by a set in Ω. Clearly $\mathrm{supp}(\Omega) \subset \{x \in \mathbb{A} \mid \alpha(x) = \alpha(\Omega) \quad \forall \alpha \in \Delta_\Omega\}$, so $\hat{N}_\Omega \supset N_\Omega$. As $\hat{N}_\Omega$ normalizes P_Ω (and all the groups previously defined), we have $\hat{P}_\Omega = P_\Omega.\hat{N}_\Omega$. Clearly, $\hat{P}_\Omega \cap N(K) = \hat{N}_\Omega$ and $\hat{P}_\Omega \supset U_\Omega^{pm+}, U_\Omega^{nm-}$.

When Ω is narrow, $\hat{P}_\Omega = U_\Omega^{pm+}.U_\Omega^{-}.\hat{N}_\Omega = U_\Omega^{nm-}.U_\Omega^{+}.\hat{N}_\Omega$.

This group should be the fixator of Ω for the action of $G(K)$ on the "ugly-building" we shall build now. But this will be proved only for some Ω, see 4.2 below.

Examples 3.14. — An explicit computation: Suppose Ω reduced to the special point 0, the origin of $V = \mathbb{A}$ chosen as in Remark 3.2.2). Then $f_0(\alpha) = 0, \forall \alpha$, $\mathfrak{g}_0 = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}$ and $M_0 = M \otimes_{\mathbb{C}} \mathcal{O}$. Hence, the definition of the ind-group structure of G [9, 7.4.6 and 7.4.7] tells us that $\widetilde{P}_0 \subset G(\mathcal{O})$. Moreover, $Lie(G) = \mathfrak{g}$ and the highest or lowest weight modules are defined by morphisms of ind-varieties [loc. cit.; 7.4.E(6) and 7.4.13] so $\widetilde{P}_0 = G(\mathcal{O})$. Now $\overline{\mathfrak{g}}_0 = \mathfrak{g} \otimes_{\mathbb{C}} \kappa$ and (as 0 is special) $\hat{N}_0 = N_0^{\min}$, $(\overline{\mathfrak{g}}'_0)^\perp = \mathfrak{c} \otimes_{\mathbb{C}} \kappa$ where $\mathfrak{c}$ is the center of $\mathfrak{g}$; so $G(\mathcal{O}) = \widetilde{P}_0 = P_0 = \hat{P}_0 \quad (= G_0$ with the notation of 4.1).

In the classical case of reductive groups, W_Ω is always equal to $W_\Omega^{\min}$. If Ω is narrow (*i.e.* included in a closed-face), $P_\Omega = P_\Omega^{\min}$ and $\hat{P}_\Omega$ are as defined by Bruhat and Tits (cf. Remark 3.5). In particular, $\hat{P}_x$ is the same as in Bruhat-Tits and the following definition gives the (pretty) Bruhat-Tits building.

3.8. The hovel and its apartments

Definition 3.15. — *The hovel $\mathcal{I} = \mathcal{I}(G, K)$ of G over K is the quotient of the set $G(K) \times \mathbb{A}$ by the relation:*

$$(g, x) \sim (h, y) \quad \Leftrightarrow \quad \exists n \in N \text{ such that } y = \nu(n)x \text{ and } g^{-1}hn \in \hat{P}_x.$$

One proves easily [3, 7.4.1] that $\sim$ is an equivalence relation. Moreover, $\hat{P}_x \cap N(K) = \hat{N}_x$. So, the map $x \mapsto cl(1, x)$ identifies $\mathbb{A}$ with its image $A_f = A(T, K)$, the *apartment of T in $\mathcal{I}(G, K)$*.

The left action of $G(K)$ on $G(K) \times \mathbb{A}$ descends to an action on $\mathcal{I}$. The *apartments* of $\mathcal{I}$ are the $g.A_f$ for $g \in G(K)$. The action of $N(K)$ on $\mathbb{A} = A_f$ is through ν; in particular, H fixes (pointwise) A_f. By construction, the fixator of $x \in \mathbb{A}$ is $\hat{P}_x$ and, for $g \in G(K)$, one has $gx \in \mathbb{A} \Leftrightarrow g \in N(K)\hat{P}_x$.

From the definition of the groups $\hat{P}_x$, it is clear that, for $\alpha \in \Phi$ and $u \in K$, $x_\alpha(u)$ fixes $D(\alpha, \omega(u))$. Hence, for $k \in \mathbb{Z}$, the group $H.U_{\alpha,k}$ fixes $D(\alpha, k)$.

4. The hovel, first properties

First, we define the notion of good fixator for a filter Ω of $\mathbb{A}$. It formalizes the fact that the fixator G_Ω of Ω for the action of $G(K)$ on $\mathcal{I}$ has a nice decomposition and the fact that G_Ω acts transitively on the apartments containing Ω (4.1). Thanks to a technical proposition (Proposition 4.3), we can show, in particular, that faces, sectors, sector-germs, walls and half-apartments in $\mathbb{A}$ do have a good fixator (4.2). This, in turn, gives a lot of applications (4.3), like the retraction associated to a sector-germ. We finish this section with the structure of the residue buildings (4.5).

4.1. Good fixators

When $\Omega \subset \Omega' \subset \mathbb{A}$, then $\hat{P}_\Omega \supset \hat{P}_{\Omega'}$. As $\hat{P}_x$ is the fixator of $x \in \mathbb{A}$, $\hat{P}_\Omega$ is included in the fixator G_Ω of Ω (for the action of $G(K)$ on $\mathcal{I} \supset \mathbb{A}$). Actually, when Ω is a set $G_\Omega = \bigcap_{x\in\Omega} \hat{P}_x$, and when Ω is a filter $G_\Omega = \bigcup_{S\in\Omega} G_S$.

We have $G_\Omega \cap N(K) = \hat{N}_\Omega$ and $G_\Omega \supset \hat{P}_\Omega$ which contains $U_\Omega^{pm+}.U_\Omega^{nm-}.\hat{N}_\Omega$ and $U_\Omega^{nm-}.U_\Omega^{pm+}.\hat{N}_\Omega$.

For Ω a filter of subsets in $\mathbb{A}$, the subset of $G(K)$ consisting of the $g \in G(K)$ such that $g.\Omega \subset \mathbb{A}$ is: $G(\Omega \subset \mathbb{A}) = \bigcup_{S\in\Omega} (\bigcap_{x\in S} N(K).\hat{P}_x)$. Indeed $g.\Omega \subset \mathbb{A} \Leftrightarrow \exists S \in \Omega,\ g.S \subset \mathbb{A} \Leftrightarrow \exists S \in \Omega,\ \forall x \in S,\ gx \in \mathbb{A} \Leftrightarrow$ (by 3.15) $\exists S \in \Omega,\ \forall x \in S,\ g \in N(K).\hat{P}_x$.

Definition 4.1. — *Consider the following properties:*

(GF+) $G_\Omega = \hat{P}_\Omega = U_\Omega^{pm+}.U_\Omega^{nm-}.\hat{N}_\Omega$,

(GF−) $G_\Omega = \hat{P}_\Omega = U_\Omega^{nm-}.U_\Omega^{pm+}.\hat{N}_\Omega$,

(TF) $G(\Omega \subset \mathbb{A}) = N(K).G_\Omega$.

We say that Ω in $\mathbb{A}$ has a good fixator if it satisfies these three properties.

We say that Ω in $\mathbb{A}$ has an half-good fixator if it satisfies (TF) and (GF+) or (GF−).

We say that Ω in $\mathbb{A}$ has a transitive fixator if it satisfies (TF).

By point a) in the proof of Proposition 3.4, this definition doesn't depend on the choice of Δ^+ in its W^v−conjugacy class and $N(K)$ permutes the filters with good fixators and the corresponding fixators. By 3.13 and 3.15, a point has a good fixator.

In the classical case of reductive groups, every Ω has a good fixator and $\hat{P}_\Omega$ is as defined by Bruhat and Tits [3, 7.1.8, 7.1.11, 7.4.8].

Remark 4.2. — If Ω in $\mathbb{A}$ has a transitive fixator. Then G_Ω is transitive on the apartments containing Ω: if $g \in G(\Omega \subset \mathbb{A})$, there exists $n \in N(K)$ such that $g|_\Omega = n|_\Omega$; moreover if $g.\mathbb{A} \supset \Omega$, then $g^{-1}\Omega \subset \mathbb{A}$ and $g^{-1} = np \in N(K).G_\Omega$, so $g.\mathbb{A} = p^{-1}n^{-1}.\mathbb{A} = p^{-1}.\mathbb{A}$. In particular G_Ω and all invariant subgroups of G_Ω do not depend of the particular choice of the apartment A containing Ω.

PROPOSITION 4.3. — 1) *Suppose $\Omega \subset \Omega' \subset cl(\Omega)$. If Ω in $\mathbb{A}$ has a good (or half-good) fixator, then this also holds for Ω' and $G_\Omega = \hat{N}_\Omega.G_{\Omega'}$, $N(K).G_\Omega = N(K).G_{\Omega'}$. In particular, any apartment containing Ω contains its enclosure $cl(\Omega)$.*

Conversely, if $supp(\Omega) = \mathbb{A}$ (or $supp(\Omega') = supp(\Omega)$, hence $\hat{N}_{\Omega'} = \hat{N}_\Omega$), Ω has an half-good fixator and Ω' has a good fixator, then Ω has a good fixator.

2) *If a filter Ω in $\mathbb{A}$ is generated by a family $\mathcal{F}$ of filters with good (or half-good) fixators, then Ω has a good (or half-good) fixator $G_\Omega = \bigcup_{F\in\mathcal{F}} G_F$.*

3) *Suppose that the filter Ω in $\mathbb{A}$ is the union of an increasing sequence $(F_i)_{i\in\mathbb{N}}$ of filters with good (or half-good) fixators and that, for some i, the space* $\mathrm{supp}(F_i)$ *has a finite fixator W_0 in W_Y, then Ω has a good (or half-good) fixator $G_\Omega = \bigcap_{i\in\mathbb{N}} G_{F_i}$.*

4) *Let Ω and Ω' be two filters in $\mathbb{A}$. Suppose Ω' satisfies (GF+) (resp. (GF+) and (TF)) and that there exist a finite number of positive, closed, vectorial chambers $\overline{C_1^v}, \cdots, \overline{C_n^v}$ such that: $\Omega \subset \cup_{i=1,n}\ \Omega' + \overline{C_i^v}$. Then $\Omega \cup \Omega'$ satisfies (GF+) (resp. (GF+) and (TF)) and $G_{\Omega\cup\Omega'} = G_\Omega \cap G_{\Omega'}$.*

Remark 4.4. — In 4) above, the same results are true when changing $+$ to $-$.

If Ω' has a good fixator, $\Omega \subset \cup_{i=1,n}\ \Omega' + \overline{C_i^v}$ and $\Omega \subset \cup_{i=1,n}\ \Omega' - \overline{C_i^v}$, then $\Omega \cup \Omega'$ has a good fixator.

If Ω satisfies (GF$-$), Ω' satisfies (GF+), Ω or Ω' satisfies (TF), $\Omega \subset \cup_{i=1,n}\ \Omega' + \overline{C_i^v}$ and $\Omega' \subset \cup_{i=1,n}\ \Omega - \overline{C_i^v}$, then $\Omega \cup \Omega'$ has a good fixator.

Proof. —

1) When $\Omega \subset \Omega' \subset cl(\Omega)$, we always have $U_\Omega^{pm+} = U_{\Omega'}^{pm+} = U_{cl(\Omega)}^{pm+}$, $U_\Omega^{nm-} = U_{\Omega'}^{nm-} = U_{cl(\Omega)}^{nm-}$, $G_{\Omega'} \subset G_\Omega$, $G(\Omega' \subset \mathbb{A}) \subset G(\Omega \subset \mathbb{A})$ and $\hat{N}_{\Omega'} = N(K) \cap G_{\Omega'} \subset \hat{N}_\Omega$ (with equality when $supp(\Omega') = supp(\Omega)$); so the first assertion of 1) is clear. The second assertion is a consequence of Remark 4.2.

For the last assertion we know that $G_{\Omega'} = U_\Omega^{pm+}.U_\Omega^{nm-}.\hat{N}_{\Omega'} = U_\Omega^{nm-}.U_\Omega^{pm+}.\hat{N}_{\Omega'}$, $\hat{N}_{\Omega'} = \hat{N}_\Omega$ and $G_{\Omega'} = G_\Omega$; so the fixator G_Ω is good.

2) If Ω is generated by the family $\mathcal{F}$ of filters, we have

$$G_\Omega = \bigcup_{F\in\mathcal{F}} G_F,\ U_\Omega^{pm+} = \bigcup_{F\in\mathcal{F}} U_F^{pm+},$$
$$U_\Omega^{nm-} = \bigcup_{F\in\mathcal{F}} U_F^{nm-},\ \hat{N}_\Omega = \bigcup_{F\in\mathcal{F}} \hat{N}_F$$

and $G(\Omega\subset\mathbb{A}) = \bigcup_{F\in\mathcal{F}} G(F\subset\mathbb{A})$; so 2) is clear.

3) If Ω in $\mathbb{A}$ is the union of an increasing sequence $(F_i)_{i\in\mathbb{N}}$ of filters, we have $G_\Omega = \bigcap_{i\in\mathbb{N}} G_{F_i}$, $U_\Omega^{pm+} = \bigcap_{i\in\mathbb{N}} U_{F_i}^{pm+}$, $U_\Omega^{nm-} = \bigcap_{i\in\mathbb{N}} U_{F_i}^{nm-}$ and $G(\Omega\subset\mathbb{A}) = \bigcap_{i\in\mathbb{N}} G(F_i\subset\mathbb{A})$. By hypothesis we may suppose that all $\mathrm{supp}(F_i)$ have the same finite fixator W_0, so, $\hat{N}_\Omega = \hat{N}_{F_i} = W_0.H$.

If $g\in\bigcap_{i\in\mathbb{N}} G_{F_i} = \bigcap_{i\in\mathbb{N}} U_{F_i}^{pm+}.U_{F_i}^{nm-}.H.W_0$, by extracting a subsequence, there exists $n_0\in N(K)$ such that $gn_0^{-1}\in\bigcap_{i\in\mathbb{N}} U_{F_i}^{pm+}.U_{F_i}^{nm-}.H$, and, because $U^\pm(K)\cap B^\mp(K) = \{1\}$ (2.1.7), this intersection is equal to $U_\Omega^{pm+}.U_\Omega^{nm-}.H$. So, $G_\Omega = U_\Omega^{pm+}.U_\Omega^{nm-}.\hat{N}_\Omega$.

If $g\in G(\Omega\subset\mathbb{A}) = \bigcap_{i\in\mathbb{N}} N(K)G_{F_i}$, then, for all i, $g\in w_iG_{F_i}$ for some $w_i\in\hat{W}$, unique modulo W_0 as $G_{F_i}\cap N(K) = \hat{N}_{F_i} = W_0.H$. Extracting a subsequence, we may suppose that w_i is independent on i, so $g\in w_i.(\bigcap_j G_{F_j}) = w_i.G_\Omega$ and $G(\Omega\subset\mathbb{A})\subset N(K).G_\Omega$.

4) By induction, we may suppose $\Omega\subset\Omega' + \overline{C_1^v}$. We may also assume that $\overline{C_1^v}$ is the closed positive fundamental chamber $\overline{C_f^v}$.

Suppose (GF+) and (TF) for Ω'. Let $u\in U_{\Omega'}^{pm+}$, $v\in U_{\Omega'}^{nm-}$ and $n\in N(K)$ be such that $uvn\in(G_\Omega.N(K))\cap(G_{\Omega'}.N(K))$; we now replace Ω and Ω' by appropriate sets in these filters. Clearly, $U_{\Omega'}^{pm+} = U_{\Omega'+\overline{C}_f^v}^{pm+}\subset U_{\Omega\cup\Omega'}^{pm+}\subset U_\Omega^{pm+}$. So, for all $x\in\Omega$, $vn\in G_x.N(K)$ and, as a point has a good fixator, we may write $vn = v'_xu'_xn'_x$ with $v'_x\in U_x^{nm-}$, $u'_x\in U_x^{pm+}$ and $n'_x\in N(K)$. Hence $n'_xn^{-1} = (u'_x)^{-1}(v'_x)^{-1}v$ and, by Birkhoff (2.1.7), $n'_x = n$, $u'_x = 1$, $v'_x = v\in U_{\Omega'}^{nm-}\cap U_x^{nm-} = U_{\Omega'\cup\{x\}}^{nm-}$. Now, we have $u\in U_{\Omega\cup\Omega'}^{pm+}$ and $v\in\cap_{x\in\Omega} U_{\Omega'\cup\{x\}}^{nm-} = U_{\Omega\cup\Omega'}^{nm-}$, so $uvn\in U_{\Omega\cup\Omega'}^{pm+}.U_{\Omega\cup\Omega'}^{nm-}.N(K)\subset(G_\Omega\cap G_{\Omega'}).N(K)$.

Suppose (GF+) for Ω'. Let uvn as above be in $G_\Omega\cap G_{\Omega'}$, we have still the same results, but moreover $n\in\hat{N}_{\Omega'}$ and $n'_x\in\hat{N}_x$. So $n = n'_x\in\hat{N}_{\Omega'\cup\{x\}}$, $\forall x\in\Omega$, hence $n\in\hat{N}_{\Omega\cup\Omega'}$. Therefore we get $uvn\in U_{\Omega\cup\Omega'}^{pm+}.U_{\Omega\cup\Omega'}^{nm-}.\hat{N}_{\Omega\cup\Omega'}$.

□

4.2. Examples of filters with good fixators

1) If $x \leqslant y$ in $\mathbb{A}$, then $\{x,y\}$, $[x,y]$ and $cl(\{x,y\})$ have good fixators and $G_{\{x,y\}} = G_{[x,y]}$. Moreover, if $x \neq y$, $]x,y] = [x,y] \setminus \{x\}$ has a good fixator: it satisfies (GF-) and (TF) by Proposition 4.3 4) and, as $]x,y] \subset [x,y] \subset cl(]x,y])$, it has a good fixator by Proposition 4.3 1).
2) A local face in $\mathbb{A}$ has a good fixator: $\mathrm{germ}_x(x+F^v)$ is generated by the sets $F_n = (x+F^v) \cap (y_n - \overline{F^v})$ for $y_n = x + \frac{1}{n}\xi$, $\xi \in F^v$ and $n \in \mathbb{N}$; moreover (for $F^v \neq \{0\}$) $]x,y_n] \subset F_n \subset cl(]x,y_n])$, so by 1) above and Proposition 4.3 (1) and 2)) F_n and the local face have a good fixator. Now $\mathrm{germ}_x(x+F^v) \subset F(x,F^v) \subset \overline{F}(x,F^v) = cl(\mathrm{germ}_x(x+F^v))$; so, by Proposition 4.3 1), any face or closed face has a good fixator.
3) A sector in $\mathbb{A}$ has a good fixator: $x+C^v$ is the increasing union of the sets $F_n = (x+C^v) \cap cl(]x,y_n])$ where $y_n = x+n\xi$, $\xi \in C^v$ and $n \in \mathbb{N}$. Moreover these F_n have $\mathbb{A}$ as support and $]x,y_n] \subset F_n \subset cl(]x,y_n])$, so F_n and $x+C^v$ have good fixators.
4) A sector-germ has a good fixator. The fixator of $\mathfrak{S}_{\pm\infty}$ is $H.U^{\pm}(K)$, since every element in $U^{\pm}(K)$ is a finite product of elements in groups $U_\alpha(K)$ for $\alpha \in \Phi^{\pm}$.

   On the contrary, $U^{\max +}(K)$ is not the union of the $U^{\max +}_{\Omega}$ for $\Omega \in \mathfrak{S}_{+\infty}$.
5) The apartment $\mathbb{A}$ itself has a good fixator $G_{\mathbb{A}} = H$: $\mathbb{A}$ is the increasing union of $cl(\{-n\xi, n\xi\})$ for $\xi \in C^v_f$.
6) For the same reasons, a wall $M(\alpha,k)$ has a good fixator which is

$$U_{\alpha,k}.U_{-\alpha,-k}.\{1, r_{\alpha,k}\}.H.$$

7) Exercise: An half-apartment $D(\alpha,k)$ has a good fixator $HU_{\alpha,k}$. If $x_+ - x_- \in \mathcal{T}^o$, then $cl(\{F(x_-,F^v_-), F(x_+,F^v_+)\})$ has a good fixator for all vectorial faces F^v_ε (where $\varepsilon = \pm$).

4.3. Applications

1) By 4.2.5), the fixator (resp. stabilizator) of the apartment $\mathbb{A} = A_f$ is H (resp. N(K)). In particular, the maps $g \mapsto g.\mathbb{A}$ and $g \mapsto g.T.g^{-1}$ give bijections $\{$apartments of $\mathcal{I}(G,K)\} \leftrightarrow G(K)/N(K) \leftrightarrow \{$maximal split tori of $G(K)\}$.

Moreover, the action of $N(K)$ on $\mathbb{A}$ preserves the affine structure of $\mathbb{A}$, its lattice of cocharacters Y, $\mathcal{T}$ and $\mathcal{T}^o$. So, any apartment A in $\mathcal{I}(G,K)$ is endowed with a canonical structure of real affine space, an affine action of a Weyl group $W(A)$, a lattice $Y(A)$ of cocharacter points, Tits cones and a preorder relation. More generally, all structures in $\mathbb{A}$ invariant under $N(K)$ are transferred to any apartment by the $G(K)-$action: in an apartment, the notions of (spherical) face, special point, cocharacter point, wall, sector, sector-germ or filter with good fixator are well defined (independently of the apartment containing them, as they all have good fixators).

When we speak of an isomorphism between apartments, we mean an affine isomorphism exchanging the walls and the Tits cones.

2) Let A_1, A_2 be two apartments and x, y be two points in $A_1 \cap A_2$. If $x \leqslant y$ in A_1, then, by Remark 4.2 and 4.2.1), there exists $g \in \hat{P}_{cl(x,y)}$, such that $A_2 = g.A_1$, hence $A_1 \cap A_2 \supset cl(x,y)$ and $x \leqslant y$ in A_2. In particular, the relation $\leqslant$ is defined on the whole hovel $\mathcal{I}(G,K)$ (note that $x \leqslant y$ implies by definition that x and y are in a same apartment). We shall see below (6.5) that this relation is transitive, so it is a preorder-relation (reflexive, transitive, perhaps not antisymmetric).

The intersection of two apartments A_1, A_2 is *order-convex*: if $x, y \in A_1 \cap A_2$ and $x \leqslant y$, then the segment $[x,y]$ of A_1 is in $A_1 \cap A_2$ and equal to the corresponding segment in A_2. In particular, any affine subspace of A_1 whose direction meets the open Tits cone $\mathcal{T}^\circ(A_1)$ and which is contained in $A_1 \cap A_2$ is also an affine subspace of A_2.

3) For any face (or any narrow filter) F and any sector germ $\mathfrak{S}$ in $\mathcal{I}(G,K)$, there exists an apartment A containing F and $\mathfrak{S}$: Using the $G(K)-$action one may suppose $\mathfrak{S} = \mathfrak{S}_{\pm\infty}$. Now $F = g.F'$ with F' a face in $\mathbb{A}$. By the Iwasawa decomposition, $g = unv$ with $u \in U^{\pm}(K) \subset G_{\mathfrak{S}}$, $n \in N(K)$ and $v \in U_{F'} \subset G_{F'}$. So $F = un.F' \subset un.\mathbb{A} = u.\mathbb{A}$ and $\mathfrak{S} \subset u.\mathbb{A}$.

By order-convexity, any apartment containing F and $\mathfrak{S} =$ germ $(y + \overline{C^v})$ contains $F + C^v$ (and even $cl(F + C^v) \supset F + \overline{C^v}$, when F has a good fixator, by 1) and 4) of Proposition 4.3). In particular, any apartment containing x and $\mathfrak{S}$ contains the sector $\mathfrak{s}$ of direction $\mathfrak{S}$ and base point x. By 4.2.3) and Remark 4.2, any two such apartments are conjugated by $G_{\mathfrak{s}}$.

4) If $\Omega_1 = F(x, F_1^v)$ is a face of base point x and Ω_2 a narrow filter containing x, there exists an apartment A containing both of them: in an apartment A_1 containing Ω_1 we choose a vectorial chamber C^v such that $\overline{C^v} \supset F_1^v$; now an apartment A containing Ω_2 and the germ of the sector $x+C^v$ contains Ω_1 and Ω_2. If moreover Ω_2 is also a face, then $G_{\Omega_1 \cup \Omega_2}$ acts transitively on the apartments containing Ω_1 and Ω_2 by 4.2.2, Proposition 4.3 4) and Remark 4.2. Actually, one can prove that $\Omega_1 \cup \Omega_2$ has a good fixator when the faces Ω_1 and Ω_2 are of opposite signs or if one of them is spherical.

If $C = F(x, C^v)$ is a chamber (in $\mathbb{A}$) and $M(\alpha, k)$ one of its walls (with $C \subset D(\alpha, k)$), then $U_{\alpha,k} = U_{\alpha,C}$ acts transitively on the chambers C' adjacent to C along $M(\alpha, k)$: this is a consequence of 4.2.2 and Remark 4.2 as G_C may be written

$$U_{\alpha,C}.U_C^{pm}(\Phi^+ \setminus \{\alpha\}).U_C^{nm-}.H$$

and, in this decomposition, all factors but $U_{\alpha,C}$ fix the chamber C_0' in $\mathbb{A}$ adjacent to C along $M(\alpha, k)$. In particular, any such chamber C' and $D(\alpha, k)$ are contained in a same apartment.

4.4. Retraction with respect to a sector-germ

Let $\mathfrak{S}$ be a sector-germ in an apartment A of $\mathcal{I}(G, K)$. For $x \in \mathcal{I}(G, K)$, choose an apartment A' containing $\mathfrak{S}$ and x. As $\mathfrak{S}$ has a good fixator, there exists a g in $\hat{P}_{\mathfrak{S}}$ such that $A = g.A'$. If g and g' are two such elements, then $g^{-1}g'$ induces an automorphism of A' fixing the sector germ $\mathfrak{S}$, hence this automorphism is the identity: the map $A' \to A$, $y \mapsto g.y$ is unique. Moreover, $\mathfrak{S} \cup \{x\}$ has a good fixator (see Remark 4.4), so $\hat{P}_{\mathfrak{S} \cup \{x\}}$ is transitive on the possible apartments A': the point $g.x$ does not depend on the choice of A'. So, one may define $\rho_{A,\mathfrak{S}}(x) = g.x$.

Definition 4.5. — *The map $\rho = \rho_{A,\mathfrak{S}}\colon \mathcal{I} \to A$, $x \mapsto \rho_{A,\mathfrak{S}}(x)$ is the retraction of $\mathcal{I}$ onto A with center $\mathfrak{S}$. It depends only on A and $\mathfrak{S}$.*

The restriction of ρ to A is the identity. It is clear that, up to canonical isomorphisms, $\rho_{A,\mathfrak{S}}$ depends only on $\mathfrak{S}$. We set $\rho_{\pm\infty} = \rho_{\mathbb{A},\mathfrak{S}_{\pm\infty}}$.

A segment-germ $[x, y)$ for $x \neq y$ in an apartment A (cf. 2.2.2) is a narrow filter. When $x \leqslant y$ (resp. $y \leqslant x$), its enclosure is a closed-face and $[x, y)$

has a good fixator (4.2.1) and 3) of Proposition 4.3); we say that $[x,y)$ is *positive* (resp. *negative*) and that $[x,y]$ and $[x,y)$ are *generic*.

For any sector-germ $\mathfrak{S}$ and any segment-germ $[x,y)$, there exists an apartment containing $\mathfrak{S}$ and $[x,y)$, *i.e.* containing $[x,z]$ for some $z \in [x,y] \setminus \{x\}$.

A segment $[x,y]$ in an apartment is compact and, for $z \in [x,y]$, the set $[z,x) \cup [z,y)$ is an open neighbourhood of z. So, if $\mathfrak{S}$ is a sector-germ, there exist an integer n, points $x_0 = x, x_1, \ldots, x_n = y \in [x,y]$ and apartments $A_1, A_2, \ldots, A_n$ such that A_i contains $\mathfrak{S}$ and $[x_{i-1}, x_i]$. As a consequence, for all apartments A' containing $\mathfrak{S}$, $\rho_{A',\mathfrak{S}}([x,y])$ is the piecewise linear path $[\rho x_0, \rho x_1] \cup [\rho x_1, \rho x_2] \cup \cdots \cup [\rho x_{n-1}, \rho x_n]$.

We shall give a better description of this piecewise linear path when $x \leqslant y$ in the last section.

Remark 4.6. — The fixator of some spherical sector-face-germ $\mathfrak{F} = \mathrm{germ}(x + F^v)$ contains clearly the group $P(F^\mu)$ associated in [17] to a microaffine face $F^\mu = F^v \times \mathcal{F}$ (for some $\mathcal{F}$ containing x); and it was proved in [loc. cit.; 3.5] that $G = P(F^\mu).N.P(E^\mu)$ for any microaffine faces F^μ and E^μ of the same sign. Actually, using Proposition 3.6 above, the proof of this result is still valid if only one among E^v and F^v is spherical and the signs of E^v and F^v may be opposite. So, any two sector-face-germs in $\mathcal{I}$ are contained in a same apartment, if at least one of these sector-face-germs is spherical. For an abstract definition of affine hovels, this property is used in [20] as a good substitute to axioms (A3) and (A4) in Tits' definition of affine buildings [21], see also [16, Appendix 3].

4.5. Residue buildings

Let us denote the set of all positive (resp. negative) segment-germs $[x,y)$ with $x < y$ (resp. $y < x$) by $\mathcal{I}_x^+$ (resp. $\mathcal{I}_x^-$). The set $\mathcal{I}_x^+$ (resp. $\mathcal{I}_x^-$) can be given two structures of a building. An apartment $\mathfrak{a}_+$ in $\mathcal{I}_x^+$ (resp. $\mathfrak{a}_-$ in $\mathcal{I}_x^-$) is the intersection $A \cap \mathcal{I}_x^+$ (resp. $A \cap \mathcal{I}_x^-$) of an apartment A of $\mathcal{I}$ containing x with $\mathcal{I}_x^+$ (resp. $\mathcal{I}_x^-$) (or, more precisely, the set of all $[x,y)$ for $y \in A$ and $x < y$ (resp. $x > y$)). Now, on any apartment $\mathfrak{a}_\pm$, one can put two structures of a Coxeter complex:

- the restricted one, modelled on $(W_x^{\min}, S_x)$, where $W_x^{\min}$ is the subgroup of W generated by all reflections with respect to true walls passing through x, and where

$$S_x = \{s_H \mid H \text{ is a wall of } F(x, C_f^v) \text{ containing } x\}$$

may be infinite. We restrain the action of $W_x^{\min}$ on $\mathfrak{a}_\pm$. The faces of this structure are the faces $F(x,F^v)$ with F^v positive (resp. negative) (or, more precisely, the set of all segment-germs $[x,y)$ contained in $F(x,F^v)$).

- the unrestricted one, modelled on (W^v,S^v), where we force x to be a special point and consider the faces $\mathrm{germ}_x(x+F^v)$ (local-face in x) with F^v positive (resp. negative) (note that $\mathrm{germ}_x(x+F^v) \subset F(x,F^v)$). So we add new (ghost) walls $M(\alpha,k)$ for $\alpha \in \Phi$ and $k \in \mathbb{R}\setminus\mathbb{Z}$, $\alpha(x)+k=0$.

Proposition 4.7. — *The set $\mathcal{I}_x^+$ or $\mathcal{I}_x^-$, endowed with its apartments with their restricted (resp. unrestricted) structures of Coxeter complex, is a building.*

Proof. — We have to verify the last two axioms of a building (as in [2, IV.1] or [15, 2.4.1]). We focus on the positive case, the negative one is obtained in the same way. In both Coxeter structures, 4.3.4) shows that, given two faces F_1 and F_2, there exists an apartment containing x and both of them. Further, the group $G_{F_1\cup F_2}$ acts transitively on the apartments containing $F_1\cup F_2$. Hence, for any two such apartments A and A', there exists an element $g \in G_{F_1\cup F_2}$ such that $A' = g\cdot A$ which also gives an isomorphism $\mathfrak{a}_+ \simeq \mathfrak{a}'_+$ fixing $F_1\cup F_2$. □

Note that the unrestricted building structure can be thick only when it coincides with the restricted one, *i.e.* when x is special (thick means that any panel is a face of at least three chambers). The buildings $\mathcal{I}_x^\pm$ may be spherical for the restricted structures, as $W_x^{\min}$ may be finite when x is not special.

Now, we consider $\mathcal{I}_x^\pm$ endowed with the unrestricted structure. On the set of chambers $Ch(\mathcal{I}_x^\varepsilon)$ of $\mathcal{I}_x^\varepsilon$ (for a sign $\epsilon\in\{+,-\}$), we have a distance $d_\varepsilon : Ch(\mathcal{I}_x^\pm)\times Ch(\mathcal{I}_x^\pm) \to W^v$ defined as follows, cf. [15, 2.2 to 2.4].

If $c,c' \in Ch(\mathcal{I}_x^\varepsilon)$, choose an apartment A containing c,c' and a chamber $c_0 = \mathrm{germ}_x(x+\varepsilon C_0)$ in $\mathfrak{a}_\varepsilon = A\cap\mathcal{I}_x^\varepsilon$; identify (A,C_0) to (A_f,C_f^v); this enables us to identify $W^v(A)$ to $W^v = W^v(A_f)$. Now, if $c=wc_0$ and $c'=w'c_0$ for some $w,w'\in W^v$, then $d_\varepsilon(c,c') = w^{-1}w'$. Note that if we choose $c_0=c$, then $c' = d_\varepsilon(c,c').c$.

Now, we define a codistance

$$d_x^* : \big(Ch(\mathcal{I}_x^+)\times Ch(\mathcal{I}_x^-)\big) \cup \big(Ch(\mathcal{I}_x^-)\times Ch(\mathcal{I}_x^+)\big) \to W^v$$

in the following way. If $(c,e)\in Ch(\mathcal{I}_x^\varepsilon)\times Ch(\mathcal{I}_x^{-\varepsilon})$, by 4.3.4), there exists an apartment A containing x, c and e, unique up to isomorphism. If $c' = \mathrm{germ}_x(x+C')$ is a chamber in A, we denote the chamber opposite c' in A

by $-c'$, *i.e.* $-c' = \mathrm{germ}_x(x - C')$. Choose a chamber $c_0 = \mathrm{germ}_x(x \pm \varepsilon C_0)$ in $\mathfrak{a}_{\pm\varepsilon} = A \cap \mathcal{I}_x^{\pm\varepsilon}$ and identify (A, C_0) to (A_f, C_f^v). If $c = \pm w_1.c_0$ and $e = \mp w_2.c_0$, the codistance between c and e is then $d_x^*(c,e) = d_{-\varepsilon}(-c,e) = d_\varepsilon(c,-e) = w_1^{-1}w_2$. It does not depend on the choices.

PROPOSITION 4.8. — *The two buildings $\mathcal{I}_x^+$ and $\mathcal{I}_x^-$, endowed with their unrestricted structures of buildings and the codistance d_x^*, form a twinned pair of buildings.*

N.B. — With analogue arguments, one shows that this still holds if $\mathcal{I}_x^+$ and $\mathcal{I}_x^-$ are endowed with the restricted structures.

Proof. — We have to check the axioms of twinning as given in [23, 2.2], see also [15, 2.5.1].

Indeed, the first axiom (Tw1) is fulfilled: $d_x^*(e,c) = w_2^{-1}w_1 = d_x^*(c,e)^{-1}$. Let now $c \in Ch(\mathcal{I}_x^\varepsilon)$ and $e, e' \in Ch(\mathcal{I}_x^{-\varepsilon})$ be chambers such that $d_x^*(c,e) = w$ and $d_{-\epsilon}(e,e') = s \in S^v$ with $\ell(ws) = \ell(w) - 1$. Let A be an apartment containing x, c and e and choose $c_0 = -c$; since $\ell(ws) = \ell(w) - 1$, the wall H generated by the panel of e of type $\{s\}$ separates the latter from $-c$. In other words, c and e are on the same side of H. Therefore, by 4.3.4) there exists an apartment A' containing c, e and e'. In this apartment, $e = w.(-c)$ and $e' = (wsw^{-1}).e$, so, $e' = ws.(-c)$ and $d_x^*(c,e') = ws$. This is the second axiom (Tw2).

To check the third axiom (Tw3), let again c and e be two chambers such that $d_x^*(c,e) = w$, and let $s \in S^v$. In an apartment A containing c and e, the chamber h adjacent to e along the panel of type $\{s\}$ satisfies $d_{-\epsilon}(e,h) = s$ and $d_x^*(c,h) = ws$. □

An apartment A of $\mathcal{I}$ containing x gives a twin apartment $\mathfrak{a} = \mathfrak{a}_+ \cup \mathfrak{a}_-$, where $\mathfrak{a}_\pm = A \cap \mathcal{I}_x^\pm$. If c_0 is a chamber in $\mathfrak{a}$, there is (as in any twin building) a retraction ρ of center c_0 of $\mathcal{I}_x^+$ onto $\mathfrak{a}_+$ and of $\mathcal{I}_x^-$ onto $\mathfrak{a}_-$; it preserves the distances or codistances to c_0.

5. Littelmann paths

In this paragraph, we give a brief account and some new results on Littelmann's theory of paths [10], [11], [12]. First, we recall the definitions of $\lambda-$paths, billiard paths, LS paths, and Hecke paths; we then compare the last two notions (see Section 5.1). In analogy with [5] where the dimension of a gallery is defined, we introduce some statistics on paths used to characterize the LS paths (see Section 5.2 and 5.3). Note that the symmetrizability of $\mathfrak{g}$ assumed since 3.7 is useless in this section.

5.1. $\lambda-$paths

We consider piecewise linear continuous paths $\pi : [0,1] \to \mathbb{A}$ such that each (existing) tangent vector $\pi'(t)$ is in an orbit $W^v.\lambda$ of some $\lambda \in \overline{C_f^v}$ under the vectorial Weyl group W^v. Such a path is called a $\lambda-$*path*; it is increasing with respect to the preorder relation of 2.2.1. If $\pi(0)$, $\pi(1)$ and λ are in Y, we say that π is "in Y".

For any $t \neq 0$ (resp. $t \neq 1$), we let $\pi'_-(t)$ (resp. $\pi'_+(t)$) denote the derivative of π at t from the left (resp. from the right). Further, we define $w_\pm(t) \in W^v$ to be the smallest element in its $W^v_\lambda-$class such that $\pi'_\pm(t) = w_\pm(t).\lambda$ (where W^v_λ is the fixator in W^v of λ). Moreover, we denote by $\pi_-(t) = \pi(t) - [0,1)\pi'_-(t) = [\pi(t), \pi(t-\varepsilon))$ (resp. $\pi_+(t) = \pi(t) + [0,1)\pi'_+(t) = [\pi(t), \pi(t+\varepsilon))$ (for $\varepsilon > 0$ small) the positive (resp. negative) segment-germ of π at t (cf. 2.2.2).

The reverse path $\overline{\pi}$ defined by $\overline{\pi} = \pi(1-t)$ has symmetric properties, it is a $(-\lambda)-$path.

If, for all t, $w_+(t) \in W^v_{\pi(t)}.w_-(t)$, we shall say that π is a *billiard path*. This corresponds to what is stated in [8, Lemma 4.4], but seems stronger than the definition given in [loc. cit.; 2.5] which looks more like our definition of $\lambda-$path.

For any choices of $\lambda \in \overline{C_f^v}$, $\pi_0 \in \mathbb{A}$, $r \in \mathbb{N} \setminus \{0\}$ and sequences $\underline{\tau} = (\tau_1, \tau_2, \dots, \tau_r)$ of elements in W^v/W^v_λ and $\underline{a} = (a_0 = 0 < a_1 < a_2 < \cdots < a_r = 1)$ of elements in $\mathbb{R}$, we define a $\lambda-$path $\pi = \pi(\lambda, \pi_0, \underline{\tau}, \underline{a})$ by the formula:

$$\pi(t) = \pi_0 + \sum_{i=1}^{j-1} (a_i - a_{i-1})\tau_i(\lambda) + (t - a_{j-1})\tau_j(\lambda) \quad \text{for} \quad a_{j-1} \leqslant t \leqslant a_j.$$

Any $\lambda-$path may be defined in this way. We shall always assume $\tau_j \neq \tau_{j+1}$.

5.1.1. LS and Hecke paths

We consider now more specific paths.

DEFINITION 5.1. — [10] *A Lakshmibai-Seshadri path (or LS path) of shape $\lambda \in Y^+$ is a $\lambda-$path $\pi = \pi(\lambda, \pi_0, \underline{\tau}, \underline{a})$ starting in $\pi_0 \in Y$ and such that: for all $j = 1, \dots, r-1$, there exists an a_j-chain from τ_j to τ_{j+1} i.e. a sequence of cosets in W^v/W^v_λ:*

$$\sigma_{j,0} = \tau_j \ , \ \sigma_{j,1} = r_{\beta_{j,1}}\tau_j \ , \dots , \ \sigma_{j,s_j} = r_{\beta_{j,s_j}} \dots r_{\beta_{j,1}}\tau_j = \tau_{j+1}$$

where $\beta_{j,1}, \dots, \beta_{j,s_j}$ are positive real roots such that, for all $i = 1, ..., s_j$:

i) $\sigma_{j,i} < \sigma_{j,i-1}$, *for the Bruhat-Chevalley order on* W^v/W^v_λ,

ii) $a_j\beta_{j,i}(\sigma_{j,i}(\lambda)) \in \mathbb{Z}$,

iii) $\ell_\lambda(\sigma_{j,i}) = \ell_\lambda(\sigma_{j,i-1}) - 1$, *here* $\ell_\lambda(-)$ *is the length in* W^v/W^v_λ.

N.B. — Actually Littelmann requires the following additional condition

iv) π *is normalized i.e.* $\pi_0 = 0$.

Definition 5.2. — [8, 3.27] *A Hecke path of shape* λ *is a* $\lambda-$*path such that, for all* $t \in [0,1] \setminus \{0,1\}$, $\pi'_+(t) \leqslant_{W^v_{\pi(t)}} \pi'_-(t)$, *which means that there exists a* $W^v_{\pi(t)}-$*chain from* $\pi'_-(t)$ *to* $\pi'_+(t)$, *i.e. finite sequences* $(\xi_0 = \pi'_-(t), \xi_1, \dots, \xi_s = \pi'_+(t))$ *of vectors in* V *and* $(\beta_1, \dots, \beta_s)$ *of positive real roots such that, for all* $i = 1, \dots, s$*:*

v) $r_{\beta_i}(\xi_{i-1}) = \xi_i$,

vi) $\beta_i(\xi_{i-1}) < 0$,

vii) $r_{\beta_i} \in W^v_{\pi(t)}$ *i.e.* $\beta_i(\pi(t)) \in \mathbb{Z}$*:* $\pi(t)$ *is in a wall of direction* Ker(β_i).

Remarks 5.3. — Conditions v) and vii) tell us that π is a billiard path. More precisely, the path is folded at $\pi(t)$ by applying successive reflections along the walls $M(\beta_i, -\beta_i(\pi(t)))$. Moreover condition vi) tells us that the path is "positively folded" (cf. [5]).

The definition of affine paths in [Littelmann-98] is a little bit different; in particular, it is stable by concatenation.

5.1.2. LS versus Hecke

Let $\pi = \pi(\lambda, \pi_0, \underline{\tau}, \underline{a})$ be a $\lambda-$path. The conditions in Definition 5.2 are trivially satisfied for $t \neq a_1, \dots, a_{r-1}$. So, we compare conditions i), ii), iii) to conditions v), vi), vii) at $t = a_j$, $1 \leqslant j \leqslant r-1$, for $s = s_j$ and $\beta_i = \beta_{j,i}$. As $\pi'_-(t) = \tau_j(\lambda)$, the condition v) tells us that $\xi_i = \sigma_{j,i}(\lambda)$.

Lemma 5.4. — *Conditions i) and vi) are equivalent. If they are satisfied (for all* $i = 1, \dots, s_j$*), then* $w_+(t) < w_-(t)$ *in the Bruhat-Chevalley order of* W^v/W^v_λ.

Proof. — This is clear as $\sigma_{j,i} = r_{\beta_{j,i}}\sigma_{j,i-1}$ and $\xi_{i-1} = \sigma_{j,i-1}(\lambda)$. □

Remark 5.5. — When λ is in Y, the conditions i) and ii) tell us that $a_j \in \mathbb{Q}$ (as required by Littelmann for LS paths).

Lemma 5.6. — *Suppose that* $\pi_0 \in Y$, $\lambda \in Y^+$ *and that conditions ii) are satisfied for* $1 \leqslant j' < j$ *and* $1 \leqslant i \leqslant s_{j'}$. *Then the set of conditions ii) for* $1 \leqslant i \leqslant s_j$ *(and this* j*) is equivalent to the set of conditions vii) for*

$1 \leqslant i \leqslant s_j$ (and $t = a_j$). If $\pi_0 \in Y$, $\lambda \in Y^+$ and conditions ii) (or vii)) are satisfied for all $1 \leqslant j \leqslant r-1$ and $1 \leqslant i \leqslant s_j$, then $\pi(1) \in Y$, hence π is in Y.

Proof. — From the definition, one has

$$\pi(a_j) = \pi_0 + \sum_{i=1}^{j} (a_i - a_{i-1})\tau_i(\lambda) = \pi_0 + a_j\tau_j(\lambda) + \sum_{i=1}^{j-1} a_i(\tau_i(\lambda) - \tau_{i+1}(\lambda))$$

and (with the $\sigma_{j,i}$ as in Definition 5.1):

$$a_j(\tau_{j+1}(\lambda) - \tau_j(\lambda)) = \sum_{i=1}^{s_j} a_j(\sigma_{j,i}(\lambda) - \sigma_{j,i-1}(\lambda)) = \sum_{i=1}^{s_j} a_j\beta_{j,i}(\sigma_{j,i}(\lambda)){\beta^\vee}_{j,i}.$$

Hence, the conditions (ii) for $1 \leqslant i \leqslant s_j$ imply that $a_j(\tau_{j+1}(\lambda) - \tau_j(\lambda)) \in Q^\vee \subset Y$. In particular, conditions ii) for all i, j imply that $\pi(1) \in Y$.

One has $a_j\beta_{j,i}(\sigma_{j,i}(\lambda)) = r_{\beta_{j,1}} \cdots r_{\beta_{j,i}}(\beta_{j,i})(a_j\tau_j(\lambda))$, so the condition ii) above for $i = 1, \dots, s_j$ may be written:

$$r_{\beta_{j,1}} \cdots r_{\beta_{j,i}}(\beta_{j,i})(a_j\tau_j(\lambda)) \in \mathbb{Z}.$$

It is easy to verify that these conditions, for all $i = 1, \dots, s_j$, mean that the roots $\beta_{j,i}$ satisfy $\beta_{j,i}(a_j\tau_j(\lambda)) \in \mathbb{Z}$. If we assume ii) for each $j' < j$ and $i \leqslant s_{j'}$, this is equivalent to $\beta_{j,i}(\pi(a_j)) \in \mathbb{Z}$. □

Any LS path π is a Hecke path in Y. The reverse path $\overline{\pi}$ has symmetric properties. The reverse path of a Hecke path in Y has also symmetric properties.

Conversely, any Hecke path π of shape λ in Y is not far from being a LS path. Condition iii) only is missing. Actually, by condition i) one has $s_j \leqslant \ell_\lambda(\tau_j) - \ell_\lambda(\tau_{j+1})$; so condition iii) is equivalent to $s_j = \ell_\lambda(\tau_j) - \ell_\lambda(\tau_{j+1})$. Hence π is a LS path if and only if the $W^v_{\pi(t)}-$chains are of maximal lengths. See [8, Proposition 3.24] for a more precise statement.

5.2. Statistics on paths

We define two statistics on $\lambda-$paths and compare them with the one one would have defined inspired by [5].

5.2.1. Dual and co-dimension

Definition 5.7. — *The dual dimension of a $\lambda-$path π, denoted by* $\mathrm{ddim}(\pi)$ *and the codimension of* π*, denoted by* $\mathrm{codim}(\pi)$*, are the non negative integers:*

$$\mathrm{ddim}(\pi)=\sum_{t>0}\ell_{\pi(t)}(w_-(t)),\qquad \mathrm{codim}(\pi)=\sum_{t<1}\ell_{\pi(t)}(w_+(t)),$$

where $\ell_{\pi(t)}(\ )$ is the relative length function on the Coxeter group W^v with respect to $W^v_{\pi(t)}$ defined as follows: $\ell_{\pi(t)}(w)$ is the number of walls $M(\alpha)$ for $\alpha\in\Phi^+_{\pi(t)}$ separating the fundamental chamber C^v_f from wC^v_f; it coincides with the usual length on $W^v_{\pi(t)}$.

It seems that the sums are infinite, but, actually, there are only a finite number of possible $w_-(t)$ or $\pi'_-(t)=w_-(t)\lambda$ (resp. $w_+(t)$ or $\pi'_+(t)=w_+(t)\lambda$). Moreover, for any t, $\ell_{\pi(t)}(w_-(t))$ (resp. $\ell_{\pi(t)}(w_+(t))$) is the number of roots $\beta\in\Phi^+_{\pi(t)}$ such that $\beta(\pi'_-(t))<0$ (resp. $\beta(\pi'_+(t))<0$). Hence $\mathrm{ddim}(\pi)$ (resp. $\mathrm{codim}(\pi)$) is the number of pairs $(t,M(\beta,k))$ consisting of a $t>0$ (resp. $t<1$) and a wall associated to $\beta\in\Phi^+$ such that $\pi(t)=\overline{\pi}(1-t)\in M(\beta,k)$ and $\pi(t-\varepsilon)=\overline{\pi}(1-t+\varepsilon)\in D^\circ(\beta,k)=\mathbb{A}\setminus D(-\beta,-k)$ (resp. $\pi(t+\varepsilon)\notin D(\beta,k)$), for all small $\varepsilon>0$; this number is clearly finite.

To be short, $\mathrm{ddim}(\pi)$ is the number (with multiplicities) of all walls positively leaved by the reverse path $\overline{\pi}$ (load-bearing walls for $\overline{\pi}$ as in [5]); and $\mathrm{codim}(\pi)$ is the number (with multiplicities) of all walls negatively leaved by π.

In the following, for $\beta\in\Phi^+$ and π a $\lambda-$path, we define $pos_\beta(\pi)$ (resp. $neg_\beta(\pi)$) as the number (with multiplicities) of walls of direction $\mathrm{Ker}(\beta)$ leaved positively (resp. negatively) by π. Hence:

$$\mathrm{ddim}(\pi)=\sum_{\beta>0}pos_\beta(\overline{\pi})\quad\text{and}\quad\mathrm{codim}(\pi)=\sum_{\beta>0}neg_\beta(\pi)$$

5.2.2. Classical case

Let π be a $\lambda-$path in Y and set $\nu=\pi(1)-\pi(0)$. If Φ is finite, [5] suggests us to define the dimension of π as: $\dim(\pi)=\sum_{\beta>0}pos_\beta(\pi)$ (so, $\mathrm{ddim}(\pi)=\dim(\overline{\pi})$) and to prove (for Hecke paths) that $\dim(\pi)\leqslant\rho(\lambda+\nu)$ where $\rho_{\Phi^+}=\rho$ is defined by $2\rho=\sum_{\beta>0}\ \beta$.

Actually, $\beta(\nu)=pos_\beta(\pi)-pos_\beta(\overline{\pi})=neg_\beta(\overline{\pi})-neg_\beta(\pi)$. So, $\dim(\pi)\leqslant\rho(\lambda+\nu)$ if and only if $\sum_{\beta>0}pos_\beta(\pi)\leqslant\rho(\lambda-\nu)+\sum_{\beta>0}\ \beta(\nu)=\rho(\lambda-\nu)+\sum_{\beta>0}pos_\beta(\pi)-\sum_{\beta>0}pos_\beta(\overline{\pi})$ if and only if $\mathrm{ddim}(\pi)\leqslant\rho(\lambda-\nu)$.

First, one has $pos_\beta(\pi)+neg_\beta(\pi) = neg_\beta(\overline{\pi})+pos_\beta(\overline{\pi})$. Further, $\dim(\pi)+\mathrm{codim}(\pi) = \sum_{\beta>0}(pos_\beta(\pi)+neg_\beta(\pi)) = \sum_{\beta>0}(neg_\beta(\overline{\pi})+pos_\beta(\overline{\pi}))$ is the number of pairs $(t, M(\beta,k))$ consisting of a $t<1$ (resp. $t>0$) such that $\pi(t)\in M(\beta,k)$ and $\pi_+(t)\not\subset M(\beta,k)$ (resp. $\pi_-(t)\not\subset M(\beta,k)$). This number is invariant if we replace π by π_1 defined by: $\pi_1(t)=\pi(t)$ for $t\leqslant t_1$ and $\pi_1(t)=w\pi(t)$ for $t\geqslant t_1$, for some $t_1\in[0,1]$ and $w\in W^{\min}_{\pi(t_1)}$. In addition, any billiard path of shape λ is obtained by a sequence of such transformations starting from the straight $\lambda-$path π_λ ($\pi_\lambda(t)=t\lambda$). So, $\dim(\pi)+\mathrm{codim}(\pi) = \dim(\pi_\lambda)+\mathrm{codim}(\pi_\lambda) = \dim(\pi_\lambda) = \sum_{\beta>0}\ \beta(\lambda) = \rho(2\lambda)$. Therefore, for any billiard path π in Y, $\dim(\pi)\leqslant\rho(\lambda+\nu)$ if and only if $\mathrm{codim}(\pi)\geqslant\rho(\lambda-\nu)$.

5.3. A new characterization of LS paths

The goals of this section are to prove, in case Φ is infinite, the inequalities $\mathrm{codim}(\pi)\geqslant\rho(\lambda-\nu)\geqslant\mathrm{ddim}(\pi)$ for Hecke paths in Y, and to obtain a new characterization of LS paths. We choose $\rho_{\Phi+}=\rho\in X$ such that $\rho(\alpha^\vee)=1$ for all simple roots α. It is clear that $\lambda-\nu$ is a linear combination of coroots; so $\rho(\lambda-\nu)$ does not depend on the choice of ρ.

5.3.1. The characterization

Proposition 5.8. — *Let π be a Hecke path of shape λ in Y and $\nu=\pi(1)-\pi(0)$. Then*

$$\mathrm{ddim}(\pi)\leqslant\rho(\lambda-\nu)\leqslant\mathrm{codim}(\pi)\quad\text{and}\quad\mathrm{ddim}(\pi)+\mathrm{codim}(\pi)=2\rho(\lambda-\nu)$$

with equality if and only if π is a LS path.

The proof of Proposition 5.8 follows the same strategy as the proof of Proposition 4 in [5] and occupies the next three subsections.

Corollary 5.9. — *Let y_0, $y_1\in Y$ and $\lambda\in Y^+$. Then the number of Hecke paths π of shape λ starting in y_0 and ending in y_1 is finite.*

N.B. — Using Littelmann's path model, it was already clear that the number of LS paths satisfying the same conditions is finite, but our proof is purely combinatorial.

Proof. — By Proposition 5.8 and the definition of codim, $\ell_{\pi(0)}(w_+(0))\leqslant\mathrm{codim}(\pi)\leqslant2\rho(\lambda-\nu)$, with $\nu=y_1-y_0$. As $\pi(0)$ is a special point, this means that there is a finite number of possible $w_+(0)$. So there is a finite number of possible $w_\pm(t)$, or $\sigma_{j,i}$, or $\beta_{j,i}$, or a_j satisfying conditions i) and ii) of Definition 5.1. In conclusion, the number of Hecke paths $\pi = \pi(\lambda, y_0, \underline{\tau}, \underline{a})$ is finite (perhaps zero). □

5.3.2. The operator $\tilde{e}_\alpha$

Definition 5.10. — *Let π be a $\lambda-$path and α a simple root. Set $Q = \min\{\alpha(\pi([0,1])) \cap \mathbb{Z}\}$, the minimal integral value attained by the function $\alpha(\pi(\ ))$ and let q be the greatest number in $[0,1]$ such that $\alpha(\pi([0,q])) \geqslant Q$. If $q < 1$ (i.e. if $Q > \min\{\alpha(\pi([0,1]))\}$), let $\theta > q$ be such that*

$$\alpha(\pi(q)) = \alpha(\pi(\theta)) = Q \quad \text{ and } \quad \alpha(\pi(t)) < Q \text{ for } q < t < \theta.$$

We cut the path π into three parts in the following way. Let π_1, π_2 and π_3 be the paths defined by

$$\pi_1(t) = \pi(tq); \quad \pi_2(t) = \pi(q+t(\theta-q))-\pi(q); \quad \pi_3(t) = \pi(\theta+t(1-\theta))-\pi(\theta)$$

for $t \in [0,1]$. Then, by definition, $\pi = \pi_1 * \pi_2 * \pi_3$, where $*$ means the concatenation of paths as defined in [10, 1.1]. The path $\tilde{e}_\alpha\pi$ is equal to $\pi_1 * r_\alpha(\pi_2) * \pi_3$. After a suitable reparametrization $\tilde{e}_\alpha\pi$ is a $\lambda-$path in Y.

We use also the operators e_α and f_α (α simple) defined by Littelmann in [10, 1.2 and 1.3]. We do not recall the complete definition here, but note that when they exist, $e_\alpha\pi = \pi_1 * r_\alpha(\pi_2) * \pi_3$ (resp. $f_\alpha\pi = \pi_1 * r_\alpha(\pi_2) * \pi_3$), where the path π is cut into well-defined parts $\pi = \pi_1 * \pi_2 * \pi_3$. Further, $e_\alpha\pi(1) = \pi(1) + \alpha^\vee$ and $f_\alpha\pi(1) = \pi(1) - \alpha^\vee$. After a suitable reparametrization, $e_\alpha\pi$ and $f_\alpha\pi$ are $\lambda-$paths in Y. More importantly, Littelmann obtains a characterization of LS paths by using these operators. He proves [10, 5.6] that a $\lambda-$path π with $\pi(0) = 0$ is a LS path if, and only if, there exist some simple roots $\alpha_{i_1}, ..., \alpha_{i_s}$ such that

$$e_{\alpha_{i_1}} \circ \cdots \circ e_{\alpha_{i_s}}(\pi) = \pi_\lambda,$$

where for all $t \in [0,1]$, $\pi_\lambda(t) = t\lambda$.

Lemma 5.11. — *i) If π is a Hecke path in Y and $e_\alpha\pi$ (resp. $\tilde{e}_\alpha\pi$) is defined, then $\operatorname{ddim}(e_\alpha\pi) = \operatorname{ddim}(\pi) - 1$ and $\operatorname{codim}(e_\alpha\pi) = \operatorname{codim}(\pi) - 1$ (resp. $\operatorname{ddim}(\tilde{e}_\alpha\pi) = \operatorname{ddim}(\pi)+1$ and $\operatorname{codim}(\tilde{e}_\alpha\pi) = \operatorname{codim}(\pi)-1$), and, similarly, if $f_\alpha\pi$ is defined, then $\operatorname{ddim}(f_\alpha\pi) = \operatorname{ddim}(\pi) + 1$ and $\operatorname{codim}(f_\alpha\pi) = \operatorname{codim}(\pi) + 1$.*

ii) If π is a Hecke path in Y such that $\tilde{e}_\alpha\pi$ is defined, then $\tilde{e}_\alpha\pi$ is again a Hecke path in Y.

iii) If π is a Hecke path in Y such that $\tilde{e}_\alpha\pi$ is not defined but $e_\alpha\pi$ (resp. $f_\alpha\pi$) is, then $e_\alpha\pi$ (resp. $f_\alpha\pi$) is again a Hecke path in Y.

We prove the Lemma in Section 5.3.4.

5.3.3. Proof of Proposition 5.8

By translation, we may (and shall often) suppose π normalized, *i.e.* $\pi(0) = 0$. It is clear that $\mathrm{ddim}(\pi_\lambda) = \mathrm{codim}(\pi_\lambda) = 0$. As a corollary of i) and the characterization of LS paths, if π is a LS λ−path then $\mathrm{ddim}(\pi) = \mathrm{codim}(\pi) = \rho(\lambda - \nu)$. The other implication is obtained by induction on $\rho(\lambda - \nu)$. We suppose $\pi(0) = 0$. There is only one λ−path π such that $\pi(1) = \lambda$; it is π_λ. And in this case, $\mathrm{ddim}(\pi_\lambda) = 0$.

If $\nu \neq \lambda$, then $w_+(0) \neq \mathrm{id}$ and there exists a simple root α such that $e_\alpha\pi$ or $\tilde{e}_\alpha\pi$ is defined. If, for all β simple, $\tilde{e}_\beta\pi$ is not defined, then the claim follows immediately by induction and by Lemma 5.11. Otherwise, we apply all possible operators $\tilde{e}_\beta$ to π to end up with a Hecke path η such that $\eta(1) = \pi(1) = \nu$, $\mathrm{ddim}(\eta) = \mathrm{ddim}(\pi) + k$, $\mathrm{codim}(\eta) = \mathrm{codim}(\pi) - k$ $(k > 0)$ and there still exists α such that $e_\alpha\eta$ is defined. But then, by induction, $\mathrm{ddim}(\eta) - 1 = \mathrm{ddim}(e_\alpha\eta) \leqslant \rho(\lambda - e_\alpha\eta(1)) = \rho(\lambda - \nu) - 1$, which implies that $\mathrm{ddim}(\pi) < \rho(\lambda - \nu)$. Moreover, $\mathrm{codim}(\pi) + \mathrm{ddim}(\pi) = \mathrm{codim}(\eta) + \mathrm{ddim}(\eta) = \mathrm{codim}(e_\alpha\eta) + \mathrm{ddim}(e_\alpha\eta) + 2 = 2\rho(\lambda - e_\alpha\eta(1)) + 2 = 2\rho(\lambda - \nu)$ (by induction).

Suppose now that $\mathrm{ddim}(\pi) = \rho(\lambda - \nu) > 0$, then for dimension reasons, $\tilde{e}_\alpha\pi$ is never defined. But $e_\alpha\pi$ is and $\mathrm{ddim}(e_\alpha\pi) = \rho(\lambda - \nu) - 1$. Repeating the same argument leads to a sequence of simple roots $\alpha_{i_1}, ..., \alpha_{i_s}$ such that $e_{\alpha_{i_1}} \circ \cdots \circ e_{\alpha_{i_s}}(\pi) = \pi_\lambda$, in other words, π is a LS path. This proves the proposition. □

Remark 5.12. — This proof implies also that a Hecke path in Y is LS if and only if, for all simple roots $\alpha_j, \alpha_{i_1}, ..., \alpha_{i_s}$, the minimum of $\alpha_j(e_{\alpha_{i_1}} \circ \cdots \circ e_{\alpha_{i_s}}(\pi))$ is in $\mathbb{Z}$, cf. [11, 4.5].

5.3.4. Proof of Lemma 5.11

We suppose $\pi(0) = 0$. Let us start with the operator e_α and dual dimensions. The paths π and $e_\alpha\pi$ are cut into three parts, meaning that

$$\begin{cases} \pi(t) = \pi_1(t), e_\alpha\pi(t) = \pi_1(t) & \text{if } 0 \leqslant t \leqslant 1/3 \\ \pi(t) = \pi_2(t) + \pi(1/3), e_\alpha\pi(t) = r_\alpha(\pi_2)(t) + \pi_1(1/3) & \text{if } 1/3 \leqslant t \leqslant 2/3 \\ \pi(t) = \pi_3(t) + \pi(2/3), e_\alpha\pi(t) = \pi_3(t) + e_\alpha\pi(2/3) & \text{if } 2/3 \leqslant t \leqslant 1. \end{cases}$$

For the first part of π, that is for $t \leqslant 1/3$, there is nothing to prove. Because α is a simple root, if $1/3 < t \leqslant 2/3$, the relative position of

$\pi_-(t)$ with respect to a wall $M(\beta,k)$ (with $\beta\neq\alpha$) is the same as the relative position of $e_\alpha\pi_-(t)$ with respect to $r_\alpha M(\beta,k)$. Further, if $2/3<t\leqslant 1$, $e_\alpha\pi(t)=\pi(t)+\alpha^\vee$. So, again, up to translation the relative positions are the same. It remains to check the positions relatively to the walls $M(\alpha,-Q),M(\alpha,-Q-2)$ at $t=2/3$. But $2/3$ is the smallest real number t such that $\pi(t)\in M(\alpha,-Q)$, therefore $\pi_-(2/3)\not\subset D(-\alpha,Q)$, $e_\alpha\pi(2/3)\in M(-\alpha,Q+2)$ and $e_\alpha\pi_-(2/3)\subset D(-\alpha,Q+2)$. Therefore, $\mathrm{ddim}(e_\alpha\pi)=\mathrm{ddim}(\pi)-1$.

For the formulas $\mathrm{ddim}(\tilde e_\alpha\pi)=\mathrm{ddim}(\pi)+1$ and $\mathrm{ddim}(f_\alpha\pi)=\mathrm{ddim}(\pi)+1$, similar arguments show that it suffices to look at the case $t=2/3$ in the corresponding cuts of the path π. For the operator $\tilde e_\alpha$, one has $\pi(2/3)=\tilde e_\alpha\pi(2/3)\in M(-\alpha,Q)$ and $\pi_-(2/3)\subset D(-\alpha,Q)$ whereas $\tilde e_\alpha\pi_-(2/3)\not\subset D(-\alpha,Q)$. This proves the formula for the operator $\tilde e_\alpha$. And for the operator f_α one has: $\pi(2/3)\in M(-\alpha,Q+1)$, $\pi_-(2/3)\subset D(-\alpha,Q+1)$ whereas $f_\alpha\pi(2/3)\in D(-\alpha,Q-1)$ and $f_\alpha\pi_-(2/3)\not\subset D(-\alpha,Q-1)$. The proof of i) for the dual dimensions is then complete.

For the codimensions, similar arguments show that it suffices to look at the case $t=1/3$ and the root α. For the operator e_α, $e_\alpha\pi(1/3)=\pi(1/3)\in M(\alpha,-Q-1)$, $\pi_+(1/3)\not\subset D(\alpha,-Q-1)$, $e_\alpha\pi_+(1/3)\not\subset D(-\alpha,Q+1)$; therefore $\mathrm{codim}(e_\alpha\pi)=\mathrm{codim}(\pi)-1$. For the operator f_α (resp. $\tilde e_\alpha$), $f_\alpha\pi(1/3)=\pi(1/3)\in M(\alpha,-Q)$ (resp. $\tilde e_\alpha\pi(1/3)=\pi(1/3)\in M(\alpha,-Q)$), $\pi_+(1/3)\not\subset D(-\alpha,Q)$ and $f_\alpha\pi_+(1/3)\not\subset D(\alpha,-Q)$ (resp. $\pi_+(1/3)\not\subset D(\alpha,-Q)$ and $f_\alpha\pi_+(1/3)\not\subset D(-\alpha,Q)$), therefore $\mathrm{codim}(f_\alpha\pi)=\mathrm{codim}(\pi)+1$ (resp. $\mathrm{codim}(\tilde e_\alpha\pi)=\mathrm{codim}(\pi)-1$).

Concerning ii), using the same arguments again, one has to take only care of the places $t=1/3$ and $t=2/3$ in the path π. For $t=1/3$, the $W^v_{\pi(1/3)}-$chain for $\tilde e_\alpha\pi$ is obtained from the one for π just by adding $\xi_{s+1}=\tilde e_\alpha\pi'_+(1/3)=r_\alpha(\pi'_+(1/3))$ and $\beta_{s+1}=\alpha$; as $\alpha(\pi'_+(1/3))<0$ the conditions are satisfied. For $t=2/3$, the $W^v_{\pi(2/3)}-$chain for $\tilde e_\alpha\pi$ is obtained from the one for π just by adding $\xi_{-1}=\tilde e_\alpha\pi'_-(2/3)=r_\alpha(\pi'_-(2/3))$ and $\beta_0=\alpha$; as $\alpha(\pi'_-(2/3))>0$ the conditions are satisfied (after a shift of the indices of the chain). Therefore, $\tilde e_\alpha\pi$ is a Hecke path and ii) is proved.

It remains to prove iii). Let us start with e_α. Once again, it suffices to check the values $t=1/3$ and $t=2/3$. The situation around the point $\pi(1/3)$ is the same as above. Because $\tilde e_\alpha\pi$ is not defined, $\alpha(\pi'_+(2/3))\geqslant 0$. Let $(\xi_0,\dots,\xi_s)$, $(\beta_1,\dots,\beta_s)$ be the $W^v_{\pi(2/3)}-$chain from $\pi'_-(2/3)$ to $\pi'_+(2/3)$. If

$\alpha = \beta_u$, $1 \leqslant u \leqslant s$ (and u is minimal for this property), then

$$(r_\alpha\xi_0, r_\alpha\xi_1, \dots, r_\alpha\xi_{u-1} = \xi_u, \xi_{u+1}, \dots, \xi_s),$$
$$(r_\alpha\beta_1, \dots, r_\alpha\beta_{u-1}, \beta_{u+1}, \dots, \beta_s)$$

is the $W^v_{e_\alpha\pi(2/3)}$−chain from $e_\alpha\pi'_-(2/3)$ to $e_\alpha\pi'_+(2/3)$. If no such u exists and $\alpha(\pi'_+(2/3)) > 0$ (resp. $\alpha(\pi'_+(2/3)) = 0$), then this chain is $(r_\alpha\xi_0, \dots, r_\alpha\xi_s, \xi_{s+1} = \xi_s)$, $(r_\alpha\beta_1, \dots, r_\alpha\beta_s, \beta_{s+1} = \alpha)$ (resp. $(r_\alpha\xi_0, \dots, r_\alpha\xi_s = \xi_s)$, $(r_\alpha\beta_1, \dots, r_\alpha\beta_s)$). This proves that $e_\alpha\pi$ is still a Hecke path. The proof for $f_\alpha\pi$ follows similar lines and is left to the reader! $\square$

6. Segments in the hovel

This section contains the most important application of the definition of the hovel $\mathcal{I}$. We first prove that the retraction of any segment $[x, y]$ (with $x \leqslant y$) in $\mathcal{I}$ is a Hecke path in $\mathbb{A}$ (see Theorem 6.2). Then, we give a parametrization of all segments retracting on a given Hecke path sharing the same end (Theorem 6.3 and Corollary 6.5). The algebraic structure of the set of parameters is studied in 6.3 and allows us to define a generalization of Mirković-Vilonen cycles. Then, we state another characterization of LS paths in terms of a new statistic, but depending on extra data and not solely on the path (6.4). To finish, we prove a result on the structure of $\mathcal{I}$ (Theorem 6.9).

The field K is as in Section 3.1. Note however that, in the classical case where G is a split reductive group, all what follows holds for any field K endowed with a discrete valuation; we just have to use the Bruhat-Tits building instead of the hovel constructed in Section 3.

6.1. Retracting segments

We consider a negative sector germ $\mathfrak{S}$ and denote by ρ the retraction of center $\mathfrak{S}$ without specifying on which apartment A (containing $\mathfrak{S}$) ρ maps $\mathcal{I}$, as ρ does not depend on A up to canonical isomorphisms. Actually we identify any pair $(A, \mathfrak{S})$ of an apartment A containing $\mathfrak{S}$ with the fundamental pair $(A_f = \mathbb{A}, \mathfrak{S}_{-\infty})$, this is well determined up to translation.

We consider two points x, y in the hovel with $x \leqslant y$. The segment $[x, y]$ is the image of the path $\pi : [0, 1] \longrightarrow \mathcal{I}$ defined by $\pi(t) = x + t(y - x)$ in any apartment containing x and y (4.3.2). As each segment in $[x, y]$ has a good fixator, the derivative $\pi'(t)$ is independent of the apartment containing a neighbourhood of $\pi(t)$ in $[x,\pi(t)]$ or $[\pi(t),y]$, up to the Weyl group W^v.

We saw in 4.4 that the image $\rho\pi$ is a piecewise linear continuous path in A. By the previous paragraph, there exists a λ in the fundamental closed-chamber $\overline{C^v_f}$ such that $\rho\pi'(t) = w_t.\lambda$ for each $t \in [0,1]$ (different from $\pi^{-1}(x_i)$ for x_i as in 4.4) and some $w_t \in W^v$ (chosen minimal with this property). Hence $\rho\pi$ is a $\lambda-$path (in particular the map ρ is increasing with respect to the "preorder" of 4.3.2) and may be described as $\pi(\lambda,\rho\pi(0),\underline{\tau},\underline{a})$. We shall prove that $\rho\pi$ is a Hecke path and often a LS path.

We choose some $t \in]0,1[= [0,1]\setminus\{0,1\}$ and we set $z=\rho\pi(t)$. We denote by $\rho\pi'_-$ (resp. $\rho\pi'_+$) the left (resp. right) derivative of $\rho\pi$ in t and w_- (resp. w_+) the minimal element in W^v such that $\rho\pi'_- = w_-\lambda$ (resp. $\rho\pi'_+ = w_+\lambda$).

Proposition 6.1. — *We have $\rho\pi'_+ \leqslant_{W^v_z} \rho\pi'_-$ (cf. Definition 5.2) and $w_+ \leqslant w_-$ in the Bruhat-Chevalley order of W^v/W^v_λ. More precisely, there exist $s \in \mathbb{N}$ and a sequence $\beta_1,\dots,\beta_s$ of positive real roots such that:*

– for $1 \leqslant i \leqslant s$, there exists a wall of direction $\mathrm{Ker}(\beta_i)$ containing $z=\rho\pi(t)$,

– if one defines $\xi_0 = \rho\pi'_-$, $\xi_1 = r_{\beta_1}.\xi_0$, …, $\xi_s = r_{\beta_s}.\dots.r_{\beta_1}.\xi_0$, one has $\xi_s = \rho\pi'_+$ and $\beta_k(\xi_{k-1}) < 0$ for $1 \leqslant k \leqslant s$,

– if one defines $\sigma_0 = w_-$, $\sigma_1 = r_{\beta_1}.w_-$, …, $\sigma_s = r_{\beta_s}\cdots r_{\beta_1}.w_-$, then $\rho\pi'_+ = \sigma_s\lambda$ and, for $1 \leqslant k \leqslant s$, one has $\sigma_k < \sigma_{k-1}$ in the Bruhat-Chevalley order of W^v/W^v_λ,

– there exists in $\mathfrak{a}_+ = A\cap\mathcal{I}^+_z$ an (unrestricted) gallery $\delta=(c_0,c'_1,\dots,c'_n)$ from $c_0 = \mathrm{germ}_z(z+C^v_f)$ to $c'_n = \mathrm{germ}_z(z+w_+C^v_f) \supset z+[0,1)\rho\pi'_+$, the type of which is associated to a (given) reduced decomposition of w_-. The panels along which this gallery is folded (actually, positively folded: see the proof) are successively the walls $z+\mathrm{Ker}(\beta_1)$, …, $z+\mathrm{Ker}(\beta_s)$.

Proof. — Let A^0 be an apartment containing $[x,y]$; set $\pi_- = [\pi(t),x)$ and $\pi_+ = [\pi(t),y)$. By 4.3.3 and 4.4 there exist apartments A^+ and A^- containing the sector $\mathfrak{s}$ (of direction $\mathfrak{S}$ and base point $\pi(t)$) and respectively π_+ and π_-. We choose A^- for the image A of ρ, so $\pi(t)=\rho\pi(t)=z$ and $\pi_- = \rho\pi_- = z-[0,1)\pi'_-$.

As A^0 and A^- contain π_-, there exists $g \in \hat{P}_{\pi_-}$ such that $A^0 = g.A^-$. In the decomposition $\hat{P}_{\pi_-} = U^{nm-}_{\pi_-}.U^{pm+}_{\pi_-}.\hat{N}_{\pi_-}$, the group $\hat{N}_{\pi_-}$ fixes π_- and stabilizes A^-, so, one has $A^0 = u^-u^+A^-$ with $u^- \in U^{nm-}_{\pi_-}$ and $u^+ \in U^{pm+}_{\pi_-}$. Let us consider the apartment $A^1 = (u^-)^{-1}A^0 = u^+A^-$; it contains $\pi_- = (u^-)^{-1}(\pi_-)$ and $\pi^1_+ = (u^-)^{-1}(\pi_+)$, which are opposite segment germs; moreover $\rho(\pi^1_+) = \rho(\pi_+)$. On the other side, A^1 contains the chamber

$C_0 = F(z, C^v_f)$ in A^-, which is opposite $\mathfrak{s}$. We replace in the following π_+ by π^1_+ and A^0 by A^1.

In A^-, $\pi_- \in w_-\mathfrak{s}$, hence, the germ opposite π_- is in w_-C_0. So, in A^1, $\pi^1_+ \in w^1_-C_0$, where w^1_- corresponds to w_- in the identification of $W^v(A^-)$ and $W^v(A^1)$ via u^+.

We choose in $\mathfrak{a}^1_+ = A^1 \cap \mathcal{I}^+_z$ (an apartment of $\mathcal{I}^+_z$ endowed with the unrestricted building structure) a minimal gallery $m = (c_0, c_1, ..., c_n)$ between $c_0 = \mathrm{germ}_z(z + C^v_f)$ and $c_n \supset \pi^1_+ = z + [0,1)w^1_-\lambda$ of type $\tau = (i_1, ..., i_n)$, $i_j \in I$; hence $w^1_- = r_{i_1} \cdots r_{i_n}$ is a reduced decomposition. The restriction of ρ to the residue twin building $(\mathcal{I}^+_z, \mathcal{I}^-_z, d^*_z)$ (with the unrestricted structure) preserves the codistance to $\bar{\mathfrak{s}} = \mathrm{germ}_z(\mathfrak{s})$, which is a chamber in $\mathcal{I}^-_z$. Therefore, this restriction is the retraction $\rho_z : \mathcal{I}^+_z \to \mathfrak{a}^-_+ = A^- \cap \mathcal{I}^+_z$ of centre $\bar{\mathfrak{s}}$. We have $\rho\pi^1_+ = \rho_z\pi^1_+$.

The retracted gallery $\delta = \rho_z(m) = (c_0, c'_1 = \rho_z(c_1), ..., c'_n = \rho_z(c_n))$ in $A = A^-$ is a positively folded gallery, meaning that $\rho_z(c_j) = \rho_z(c_{j+1})$ implies that $\rho_z(c_j)$ is on the positive side of the wall H_j spanned by the panel of type $\{i_j\}$ of $\rho_z(c_j)$ (note that H_j is a wall for the unrestricted structure). Otherwise, suppose that $\rho_z(c_j) = \rho_z(c_{j+1})$ is on the negative side of H_j. Then, because $\bar{\mathfrak{s}}$ is the opposite fundamental chamber in z, it is always on the negative side of H_j. Further, let $\mathfrak{a}$ be a twin apartment containing $\bar{\mathfrak{s}}$ and c_j, as the retraction preserves the codistance to $\bar{\mathfrak{s}}$, we also have that $\bar{\mathfrak{s}}$ and c_j are on the same side of the wall spanned by the panel of type $\{i_j\}$ of c_j in $\mathfrak{a}$. Therefore, 4.3.4) implies that, modifying the latter if needed, we can assume that c_{j+1} is still in $\mathfrak{a}$. But, on one side, $c_j \neq c_{j+1}$ then, computing in $\mathfrak{a}$, $\ell(d^*_z(\bar{\mathfrak{s}}, c_{j+1})) = \ell(d^*_z(\bar{\mathfrak{s}}, c_j)) - 1$; on the other side, $\ell(d^*_z(\bar{\mathfrak{s}}, c_{j+1})) = \ell(d^*_z(\bar{\mathfrak{s}}, \rho_z c_{j+1})) = \ell(d^*_z(\bar{\mathfrak{s}}, \rho_z c_j)) = \ell(d^*_z(\bar{\mathfrak{s}}, c_j))$. Contradiction!

If the wall H^1_j separating c_j from c_{j+1} in $\mathfrak{a}^1_+$ is a ghost wall *i.e.* not a true wall (for the restricted structure), then the enclosure of c_j in the hovel contains c_{j+1} and there is an apartment of $\mathcal{I}$ containing $\mathfrak{s}$, c_j and c_{j+1}, so $\rho_z(c_j) \neq \rho_z(c_{j+1})$.

Let us now denote by $j_1, ..., j_s$ the indices such that $c'_j = \rho_z(c_j) = \rho_z(c_{j+1}) = c'_{j+1}$. For any $k \in \{1, ..., s\}$, H_{j_k} is a true wall spanned by the panel $\rho(H^1_{j_k} \cap c_j)$ and we denote the positive real root associated with H_{j_k} by β_k (i.e. H_{j_k} is of direction $\mathrm{Ker}(\beta_k)$). Actually, the gallery δ is obtained from the minimal gallery $\delta^0 = (c^0_0 = c_0, c^0_1, \ldots, c^0_n)$ of type $(i_1, ..., i_n)$ beginning in c_0, ending in $c^0_n = w_-(c_0)$ and staying inside A^- by applying successive (positive) foldings along the walls associated to the indices $\{j_1, ..., j_s\}$, starting with H_{j_1}, then folding along H_{j_2}... At each step, one gets a positively folded gallery $\delta^k = (c_0^k = c_0, c_1^k, \ldots, c_n^k)$ ending closer and closer to the chamber c_0. So, this proves the last assertion of the proposition.

Let us denote $\xi_0 = \pi'_- = \rho\pi'_- = w_-\lambda$ (in A_-) and $\xi_k = r_{\beta_k} \cdots r_{\beta_1}\xi_0$. As $\rho_z(c_n) \supset \rho\pi_+ = z + [0,1)\rho\pi'_+$ and $\rho\pi'_+ \in W^v\rho\pi'_-$ one has $\xi_s = \rho\pi'_+$, and more generally, $z + [0,1)\xi_k \subset c_n^k$. As δ^0 is a minimal gallery from c_0 to $z + [0,1)\pi'_-$, $c_{j+1}^0, \ldots, c_n^0$ and $z + [0,1)\pi'_-$ are on the same side of any wall separating c_0^0 from c_{j+1}^0; in particular, $(c_{j_k+1}^k, \ldots, c_n^k)$ is a minimal gallery, entirely on the same side of H_{j_k} and $z + [0,1)\xi_k \not\subset H_{j_k}$. But $c_{j_k}^k = \rho_z(c_{j_k}) = \rho_z(c_{j_k+1}) = c_{j_k+1}^k$ and we saw that this chamber is on the positive side of the wall H_{j_k} (of direction $\mathrm{Ker}(\beta_k)$). So, $c_{j_k+1}^k, \ldots, c_n^k$ are on the positive side of H_{j_k}; this means that $\beta_k(\xi_k) > 0$ *i.e.* $\beta_k(\xi_{k-1}) < 0$. Hence, the sequences $(\xi_0, \xi_1, \ldots, \xi_s)$ and $(\beta_1, \ldots, \beta_s)$ give a W_z^v−chain from $\pi'_- = \rho\pi'_-$ to $\rho\pi'_+$. This proves the proposition, in view of Lemma 5.4. □

THEOREM 6.2. — *Let $\pi = [x,y]$ be a segment in an apartment A' with $x < y$, and ρ the retraction of $\mathcal{I}$ with center the fundamental sector-germ $\mathfrak{S}_{-\infty}$ onto an apartment A. Then the retracted segment $\rho\pi$ is a Hecke path in A.*

If moreover x and y are cocharacter points (i.e. $x, y \in Y(A')$), then $\rho\pi$ is a Hecke path in $Y(A)$.

Proof. — The path $\rho\pi$ is Hecke by Proposition 6.1 and Definition 5.2. If $x, y \in Y(A')$, computing in A', $\lambda = W^v(y-x) \cap \overline{C_f^v}$ is in $Y^+(A')$. Moreover, by 4.3.1, $\rho(Y(A')) \subset Y(A)$, so $\rho\pi$ is in $Y(A)$. □

6.2. Segments retracting on a given Hecke path

If we consider all segments $\pi = [x,y]$ from some $x \in \mathbb{A}$ to some $y \in \mathcal{I}$ (*i.e.* $\pi(t) = x + t(y-x)$) whose retraction $\rho\pi$ is a given Hecke path π_1 in A starting at x, then, there are too many of them. For example, take for π_1 the path $t \mapsto t\lambda$ (with λ dominant), then, already at 0, one has infinitely many choices to define a segment starting at 0 and retracting onto π_1. Therefore, in this subsection, we fix $y = \pi(1)$.

More precisely, let $y \in \mathcal{I}$ and π_1 a Hecke path in A with $\pi_1(1) = \rho(y)$, we define $S(\pi_1, y)$ as the set of all segments $\pi = [x,y]$ in $\mathcal{I}$ such that $\pi_1 = \rho\pi$.

THEOREM 6.3. — *The set $S(\pi_1, y)$ is nonempty and is parametrized by exactly $N = \mathrm{ddim}(\pi_1)$ parameters in the residue field κ. More precisely, the set $P(\pi_1, y)$ of parameters is a finite union of subsets of κ^N, each being a product of N factors either equal to κ or to κ^*.*

Remark 6.4. — In particular, $P(\pi_1, y)$ is a Zariski open subset of κ^N stable under the natural action of the torus $(\kappa^*)^N$, in other words, a quasi-affine toric variety.

Proof. — We shall prove that, for $t \in [0,1]$, the segments $\pi^t : [t,1] \to \mathcal{I}$ with $\pi^t(1) = y$, retracting onto $\pi_1^t = {\pi_1}_{|_{[t,1]}}$, are parametrized by exactly $\sum_{t'>t} \ell_{\pi_1(t)}(w_-(t))$ parameters. This is clear for $t = 1$. Suppose the result true for some t. So, $\pi(t)$ is given and we shall prove now that the number of parameters for the choice of the segment-germ $\pi_-(t)$ of origin $\pi(t)$ is $\ell_{\pi_1(t)}(w_-(t))$. This result and arguments after the Definition 5.7 imply that $\pi_-(t)$ determines $\pi_{|_{[t',t]}}$ where t' $(< t)$ is 0 or the next number in $[0,1]$ such that $\overline{\pi_1}$ leaves positively a wall in t'.

Now, $\pi(t)$ is given and we want to find out how many parameters govern the choice of $\pi_-(t)$. We choose A so that it contains $\mathfrak{S}$ and $\pi_+(t)$; so $\pi_+(t) = \rho\pi_+(t)$ and we set $z = \pi(t)$. There is an apartment A^- containing $\mathfrak{S}$ and $\pi_-(t)$, thus, ρ is an isomorphism from A^- to A. The (unrestricted) chamber $c_0^- = \mathrm{germ}_z(z - C_f^v)$ (cf. 4.5) is in $A \cap A^-$. We choose a reduced decomposition $w_-(t) = r_{i_1}.\dots.r_{i_n}$ in W^v. The associated minimal (unrestricted) gallery of type $(i_1, \dots, i_n)$ from c_0^- to $\pi_-(t)$ is denoted by $m^- = (c_0^-, c_1^-, \dots, c_n^-)$. Clearly, $\pi_-(t)$ is entirely determined by the gallery m^-, so it seems to depend on $n = \ell(w_-(t))$ parameters in κ. But actually, if the wall separating c_{j-1}^- from c_j^- (or ρc_{j-1}^- from ρc_j^-) is a ghost wall *i.e.* not a true wall (see Section 4.5), the chamber c_j^- is determined by c_{j-1}^-; whereas, if this wall is true (some $M(\alpha, k)$ for some $\alpha \in \Phi^-$), then the choice of c_j^- depends on an element in $U_{\alpha,k}^\times = U_{\alpha,k}/U_{\alpha,>k} \simeq \kappa$ (cf. 4.3.4). Hence, the true number of parameters is $\ell_{\pi_1(t)}(w_-(t))$.

But, we forgot to check that $\pi_-(t)$ is opposite $\pi_+(t)$. Actually, as we shall see now, removing at most one value for each parameter, this condition is fulfilled. This proves the first part of the theorem.

As π_1 is a Hecke path, conditions i), v), vi) and vii) of Definitions 5.1 and 5.2 are satisfied for some roots $\beta_1, \dots, \beta_s$ and we can use the results of Section 5.1.2. Let $\delta^0 = (c_0, c_1^0, \dots, c_n^0)$ be the minimal gallery of type $(i_1, \dots, i_n)$ in $\mathfrak{a}_+ = A \cap \mathcal{I}_x^+$ starting from $c_0 = \mathrm{germ}_z(z + C_f^v)$; its end $c_n^0 = w_- c_0$ contains $z + [0,1)\pi'_-(t)$. We shall fold this gallery stepwise. Since $r_{\beta_1} w_- < w_-$, the wall $z + \mathrm{Ker}(\beta_1)$ separates c_0 from c_n^0: it is the wall between some adjacent chambers c_{j-1}^0 and c_j^0. We define $\delta^1 = (c_0, c_1^1 = c_1^0, \dots, c_{j-1}^1 = c_{j-1}^0, c_j^1 = r_{\beta_1} c_j^0, \dots, c_n^1 = r_{\beta_1} c_n^0)$, so $c_{j-1}^1 = c_j^1$ and $c_n^1 = r_{\beta_1} w_- c^0$. But $r_{\beta_2} r_{\beta_1} w_- < r_{\beta_1} w_-$, so the wall $z + \mathrm{Ker}(\beta_2)$ separates c_0 from c_n^1: it is the wall between some adjacent chambers c_{k-1}^1 and c_k^1. We define $\delta^2 = (c_0, c_1^2 = c_1^1, \dots, c_{k-1}^2 = c_{k-1}^1, c_k^2 = r_{\beta_2} c_k^1, \dots, c_n^2 = r_{\beta_2} c_n^1)$. At the end of this procedure, we get a gallery $\delta^s = (c_0, c_1^s, \ldots, c_n^s)$ of type $(i_1, \ldots, i_n)$ in A starting from c_0 and ending in $c_n^s = w_+(t)c_n^0 \supset \pi_+(t)$. Moreover, this gallery is positively folded along true walls.

As $\pi'_+(t) \in W^v\pi'_-(t)$, to prove that $\pi_-(t) = z - [0,1)\pi'_-(t)$ and $\pi_+(t) = z + [0,1)\pi'_+(t)$ are opposite segment-germs, it suffices to prove that c_n^s and c_n^- are opposite chambers. For this, we prove that, except perhaps for one choice of each parameter, c_j^s and c_j^- are opposite for $0 \leqslant j \leqslant n$. This is true for $j = 0$. Suppose c_{j-1}^- opposite c_{j-1}^s. Then c_j^- (resp. c_j^s) is adjacent to c_{j-1}^- (resp. c_{j-1}^s) along an (unrestricted) panel of type i_j. If the wall containing these two panels is not true (*i.e.* restricted), then c_j^- and c_j^s are automatically opposite. Now, if this wall is true, by 4.5 and the general properties of twin buildings (see [15, 2.5.1]) among the chambers adjacent (or equal) to c_{j-1}^- along the panel of type i_j, there is a unique chamber not opposite c_j^s. Hence, all but (perhaps) one choice for c_j^- is opposite c_j^s; and the corresponding parameter has to be chosen in κ or in κ^*. Therefore, the set $S(\pi_1, y)$ is nonempty.

Let us have a closer look at the set of parameters. Choose $\pi \in S(\pi_1, y)$ and $t \in [0,1]$. We show now that $\pi_-(t)$ is obtained with the above procedure. We have the gallery $(c_0^-, \ldots, c_n^-)$ as above in A^-. We choose the apartment A containing $\mathfrak{S}$ and a chamber $c_n \supset \pi_+(t)$ opposite c_n^-. Using the same properties of twin buildings, we find a gallery $\delta = (c_0, c_1, \ldots, c_n)$ of type $(i_1, \ldots, i_n)$ in A, folded only along true walls, and such that, for all j, c_j and c_j^- are opposite. In particular, c_0 is as defined above. So, using δ instead of δ^s, $\pi_-(t)$ is defined as before. Moreover, the number of possibilities for δ is finite. Hence, the set of parameters for $S(\pi_1, y)$ is a finite union of subsets of κ^N, each being a product of N factors either κ or κ^*. $\square$

Corollary 6.5. — *Suppose π_1 is a Hecke path in $Y(A)$. Then the number* $\mathrm{ddim}(\pi_1)$ *of parameters for $S(\pi_1, y)$ is at most $\rho_{\Phi^+}(\lambda - \pi_1(1) + \pi_1(0))$, with equality if and only if π_1 is a LS path.*

Proof. — This is a simple consequence of Proposition 5.8 and Theorem 6.3. $\square$

6.3. Algebraic structure of $S(\pi_1, y)$ and Mirković-Vilonen cycles

To simplify notation, we suppose that $y = 0$ in A and (as before) $\lambda \in Y^+$. Moreover, we suppose that $K = \mathbb{C}(\!(\varpi)\!)$.

1) The set $\mathcal{G}_\lambda$ of segments π in $\mathcal{I}$ of shape λ and ending in 0 may be identified with the set of its starting points $\pi(0)$ *i.e.* with $G_0.(-\lambda) =$

$G(\mathcal{O}).(-\lambda)$. For $\nu \in Y$, let us define $\mathcal{G}_{\lambda,\nu}$ as the subset of $\mathcal{G}_\lambda$ consisting of the segments π with $\rho(\pi(0)) = -\nu$. Thus, $\mathcal{G}_{\lambda,\nu}$ is identified with $U^-(K).(-\nu) \cap G_0.(-\lambda)$. As $-\lambda \in G(K).0$, we can see $\mathcal{G}_{\lambda,\nu}$ as a subset of the affine grassmannian $\mathcal{G} = G(K)/G(\mathcal{O})$, cf. Example 3.14. We shall view the algebraic structure of $\mathcal{G}_{\lambda,\nu}$ as inherited from $U^-(K)$.

By Theorem 6.2 and Corollary 5.9, $\mathcal{G}_{\lambda,\nu}$ is the finite (disjoint) union of the subsets $S(\pi_1, 0)$ for π_1 a Hecke path of shape λ in A from $-\nu$ to 0.

2) Now, we better describe the parameters for $S(\pi_1, 0)$ found in Theorem 6.3. Let $0 < t_1 < \cdots < t_m \leqslant 1$ be the values of t such that $n_i = \ell_{\pi_1(t_i)}(w_-(t_i)) > 0$ and $t_0 = 0$, $t_{m+1} = 1$. For $1 \leqslant i \leqslant m$ given, there exist negative roots $\alpha_{i,j}$ and integers $k_{i,j}$, $1 \leqslant j \leqslant n_i$, such that $M(\alpha_{i,n_i}, k_{i,n_i}), \ldots, M(\alpha_{i,1}, k_{i,1})$ are the true walls successively crossed by a minimal gallery from $c_0^- = \mathrm{germ}_{\pi_1(t_i)}(\pi_1(t_i) - C_f^v)$ to $\pi_{1-}(t_i)$. Further, for any $a \in \mathbb{C}$, let us set $x_{i,j}(a) = x_{\alpha_{i,j}}(a\varpi^{k_{i,j}}) \in U^\times_{\alpha_{i,j},k_{i,j}}$. Moreover, let $\pi \in S(\pi_1, 0)$ and $g \in U^-(K)$ such that $\pi(t_i) = g\pi_1(t_i)$; since $g^{-1}\pi_-(t_i)$ is also the end of a minimal gallery from c_0^- of the same type, for any $t \in [t_{i-1}, t_i[$,

$$g^{-1}\pi(t) = x_{i,n_i}(a_{i,n_i}) \cdots x_{i,1}(a_{i,1})\pi_1(t)$$

for some parameters $a_{i,n_i}, ..., a_{i,1}$ that have to be chosen in $\mathbb{C}$ or $\mathbb{C}^*$ according to the proof of Theorem 6.3.

Iterating this procedure, one obtains that if $\pi \in S(\pi_1, 0)$, then there exists some $(a_{i,j}) \in P(\pi_1, 0) \subset \mathbb{C}^N$ such that

$$\pi(0) = (\prod_{i \geqslant 1; j \leqslant n_i} x_{i,j}(a_{i,j})).(-\nu)$$

where the product is taken in lexicographical order from right to left. More generally, for $t_{i_0-1} \leqslant t \leqslant t_{i_0}$,

$$\pi(t) = (\prod_{i \geqslant i_0; j \leqslant n_i} x_{i,j}(a_{i,j})).(\pi_1(t)).$$

Thus, we define a map

$$\mu : \mathbb{C}^N \supset P(\pi_1, 0) \to U^-(K), (a_{i,j})_{i \leqslant m; j \leqslant n_i} \mapsto \prod_{i \leqslant m; j \leqslant n_i} x_{i,j}(a_{i,j})$$

such that the composition

$$\overline{\mu} = \mathrm{proj} \circ \mu : \mathrm{\mathbb{C}^N} \supset \mathrm{P}(\pi_1, 0) \to \mathrm{U^-(K)/U^-(K)}_{-\nu}$$

is injective. But,

$$U^-(K) \subset U^{nmax-}(K) = \prod_{\alpha\in\Delta^-} U_\alpha(K) = \prod_{\alpha\in\Delta^-} \mathbb{C}((\varpi))$$

and, as μ involves finitely many groups $U_{\alpha,k}$ with $\alpha \in \Phi^-$, there exists y in Y such that the image of μ is contained in $U^-(K)_y \subset U^{nmax-}(K)_y = \prod_{\alpha\in\Delta^-} U_{\alpha,-\alpha(y)} = \prod_{\alpha\in\Delta^-} \mathbb{C}[[\varpi]]$. This last group has the structure of a pro-group in the sense of [9] and the map μ is clearly a morphism for this algebraic structure.

3) Hence, $\mathcal{G}_{\lambda,\nu}$ is a finite (disjoint!) union of sets $S(\pi_1, 0)$ each in bijection with a quasi-affine irreducible variety $P(\pi_1, 0)$ and these sets are indexed by the Hecke paths π_1 of shape λ from $-\nu$ to 0 in A. The maximal dimension of these varieties is $\rho_{\Phi^+}(\lambda-\nu)$, and the varieties of maximal dimension correspond to LS paths from $-\nu$ to 0 in A. A Mirković-Vilonen cycle inside $\mathcal{G}_{\lambda,\nu}$ should be the closure of a set $S(\pi_1, 0)$ (for π_1 a LS path) and $P(\pi_1, 0)$ should be isomorphic to a dense open subvariety of this cycle.

This holds in the classical case of reductive groups. These cycles in $\mathcal{G}_{\lambda,\nu}$ are dense in the Mirković-Vilonen cycles corresponding to $-\lambda$ and $-\nu$ and described by using the reverses of the paths above, cf. [5].

6.4. Another characterization of LS paths

Suppose π_1 is a Hecke path of shape λ in the apartment A. For each t, $0 < t < 1$, let $w_-(t)$ (resp. $w_+(t)$) be the minimal element in W^v such that $\pi'_{1-}(t) = w_-(t)\lambda$ (resp. $\pi'_{1+}(t) = w_+(t)\lambda$). By Proposition 6.1 and Theorem 6.3, there exists an unrestricted gallery $\delta_t = (d_0, \dots, d_n)$ in $\mathfrak{a}_+ = A \cap \mathcal{I}^+_{\pi_1(t)}$, of type $(i_1, \dots, i_n)$ associated to a (given) reduced decomposition of $w_-(t)$, starting from $d_0 = c_0 = \mathrm{germ}_{\pi_1(t)}(\pi_1(t) + C^v_f)$ and ending in $d_n \supset \pi_1(t) + [0,1)\pi'_{1+}(t)$. Moreover, this gallery may be taken positively folded along true walls. For each t, we choose such a gallery and we set $\tilde\pi_1 = (\pi_1, (\delta_t)_{0<t<1})$.

The gallery δ_t is minimal for almost all t (when $\pi'_{1-}(t) = \pi'_{1+}(t)$). Let us define $neg(\delta_t)$ as the number of all unrestricted walls H_j (containing the panel of type i_j in d_j or d_{j-1}) which are true walls and separate d_j from d_0. Actually, as δ_t is positively folded, such an H_j separates d_j from d_{j-1} *i.e.* $d_j \neq d_{j-1}$.

Definition 6.6. — *The codimension of $\tilde{\pi}_1$ is:*

$$\operatorname{codim}(\tilde{\pi}_1) = \ell_{\pi(0)}(w_+(0)) + \sum_{0<t<1} neg(\delta_t).$$

By the same arguments as for Definition 5.7, $\operatorname{codim}(\tilde{\pi}_1)$ is a nonnegative integer; actually, $\operatorname{codim}(\tilde{\pi}_1) \leqslant \ell_{\pi(0)}(w_+(0)) + \operatorname{ddim}(\pi_1) - \ell_{\pi(1)}(w_-(1))$.

Proposition 6.7. — *Let π_1 be a Hecke path in Y. For each choice of $\tilde{\pi}_1$, $\operatorname{codim}(\tilde{\pi}_1) \geqslant \operatorname{codim}(\pi_1)$. Further, π_1 is a LS path if and only if there is equality for (at least) one choice of $\tilde{\pi}_1$.*

Remark 6.8. — Therefore, $\operatorname{codim}(\tilde{\pi}_1) \geqslant \operatorname{codim}(\pi_1) \geqslant \rho_{\Phi^+}(\lambda - \nu) \geqslant \operatorname{ddim}(\pi_1)$, with equalities if and only if π_1 is a LS-path (for good choices of $\tilde{\pi}_1$).

Proof. — It is clear that any true wall H separating d_0 from $\pi_{1+}(t) = \pi_1(t) + [0,1)\pi'_{1+}(t)$ is among the walls H_j, and, if j is chosen maximal for this property, H separates d_0 from d_j. So $\ell_{\pi_1(t)}(w_+(t)) \leqslant neg(\delta_t)$ for $0 < t < 1$ and $\operatorname{codim}(\tilde{\pi}_1) \geqslant \operatorname{codim}(\pi_1)$.

Suppose $\ell_{\pi_1(t)}(w_+(t)) = neg(\delta_t)$, then every true wall H separating d_0 from $\pi_{1+}(t)$ is leaved negatively once and only once by the gallery δ_t; in particular δ_t cannot be negatively folded along such a wall and cannot cross it positively. Moreover δ_t cannot leave negatively any other true wall. As, by hypothesis, δ_t may only be folded along a true wall, this gallery remains inside the (unrestricted) enclosure of d_0 and $\pi_{1+}(t)$. The number of foldings of δ_t is $s = \ell_{\pi_1(t)}(w_-(t)) - \ell_{\pi_1(t)}(w_+(t))$ and δ_t is positively folded. One can now argue as at the end of the proof of Proposition 6.1. One obtains positive roots $\beta_1, \dots, \beta_s$ such that conditions (i) and (ii) of Definition 5.1 are fulfilled; condition (iii) is then a consequence of $s = \ell_{\pi_1(t)}(w_-(t)) - \ell_{\pi_1(t)}(w_+(t))$ and π_1 is a LS path.

Conversely, if π_1 is a LS path, the construction of δ_t as in Theorem 6.3 may be performed by using a set $(\beta_1, \dots, \beta_s)$ of positive roots with $s = \ell_{\pi_1(t)}(w_-(t)) - \ell_{\pi_1(t)}(w_+(t))$. This gallery δ_t is folded exactly s times (positively and along true walls), its length is $n = \ell_{\pi_1(t)}(w_-(t))$; so, once we get rid of the stutterings, we get a minimal gallery δ_t^{ns} from c_0 to $\pi_{1+}(t) = \pi_1(t) + [0,1)w_+(t)\lambda$. Hence, as the foldings were positive,

$$neg(\delta_t) = neg(\delta_t^{ns}) = \ell_{\pi_1(t)}(w_+(t)),$$

so $\operatorname{codim}(\tilde{\pi}_1) = \operatorname{codim}(\pi_1)$. □

6.5. Preorder relation on the hovel

THEOREM 6.9. — *On the hovel $\mathcal{I}$, the relation $\leqslant$ (defined in 4.3.2) is a preorder relation. More precisely, if three different points x, y and z in $\mathcal{I}$ are such that $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$ and, in particular, x and z are in a same apartment.*

Remark 6.10. — This result precises the structure of the hovel $\mathcal{I}$. It is a generalization of Lemme 7.3.6 in [3]. It may also be seen as a generalization of the Cartan decomposition proved by Garland for $p-$adic loop groups [4], even if it is weaker than this decomposition in the affine case. As Garland asserts, the Cartan decomposition holds only after some twisting; this is more or less equivalent to the fact that not any two points in $\mathcal{I}$ are in a same apartment.

More precisely, let us look at the simplest affine Kac-Moody group $G = \mathsf{SL}_2^{(1)}$. If $K = \mathbb{C}(\!(\varpi)\!)$ and $\mathcal{O} = \mathbb{C}[\![\varpi]\!]$, then, up to the center (which is in $T(K)$), the group $G(K) = \mathsf{SL}_2^{(1)}(\mathbb{C}(\!(\varpi)\!))$ is a semidirect product $G(K) = K^* \ltimes \mathsf{SL}_2(K[u, u^{-1}])$, with $K^* \subset T(K)$. We saw in Example 3.14 that $G_0 = \hat{P}_0 = G(\mathcal{O}) = \mathcal{O}^* \ltimes \mathsf{SL}_2(\mathcal{O}[u, u^{-1}])$ (up to the center). The Cartan decomposition would tell that $G(K) = G(\mathcal{O})T(K)G(\mathcal{O})$, hence $\mathsf{SL}_2(K[u, u^{-1}]) = \mathsf{SL}_2(\mathcal{O}[u, u^{-1}])T_1(K)\mathsf{SL}_2(\mathcal{O}[u, u^{-1}])$, where $T_1(K)$ is the torus of diagonal matrices in $\mathsf{SL}_2(K)$. The four coefficients of a matrix in the right hand side span the same sub$-\mathcal{O}[u, u^{-1}]-$module of $K[u, u^{-1}]$ as a matrix in $T_1(K)$. So, this module is generated by h and h^{-1}, i.e. by a single element of K^*. But, the $\mathcal{O}[u, u^{-1}]-$module spanned by the coefficients of $g = \begin{pmatrix} 1 & \varpi^{-1}(1+u) \\ 0 & 1 \end{pmatrix}$ is not generated by a single element. Hence, Cartan decomposition fails and the points 0 and $g.0$ are not in a same apartment. If they were in an apartment A', then $A' = h_1.A_f$ and $g^{-1}.A' = h_2.A_f$ for $h_1, h_2 \in G_0$ (by Remark 4.2), hence $h_1^{-1}gh_2 \in N(K)$ (by 4.3.1), contradiction!

LEMMA 6.11. — *In the situation of Theorem 6.9, there exists an apartment A containing x and $[y, z)$. Moreover, this apartment is unique up to isomorphism.*

Proof. — In an apartment A_1 containing x and y, there exists a vectorial chamber C^v such that $x \in y + \overline{C^v}$. Moreover, there exists an apartment A containing $\mathfrak{S} = \mathrm{germ}(y + \overline{C^v})$ and $[y, z)$; this apartment also contains $y + \overline{C^v} \ni x$ (cf. 4.3.3). The uniqueness is a consequence of 4.2.2, 4.2.4, Proposition 4.3 4) and Remark 4.2. $\square$

Proof of the theorem. —

1) For $z' \in [y,z[$ such that $x \leqslant z'$, we choose an apartment A containing $[z',x)$ and $[z',z)$ (4.3.4); this apartment has an associated system of real roots $\Phi(A)$ and we define the finite set $\Phi(z')$ of the roots $\alpha \in \Phi(A)$ such that $\alpha(z') > \alpha(x_1)$ and $\alpha(z') > \alpha(z_1)$ for some $x_1 \in [x,z'] \cap A$ and some $z_1 \in [z,z'] \cap A$. As $[z',x)$ and $[z',z)$ are generic, 4.3.4) shows that $\Phi(z')$ depends, up to isomorphism, only on $[z',x)$ and $[z',z)$ but not on A.

   By Lemma 6.11 (mutatis mutandis), there exists an apartment $A_{z'}$ containing z and $[z',x)$ and this apartment is unique up to isomorphism. We define $N_{z'}$ as the finite number of walls (in $A_{z'}$) of direction $\mathrm{Ker}(\alpha)$ for some $\alpha \in \Phi(z')$ and separating z' from z (in a strict sense). We shall argue by induction on $(|\Phi(y)|, N_y)$ (with lexicographical order).

2) By Lemma 6.11, there is an apartment A_1 containing x and $[y,z)$. We choose a vectorial chamber C^v in A_1 such that its associated system of positive roots $\Phi^+(C^v)$ contains the roots $\alpha \in \Phi(A_1)$ such that $\alpha(y) > \alpha(x)$ or $\alpha(y) = \alpha(x)$ and $\alpha(z_1) > \alpha(y)$ (for some $z_1 \in [y,z] \cap A_1$); in particular $[x,y] \subset y - \overline{C^v}$. Now if $\alpha \in \Phi^+(C^v)$ is such that $\alpha(z_1) < \alpha(y)$ (for some $z_1 \in [y,z] \cap A_1$) then $\alpha(y) > \alpha(x)$; hence $\Phi(y)$ (computed in A_1) is the set of roots $\alpha \in \Phi^+(C^v)$ such that $\alpha(z_1) < \alpha(y)$ (for some $z_1 \in [y,z] \cap A_1$).

   Let $\mathfrak{S}$ be the sector-germ associated to $-C^v$ in A_1 and ρ the retraction of center $\mathfrak{S}$ onto A_1.

3) Suppose $z_1 \in]y,z]$ is such that no wall (in A_y or any apartment containing z_1 and $[y,x)$) of direction $\mathrm{Ker}\alpha$ for some $\alpha \in \Phi(y)$ separates y from z_1. We shall prove that the enclosure of $\mathfrak{S}$ and z_1 contains y and x, so, there is an apartment containing x, y, z_1 and $\mathfrak{S}$. Hence, the theorem is true if $z_1 = z$ and this gives the first step of the induction when $\Phi(y) = \emptyset$ or $N_y = 0$.

   As in 4.4) we get a sequence $y_0 = y, y_1, \dots, y_n = z_1 \in [y,z_1]$ and apartments $A_1, A_2, \dots, A_n$ such that A_i contains $\mathfrak{S}$ and $[y_{i-1}, y_i]$. The characterization of $\Phi(y)$ in 2) above and the hypothesis on the walls prove that y is in the enclosure of y_1 and $\mathfrak{S}$, then x is also in this enclosure. So A_2, which contains y_1 and $\mathfrak{S}$ contains also y and x. We can replace y_1 by y_2 and A_1 by A_2; by induction on n we obtain the result of 3).

4) We choose for $z_1 \in]y,z]$ the point satisfying the hypothesis of 3) which is the nearest to z, it exists as $\Phi(y)$ and N_y are finite. We may

(and do) suppose $z_1 \neq z$. We choose for A_1 the apartment containing x, y, z_1 and the $\mathfrak{S}$ explained in 3). An apartment A_2 containing $\mathfrak{S}$ and $[z_1, z)$ is sent isomorphically by ρ onto A_1. This enables us to identify $\Phi(z_1)$ with the set $\Phi'(z_1)$ of the roots $\alpha \in \Phi(A_1)$ such that $\alpha(z_1) > \alpha(x)$ (hence $\alpha \in \Phi^+(C^v)$) and $\alpha(z_1) > \alpha(\rho z_2)$ (for some $z_2 \in [z_1, z]$ near z_1). By Proposition 6.1, $\rho([z_1, z)) = z_1 + [0,1)w^+\lambda$, $[y, z_1) = y + [0,1)w^-\lambda$ for some $\lambda \in \overline{C^v}$ and $w^+, w^- \in W^v$ such that $w^+ \leqslant w^-$. In particular, for $\alpha \in \Phi^+(C^v)$, $\alpha(z_1) > \alpha(\rho z_2)$ means $\alpha(w^+\lambda) < 0$, so $\Phi'(z_1) \subset \{\alpha \in \Phi^+(C^v) \mid \alpha(w^+\lambda) < 0\}$ and (as w^+ is chosen minimal) this set is of cardinal $\ell(w^+)$. Now we saw in 2) that $\Phi(y) = \{\alpha \in \Phi^+(C^v) \mid \alpha(w^-\lambda) < 0\}$. Hence, as $w^+ \leqslant w^-$, $|\Phi'(z_1)| \leqslant \ell(w^+) \leqslant \ell(w^-) \leqslant |\Phi(y)|$.

If $|\Phi'(z_1)| < |\Phi(y)|$ the theorem is true by induction. Otherwise, the four numbers above are equal; in particular, as $w^+ \leqslant w^-$, one has $w^+ = w^-$ and $\Phi'(z_1) = \Phi(y)$.

We consider the segment $[y, z]$ as a linear path $\pi : [0,1] \to [y, z]$, $\pi(0) = y$, $\pi(1) = z$, $z_1 = \pi(t_1)$ and $\pi'(t) \in W^v\lambda$, $\forall t$. The number N_{z_1} is calculated in an apartment A_{z_1} containing z and $[z_1, x)$ using $\Phi(z_1)$ and $[z_1, z]$. We may suppose A_{z_1} also containing $\mathrm{germ}_{z_1}(z_1 - C^v)$; then there is an isomorphism from $(A_{z_1}, \Phi(z_1), [z_1, z])$ to $(A_1, \Phi'(z_1), [z_1, Z])$ where $Z = z_1 + (1 - t_1)w^+\lambda$, so N_{z_1} may be computed with this last triple. Arguing the same way, we see that N_y may be calculated with the triple $(A_1, \Phi(y), [y, Z'])$ with $Z' = y + w^-\lambda$. Actually, $z_1 = y + t_1.w^-\lambda$ and we saw that $\Phi'(z_1) = \Phi(y)$ and $w^+\lambda = w^-\lambda$, so $Z' = Z$. Moreover, by the choice of z_1, there is a wall of direction $\mathrm{Ker}(\alpha)$ for some $\alpha \in \Phi(y)$ containing z_1. Hence, $N_{z_1} < N_y$, and this proves the theorem by induction.

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