ANNALES

DE

L'INSTITUT FOURIER

Hélène ESNAULT & Phùng Hô HAI

**The fundamental groupoid scheme and applications**

Tome 58, nº 7 (2008), p. 2381-2412.

<http://aif.cedram.org/item?id=AIF_2008__58_7_2381_0>

Ann. Inst. Fourier, Grenoble
**58**, 7 (2008) 2381-2412

# THE FUNDAMENTAL GROUPOID SCHEME AND APPLICATIONS

**by Hélène ESNAULT & Phùng Hô HAI (*)**

Abstract. — We define a linear structure on Grothendieck's arithmetic fundamental group $\pi_1(X,x)$ of a scheme $X$ defined over a field $k$ of characteristic 0. It allows us to link the existence of sections of the Galois group $\mathrm{Gal}(\bar{k}/k)$ to $\pi_1(X,x)$ with the existence of a neutral fiber functor on the category which linearizes it. We apply the construction to affine curves and neutral fiber functors coming from a tangent vector at a rational point at infinity, in order to follow this rational point in the universal covering of the affine curve.

Résumé. — Nous définissons une structure linéaire sur le groupe fondamental arithmétique $\pi_1(X,x)$ d'un schéma $X$ défini sur un corps $k$ de caractéristique 0. Cela nous permet de lier l'existence de sections du groupe de Galois $\mathrm{Gal}(\bar{k}/k)$ vers $\pi_1(X,x)$ à l'existence d'un foncteur neutre sur la catégorie qui linéarise ce dernier. Nous appliquons cette construction à une courbe affine et aux foncteurs neutres qui proviennent d'un vecteur tangent à l'infini. Nous pouvons ainsi suivre ce point rationnel dans le revêtement universel de la courbe affine.

## 1. Introduction

For a connected scheme $X$ over a field $k$, Grothendieck defines in [6, Section 5] the arithmetic fundamental group as follows. He introduces the category $\mathsf{ECov}(X)$ of finite étale coverings $\pi : Y \to X$, the Hom-Sets being $X$-morphisms. The choice of a geometric point $x \in X$ defines a fiber functor $\pi \xrightarrow{\omega_x} \pi^{-1}(x)$ with values in the category $\mathsf{FSets}$ of finite sets. He defines the arithmetic fundamental group with base point $x$ to be the automorphism group $\pi_1(X,x) = \mathrm{Aut}(\omega_x)$ of the fiber functor. It is an abstract group, endowed with the pro-finite topology stemming from its finite images in the permutation groups of $\pi^{-1}(x)$ as $\pi$ varies. The main theorem

*Keywords:* Finite connection, tensor category, tangential fiber functor.
*Math. classification:* 14F05, 14L17, 18D10.
(*) Partially supported by the DFG Leibniz Preis and the DFG Heisenberg program.

is the equivalence of categories $\mathsf{ECov}(X) \xrightarrow{\omega_x} \pi_1(X,x)\text{-}\mathsf{FSets}$ between the étale coverings and the finite sets acted on continuously by $\pi_1(X,x)$. This equivalence extends to pro-finite objects on both sides. When applied to the set $\pi_1(X,x)$, acted on by itself as a group via translations, it defines the universal pro-finite étale covering $\widetilde{\pi}_x : \widetilde{X}_x \to X$ based at $x$. It is a Galois covering of group $\pi_1(X,x)$ and $\widetilde{\pi}_x^{-1}(x) = \pi_1(X,x)$. Furthermore, the embedding of the sub-category $\mathsf{ECov}(k)$ of $\mathsf{ECov}(X)$, consisting of the étale coverings $X \times_k \mathrm{Spec}(L) \to X$ obtained from base change by a finite field extension $L \supset k$, with $L$ lying in the residue field $\kappa(x)$ of $x$, induces an augmentation map $\pi_1(X,x) \xrightarrow{\epsilon} \mathrm{Gal}(\bar{k}/k)$. Here $\bar{k}$ is the algebraic closure of $k$ in $\kappa(x)$. Then $\epsilon$ is surjective when $k$ is separably closed in $H^0(X,\mathcal{O}_X)$. Grothendieck shows that the kernel of the augmentation is identified with $\pi_1(X\times_k \bar{k})$, thus one has an exact sequence

$$1 \to \pi_1(X\times_k \bar{k}, x) \to \pi_1(X,x) \xrightarrow{\epsilon} \mathrm{Gal}(\bar{k}/k). \tag{1.1}$$

Via the equivalence of categories, one has a factorization

$$\begin{tikzcd} \widetilde{X}_x \arrow[r] \arrow[dr, "\widetilde{\pi}_x"'] & X\times_k \bar{k} \arrow[d] \\ & X \end{tikzcd} \tag{1.2}$$

identifying the horizontal map with the universal pro-finite étale covering of $X\times_k \bar{k}$, based at $x$ viewed as a geometric point of $X \times_k \bar{k}$. In particular, $\widetilde{X}_x$ is a $\bar{k}$-scheme.

The aim of this article is to linearize Grothendieck's construction in the case where $X$ is smooth and $k$ has characteristic 0. There is a standard way to do this using local systems of $\mathbb{Q}$-vector spaces on $X_{\text{ét}}$, which is a $\mathbb{Q}$-linear abelian rigid tensor category ([1, Section 10]). Rather than doing this, we go to the $\mathcal{D}$-module side of the Riemann-Hilbert correspondence, where we have more flexibility for the choice of a fiber functor. Before describing more precisely our construction, the theorems and some applications, let us first recall Nori's theory.

For a proper, reduced scheme $X$ over a perfect field $k$, which is connected in the strong sense that $H^0(X,\mathcal{O}_X) = k$, Nori defines in [11, Chapter II] the fundamental group scheme as follows. He introduces the category $\mathcal{C}^N(X)$ of essentially finite bundles, which is the full sub-category of the coherent category on $X$, spanned by Weil-finite bundles $V$, that is for which there are two polynomials $f, g \in \mathbb{N}[T]$, $f \neq g$, with the property that $f(V)$ is isomorphic to $g(V)$. It is a $k$-linear abelian rigid tensor category. The choice of a rational point $x \in X(k)$ defines a fiber functor $V \xrightarrow{\rho_x} V|_x$ with values in the category of vector spaces $\mathsf{Vec}_k$ over $k$, thus endows $\mathcal{C}^N(X)$ with the structure of a neutral Tannaka category. Nori defines his fundamental group scheme with base point $x$ to be the automorphism group $\pi^N(X,x) = \mathrm{Aut}^{\otimes}(\rho_x)$ of $\rho_x$. It is a $k$-affine group scheme, and its affine structure is pro-finite in the sense that its images in $GL(V|_x)$ are 0-dimensional (*i.e.* finite) group schemes over $k$ for all objects $V$ of $\mathcal{C}^N(X)$. Tannaka duality ([3, Theorem 2.11]) asserts the equivalence of categories $\mathcal{C}^N(X) \xrightarrow{\rho_x} \mathrm{Rep}(\pi^N(X,x))$. The duality extends to the ind-categories on both sides. When applied to $k[\pi^N(X,x)]$ acted on by $\pi^N(X,x)$ via translations, it defines a principal bundle $\pi_{\rho_x} : X_{\rho_x} \to X$ under $\pi^N(X,x)$. One has $\pi_{\rho_x}^{-1}(x) = \pi^N(X,x)$. In particular, the $k$-rational point $1 \in \pi^N(X,x)(k)$ is a lifting in $X_{\rho_x}$ of the $k$-rational point $x \in X(k)$ . On the other hand, one has the base change property for $\mathcal{C}^N(X)$. Any $\bar{k}$-point $\bar{x} \to X$ lifting $x$ yields a base change isomorphism $\pi^N(X,x) \times_k \bar{k} \cong \pi^N(X \times_k \bar{k}, \bar{x})$, where $k \subset \bar{k}$ is given by $\bar{x} \to x$. Denoting by $\rho_{\bar{x}}$ the fiber functor $V \xrightarrow{\rho_{\bar{x}}} V|_{\bar{x}}$ on $\mathcal{C}^N(X \times_k \bar{k})$, it implies an isomorphism $(X \times_k \bar{k})_{\rho_{\bar{x}}} \cong X_{\rho_x} \times_k \bar{k}$. Given those two parallel descriptions of $\widetilde{\pi}_x : \widetilde{X}_x \to X$ and $\pi_{\rho_x} : X_{\rho_x} \to X$ in Grothendieck's and in Nori's theories, together with the base change property of $\mathcal{C}^N(X)$, one deduces $\pi_1(X \times_k \bar{k}, x) \cong \pi^{\acute{e}t}(X \times_k \bar{k}, x)(\bar{k})$, where $\pi^{\acute{e}t}(X,x)$ is the quotient of $\pi^N(X,x)$ obtained by considering the quotient prosystem of étale group schemes (see [4, Remarks 3.2, 2)]). So in particular, if $k$ has characteristic zero, one has $\pi_1(X \times_k \bar{k}, x) \cong \pi^N(X \times_k \bar{k}, x)(\bar{k})$. We conclude that $\pi_{\rho_{\bar{x}}} = \widetilde{\pi}_{\bar{x}}$. So (1.2) has the factorization

$$\begin{tikzcd} \widetilde{X}_{\bar{x}} \arrow[r] \arrow[d] \arrow[dr, "\widetilde{\pi}_{\bar{x}}"'] & X \times_k \bar{k} \arrow[d] \\ X_{\rho_x} \arrow[r, "\pi_{\rho_x}"'] & X \end{tikzcd} \tag{1.3}$$

(See Theorem 2.11 and Remarks 2.13). Summarizing, we see that Grothendieck's construction is very closed to a Tannaka construction, except that his category $\mathsf{ECov}(X)$ does not have a $k$-structure. On the other hand, the comparison of $\pi_1(X,x)$ with $\pi^N(X,x)$, aside of the assumptions under which $\pi^N(X,x)$ is defined, that is $X$ proper, reduced and strongly connected, and $x \in X(k)$, requests the extra assumption that $k$ be algebraically closed. In particular, the sub-category $\mathsf{ECov}(k)$ of $\mathsf{ECov}(X)$ is not seen by Nori's construction, so the augmentation $\epsilon$ in (1.1) is not seen either, and there is no influence of the Galois group of $k$. The construction only yields a $k$-structure on $\pi_1(X \times_k \bar{k}, x)$ under the assumption that $x$ is a rational point. However, when this assumption is fulfilled, it yields a $k$-form of the universal pro-finite étale covering which carries a rational pro-point.

The purpose of this article is to reconcile the two viewpoints, using Deligne's more evolved Tannaka formalism as developed in [2]. From now on, we will restrict ourselves to the characteristic 0 case. We illustrate the first idea on the simplest possible example $X = x = \mathrm{Spec}(k)$. Then certainly $\mathcal{C}^N(X)$ is the trivial category, that is every object is isomorphic to a direct sum of the trivial object, thus $\pi^N(X, x) = \{1\}$. Let us fix $\bar{x} \to x$ corresponding to the choice of an algebraic closure $k \subset \bar{k}$. It defines the fiber functor $\mathcal{C}^N(X) \xrightarrow{\rho_{\bar{x}}} \mathsf{Vec}_{\bar{k}}$ which assigns $V|_{\bar{x}} = V \otimes_k \bar{k}$ to $V$. It is a non-neutral fiber functor. It defines a groupoid scheme $\mathrm{Aut}^{\otimes}(\rho_{\bar{x}})$ over $k$, acting transitively on $\bar{x}$ via $(t, s) : \mathrm{Aut}^{\otimes}(\rho_{\bar{x}}) \to \bar{x} \times_k \bar{x}$. In fact $\mathrm{Aut}^{\otimes}(\rho_{\bar{x}}) = \bar{x} \times_k \bar{x}$ and Deligne's Tannaka duality theorem [2, Théorème 1.12] asserts that the trivial category $\mathcal{C}^N(X)$ is equivalent to the representation category of the trivial $k$-groupoid scheme $\bar{x} \times_k \bar{x}$ acting on $\bar{x}$. We define $(\bar{x} \times_k \bar{x})_s$ to be $\bar{x} \times_k \bar{x}$ viewed as a $\bar{k}$-scheme by means of the right projection $s$. Galois theory then implies that the $\bar{k}$-points of $(\bar{x} \times_k \bar{x})_s$ form a pro-finite group which is the Galois group of $\mathrm{Gal}(\bar{k}/k)$ where the embedding $k \subset \bar{k}$ corresponds to $\bar{x} \to x$. Thus the geometry here is trivial, but the arithmetic is saved via the use of a non-neutral fiber functor. This is the starting point of what we want to generalize.

In order to have a larger range of applicability to not necessarily proper schemes, we extend in section 2 Nori's definition to smooth schemes of finite type $X$ over $k$ which have the property that the field $k$, which we assume to be of characteristic 0, is exactly the field of constants of $X$, that is $k = H^0_{DR}(X)$. In particular, it forces the augmentation $\epsilon$ in (1.1) to be surjective. We define the category $\mathsf{FConn}(X)$ to be the category of finite flat connections, that is of bundles $V$ with a flat connection $\nabla : V \to \Omega^1_X \otimes_{\mathcal{O}_X} V$, such that $(V, \nabla)$ is a sub-quotient connection of a Weil-finite connection. The latter means that there are $f, g \in \mathbb{N}[T], f \neq g$ so that $f((V, \nabla))$ is isomorphic to $g((V, \nabla))$. The Hom-Sets are flat morphisms. $\mathsf{FConn}(X)$ is a $k$-linear rigid tensor category (see Definitions 2.1 and 2.4). Theorem 2.15 shows that if $X$ is smooth proper, with $k = H^0(X, \mathcal{O}_X)$ of characteristic 0, then the forgetful functor $\mathsf{FConn}(X) \to \mathcal{C}^N(X),\ (V, \nabla) \mapsto V$ is an equivalence of $k$-linear abelian rigid tensor categories.

A fiber functor $\rho : \mathsf{FConn}(X) \to \mathsf{QCoh}(S)$ in the quasi-coherent category of a scheme $S$ over $k$ endows $\mathsf{FConn}(X)$ with a Tannaka structure. One defines $\Pi = \mathrm{Aut}^{\otimes}(\rho)$ to be the groupoid scheme over $k$ acting transitively on $S$, with diagonal $S$-group scheme $\Pi^{\Delta}$. The fiber functor $\rho$ establishes an equivalence $\mathsf{FConn}(X) \xrightarrow{\cong\ \rho} \mathrm{Rep}(S : \Pi)$ ([2, Théorème 1.12]). Our first central theorem says that one can construct a universal covering in this abstract setting, associated to $S$ and $\rho$, as a direct generalization of the easily defined $\pi_{\rho_x} : X_{\rho_x} \to X$ sketched in (1.3).

Theorem 1.1 (See precise statement in Theorem 2.7). — *There is a diagram of $k$-schemes*

(1.4)
$$\begin{tikzcd} & X_\rho \arrow[dl, "p_\rho"'] \arrow[d, "s_\rho"] \arrow[dr, "\pi_\rho"] & \\ S & S\times_k X \arrow[l, "p_1"] \arrow[r, "p_2"'] & X \end{tikzcd}$$

*with the following properties.*

1) *$s_\rho$ is a $\Pi^\Delta$-principal bundle, that is*
$$X_\rho \times_{S\times_k X} X_\rho \cong X_\rho \times_S \Pi^\Delta.$$
2) $R^0(p_\rho)_{*DR}(X_\rho/S) := R^0(p_\rho)_*(\Omega^\bullet_{X_\rho/S}) = \mathcal{O}_S$.
3) *For all objects $N = (W, \nabla)$ in $\mathsf{FConn}(X)$, the connection*
$$\pi_\rho^* N : \pi_\rho^* W \to \pi_\rho^* W \otimes_{\mathcal{O}_{X_\rho}} \Omega^1_{X_\rho/S}$$
*relative to $S$ is endowed with a functorial isomorphism with the relative connection*
$$p_\rho^*\rho(N) = (p_\rho^{-1}\rho(N) \otimes_{p_\rho^{-1}\mathcal{O}_S} \mathcal{O}_{X_\rho}, 1\otimes d)$$
*which is generated by the relative flat sections $p_\rho^{-1}\rho(N)$.*
4) *One recovers the fiber functor $\rho$ via $X_\rho$ by an isomorphism*
$$\rho(N) \cong R^0(p_\rho)_{*DR}(X_\rho/S, \pi_\rho^* N)$$
*which is compatible with all morphisms in $\mathsf{FConn}(X)$. In particular, the data in (1.4) are equivalent to the datum $\rho$ (which defines $\Pi$).*

Furthermore this construction is functorial in $S$. While applied to $S = X$, $\rho((V,\nabla)) = V$, $X_\rho$ is nothing but $\Pi$ with $(p_\rho, \pi_\rho) = (t, s)$ (Definition 2.8, 2)). While applied to $S = \mathrm{Spec}(\bar{k})$, $\rho((V,\nabla)) := \rho_{\bar{x}}((V,\nabla)) = V|_{\bar{x}}$ for a geometric point $\bar{x} \to X$ with residue field $\bar{k}$, then $\pi_{\rho_{\bar{x}}}$ is an étale pro-finite covering, which will turn out to be Grothendieck's universal pro-finite étale covering. Let us be more precise.

Theorem 1.1 is proven by showing that $s_*\mathcal{O}_\Pi$ is a representation of the groupoid scheme $\Pi$, by analyzing some of its properties, and by translating them via Tannaka duality. It invites one to consider the scheme $\Pi$ considered as a $S$-scheme via $s$, which we denote by $\Pi_s$. In case $S = \mathrm{Spec}(\bar{k})$, $\rho = \rho_{\bar{x}}$, we denote $\mathrm{Aut}^{\otimes}(\rho_{\bar{x}})$ by $\Pi(X, \bar{x})$. We show

Theorem 1.2 (See Theorem 4.4). — *The rational points* $\Pi(X, \bar{x})_s(\bar{k})$ *carry a structure of a pro-finite group. One has an exact sequence of pro-finite groups*

$$1 \to \Pi(\bar{X}, \bar{x})(\bar{k}) \to \Pi(X, \bar{x})_s(\bar{k}) \to (\mathrm{Spec}(\bar{k}) \times_k \mathrm{Spec}(\bar{k}))_s(\bar{k}) \to 1 \tag{1.5}$$

*together with an identification of exact sequences of pro-finite groups*

(1.6)
$$\begin{tikzcd}
1 \arrow[r] & \Pi(\bar{X}, \bar{x})(\bar{k}) \arrow[r] \arrow[d, "="'] & \Pi(X, \bar{x})_s(\bar{k}) \arrow[r] \arrow[d, "="'] & (\mathrm{Spec}(\bar{k}) \times_k \mathrm{Spec}(\bar{k}))_s(\bar{k}) \arrow[r] \arrow[d, "="'] & 1 \\
1 \arrow[r] & \pi_1(\bar{X}, \bar{x}) \arrow[r] & \pi_1(X, \bar{x}) \arrow[r, "\epsilon"] & \mathrm{Gal}(\bar{k}/k) \arrow[r] & 1
\end{tikzcd}$$

A corollary is that $\pi_{\rho_{\bar{x}}} : X_{\rho_{\bar{x}}} \to X$ is precisely the universal pro-finite étale covering of $X$ based at $\bar{x}$ (see Corollary 4.5). One interesting consequence of Theorem 1.2 is that it yields a Tannaka interpretation of the existence of a splitting of the augmentation $\epsilon$. Indeed, (1.5) does not have to do with the choice of the category $\mathsf{FConn}(X)$. More generally, if $(t, s) : \Pi \to \mathrm{Spec}(\bar{k}) \times_k \mathrm{Spec}(\bar{k})$ is a $k$-groupoid scheme acting transitively upon $\mathrm{Spec}(\bar{k})$, then we show

Theorem 1.3 (See Theorem 3.2).

1) *There exists a group structure on* $\Pi_s(\bar{k})$ *such that the map*
$$(t, s)|_{\Pi_s} : \Pi_s(\bar{k}) \to (\mathrm{Spec}(\bar{k}) \times_k \mathrm{Spec}(\bar{k}))_s(\bar{k}) \cong \mathrm{Gal}(\bar{k}/k)$$
*is a group homomorphism.*
2) *Splittings of* $(t, s)|_{\Pi_s} : \Pi_s(\bar{k}) \to (\mathrm{Spec}(\bar{k}) \times_k \mathrm{Spec}(\bar{k}))_s(\bar{k}) \cong \mathrm{Gal}(\bar{k}/k)$ *as group homomorphisms are in one to one correspondence with splittings of* $(t, s) : \Pi \to \mathrm{Spec}(\bar{k}) \times_k \mathrm{Spec}(\bar{k})$ *as* $k$*-affine groupoid scheme homomorphisms.*
3) *There is a one to one correspondence between neutral fiber functors of* $\mathrm{Rep}(\bar{k} : \Pi)$*, up to natural equivalence, and splittings of* $(t, s)$ *up to an inner conjugation of* $\Pi$ *given by an element of* $\Pi^{\Delta}(\bar{k})$.

In particular we show that splittings of $\epsilon$ in (1.6) up to conjugation with $\pi_1(\bar{X}, \bar{x})$ are in one to one correspondence with neutral fiber functors of $\mathsf{FConn}(X) \to \mathsf{Vec}_k$ (see Corollary 4.6).

We now discuss an application. Grothendieck, in his letter to G. Faltings dated June 27-th, 1983 [7], initiated a program to recognize hyperbolic algebraic curves defined over function fields $k$ over $\mathbb{Q}$ via the exact sequence (1.1). His anabelian conjectures have essentially been proven, with the notable exception of the so-called section conjecture. It predicts that if $X$ is a smooth projective curve of genus $\geqslant 2$ defined over $k$ of finite type over $\mathbb{Q}$, then conjugacy classes of sections of $\epsilon$ in (1.1) are in one to one correspondence with rational points of $X$.

In fact, Grothendieck conjectures a precise correspondence also for affine hyperbolic curves. If $X$ is affine and hyperbolic, the correspondence is not one-to one, there are many sections that correspond to a rational point at infinity. More precisely, let $X^\wedge$ be the smooth compactification of $X$, $\widetilde{X^\wedge}$ be the normalization of $X^\wedge$ in $\widetilde{X}_{\bar{x}}$:

$$\begin{tikzcd} \widetilde{X}_{\bar{x}} \arrow[r, hook] \arrow[d] & \widetilde{X^\wedge} \arrow[d] \\ X \arrow[r, hook] & X^\wedge \end{tikzcd} \tag{1.7}$$

Then the conjecture says that each section of $\epsilon$ defines a $k$-form of $\widetilde{X^\wedge}$ which carries a unique rational pro-point, which maps to a rational point $y$ of $X^\wedge$, lying either on $X$ or at infinity, *i.e.* on $X^\wedge \setminus X$.

In view of our construction, the $k$-form and its rational pro-point mapping to $y \in X$ is nothing but the pro-scheme $X_{\rho_y}$ associated to the fiber functor at $y$ and the resulting rational pro-point.

Section 5 is devoted to the points at infinity. We use the tangential (neutral) fiber functors $\eta : \mathsf{FConn}(X) \to \mathsf{Vec}_k$ on $\mathsf{FConn}(X)$ as defined by Deligne [1, Section 15], and Katz [9, Section 2.4]. They factor through the category of finite connections on the local field of a rational point at $\infty$, and are defined by a tangent vector. The construction yields $k$-forms with pro-points that descend to points at infinity.

Theorem 1.4 (See Theorem 5.2). — *Let $X/k$ be a smooth affine curve over a characteristic 0 field $k$, and let $y$ be a $k$-rational point at infinity. Set $X' := X \cup \{y\}$. Let $\eta : \mathsf{FConn}(X) \to \mathsf{Vec}_k$ be the neutral fiber functor defined by a tangent vector at $y$. Fix a geometric point $\bar{x} \to X$ with residue field $\bar{k}$. Then the universal pro-finite étale covering $\widetilde{X}_{\bar{x}}$ based at $\bar{x}$ has a $k$-structure $X_\eta$, i.e. $\widetilde{X}_{\bar{x}} \cong_{\bar{k}} X_\eta \times_k \bar{k}$, with the property that the $k$-rational point $y$ lifts to a $k$-rational pro-point of the normalization $(X_\eta)'$ of $X'$ in $k(X_\eta)$.*

This gives an explicit description of the easy part of the conjecture formulated in [7], page 8, by Grothendieck. So we can say that Tannaka methods, which are purely algebraic, and involve neither the geometry of $X$ nor the arithmetic of $k$, can't detect rational points on $X$. But they allow one, once one has a rational point on $X$, to follow it in the pro-system which underlies the Tannaka structure, in particular on Grothendieck's universal covering.

They also allow to generalize in a natural way Grothendieck's conjecture to general fiber functors $\rho : \mathsf{FConn}(X) \to \mathsf{QCoh}(S)$. (See Remark 2.13 2).) Theorem 1.1 2) says that $\rho$ is always cohomological in the sense that there is an isomorphic fiber functor which is cohomologically defined. Theorem 1.3 suggests to ask under which conditions on the geometry of $X$ and the arithmetic of $k$ one may say that $X$ has an $S$-point if and only if an $S$-valued $\rho$ exists. Further Theorem 1.1 allows one to ask for a generalization to $S$-pro-points of Theorem 1.4.

*Acknowledgements.* — We thank Alexander Beilinson and Gerd Faltings for clarifying discussions, for their interest and their encouragements. We thank the referee for a thorough, accurate and helpful report.

## 2. Finite connections

In this section, we introduce the category of finite connections in the spirit of Weil and follow essentially verbatim Nori's developments in [11, I, Section 2.3] for the main properties.

Definition 2.1. — *Let $X$ be a smooth scheme of finite type over a field $k$ of characteristic 0, with the property that*

$$k = H^0_{DR}(X) := H^0(X, \mathcal{O}_X)^{d=0}.$$

*The category $\mathsf{Conn}(X)$ of flat connections has objects $M = (V, \nabla)$ where $V$ is a vector bundle (i.e. a locally free coherent sheaf) and $\nabla : V \to \Omega^1_X \otimes_{\mathcal{O}_X} V$ is a flat connection. The Hom-Sets are flat morphisms $f : V \to V'$, i.e. morphisms of coherent sheaves which commute with the connection. The rank of $M$ is the rank of the underlying vector bundle $V$.*

Throughout this section, we fix a scheme $X$ as in Definition 2.1, except in Proposition 2.14, where we relax the smoothness condition but request the scheme to be proper.

Definition 2.2. — *A $k$-linear abelian category is said to be locally finite if the Hom-set of any two objects is a finite dimensional vector space* *over $k$ and each object has a decomposition series of finite length. Such a category is called finite if it is spanned by an object, i.e. each object in the category is isomorphic to a subquotient of a direct sum of copies of the mentioned object.*

PROPERTIES 2.3.

1) *Standard addition, tensor product and exact sequences of connections endow $\mathsf{Conn}(X)$ with the structure of an abelian $k$-linear rigid tensor category.*
2) $\mathsf{Conn}(X)$ *is locally finite.*

*Proof.* — This is classical, see *e.g.* [9]). □

DEFINITION 2.4. — $\mathsf{FConn}(X)$ *is defined to be the full sub-category of* $\mathsf{Conn}(X)$ *spanned by Weil-finite objects, where $M$ is Weil-finite if there are polynomials $f, g \in \mathbb{N}[T], f \neq g$, such that $f(M)$ is isomorphic to $g(M)$, where $T^n(M) := M \otimes \cdots \otimes M$ $n$-times and $nT(M) := M \oplus \cdots \oplus M$ $n$-times. Thus an object in $\mathsf{FConn}(X)$ is a sub-quotient in $\mathsf{Conn}(X)$ of a Weil-finite object.*

PROPERTIES 2.5.

1) *An object $M$ of $\mathsf{Conn}(X)$ is Weil-finite if and only if $M^\vee$ is Weil-finite.*
2) *An object $M$ in $\mathsf{Conn}(X)$ is Weil-finite if and only the collection of all isomorphism classes of indecomposable objects in all the tensor powers $M^{\otimes n}, n \in \mathbb{N} \setminus \{0\}$, is finite.*
3) *The full tensor sub-category $\langle M \rangle$, the objects of which are sub-quotients of directs sums of tensor powers of $M$, is a finite tensor category.*
4) *Let $L \supset k$ be a finite field extension, we shall denote $X \times_k \mathrm{Spec}(L)$ by $X \times_k L$ for short. Denote by $\alpha : X \times_k L \to X$ the base change morphism. Then $\alpha^* \alpha_* M \twoheadrightarrow M$, resp. $N \subset \alpha_* \alpha^* N$ for every object $M$ in $\mathsf{Conn}(X \times_k L)$ resp. for every $N$ in $\mathsf{Conn}(X)$.*

*Proof of Properties 2.5.* — For 1), $f(M) \cong g(M)$ if and only if $f(M^\vee) \cong g(M^\vee)$. For 2), one argues as in [11, I, Lemma 3.1]. One introduces the naive $K$ ring $K^{\mathrm{Conn}}(X)$ which is spanned by isomorphism classes of objects $M$ of $\mathsf{Conn}(X)$ modulo the relation $[M] \cdot [M'] = [M \otimes M']$ and $[M] + [M'] = [M \oplus M']$. Since in $\mathsf{Conn}(X)$ every object has a decomposition in a direct sum of finitely many indecomposable objects, which is unique up to isomorphism, $K^{\mathsf{Conn}}(X)$ is freely spanned by classes $[M]$ with $M$ indecomposable. Then the proof of 2) goes word by word as in loc. cit. Properties 2.3 2). Now 2) implies 3) and 4) is obvious. □

We will show in the sequel that each object of $\mathsf{FConn}(X)$ is Weil-finite using Tannaka duality.

We first recall some basic concepts of groupoid schemes. Our ongoing reference is [2].

An affine groupoid scheme $\Pi$ over $k$ acting transitively over $S$ is a $k$-scheme equipped with a faithfully flat morphism $(t,s) : \Pi \longrightarrow S \times_k S$ and

– the multiplication $m : \Pi \ {}_s\!\underset{S}{\times}\!{}_t \ \Pi \longrightarrow \Pi$, which is a morphism of $S \times_k S$-schemes,
– the unit element morphism $e : S \longrightarrow \Pi$, which is a morphism of $S \times_k S$-schemes, where $S$ is considered as an $S \times_k S$-scheme by means of the diagonal morphism,
– the inverse element morphism $\iota : \Pi \longrightarrow \Pi$ of $S \times_k S$-schemes, which interchanges the morphisms $t, s$: $t \circ \iota = s$, $s \circ \iota = t$,

satisfying the following conditions:

$$\begin{gathered} m(m \times \mathrm{Id}) = m(\mathrm{Id} \times m), \\ m(e \times \mathrm{Id}) = m(\mathrm{Id} \times e) = \mathrm{Id}, \\ m(\mathrm{Id} \times \iota) = e \circ t, \quad m(\iota \times \mathrm{Id}) = e \circ s. \end{gathered} \tag{2.1}$$

One defines the diagonal group scheme $\Pi^\Delta$ over $S$ by the cartesian diagram

$$\begin{tikzcd} \Pi^\Delta \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Pi \arrow[d, "(t,s)"] \\ S \arrow[r, "\Delta"'] & S \times_k S \end{tikzcd} \tag{2.2}$$

where $\Delta : S \to S \times_k S$ is the diagonal embedding, and the $S$-scheme $\Pi_s$ by

$$\Pi_s = S - \text{scheme } \Pi \xrightarrow{s} S. \tag{2.3}$$

For $(b,a) : T \to S \times_k S$, one defines the $T$-scheme $\Pi_{b,a}$ by the cartesian diagram

$$\begin{tikzcd} \Pi_{b,a} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Pi \arrow[d, "(t,s)"] \\ T \arrow[r, "(b,a)"'] & S \times_k S \end{tikzcd} \tag{2.4}$$

For three $T$ points $a, b, c : T \to S$ of $S$, the multiplication morphism $m$ yields the map

$$\Pi_{c,b} \times_T \Pi_{b,a} \to \Pi_{c,a}. \tag{2.5}$$

By definition, the (abstract) groupoid $\Pi(T)$ has for objects morphisms $a : T \to S$ and for morphisms between $a$ and $b$ the set $\Pi_{b,a}(T)$ of $T$-points of $\Pi_{b,a}$. The composition law is given by (2.5).

Moreover, the multiplication $m$ induces an $S \times_k S$-morphism

$$\mu : \Pi_s \underset{S}{\times} \Pi^{\Delta} = \Pi \,{}_s\!\underset{S}{\times} \Pi^{\Delta} \to \Pi \underset{S\times_k S}{\times} \Pi$$

by the following rule: for

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[dr, "a"'] & \Pi \arrow[d, "s"] \\ & S \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "g"] \arrow[dr, "a"'] & \Pi^{\Delta} \arrow[d] \\ & S \end{tikzcd}$$

with $t \circ f = b : T \to S$, one has $f \in \Pi_{ba}(T)$, $g \in \Pi_{aa}(T)$, thus one can compose $fg \in \Pi_{ba}(T)$ (as morphisms in the groupoid $\Pi(T)$) and define

$$\mu(f, g) = (f, fg).$$

One has

$$\mu \text{ defines a principal bundle structure} \tag{2.6}$$
$$\text{i.e. } \Pi_s \times_S \Pi^{\Delta} \cong \Pi \times_{S\times_k S} \Pi \ .$$

Indeed one has to check that $\mu$ is an isomorphism which is a local property on $S$. In [5, Lemma 6.5] this fact was checked for the case $S = \mathrm{Spec}(K)$, $K \supset k$ a field, the proof can be easily extended for any $k$-algebras.

By definition, $\Pi_{t,s}$ is the fiber product $\Pi \times_{S\times_k S} \Pi$, seen as a $\Pi$-scheme via the first projection. Thus the identity morphism $\mathrm{Id}_\Pi : \Pi \to \Pi$ can be seen as a morphism between the two objects $t, s : \Pi \to S$ in the groupoid $\Pi(\Pi)$. This is the universal morphism. Let $V$ be a quasi-coherent sheaf on $S$. A representation $\chi$ of $\Pi$ on $V$ is a family of maps

$$\chi_{b,a} : \Pi_{b,a}(T) \longrightarrow \mathrm{Iso}_T(a^*V \to b^*V) \tag{2.7}$$

for each $(b, a) : T \to S \times_k S$, which is compatible with the multiplication (2.5) and the base change $T' \to T$. Then $\chi$ is determined by the image of $\mathrm{Id}_\Pi$ under $\chi_{t,s}$

$$\chi_{t,s}(\mathrm{Id}_\Pi) : s^*V \xrightarrow{\cong} t^*V.$$

The category of quasi-coherent representations of $\Pi$ on $S$ is denoted by $\mathrm{Rep}(S:\Pi)$. Each representation of $\Pi$ is the union of its finite rank representations. That is $\mathrm{Rep}(S:\Pi)$ is the ind-category of the category $\mathrm{Rep}_f(S:\Pi)$ of representations on coherent sheaves on $S$. The category $\mathrm{Rep}_f(S:\Pi)$ is an abelian rigid tensor category. Its unit object is the trivial representation of $\Pi$ on $\mathcal{O}_S$. On the other hand, since $\Pi$ is faithfully flat over $S\times_k S$, one has

$$\mathrm{End}_{\mathrm{Rep}(S:\Pi)}(\mathcal{O}_S)=k. \tag{2.8}$$

Thus $\mathrm{Rep}_f(S:\Pi)$ is a $k$-linear abelian rigid tensor category.

We notice that those notations apply in particular to $\Pi=S\times_k S$, the trivial groupoid acting on $S$, and one has

$$\mathsf{Vec}_k\cong \mathrm{Rep}(S:S\times_k S). \tag{2.9}$$

In this case, $(S\times_k S)^\Delta=S$ via the diagonal embedding, and $(S\times_k S)_s$ is the scheme $S\times_k S$ viewed as a $S$-scheme via the second projection. In order not to confuse notations, we denote the left projection $S\times_k S\to S$ by $p_1$ and by $p_2$ the right one. Thus $p_1=t,\ p_2=s$.

Theorem 2.6. — *Let $\Pi$ be an affine groupoid scheme over $k$ acting transitively on a $k$-scheme $S$.*

1) *The formula* (2.5) *defines $p_{2*}\mathcal{O}_{S\times_k S}$ and $s_*\mathcal{O}_\Pi$ as objects in* $\mathrm{Rep}(S:\Pi)$.
2) *$p_{2*}\mathcal{O}_{S\times_k S}$ and $s_*\mathcal{O}_\Pi$ are algebra objects in* $\mathrm{Rep}(S:\Pi)$.
3) *The inclusion $p_{2*}\mathcal{O}_{S\times_k S}\to s_*\mathcal{O}_\Pi$ of quasi-coherent sheaves on $S$ yields a $p_{2*}\mathcal{O}_{S\times_k S}$-algebra structure on $s_*\mathcal{O}_\Pi$.*
4) *The maximal trivial sub-object of $s_*\mathcal{O}_\Pi$ in* $\mathrm{Rep}(S:\Pi)$ *is the sub-object $p_{2*}\mathcal{O}_{S\times_k S}$.*

*Proof.* — We first prove 1). Fixing $f\in\Pi_{b,a}(T)$, (2.5) yields a morphism $\Pi_{bc}\to\Pi_{ac}$ and consequently an $\mathcal{O}_T$-algebra homomorphism $a^*s_*\mathcal{O}_\Pi\to b^*s_*\mathcal{O}_\Pi$. It is easily checked that this yields a representation of $\Pi$ on $s_*\mathcal{O}_\Pi$. In particular, the groupoid $S\times_k S$ has a representation in $p_{2*}\mathcal{O}_{S\times_k S}$. The homomorphism of groupoid schemes $(t,s):\Pi\longrightarrow S\times_k S$ yields a representation of $\Pi$ in $p_{2*}\mathcal{O}_{S\times_k S}$, which is in fact trivial. For 2), the algebra structure stemming from the structure sheaves is compatible with the $\Pi$-action. Then 3) is trivial.

As for 4), one notices that this property is local with respect to $S$, thus we can assume that $S=\operatorname{Spec}R$, where $R$ is a $k$-algebra. Then $\mathcal{O}_\Pi$ is a $k$-Hopf algebroid acting on $R$. One has an isomorphism of categories

$$\mathrm{Rep}(S:\Pi)\cong\mathsf{Comod}(R,\mathcal{O}_\Pi) \tag{2.10}$$

where $\mathsf{Comod}(R, \mathcal{O}_\Pi)$ denotes the category of right $\mathcal{O}_\Pi$-comodules in $\mathsf{Mod}_R$. Under this isomorphism, the representation of $\Pi$ in $s_*\mathcal{O}_\Pi$ corresponds to the coaction of $\mathcal{O}_\Pi$ on itself by means of the coproduct $\Delta : \mathcal{O}_\Pi \longrightarrow \mathcal{O}_\Pi \ {}_s\!\underset{R}{\otimes}_t \ \mathcal{O}_\Pi$. For $M \in \mathrm{Rep}(S : \Pi)$, one has the natural isomorphism

$$\mathrm{Hom}_{\mathrm{Rep}(S:\Pi)}(M, s_*\mathcal{O}_\Pi) \cong \mathrm{Hom}_R(M, R), \ f \mapsto e \circ f \tag{2.11}$$

where $e : \mathcal{O}_\Pi \to R$ is the unit element morphism. Indeed, the inverse to this map is given by the coaction of $\delta : M \to M \underset{R}{\otimes}_t \mathcal{O}_\Pi$ as follows

$$\big(\varphi : M \to R\big) \mapsto \big((\varphi \otimes \mathrm{Id})\delta : M \to M \otimes_t \mathcal{O}_\Pi \to \mathcal{O}_\Pi\big). \tag{2.12}$$

In particular, if we take $M = R$, the trivial representation of $\Pi$, then we see that

$$\mathrm{Hom}_{\mathsf{Comod}(R:\mathcal{O}_\Pi)}(R, \mathcal{O}_\Pi) \cong R. \tag{2.13}$$

Notice that in $\mathsf{Comod}(R, \mathcal{O}_\Pi)$, one has $\mathrm{Hom}_{\mathsf{Comod}(R,\mathcal{O}_\Pi)}(R, R) \cong k$. We conclude that the maximal trivial sub-comodule of $\mathcal{O}_\Pi$ is $R \otimes_k R$. This finishes the proof. □

In order to apply the theory of Tannaka duality as developed by Deligne in [2], one needs a fiber functor. Let $S$ be a scheme defined over $k$ and let

$$\rho : \mathsf{FConn}(X) \to \mathsf{QCoh}(S) \tag{2.14}$$

be a fiber functor with values in the category of quasi-coherent sheaves over $S$, as in [2, Section 1]. Such a $\rho$ exists. For example, a very tautological one is provided by $S = X$, $\rho = \tau$ (for tautological):

$$\mathsf{FConn}(X) \to \mathsf{QCoh}(X), (V, \nabla) \mapsto V.$$

Then the functor from $S \times_k S$-schemes to $\mathsf{Sets}$, which assigns to any $(b, a) : T \to S \times_k S$ the set of natural isomorphisms $\mathsf{Iso}_T^\otimes(a^*\rho, b^*\rho)$ of fiber functors to $\mathsf{QCoh}(T)$, is representable by the affine groupoid scheme

$$(t, s) : \Pi := \mathrm{Aut}^\otimes(\rho) \to S \times_k S$$

defined over $k$ and acting transitively over $S$ ([2, Théorème 1.12]):

$$\mathsf{Iso}_T^\otimes(a^*\rho, b^*\rho) \cong \mathsf{Mor}_{S \times_k S}(T, \Pi). \tag{2.15}$$

Tannaka duality asserts in particular that $\rho$ induces an equivalence between $\mathsf{FConn}(X)$ and the category $\mathrm{Rep}_f(S : \Pi)$ of representations of $\Pi$ in coherent sheaves over $S$. This equivalence extends to an equivalence between the ind-category $\mathsf{Ind\text{-}FConn}(X)$ of connections which are union of finite sub-connections and the category $\mathrm{Rep}(S : \Pi)$ of representations of $\Pi$ in quasi-coherent sheaves on $S$.

It follows from (2.15) for $T = \Pi$, $(b, a) = (t, s)$ that there is an isomorphism

$$s^*\rho \cong t^*\rho. \tag{2.16}$$

We now translate via Tannaka duality the assertions of Theorem 2.7.

Theorem 2.7. — *Let $X$ be a smooth scheme of finite type defined over a field $k$ of characteristic 0. Let $S$ be a $k$-scheme and $\rho : \mathsf{FConn}(X) \to \mathsf{QCoh}(S)$ be a fiber functor. Let $\Pi = \mathrm{Aut}^{\otimes}(\rho)$ be the corresponding Tannaka $k$-groupoid scheme acting on $S$. Then there is a diagram of $k$-schemes*

$$\begin{tikzcd} & X_\rho \arrow[dl, "p_\rho"'] \arrow[d, "s_\rho"] \arrow[dr, "\pi_\rho"] & \\ S & S\times_k X \arrow[l, "p_1"] \arrow[r, "p_2"'] & X \end{tikzcd} \tag{2.17}$$

*with the following properties.*

1) *$s_\rho$ is a $\Pi^\Delta$-principal bundle, that is*

$$X_\rho \times_{S\times_k X} X_\rho \cong X_\rho \times_S \Pi^\Delta.$$

2) $R^0(p_\rho)_{*DR}(X_\rho/S) := R^0(p_\rho)_*(\Omega^\bullet_{X_\rho/S}) = \mathcal{O}_S$.
3) *For all objects $N = (W, \nabla)$ in $\mathsf{FConn}(X)$, the connection*

$$\pi_\rho^* N : \pi_\rho^* W \to \pi_\rho^* W \otimes_{\mathcal{O}_{X_\rho}} \Omega^1_{X_\rho/S}$$

*relative to $S$ is endowed with an isomorphism with the relative connection*

$$p_\rho^*\rho(N) = (p_\rho^{-1}\rho(N) \otimes_{p_\rho^{-1}\mathcal{O}_S} \mathcal{O}_{X_\rho}, \mathrm{Id} \otimes d_{X_\rho/S})$$

*which is generated by the relative flat sections $p_\rho^{-1}\rho(N)$. This isomorphism is compatible with all morphisms in $\mathsf{FConn}(X)$.*
4) *One recovers the fiber functor $\rho$ via $X_\rho$ by an isomorphism*

$$\rho(N) \cong R^0(p_\rho)_{*DR}(X_\rho/S, \pi_\rho^* N)$$

*which is compatible with all morphisms in $\mathsf{FConn}(X)$. In particular, the data in (2.17) are equivalent to the datum $\rho$ (which defines $\Pi$).*
5) *This construction is compatible with base change: if $u : T \to S$ is a morphism of $k$-schemes, and $u^*\rho : \mathsf{FConn}(X) \to \mathsf{QCoh}(T)$ is the*

*composite fiber functor, then one has a cartesian diagram*

$$\begin{tikzcd} X_{u^*\rho} \arrow[r] \arrow[d, "s_{u^*\rho}"'] \arrow[dr, phantom, "\square"] & X_\rho \arrow[d, "s_\rho"] \\ T \times_k X \arrow[r, "u\times 1"'] & S \times_k X \end{tikzcd} \tag{2.18}$$

*which makes 1), 2), 3) functorial.*

*Proof.* — We define $A = (B, D) \subset M = (V, \nabla)$ to be the objects in $\mathsf{Ind\text{-}FConn}(X)$ which are mapped by $\rho$ to $p_{2*}\mathcal{O}_{S\times_k S} \subset s_*\mathcal{O}_\Pi$ defined in Theorem 2.6. By construction, $B = p_{2*}\mathcal{O}_{S\times_k X}$, where $p_2 : S \times_k X \to X$. Since $p_{2*}\mathcal{O}_{S\times_k S}$ is an $\mathcal{O}_S$-algebra with trivial action of $\Pi$, it follows that $A$, as an object in $\mathsf{Ind\text{-}FConn}(X)$, is trivial and is an algebra over $(\mathcal{O}_X, d)$. The algebra structure forces via the Leibniz formula the connection $D$ to be of the form $p_{2*}(D')$, where $D'$ is a relative connection on $S \times_k X/S$. The inclusion $(\mathcal{O}_X, d) \subset (B, D)$ implies that $D'(1) = d(1) = 0$, hence $D' = d_{S\times_k X/S}$. Thus we have

$$\begin{aligned} \mathrm{Spec}_X B = S \times_k X,\ B = (p_2)_*\mathcal{O}_{S\times_k X}, \\ D = (p_2)_*(d_{S\times_k X/S} : \mathcal{O}_{S\times_k X} \longrightarrow \Omega^1_{S\times_k X/S}) \\ \rho((p_2)_*(\mathcal{O}_{S\times_k X}, d_{S\times_k X/S})) = (p_2)_*\mathcal{O}_{S\times_k S}. \end{aligned} \tag{2.19}$$

We now apply to $M \supset A$ a similar argument. Since $s_*\mathcal{O}_\Pi$ is a $p_{2*}\mathcal{O}_{S\times_k S}$-algebra as a representation of $\Pi$, $M$ is an $A$-algebra object in $\mathsf{Ind\text{-}FConn}(X)$. The geometric information yields that $V$ is a $B$-algebra. Thus, setting $X_\rho = \mathrm{Spec}_X V$, we obtain a morphism $s_\rho : X_\rho \to S \times_k X$. The algebra structure in $\mathsf{Ind\text{-}FConn}(X)$ forces $\nabla$ to be coming from a relative connection $\nabla' : \mathcal{O}_{X_\rho} \to \Omega^1_{X_\rho/S}$. Moreover the inclusion $A \subset M$ implies $\nabla'(1) = D(1) = 0$, thus $\nabla' = d_{X_\rho/S}$. Summarizing, Theorem 2.6, 3) translates as follows

$$\begin{tikzcd} X_\rho = \mathrm{Spec}_X(V) \arrow[r, "s_\rho"] \arrow[dr, "\pi_\rho"'] & S \times_k X \arrow[d, "p_2"] \\ & X \end{tikzcd} \tag{2.20}$$

$$\begin{aligned} V = (\pi_\rho)_*\mathcal{O}_{X_\rho},\ \nabla = (\pi_\rho)_*(d_{X_\rho/S} : \mathcal{O}_{X_\rho} \longrightarrow \Omega^1_{X_\rho/S}), \\ \rho((\pi_\rho)_*(\mathcal{O}_{X_\rho}, d_{X_\rho/S})) = s_*\mathcal{O}_\Pi. \end{aligned}$$

Now Theorem 2.6, 1) together with (2.6) shows 1). To show 2) we first notice that $\mathsf{Ind\text{-}FConn}(X)$ is a full sub-category of the category of flat connections on $X$. Therefore, by Theorem 2.6 4), $(B, D)$ is the maximal trivial sub-connection of $(V, \nabla)$ in the latter category. So one obtains that $H^0_{DR}(X, M) = B$, or said differently, $R^0(p_\rho)_{*DR}(X_\rho/S) := R^0(p_\rho)_*(\Omega^\bullet_{X_\rho/S}) = \mathcal{O}_S$.

We prove 3). For $N$ in $\mathsf{FConn}(X)$, one has $s^*\rho(N) \cong t^*\rho(N)$ by (2.16). Since $\rho(N)$ is locally free ([2, 1.6]), projection formula applied to $s$ yields $\rho(N) \otimes_{\mathcal{O}_S} s_*\mathcal{O}_\Pi \cong s_*t^*\rho(N)$ as representations of $\Pi$. The Tannaka dual of the left hand side is $(\pi_{\rho *}\pi_\rho^* N$ while the Tannaka dual of the right hand side is $(\pi_\rho)_*(p_\rho^{-1}\rho(N) \otimes_{p_\rho^{-1}\mathcal{O}_S} \mathcal{O}_{X_\rho}, 1 \otimes d_{X_\rho/S})$. This shows 3).

For Claim 4), we just notice that by projection formula applied to the locally free bundle $\rho(N)$ and the connection $\pi_\rho^* N$ relative to $S$, one has

$$\begin{aligned} R^0(p_\rho)_{*DR}((p_\rho^{-1}\rho(N) \otimes_{p_\rho^{-1}\mathcal{O}_S} \mathcal{O}_{X_\rho}, 1 \otimes d_{X_\rho/S})) \\ = \rho(N) \otimes_{\mathcal{O}_S} R^0(p_\rho)_{*DR}(\mathcal{O}_{X_\rho}, d_{X_\rho/S}) \end{aligned}$$

By 2), this expression is equal to $\rho(N)$, while by 3), it is isomorphic to $R^0(p_\rho)_{*DR}(X_\rho/S, \pi_\rho^* N)$. This shows 4). Finally, the functoriality in 5) is the translation of the base change property [2, (3.5.1)]. $\square$

Definition 2.8.

1) *We fix an embedding $k \subset \bar{k}$ which defines* $\mathrm{Spec}(\bar{k})$ *as a $k$-scheme. Let $\rho : \mathsf{FConn}(X) \to \mathsf{Vec}_{\bar{k}}$ be a fiber functor. Then $\pi_\rho : X_\rho \to X$ in* (2.13) *has the factorization*

(2.21)
$$\begin{tikzcd} X_\rho \arrow[r, "s_\rho"] \arrow[dr, "\pi_\rho"'] & \bar{k} \times_k X \arrow[d, "p_2"] \\ & X \end{tikzcd}$$

*where $s_\rho$ is a principal bundle over $X$ under the $\bar{k}$-pro-finite group scheme $\Pi^\Delta$. In particular it is a pro-finite étale covering. Thus $\pi_\rho$ is an (infinite) étale covering of $X$ which we call* the universal covering of $X$ associated to $\rho$.

2) *Let $\rho = \tau$ be the tautological functor which assigns to a connection $(V, \nabla)$ its underlying bundle $V$. Then by definition*

$$\big(X_\tau \xrightarrow{s_\tau} X \times_s X\big) = \big(\Pi \xrightarrow{(t,s)} X \times_k X\big),$$

*with* $\Pi = \mathrm{Aut}^\otimes(\tau)$. *Since this groupoid plays a special rôle, we denote it by* $\Pi(X, \tau)$ *and call it* the total fundamental groupoid scheme of $X$.

PROPOSITION 2.9. — *Let $\rho : \mathsf{FConn}(X) \longrightarrow \mathsf{Vec}_k$ be a neutral fiber functor. Then for all finite field extensions $L \supset k$, $\rho$ lifts uniquely to $\rho_L : \mathsf{FConn}(X \times_k L) \to \mathsf{Vec}_L$.*

*Proof.* — Since $\rho$ is a neutral functor, $\Pi = \mathrm{Aut}^{\otimes}(\rho)$ is a group scheme over $k$. Tannaka duality reads $\mathsf{FConn}(X) \xrightarrow{\rho} \mathrm{Rep}(\Pi)$. By Property 2.5, 4), $\mathsf{FConn}(X \times_k L)$ is the $L$-base change of $\mathsf{FConn}(X)$ in the sense of [8, Corollary 5.6]. So Tannaka duality lifts to the duality $\rho_L : \mathsf{FConn}(X) \to \mathrm{Rep}_L(\Pi \times_k L)$ which fits in the following commutative diagram

$$\begin{tikzcd} \mathsf{FConn}(X) \arrow[r, "\rho", "\cong"'] \arrow[d] & \mathrm{Rep}_k(\Pi) \arrow[d] \\ \mathsf{FConn}(X \times_k L) \arrow[r, "\rho_L"'] & \mathrm{Rep}_L(\Pi \times_k L). \end{tikzcd} \tag{2.22}$$

We define the functor $\rho_L$ as follows. Let $\pi$ be the projection $X \times_k L \to X$. Then for any connection $N \in \mathsf{FConn}(X \times_k L)$ one has the canonical projection $\pi^*\pi_* N \to N$ of connections in $\mathsf{FConn}(X \times_k L)$. Consequently, by considering the dual connection $N^\vee$, one obtains an injection $N \to (\pi^*\pi_* N^\vee)^\vee \cong \pi^*(\pi_* N^\vee)^\vee$. Thus $N$ is uniquely determined by a morphism $f_N \in H^0_{DR}(X \times_k L, \pi^*((\pi_* N)^\vee \otimes (\pi_* N^\vee)^\vee))$. Via the base change isomorphism

$$H^0_{DR}(X \times_k L, \pi^* M) \xrightarrow{\cong} H^0_{DR}(X, M) \otimes_k L, \tag{2.23}$$

one can interpret $f_N$ as an element of

$$H^0_{DR}(X, (\pi_* N)^\vee \otimes (\pi_* N^\vee)^\vee) \otimes_k L \cong \mathrm{Hom}_{\mathsf{FConn}(X)}\, \pi_* N, (\pi_* N^\vee)^\vee) \otimes_k L$$

We define now the functor $\rho_L$ on objects of $\mathsf{FConn}(X \times_k L)$ of the form $\pi^* M$ by setting $\rho_L(\pi^* M) = \rho(M) \otimes_k L$ and (2.23) allows us to define $\rho_L$ on morphisms from $\pi^* M$ to $\pi^* M'$.

Next, we extend $\rho_L$ to an arbitrary object $N$ by defining $\rho_L(N)$ to be the image of $\rho_L(f_N)$ in $\rho_L((\pi_* N^\vee)^\vee)$.

As for unicity, if $\rho'$ is another lifting of $\rho$, then it has to agree on $f_N$ with $\rho_L$, thus has to agree with $\rho_L$ for all $N$. □

*Notation 2.10.* — With notations as in Proposition 2.9, we denote by $\rho_L$ the unique lifting of $\rho$ to $X \times_k L$ and by $\rho \times_k \bar{k}$ the induced lifting on $X \times_k \bar{k}$, once an algebraic closure $k \subset \bar{k}$ of $k$ has been fixed.

THEOREM 2.11. — *Let $X$ be a smooth scheme of finite type defined over a field $k$ of characteristic 0. We fix an embedding $k \subset \bar{k}$ which* *defines* $\mathrm{Spec}(\bar{k})$ *as a* $k$*-scheme. Assume there is a neutral fiber functor* $\rho : \mathsf{FConn}(X) \to \mathsf{Vec}_k$. *Then one has an isomorphism*

(2.24)
$$\begin{tikzcd} X_\rho \times_k \bar{k} \arrow[rr, "\cong"] \arrow[dr, "s_\rho \times_k \bar{k}"'] & & X_{\rho \times_k \bar{k}} \arrow[dl, "s_{\rho \times_k \bar{k}}"] \\ & X \times_k \bar{k} = \bar{k} \times_k X & \end{tikzcd}$$

*In particular, this yields a cartesian diagram*

(2.25)
$$\begin{tikzcd} X_{\rho \times_k \bar{k}} \cong \bar{k} \times_k X_\rho \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \bar{k} \times_k X \arrow[d, "p_2"] \\ X_\rho \arrow[r, "s_\rho"'] & X \end{tikzcd} .$$

*Proof.* — If $\Pi = \mathrm{Aut}^{\otimes}(\rho)$ is as in the proof of Proposition 2.9, then by construction, $\Pi \times_k \bar{k} = \mathrm{Aut}^{\otimes}(\rho) \times_k \bar{k}$. Thus base change as in Theorem 2.7, 5) implies the wished base change property for $X_\rho$. $\square$

We now come back to the converse of Properties 2.5, 3).

Corollary 2.12. — *Let* $X$ *be as in Definition 2.1. Then* $\mathsf{FConn}(X)$ *is a semi-simple category. Consequently any object of* $\mathsf{FConn}(X)$ *is Weil-finite.*

*Proof.* — We have to show that an exact sequence $\epsilon : 0 \to M \to P \to N \to 0$ splits, or equivalently, that the corresponding cohomology class in $H^1_{DR}(X, N^\vee \otimes M)$, also denoted by $\epsilon$, vanishes. Since de Rham cohomology fulfills base change, $\epsilon$ vanishes if and only if $\epsilon \otimes_k \bar{k}$ vanishes in $H^1_{DR}(X \times_k \bar{k}, N^\vee \otimes M)$. Thus we may assume that $k = \bar{k}$. Then we can find a $k$-rational point $x \in X(k)$ and $\mathsf{FConn}(X)$ is a (neutral) Tannaka category with respect to the fiber functor $\rho_x((V, \nabla)) = V|_x$. The exact sequence $\epsilon : 0 \to M \to P \to N \to 0$ lies in the category $\langle M \oplus N \oplus P \rangle$, which is finite by Properties 2.5, 3). Hence its Tannaka group $H$ is a finite group scheme over $\bar{k}$, and one has a surjective factorization

(2.26)
$$\begin{tikzcd} X_\rho \arrow[r, two heads] \arrow[dr, "\pi_\rho"'] & X_H \arrow[d, "\pi_H"] \\ & X \end{tikzcd}$$

where $\pi_H$ is a principal bundle under $H$. As $\pi_H^*$ is injective on de Rham cohomology, we are reduced to the case where $N = M = (\mathcal{O}_X, d), P \cong (\mathcal{O}_X, d) \oplus (\mathcal{O}_X, d)$. Then necessarily $\epsilon$ splits. $\square$

*Remarks 2.13.*

1) The notations are as in Theorem 2.7. Assume that $\rho$ is a neutral fiber functor. Then $\Pi = \mathrm{Aut}^{\otimes}(\rho)$ is an affine pro-finite $k$-group scheme, $S = \mathrm{Spec}(k)$ and by definition $s_*\mathcal{O}_\Pi = k[\Pi]$. Then (2.20) yields in particular

$$\rho((\pi_\rho)_*(\mathcal{O}_{X_\rho}, d_{X_\rho/k})) = k[\Pi]. \tag{2.27}$$

Let us assume that $\rho = \rho_x$, where $x \in X(k)$ and $\rho_x((V,\nabla)) = V|_x$. Then (2.27) reads

$$(\pi_{\rho_x})_*(\mathcal{O}_{X_{\rho_x}})|_x = k[\Pi]. \tag{2.28}$$

Here the $k$-structure on the $k$-group scheme $\Pi$ is via the residue field $k$ of $x$. Thus in particular,

$$1 \in \Pi(k) = \pi_{\rho_x}^{-1}(x)(k) \subset X_{\rho_x}(k) \tag{2.29}$$

and the rational point $x$ of $X$ lifts all the way up to the universal covering of $X$ associated to $\rho_x$.

2) Theorem 2.7, 4) says in loose terms that all fiber functors of $\mathsf{FConn}(X)$ are cohomological, as they are canonically isomorphic to a 0-th relative de Rham cohomology. On the other hand, there are not geometric in the sense that they do not come from rational points of $X$ or of some compactification (for the latter, see discussion in section 5). A simple example is provided by a smooth projective rational curve defined over a number field $k$, and without any rational point. Then $\mathsf{FConn}(X)$ is trivial, thus has the neutral fiber functor $M = (V,\nabla) \mapsto H^0_{DR}(X,V)$. Yet there are no rational points anywhere around.

Recall the definition of Nori's category of Nori finite bundles ([11, Chapter II]). Let $X$ be a proper reduced scheme defined over a perfect field $k$. Assume $X$ is connected in the sense that $H^0(X,\mathcal{O}_X) = k$. Then the category $\mathcal{C}^N(X)$ of Nori finite bundles is the full sub-category of the quasi-coherent category $\mathsf{QCoh}(X)$ consisting of bundles which in $\mathsf{QCoh}(X)$ are sub-quotients of Weil-finite bundles, where a bundle is Weil-finite when there are polynomials $f, g \in \mathbb{N}[T], f \neq g$ so that $f(M)$ is isomorphic to $g(M)$.

Let $\omega : \mathcal{C}^N(X) \longrightarrow \mathsf{QCoh}(X)$ be a fiber functor, where $S$ is a scheme over $k$. Then one can redo word by word the whole construction of Theorem 2.7 with $(\mathsf{FConn}(X), \rho)$ replaced by $(\mathcal{C}^N(X), \omega)$, as it goes purely via Tannaka duality. We denote by $\Pi^N = \mathrm{Aut}^{\otimes}(\omega)$ the $k$-groupoid scheme acting transitively on $S$. One obtains the following.

Proposition 2.14. — *Let $X$ be a connected proper reduced scheme of finite type over a perfect field $k$. Let $\omega$ be a fiber functor $\mathcal{C}^N(X) \longrightarrow \mathsf{QCoh}(S)$ where $S$ is a $k$-scheme and let $\Pi^N = \mathrm{Aut}^{\otimes}(\omega)$ be the corresponding Tannaka groupoid scheme acting on $S$. Then there is a diagram of $k$-schemes*

(2.30)
$$\begin{tikzcd} & X^N_\omega \arrow[ddl, "p^N_\omega"'] \arrow[d, "s^N_\omega"] \arrow[ddr, "\pi^N_\omega"] & \\ & S\times_k X \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\ S & & X \end{tikzcd}$$

*with the following properties.*

1') *$s^N_\omega$ is a $(\Pi^N)^\Delta$-principal bundle, that is*
$$X^N_\omega \times_{S\times_k X} X^N_\omega \cong X^N_\omega \times_S (\Pi^N)^\Delta.$$

2') *$R^0(p^N_\omega)_*(\mathcal{O}_{X^N_\omega}) = \mathcal{O}_S$.*

3') *For all objects $V$ in $\mathcal{C}^N(X)$, the bundle $(\pi^N_\omega)^*V$ is endowed with an isomorphism with the bundle $(p^N_\omega)^*\omega(V)$ which is compatible with all morphisms in $\mathcal{C}^N(X)$.*

4') *One recovers the fiber functor $\omega$ via $X^N_\rho$ by an isomorphism*
$$\omega(M) \cong R^0(p^N_\omega)_*(X_\omega/S, \pi^*_\omega M)$$
*which is compatible with all morphisms in $\mathcal{C}^N(X)$. In particular, the data in (2.30) are equivalent to the datum $\omega$ (which defines $\Pi^N$).*

5') *This construction is compatible with base change: if $u : T \to S$ is a morphism of $k$-schemes, and $u^*\omega : \mathcal{C}^N(X) \to \mathsf{QCoh}(T)$ is the composite fiber functor, then one has a cartesian diagram*

(2.31)
$$\begin{tikzcd} X^N_{u^*\omega} \arrow[r] \arrow[d, "s^N_{u^*\omega}"'] \arrow[dr, phantom, "\square"] & X^N_\omega \arrow[d, "s^N_\omega"] \\ T\times_k X \arrow[r, "u\times 1"'] & S\times_k X \end{tikzcd}$$

*which make 1), 2), 3) functorial.*

6') *(see Definition 2.8, 2). If $\omega$ is the tautological fiber functor $\iota$ defined by $\iota(V) = V$, then*
$$\big(X^N_\iota \xrightarrow{s_\tau} X\times_s X\big) = \big(\Pi^N \xrightarrow{(t,s)} X\times_k X\big).$$

THEOREM 2.15. — *Let $X$ be a smooth proper scheme of finite type over a field $k$ of characteristic 0 with $k = H^0(X, \mathcal{O}_X)$. Then the functor $F : \mathsf{FConn}(X) \to \mathcal{C}^N(X)$, $F((V,\nabla)) = V$ is an equivalence of Tannaka categories.*

*Proof.* — Since both categories $\mathsf{FConn}(X)$ and $\mathcal{C}^N(X)$ satisfy the base change property in the sense of Property 2.5, 4), we may assume that $k = \bar{k}$.

Let $V$ be a finite bundle, we want to associate to it a connection $\nabla_V$, such that $(V, \nabla_V)$ is a finite connection. Denote by $\mathcal{T}$ the full tensor sub-category of $\mathcal{C}^N(X)$ generated by $V$. It is a finite category. We apply the construction in Proposition 2.14 to $\omega = \iota : \mathcal{T} \longrightarrow \mathsf{QCoh}(X)$, defined by $\iota(V) = V$. Then (2.30) reads

(2.32)
$$\begin{tikzcd} & X_{\mathcal{T}} = \Pi_{\mathcal{T}} \arrow[dl, "t"'] \arrow[d] \arrow[dr, "s"] & \\ X & X \times_k X \arrow[l, "p_1"] \arrow[r, "p_2"'] & X \end{tikzcd}$$

where $t, s$ are the structure morphisms. The scheme $X_{\mathcal{T}}$ is a principal bundle over $X \times_k X$ under the finite $X$-group scheme $(\Pi_{\mathcal{T}})^{\Delta}$. For an object $V$ in $\mathcal{T}$, 3') becomes a functorial isomorphism

$$t^*V \cong s^*V. \tag{2.33}$$

Let $d_{\mathrm{rel}}$ denote the relative differential $\mathcal{O}_{X_{\mathcal{T}}} \to \Omega^1_{X_{\mathcal{T}}}/s^*\Omega^1_X$ on $X_{\mathcal{T}}/X$ with respect to the morphism $s$. Then the bundle $s^*V$ carries the canonical connection

$$s^*V = s^{-1}V \otimes_{s^{-1}\mathcal{O}_X} \mathcal{O}_{X_{\mathcal{T}}} \xrightarrow{\mathrm{Id}_V \otimes d_{\mathrm{rel}}} s^{-1}V \otimes_{s^{-1}\mathcal{O}_X} (\Omega^1_{X_{\mathcal{T}}}/s^*\Omega^1_X). \tag{2.34}$$

Then (2.33) implies that (2.34) can be rewritten as

$$t^*V \xrightarrow{\mathrm{Id}_V \otimes d_{\mathrm{rel}}} t^*(V) \otimes_{\mathcal{O}_{X_{\mathcal{T}}}} t^*(\Omega^1_X). \tag{2.35}$$

Applying $R^0t_*$ to (2.35), the property 2') together with projection formula implies that one obtains a connection

$$\nabla_V := R^0t_*(\mathrm{Id}_V \otimes d_{\mathrm{rel}}) : V \to V \otimes_{\mathcal{O}_X} \Omega^1_X. \tag{2.36}$$

Integrability of $\mathrm{Id}_V \otimes d_{\mathrm{rel}}$ implies integrability of $\nabla_V$. The compatibility of $\mathrm{Id}_V \otimes d_{\mathrm{rel}}$ with morphisms in $\mathcal{C}^N(X)$ implies the compatibility of $\nabla_V$ with morphisms in $\mathcal{C}^N(X)$ as well. We conclude that $\nabla_V$ defines a functor

$$\mathcal{C}^N(X) \to \mathsf{Conn}(X),\ V \mapsto (V, \nabla_V). \tag{2.37}$$

It remains to show that $(V, \nabla_V)$ is in $\mathsf{FConn}(X)$. To this aim, we fix a $k$-rational point $x : \mathrm{Spec}(k) \to X$ and consider the restriction of (2.32) to $X_{\mathcal{T},x} \to X \times_k x$, where $X_{\mathcal{T},x} = s^{-1}x$. The restriction of $t$ to $X_{\mathcal{T},x}$ will be denoted by $t_x : X_{\mathcal{T},x} \to X$. This is a principal bundle under the finite $k$-group scheme $(\Pi_\tau)^\Delta|_x$, the fiber of $(\Pi_\tau)^\Delta \to X$ above $x$. Since the connection $\mathrm{Id}_V \otimes d_{\mathrm{rel}}$ in (2.34), or equivalently (2.35) is relative to the $X$-factor on the right, we can restrict it to $X_{\mathcal{T},x}$ to obtain a connection

$$(\mathrm{Id}_V \otimes d_{\mathrm{rel}})|_{X_{\mathcal{T},x}} : t_x^* V \xrightarrow{\mathrm{Id}_V \otimes d_{\mathrm{rel}}} t_x^*(V) \otimes_{\mathcal{O}_{X_{\mathcal{T},x}}} t_x^*(\Omega^1_X).$$

Since $t^*V$ was generated by the relatively flat sections $s^{-1}V$ of $\mathrm{Id}_V \otimes d_{\mathrm{rel}}$, $t_x^*V$ is generated by the relative flat sections $V|_x$ of $(\mathrm{Id}_V \otimes d_{\mathrm{rel}})|_{X_{\mathcal{T},x}}$. On the other hand, by construction

$${t_x}^*(V, \nabla_V) = (\mathrm{Id}_V \otimes d_{\mathrm{rel}})|_{X_{\mathcal{T},x}}. \tag{2.38}$$

Thus $(V, \nabla_V)$ is trivializable when pulled back to the principal bundle $t_x : X_{\mathcal{T},x} \to X$ under the finite group scheme $(\Pi_\tau)^\Delta|_x$. Hence $(V, \nabla_V)$ is a sub-connection of a direct sum of copies of the connection $(W, \nabla_W) := (t_x)_*(\mathcal{O}_{X_{\mathcal{T},x}}, d_{\mathrm{rel}})$, which is known to be finite as it fulfills the equation

$$(W, \nabla_W)^{\otimes 2} \cong (W, \nabla_W)^{\oplus \mathrm{rank} W}. \tag{2.39}$$

Hence $(V, \nabla)$ is finite. We conclude that (2.37) has image in $\mathsf{FConn}(X)$. Since the composite functor

$$\mathsf{FConn}(X) \xrightarrow{F\ \hookrightarrow} \mathcal{C}^N(X) \xrightarrow{(2.37)\ \hookrightarrow} \mathsf{FConn}(X) \tag{2.40}$$

is the identity, and both $F$ and (2.37) are full. This finishes the proof. $\square$

## 3. Splitting of groupoid schemes

*Notation 3.1.* — Throughout this section, we will use the following notations. $k$ is a field of characteristic 0, endowed with an algebraic closure $\bar{k} \supset k$. $\mathcal{C}$ is an abelian $k$-linear rigid tensor category, endowed with a fiber functor $\rho : \mathcal{C} \to \mathsf{Vec}_{\bar{k}}$. By [2, Théorème 1.12], the groupoid scheme $\Pi = \mathrm{Aut}^{\otimes}(\rho)$ over $k$, acting on $\mathrm{Spec}(\bar{k})$, via $(t, s) : \Pi \to \mathrm{Spec}(\bar{k}) \times_k \mathrm{Spec}(\bar{k})$, is affine and acts transitively. $\mathcal{C}$ is equivalent via the fiber functor to the category of finite dimensional representations of $\Pi$:

$$\mathcal{C} \xrightarrow{\rho\ \cong} \mathrm{Rep}_f(\mathrm{Spec}(\bar{k}) : \Pi). \tag{3.1}$$

As in (2.3), we denote by $\Pi_s$ the scheme $\Pi$ viewed as a $\bar{k}$-scheme via the projection $s : \Pi \to \mathrm{Spec}(\bar{k})$. So

$$\Pi_s(\bar{k}) = \left\{ \begin{tikzcd} \mathrm{Spec}(\bar{k}) \arrow[r] \arrow[dr, "\mathrm{Id}"'] & \Pi \arrow[d, "s"] \\ & \mathrm{Spec}(\bar{k}) \end{tikzcd} \right\}. \tag{3.2}$$

For $\mathcal{C}$ the trivial category, the objects of which being sums of finitely many copies of the unit object, $\Pi$ is the groupoid scheme $\mathrm{Spec}(\bar{k}) \times_k \mathrm{Spec}(\bar{k})$ while $\Pi_s(\bar{k})$ is identified via Galois theory with the pro-finite group $\mathrm{Gal}(\bar{k}/k)$.

Theorem 3.2.

1) *There exists a group structure on* $\Pi_s(\bar{k})$ *such that the map*

$$(t,s)|_{\Pi_s} : \Pi_s(\bar{k}) \to (\mathrm{Spec}(\bar{k}) \times_k \mathrm{Spec}(\bar{k}))_s(\bar{k}) \cong \mathrm{Gal}(\bar{k}/k)$$

*is a group homomorphism, the kernel of which is* $\Pi^{\Delta}(\bar{k})$.

2) *Splittings of*

$$(t,s)|_{\Pi_s} : \Pi_s(\bar{k}) \to (\mathrm{Spec}(\bar{k}) \times_k \mathrm{Spec}(\bar{k}))_s(\bar{k}) \cong \mathrm{Gal}(\bar{k}/k)$$

*as group homomorphisms are in one to one correspondence with splittings of* $(t,s) : \Pi \to \mathrm{Spec}(\bar{k}) \times_k \mathrm{Spec}(\bar{k})$ *as* $k$*-affine groupoid scheme homomorphisms.*

3) *There is a one to one correspondence between neutral fiber functors of* $\mathcal{C}$*, up to natural equivalence, and splittings of* $(t,s)$ *up to an inner conjugation of* $\Pi$ *given by an element of* $\Pi^{\Delta}(\bar{k})$.

*Proof.* — Let $\theta \in \Pi_s(\bar{k})$, set $\gamma = t\theta : \mathrm{Spec}(\bar{k}) \xrightarrow{\theta} \Pi_s \xrightarrow{t} \mathrm{Spec}(\bar{k})$. As the morphisms are all over $\mathrm{Spec}(k)$, $\gamma$ is an element of $\mathrm{Gal}(\bar{k}/k)$. By definition, $\theta$ is a tensor isomorphism between $\rho$ and $\gamma^*(\rho)$, where $\gamma^*(\rho)$ is the $\mathrm{Spec}(\bar{k})$-valued fiber functor on $\mathcal{C}$ which is the composite of $\rho : \mathcal{C} \to \mathsf{Vec}_{\bar{k}}$ with $\gamma^* : \mathsf{Vec}_{\bar{k}} \to \mathsf{Vec}_{\bar{k}}$, $V \mapsto V \otimes_\gamma \bar{k}$:

$$\rho \xrightarrow{\theta\ \cong} \gamma^* \circ \rho. \tag{3.3}$$

Thus given $\theta'$ with image $\gamma'$, the group structure is simply given by the compositum

$$\theta \cdot \theta' : \rho \xrightarrow{\theta'\ \cong} (\gamma')^* \circ \rho \xrightarrow{(\gamma')^*\theta \cong} (\gamma')^* \circ \gamma^* \circ \rho = (\gamma \circ \gamma')^* \circ \rho. \tag{3.4}$$

The inverse to $\theta$ is $\gamma^*(\theta^{-1})$. This shows 1).

Assume that one has a splitting $\sigma_s : \mathrm{Gal}(\bar{k}/k) \to \Pi_s(\bar{k})$ of the map $(t,s)|_{\Pi_s(\bar{k})} : \Pi_s(\bar{k}) \to \mathrm{Gal}(\bar{k}/k)$. This means that to any $\gamma$ in $\mathrm{Gal}(\bar{k}/k)$, one assigns a map $\sigma_s(\gamma) : \mathrm{Spec}(\bar{k}) \to \Pi_s(\bar{k})$, such that

$$\sigma_s(\gamma)\cdot\sigma_s(\gamma') = \sigma_s(\gamma\circ\gamma'); \quad t(\sigma(\gamma)) = \gamma.$$

As in the discussion above, $\sigma_s(\gamma)$ defines a natural isomorphism $\sigma_s(\gamma) : \rho \to \rho\otimes_\gamma \bar{k}$. For a $\bar{k}$-vector space $V$ we identify $V\otimes_\gamma \bar{k}$ with $V$ by setting $v\otimes_\gamma a \mapsto v\gamma(a)$. Consequently $\sigma_s(\gamma)$ yields a natural action of $\mathrm{Gal}(\bar{k}/k)$ on $\rho$:

$$\gamma : \rho \xrightarrow{\sigma_s(\gamma)} \rho\otimes_\gamma \bar{k} \cong \rho,\ \gamma\in \mathrm{Gal}(\bar{k}/k). \tag{3.5}$$

Here "natural action" means that it commutes with morphism in $\mathcal{C}$. Galois theory applied to the values of the fiber functor $\rho$ implies that there exists a $k$-form $\rho_0$ for $\rho$: $\rho = \rho_0\otimes_k \bar{k}$. Now the Tannaka construction of $\Pi$ from $\rho$ [2, Section 4.7] tells us that there exists a homomorphism of groupoid schemes acting on $\mathrm{Spec}(\bar{k})$: $\Pi \longrightarrow \mathrm{Spec}(\bar{k})\times_k \mathrm{Spec}(\bar{k})$. The converse claim is trivial. This finishes the proof of 2).

We turn now to the proof of 3). Notice that the structure map $(t,s)$ considered as a homomorphism of groupoid schemes corresponds through Tannaka duality to the tautological fully faithful functor $\beta : \mathsf{Vec}_k^f \to \mathcal{C}$, $V \mapsto V\otimes_k I$, which has the property that $\rho\circ\beta$ is the base change $\tau : \mathsf{Vec}_k^f \to \mathsf{Vec}_{\bar{k}}$, $V\mapsto V\otimes_k \bar{k}$. Here $\mathsf{Vec}_k^f$ means the category of finite dimensional $k$-vector spaces. (See [3, Proof of Theorem 2.11]). Then splittings of $(t,s)$ as homomorphisms of $k$-groupoid schemes acting on $\mathrm{Spec}(\bar{k})$ are in one to one correspondence with splittings $\sigma$ of the functor $\beta$ which are compatible with $\tau$ and $\rho$. This means one has a functor $\sigma : \mathcal{C}\to \mathsf{Vec}_k^f$ such that $\sigma\circ\beta = \mathrm{Id}$, together with an isomorphism of tensor functor $d : \tau\circ\sigma \to \rho$. So the following diagram

$$\begin{tikzcd} \mathsf{Vec}_k^f \arrow[dr, "\tau"'] & & \mathcal{C} \arrow[ll, "\sigma"'] \arrow[dl, "\rho"] \\ & \mathsf{Vec}_{\bar{k}} & \end{tikzcd}$$

commutes up to $d$.

By [2, Proposition 8.11], the isomorphism functor $\mathrm{Iso}^{\otimes}(\rho,\tau\circ\sigma)$ is representable by a torsor over $\mathrm{Spec}(\bar{k})$ under the $\bar{k}$-group scheme $\Pi^{\Delta}$. Thus, through Tannaka duality, neutral fiber functors of $\mathcal{C}$ are in one to one correspondence with splittings to $(t,s)$ up to an inner conjugation of $\Pi$ given by an element of $\Pi^{\Delta}(\bar{k})$. This finishes the proof of 3). $\square$

## 4. Grothendieck's arithmetic fundamental group

For a scheme $X$ defined over a field $k$ Grothendieck introduces in [6, Section 5] the category $\mathsf{ECov}(X)$ of finite étale coverings $\pi : Y \to X$, with Hom-Sets being $X$-morphisms. The choice of a geometric point $\bar{x}$ of $X$ defines a fiber functor $\pi \mapsto \pi^{-1}(\bar{x})$ with value in the category of finite sets. The automorphism group of this functor is called the *arithmetic fundamental group of $X$ with base point $\bar{x}$* and is denoted by $\pi_1(X, \bar{x})$. It is an abstract group, endowed with the pro-finite topology stemming from all its finite quotients. The main theorem claims an equivalence between finite sets with continuous $\pi_1(X, \bar{x})$-action and finite étale coverings of $X$. The equivalence also extends to an equivalence between pro-finite sets with continuous $\pi_1(X, \bar{x})$-action and pro-finite étale coverings of $X$. In particular, the action of $\pi_1(X, \bar{x})$ on itself via translations defines the *universal pro-finite étale covering of $X$ based at $\bar{x}$*. It will be denoted by $\widetilde{\pi}_{\bar{x}} : \widetilde{X}_{\bar{x}} \to X$. By definition, $\widetilde{\pi}_{\bar{x}}^{-1}(\bar{x}) = \pi_1(X, \bar{x})$.

This section is devoted to the comparison between Grothendieck's arithmetic fundamental group and our fundamental groupoid scheme, as well as between the universal pro-finite étale covering and a special case of the covering constructed in Theorem 2.7.

*Notation 4.1.* — In this section, $X$ is again a smooth scheme of finite type over a field $k$ of characteristic 0, with the property $k = H^0_{DR}(X)$. For a fiber functor $\rho$ of $\mathsf{FConn}(X)$ we denote by

$$\Pi(X, \rho) = \mathrm{Aut}^{\otimes}(\rho)$$

the corresponding Tannaka $k$-groupoid scheme. (As compared to the notations of section 2 where we simply used the notation $\Pi$, we emphasize here $\rho$). For a geometric point $\bar{x} : \mathrm{Spec}(\bar{k}) \to X$, we denote by $\rho_{\bar{x}} : \mathsf{FConn}(X) \to \mathsf{Vec}_{\bar{k}}$ the fiber functor that assigns to a connection the fiber of the underlying bundle at $\bar{x}$. We simplify the notation by setting

$$\Pi(X, \bar{x}) := \Pi(X, \rho_{\bar{x}}).$$

We call this $k$-groupoid scheme acting on $\mathrm{Spec}(\bar{k})$ the *fundamental groupoid scheme of $X$ with base point $\bar{x}$*.
Recall that the embedding $k \subset \bar{k}$ here is defined by the residue field of $\bar{x}$. Let $L$ be a finite field extension of $k$ in $\bar{k}$. We define the $L$-base change

$\Pi(X,\bar{x})_L$ of $\Pi(X,\bar{x})$ by the following cartesian product

$$\begin{tikzcd}
\Pi(X,\bar{x})_L \arrow[r, "q"] \arrow[d] \arrow[dr, phantom, "\square"] & \Pi(X,\bar{x}) \arrow[d] \\
\operatorname{Spec} L \arrow[r, "\Delta"'] & \operatorname{Spec}(L)\times_k \operatorname{Spec}(L)
\end{tikzcd}$$

where the morphism $\Pi(X,\bar{x}) \to \operatorname{Spec}(L)\times_k \operatorname{Spec}(L)$ is the composition of $(t,s)$ with the projection $\operatorname{Spec}(\bar{k})\times_k \operatorname{Spec}(\bar{k}) \to \operatorname{Spec}(L)\times_k \operatorname{Spec}(L)$. Then $\Pi(X,\bar{x})_L$ is an $L$-groupoid scheme acting on $\operatorname{Spec}(\bar{k})$ and we have the following commutative diagram

(4.1)
$$\begin{tikzcd}
\Pi(X,\bar{x})_L \arrow[r] \arrow[d, "{(t,s)}"'] & \Pi(X,\bar{x}) \arrow[r] \arrow[d, "{(t,s)}"'] & \operatorname{Spec}(L)\times_k \operatorname{Spec}(L) \arrow[d, "\mathrm{Id}"'] \\
\operatorname{Spec}(\bar{k})\times_L \operatorname{Spec}(\bar{k}) \arrow[r] & \operatorname{Spec}(\bar{k})\times_k \operatorname{Spec}(\bar{k}) \arrow[r] & \operatorname{Spec}(L)\times_k \operatorname{Spec}(L)
\end{tikzcd}$$

which is an exact sequence of groupoid schemes in the sense of [8].

Lemma 4.2. — *For all finite extensions $L \supset k$ with $L \subset \bar{k}$, one has a canonical isomorphism*

$$\Pi(X\times_k L,\bar{x}) \cong \Pi(X,\bar{x})_L$$

*which implies that $\Pi(X\times_k \bar{k},\bar{x}) \cong \Pi(X,\bar{x})^\Delta$. Consequently the following diagram is an exact sequence of groupoid schemes*

(4.2)
$$\begin{tikzcd}
\Pi(X\times_k \bar{k},\bar{x}) \arrow[r] \arrow[d] & \Pi(X,\bar{x}) \arrow[r] \arrow[d, "{(t,s)}"'] & \operatorname{Spec}(\bar{k})\times_k \operatorname{Spec}(\bar{k}) \arrow[d, "\mathrm{Id}"'] \\
\operatorname{Spec}(\bar{k}) \arrow[r, "\Delta"'] & \operatorname{Spec}(\bar{k})\times_k \operatorname{Spec}(\bar{k}) \arrow[r, "\mathrm{Id}"'] & \operatorname{Spec}(\bar{k})\times_k \operatorname{Spec}(\bar{k}),
\end{tikzcd}$$

*where $\Delta$ is the diagonal embedding.*

*Proof.* — We apply the base change property 2.5, 4) and [8, Corollary 5.11] to obtain $\Pi(X\times_k L,\bar{x}) \cong \Pi(X,\bar{x})_L$. Taking the limit on all $L$, one obtains $\Pi(X\times_k \bar{k},\bar{x}) \cong \Pi(X\times_k \bar{k},\bar{x})^\Delta$. $\square$

In the sequel, we use the simpler notation $\bar{X} := X\times_k \bar{k}$.

Applying the construction of Theorem 2.7 to the category $\mathsf{FConn}(\bar{X})$ equipped with the fiber functor at $\bar{x}$, ones obtains an étale cover

$$\pi_{\rho_{\bar{x}}} : (\bar{X})_{\rho_{\bar{x}}} \longrightarrow \bar{X} \tag{4.3}$$

LEMMA 4.3. — *The covering $\pi_{\rho_{\bar{x}}} : (\bar{X})_{\rho_{\bar{x}}} \longrightarrow \bar{X}$ is the universal pro-finite étale covering $\widetilde{\pi}_{\bar{x}} : \widetilde{(\bar{X})}_{\bar{x}} \to \bar{X}$ based at $\bar{x}$. In particular, one has the identification $\bar{\kappa} : \Pi(\bar{X},\bar{x})(\bar{k}) \xrightarrow{=} \pi_1(\bar{X},\bar{x})$ which decomposes as*

$$\bar{\kappa} : \Pi(\bar{X},\bar{x})(\bar{k}) = \pi_{\rho_{\bar{x}}}^{-1}(\bar{x}) = \widetilde{\pi}_{\bar{x}}^{-1}(\bar{x}) = \pi_1(\bar{X},\bar{x}). \tag{4.4}$$

*Proof.* — Since $\bar{X}$ is defined over an algebraically closed field $\bar{k}$, its universal pro-finite étale covering based at $\bar{x}$ is the pro-limit of finite Galois coverings $Y \xrightarrow{G} \bar{X}$ with $H^0_{DR}(Y) = \bar{k}$, which over $\bar{k}$ simply means $Y$ is connected. On the other hand, according to Theorem 2.7, $\pi_{\rho_{\bar{x}}}$ is a principal bundle under the $\bar{k}$-group scheme $\Pi(\bar{X},\bar{x})$. In fact, any finite full tensor sub-category $\mathcal{T}$ of $\mathsf{FConn}(X)$ defines a principal bundle $(\bar{X})_{\mathcal{T},\rho_{\bar{x}}}$ under the $\bar{k}$-group scheme $\Pi_{\mathcal{T}}(\bar{X},\bar{x})$, hence an étale Galois covering of $X$ under the finite group $\Pi_{\mathcal{T}}(\bar{X},\bar{x})(\bar{k})$ (since in characteristic 0, a finite group scheme is étale). Also according to Theorem 2.7, 2), $H^0_{DR}((\bar{X})_{\mathcal{T},\rho_{\bar{x}}}) = \bar{k}$, thus $(\bar{X})_{\mathcal{T},\rho_{\bar{x}}}$ is connected. Since $\mathsf{FConn}(\bar{X})$ is the union of its finite sub-categories, $\pi_{\rho_{\bar{x}}} : (\bar{X})_{\rho_{\bar{x}}} \to \bar{X}$ is the pro-limit of the $\pi_{\mathcal{T},\rho_{\bar{x}}} : (\bar{X})_{\mathcal{T},\rho_{\bar{x}}} \to \bar{X}$. Furthermore, by construction, $\pi^{-1}_{\mathcal{T},\rho_{\bar{x}}}(\bar{x}) = \Pi_{\mathcal{T}}(\bar{X},\bar{x})$ as a $\bar{k}$-group scheme, where $\bar{k}$ is the residue field of $\bar{x}$.

Conversely, let $p : Y \to X$ be a Galois covering with Galois group $G$, with $\pi_1(X,\bar{x}) \twoheadrightarrow G$, thus with $Y$ connected. Considering $G$ as a $\bar{k}$-algebraic constant group, then $p : Y \to X$ is a principal bundle under $G$ with $G = p^{-1}(\bar{x})$. Then $M := p_*(\mathcal{O}_Y, d)$ is finite as it fulfills the relation $M^{\otimes 2} \cong M^{\oplus \deg(p)}$. If we denote by $\mathcal{T}$ the full tensor sub-category generated by $p_*(\mathcal{O}_Y, d)$ then $G \cong \Pi_{\mathcal{T}}(\bar{X},\bar{x})(\bar{k})$ and $Y \cong (\bar{X})_{\mathcal{T},\rho_{\bar{x}}}$. Thus $\pi_{\rho_{\bar{x}}}$ is the universal pro-finite étale covering of $\bar{X}$ based at $\bar{x}$. This shows (4.4) and finishes the proof. □

THEOREM 4.4. — *Let $X/k$ be smooth scheme with $H^0_{DR}(X) = k$. Let $\bar{x} \to X$ be a geometric point with residue field $\bar{k}$. Then* (4.2) *induces an exact sequence of pro-finite groups*

$$1 \to \Pi(\bar{X},\bar{x})(\bar{k}) \to \Pi(X,\bar{x})_s(\bar{k}) \to (\mathrm{Spec}(\bar{k}) \times_k \mathrm{Spec}(\bar{k}))_s(\bar{k}) \to 1. \tag{4.5}$$

*Furthermore, the identity $\bar{\kappa}$ of Lemma 4.3 extends to an identity of exact sequences of pro-finite groups*

(4.6)

$$\begin{tikzcd}
1 \arrow[r] & \Pi(\bar{X},\bar{x})(\bar{k}) \arrow[r] \arrow[d, "\bar{\kappa}"', "="] & \Pi(X,\bar{x})_s(\bar{k}) \arrow[r] \arrow[d, "\kappa"', "="] & (\mathrm{Spec}(\bar{k})\times_k \mathrm{Spec}(\bar{k}))_s(\bar{k}) \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & \pi_1(\bar{X},\bar{x}) \arrow[r] & \pi_1(X,\bar{x}) \arrow[r, "\epsilon"] & \mathrm{Gal}(\bar{k}/k) \arrow[r] & 1
\end{tikzcd}$$

*Proof.* — Let $\theta \in \Pi(X,\bar{x})_s(\bar{k})$, with image $\gamma \in \mathrm{Gal}(\bar{k}/k)$. Then by (3.3), $\theta$ is a tensor isomorphism between $\rho_{\bar{x}}$ and $\gamma^* \circ \rho_{\bar{x}}$. On the other hand, if $\pi : Y \to X$ is a Galois covering under finite quotient $H$ of $\pi_1(X,\bar{x})$, then $\pi_*(\mathcal{O}_Y, d) \in \mathsf{FConn}(X)$. Thus $\theta$ yields an isomorphism between $\mathcal{O}_{\pi^{-1}(\bar{x})}$ and $\gamma^*(\mathcal{O}_{\pi^{-1}(\bar{x})})$. This implies in particular that $\theta$ yields an automorphism of the set $\pi^{-1}(\bar{x})$, that is an element in $H$. Since, for another finite quotient $K$ of $\pi_1(X,\bar{x})$, with $K \twoheadrightarrow H$, the construction is compatible in the pro-system $\pi_1(X,\bar{x})$, and we obtain from $\theta$ an element in $\pi_1(X,\bar{x})$. This defines the homomorphism $\kappa$. As $\bar{\kappa}$ is an isomorphism, $\kappa$ is an isomorphism as well. □

Applying the construction in Theorem 2.7 to the fiber functor $\rho_{\bar{x}}$ one obtains an étale cover $\pi_{\rho_{\bar{x}}} : X_{\rho_{\bar{x}}} \longrightarrow X$.

Corollary 4.5. — *The étale cover $\pi_{\rho_{\bar{x}}} : X_{\rho_{\bar{x}}} \longrightarrow X$ is the universal pro-finite étale covering of $X$ based at $\bar{x}$.*

*Proof.* — Let $\widetilde{\pi}_{\bar{x}} : \widetilde{X}_{\bar{x}} \to X$ be the universal pro-finite étale covering based at $\bar{x}$. Then, $\bar{x}$ lifts to the 1-element in $\widetilde{\pi}_{\bar{x}}^{-1}(\bar{x}) = \pi_1(X,\bar{x})$ which we denote by $\widetilde{x}$. Thus $\widetilde{x} \in \widetilde{X}_{\bar{x}}(\bar{k})$. By Theorem 4.4, $\widetilde{X}_{\bar{x}}$ and $X_{\rho_{\bar{x}}}$ are both Galois covers of $X$ under the same pro-finite group via $\kappa$. Thus an isomorphism between the two covers over $X$ is uniquely determined by the image of $\widetilde{x}$, which we determine to be the 1-element in $\Pi(X,\bar{x})(\bar{k}) = \rho_{\bar{x}}^{-1}(\bar{x}) \subset X_{\rho_{\bar{x}}}(\bar{k})$. □

We deduce now the main corollary of the sections 3 and 4.

Corollary 4.6. — *Let $X/k$, $\bar{x}, \rho_{\bar{x}}$ be as in Theorem 4.4. Then there is a one to one correspondence between splittings of $\epsilon$ as pro-finite groups up to conjugation by $\pi_1(\bar{X},\bar{x})$ and neutral fiber functors $\mathsf{FConn}(X) \to \mathsf{Vec}_k$, up to natural equivalence.*

*Proof.* — By Theorem 4.4, a splitting of $\epsilon$ of pro-finite groups is equivalent to a splitting of $\Pi(X,\rho_x)_s(\bar{k}) \to \mathrm{Gal}(\bar{k}/k)$. By Theorem 3.2, 2), the latter is equivalent to a splitting of $(t,s) : \Pi(X,\rho_{\bar{x}}) \to \bar{x} \times_k \bar{x}$ as $k$-groupoid schemes acting on $\bar{x}$. By Theorem 3.2, 3), such splittings are, up to conjugation by $\Pi(\bar{X},\bar{x})(\bar{k})$, in one to one correspondence with neutral fiber functors on $\mathsf{FConn}(X)$, up to natural equivalence. □

## 5. An application toward Grothendieck's section conjecture

In his letter to G. Faltings [7], Grothendieck conjectures that if $X$ is a smooth projective curve of genus $\geqslant 2$ defined over $k$ of finite type over

$\mathbb{Q}$, then conjugacy classes of sections of $\epsilon$ in (4.6) are in one to one correspondence with rational points of $X$. In fact, it follows immediately from the definition of $\pi_1^{\text{ét}}(X)$ that each rational point of $X$ yields a conjugacy class of sections of $\epsilon$. Grothendieck suggests an explicit way to relate to a conjugacy class of sections of $\epsilon$ a rational point of $X$, which we shall recall now. Fix a $\bar{k}$-point $\bar{x}$ of $X$ and let $\widetilde{X}_{\bar{x}}$ be the universal pro-étale covering of $X$ based at $\bar{x}$. Consider $\widetilde{X}_{\bar{x}}$ as a $\bar{k}$-scheme. The given splitting of $\epsilon$ defines an action of $\mathrm{Gal}(\bar{k}/k)$ on $\widetilde{X}_{\bar{x}}$. The conjecture is that the set of $\bar{k}$-points of $\widetilde{X}_{\bar{x}}$, which are invariant under the action of $\mathrm{Gal}(\bar{k}/k)$, consists of a unique point, which descends to a rational point of $X$.

In terms of Remark 2.13, the conjecture says that the action of $\mathrm{Gal}(\bar{k}/k)$ on $\widetilde{X}_{\bar{x}}$, considered as a $\bar{k}$-scheme, defines a $k$-form $X_\rho$ together with a $k$-pro-point, which descends to a $k$-point of $X$. It follows from Corollary 4.6 that if Grothendieck's conjecture is true then the set of $k$-rational points of a smooth projective curve $X$ of genus $g \geqslant 2$ over $k$ of finite type over $\mathbb{Q}$ is in bijection with the set of neutral fiber functors $\mathsf{FConn}(X) \to \mathsf{Vec}_k$. In other words, if Grothendieck's conjecture was too optimistic, there would be a smooth projective curve $X$ of genus $\geqslant 2$, and an *exotic* neutral fiber functor $\rho$ on $\mathsf{FConn}(X)$ which is not *geometric*, *i.e.* not of the shape $\rho = \rho_x,\ \rho_x((V,\nabla)) = V|_x$ for some rational point $x \in X(k)$.

Grothendieck goes further to consider the case when $X$ is affine. Let $\bar{x}$ be a $\bar{k}$-point of $X$, $X^\wedge$ be the smooth compactification of $X$ and $\widetilde{X^\wedge}$ be the normalization of $X^\wedge$ in $\widetilde{X}_{\bar{x}}$:

(5.1)
$$\begin{tikzcd} \widetilde{X}_{\bar{x}} \arrow[r, hook] \arrow[d] & \widetilde{X^\wedge} \arrow[d] \\ X \arrow[r, hook] & X^\wedge \end{tikzcd}$$

Also in this case he conjectures that the action of $\mathrm{Gal}(\bar{k}/k)$ on $\widetilde{X^\wedge}(\bar{k})$, given by a splitting of $\epsilon$, has a unique fix point, which descends to a rational point of $X^\wedge$. A section of $\epsilon$ is said to be at infinity if the corresponding fix point in $\widetilde{X^\wedge}$ lies in $\widetilde{X^\wedge} \smallsetminus \widetilde{X}_{\bar{x}}$, and hence descends to a rational point lying at $\infty$ in $X^\wedge \smallsetminus X$. To a given rational point at infinity in $X^\wedge$ there are associated infinitely many sections at infinity.

The aim of this section is to apply our construction in the previous sections to understand the last claims of the conjecture. Let $y$ be a rational point of $X^\wedge \smallsetminus X$. We construct $k$-forms of $\widetilde{X}_{\bar{x}}$ such that the normalization of $X^\wedge$ in it has a $k$-pro-point lying above $y$. It suffices to consider $X' = X \cup \{y\}$ instead of $X^\wedge$.

We first construct from $y$ a neutral fiber functor $\eta$ by means of the Deligne-Katz functor. Let $\widehat{\mathcal{O}}_y$ be the local formal ring at $y$, and $\widehat{K}_y$ be its field of fractions. We choose a local parameter so $\widehat{\mathcal{O}}_y \cong k[[t]]$ and $\widehat{K}_y \cong k((t))$.

Let $D^\times := \mathrm{Spec}(k((t))) \subset D := \mathrm{Spec}(k[[t]])$ be the punctured formal disk embedded in the formal disk. In [9, Section 2.1], Katz defines the $k$-linear rigid tensor category $\mathsf{Conn}(D^\times)$ of $t$-adically continuous connections on $D^\times$ as follows. Objects are pairs $M = (V, \nabla)$, where $V$ is a finite dimensional vector space on $k((t))$ and $\nabla : V \to V \otimes_{k((t))} \omega_{k((t))}$ a $t$-adically continuous connection, with $\omega_{k((t))} \cong k((t))dt$. The Hom-Sets are flat morphisms. We define $\mathsf{FConn}(D^\times)$ in the obvious way, as being the full sub-category of $\mathsf{Conn}(D^\times)$ spanned by Weil-finite objects, with definition as in Definition 2.4. One has the restriction functor

$$\mathsf{FConn}(X) \xrightarrow{\mathsf{rest}} \mathsf{FConn}(D^\times). \tag{5.2}$$

Let us recall the constructions of Deligne [1, Section 15, 28-36], and Katz [9, Section 2.4], which are depending on the $t$ chosen. There is a functor of $k$-linear abelian rigid tensor categories

$$\mathsf{FConn}(D^\times) \xrightarrow{DK} \mathsf{FConn}(\mathbb{G}_m),\ \mathbb{G}_m := \mathrm{Spec}(k[t, t^{-1}]) \tag{5.3}$$

which is the restriction to $\mathsf{FConn}$ of a functor defined on the categories of regular singular connections on $D^\times$ and $\mathbb{G}_m$ (by Deligne), and even of all connections (by Katz). It is characterized by the property that it is additive, functorial for direct images by $\mathrm{Spec}(k'((u))) \to D^\times$, $u^n = t$, where $k' \supset k$ is a finite field extension which contains the $n$-th rooths of unity, and by its value on nilpotent connections. The choice of a rational point $a \in \mathbb{G}_m(k)$, for example $a = 1$, defines a fiber functor $\rho_{\mathbb{G}_m} : \mathsf{FConn}(\mathbb{G}_m) \to \mathsf{Vec}_k$ by assigning $V|_a$ to $(V, \nabla)$. We denote the resulting functor

$$\mathsf{FConn}(D^\times) \overset{DK}{\to} \mathsf{FConn}(\mathbb{G}_m) \overset{\rho_{\mathbb{G}_m}}{\to} \mathsf{Vec}_k$$

by $\varphi$. The composite functor $\mathsf{FConn}(X) \overset{\mathsf{rest}}{\to} \mathsf{FConn}(D^\times) \to \mathsf{Vec}_k$ will be denoted by $\eta$:

$$\begin{tikzcd} \mathsf{FConn}(X) \arrow[d, "\mathsf{rest}"'] \arrow[dr, "\eta"] & \\ \mathsf{FConn}(D^\times) \arrow[r, "\varphi"'] & \mathsf{Vec}_k. \end{tikzcd} \tag{5.4}$$

Recall that Deligne's fiber functor, that is on regular singular connections, depends only on the tangent vector underlying the local parameter, *i.e.*

on the image of $t$ in $\langle t\rangle/\langle t^2\rangle$, so a fortiori on finite connections, only the tangent vector enters the construction.

We apply the construction of Theorem 2.7 to the pair $(\mathsf{FConn}(X), \eta)$ to obtain a morphism $X_\eta \to X$. The construction of Theorem 2.7, while applied to $D^\times = \mathrm{Spec}(k((t)))$ and the pair $(\mathsf{FConn}(D^\times), \varphi)$, yields a covering $D^\times_\varphi \to D^\times$. Indeed, as the connections are $t$-adically continuous, the finite sub-category spanned by a finite connection $M$ yields a finite field extension $k((t)) \subset L_M$, which defines $D^\times_M := \mathrm{Spec}(L_M)$, and $D^\times_\varphi = \varprojlim_M D^\times_M$. Theorem 2.7, 2) applies here to give $H^0_{DR}(D^\times_\varphi) = k$. Thus, (5.4) implies that we have the following diagram

$$\begin{tikzcd} D^\times_\varphi \arrow[r] \arrow[d] & X_\eta \times_X D^\times \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_\eta \arrow[d] \\ D^\times \arrow[r, "="'] & D^\times \arrow[r] & X \end{tikzcd} \tag{5.5}$$

We now introduce the partial compactification in $y$. Since the morphism $D^\times_\varphi \to D^\times$ is indeed a pro-system of $\mathrm{Spec}(L_M)$ of $D^\times$, we define its compactification as the corresponding pro-system of the normalizations of $D$ in $L_M$. Similarly the compactification of $X_\eta \to X$ is the pro-system of the normalizations of $X'$ in the function fields of the coverings. We denote by $'$ the compactifications. This yields

$$(D^\times_\varphi)' \to D,\ (X_\eta)' \to X', \tag{5.6}$$

and the diagram

$$\begin{tikzcd} (D^\times_\varphi)' \arrow[r] \arrow[d] & (X_\eta)' \times_{X'} D \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (X_\eta)' \arrow[d] \\ D \arrow[r, "="'] & D \arrow[r] & X'. \end{tikzcd} \tag{5.7}$$

Note the rational point $y \in X'(k)$ lies in $D(k)$, with defining maximal ideal $\langle t\rangle$.

Lemma 5.1. — *The point $y \in D(k)$ lifts to a pro-point $y'_\varphi \in (D^\times_\varphi)'(k)$, and thus defines a rational pro-point $y'_\eta$ in $(X_\eta)'(k)$.*

*Proof.* — Étale coverings of $D^\times$ are disjoint unions of coverings of the shape $\mathrm{Spec}(k'((u)))$ for some finite field extension $k' \supset k$ and $u^n = t$ for some $n \in \mathbb{N} \setminus \{0\}$. Notice that $H^0_{DR}(\mathrm{Spec}(k'((u)))) = k'$. On the other hand, Theorem 2.7, 2) implies that $H^0_{DR}(D^\times_\varphi) = k$. This means that in the pro-system defining $D^\times_\varphi$, there are only connected coverings with $k' = k$.

Hence $y \in D(k)$ lifts to $y'_\varphi \in (D^\times_\varphi)'(k)$. Its image $y'_\eta$ in $(X_\eta)'(k)$ is the required pro-point. $\square$

On the other hand, the two $\mathsf{Vec}_{\bar k}$-valued fiber functors $\eta \times_k \bar k$ and $\rho_{\bar x}$ on $X \times_k \bar k$ are equivalent, hence one obtains

$$\widetilde{X}_{\bar x} \cong X_\eta \times_k \bar k \text{ as } \bar k-\text{schemes.} \tag{5.8}$$

We summarize:

Theorem 5.2. — *Let $X/k$ be a smooth affine curve over a characteristic 0 field $k$. Let $y$ be a $k$-rational point at infinity. Set $X' := X \cup \{y\}$. Fix a geometric point $\bar x \to X$ with residue field $\bar k$. Then the universal pro-finite étale covering $\widetilde{X}_{\bar x}$ based at $\bar x$ has a $k$-structure $X_\eta$, i.e. $\widetilde{X}_{\bar x} \cong_{\bar k} X_\eta \times_k \bar k$, with the property that the $k$-rational point $y$ lifts to a $k$-rational pro-point of the normalization $(X_\eta)'$ of $X'$ in $k(X_\eta)$.*

## BIBLIOGRAPHY


[1] P. Deligne, "Le groupe fondamental de la droite projective moins trois points", in *Galois groups over* **Q** *(Berkeley, CA, 1987)*, Math. Sci. Res. Inst. Publ., vol. 16, Springer, New York, 1989, p. 79-297.

[2] ———, "Catégories tannakiennes", in *The Grothendieck Festschrift, Vol. II*, Progr. Math., vol. 87, Birkhäuser, Boston, MA, 1990, p. 111-195.

[3] P. Deligne & J. Milne, "Tannakian Categories", *Lectures Notes in Mathematics, Springer-Verlag* **900** (1982), p. 101-228.

[4] H. Esnault, P. Hai & X.-T. Sun, "On Nori group scheme", *Progress in Math., Birkhäuser* **265** (2007), p. 375-396.

[5] H. Esnault & P. H. Hai, "The Gauss-Manin connection and Tannaka duality", *Int. Math. Res. Not.* (2006), p. 35, Art. ID 93978.

[6] A. Grothendieck, "Revêtements étales et groupe fondamental, SGA 1", *Lectures Notes in Mathematics, Springer-Verlag* **224** (1971), p. xxii + 447.

[7] A. Grothendieck, "Brief an G. Faltings", in *Geometric Galois actions, 1*, London Math. Soc. Lecture Note Ser., vol. 242, Cambridge Univ. Press, Cambridge, 1997, With an English translation on pp. 285–293, p. 49-58.

[8] P. Hai, "A construction of a quotient category", 24 pages, preprint, 2006.

[9] N. M. Katz, "On the calculation of some differential Galois groups", *Invent. Math.* **87** (1987), no. 1, p. 13-61.

[10] M. V. Nori, "On the representations of the fundamental group", *Compositio Math.* **33** (1976), no. 1, p. 29-41.

[11] ———, "The fundamental group-scheme", *Proc. Indian Acad. Sci. Math. Sci.* **91** (1982), no. 2, p. 73-122.


Manuscrit reçu le 26 avril 2007,
révisé le 7 avril 2008,
accepté le 25 avril 2008.

Hélène ESNAULT & Phùng Hô HAI
Universität Duisburg-Essen
Mathematik
45117 Essen (Germany)
esnault@uni-due.de
hai.phung@uni-duisburg-essen.de