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HOMOGENEOUS BUNDLES AND THE FIRST EIGENVALUE OF SYMMETRIC SPACES

by Leonardo BILIOTTI & Alessandro GHIGI

ABSTRACT. — In this note we prove the stability of the Gieseker point of an irreducible homogeneous bundle over a rational homogeneous space. As an application we get a sharp upper estimate for the first eigenvalue of the Laplacian of an arbitrary Kähler metric on a compact Hermitian symmetric spaces of ABCD–type.

RÉSUMÉ. — On montre que le point de Gieseker d'un fibré homogène irréductible sur un espace homogène rationnel est stable. On en déduit une majoration optimale de la première valeur propre du laplacien d'une métrique Kählérienne quelconque sur un espace symétrique Hermitien compact du type ABDC.

1. Introduction

Let X be a compact complex manifold and let E be a holomorphic vector bundle of rank r over X. The *Gieseker point* of E is the map

$$T_E : \Lambda^r H^0(X, E) \longrightarrow H^0(X, \det E) \tag{1.1}$$

that sends an element $s_1 \wedge \cdots \wedge s_r \in \Lambda^r H^0(X, E)$ to the section $x \mapsto s_1(x)\wedge\cdots\wedge s_r(x)$ of $H^0(X, \det E)$. This map was first considered by Gieseker in his work [10] in order to construct the moduli space of vector bundles on a projective manifold. He proved that for the set of Gieseker stable bundles E with fixed rank and Chern classes on a polarised (X, H) there is a uniform k_0 such that for $k > k_0$, $T_{E(k)} = T_{E\otimes H^{\otimes k}}$ is a stable vector (in the sense of geometric invariant theory) with respect to the action of $\mathrm{SL}\big(H^0(X, E(k))\big)$ on $\mathrm{Hom}\big(\Lambda^r H^0(X, E(k)), H^0(X, \det E(k))\big)$.

In this paper we consider the Gieseker point of homogeneous bundles over rational homogeneous spaces. Such bundles are known to be Mumford-Takemoto stable [23], [24] and this implies they are Gieseker stable [16,

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p. 191]. So we already know that after twisting with a sufficiently ample line bundle their Gieseker point is stable. The interest here is in the stability of T_E itself, without allowing any twist. Our result is the following.

THEOREM 1.1. — *Let $E \to X$ be an irreducible homogeneous vector bundle over a rational homogeneous space $X = G/P$. If $H^0(E) \neq 0$, then T_E is stable.*

We give two proofs of this result. The first is algebraic and uses a criterion of Luna for an orbit to be closed. This proof works over any algebraically closed field of characteristic zero. The second proof uses invariant metrics and relies on a result by Xiaowei Wang [25].

Our interest for this result is connected with a problem in Kähler geometry. Consider a compact Kähler manifold X and fix a Kähler class $a \in H^2(X)$. Bourguignon, Li and Yau [4], gave an upper bound for the first eigenvalue of the Laplacian $\Delta_g : C^\infty(X) \to C^\infty(X)$ relative to any Kähler metric g whose Kähler form ω_g lies in the class a. The bound depends on the numerical invariants (h^0 and degree) of a globally generated line bundle L over X. To get the best estimate one has to choose appropriately the bundle. As is shown in [4], if $X = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(1)$ one gets the upper bound 2, which is optimal since it is achieved by the Fubini-Study metric.

In paper [2] Arezzo, Loi and the second author, generalised this result, by substituting a vector bundle E to the line bundle L. In this case one gets the same kind of estimate, but the vector bundle E must satisfy an additional condition, namely its Gieseker T_E point must be stable. By this method one gets an upper bound for λ_1 on the complex Grassmannian [2]. Such a bound is optimal since it is achieved by the symmetric metric. It is important to notice that if one twists the bundle E by a positive line bundle H, the estimate gotten from the twisted bundle $E(k)$ is very rough. In fact the estimate blows up as $k \to \infty$ (see (3.2) below). So it is important to obtain some information on the stability of T_E without twisting the bundle E.

The main motivation for the present work was to extend the estimate for λ_1 to other Hermitian symmetric spaces of the compact type using appropriate homogeneous bundles. We are able to prove the following.

THEOREM 1.2. — *Let X be a compact irreducible Hermitian symmetric space of ABCD-type. Then*

$$\lambda_1(X, g) \leqslant 2$$

for any Kähler metric g whose Kähler class ω_g lies in $2\pi\,\mathrm{c}_1(X)$. This bound is attained by the symmetric metric.

It should be mentioned here that El Soufi and Ilias [7, Rmk. 1, p. 96] have proved that the symmetric metric is a critical point (in suitable sense) for the functional λ_1 on the set of *all* Riemannian metrics with fixed volume.

Curiously in the two exceptional examples (E-type) the best estimate gotten by this method is strictly larger than 2, which is λ_1 of the symmetric metric.

Theorem 1.3. — *If $X = E_6/P(\alpha_1)$ (resp. $X = E_7/P(\alpha_7)$) then*

$$\lambda_1(X,g) \leqslant \frac{36}{17} \qquad \textit{resp.} \quad \lambda_1(X,g) \leqslant \frac{133}{53}$$

for any Kähler metric g whose Kähler class ω_g lies in $2\pi\,\mathrm{c}_1(X)$.

It would be interesting to understand if this is a deficiency of the method, or if these symmetric spaces do in fact support metrics with λ_1 larger than 2.

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2. Stability of the Gieseker point

Let X be a compact complex manifold and let $E \to X$ be a holomorphic vector bundle of rank r. Set

$$V = H^0(X,E), \quad V' = H^0(X, \det E) \quad \mathbb{W} = \mathrm{Hom}(\Lambda^r V, V'). \tag{2.1}$$

The algebraic group $\mathrm{GL}(V)$ acts linearly on V hence on $\Lambda^r V$. It therefore also acts on $\mathbb{W}$. For $a \in \mathrm{GL}(V)$ let $\Lambda^r a$ be the induced map on $\Lambda^r V$. The action of $\mathrm{GL}(V)$ on $\mathbb{W}$ is given by

$$a.T := T \circ (\Lambda^r a)^{-1}. \tag{2.2}$$

Consider the above action restricted to the subgroup $\mathrm{SL}(V) \subset \mathrm{GL}(V)$. According to the terminology of geometric invariant theory, a point $T \in \mathbb{W}$ is *stable* (for this restricted action) if the orbit of $\mathrm{SL}(V)$ through T is closed in $\mathbb{W}$ and the stabiliser of T inside $\mathrm{SL}(V)$ is finite. We denote by $\widetilde{S}$ and S the stabilisers of T_E in $\mathrm{GL}(V)$ and $\mathrm{SL}(V)$ respectively:

$$\widetilde{S} = \{a \in \mathrm{GL}(V) : a.T_E = T_E\} \qquad S = \{a \in \mathrm{SL}(V) : a.T_E = T_E\}. \tag{2.3}$$

For $x \in X$ put

$$V_x = \{s \in V : s(x) = 0\} \qquad V'_x = \{t \in V' : t(x) = 0\}.$$

The following two simple lemmata will be used in the (algebraic) proof of Theorem 1.

Lemma 2.1. — *Let E be globally generated of rank r. Then a section $s \in V$ belongs to V_x if and only if for any choice of $r-1$ sections $s_2, \dots, s_r$ of E the section $T_E(s, s_2, \dots, s_r)$ of $\det E$ lies in V'_x.*

The proof is immediate. Let $\mathrm{Aut}(E)$ be the group of holomorphic bundle automorphisms of E. If $f \in \mathrm{Aut}(E)$ and $x \in X$, the map $f_x : E_x \to E_x$ is a linear isomorphism. The function $x \mapsto \det f_x$ is holomorphic hence a (nonzero) constant and $f \mapsto \det f$ is a character of $\mathrm{Aut}(E)$. For $f \in \mathrm{Aut}(E)$ and $s \in H^0(X, E)$ put

$$\varepsilon(f)(s) := f \circ s. \tag{2.4}$$

This defines a representation $\varepsilon : \mathrm{Aut}(E) \to \mathrm{GL}(V)$.

Lemma 2.2. — *Let E be globally generated. Then $\varepsilon(\{f \in \mathrm{Aut}(E) : \det f = 1\}) = \widetilde{S}$.*

Proof. — For $f \in \mathrm{Aut}(E)$

$$\begin{aligned}
&\big(\varepsilon(f).T_E\big)(s_1, \dots, s_r)(x) = \big(\varepsilon(f)^{-1} s_1\big)(x) \wedge \cdots \wedge \big(\varepsilon(f)^{-1} s_r\big)(x) \\
&= \big(f_x^{-1} s_1(x)\big) \wedge \cdots \wedge \big(f_x^{-1} s_r(x)\big) = \big(\det f\big)^{-1} \cdot s_1(x) \wedge \ldots \wedge s_r(x) \\
&= \big(\det f\big)^{-1} \cdot T_E(s_1, \dots, s_r)(x).
\end{aligned}$$

So

$$\varepsilon(f).T_E = \big(\det f\big)^{-1} \cdot T_E. \tag{2.5}$$

If $\det f = 1$ then $\varepsilon(f).T_E = T_E$. This proves that $\varepsilon(\{f \in \mathrm{Aut}(E) : \det f = 1\}) \subset \widetilde{S}$. Conversely, let $a \in \widetilde{S}$. We claim that $a(V_x) = V_x$ for any $x \in X$. Indeed let $s \in V_x$. Then for any $s_2, \dots, s_r \in V$

$$\begin{aligned}
&T_E(as, s_2, \dots, s_r) = T_E(as, aa^{-1}s_2, \dots, aa^{-1}s_r) \\
&= (a^{-1}.T_E)(s, a^{-1}s_2, \dots, a^{-1}s_r) \\
&= T_E(s, a^{-1}s_2, \dots . a^{-1}s_r)(x) = 0.
\end{aligned}$$

So $as \in V_x$ by Lemma 2.1, and indeed $a(V_x) = V_x$ as claimed. We get therefore an induced isomorphism

$$f_x : E_x \cong V/V_x \to V/V_x \cong E_x.$$

By construction $f_x(s(x)) = (a\,s)(x)$. Since E is globally generated this ensures that f is holomorphic so $f \in \mathrm{Aut}(E)$ and $\varepsilon(f) = a$. By (2.5) $\big(\det f\big)^{-1} \cdot T_E = \varepsilon(f).T_E = a.T_E = T_E$. Since E is globally generated, $T_E \neq 0$ and it follows that $\det f = 1$. □

We recall two results that will be needed in the following.

Theorem 2.3. — *Let H be a reductive group and $K \subset H$ a reductive subgroup. Let X be an affine H-variety. If $x \in X$ is a fixed point of K the orbit Hx is closed if and only if the orbit $N_H(K)x$ is closed.*

This criterion is due to Luna [18, Cor. 1] and is based on the Slice Theorem. For a complex analytic proof of the Slice Theorem see [12].

We recall that a *rational homogeneous space* is a projective variety X of the form G/P with G a simply connected complex semisimple Lie group and P a parabolic subgroup without simple factors. (See for example [1], [22], [3]). Such spaces are also called *generalised flag manifolds*. A homogeneous vector bundle E over X is of the form $E = G \times_P U$ where U is a representation of P. If the representation is irreducible the vector bundle itself is called *irreducible*.

Theorem 2.4 ([23, Thm. 1]). — *Let $E \to X$ be an irreducible homogeneous vector bundle over a rational homogeneous space X. Then E is simple, i.e.,* $\mathrm{Aut}(E) = \mathbb{C}^* \cdot I_E$.

First proof of Theorem 1.1. — Let $X = G/P$ be as above. By Bott-Borel-Weil theorem (Thm. 3.3 below) the hypothesis $H^0(E) \neq 0$ already ensures that E is globally generated, so both V and V' have positive dimension and $T_E \not\equiv 0$. By the same theorem G acts irreducibly on both V and V'. Denote by $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(V')$ these representations. Since G is semisimple all the characters of G are trivial. In particular any representation of G on a vector space U has image contained in $\mathrm{SL}(U)$. So in fact $\rho : G \to \mathrm{SL}(V)$ and $\sigma : G \to \mathrm{SL}(V')$. The Gieseker point is G-equivariant, that is

$$T_E(\Lambda^r \rho(g)(u)) = \sigma(g)(T(u)) \qquad u \in \Lambda^r V. \tag{2.6}$$

Set $H = \mathrm{SL}(V) \times G$ and define a representation $\varpi : H \to \mathrm{GL}(\mathbb{W})$ by

$$\varpi(a, g)T = \sigma(g) \circ T \circ \Lambda^r a^{-1}.$$

Let $\tau : G \to H$ be the morphism $\tau(g) = (\rho(g), g)$ and let K be the image of τ. $K \subset H$ is a closed reductive subgroup and by (2.6) T_E is a fixed point of K acting via ϖ. We claim that the normaliser $N_H(K)$ is a finite extension of K. In fact, denote by Ad the conjugation on H. Given $n \in N_H(K)$ put

$$\phi(n) = \mathrm{Ad}(n)_{|K} : K \to K. \tag{2.7}$$

Let $\mathrm{Aut}(K)$ denote the group of automorphisms of K and $\mathrm{Inn}(K)$ the subgroup of inner automorphisms. Then $\phi : N_H(K) \to \mathrm{Aut}(K)$ is a morphism of groups. Since K is semisimple $\mathrm{Aut}(K)$ is a finite extension of $\mathrm{Inn}(K)$ (see [13, p. 423] or [20, Thm. 1 p. 203]). Put $N' = \phi^{-1}(\mathrm{Inn}(K))$. Then $N' \lhd N_H(K)$ and

$$N_H(K)/N' \hookrightarrow \mathrm{Aut}(K)/\operatorname{Inn}(K).$$

Therefore $N_H(K)$ is a finite extension of N' and it is enough to prove that N' is a finite extension of K. Indeed if $n \in N'$ there is some $k \in K$ such that $nk'n^{-1} = kk'k^{-1}$ for any $k' \in K$. So $k^{-1}n$ centralises K. If $k^{-1}n = (a, g)$ (with $a \in \mathrm{SL}(V)$ and $g \in G$) this means that for any $g' \in G$ we have

$$a\rho(g') = \rho(g')a \qquad\qquad gg' = g'g.$$

The second formula says that $g \in Z(G)$. The first formula says that $a : V \to V$ commutes with the representation ρ. Since this is irreducible Schur lemma implies that $a = \varepsilon I$ for some $\varepsilon \in \mathbb{C}^*$. But $a \in \mathrm{SL}(V)$, so $\varepsilon^p = 1$ where $p = \dim V$. Denote by U_p the group of p–roots of unity. Then $k^{-1}n = (\varepsilon, g) \in U_p \times Z(G)$. This proves that the composition

$$U_p \times Z(G) \to N' \to N'/K$$

is onto. Since $Z(G)$ is finite, it follows that N' and $N_H(K)$ are finite extensions of K. Now $T_E \in \mathbb{W}$ is a fixed point of K and $N_H(K)$ is a finite extension of K, so the orbit $N_H(K).T_E$ is a finite set, hence it is closed. Notice that both H and K are reductive. We can therefore apply Luna's criterion (Thm. 2.3) to the effect that the orbit $H.T_E$ is closed. To finish we claim that $H.T_E = \mathrm{SL}(V).T_E$. Since the action (2.2) of $\mathrm{SL}(V)$ and the restriction of ϖ to $\mathrm{SL}(V) \times \{1\} \subset H$ agree, the inclusion $H.T_E \supset \mathrm{SL}(V).T_E$ is obvious. For the other let $h = (a, g) \in H$. Then

$$\begin{aligned}\varpi(h)T_E &= \sigma(g) \circ T_E \circ \Lambda^r a^{-1} = \sigma(g) \circ T_E \circ \Lambda^r \big(a\rho(g)^{-1} \cdot \rho(g)\big)^{-1}\\ &= \sigma(g) \circ T_E \circ \Lambda^r \rho(g)^{-1} \circ \Lambda^r(\rho(g)a^{-1})\\ &= T_E \circ \Lambda^r(\rho(g)a^{-1}) = \varpi(a\rho(g)^{-1}, 1)T_E\end{aligned}$$

and $a\rho(g^{-1}) \in \mathrm{SL}(V)$. Therefore $H.T_E \subset \mathrm{SL}(V).T_E$ so the two orbits coincide. This shows that the orbit of T_E is closed. Let S and $\widetilde{S}$ be the stabilisers defined as in (2.3). By Theorem 2.4, $\operatorname{Aut}(E) = \mathbb{C}^* \cdot I_E$, therefore $\{f \in \operatorname{Aut}(E) : \det f = 1\}$ is finite, which implies, by Lemma 2.2, that $\widetilde{S}$ and a fortiori S are finite. □

We remark that this proof works over any algebraically closed field of characteristic zero.

We come now to the second proof of this result. Recall that if E is a globally generated bundle on X and $\boldsymbol{s} = \{s_1, \ldots, s_N\}$ is a basis of $H^0(X, E)$ there is an induced map $\phi_{\boldsymbol{s}} : X \to G(r, N)$. Consider on $G(r, N)$ the standard symmetric Kähler structure which coincides with the pull-back of the Fubini-Study metric via the Plücker embedding. Denote by $\mu : G(r, N) \to \mathfrak{su}(N)$ the moment map for the standard action of $\mathrm{SU}(N)$ on $G(r, N)$.

Theorem 2.5 ([25, Thm. 3.1]). — *Let (X^m, ω) be a compact Kähler manifold and let E be a globally generated bundle on X. Then T_E is stable if and only if there is a basis $\boldsymbol{s}$ of $H^0(X, E)$ such that*

$$\int_X \mu\big(\phi_{\boldsymbol{s}}(x)\big)\omega^m(x) = 0. \tag{2.8}$$

For the reader's convenience we briefly sketch the proof.

Proof. — Fix an arbitrary Hermitian metric h on E and consider on V the corresponding L^2–scalar product. Let $\boldsymbol{s}$ be an orthonormal basis with respect to this product. On the line bundle $\det E$ consider the metric $\phi_{\boldsymbol{s}}^* h_G$ where h_G is the metric on $\mathcal{O}_{G(r,N)}(1)$. Consider on V' the corresponding L^2–scalar product. Finally denote by $\langle \cdot\,, \cdot\rangle_{\mathbb{W}}$ the Hermitian inner product on $\mathbb{W}$, $\|\cdot\|_{\mathbb{W}}$ being the corresponding norm. Since we have fixed a basis we may identify $\mathrm{SL}(V)$ with $\mathrm{SL}(N, \mathbb{C})$. For $g \in \mathrm{SL}(N, \mathbb{C})$ set $\nu(g) = \log ||g^{-1}.T_E||_{\mathbb{W}}$. We consider ν as a function on $\mathrm{SL}(N, \mathbb{C})/\mathrm{SU}(N)$. On this space Wang introduces another functional

$$L(g) := \int_M \left(\sum_I ||(g^{-1}T_E)(s_I)(x)||^2_{\phi_{\boldsymbol{s}}^* h_G}\right)\frac{\omega^n}{n!}(x),$$

which is strictly convex on $\mathrm{SL}(N, \mathbb{C})/\mathrm{SU}(N)$ [25, Lemma 3.5]. (Here $s_I = s_{i_1} \wedge \cdots \wedge s_{i_r} \in \Lambda^r V$). Critical points of L correspond to $g \in \mathrm{SL}(N)$ such that the basis $\{gs_1, \ldots, gs_N\}$ satisfies (2.8). For some constants $C_2, C_4 \in \mathbb{R}$ and $C_1, C_3 > 0$ the inequalities

$$L \geqslant C_1\nu + C_2 \geqslant C_3 L + C_4 \tag{2.9}$$

hold on $\mathrm{SL}(N, \mathbb{C})/\mathrm{SU}(N)$. The first is proved by Wang [25, p. 406]. The second is simply an application of Jensen inequality to the convex function

$-\log$. If T_E is stable, then ν is proper by the Kempf-Ness theorem [15]. Hence L is proper too, so admits a minimum and there is a basis $\boldsymbol{s'}$ such that (2.8) is satisfied. On the other hand if there is such a basis, L has a minimum and being strictly convex this means it is proper. By (2.9), ν is proper as well and, again by Kempf-Ness theorem, this implies that T_E is stable. It should be noted that the identification of the moment map for a projective action with the differential of a convex functional is standard in analytic Geometric Invariant Theory [19, Ch. 8], [6, § 6.5], [11]. □

Second proof of Theorem 1.1. — Let K be a compact form of $G = \mathrm{Aut}(X)$. By averaging on K we can find K-invariant metrics ω and h on X and E respectively. Let $\boldsymbol{s}$ be a basis of $H^0(X, E)$ that is orthonormal with respect to the L^2-scalar product obtained using h and ω. By Bott-Borel-Weil theorem (Thm. 3.3 below) G and hence K act irreducibly on $H^0(X, E) \cong \mathbb{C}^N$. Denote by $\sigma : K \to \mathrm{SU}(N)$ this representation (recall that K is semisimple). Then $\mu \circ \phi_{\boldsymbol{s}}$ is K-equivariant and

$$B = \int_X \mu\big(\phi_{\boldsymbol{s}}(x)\big)\omega^m(x)$$

is a fixed point of $\mathrm{Ad}(\sigma(K)) \subset \mathrm{GL}(\mathfrak{su}(N))$, that is $\sigma(k)B = B\sigma(k)$ for $k \in K$. By Schur lemma this implies that $B = \lambda I$, so $B = 0$ since $B \in \mathfrak{su}(N)$. By Theorem 2.5 the Gieseker point T_E is stable. □

In order to clarify the meaning of the above result it might be good to notice that together with the numerical criteria of [10] it allows an easy proof of the Gieseker stability (see *e.g.* [16, p. 189]) of irreducible homogeneous bundles. We sketch this argument, although a stronger result (Mumford-Takemoto stability) is well-known (see [22, p. 65] and references therein).

Proposition 2.6 ([10, Prop. 2.3]). — *Let $T \in \mathbb{W}$ be a stable point. Let $V'' \subset V$ be a subspace and let d a number $1 \leqslant d < r$. Assume that for any $d+1$ vectors $v_1, \ldots, v_{d+1} \in V''$, $T(v_1, \ldots, v_d, v_{d+1}, \cdots) \equiv 0$. Then $\dim V'' < (d/r) \cdot \dim V$. If T is only semistable, then equality can hold.*

In [10] there is a proof in the semistable case, which works as well in the stable case.

Corollary 2.7. — *Let $E \to X$ be an irreducible homogeneous vector bundle of rank r over a rational homogeneous space $X = G/P$. If $H^0(E) \neq 0$, and $F \subset E$ is a subsheaf of rank d, then $h^0(F) < (d/r) \cdot h^0(E)$.*

Fix now an irreducible homogeneous bundle E of rank r and let $F \subset E$ be a subsheaf of rank d, with $0 < d < r$. Let H be any polarisation on X.

Since any line bundle is homogeneous, $E(k) = E \otimes H^{\otimes k}$ is homogeneous. By Serre Theorem there is a k_0 such that for $k \geqslant k_0$

$$H^i(X, F(k)) = H^i(X, E(k)) = \{0\} \qquad i > 0$$

and both $E(k)$ and $F(k)$ are globally generated. By Theorem 1.1, $T_{E(k)}$ is stable, so by the above corollary $\chi(X, F(k)) = h^0(X, F(k)) < (d/r) \cdot h^0(X, E(k)) = (d/r) \cdot \chi(X, E(k))$. This proves that any irreducible homogeneous bundle is Gieseker stable with respect to any polarisation.

3. The first eigenvalue of Hermitian symmetric spaces

Here we want to apply the previous stability result to a problem in spectral geometry. Let X be a projective manifold and L an ample line bundle on X. Let $\mathcal{K}(L)$ be the set of Kähler metrics g with Kähler form ω_g lying in the class $2\pi\, \mathrm{c}_1(L)$. For g in $\mathcal{K}(L)$ let Δ_g be the Laplacian on functions,

$$\Delta_g f = -d^* df = 2\, g^{i\bar{j}} \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}.$$

It is well-known that Δ_g is a negative definite elliptic operator and has therefore discrete spectrum: denote its eigenvalues by $0 > -\lambda_1(g) > -\lambda_2(g) > \cdots$. The following result of Lichnerowicz relates λ_1 to Kähler-Einstein geometry.

Theorem 3.1 ([9, Cor. 2.4.4, p. 42]). — *If X is a Fano manifold and g_{KE} is a Kähler-Einstein metric, i.e.,* $\mathrm{Ric}(g_{KE}) = g_{KE}$*, then $\lambda_1(g_{KE}) = 2$ if* $\mathrm{Aut}(X)$ *has positive dimension and $\lambda_1(g_{KE}) > 2$ otherwise.*

We are interested in upper estimates for $\lambda_1(g)$ of general metrics in the class $\mathcal{K}(L)$. Bourguignon, Li and Yau [4] first studied this problem and showed that the supremum

$$I(L) = \sup_{\mathcal{K}(L)} \lambda_1(g) \tag{3.1}$$

is always finite. (This heavily depends on the restriction to Kähler metrics, see [5]). They gave an explicit upper bound for $I(L)$ in terms of numerical invariants of a globally generated line bundle E. For $(X, L) = (\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1))$ they were able to show that $I(L) = 2$. The following criterion, due to Arezzo, Loi and the second author, is an extension of Bourguignon, Li and Yau's theorem. It allows to attack this problem using holomorphic vector bundles instead of just line bundles.

THEOREM 3.2 ([2, Thm. 1.1]). — *Let (X, L) be a polarised manifold and E a holomorphic vector bundle of rank r over X. Assume that E is globally generated and nontrivial and put*

$$J(E,L) := \frac{2\dim_{\mathbb{C}} X \cdot h^0(E)\,\langle c_1(E)\cup c_1(L)^{m-1}, [X]\rangle}{r\,(h^0(E)-r)\,\langle c_1(L)^m, [X]\rangle}. \tag{3.2}$$

If the Gieseker point T_E is stable, then

$$I(L) \leqslant J(E,L). \tag{3.3}$$

The result of [2] is slightly more general since there is no projectivity assumption on X.

We want to apply this result to the case where $X = G/P$ is a rational homogeneous space and E is homogeneous. In this case $J(E, L)$ can be computed, at least in principle, in terms of Lie algebra data. To proceed we fix the following (standard) notation. (See *e.g.* [8, Ch. 1],[22], [3]). G is a simply connected complex semisimple Lie group, $\mathfrak{g} = \operatorname{Lie} G$, $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, $l = \dim\mathfrak{h}$ is the rank of G, B is the Killing form of $\mathfrak{g}$, Δ is the root system of $(\mathfrak{g}, \mathfrak{h})$, Δ_+ is a system of positive roots, $\Delta_- = -\Delta_+$, $\Pi = \{\alpha_1, \dots, \alpha_l\}$ is the set of simple roots, $\varpi_1, \dots, \varpi_l$ denote the fundamental weights. $\Lambda = \mathbb{Z}\varpi_1 \oplus\cdots\oplus \mathbb{Z}\varpi_l \subset \mathfrak{h}^*$ is the weight lattice of $\mathfrak{g}$ relative to the Cartan subalgebra $\mathfrak{h}$. For $\alpha \in \Delta$ let $H_\alpha \in \mathfrak{h}$ be such that $\alpha(X) = B(X, H_\alpha)$. $\mathfrak{b}$ is the standard *negative* Borel subalgebra:

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_-} \mathfrak{g}_\alpha. \tag{3.4}$$

Parabolic subalgebras containing $\mathfrak{b}$ are of the form

$$\mathfrak{p}(\Sigma) = \mathfrak{b} \oplus \bigoplus_{\alpha\in\operatorname{span}(\Pi-\Sigma)\cap\Delta_+} \mathfrak{g}_\alpha \tag{3.5}$$

where Σ is some subset of Π. For example $\Sigma = \Pi$ corresponds to $\mathfrak{b}$, $\Sigma = \varnothing$ to $\mathfrak{g}$ and maximally parabolic subalgebras are of the form $\mathfrak{p}(\alpha_k)$. The algebra $\mathfrak{p}(\Sigma)$ admits a Levi decomposition $\mathfrak{p}(\Sigma) = \mathfrak{l}(\Sigma) \oplus \mathfrak{u}(\Sigma)$, where $\mathfrak{u}(\Sigma)$ is the nilpotent radical and

$$\mathfrak{l}(\Sigma) = \mathfrak{h} \oplus \bigoplus_{\alpha\in\operatorname{span}(\Pi-\Sigma)\cap\Delta} \mathfrak{g}_\alpha$$

is the reductive part. This latter admits a further decomposition $\mathfrak{l}(\Sigma) = \mathfrak{z}(\Sigma) \oplus \mathfrak{s}(\Sigma)$, $\mathfrak{z}(\Sigma)$ being the center and $\mathfrak{s}(\Sigma)$ being semisimple. Moreover

$$\mathfrak{z}(\Sigma) = \bigcap_{\alpha\in\Pi-\Sigma} \ker\alpha \subset \mathfrak{h}$$

and

$$\mathfrak{s}(\Sigma) = \mathfrak{h}'(\Sigma) \oplus \bigoplus_{\alpha \in \mathrm{span}(\Pi - \Sigma) \cap \Delta} \mathfrak{g}_\alpha$$

where $\mathfrak{h}'(\Sigma) = \mathrm{span}\{H_\alpha : \alpha \in \Pi - \Sigma\} \subset \mathfrak{h}$ is a Cartan subalgebra for $\mathfrak{s}(\Sigma)$ and $\mathfrak{h} = \mathfrak{z}(\Sigma) \oplus \mathfrak{h}'(\Sigma)$. We denote by $B, P(\Sigma), L(\Sigma), U(\Sigma), Z(\Sigma), S(\Sigma)$ the corresponding closed subgroups of G. Note that $S(\Sigma)$ is simply connected. One can describe $\mathfrak{p}(\Sigma), P(\Sigma)$ and the homogeneous space $G/P(\Sigma)$ by the Dynkin diagram of G with the nodes corresponding to roots in Σ crossed.

A weight $\lambda = \sum_i m_i \varpi_i \in \Lambda$ is *dominant for* G or simply *dominant* if $m_i \geqslant 0$ for any i. It is said to be *dominant with respect to* $\mathfrak{p}(\Sigma)$ if $m_i \geqslant 0$ for any index i such that $\alpha_i \notin \Sigma$. By highest weight theory, the irreducible representations of G are parametrised by dominant weights, while irreducible representations of a parabolic subgroup $P(\Sigma)$ are parametrised by weights that are dominant with respect to $\mathfrak{p}(\Sigma)$. If λ is dominant we let W_λ denote the irreducible representation of G with highest weight λ. If λ is dominant for $\mathfrak{p}(\Sigma)$ we let V_λ denote the irreducible representation of $P(\Sigma)$ with highest weight λ. We let moreover E_λ denote the homogeneous vector bundle on $X = G/P(\Sigma)$ defined by the representation V_λ, that is $E_\lambda = G \times_{P(\Sigma)} V_\lambda$.

THEOREM 3.3 (Bott-Borel-Weil). — *If $\lambda \in \Lambda$ is dominant for G, then*

$$H^0(X, E_\lambda) = W_\lambda.$$

Otherwise $H^0(X, E_\lambda) = \{0\}$.

Bott's version of the theorem is much more general, but this partial statement is enough for what follows. We also remark that if one chooses $\mathfrak{b}$ to be the Borel subalgebra with *positive* instead of negative roots, which is customary for example in the usual picture of $\mathbb{P}^n$ as the set of lines in $\mathbb{C}^{n+1}$, then one has to consider *lowest weights* instead of highest ones. This amounts to dualize both representations. With this choice the statement of the theorem becomes $H^0(X, E^*_\lambda) = \big(W_\lambda\big)^*$. (The book [3] follows this convention).

Recall that the set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ is a basis of $\Lambda \otimes \mathbb{Q}$. For a weight $\lambda \in \Lambda$, let $\lambda = \sum_i \xi_i(\lambda)\alpha_i$ be its expression in this basis. We say that the (rational) number $\xi_i(\lambda)$ is the *coefficient of α_i in λ*. We denote by λ_{ad} the highest weight of the adjoint representation of G (that is the largest root).

LEMMA 3.4. — *Let $X = G/P(\alpha_k)$. The bundle E_{ϖ_k} associated to the fundamental weight ϖ_k is a very ample line bundle over X. Moreover*

$\mathrm{Pic}(X) \cong H^2(X,\mathbb{Z}) = \mathbb{Z}\,\mathrm{c}_1(E_{\varpi_k})$. *For any weight* $\lambda \in \Lambda$ *that is dominant for* $P(\alpha_k)$

$$\mathrm{c}_1(E_\lambda) = \dim V_\lambda \, \frac{\xi_k(\lambda)}{\xi_k(\varpi_k)} \, \mathrm{c}_1(E_{\varpi_k}). \tag{3.6}$$

(For the proof see *e.g.* [23, §5.2], [22, p. 56]).

In the following statement we summarise what we need of the structure theory of Hermitian symmetric spaces.

Theorem 3.5. — *An irreducible Hermitian (globally) symmetric space of the compact type is a rational homogeneous space. Moreover a rational homogeneous space* $X = G/P$ *is symmetric if and only if the representation of* $\mathfrak{p}$ *on* $\mathfrak{g}/\mathfrak{p}$ *induced from the adjoint representation of* $\mathfrak{g}$ *is irreducible. The actual possibilities are explicitly listed in Table 1.*

The characterisation in terms of irreducibility of $\mathfrak{g}/\mathfrak{p}$ is due to Kobayashi and Nagano [17, Thm. A] (see also [3, p. 26]).

	Klein form		Type
Grassmannian	$G_{k,n} = \mathrm{SL}(n)/P(\alpha_k)$	$n \geqslant 2$	AIII
Odd quadrics	$Q_{2n-1} = \mathrm{Spin}(2n+1)/P(\alpha_1)$	$n \geqslant 2$	BI
Even quadrics	$Q_{2n-2} = \mathrm{Spin}(2n)/P(\alpha_1)$	$n \geqslant 3$	DI
Spinor variety	$X = \mathrm{Spin}(2n)/P(\alpha_n)$	$n \geqslant 4$	DIII
Lagrangian Grassmannian	$X = \mathrm{Sp}(n,\mathbb{C})/P(\alpha_n)$	$n \geqslant 2$	CI
	$X = E_6/P(\alpha_1)$		EIII
	$X = E_7/P(\alpha_7)$		EVII

Table 3.1. Irreducible Hermitian symmetric spaces of the compact type.

Proposition 3.6. — *Let* $X = G/P(\alpha_k)$ *be a compact irreducible Hermitian symmetric space and let* $\lambda \in \Lambda$ *be a nontrivial dominant weight. Then*

$$J(E_\lambda, -K_X) = \frac{2\dim W_\lambda}{\dim W_\lambda - \dim V_\lambda} \cdot \frac{\xi_k(\lambda)}{\xi_k(\lambda_{\mathrm{ad}})}. \tag{3.7}$$

Proof. — The tangent bundle to $X = G/P$ is the homogeneous bundle obtained from the representation of P on $\mathfrak{g}/\mathfrak{p}$. For symmetric X this is irreducible by Theorem 3.5, so Bott-Borel-Weil theorem and Lemma 3.4 apply. Since $H^0(X,TX) = \mathfrak{g} = W_{\lambda_{\mathrm{ad}}}$ (see [1, p. 75, p. 131]), $\mathfrak{g}/\mathfrak{p} = V_{\lambda_{\mathrm{ad}}}$

and $TX = E_{\lambda_{\rm ad}}$. Set $m = \dim X = \dim V_{\lambda_{\rm ad}}$. By Lemma 3.4

$$\mathrm{c}_1(-K_X) = \mathrm{c}_1(TX) = m \cdot \frac{\xi_k(\lambda_{\rm ad})}{\xi_k(\varpi_k)}\, \mathrm{c}_1(E_{\varpi_k}),$$

$$\mathrm{c}_1(E_\lambda) = \dim V_\lambda \cdot \frac{\xi_k(\lambda)}{\xi_k(\varpi_k)}\, \mathrm{c}_1(E_{\varpi_k}),$$

$$\frac{\langle c_1(E_\lambda) \cup c_1(-K_X)^{m-1}, [X]\rangle}{\langle c_1(-K_X)^m, [X]\rangle} = \frac{\dim V_\lambda}{m} \cdot \frac{\xi_k(\lambda)}{\xi_k(\lambda_{\rm ad})}.$$

The rank of E_λ is $\dim V_\lambda$, while $h^0(X, E_\lambda) = \dim W_\lambda$ by Bott-Borel-Weil theorem. Therefore

$$\begin{aligned} J(E_\lambda, -K_X) &= \frac{2\, m\, h^0(E_\lambda)}{r\,(h^0(E_\lambda) - r)} \cdot \frac{\langle c_1(E_\lambda) \cup c_1(-K_X)^{m-1}, [X]\rangle}{\langle c_1(-K_X)^m, [X]\rangle} \\ &= \frac{2\, m\, \dim W_\lambda}{\dim V_\lambda(\dim W_\lambda - \dim V_\lambda)} \cdot \frac{\dim V_\lambda}{m} \cdot \frac{\xi_k(\lambda)}{\xi_k(\lambda_{\rm ad})} \\ &= \frac{2\, \dim W_\lambda}{\dim W_\lambda - \dim V_\lambda} \cdot \frac{\xi_k(\lambda)}{\xi_k(\lambda_{\rm ad})}. \end{aligned}$$

□

We are now ready for the proof of theorems 1.2 and 1.3.

Proof of Theorem 1.2. — Let X be a compact irreducible Hermitian symmetric space. Denote by g_{KE} the symmetric (Kähler-Einstein) metric with Kähler form in $2\pi c_1(X)$. We need to show that

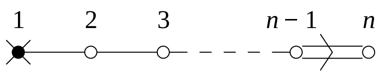
$$I(-K_X) = 2 = \lambda_1(g_{KE}). \tag{3.8}$$

The second equality follows from Theorem 3.1. So $I(-K_X) \geqslant 2$ by definition (3.1). It is enough to prove that $I(-K_X) \leqslant 2$. For each space in the first five families in Table 3 we find a homogeneous bundle $E_\lambda \to X$ such that $J(E_\lambda, -K_X) = 2$. The result then follows applying Theorem 3.2. The relevant information regarding weights and degrees can be found for example in [14, p. 66, p. 69].

1. The case of the Grassmannians (type $AIII$) is settled by hand in [2, Thm. 1.3]. The vector bundle E is the dual of the universal subbundle. If we choose the Borel group as in (3.4) then $E = E_{\varpi_1}$.

2. For odd quadrics the Dynkin diagram is:

$Q_{2n-1} = \mathrm{Spin}(2n+1)/P(\alpha_1)$
Type BI

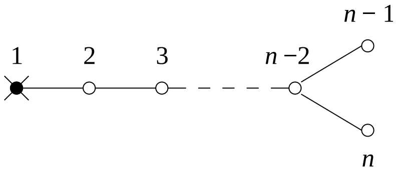


The largest root is $\lambda_{\rm ad} = \varpi_2$. Put $\lambda = \varpi_n$. Then W_λ is the spin representation, while V_λ corresponds to the spin representation of the semisimple part $S(\alpha_1) \cong \mathrm{Spin}(2n-1)$ of $P(\alpha_1)$. The bundle E_λ is the *spinor bundle* studied *e.g.* by Ottaviani [21]. Of course $\dim W_\lambda = 2^n$, $\dim V_\lambda = 2^{n-1}$. Finally $\xi_1(\lambda) = \xi_1(\varpi_n) = 1/2$, $\xi_1(\lambda_{\mathrm{ad}}) = \xi_1(\varpi_2) = 1$, so

$$J(E_\lambda, -K_X) = \frac{2 \dim W_\lambda}{\dim W_\lambda - \dim V_\lambda} \cdot \frac{\xi_1(\lambda)}{\xi_1(\lambda_{\mathrm{ad}})} = 2 \cdot \frac{2^n}{2^n - 2^{n-1}} \cdot \frac{1/2}{1} = 2.$$

3. The situation is very similar for even quadrics. The Dynkin diagram is:

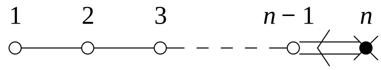
$Q_{2n-2} = \mathrm{Spin}(2n)/P(\alpha_1)$
Type DI



The largest root is again $\lambda_{\mathrm{ad}} = \varpi_2$. We take W_λ to be either one of the half-spin representation. (E_λ is one of the two spinor bundles on Q_{2n-2}, [21]). Say $W_\lambda = \mathcal{S}_+$. Then $\lambda = \varpi_n$ and V_λ is the half-spin representation $\mathcal{S}_+$ of $S(\alpha_1) \cong \mathrm{Spin}(2n-2)$. Now $\dim W_\lambda = 2^{n-1}$, $V_\lambda = 2^{n-2}$, $\xi_1(\varpi_n) = 1/2$, $\xi_1(\varpi_2) = 1$, so again $J(E_\lambda, -K_X) = 2$.

4. For the Lagrangian Grassmannian the Dynkin diagram is:

$X = \mathrm{Sp}(n, \mathbb{C})/P(\alpha_n)$
Type CI

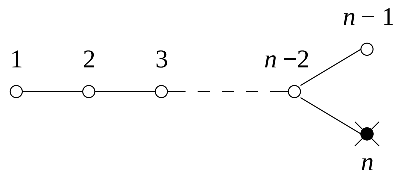


The highest weight of the adjoint representation is $\lambda_{\mathrm{ad}} = 2\varpi_1$. W_{ϖ_1} is the standard representation of $\mathrm{Sp}(n, \mathbb{C})$ on $\mathbb{C}^{2n}$. The semisimple part of $P(\alpha_n)$ is $S(\alpha_n) = \mathrm{SL}(n)$, so V_{ϖ_1} is the standard representation of $\mathrm{SL}(n)$ on $\mathbb{C}^n$. So choosing $E = E_{\varpi_1}$ we get

$$J(E, -K_X) = 2 \cdot \frac{2n}{2n - n} \cdot \frac{\xi_n(\varpi_1)}{\xi_n(2\varpi_1)} = 2.$$

5. For the Spinor varieties the Dynkin diagram is:

$\mathrm{Spin}(2n)/P(\alpha_n)$
Type DIII



Take $E = E_{\varpi_1}$. W_{ϖ_1} is the standard representation of $\mathrm{Spin}(2n)$. The semisimple part of $P(\alpha_n)$ is $S(\alpha_n) = \mathrm{SL}(n)$, so V_{ϖ_1} is the standard representation of $\mathrm{SL}(n)$ on $\mathbb{C}^n$. The largest root is $\lambda_{\mathrm{ad}} = \varpi_2$, $\xi_n(\varpi_1) = 1/2$,

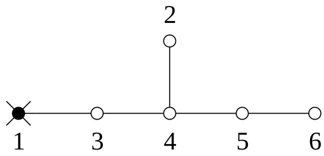
$\xi_n(\varpi_2) = 1$. So

$$J(E, -K_X) = 2 \cdot \frac{2n}{2n-n} \cdot \frac{1/2}{1} = 2.$$

□

Proof of Theorem 1.3. — 1. For $X = E_6/P(\alpha_1)$ the Dynkin diagram (with Bourbaki numbering) is:

$X = E_6/P(\alpha_1)$
Type EIII



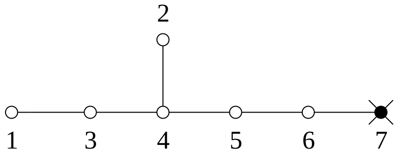
The largest root is $\lambda_{\mathrm{ad}} = \varpi_2$. An easy computation gives $J(E_{\varpi_6}, -K_X) = 36/17$ and $J(E_{\varpi_2}, -K_X) = 78/31$. If $\lambda = \sum_i a_i \varpi_i$, then

$$J(E_\lambda, -K_X) \geqslant 2\frac{\xi_1(\lambda)}{\xi_1(\varpi_2)} = \frac{8}{3}a_1 + 2a_2 + \frac{10}{3}a_3 + 4a_4 + \frac{8}{3}a_5 + \frac{4}{3}a_6.$$

The right hand side is $< 36/17$ if and only if $\lambda = \varpi_2$ or $\lambda = \varpi_6$. Therefore the best estimate is gotten with $\lambda = \varpi_6$.

2. For $X = E_7/P(\alpha_7)$ the Dynkin diagram (with Bourbaki numbering) is:

$X = E_7/P(\alpha_7)$
Type EVII



The largest root is $\lambda_{\mathrm{ad}} = \varpi_1$. We have $J(E_{\varpi_1}, -K_X) = 133/53$. If $\lambda = \sum_i a_i \varpi_i$, then

$$J(E_\lambda, -K_X) \geqslant 2\frac{\xi_7(\lambda)}{\xi_7(\varpi_1)} = 2\xi_7(\lambda) = 2a_1 + 3a_2 + 4a_3 + 6a_4 + 5a_5 + 4a_6 + 3a_7.$$

The right hand side is $< 133/53$ if and only if $\lambda = \varpi_1$. Therefore the best estimate is gotten with $\lambda = \varpi_1$. □

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