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BANACH ALGEBRAS OF PSEUDODIFFERENTIAL OPERATORS AND THEIR ALMOST DIAGONALIZATION

by Karlheinz GRÖCHENIG & Ziemowit RZESZOTNIK (*)

Abstract. — We define new symbol classes for pseudodifferential operators and investigate their pseudodifferential calculus. The symbol classes are parametrized by commutative convolution algebras. To every solid convolution algebra $\mathcal{A}$ over a lattice Λ we associate a symbol class $M^{\infty,\mathcal{A}}$. Then every operator with a symbol in $M^{\infty,\mathcal{A}}$ is almost diagonal with respect to special wave packets (coherent states or Gabor frames), and the rate of almost diagonalization is described precisely by the underlying convolution algebra $\mathcal{A}$. Furthermore, the corresponding class of pseudodifferential operators is a Banach algebra of bounded operators on $L^2(\mathbb{R}^d)$. If a version of Wiener's lemma holds for $\mathcal{A}$, then the algebra of pseudodifferential operators is closed under inversion. The theory contains as a special case the fundamental results about Sjöstrand's class and yields a new proof of a theorem of Beals about the Hörmander class $S^0_{0,0}$.

Résumé. — Nous étudions une nouvelle classe de symboles pour les opérateurs pseudo-différentiels et leurs calculs symboliques. À chaque algèbre $\mathcal{A}$ commutative par rapport aux convolutions sur un réseau Λ correspond une classe de symboles $M^{\infty,\mathcal{A}}$. Chaque opérateur pseudo-différentiel dans $M^{\infty,\mathcal{A}}$ est presque diagonale par rapport aux états cohérents, et sa décroissance hors de la diagonale est décrite par l'algèbre $\mathcal{A}$. Les opérateurs pseudo-différentiels avec des symboles dans $M^{\infty,\mathcal{A}}$ sont bornés sur $L^2(\mathbb{R}^d)$ et constituent une algèbre de Banach. Si une version du lemme de Wiener s'applique à $\mathcal{A}$, alors l'algèbre d'opérateurs pseudo-différentiels est fermée par rapport à l'inversion des opérateurs. La théorie contient comme un cas spécial la théorie de J. Sjöstrand et fournit une nouvelle démonstration d'un théorème de Beals sur les symboles de Hörmander dans $S^0_{0,0}$.

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1. Introduction

We study pseudodifferential operators with symbols that are defined by their time-frequency distribution (phase-space distribution). The first symbol class of this type was introduced by Sjöstrand [39, 40] whose work has inspired an important line of research by Boulkhemair, Toft, and others [9, 10, 44, 45, 46, 35]. Independently, an alternative approach with time-frequency methods was developed in [28, 29, 24, 26, 27]. The starting point of the time-frequency approach is the observation that the Sjöstrand class coincides with one of the so-called modulation spaces that were introduced by Feichtinger already in 1983 [17]. The time-frequency approach added several new insights and generalizations to pseudodifferential operators with non-smooth symbols. Recently Sjöstrand has again taken up the study of pseudodifferential operators with non-smooth symbols and substantially generalized the original definition [41]. His motivation is to study weighted symbol spaces, the boundedness and algebra properties of the corresponding pseudodifferential operators.

In this paper we also study extensions of the original Sjöstrand class. Our goal is to understand better the following fundamental questions:

- Which properties of the generalized Sjöstrand class are responsible for the boundedness of the corresponding pseudodifferential operators on $L^2(\mathbb{R}^d)$ and on other function spaces?
- Which properties of the symbol class imply the algebra property of the operators?
- Which properties of the symbol class yield the spectral invariance property and thus a strong form of the functional calculus?

We introduce a family of symbol classes which in general may contain non-smooth symbols. This family is parametrized by Banach algebras with respect to convolution on a lattice $\Lambda \subseteq \mathbb{R}^{2d}$. To each such Banach algebra $\mathcal{A}$ we associate a symbol class $\widetilde{M}^{\infty,\mathcal{A}}$ (see Definition 4.1), and we analyze the properties of the corresponding symbol class. Our main theme is how properties of the Banach algebra are inherited by the corresponding operators, and our results answer the above questions in the context of the symbol classes $\widetilde{M}^{\infty,\mathcal{A}}$. Roughly, the result may be summarized as follows:

(a) The algebra property of $\mathcal{A}$ implies that the corresponding class of operators is closed under composition. Thus we obtain new Banach algebras of pseudodifferential operators.

(b) If $\mathcal{A}$ acts boundedly on a solid sequence space $\mathcal{Y}$, then the corresponding pseudodifferential operators are bounded on a natural function space associated to $\mathcal{Y}$, a so-called modulation space. In particular, we obtain the L^2-boundedness of pseudodifferential operators in this class.

(c) If $\mathcal{A}$ is closed under inversion, then the corresponding class of operators is also closed under inversion. The inverse of a pseudodifferential operator in this class is again a pseudodifferential operator in this class. This type of result goes back to Beals [4] and represents a strong form of functional calculus.

To be specific, we formulate a special case of our main results explicitly for the algebras $\mathcal{A} = \ell^\infty_s$, which are defined by the norm $\|\mathbf{a}\|_{\ell^\infty_s} = \sup_{k\in\mathbb{Z}^{2d}} |\mathbf{a}(k)|(1+|k|)^s$ (to make $\mathcal{A}$ into a Banach algebra, we need $s > 2d$). In this case, the corresponding symbol class $\widetilde{M}^{\infty,\mathcal{A}}$ is defined by the norm

$$\|\sigma\|_{\widetilde{M}^{\infty,\mathcal{A}}} := \sup_{z,\zeta\in\mathbb{R}^{2d}} |(\sigma\cdot\Phi(\cdot - z))\widehat{\ }(\zeta)|(1+|\zeta|)^s\,, \tag{1.1}$$

(where Φ is the Gaussian) and coincides with the standard modulation space $M^\infty_{1\otimes v_s}$.

The fundamental result concerns the almost diagonalization of operators with symbols in $\widetilde{M}^{\infty,\mathcal{A}}$ with respect to time-frequency shifts (phase space shifts). For $z = (x,\xi) \in \mathbb{R}^{2d}$ let $\pi(z)f(t) = e^{2\pi i\xi\cdot t}f(t-x)$ be the corresponding time-frequency shift.

Theorem A. — *(Almost diagonalization) Let g be a nonzero Schwartz function. A symbol belongs to the class $\widetilde{M}^{\infty,\mathcal{A}}$ with $\mathcal{A} = \ell^\infty_s$, if and only if*

$$|\langle\sigma^w\pi(z)g, \pi(w)g\rangle| \leqslant C(1+|w-z|)^{-s} \qquad \forall w,z\in\mathbb{R}^{2d}\,.$$

Remarkably, the almost diagonalization of the corresponding pseudodifferential operators characterizes the symbol class completely.

Theorem B. — *(Boundedness) If $\sigma\in\widetilde{M}^{\infty,\mathcal{A}}$, then σ^w is bounded on a whole class of distribution spaces, the so-called modulation spaces. In particular, σ^w is bounded on $L^2(\mathbb{R}^d)$.*

Theorem C. — *(Algebra Property) If $\sigma_1,\sigma_2\in\widetilde{M}^{\infty,\mathcal{A}}$, then $\sigma_1^w\sigma_2^w = \tau^w$ for some $\tau\in\widetilde{M}^{\infty,\mathcal{A}}$. Thus the operators with symbols in $\widetilde{M}^{\infty,\mathcal{A}}$ form a Banach algebra with respect to composition.*

Theorem D. — *(Inverse-Closedness) If $\sigma\in\widetilde{M}^{\infty,\mathcal{A}}$ and σ^w is invertible on $L^2(\mathbb{R}^d)$, then the inverse operator $(\sigma^w)^{-1} = \tau^w$ possesses again a symbol in $\widetilde{M}^{\infty,\mathcal{A}}$.*

If the algebra $\mathcal{A}$ is the convolution algebra $\ell^1(\mathbb{Z}^{2d})$, then the corresponding symbol class $\widetilde{M}^{\infty,\mathcal{A}}$ (defined in 4.1) coincides with the Sjöstrand class

$M^{\infty,1}$, and we recover the main results of [39, 40]. The context of the symbol classes $\widetilde{M}^{\infty,\mathcal{A}}$ reveals the deeper reasons for why Sjöstrand's fundamental results hold: for instance, the L^2-boundedness of pseudodifferential operators with a symbol in $M^{\infty,1}$ can be traced back to the convolution relation $\ell^1 * \ell^2 \subseteq \ell^2$. Finally, Wiener's Lemma for absolutely convergent Fourier series is at the heart of the "Wiener algebra property" of the Sjöstrand class.

The L^2-boundedness and the algebra property (Theorems B and C) can also be derived with Sjöstrand's methods. In addition, the time-frequency approach yields the boundedness on a much larger class of function and distribution spaces, the modulation spaces. These function spaces play the role of smoothness spaces in time-frequency analysis, they may be seen as the analogues of the Sobolov spaces in the classical theory. Theorem D lies deeper and requires extended use of Banach algebra concepts.

As a consequence of these theorems we will give a new treatment of the Hörmander class $S^0_{0,0}$. On the one hand, we will characterize $S^0_{0,0}$ by the almost diagonalization properties with respect to time-frequency shifts, and on the other hand, we will provide a completely new proof of Beals' theorem on the inverse-closedness of $S^0_{0,0}$ [4].

As another application of our main results, we will investigate the action of pseudodifferential operators on time-frequency molecules and the almost diagonalization of pseudodifferential operators with respect to time-frequency molecules. This topic is hardly explored yet, and ours seems to be the first results in pseudodifferential operator theory.

Throughout the paper we use methods from time-frequency analysis (phase-space analysis) and Banach algebra methods. We draw on properties of the short-time Fourier transform and the Wigner distribution and the theory of the associated function spaces, the modulation spaces. For the investigation of the almost diagonalization of pseudodifferential operators we will use the well-developed theory of Gabor frames, which provide a kind of non-orthogonal phase-space expansions of distributions.The main sources are the books [21, 24]. Another set of tools comes from the theory of Banach algebras. The inverse-closedness as expressed in Theorem D lies quite deep and requires some new results on Banach algebras with respect to convolution. Although the proof techniques are classical, these arguments seem unusual in the theory of pseudodifferential operators; we therefore give the proofs in the Appendix.

The paper is organized as follows: In Section 2 we collect the preliminary definitions of time-frequency analysis. Section 3 presents the new Banach algebra results that are crucial for the main theorems. In Section 4 we introduce the generalized Sjöstrand classes and show that the corresponding pseudodifferential operators are almost diagonalized with respect to Gabor frames. In Section 5 we study the boundedness properties of these operators on modulation spaces, the algebra property, and the functional calculus. Section 6 gives an application to the Hörmander class $S^0_{0,0}$ and a new proof of Beals' result in [4]. The final Section 7 is devoted to time-frequency molecules and the almost diagonalization of operators in the Sjöstrand class with respect to molecules. In the Appendix we give the proofs of the Banach algebra results of Section 3.

2. Preliminaries

In this section we summarize the main definitions and results from time-frequency analysis needed here.

Time-Frequency Shifts. We combine time $x \in \mathbb{R}^d$ and frequency $\xi \in \mathbb{R}^d$ into a single point $z = (x, \xi)$ in the "time-frequency" plane $\mathbb{R}^{2d}$. Likewise we combine the operators of translation and modulation to a *time-frequency shift* and write

$$\pi(z)f(t) = M_\xi T_x f(t) = e^{2\pi i \xi \cdot t} f(t-x).$$

The *short-time Fourier transform (STFT)* of a function/distribution f on $\mathbb{R}^d$ with respect to a window g is defined by

$$\begin{aligned} V_g f(x,\xi) &= \int_{\mathbb{R}^d} f(t)\overline{g(t-x)} e^{-2\pi i t\cdot\xi}\, dt \\ &= \langle f, M_\xi T_x g\rangle = \langle f, \pi(z) g\rangle\,. \end{aligned}$$

The short-time Fourier transform of a symbol $\sigma(x,\xi)$, for $x,\xi \in \mathbb{R}^d$, is a function on $\mathbb{R}^{4d}$ and will be denoted by $\mathcal{V}_\Phi \sigma(z,\zeta)$ for $z,\zeta \in \mathbb{R}^{2d}$ in order to distinguish it from the STFT of a function on $\mathbb{R}^d$.

To compare STFTs with respect to different windows, we will make use of the pointwise estimate

$$|V_h f(z)| \leqslant |\langle k, g\rangle|^{-1} \left(|V_g f| * |V_h k|\right)(z)\,. \tag{2.1}$$

which holds under various assumptions on f, g, h, k, see e.g. [24, Lemma 11.3.3].

Modulation Spaces. Let $\varphi(t) = e^{-\pi t\cdot t}$ be the Gaussian. Then the modulation space $M^{p,q}_m(\mathbb{R}^d)$, $1 \leqslant p, q \leqslant \infty$ is defined by measuring the norm of the STFT in the weighted space $L^{p,q}_m(\mathbb{R}^{2d})$, that is

$$\|f\|_{M^{p,q}_m(\mathbb{R}^d)} = \|V_\varphi f\|_{L^{p,q}_m(\mathbb{R}^{2d})} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_\varphi f(x,\xi)|^p m(x,\xi)^p\,dx\right)^{q/p} d\xi\right)^{1/q}.$$

One of the basic results about modulation spaces is the independence of this definition from the particular test function chosen in the short-time Fourier transform. Precisely, if the weight satisfies the condition $m(z_1 + z_2) \leqslant Cv(z_1)m(z_2)$ for $z_1, z_2 \in \mathbb{R}^{2d}$ and some submultiplicative function v, and if $g \in M^1_v(\mathbb{R}^d), g \neq 0$, i.e., $\int_{\mathbb{R}^{2d}} |V_\varphi g(z)|\, v(z)\, dz < \infty$, then $\|V_g f\|_{L^{p,q}_m}$ is an equivalent norm on $M^{p,q}_m$ [24, Thm. 11.3.7].

We shall be mainly concerned with the space $M^{\infty,q}_{1\otimes v}(\mathbb{R}^{2d})$. For weights v of polynomial growth ($v(\zeta) = \mathcal{O}(|\zeta|^N)$ for some $N > 0$) the space $M^{\infty,q}_{1\otimes v}(\mathbb{R}^{2d})$ consists of all $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ such that the norm

$$\|\sigma\|_{M^{\infty,q}_{1\otimes v}(\mathbb{R}^{2d})} = \left(\int_{\mathbb{R}^{2d}}\left(\operatorname*{ess\,sup}_{z\in\mathbb{R}^{2d}} |\mathcal{V}_\Phi \sigma(z,\zeta)|\, v(\zeta)\right)^q d\zeta\right)^{\frac{1}{q}} \tag{2.2}$$

is finite. For $q = \infty$, the norm is given by

$$\|\sigma\|_{M^{\infty,\infty}_{1\otimes v}(\mathbb{R}^{2d})} = \operatorname*{ess\,sup}_{\zeta\in\mathbb{R}^{2d}} \operatorname*{ess\,sup}_{z\in\mathbb{R}^{2d}} |\mathcal{V}_\Phi \sigma(z,\zeta)|\, v(\zeta).$$

For the specific weight $v_s(\zeta) = \langle\zeta\rangle^s = (1+|\zeta|^2)^{\frac{s}{2}}$, $s \geqslant 0$, the space $M^{\infty,q}_{1\otimes v_s}(\mathbb{R}^{2d})$ shall be denoted by $M^{\infty,q}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d})$. Also, the space $M^{p,p}_m$ shall be denoted by M^p_m.

Weyl Calculus. The *Wigner distribution* of $f, g \in L^2(\mathbb{R}^d)$ is defined as

$$W(f,g)(x,\xi) = \int_{\mathbb{R}^d} f\Big(x+\frac{t}{2}\Big)\overline{g\Big(x-\frac{t}{2}\Big)}e^{-2\pi i t\cdot\xi}\,dt.$$

The Weyl transform σ^w of a symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ is defined by the sesquilinear form

$$\langle \sigma^w f, g\rangle = \langle \sigma, W(g,f)\rangle \qquad f, g \in \mathcal{S}(\mathbb{R}^d)\,. \tag{2.3}$$

Usually, the Weyl transform is defined by the integral operator

$$\sigma^w f(x) = \int_{\mathbb{R}^d} \sigma\Big(\frac{x+y}{2}, \xi\Big) e^{2\pi i(x-y)\cdot\xi} f(y)\,dyd\xi\,, \tag{2.4}$$

but this definition is somewhat restrictive, and we will not need this particular formula.

The composition of two Weyl transforms defines a twisted product between symbols via

$$\sigma^w \tau^w = (\sigma \,\sharp\, \tau)^w\,,$$

Usually the analysis of the twisted product is based on the formula [21, 33]

$$(\sigma \sharp \tau)(x,\xi) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma(u,\zeta)\tau(v,\eta)\, e^{4\pi i[(x-u)\cdot(\xi-\eta)-(x-v)\cdot(\xi-\zeta)]}\, dudvd\eta d\zeta\,,$$

but it is a distinctive feature of our approach that we will not need the explicit formula.

3. Discrete Banach Algebras

In this section we present the necessary Banach algebra methods. We have not found them in the literature, and will give the proofs in the appendix.

Throughout this section $\mathcal{A}$ denotes a solid involutive Banach algebra with respect to convolution and indexed by a discrete subgroup Λ of $\mathbb{R}^d$ (in the remaining parts of the paper $\Lambda \subset \mathbb{R}^{2d}$). The elements of $\mathcal{A}$ are sequences $\mathbf{a}(\lambda), \lambda \in \Lambda$, where $\Lambda = A\mathbb{Z}^d$ is a discrete subgroup of full rank (a lattice) in $\mathbb{R}^d$ (thus $\det A \neq 0$). The involution is defined as $\mathbf{a}^*(\lambda) = \overline{\mathbf{a}(-\lambda)}$. The norm of $\mathcal{A}$ satisfies the usual inequalities $\|\mathbf{a}^*\|_{\mathcal{A}} \leqslant \|\mathbf{a}\|_{\mathcal{A}}$ and

$$\|\mathbf{a} * \mathbf{b}\|_{\mathcal{A}} \leqslant \|\mathbf{a}\|_{\mathcal{A}}\|\mathbf{b}\|_{\mathcal{A}} \qquad \text{for all } \mathbf{a}, \mathbf{b} \in \mathcal{A}\,. \tag{3.1}$$

Furthermore, the solidity of $\mathcal{A}$ says that if $|\mathbf{a}(\lambda)| \leqslant |\mathbf{b}(\lambda)|$ for all $\lambda \in \Lambda$ and $\mathbf{b} \in \mathcal{A}$, then also $\mathbf{a} \in \mathcal{A}$ and $\|\mathbf{a}\|_{\mathcal{A}} \leqslant \|\mathbf{b}\|_{\mathcal{A}}$.

Due to the solidity, we may assume without loss of generality that the standard sequences δ_λ defined by $\delta_\lambda(\mu) = 1$ for $\lambda = \mu$ and $\delta_\lambda(\mu) = 0$ for $\lambda \neq \mu$ belong to $\mathcal{A}$, otherwise we switch to the sublattice $\Lambda' = \{\lambda \in \Lambda : \delta_\lambda \in \mathcal{A}\}$.

The solidity is a strong assumption, as is demonstrated by the following result.

Theorem 3.1. — *Let $\mathcal{A}$ be an involutive Banach algebra with respect to convolution over a lattice $\Lambda \subset \mathbb{R}^d$. If $\mathcal{A}$ is solid, then $\mathcal{A}$ is continuously embedded in $\ell^1(\Lambda)$.*

Thus $\ell^1(\Lambda)$ is the maximal solid convolution algebra on the discrete group Λ.

Example 3.2. — Let v be a non-negative function on Λ and let $\ell^q_v(\Lambda)$ be defined by the norm $\|\mathbf{a}\|_{\ell^q_v} = \|\mathbf{a}v\|_q$. Then $\ell^1_v(\Lambda)$ is a Banach algebra if and only if $v(\lambda+\mu) \leqslant C\, v(\lambda)v(\mu), \lambda, \mu \in \Lambda$ (v is submultiplicative), and $\ell^\infty_v(\Lambda)$ is a Banach algebra, if and only if $v^{-1} * v^{-1} \leqslant Cv^{-1}$ (v is subconvolutive) [14]. The standard family of weights is given as $\langle\lambda\rangle^s$, where $\langle\lambda\rangle = \sqrt{1+|\lambda|^2}$. The corresponding solid Banach algebras are $\ell^1_{\langle\cdot\rangle^s}(\Lambda)$, for $s \geqslant 0$ and $\ell^\infty_{\langle\cdot\rangle^s}(\Lambda)$, for $s > d$.

Another example of a solid Banach algebra is the so-called Krein algebra $\ell^1(\Lambda) \cap \ell^2_{d/2}(\Lambda)$ with the norm $\|\mathbf{a}\|_1 + \left(\sum_{\lambda\in\Lambda} |a_\lambda|^2 |\lambda|^d\right)^{1/2}$, see [11].

For certain weight sequences v the weighted ℓ^q-space $\ell^q_v(\Lambda)$ is also a convolution algebra, see [14].

To every solid Banach algebra under convolution we can attach a Banach algebra of matrices. Indeed, using an idea of Baskakov [3], we define an algebra of matrices that are "dominated by convolution operators in $\mathcal{A}$".

Definition 3.3. — *Let A be a matrix on Λ with entries $a_{\lambda\mu}$, for $\lambda, \mu \in \Lambda$, and let $\mathbf{d}_A$ be the sequence with entries $\mathbf{d}_A(\mu)$ defined by*

$$\mathbf{d}_A(\mu) = \sup_{\lambda\in\Lambda} |a_{\lambda,\lambda-\mu}| \,. \tag{3.2}$$

We say that the matrix A belongs to $\mathcal{C}_\mathcal{A}$, if $\mathbf{d}_A$ belongs to $\mathcal{A}$. The norm in $\mathcal{C}_\mathcal{A}$ is given by

$$\|A\|_{\mathcal{C}_\mathcal{A}} = \|\mathbf{d}_A\|_\mathcal{A} \,. \tag{3.3}$$

Note that $\mathbf{d}_A(\mu)$ is the supremum of the entries in the μ-th diagonal of A, thus the $\mathcal{C}_\mathcal{A}$-norm describes a form of the off-diagonal decay of a matrix.

We first list some elementary properties of $\mathcal{C}_\mathcal{A}$.

Lemma 3.4. — *Assume that $\mathcal{A}$ is a solid Banach algebra under convolution. Then:*

(i) *$\mathcal{C}_\mathcal{A}$ is a Banach algebra under matrix multiplication (or equivalently, the composition of the associated operators).*

(ii) *Let $\mathcal{Y}$ be a solid Banach space of sequences on Λ. If $\mathcal{A}$ acts boundedly on a solid space $\mathcal{Y}$ by convolution ($\mathcal{A} * \mathcal{Y} \subseteq \mathcal{Y}$), then $\mathcal{C}_\mathcal{A}$ acts boundedly on $\mathcal{Y}$, i.e.,*

$$\|A\mathbf{c}\|_\mathcal{Y} \leqslant \|A\|_{\mathcal{C}_\mathcal{A}} \|\mathbf{c}\|_\mathcal{Y} \qquad \text{for all } A \in \mathcal{C}_\mathcal{A}, \mathbf{c} \in \mathcal{Y} \,. \tag{3.4}$$

(iii) *In particular, since $\mathcal{A} \subseteq \ell^1(\Lambda)$, we may identify $\mathcal{C}_\mathcal{A}$ as a (Banach) subalgebra of $\mathcal{B}(\ell^2(\Lambda))$.*

Proof. — (i) Let $A = (a_{\lambda\mu}), B = (b_{\lambda\mu}) \in \mathcal{C}_\mathcal{A}$, then by definition $|a_{\lambda\mu}| = |a_{\lambda,\lambda-(\lambda-\mu)}| \leqslant d_A(\lambda - \mu)$. Consequently matrix multiplication is dominated by convolution in the sense that

$$\begin{aligned} |(AB)_{\lambda,\lambda-\mu}| &\leqslant \sum_{\nu\in\Lambda} |a_{\lambda\nu}|\,|b_{\nu,\lambda-\mu}| \\ &\leqslant \sum_{\nu\in\Lambda} \mathbf{d}_A(\lambda-\nu)\,\mathbf{d}_B(\nu-(\lambda-\mu)) \\ &= (\mathbf{d}_A * \mathbf{d}_B)(\mu)\,, \end{aligned}$$

and so $\mathbf{d}_{AB}(\mu) = \sup_{\lambda\in\Lambda} |(AB)_{\lambda,\lambda-\mu}| \leqslant (\mathbf{d}_A * \mathbf{d}_B)(\mu)$. Since $\mathbf{d}_A, \mathbf{d}_B \in \mathcal{A}$ and $\mathcal{A}$ is a Banach algebra under convolution, we find that

$$\|AB\|_{\mathcal{C}_\mathcal{A}} = \|\mathbf{d}_{AB}\|_\mathcal{A} \leqslant \|\mathbf{d}_A * \mathbf{d}_B\|_\mathcal{A} \leqslant \|\mathbf{d}_A\|_\mathcal{A}\,\|\mathbf{d}_B\|_\mathcal{A} = \|A\|_{\mathcal{C}_\mathcal{A}}\,\|B\|_{\mathcal{C}_\mathcal{A}}\,.$$

(ii) Since $|a_{\lambda\mu}| \leqslant \mathbf{d}_A(\lambda-\mu)$, we obtain the pointwise inequality

$$|A\mathbf{c}(\lambda)| = \Big|\sum_{\mu\in\Lambda} a_{\lambda\mu}\mathbf{c}(\mu)\Big| \leqslant \sum_{\mu\in\Lambda} \mathbf{d}_A(\lambda-\mu)|\mathbf{c}(\mu)| = \big(\mathbf{d}_A * |\mathbf{c}|\big)(\lambda)\,.$$

Using the hypothesis on $\mathcal{Y}$, we conclude that

$$\|A\mathbf{c}\|_\mathcal{Y} \leqslant \|\mathbf{d}_A * |\mathbf{c}|\,\|_\mathcal{Y} \leqslant \|\mathbf{d}_A\|_\mathcal{A}\,\|\,|\mathbf{c}|\,\|_\mathcal{Y} = \|A\|_{\mathcal{C}_\mathcal{A}}\,\|\mathbf{c}\,\|_\mathcal{Y}\,,$$

since $\mathcal{Y}$ is solid and $\mathcal{A}$ acts on $\mathcal{Y}$ by convolution.

(iii) follows by choosing $\mathcal{Y} = \ell^2(\Lambda)$ and by Young's inequality $\ell^1(\Lambda) * \ell^2(\Lambda) \subseteq \ell^2(\Lambda)$. □

Whereas $\mathcal{A}$ is a commutative Banach algebra, $\mathcal{C}_\mathcal{A}$ is highly non-commutative, the transition from $\mathcal{A}$ to $\mathcal{C}_\mathcal{A}$ can be thought of as a non-commutative extension of convolution algebras of sequences on Λ.

One of the main questions about the matrix algebra $\mathcal{C}_\mathcal{A}$ is whether the inverse of a matrix in $\mathcal{C}_\mathcal{A}$ is again in $\mathcal{C}_\mathcal{A}$, or in other words, we ask whether the off-diagonal decay described by $\mathcal{A}$ is preserved under inversion.

We recall the following definition. Let $\mathcal{A} \subseteq \mathcal{B}$ be two Banach algebras with a common unit element. Then $\mathcal{A}$ is *inverse-closed* in $\mathcal{B}$, if $\mathbf{a} \in \mathcal{A}$ and $\mathbf{a}^{-1} \in \mathcal{B}$ implies that $\mathbf{a}^{-1} \in \mathcal{A}$.

In the following, we identify an element $\mathbf{a} \in \mathcal{A} \subseteq \ell^1(\Lambda)$ with the corresponding convolution operator $C_\mathbf{a}\mathbf{b} = \mathbf{a} * \mathbf{b}$. In this way, we may treat $\mathcal{A}$ as a Banach subalgebra of $\mathcal{B}(\ell^2(\Lambda))$, the algebra of bounded operators on $\ell^2(\Lambda)$.

The following theorem gives a complete characterization of when the non-commutative extension $\mathcal{C}_\mathcal{A}$ is inverse-closed in $\mathcal{B}(\ell^2(\Lambda))$. For $\mathcal{A} = \ell^1_v(\mathbb{Z}^d)$ and $\mathcal{A} = \ell^\infty_v(\mathbb{Z}^d)$ this characterization is due to Baskakov [3]. Our formulation is new and reveals more clearly what the main conditions are. Recall that the spectrum $\widehat{\mathcal{A}}$ of a commutative Banach algebra $\mathcal{A}$ consists of all multiplicative linear functionals on $\mathcal{A}$. We denote the standard basis of $\ell^2(\Lambda)$ by $\delta_\lambda, \lambda \in \Lambda$.

The following theorem is crucial for the functional calculus of pseudodifferential operators.

Theorem 3.5. — *Assume that $\mathcal{A}$ is a solid convolution algebra of sequences on a lattice $\Lambda \subset \mathbb{R}^d$. Then the following are equivalent:*

(i) *$\mathcal{A}$ is inverse-closed in $\mathcal{B}(\ell^2)$.*
(ii) *$\mathcal{C}_\mathcal{A}$ is inverse-closed in $\mathcal{B}(\ell^2)$.*
(iii) *The spectrum $\widehat{\mathcal{A}} \simeq \mathbb{T}^d$.*
(iv) *The weight $\omega(\lambda) = \|\delta_\lambda\|_\mathcal{A}$ satisfies the GRS-condition*

$$\lim_{n\to\infty} \omega(n\lambda)^{1/n} = 1$$

for all $\lambda \in \Lambda$.

The main point of condition (iv) is that we have an easy condition to check whether the non-commutative matrix algebra $\mathcal{C}_\mathcal{A}$ is inverse-closed in $\mathcal{B}(\ell^2)$. We defer the proof of this theorem to the appendix, because this paper is on pseudodifferential operators and Banach algebras are only a tool. The proof is an extension and re-interpretation of Baskakov's argument.

Example 3.6. — By a theorem of Gelfand, Raikov, and Shilov [23], the weighted convolution algebra $\ell^1_v(\Lambda)$ is inverse-closed in $\mathcal{B}(\ell^2(\Lambda))$, if and only if the weight v satisfies the (so-called GRS) condition $\lim_{n\to\infty} v(n\lambda)^{1/n} = 1$ for all $\lambda \in \Lambda$. Thus Theorem 3.5 implies that the non-commutative matrix algebra $\mathcal{C}_{\ell^1_v(\Lambda)}$ is inverse-closed in $\mathcal{B}(\ell^2(\Lambda))$, if and only if the weight satisfies the GRS-condition. This is the result of Baskakov [3]. Since the polynomial weight $\langle\lambda\rangle^s$ satisfies the GRS-condition when $s \geqslant 0$, the algebras $\ell^1_{\langle\cdot\rangle^s}(\Lambda)$ and $\mathcal{C}_{\ell^1_{\langle\cdot\rangle^s}(\Lambda)}$ are inverse-closed for $s \geqslant 0$. Similarly, the algebras $\ell^\infty_v(\Lambda)$ and $\mathcal{C}_{\ell^\infty_v(\Lambda)}$ are inverse-closed if v satisfies the GRS-condition and $v^{-1} \in \ell^1(\Lambda)$. Therefore the standard algebras $\ell^\infty_{\langle\cdot\rangle^s}(\Lambda)$ and $\mathcal{C}_{\ell^\infty_{\langle\cdot\rangle^s}(\Lambda)}$ (the Jaffard class) are inverse-closed for $s > d$, see [34, 3, 30].

As a consequence of Theorem 3.5 we draw the following.

Corollary 3.7 (Spectral Invariance). — *Assume that $\widehat{\mathcal{A}} \simeq \mathbb{T}^d$. Then*

$$\mathrm{Sp}_{\mathcal{B}(\ell^2)}(A) = \mathrm{Sp}_{\mathcal{C}_\mathcal{A}}(A) \qquad \text{for all } A \in \mathcal{C}_\mathcal{A}\,. \tag{3.5}$$

Moreover, if $\mathcal{A}$ acts boundedly on a solid sequence space $\mathcal{Y}$, then

$$\mathrm{Sp}_{\mathcal{B}(\mathcal{Y})}(A) \subseteq \mathrm{Sp}_{\mathcal{B}(\ell^2)}(A) = \mathrm{Sp}_{\mathcal{C}_\mathcal{A}}(A) \qquad \text{for all } A \in \mathcal{C}_\mathcal{A}\,. \tag{3.6}$$

Proof. — The spectral identity (3.5) is just a reformulation of the fact that $\mathcal{C}_\mathcal{A}$ is inverse-closed in $\mathcal{B}(\ell^2)$. If $A \in \mathcal{C}_\mathcal{A}$ is invertible on ℓ^2, then also

$A^{-1} \in \mathcal{C}_{\mathcal{A}}$. Thus by Lemma 3.4 A^{-1} is bounded on $\mathcal{Y}$. Consequently, if $\lambda \notin \mathrm{Sp}_{\mathcal{B}(\ell^2)}(A)$, then $\lambda \notin \mathrm{Sp}_{\mathcal{B}(\mathcal{Y})}$, which is the inclusion (3.6). □

4. Almost diagonalization

In this section we introduce the general symbol classes $\widetilde{M}^{\infty,\mathcal{A}}$ that are parametrized by a solid convolution algebra $\mathcal{A}$. The main result explains how pseudodifferential operators with symbols in $\widetilde{M}^{\infty,\mathcal{A}}$ are almost diagonalized.

In order to define the symbol classes we recall the basic information about amalgam spaces and Gabor frames. The definitions are adapted to our needs, for the general theory of amalgam spaces we refer to [22, 16], for Gabor frames to [24, 12].

Amalgam spaces. Fix a Banach algebra $\mathcal{A}$ of sequences on a lattice $\Lambda \subseteq \mathbb{R}^{2d}$ and a relatively compact fundamental domain C containing the origin. We say that a function $F \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{2d})$ belongs to the associated amalgam space $W(\mathcal{A})$, if the sequence $\mathbf{a}$ of local suprema

$$\mathbf{a}(\lambda) = \operatorname*{ess\,sup}_{\zeta \in \lambda + C} F(\zeta)$$

belongs to $\mathcal{A}$. The norm on $W(\mathcal{A})$ is given by $\|F\|_{W(\mathcal{A})} = \|\mathbf{a}\|_{\mathcal{A}}$. For completeness we mention that this definition is independent of the lattice Λ and the fundamental domain C. We will use frequently that $W(\mathcal{A})$ is an involutive Banach algebra with respect to convolution on $\mathbb{R}^{2d}$ [15].

Gabor Frames. Let $\Lambda = A\mathbb{Z}^{2d}$ be a lattice with $|\det A| < 1$. We say that a set $\mathcal{G}(g, \Lambda) = \{\pi(\lambda) g : \lambda \in \Lambda\}$ forms a tight Gabor frame (with constant 1) for $L^2(\mathbb{R}^d)$ if

$$\|f\|_2^2 = \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda) g\rangle|^2 \qquad \text{for all } f \in L^2(\mathbb{R}^d)\,. \tag{4.1}$$

As a consequence every $f \in L^2(\mathbb{R}^d)$ possesses the *tight frame expansion*

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g\rangle \pi(\lambda) g\,,$$

with unconditional convergence in $L^2(\mathbb{R}^d)$.

In our considerations it will be important to use a frame with a window g that satisfies an additional assumption $V_g g \in W(\mathcal{A})$. We will take the existence of a tight frame with $V_g g \in W(\mathcal{A})$ for granted and will not worry about the subtle existence problem. It is known that tight Gabor frames with $g \in L^2(\mathbb{R}^d)$ exist for every lattice $\Lambda = A\mathbb{Z}^{2d}$ with $|\det A| < 1$ [5]. If $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ and $\alpha\beta < 1$ (or the symplectic image of such a lattice), then for every $\delta > 0, 0 < b < 1$ there exist tight Gabor frames with $|V_g g(z)| \leqslant Ce^{-\delta|z|^b}$(e.g., [30]).

Symbol Classes. Now we define a new family of symbol classes. As in [26] the grand symbol $\mathcal{G}(\sigma)$ of a symbol σ is given by

$$\mathcal{G}(\sigma)(\zeta) := \operatorname*{ess\,sup}_{z\in\mathbb{R}^{2d}} |\mathcal{V}_\Phi \sigma(z,\zeta)|\,,$$

where $\Phi(z) = e^{-\pi z\cdot z/2}$ is the Gaussian or some equivalent window (see below). Furthermore, set $j(\zeta) = (\zeta_2, -\zeta_1)$ for $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$.

Definition 4.1. — *A symbol σ belongs to the generalized Sjöstrand class $\widetilde{M}^{\infty,\mathcal{A}}(\mathbb{R}^{2d})$ if $\mathcal{G}(\sigma)\circ j \in W(\mathcal{A})$. The norm on $\widetilde{M}^{\infty,\mathcal{A}}$ is given by*

$$\|\sigma\|_{\widetilde{M}^{\infty,\mathcal{A}}} = \|\mathcal{G}(\sigma)\circ j\|_{W(\mathcal{A})}\,.$$

The tilde over M indicates that we take into account the mapping j. (It could be avoided by using the symplectic Fourier transform in addition to the ordinary Fourier transform.)

Remark 4.2. — If $\mathcal{A} = \ell^q_v(\Lambda)$, where the weight v is continuous and submultiplicative on $\mathbb{R}^{2d}$, then we obtain the standard modulation spaces

$$\widetilde{M}^{\infty,\mathcal{A}} = M^{\infty,q}_{1\otimes v\circ j^{-1}}(\mathbb{R}^{2d}), \quad 1 \leqslant q \leqslant \infty. \tag{4.2}$$

In particular, if $\mathcal{A} = \ell^1(\Lambda)$, then $\widetilde{M}^{\infty,\mathcal{A}}$ is the Sjöstrand class $M^{\infty,1}$. Thus the symbol class $\widetilde{M}^{\infty,\mathcal{A}}$ is a generalized modulation space. To prove (4.2) we observe that, by (2.2), $\sigma \in M^{\infty,q}_{1\otimes v\circ j^{-1}}(\mathbb{R}^{2d})$, if and only if $\mathcal{G}(\sigma)\circ j \in L^q_v(\mathbb{R}^{2d})$. Since v is continuous and submultiplicative, the inclusion $W(\ell^q_v(\Lambda)) \subseteq L^q_v(\mathbb{R}^d)$ holds for every lattice $\Lambda \subset \mathbb{R}^{2d}$. This shows that $\widetilde{M}^{\infty,\mathcal{A}} \subseteq M^{\infty,q}_{1\otimes v\circ j^{-1}}(\mathbb{R}^{2d})$. Conversely, if $\sigma \in M^{\infty,q}_{1\otimes v\circ j^{-1}}(\mathbb{R}^{2d})$, then $\mathcal{G}(\sigma) \in W(\ell^q_{v\circ j^{-1}}(\Lambda))$ for every lattice $\Lambda \subset \mathbb{R}^{2d}$ by [24, Thm 12.2.1], and we have equality in (4.2).

To consolidate Definition 4.1, we need to establish its independence of the particular window Φ. This is done as in the case of modulation spaces in [24, Ch. 11.3,11.4], but requires a few adjustments.

Lemma 4.3. — *Let $\Psi(z) = e^{-\pi z\cdot z/2}$ be the Gaussian. If a window $\Phi \in L^2(\mathbb{R}^{2d})$ satisfies the condition*

$$F(\zeta) := \int_{\mathbb{R}^{2d}} |\mathcal{V}_\Psi \Phi(z, j(\zeta))|dz \in W(\mathcal{A})\,, \tag{4.3}$$

then the definition of $\widetilde{M}^{\infty,\mathcal{A}}$ does not depend on the window.

Proof. — We use the pointwise estimate (2.1) in the form

$$|\mathcal{V}_\Psi\sigma(z,\zeta)| \leqslant \frac{1}{\|\Phi\|_2^2}\int_{\mathbb{R}^{2d}} |\mathcal{V}_\Phi\sigma(z-u,\eta)|\,|\mathcal{V}_\Psi\Phi(u,\zeta-\eta)|\,dud\eta\,.$$

After taking the supremum over z and inserting (4.3), we obtain

$$\mathcal{G}_\Psi(\sigma)(\zeta) \leqslant \Big(\mathcal{G}_\Phi(\sigma) * (F\circ j^{-1})\Big)(\zeta)\,,$$

where $\mathcal{G}_\Psi$ is the grand symbol with respect to the window Ψ. Consequently, since $W(\mathcal{A})$ is a Banach algebra, we obtain

$$\|\mathcal{G}_\Psi(\sigma)\circ j\|_{W(\mathcal{A})} \leqslant \|\mathcal{G}_\Phi(\sigma)\circ j\|_{W(\mathcal{A})}\,\|F\|_{W(\mathcal{A})}\,.$$

By interchanging the roles of Φ and Ψ, we obtain the norm equivalence

$$\|\mathcal{G}_\Phi(\sigma)\circ j\|_{W(\mathcal{A})} \asymp \|\mathcal{G}_\Psi(\sigma)\circ j\|_{W(\mathcal{A})} = \|\sigma\|_{\widetilde{M}^{\infty,\mathcal{A}}}\,.$$

□

Next, to establish the link between pseudodifferential operators and the generalized Sjöstrand classes $\widetilde{M}^{\infty,\mathcal{A}}$, we will need windows of the form $\Phi = W(g,g)$ for a suitable function g on $\mathbb{R}^d$. According to Lemma 4.3 we need to determine a class of functions g such that $\Phi = W(g,g)$ satisfies condition (4.3). This is explained in the following lemma.

Lemma 4.4. — *If $V_g g \in W(\mathcal{A})$, then $\Phi = W(g,g)$ satisfies condition* (4.3).

Proof. — We first show that the hypothesis implies that $V_\varphi g \in W(\mathcal{A})$. Assume first that $\langle g,\varphi\rangle \neq 0$, then by (2.1)

$$|V_\varphi g| \leqslant |\langle g,\varphi\rangle|^{-1}\left(|V_g g| * |V_\varphi\varphi|\right),$$

thus $\|V_\varphi g\|_{W(\mathcal{A})} \leqslant |\langle g,\varphi\rangle|^{-1}\,\|V_g g\|_{W(\mathcal{A})}\,\|V_\varphi\varphi\|_{W(\mathcal{A})} < \infty$. If $\langle g,\varphi\rangle = 0$, then choose a φ_1 (e.g., a linear combination of Hermite functions), such that $\langle\varphi_1,g\rangle \neq 0$ and $\langle\varphi_1,\varphi\rangle \neq 0$ and apply the above argument twice.

Next observe that $e^{-\pi z\cdot z/2} = W(\varphi,\varphi)(z)$, $z\in\mathbb{R}^{2d}$, where $\varphi(t) = e^{-\pi t\cdot t}$, and we use the "magic formula" for the STFT of a Wigner distribution [24, Lemma 14.5.1]:

$$|\mathcal{V}_{W(\varphi,\varphi)}W(g,g)(z,\zeta)| = |V_\varphi g(z+\frac{j(\zeta)}{2})|\,|V_\varphi g(z-\frac{j(\zeta)}{2})|\,.$$

Consequently,

$$F(\zeta) = \int_{\mathbb{R}^{2d}} |\mathcal{V}_{W(\varphi,\varphi)}W(g,g)(z,j(\zeta))|\,dz = \big(|V_\varphi g| * |(V_\varphi g)^*|\big)(\zeta)$$

where $|(V_\varphi g)^*(z)| = |V_\varphi g(-z)| = |V_g\varphi(z)|$. Since both $V_\varphi g \in W(\mathcal{A})$ and $V_g\varphi \in W(\mathcal{A})$, we have $F\in W(\mathcal{A})$, as claimed. □

Almost diagonalization. An important identity that establishes a relation between the short-time Fourier transform of a symbol σ and its Weyl transform σ^w is the following.

Lemma 4.5 (Lemma 3.1[27]). — *Assume that $\sigma \in M^\infty(\mathbb{R}^{2d})$ and $g \in M^1(\mathbb{R}^d)$. If we choose the window Φ to be the Wigner distribution $\Phi = W(g,g)$, then*

$$|\langle \sigma^w \pi(z)g, \pi(w)g\rangle| = \Big|\mathcal{V}_\Phi\sigma\Big(\frac{w+z}{2}, j(w-z)\Big)\Big| \tag{4.4}$$

for all $w, z \in \mathbb{R}^{2d}$.

Read backwards, this formula yields

$$|\mathcal{V}_\Phi\sigma(u,v)| = \Big|\Big\langle \sigma^w \pi\Big(u - \frac{j^{-1}(v)}{2}\Big)g, \pi\Big(u + \frac{j^{-1}(v)}{2}\Big)g\Big\rangle\Big|. \tag{4.5}$$

Both formulas (4.4) and (4.5) hold pointwise.

In order to check the membership of a symbol in $\widetilde{M}^{\infty,\mathcal{A}}$, we need at least that the window $\Phi = W(g,g)$ used to measure the $\widetilde{M}^{\infty,\mathcal{A}}$-norm satisfies condition (4.3). This is guaranteed by Lemma 4.4. In the remainder of the paper the condition $V_g g \in W(\mathcal{A})$ will thus be the standing assumption on g. In particular, we may use formulas (4.4) and (4.5), because $V_g g \in W(\mathcal{A})$ implies that $g \in M^1$ and $\widetilde{M}^{\infty,\mathcal{A}} \subseteq M^\infty$.

Our main theorem is a far-reaching extension of Theorem 3.2 in [27].

Theorem 4.6. — *Let $\mathcal{A}$ be a solid Banach algebra with involution on a lattice $\Lambda \subseteq \mathbb{R}^{2d}$. Fix a window g such that $V_g g \in W(\mathcal{A})$ and let $\mathcal{G}(g,\Lambda)$ be a tight Gabor frame for $L^2(\mathbb{R}^d)$. Then the following are equivalent for a distribution $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$.*

(i) $\sigma \in \widetilde{M}^{\infty,\mathcal{A}}$.

(ii) *There exists a function $H \in W(\mathcal{A})$ such that*

$$|\langle \sigma^w \pi(z)g, \pi(w)g\rangle| \leqslant H(w-z) \quad \text{for } w, z \in \mathbb{R}^{2d}. \tag{4.6}$$

(iii) *There exists a sequence $\mathbf{h} \in \mathcal{A}$ such that*

$$|\langle \sigma^w \pi(\mu)g, \pi(\lambda)g\rangle| \leqslant \mathbf{h}(\lambda-\mu) \quad \text{for } \lambda, \mu \in \Lambda. \tag{4.7}$$

Proof. — The equivalence of (i) and (ii) follows easily by using formulae (4.4) and (4.5).

(i) $\Rightarrow$ (ii). By (4.4) we have

$$|\langle \sigma^w \pi(z)g, \pi(w)g\rangle| = \Big|\mathcal{V}_\Phi\sigma\Big(\frac{w+z}{2}, j(w-z)\Big)\Big| \leqslant \mathcal{G}(\sigma)(j(w-z)). \tag{4.8}$$

Since $\mathcal{G}(\sigma)\circ j \in W(\mathcal{A})$, we may take $H = \mathcal{G}(\sigma)\circ j$ as the dominating function in (4.6).

(ii) $\Rightarrow$ (i). By (4.5) and (4.6) we have

$$\begin{aligned}\mathcal{G}(\sigma)(j(\zeta)) &= \operatorname*{ess\,sup}_{z\in\mathbb{R}^{2d}} |\mathcal{V}_\Phi\sigma(z,j(\zeta))| \\ &= \operatorname*{ess\,sup}_{z\in\mathbb{R}^{2d}} \Big|\Big\langle \sigma^w\pi\Big(z-\frac{\zeta}{2}\Big)g,\pi\Big(z+\frac{\zeta}{2}\Big)g\Big\rangle\Big| \leqslant H(\zeta).\end{aligned} \tag{4.9}$$

Thus, $\mathcal{G}(\sigma)\circ j \leqslant H$. Since $H\in W(\mathcal{A})$ and $\mathcal{A}$ is solid we get that $\mathcal{G}(\sigma)\circ j\in W(\mathcal{A})$.

(i) $\Rightarrow$ (iii). This implication follows from formule (4.4) as well. Indeed, for $\lambda,\mu\in\Lambda$ we have, as in (4.8), that

$$\big|\langle\sigma^w\pi(\mu)g,\pi(\lambda)g\rangle\big| \leqslant \mathcal{G}(\sigma)(j(\lambda-\mu)).$$

Let $\mathbf{h}(\lambda) = \operatorname{ess\,sup}_{\zeta\in\lambda+C}\mathcal{G}(\sigma)(j(\zeta))$. Since the fundamental domain C is assumed to contain the origin, we have $\mathcal{G}(\sigma)(j(\lambda))\leqslant\mathbf{h}(\lambda)$. Therefore, the above inequality can be extended to

$$\big|\langle\sigma^w\pi(\mu)g,\pi(\lambda)g\rangle\big| \leqslant \mathcal{G}(\sigma)(j(\lambda-\mu)) \leqslant \mathbf{h}(\lambda-\mu). \tag{4.10}$$

Since $\sigma\in\widetilde{M}^{\infty,\mathcal{A}}$ means that $\mathbf{h}\in\mathcal{A}$, we obtain (4.7).

(iii) $\Rightarrow$ (ii). This implication is more technical. Only here we use the assumption that $\mathcal{G}(g,\Lambda)$ is a tight Gabor frame for $L^2(\mathbb{R}^{2d})$. Consider the tight frame expansion of $\pi(u)g$

$$\pi(u)g = \sum_{\nu\in\Lambda}\langle\pi(u)g,\pi(\nu)g\rangle\pi(\nu)g\,, \tag{4.11}$$

for every $u\in\mathbb{R}^{2d}$. Since we assume that $V_gg\in W(\mathcal{A})$, the sequence $\boldsymbol{\alpha}$ of local suprema

$$\begin{aligned}\boldsymbol{\alpha}(\nu) &= \sup_{\zeta\in C}|V_gg(\nu+\zeta)| = \sup_{\zeta\in C}|\langle g,\pi(\nu+\zeta)g\rangle| \\ &= \sup_{\zeta\in C}|\langle\pi(\zeta)g,\pi(-\nu)g\rangle|\,,\quad \nu\in\Lambda\,,\end{aligned} \tag{4.12}$$

belongs to $\mathcal{A}$.

For given $z,w\in\mathbb{R}^{2d}$, we write them uniquely as $z=\mu+u'$, $w=\lambda+u$, where $\lambda,\mu\in\Lambda$ and $u,u'\in C$. Inserting the expansions (4.11) and the definition of $\boldsymbol{\alpha}$ in the matrix entries, we find that (4.7) yields the following estimate

$$\begin{aligned}&|\langle\sigma^w\pi(\mu+u')g,\pi(\lambda+u)g\rangle| = |\langle\sigma^w\pi(\mu)\pi(u')g,\pi(\lambda)\pi(u)g\rangle| \\ &\leqslant \sum_{\nu,\nu'\in\Lambda}|\langle\sigma^w\pi(\mu+\nu')g,\pi(\lambda+\nu)g\rangle|\,|\langle\pi(u')g,\pi(\nu')g\rangle|\,|\langle\pi(u)g,\pi(\nu)g\rangle| \\ &\leqslant \sum_{\nu,\nu'\in\Lambda}\mathbf{h}(\lambda-\mu+\nu-\nu')\boldsymbol{\alpha}(-\nu')\boldsymbol{\alpha}(-\nu) \\ &= (\mathbf{h}*\boldsymbol{\alpha}*\boldsymbol{\alpha}^*)(\lambda-\mu)\,,\end{aligned}$$

where $\boldsymbol{\alpha}^*(\lambda) = \overline{\boldsymbol{\alpha}(-\lambda)}$ is the involution on $\mathcal{A}$. Since $\boldsymbol{\alpha} \in \mathcal{A}$, $\boldsymbol{\alpha}^* \in \mathcal{A}$ and $\mathbf{h} \in \mathcal{A}$, we see that $\mathbf{h} * \boldsymbol{\alpha} * \boldsymbol{\alpha}^* \in \mathcal{A}$ as well.

To find the dominating function H postulated in (4.6), we set

$$H(\zeta) = \sum_{\nu \in \Lambda} (\mathbf{h} * \boldsymbol{\alpha} * \boldsymbol{\alpha}^*)(\nu)\, \chi_{C-C}(\zeta - \nu)\,, \tag{4.13}$$

where $\zeta \in \mathbb{R}^{2d}$. Our previous estimate says that

$$\begin{aligned} |\langle \sigma^w \pi(z) g, \pi(w) g\rangle| &\leqslant (\mathbf{h} * \boldsymbol{\alpha} * \boldsymbol{\alpha}^*)(\lambda - \mu) \\ &= (\mathbf{h} * \boldsymbol{\alpha} * \boldsymbol{\alpha}^*)(\lambda - \mu)\chi_{C-C}(u - u') \\ &\leqslant \sum_{\nu \in \Lambda} (\mathbf{h} * \boldsymbol{\alpha} * \boldsymbol{\alpha}^*)(\nu)\, \chi_{C-C}(\lambda - \mu + u - u' - \nu) \\ &= H(\lambda - \mu + u - u') = H(w - z)\,, \end{aligned}$$

and H satisfies (4.6).

To finish, we need to show that $H \in W(\mathcal{A})$. Let $\boldsymbol{\beta}(\nu) = \sup_{\zeta \in C} \chi_{C-C}(\zeta + \nu)$. Then $\boldsymbol{\beta} \in \mathcal{A}$, since $\boldsymbol{\beta}$ is finitely supported. Keeping $\lambda \in \Lambda$ and $u \in C$, we estimate as follows:

$$\begin{aligned} H(\lambda + u) &= \sum_{\nu \in \Lambda} (\mathbf{h} * \boldsymbol{\alpha} * \boldsymbol{\alpha}^*)(\nu)\, \chi_{C-C}(\lambda + u - \nu) \\ &\leqslant \sum_{\nu \in \Lambda} (\mathbf{h} * \boldsymbol{\alpha} * \boldsymbol{\alpha}^*)(\nu)\, \sup_{u \in C} \chi_{C-C}(\lambda + u - \nu) \\ &= \sum_{\nu \in \Lambda} (\mathbf{h} * \boldsymbol{\alpha} * \boldsymbol{\alpha}^*)(\nu) \boldsymbol{\beta}(\lambda - \nu) = (\mathbf{h} * \boldsymbol{\alpha} * \boldsymbol{\alpha}^* * \boldsymbol{\beta})(\lambda). \end{aligned}$$

Thus, $\sup_{u \in C} H(\lambda + u) \leqslant (\mathbf{h} * \boldsymbol{\alpha} * \boldsymbol{\alpha}^* * \boldsymbol{\beta})(\lambda)$ and $\mathbf{h} * \boldsymbol{\alpha} * \boldsymbol{\alpha}^* * \boldsymbol{\beta} \in \mathcal{A}$. Consequently, $H \in W(\mathcal{A})$ and

$$\|H\|_{W(\mathcal{A})} \leqslant \|\mathbf{h} * \boldsymbol{\alpha} * \boldsymbol{\alpha}^* * \boldsymbol{\beta}\|_{\mathcal{A}} \leqslant C \|\mathbf{h}\|_{\mathcal{A}}, \tag{4.14}$$

with $C = \|\boldsymbol{\alpha} * \boldsymbol{\alpha}^* * \boldsymbol{\beta}\|_{\mathcal{A}}$. □

Theorem 4.6 provides us with an almost diagonalization of the operator σ^w with symbol $\sigma \in \widetilde{M}^{\infty, \mathcal{A}}$ with respect to any Gabor system $\mathcal{G}(g, \Lambda)$ that forms a tight frame.

The almost diagonalization result gives a characterization of the symbol class $\widetilde{M}^{\infty, \mathcal{A}}$ in two versions, a continuous one and a discrete one. We elaborate the discrete version. Let $M(\sigma)$ be the matrix with entries

$$M(\sigma)_{\lambda, \mu} = \langle \sigma^w \pi(\mu) g, \pi(\lambda) g\rangle\,, \qquad \lambda, \mu \in \Lambda\,.$$

Clearly, $M(\sigma)$ depends on g, and we assume that the window satisfies the assumptions of Theorem 4.6.

THEOREM 4.7. — *$\sigma \in \widetilde{M}^{\infty,\mathcal{A}}$ if and only if $M(\sigma) \in \mathcal{C}_\mathcal{A}$. Moreover,*

$$c\|\sigma\|_{\widetilde{M}^{\infty,\mathcal{A}}} \leqslant \|M(\sigma)\|_{\mathcal{C}_\mathcal{A}} \leqslant \|\sigma\|_{\widetilde{M}^{\infty,\mathcal{A}}}\,. \tag{4.15}$$

Proof. — In view of Theorem 4.6, specifically $(i) \Leftrightarrow (iii)$, we need to prove only the norm equivalence. From estimate (4.10) it follows immediately, that

$$\|M(\sigma)\|_{\mathcal{C}_\mathcal{A}} \leqslant \|\mathbf{h}\|_\mathcal{A} = \|\mathcal{G}(\sigma)\circ j\|_{W(\mathcal{A})} = \|\sigma\|_{\widetilde{M}^{\infty,\mathcal{A}}}.$$

To deduce the converse inequality, we use (4.9) to conclude that $\|\mathcal{G}(\sigma)\circ j\|_{W(\mathcal{A})} \leqslant \|H\|_{W(\mathcal{A})}$ for every H having the dominating property of (4.6). In (4.13) we have described an explicit dominating function H derived from an arbitrary sequence $\mathbf{h}$ satisfying (4.7). Clearly, we can choose a sequence $\mathbf{h}$ such that $\|\mathbf{h}\|_\mathcal{A} = \|M(\sigma)\|_{\mathcal{C}_\mathcal{A}}$. Since $\|H\|_{W(\mathcal{A})} \leqslant C\|\mathbf{h}\|_\mathcal{A}$ by (4.14), we obtain that

$$\|\sigma\|_{\widetilde{M}^{\infty,\mathcal{A}}} = \|\mathcal{G}(\sigma)\circ j\|_{W(\mathcal{A})} \leqslant \|H\|_{W(\mathcal{A})} \leqslant C\|\mathbf{h}\|_\mathcal{A} = C\|M(\sigma)\|_{\mathcal{C}_\mathcal{A}}.$$

□

The matrix $M(\sigma)$ is the matrix of the operator σ^w with respect to the tight Gabor frame $\mathcal{G}(g,\Lambda)$. Indeed, let $f = \sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)g\rangle\pi(\lambda)g$ be the tight frame expansion of $f \in L^2(\mathbb{R}^d)$. It has the coefficients

$$V_g^\Lambda f(\lambda) = \langle f,\pi(\lambda)g\rangle\,, \tag{4.16}$$

where $V_g^\Lambda f$ is just the restriction of the STFT of f to the lattice Λ. Clearly, one has

$$\langle\sigma^w f,\pi(\lambda)g\rangle = \sum_{\mu\in\Lambda}\langle f,\pi(\mu)g\rangle\,\langle\sigma^w\pi(\mu)g,\pi(\lambda)g\rangle\,,$$

or in other words

$$V_g^\Lambda(\sigma^w f) = M(\sigma)V_g^\Lambda f\,. \tag{4.17}$$

This commutation relation can be depicted by the diagram

$$\begin{array}{ccc} L^2(\mathbb{R}^d) & \xrightarrow{\sigma^w} & L^2(\mathbb{R}^d) \\ \downarrow V_g^\Lambda & & \downarrow V_g^\Lambda \\ \ell^2(\Lambda) & \xrightarrow{M(\sigma)} & \ell^2(\Lambda) \end{array} \tag{4.18}$$

For a symbol $\sigma \in \widetilde{M}^{\infty,\mathcal{A}}$, the operator σ^w is always bounded on $L^2(\mathbb{R}^d)$, because $\mathcal{A} \subseteq \ell^1(\Lambda)$ by Theorem 3.1, and thus $\widetilde{M}^{\infty,\mathcal{A}} \subset M^{\infty,1} \subset \mathcal{B}(L^2(\mathbb{R}^d))$. By Sjöstrand's fundamental boundedness result every σ^w with $\sigma \in M^{\infty,1}$ is bounded on $L^2(\mathbb{R}^d)$. Thus the diagram makes perfect sense for $\sigma \in \widetilde{M}^{\infty,\mathcal{A}}$. Clearly, if $M(\sigma)$ is bounded on $\ell^2(\Lambda)$, then σ^w is bounded on $L^2(\mathbb{R}^d)$. In particular, if $M(\sigma)$ satisfies Schur's test, then σ^w is bounded on $L^2(\mathbb{R}^d)$

(see also [8] and [41]). However, in this case, it is not clear how to recognize or characterize the symbol of such an operator, quite in contrast to Theorem 4.6

If $g \in M^{1,1}$ and $\mathcal{G}(g,\Lambda)$ is a Gabor frame, then the range of V_g^Λ is always a proper closed subspace of $\ell^2(\Lambda)$ [20]. Therefore the matrix $M(\sigma)$ is not uniquely determined by the diagram (4.18). The next lemma taken from [27] explains the additional properties of $M(\sigma)$.

Lemma 4.8. — *If σ^w is bounded on $L^2(\mathbb{R}^d)$, then $M(\sigma)$ is bounded on $\ell^2(\Lambda)$ and maps $\operatorname{ran} V_g^\Lambda$ into $\operatorname{ran} V_g^\Lambda$ with $\ker M(\sigma) \supseteq (\operatorname{ran} V_g^\Lambda)^\perp$.*

Let T be a matrix such that $V_g^\Lambda(\sigma^w f) = T V_g^\Lambda f$ for all $f \in L^2(\mathbb{R}^d)$. If $\ker T \supseteq (\operatorname{ran} V_g \Lambda)^\perp$, then $T = M(\sigma)$.

Note that $M(\sigma)$ is never invertible, because its kernel always contains the nontrivial subspace $(\operatorname{ran} V_g^\Lambda)^\perp$.

5. Properties of operators with symbols in $\widetilde{M}^{\infty,\mathcal{A}}$

In this section, we study several properties and applications of the symbol class $\widetilde{M}^{\infty,\mathcal{A}}$. The common theme is how properties of the algebra $\mathcal{A}$ are inherited by properties of operators with symbols in $\widetilde{M}^{\infty,\mathcal{A}}$. We investigate the algebra property, the inverse-closedness, and the boundedness of pseudodifferential operators on distribution spaces.

5.1. The Algebra Property

Let us first consider the algebra property. In Theorem 4.6 we have used consistently and crucially that $\mathcal{A}$ is an algebra. This property is inherited by $\widetilde{M}^{\infty,\mathcal{A}}$.

Theorem 5.1. — *$\widetilde{M}^{\infty,\mathcal{A}}$ is a Banach algebra with respect to the twisted product $\sharp$ defined in* (2).

Proof. — We assume that $\mathcal{G}(g,\Lambda)$ is a tight Gabor frame with $V_g g \in W(\mathcal{A})$. Let $\sigma,\tau \in \widetilde{M}^{\infty,\mathcal{A}}$. By using (4.18) several times, we get that

$$\begin{aligned} M(\sigma \sharp \tau) V_g^\Lambda f &= V_g^\Lambda((\sigma \sharp \tau)^w f) = V_g^\Lambda(\sigma^w \tau^w f) \\ &= M(\sigma)\big(V_g^\Lambda(\tau^w f)\big) = M(\sigma) M(\tau) V_g^\Lambda f\,. \end{aligned}$$

Therefore the operators $M(\sigma \sharp \tau)$ and $M(\sigma)M(\tau)$ coincide on $\operatorname{ran} V_g^\Lambda$. Since σ^w, τ^w and $(\sigma \sharp \tau)^w$ are bounded on $L^2(\mathbb{R}^d)$, we can use Lemma 4.8 to conclude that both $M(\sigma \sharp \tau)$ and $M(\sigma)M(\tau)$ are zero on $(\operatorname{ran} V_g^\Lambda)^\perp$. Thus, we get the following matrix identity

$$M(\sigma \sharp \tau) = M(\sigma)M(\tau)\,. \tag{5.1}$$

Since by Theorem 4.7 both $M(\sigma)$ and $M(\tau)$ are in $\mathcal{C}_\mathcal{A}$, the algebra property of $\mathcal{C}_\mathcal{A}$ implies that $M(\sigma \sharp \tau) \in \mathcal{C}_\mathcal{A}$. By Theorem 4.7 once again, we deduce that $\sigma \sharp \tau \in \widetilde{M}^{\infty,\mathcal{A}}$. The norm estimate follows from

$$\begin{aligned}\|\sigma \sharp \tau\|_{\widetilde{M}^{\infty,\mathcal{A}}} &\leqslant C\|M(\sigma \sharp \tau)\|_{\mathcal{C}_\mathcal{A}} \leqslant C\,\|M(\sigma)\|_{\mathcal{C}_\mathcal{A}}\,\|M(\tau)\|_{\mathcal{C}_\mathcal{A}} \\ &\leqslant C\,\|\sigma\|_{\widetilde{M}^{\infty,\mathcal{A}}}\,\|\tau\|_{\widetilde{M}^{\infty,\mathcal{A}}}\,,\end{aligned}$$

and so $\widetilde{M}^{\infty,\mathcal{A}}$ is a Banach algebra. □

5.2. Boundedness

Let $\mathcal{Y}$ be a solid space of sequences on the lattice $\Lambda \subset \mathbb{R}^{2d}$. If $\mathcal{A}$ acts boundedly on $\mathcal{Y}$ under convolution, then it is natural to expect that the class $\widetilde{M}^{\infty,\mathcal{A}}$ acts boundedly on a suitable function space associated to $\mathcal{Y}$. The appropriate function spaces are the (generalized) modulation spaces. We present the definition of modulation spaces that is most suitable for our purpose.

Definition 5.2. — *Let $\mathcal{Y}$ be a solid Banach space of sequences on Λ such that the finitely supported sequences are dense in $\mathcal{Y}$. Let $\mathcal{G}(g,\Lambda)$ be a tight frame for $L^2(\mathbb{R}^d)$ such that $V_g g \in W(\mathcal{A})$. Let $\mathcal{L}_0$ be the span of all finite linear combinations $f = \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g$.*

Now we define a norm on $\mathcal{L}_0$ by

$$\|f\|_{M(\mathcal{Y})} = \|V_g^\Lambda f\|_\mathcal{Y}\,. \tag{5.2}$$

The modulation space $M(\mathcal{Y})$ is the norm completion of $\mathcal{L}_0$ with respect to the $M(\mathcal{Y})$-norm.

Using this definition, the following general boundedness theorem is very easy.

Theorem 5.3. — *Let $\mathcal{A}$ be a solid involutive Banach algebra with respect to convolution and let $\mathcal{Y}$ be a solid Banach space of sequences on Λ. If convolution of $\mathcal{A}$ on $\mathcal{Y}$ is bounded, $\mathcal{A} * \mathcal{Y} \subseteq \mathcal{Y}$, and if $\sigma \in \widetilde{M}^{\infty,\mathcal{A}}$, then σ^w is bounded on $M(\mathcal{Y})$. The operator norm can be estimated uniformly by*

$$\|\sigma^w\|_{M(\mathcal{Y})\to M(\mathcal{Y})} \leqslant \|M(\sigma)\|_{\mathcal{C}_\mathcal{A}} \leqslant \|\sigma\|_{\widetilde{M}^{\infty,\mathcal{A}}}\,.$$

Proof. — For $f \in \mathcal{L}_0$ we use the commutative diagram (4.17) and (3.4) and obtain the boundedness in a straightforward manner from the estimate

$$\begin{aligned} \|\sigma^w f\|_{M(\mathcal{Y})} &= \|V_g^\Lambda(\sigma^w f)\|_{\mathcal{Y}} = \|M(\sigma)V_g^\Lambda f\|_{\mathcal{Y}} \\ &\leqslant \|M(\sigma)\|_{\mathcal{C}_\mathcal{A}} \|V_g^\Lambda f\|_{\mathcal{Y}} = \|M(\sigma)\|_{\mathcal{C}_\mathcal{A}} \|f\|_{M(\mathcal{Y})}. \end{aligned}$$

Since $M(\mathcal{Y})$ is defined as the closure of $\mathcal{L}_0$, the above estimate extends to all $f \in M(\mathcal{Y})$. □

The proof looks rather trivial, but it hides many issues. In fact, our definition of the modulation space $M(\mathcal{Y})$ is based on the main result of time-frequency analysis and coorbit theory. The general definition of a modulation space [24, Ch. 11.4] starts with the solid function space $W(\mathcal{Y})$ on $\mathbb{R}^{2d}$. The modulation space $M(\mathcal{Y})$ then consists of all distributions f such that $V_g f \in W(\mathcal{Y})$. If $\mathcal{Y}$ is a weighted sequence space ℓ^p_v or $\ell^{p,q}_m$ for $1 \leqslant p, q < \infty$, one obtains the standard modulation spaces $M^{p,q}_m$ discussed in Section 2. The main result of time-frequency analysis shows that the continuous definition can be discretized. If $\{\pi(\lambda)g : \lambda \in \Lambda\}$ is a frame for $L^2(\mathbb{R}^d)$ with a "nice" window g, then $f \in M(\mathcal{Y})$ if and only if $V_g^\Lambda \in \mathcal{Y}$; and this is the definition we are using. This characterization also implies the independence of the symbol class $\widetilde{M}^{\infty,\mathcal{A}}$ of the Gabor frame $\mathcal{G}(g,\Lambda)$ in Definition 4.1.

However, the above characterization of modulation spaces by means of Gabor frames involves the major results of time-frequency analysis and is far from trivial. See [18, 19] and [24, Chs. 11-13] for a detailed account.

5.3. Spectral Invariance

Next we discuss the invertibility of pseudodifferential operators and the symbol of the inverse operator. We would like to characterize the inverse of σ^w in terms of the matrix $M(\sigma)$. This point is a bit more sublte, because the invertibility of σ^w on $L^2(\mathbb{R}^d)$ does not guarantee the invertibility of $M(\sigma)$ on $\ell^2(\Lambda)$ (see Lemma 4.8). To go around this difficulty, we need the notion of a pseudo-inverse.

Recall that an operator $A : \ell^2 \to \ell^2$ is pseudo-invertible, if there exists a closed invariant subspace $\mathcal{R} \subseteq \ell^2$, such that A is invertible on $\mathcal{R}$, $\operatorname{ran} A = \mathcal{R}$ and $\ker A = \mathcal{R}^\perp$. The unique operator $A^\dagger$ that satisfies $A^\dagger A h = A A^\dagger h = h$ for $h \in \mathcal{R}$ and $\ker A^\dagger = \mathcal{R}^\perp$ is called the (Moore-Penrose) pseudo-inverse of A. The following lemma is an important consequence of Theorem 3.5 and is taken from [27].

Lemma 5.4 (Pseudo-inverses). — *If $\mathcal{A}$ is inverse-closed in $\mathcal{B}(\ell^2)$ and $A \in \mathcal{C}_\mathcal{A}$ has a (Moore-Penrose) pseudo-inverse $A^\dagger$, then $A^\dagger \in \mathcal{C}_\mathcal{A}$.*

Proof. — By means of the Riesz functional calculus [38] the pseudo-inverse can be written as

$$A^\dagger = \frac{1}{2\pi i}\int_C \frac{1}{z}\,(z\mathrm{I}-A)^{-1}\,dz\,,$$

where C is a suitable path surrounding $\mathrm{Sp}_{\mathcal{B}(\ell^2)}(A)\setminus\{0\}$. Theorem 3.5 implies that $\mathcal{C}_{\mathcal{A}}$ is inverse-closed in $\mathcal{B}(\ell^2)$. Hence $(z\mathrm{I}-A)^{-1}\in\mathcal{C}_{\mathcal{A}}$, and by (3.5) this formula makes sense in $\mathcal{C}_{\mathcal{A}}$. Consequently, $A^\dagger\in\mathcal{C}_{\mathcal{A}}$. □

THEOREM 5.5. — *If $\mathcal{A}$ is inverse-closed in $\mathcal{B}(\ell^2(\Lambda))$, then the class of pseudodifferential operators with symbols in $\widetilde{M}^{\infty,\mathcal{A}}$ is inverse-closed in $\mathcal{B}(L^2(\mathbb{R}^d))$. In the standard formulation, if $\sigma\in\widetilde{M}^{\infty,\mathcal{A}}$ and σ^w is invertible on $L^2(\mathbb{R}^d)$, then $(\sigma^w)^{-1}=\tau^w$ for some $\tau\in\widetilde{M}^{\infty,\mathcal{A}}$.*

Proof. — Assume that σ^w is invertible on $L^2(\mathbb{R}^d)$ for some $\sigma\in\widetilde{M}^{\infty,\mathcal{A}}$. Let $\tau\in\mathcal{S}'(\mathbb{R}^{2d})$ be the unique distribution such that $\tau^w=(\sigma^w)^{-1}$. We need to show that $\tau\in\widetilde{M}^{\infty,\mathcal{A}}$.

Since τ^w is bounded on $L^2(\mathbb{R}^d)$, Lemma 4.8 implies that the matrix $M(\tau)$ is bounded on $\ell^2(\Lambda)$ and maps $\operatorname{ran}V_g^\Lambda$ into $\operatorname{ran}V_g^\Lambda$ with $\ker M(\tau)\supseteq(\operatorname{ran}V_g^\Lambda)^\perp$.

If $f\in L^2(\mathbb{R}^d)$, then by (4.17) we have

$$M(\tau)M(\sigma)V_g^\Lambda f = M(\tau)V_g^\Lambda(\sigma^w f) = V_g^\Lambda(\tau^w\sigma^w f) = V_g^\Lambda f\,.$$

This means that $M(\tau)M(\sigma)=\mathrm{Id}$ on $\operatorname{ran}V_g^\Lambda$ and that $M(\tau)M(\sigma)=0$ on $(\operatorname{ran}V_g^\Lambda)^\perp$. Likewise, $M(\sigma)M(\tau)=\mathrm{Id}_{\operatorname{ran}V_g^\Lambda}$ and $\ker M(\tau)=(\operatorname{ran}V_g^\Lambda)^\perp$. Thus, we conclude that $M(\tau)=M(\sigma)^\dagger$.

By Theorem 4.7, the hypothesis $\sigma\in\widetilde{M}^{\infty,\mathcal{A}}$ implies that $M(\sigma)$ belongs to the matrix algebra $\mathcal{C}_{\mathcal{A}}$. Consequently, by Lemma 5.4, we also have that $M(\tau)=M(\sigma)^\dagger\in\mathcal{C}_{\mathcal{A}}$. Using Theorem 4.6 again, we conclude that $\tau\in\widetilde{M}^{\infty,\mathcal{A}}$. □

The proof of Theorem 5.5 should be compared to the proofs of analogous statements by Beals [4] and Sjöstrand [40]. The key element in our proof is Theorem 3.5 about the inverse-closedness of the matrix algebra $\mathcal{C}_{\mathcal{A}}$. The hard work in Theorem 5.5 is thus relegated to the theory of Banach algebras.

Example 5.6. — Since the algebras $\ell^q_v(\Lambda)$, $q=1,\infty$ of Examples 3.2 and 3.6 are inverse-closed, the corresponding classes of pseudodifferential operators are inverse-closed as well. Using (4.2) we obtain that the class $M^{\infty,q}_{1\otimes v\circ j^{-1}}(\mathbb{R}^{2d})$ is inverse-closed for $q=1,\infty$ and suitable weights v. In particular, $M^{\infty,1}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d})$ is inverse-closed for $s\geqslant 0$ and $M^{\infty,\infty}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d})$ is inverse-closed for $s>2d$.

Corollary 5.7 (Spectral Invariance on Modulation Spaces). — *Let $\mathcal{A}$ be an inverse-closed algebra that acts boundedly on a solid space $\mathcal{Y}$. If σ^w is invertible on $L^2(\mathbb{R}^d)$, then σ^w is invertible on $M(\mathcal{Y})$. That is,*

$$\mathrm{Sp}_{\mathcal{B}(M(\mathcal{Y}))}(\sigma^w) \subseteq \mathrm{Sp}_{\mathcal{B}(L^2(\mathbb{R}^d))}(\sigma^w) \qquad \text{for all } \sigma \in \widetilde{M}^{\infty,\mathcal{A}}. \tag{5.3}$$

Proof. — By Theorem 5.5, we have that $(\sigma^w)^{-1} = \tau^w$ for some $\tau \in \widetilde{M}^{\infty,\mathcal{A}}$. Thus, Theorem 5.3 implies that τ^w is bounded on $M(\mathcal{Y})$. Since $\sigma^w\tau^w = \tau^w\sigma^w = \mathrm{I}$ on $L^2(\mathbb{R}^d)$, this factorization extends to $M(\mathcal{Y})$. Therefore, $\tau^w = (\sigma^w)^{-1}$ on $M(\mathcal{Y})$. Applied to the operator $(\sigma - \lambda\mathrm{I})^w$ for $\lambda \notin \mathrm{Sp}_{\mathcal{B}(L^2)}$, we find that also $\lambda \notin \mathrm{Sp}_{\mathcal{B}(M(\mathcal{Y}))}$, and the inclusion of the spectra is proved. □

6. Hörmander's Class and Beals' Functional Calculus

The Hörmander class $S^0_{0,0}(\mathbb{R}^{2d})$ consists of smooth functions all of whose derivatives are bounded,

$$S^0_{0,0}(\mathbb{R}^{2d}) = \{f \in C(\mathbb{R}^{2d}) : |D^\alpha f| \leqslant C_\alpha\}.$$

Clearly, $S^0_{0,0}(\mathbb{R}^{2d}) = \bigcap_{n\geqslant 0} C^n(\mathbb{R}^{2d})$, where $C^n(\mathbb{R}^{2d})$ is the space of functions with n bounded derivatives. A characterization with modulation spaces was mentioned by Toft [47] (Remark 3.1 without proof) based on his results for embeddings between modulation spaces and Besov spaces.

Lemma 6.1. —

$$\bigcap_{n\geqslant 0} C^n(\mathbb{R}^d) = \bigcap_{s\geqslant 0} M^{\infty,\infty}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^d) = \bigcap_{s\geqslant 0} M^{\infty,1}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^d). \tag{6.1}$$

Hence $S^0_{0,0}(\mathbb{R}^{2d}) = \bigcap_{s\geqslant 0} M^{\infty,\infty}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d}) = \bigcap_{s\geqslant 0} M^{\infty,1}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d})$.

Proof. — We give a proof based on the technique developed in [24]. We shall prove three inclusions.

i) $\bigcap_{n\geqslant 0} C^n(\mathbb{R}^d) \subset \bigcap_{s\geqslant 0} M^{\infty,\infty}_{1\otimes\langle\cdot\rangle^s}$. By formula (14.38) of [24] every $f \in C^n(\mathbb{R}^d)$ fulfills

$$\sup_{x\in\mathbb{R}^d} |V_g f(x,\xi)| \leqslant C\|f\|_{C^n} |\xi^\beta|^{-1},$$

for all multi-indices $|\beta| \leqslant n$. Therefore, if $f \in \bigcap_{n\geqslant 0} C^n(\mathbb{R}^d)$, then for every β there is a constant C_β such that $\sup_{x\in\mathbb{R}^d} |V_g f(x,\xi)| \leqslant C_\beta|\xi^\beta|^{-1}$. Since $\langle\xi\rangle^n \leqslant \sum_{|\beta|\leqslant n} c_\beta|\xi^\beta|$ for suitable coefficients $c_\beta \geqslant 0$ (depending on n), we obtain that, for every $n \geqslant 0$,

$$\sup_{x,\xi\in\mathbb{R}^d} \langle\xi\rangle^n\, |V_g f(x,\xi)| \leqslant C_n < \infty\,, \tag{6.2}$$

whence $f \in \bigcap_{s\geqslant 0} M^{\infty,\infty}_{1\otimes\langle\cdot\rangle^s}$.

ii) $\bigcap_{s\geqslant 0} M^{\infty,\infty}_{1\otimes\langle\cdot\rangle^s} \subset \bigcap_{r\geqslant 0} M^{\infty,1}_{1\otimes\langle\cdot\rangle^r}$. If $s > r+d$, then

$$\begin{aligned}\|f\|_{M^{\infty,1}_{1\otimes\langle\cdot\rangle^r}} &= \int_{\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} |V_g f(x,\xi)|\langle\xi\rangle^r \, d\xi \\ &\leqslant \sup_{x,\xi\in\mathbb{R}^d} |V_g f(x,\xi)|\langle\xi\rangle^s \int_{\mathbb{R}^d} \langle\xi\rangle^{r-s} d\xi = C_{r-s}\, \|f\|_{M^{\infty,\infty}_{1\otimes\langle\cdot\rangle^s}} < \infty.\end{aligned}$$

The embedding $M^{\infty,\infty}_{1\otimes\langle\cdot\rangle^s} \subset M^{\infty,1}_{1\otimes\langle\cdot\rangle^r}$ for $s > r+d$ yields immediately the embedding $\bigcap_{s\geqslant 0} M^{\infty,\infty}_{1\otimes\langle\cdot\rangle^s} \subset \bigcap_{r\geqslant 0} M^{\infty,1}_{1\otimes\langle\cdot\rangle^r}$.

iii) $\bigcap_{s\geqslant 0} M^{\infty,1}_{1\otimes\langle\cdot\rangle^s} \subset \bigcap_{n\geqslant 0} C^n(\mathbb{R}^d)$. Let $f \in \bigcap_{s\geqslant 0} M^{\infty,1}_{1\otimes\langle\cdot\rangle^s}$. Since $V_g f(x,\xi) = (fT_x g)^\wedge(\xi)$, we may rewrite the $M^{\infty,1}_{1\otimes\langle\cdot\rangle^s}$-norm as

$$\|f\|_{M^{\infty,1}_{1\otimes\langle\cdot\rangle^s}} = \int_{\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} |(fT_x g)^\wedge(\xi)|\langle\xi\rangle^r \, d\xi < \infty\,. \tag{6.3}$$

By (6.3) and some basic properties of the Fourier transform, we majorize the derivatives of $fT_x g$ by

$$\begin{aligned}\|D^\alpha(fT_x g)\|_\infty &\leqslant \|\big(D^\alpha(fT_x g)\big)^\wedge\|_1 \leqslant C_\alpha \int_{\mathbb{R}^d} |(fT_x g)^\wedge(\xi)|\,|\xi^\alpha|\, d\xi \\ &\leqslant C_\alpha \int_{\mathbb{R}^d} |(fT_x g)^\wedge(\xi)|\,\langle\xi\rangle^{|\alpha|}\, d\xi \leqslant C_\alpha \|f\|_{M^{\infty,1}_{1\otimes\langle\cdot\rangle^{|\alpha|}}}\,,\end{aligned} \tag{6.4}$$

and this estimate is uniform in $x \in \mathbb{R}^d$. Next, let $g \in \mathcal{S}(\mathbb{R}^d)$ be a function that is equal to 1 on the unit ball B. By the Leibniz rule we have

$$\begin{aligned}D^\alpha(fT_x g) &= \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} D^\alpha f D^{\alpha-\beta}(T_x g) \\ &= D^\alpha f\, T_x g + \sum_{\beta<\alpha} \binom{\alpha}{\beta} D^\alpha f D^{\alpha-\beta}(T_x g)\end{aligned}$$

If $y \in x+B$, then clearly the sum with $\beta<\alpha$ vanishes and

$$D^\alpha(fT_x g)(y) = D^\alpha f\, T_x g(y) = D^\alpha f\, T_x \chi_B(y).$$

Thus, by (6.4), we get that

$$\|D^\alpha f\|_\infty = \sup_{x\in\mathbb{R}^d} \|D^\alpha f\, T_x\chi_B\|_\infty \leqslant \|D^\alpha(f\, T_x g)\|_\infty \leqslant C_\alpha \|f\|_{M^{\infty,1}_{1\otimes\langle\cdot\rangle^{|\alpha|}}}\,.$$

Consequently, if $f \in \bigcap_{s\geqslant 0} M^{\infty,1}_{1\otimes\langle\cdot\rangle^s}$, then $f \in \bigcap_{n\geqslant 0} C^n(\mathbb{R}^d)$. □

The above lemma and Theorem 4.6 yield the following characterization of the Hörmander class.

Theorem 6.2. — *Take a window $g \in \mathcal{S}(\mathbb{R}^d)$ such that $\mathcal{G}(g,\Lambda)$ is a tight Gabor frame for $L^2(\mathbb{R}^d)$. Then the following are equivalent*

(i) $\sigma \in S^0_{0,0}(\mathbb{R}^{2d})$.

(ii) $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ *and for every* $s \geqslant 0$ *there is a constant* C_s *such that*

$$|\langle \sigma^w \pi(z)g, \pi(w)g\rangle| \leqslant C_s \langle w - z\rangle^{-s} \quad \textit{for } w, z \in \mathbb{R}^{2d}. \tag{6.5}$$

(iii) $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ *and for every* $s \geqslant 0$ *there is a constant* C_s *such that*

$$|\langle \sigma^w \pi(\mu)g, \pi(\lambda)g\rangle| \leqslant C_s \, \langle \lambda - \mu\rangle^{-s} \quad \textit{for } \lambda, \mu \in \Lambda. \tag{6.6}$$

Proof. — We apply Theorem 4.6 with the algebra $\mathcal{A} = \ell^\infty_{\langle\cdot\rangle^s}(\Lambda)$ for $s > 2d$. Since $g \in \mathcal{S}(\mathbb{R}^d)$ if and only if $V_g g \in \mathcal{S}(\mathbb{R}^{2d})$ [21, 24], we have $V_g g \in W(\mathcal{A})$, and the relevant hypothesis on g is satisfied. Moreover, $\widetilde{M}^{\infty,\mathcal{A}} = M^{\infty,\infty}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d})$ by (4.2).

By Theorem 4.6, $\sigma \in M^{\infty,\infty}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d})$ for a given $s > 2d$, if and only if

$$|M(\sigma)_{\lambda\mu}| = |\langle \sigma^w \pi(\mu)g, \pi(\lambda)g\rangle| = \mathcal{O}(\langle \lambda - \mu\rangle^{-s}).$$

Using Lemma 6.1, we find that $\sigma \in S^0_{0,0}(\mathbb{R}^{2d}) = \bigcap_{s\geqslant 0} M^{\infty,\infty}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d})$, if and only if $M(\sigma)$ decays rapidly, as stated. The argument for the equivalence $(i) \Leftrightarrow (ii)$ is analogous. □

The Banach algebra approach to the Hörmander class yields a new proof of Beal's Theorem on the functional calculus in $S^0_{0,0}(\mathbb{R}^{2d})$ [4] (see also [48]).

Theorem 6.3. — *If* $\sigma \in S^0_{0,0}(\mathbb{R}^{2d})$ *and* σ^w *is invertible on* $L^2(\mathbb{R}^d)$*, then there exists a* $\tau \in S^0_{0,0}(\mathbb{R}^{2d})$*, such that* $(\sigma^w)^{-1} = \tau^w$.

Proof. — This follows from Lemma 6.1 and Example 5.6, where we have shown that the classes $M^{\infty,1}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d})$ and $M^{\infty,\infty}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^{2d})$ are inverse-closed, for $s \geqslant 0$ and $s > 2d$ respectively. □

7. Time-Frequency Molecules

The result on almost diagonalization can be extended to much more general systems than Gabor frames. These are so-called time-frequency molecules. They were introduced in [25], a different version of time-frequency molecules was investigated independently in [2].

Definition 7.1. — *Let* $\mathcal{G}(g,\Lambda)$ *be a tight Gabor frame with* $V_g g \in W(\mathcal{A})$*. We say that* $\{e_\mu \in \mathbb{R}^d : \mu \in \Lambda\}$ *forms a family of* $\mathcal{A}$*-molecules if there exists a sequence* $\mathbf{a} \in \mathcal{A}$ *such that for every* $\lambda, \mu \in \Lambda$ *we have*

$$|\langle e_\mu, \pi(\lambda)g\rangle| \leqslant \mathbf{a}(\lambda - \mu). \tag{7.1}$$

In this context a set of time-frequency shifts $\{\pi(\mu)g : \mu \in \Lambda\}$ plays the role of "atoms". Moreover, if $V_g g \in W(\mathcal{A})$, then $V_g^\Lambda g \in \mathcal{A}$ and $\{\pi(\mu)g : \mu \in \Lambda\}$ is also a set of $\mathcal{A}$-molecules.

The above definition does not depend on the choice of the window g. Assume that h also generates a tight frame $\mathcal{G}(h,\Lambda)$ and that $V_h h \in W(\mathcal{A})$.Then $V_g h \in W(\mathcal{A})$ as in the proof of Lemma 4.4. Thus, $V_g^\Lambda h \in \mathcal{A}$ and therefore the sequence $\mathbf{b}(\lambda) = |\langle h, \pi(\lambda)g\rangle|$ is in $\mathcal{A}$. We insert the tight frame expansion of $\pi(\lambda)h = \sum_{\mu\in\Lambda}\langle \pi(\lambda)h, \pi(\mu)g\rangle \pi(\mu)g$ with respect to $\mathcal{G}(g,\Lambda)$ into (7.1) and obtain that

$$\begin{aligned} |\langle e_\mu, \pi(\lambda)h\rangle| &\leqslant \sum_{\nu\in\Lambda} |\langle \pi(\lambda)h, \pi(\nu)g\rangle|\,|\langle e_\mu, \pi(\nu)g\rangle| \\ &\leqslant \sum_{\nu\in\Lambda} \mathbf{b}(\nu-\lambda)\mathbf{a}(\nu-\mu) = \mathbf{b}^* * \mathbf{a}(\lambda-\mu). \end{aligned}$$

Since $\mathbf{b}^* * \mathbf{a} \in \mathcal{A}$, (7.1) is also satisfied for h in place of g.

Although time-frequency molecules are rather different from classical molecules in analysis that are defined by support and moment conditions [43, 42], they observe a similar principle. The following statement justifies the name "molecules".

Proposition 7.2. — *Assume that $V_g g \in W(\mathcal{A})$. If the pseudodifferential operator A maps the tight Gabor frame $\{\pi(\lambda)g : \lambda \in \Lambda\}$ to a set of $\mathcal{A}$-molecules, then A is bounded on $L^2(\mathbb{R}^d)$. More generally, A is bounded on every modulation space $M(\mathcal{Y})$, whenever $\mathcal{A}$ acts continuously on $\mathcal{Y}$ by convolution.*

Proof. — Let $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ be the distributional symbol corresponding to A. We check the matrix of σ^w with respect to the Gabor frame $\mathcal{G}(g,\Lambda)$ and obtain

$$|\langle \sigma^w \pi(\mu)g, \pi(\lambda)g\rangle| = |\langle e_\mu, \pi(\lambda)g\rangle| \leqslant \mathbf{a}(\lambda-\mu), \qquad \lambda,\mu \in \Lambda\,.$$

Since $\mathbf{a} \in \mathcal{A}$, the equivalence (iii) $\Leftrightarrow$(i) of Theorem 4.6 implies that the symbol σ of A belongs to the class $\widetilde{M}^{\infty,\mathcal{A}}$. Now Theorem 5.3 shows the boundedness of A on $L^2(\mathbb{R}^d)$ and on $M(\mathcal{Y})$. □

With the notion of molecules, we may rephrase Theorem 4.6 as follows.

Corollary 7.3. — *Let $\mathcal{A}$ be a solid involutive Banach algebra with respect to convolution on a lattice $\Lambda \subseteq \mathbb{R}^{2d}$ and $\{\pi(\lambda)g : \lambda \in \Lambda\}$ be a tight Gabor frame with a window g such that $V_g g \in W(\mathcal{A})$. Then $\sigma \in \widetilde{M}^{\infty,\mathcal{A}}$, if and only if σ^w maps the time-frequency atoms $\pi(\lambda)g, \lambda \in \Lambda$, into $\mathcal{A}$-molecules.*

As a further consequence we show that Weyl operators with symbols in $\widetilde{M}^{\infty,\mathcal{A}}$ are almost diagonalized with respect to $\mathcal{A}$-molecules.

Corollary 7.4. — *Let $\{e_\lambda : \lambda \in \Lambda\}$ and $\{f_\mu : \mu \in \Lambda\}$ be two families of $\mathcal{A}$-molecules. If $\sigma \in \widetilde{M}^{\infty,\mathcal{A}}$, then there exists a sequence $\tilde{\mathbf{h}} \in \mathcal{A}$ such that*

$$|\langle \sigma^w f_\mu, e_\lambda\rangle| \leqslant \tilde{\mathbf{h}}(\lambda-\mu) \quad \text{for } \lambda,\mu \in \Lambda. \tag{7.2}$$

Proof. — The argument is similar to the final implication in the proof of Theorem 4.6. Each molecule f_μ has the tight frame expansion

$$f_\mu = \sum_{\nu\in\Lambda} \langle f_\mu, \pi(\nu)g\rangle \pi(\nu)g\,.$$

with $|\langle f_\mu, \pi(\nu)g\rangle| \leqslant \mathbf{a}'(\nu-\mu)$ for some $\mathbf{a}' \in \mathcal{A}$. Likewise for the molecules e_λ. Therefore, for every $\lambda,\mu \in \Lambda$ we have, by (4.7), that

$$\begin{aligned}|\langle \sigma^w f_\mu, e_\lambda\rangle| &= \Big| \sum_{\nu,\nu'\in\Lambda} \langle f_\mu, \pi(\nu)g\rangle\, \overline{\langle e_\lambda, \pi(\nu')g\rangle}\, \langle \sigma^w \pi(\nu)g, \pi(\nu')g\rangle \Big| \\ &\leqslant \sum_{\nu,\nu'\in\Lambda} \mathbf{a}'(\nu-\mu)\mathbf{a}(\nu'-\lambda)h(\nu'-\nu) = (\mathbf{a}' * \mathbf{a}^* * \mathbf{h})(\lambda-\mu).\end{aligned}$$

Since $\mathbf{a}' * \mathbf{a}^* * \mathbf{h} \in \mathcal{A}$, we can take $\tilde{\mathbf{h}} = \mathbf{a}' * \mathbf{a}^* * \mathbf{h}$ as the dominating sequence in (7.2). □

We shall exhibit a family of $\mathcal{A}$-molecules with polynomial decay in the time-frequency plane. For this we take $\mathcal{A}$ to be the algebra $\mathcal{A} = \ell^\infty_{\langle\cdot\rangle^s}(\Lambda)$ and $s > 2d$.

Lemma 7.5. — *Let $\mathcal{A} = \ell^\infty_{\langle\cdot\rangle^s}(\Lambda)$ and $s > 2d$ and fix a tight Gabor frame window g such that $V_g g \in W(\mathcal{A})$. Now choose a set $\{z_\mu : \mu \in \Lambda\} \subset \mathbb{R}^{2d}$ satisfying $|z_\mu - \mu| \leqslant C'$ for every $\mu \in \Lambda$ and a family $\{\varphi_\mu : \mu \in \Lambda\}$ of functions such that $|V_g\varphi_\mu(z)| \leqslant C\langle z\rangle^{-s}$ uniformly in $\mu \in \Lambda$. Then the collection*

$$e_\mu = \pi(z_\mu)\varphi_\mu\,, \quad \mu \in \Lambda$$

forms a family of $\mathcal{A}$-molecules.

Proof. — For $\lambda,\mu \in \Lambda$, the assumptions $|V_g\varphi_\mu(z)| \leqslant C\langle z\rangle^{-s}$ and $|z_\mu - \mu| \leqslant C'$ imply that

$$\begin{aligned}|\langle e_\mu, \pi(\lambda)g\rangle| &= |\langle \pi(z_\mu)\varphi_\mu, \pi(\lambda)g\rangle| = |V_g\varphi_\mu(\lambda - z_\mu)| \\ &\leqslant C\langle \lambda - z_\mu\rangle^{-s} \leqslant C''\langle\lambda-\mu\rangle^{-s},\end{aligned}$$

thus the assertion follows. □

As another application we show how pseudodifferential operators are almost diagonal with respect to local Fourier bases. To be specific, we consider the local sine bases of the form

$$\psi_{k,l}(t) = \sqrt{\frac{2}{\alpha}}\, b(t-\alpha k) \sin\Big(\frac{(2l+1)\pi}{2\alpha}(t-\alpha k)\Big).$$

The "bell function" b can be constructed to be real-valued and in $C^N(\mathbb{R})$ for given smoothness $N \in \mathbb{N}$, such that the system $\{\psi_{k,l} : k \in \mathbb{Z}, l = 0,1,2,\dots\}$ forms an orthonormal basis of $L^2(\mathbb{R})$ [1, 32]. The parametrization of all bell function starts from a function $\zeta \in C^{N-1}(\mathbb{R})$ that is real, even, with support in $[-\epsilon,\epsilon]$ for $\epsilon \in (0,\alpha/2)$ and satisfies $\int_{\mathbb{R}} \zeta(s) ds = \frac{\pi}{2}$. Now set $\theta(t) = \int_{-\infty}^t \zeta(s)\, ds$, then the bell function b is given by

$$b(t) = \sin(\theta(t)) \cos(\theta(t-\alpha)).$$

See [1, 32] for the details of this construction.

From this definition it follows that b is a real-valued, compactly supported function in $C^N(\mathbb{R})$ that satisfies

$$|V_b b(x,\xi)| \leqslant C\langle\xi\rangle^{-N}\chi_{[-K,K]}(x) \leqslant C\langle(x,\xi)\rangle^{-N}$$

for some constants $C, K > 0$ (see, e.g., [31] for a derivation of this estimate). In particular, b satisfies the standard assumption $V_b b \in W(\mathcal{A})$ with respect to the algebra $\mathcal{A} = \ell^\infty_{\langle\cdot\rangle^N}(\Lambda)$ and the lattice $\Lambda = \alpha\mathbb{Z} \times \frac{1}{2\alpha}\mathbb{Z}$.

By splitting the sine into exponentials and some algebraic manipulations, we may write the basis functions as

$$\psi_{k,l} = \frac{(-1)^{kl}}{2i}\sqrt{\frac{2}{\alpha}}\Big(\pi\Big(\alpha k, \frac{l}{2\alpha}\Big) b_+ - \pi\Big(\alpha k, -\frac{l}{2\alpha}\Big) b_-\Big), \tag{7.3}$$

where $b_\pm(t) = e^{\pm\pi i t/(2\alpha)} b(t)$. Since $|V_b b(z)| = \mathcal{O}(\langle z\rangle^{-N})$, we also have $|V_{b_\pm} b_\pm(z)| = \mathcal{O}(\langle z\rangle^{-N})$ for all choices of signs. Thus $V_{b_\pm} b_\pm \in W(\mathcal{A})$. Since the lattice Λ admits a tight Gabor frame window $g \in \mathcal{S}(\mathbb{R})$, we get that $V_g g \in W(\mathcal{A})$ and, therefore, $V_g b_\pm \in W(\mathcal{A})$. This allows us to apply Lemma 7.5 to conclude that $\{\pi(\lambda) b_\pm : \lambda \in \Lambda\}$ is a family of $\mathcal{A}$-molecules. By using the decomposition (7.3) and Corollary 7.4, we obtain the following form of almost diagonalization with respect to a local sine basis.

THEOREM 7.6. — *Assume that $b \in W(\ell^\infty_{\langle\cdot\rangle^N})$ and $\sigma \in M^{\infty,\infty}_{1\otimes\langle\cdot\rangle^s}(\mathbb{R}^2)$ for $N > s > 2$. Then there is a constant C_s such that*

$$|\langle\sigma^w \psi_{k',l'}, \psi_{k,l}\rangle| \leqslant C_s \Big(\Big\langle\Big(\alpha(k-k'), \frac{1}{2\alpha}(l-l')\Big)\Big\rangle^{-s} + \Big\langle\Big(\alpha(k-k'), \frac{1}{2\alpha}(l+l')\Big)\Big\rangle^{-s}\Big)$$

for $k, k' \in \mathbb{Z}$ and $l, l' \in \mathbb{N}\cup\{0\}$.

This is reminiscent of a result by Rochberg and Tachizawa in [37]. The same type of almost diagonalization holds for the other types of local Fourier bases (where the sine is replaced by a pattern of sines and cosines).

8. Appendix

In this appendix we sketch the proofs of Theorems 3.1 and 3.5. By a basis change for the lattice Λ, we may assume without loss of generality that $\Lambda = \mathbb{Z}^d$.

First we show that $\ell^1(\Lambda)$ is the *maximal solid involutive convolution algebra over* Λ (Theorem 3.1). We prove Theorem 3.1 by a sequence of elementary lemmas. Their common assumption is that $\mathcal{A}$ is a solid involutive Banach algebra with respect to convolution over $\mathbb{Z}^d$.

Lemma 8.1. — *$\mathcal{A}$ is continuously embedded into $\ell^2(\mathbb{Z}^d)$.*

Proof. — If $\mathbf{a} \in \mathcal{A}$, then $\mathbf{a}^* * \mathbf{a} \in \mathcal{A}$ and by solidity $(\mathbf{a}^* * \mathbf{a})(0)\delta_0 \in \mathcal{A}$. On the one hand, we have

$$(\mathbf{a}^* * \mathbf{a})(0) = \sum_{k\in\mathbb{Z}} \mathbf{a}^*(-k)\mathbf{a}(k) = \sum_{k\in\mathbb{Z}} \overline{\mathbf{a}(k)}\mathbf{a}(k) = \|\mathbf{a}\|_2^2,$$

on the other hand

$$\|\mathbf{a}\|_2^2\|\delta_0\|_{\mathcal{A}} = \|\mathbf{a}^* * \mathbf{a}(0)\delta_0\|_{\mathcal{A}} \leqslant \|\mathbf{a}^* * \mathbf{a}\|_{\mathcal{A}} \leqslant C\|\mathbf{a}\|_{\mathcal{A}}^2\,,$$

and thus $\mathcal{A}$ is continuously embedded in $\ell^2(\mathbb{Z}^d)$. □

Let $\mathcal{B} = \{\mathbf{a} \in \ell^2(\mathbb{Z}^d) : \mathcal{F}\mathbf{a} \in L^\infty(\mathbb{T}^d)\}$ with the norm $\|\mathbf{a}\|_{\mathcal{B}} = \|\mathcal{F}\mathbf{a}\|_\infty$, where the Fourier transform is given by

$$\mathcal{F}\mathbf{a}(\xi) = \sum_{k\in\mathbb{Z}^d} \mathbf{a}(k)e^{-2\pi ik\cdot\xi}.$$

Then $\mathcal{B}$ is a Banach algebra.

Lemma 8.2. — *$\mathcal{A}$ is continuously embedded in $\mathcal{B}$ and $\|\mathbf{a}\|_{\mathcal{B}} \leqslant \|\mathbf{a}\|_{\mathcal{A}}$ for $\mathbf{a} \in \mathcal{A}$.*

Proof. — Take an arbitrary $\mathbf{a} \in \mathcal{A}$. Denote the n-fold convolution $\mathbf{a} * \mathbf{a} * \cdots * \mathbf{a}$ by $\mathbf{a}^{*n}$. From Lemma 8.1 it follows that

$$\|(\mathcal{F}\mathbf{a})^n\|_2 = \|\mathbf{a}^{*n}\|_2 \leqslant C\|\mathbf{a}^{*n}\|_{\mathcal{A}} \leqslant C\|\mathbf{a}\|_{\mathcal{A}}^n,$$

for some constant C. Thus, $\|(\mathcal{F}\mathbf{a})^n\|_2^{1/n} \leqslant C^{1/n}\|\mathbf{a}\|_{\mathcal{A}}$ and by taking the limit as $n \to \infty$ we obtain the desired inequality $\|\mathcal{F}\mathbf{a}\|_\infty = \|\mathbf{a}\|_{\mathcal{B}} \leqslant \|\mathbf{a}\|_{\mathcal{A}}$. □

LEMMA 8.3. — *If* $\mathbf{b} \in \mathcal{B}$ *and* $|\mathbf{a}| \leqslant \mathbf{b}$*, then* $\mathbf{a} \in \mathcal{B}$ *and* $\|\mathbf{a}\|_{\mathcal{B}} \leqslant \|\mathbf{b}\|_{\mathcal{B}}$.

Proof. — Let $C_{\mathbf{c}}$ be the convolution operator given by $C_{\mathbf{c}}\mathbf{h} = \mathbf{h}*\mathbf{c}$ acting on $\mathbf{h} \in \ell^2(\mathbb{Z}^d)$. Then $C_{\mathbf{c}}$ is bounded on $\ell^2(\mathbb{Z}^d)$ if and only if $\mathcal{F}\mathbf{c} \in L^\infty(\mathbb{T}^d)$, and the operator norm on $\ell^2(\mathbb{Z}^d)$ is precisely $\|C_{\mathbf{c}}\|_{\mathrm{op}} = \|\mathcal{F}\mathbf{c}\|_\infty = \|\mathbf{c}\|_{\mathcal{B}}$. If $|\mathbf{a}| \leqslant \mathbf{b}$, then we have

$$|C_{\mathbf{a}}\mathbf{h}| \leqslant |\mathbf{h}| * |\mathbf{a}| \leqslant |\mathbf{h}| * \mathbf{b} = C_{\mathbf{b}}|\mathbf{h}| \qquad \text{for all } \mathbf{h} \in \ell^2(\mathbb{Z}^d)\,,$$

and so

$$\|C_{\mathbf{a}}\mathbf{h}\|_2 \leqslant \|C_{\mathbf{b}}|\mathbf{h}|\|_2 \leqslant \|C_{\mathbf{b}}\|_{\mathrm{op}}\|\mathbf{h}\|_2.$$

Therefore, $\|C_{\mathbf{a}}\|_{\mathrm{op}} \leqslant \|C_{\mathbf{b}}\|_{\mathrm{op}}$, that is, $\|\mathbf{a}\|_{\mathcal{B}} \leqslant \|\mathbf{b}\|_{\mathcal{B}}$. □

LEMMA 8.4. — *Let* $\mathbf{a}$ *be a sequence on* $\mathbb{Z}^d$ *such that* $\mathcal{F}|\mathbf{a}|$ *is well defined. Then*

$$\|\mathbf{a}\|_1 = \|\mathcal{F}|\mathbf{a}|\|_\infty. \tag{8.1}$$

Proof. — Assume first that $\mathbf{a} \in \ell^1(\mathbb{Z}^d)$. In this case $\mathcal{F}|\mathbf{a}|$ is continuous on $\mathbb{T}^d$, and therefore

$$\|\mathcal{F}|\mathbf{a}|\|_\infty \geqslant \mathcal{F}|\mathbf{a}|(0) = \|\mathbf{a}\|_1.$$

Since we always have $\|\mathcal{F}|\mathbf{a}|\|_\infty \leqslant \|\mathbf{a}\|_1$, the equality (8.1) holds.

Next assume that $\mathcal{F}|\mathbf{a}| \in L^\infty(\mathbb{T}^d)$. We show that $\mathbf{a} \in \ell^1(\mathbb{Z}^d)$, then (8.1) follows. Consider the truncation of $|\mathbf{a}|$, $\mathbf{a}_N = \mathbf{1}_{[-N,N]^d}|\mathbf{a}|$, where $\mathbf{1}_{[-N,N]^d}$ is the characteristic function of the cube $[-N,N]^d \subset \mathbb{Z}^d$ for $N \in \mathbb{N}$. Since $|\mathbf{a}_N| \leqslant |\mathbf{a}|$, Lemma 8.3 yields that $\|\mathcal{F}\mathbf{a}_N\|_\infty \leqslant \|\mathcal{F}|\mathbf{a}|\|_\infty$ for all $N \in \mathbb{N}$. As $\mathbf{a}_N \in \ell^1(\mathbb{Z}^d)$, we obtain that

$$\sum_{n\in[-N,N|^d} |\mathbf{a}(n)| = \|\mathbf{a}_N\|_1 = \|\mathcal{F}|\mathbf{a}_N|\|_\infty = \|\mathcal{F}\mathbf{a}_N\|_\infty \leqslant \|\mathcal{F}|\mathbf{a}|\|_\infty$$

for all $N \in \mathbb{N}$. This shows that $\mathbf{a} \in \ell^1(\mathbb{Z}^d)$. □

We now prove Theorem 3.1 showing the maximality of $\ell^1(\mathbb{Z}^d)$ in the class of solid involutive convolution Banach algebras over $\mathbb{Z}^d$.

THEOREM 8.5. — *Assume that* $\mathcal{A}$ *is a solid involutive Banach algebra of sequences over* $\mathbb{Z}^d$*. Then* $\mathcal{A} \subseteq \ell^1(\mathbb{Z}^d)$ *and* $\|\mathbf{a}\|_1 \leqslant \|\mathbf{a}\|_{\mathcal{A}}$ *for* $\mathbf{a} \in \mathcal{A}$.

Proof. — Let $\mathbf{a} \in \mathcal{A}$. Since $\mathcal{A}$ is solid, both $\mathbf{a}$ and $|\mathbf{a}|$ have the same norm in $\mathcal{A}$. Thus Lemmas 8.1 and 8.2 imply that

$$\|\mathbf{a}\|_{\mathcal{A}} = \||\mathbf{a}|\|_{\mathcal{A}} \geqslant \||\mathbf{a}|\|_{\mathcal{B}} = \|\mathcal{F}|\mathbf{a}|\|_\infty\,,$$

and by Lemma 8.4, $\|\mathcal{F}|\mathbf{a}|\|_\infty = \|\mathbf{a}\|_1$. This shows that $\|\mathbf{a}\|_{\mathcal{A}} \geqslant \|\mathbf{a}\|_1$ and thus $\mathcal{A}$ is embedded in $\ell^1(\mathbb{Z}^d)$. □

We next turn to the proof of Theorem 3.5. Let us recall the basic facts about the Gelfand theory of commutative Banach algebras.

(a) The convolution operator $C_{\mathbf{a}}$ defined by $C_{\mathbf{a}}\mathbf{c} = \mathbf{a}*\mathbf{c}$ for $\mathbf{a} \in \ell^1(\mathbb{Z}^d)$ is invertible, if and only if the Fourier series $\widehat{\mathbf{a}}(\xi) = \sum_{k\in\mathbb{Z}^d} \mathbf{a}(k)e^{2\pi i k\cdot\xi}$ does not vanish at any $\xi \in \mathbb{T}^d$.
(b) The Gelfand transform of $\mathbf{a} \in \ell^1(\mathbb{Z}^d)$ coincides with the Fourier series $\widehat{\mathbf{a}}(\xi)$ for $\xi \in \mathbb{T}^d$.
(c) An element a in a commutative Banach algebra is invertible, if and only if its Gelfand transform does not vanish at any point.
(d) By Theorem 3.1, we have that $\mathcal{A} \subseteq \ell^1(\mathbb{Z}^d)$, therefore $\widehat{\ell^1} \simeq \mathbb{T}^d \subseteq \widehat{\mathcal{A}}$, and the Gelfand transform of $\mathbf{a} \in \mathcal{A}$ restricted to $\mathbb{T}^d$ is just the Fourier series $\widehat{\mathbf{a}}$ of $\mathbf{a}$.

Proof of Theorem 3.5 — First Part. The equivalence of (i) and (iii) follows from the Gelfand theory for commutative Banach algebras:

(iii) $\Rightarrow$ (i). If $\widehat{\mathcal{A}} \simeq \mathbb{T}^d$, then the convolution operator $C_{\mathbf{a}}$ is invertible if and only if $\widehat{\mathbf{a}}(\xi) \neq 0$ for all $\xi \in \mathbb{T}^d$ by (a), if and only if $\mathbf{a}$ is invertible in $\mathcal{A}$ by (b) – (d).

(i) $\Rightarrow$ (iii). If $\widehat{\mathcal{A}} \not\simeq \mathbb{T}^d$, then the invertibility criteria for $C_{\mathbf{a}}$ in $\mathcal{B}(\ell^2)$ and for $\mathbf{a}$ in $\mathcal{A}$ differ, and $\mathcal{A}$ cannot be inverse-closed in $\mathcal{B}(\ell^2)$.

(ii) $\Rightarrow$ (i). The convolution operator $C_{\mathbf{a}}$ has the matrix A with entries $A_{kl} = \mathbf{a}(k-l)$. Consequently $\mathbf{d}_A(l) = \sup_k |A_{k,k-l}| = |\mathbf{a}(l)|$ and $\|A\|_{\mathcal{C}_{\mathcal{A}}} = \|\mathbf{d}_A\|_{\mathcal{A}} = \|\mathbf{a}\|_{\mathcal{A}}$. Thus $A \in \mathcal{C}_{\mathcal{A}}$ if and only if $\mathbf{a} \in \mathcal{A}$.

If $\mathcal{A}$ is not inverse-closed in $\mathcal{B}(\ell^2)$, then there exists an $\mathbf{a} \in \mathcal{A}$, such that $C_{\mathbf{a}}$ (with matrix B) is invertible on $\ell^2(\mathbb{Z}^d)$ with inverse $C_{\mathbf{b}}$, but $\mathbf{b} \notin \mathcal{A}$. This means that B cannot be in $\mathcal{C}_{\mathcal{A}}$ and so $\mathcal{C}_{\mathcal{A}}$ is not inverse-closed in $\mathcal{B}(\ell^2)$.

(i) $\Rightarrow$ (iv). Since $\mathcal{A}$ is inverse-closed in $\mathcal{B}(\ell^2)$, the spectrum of the convolution operator $C_{\mathbf{a}}$ acting on ℓ^2 coincides with the spectrum of $\mathbf{a}$ in the algebra $\mathcal{A}$, $\mathrm{Sp}_{\mathcal{A}}(\mathbf{a}) = \mathrm{Sp}_{\mathcal{B}(\ell^2)}(C_{\mathbf{a}})$. In particular, the spectral radii of the particular elements δ_λ and of C_{δ_λ} coincide. On the one hand we have

$$r_{\mathcal{A}}(\delta_\lambda) = \lim_{n\to\infty} \|\delta_\lambda^n\|_{\mathcal{A}}^{1/n} = \lim_{n\to\infty} \|\delta_{n\lambda}\|_{\mathcal{A}}^{1/n} = \lim_{n\to\infty} \omega(n\lambda)^{1/n} .$$

On the other hand, the convolution operator C_{δ_λ} on ℓ^2 is simply translation by λ and is unitary. Therefore the spectral radius of C_{δ_λ} is $r_{\mathcal{B}(\ell^2)}(C_{\delta_\lambda}) = 1$. By inverse-closedness we obtain the GRS-condition $\lim_{n\to\infty} \omega(n\lambda)^{1/n} = 1$ for all $\lambda \in \Lambda$.

(iv) $\Rightarrow$ (iii). Since $\mathbf{a} \in \mathcal{A}$ has the expansion $\mathbf{a} = \sum_\lambda \mathbf{a}(\lambda)\delta_\lambda$, we have $\|\mathbf{a}\|_{\mathcal{A}} \leqslant \sum_\lambda |\mathbf{a}_\lambda| \|\delta_\lambda\|_{\mathcal{A}} = \|\mathbf{a}\|_{\ell^1_\omega}$. Thus ℓ^1_ω is continuously embedded in $\mathcal{A}$, and by Theorem 3.1 $\mathcal{A} \subseteq \ell^1$. By a theorem of Gelfand-Naimark-Raikov [23] the spectrum of ℓ^1_ω is isomorphic to $\mathbb{T}^d$ if and only if ω satisfies the GRS-condition. If (iv) holds, then we obtain the embeddings $\mathbb{T}^d \simeq \widehat{\ell^1_\omega} \subseteq \widehat{\mathcal{A}} \subseteq \widehat{\ell^1} \simeq \mathbb{T}^d$. Thus $\mathcal{A} \simeq \mathbb{T}^d$ as claimed. □

The substance of the theorem lies in the implication (iii) $\Rightarrow$ (ii). To prove this non-trivial implication, we study the Fourier series associated to an infinite matrix $A = (a_{kl})_{k,l\in\mathbb{Z}^d}$ [13, 3].

Let $D_A(n)$ be the n-th diagonal of A, i.e., the matrix with entries

$$D_A(n)_{kl} \quad = \quad \begin{cases} a_{k,k-n} & \text{if } l = k-n \\ 0 & \text{else} \end{cases}$$

for $k,l,n \in \mathbb{Z}^d$. Furthermore, define the "modulation" $M_t, t \in \mathbb{T}^d$, acting on a sequence $\mathbf{c} = (\mathbf{c}(k))_{k\in\mathbb{Z}^d}$ by

$$\big(M(t)\mathbf{c}\big)(k) = e^{2\pi i k\cdot t}\mathbf{c}(k)\,. \tag{8.2}$$

Finally to every matrix A we associate the matrix-valued function

$$\mathbf{f}(t) = M_t A M_{-t} \qquad t \in \mathbb{T}^d\,. \tag{8.3}$$

Clearly each $M(t)$ is unitary on $\ell^2(\mathbb{Z}^d)$ and $M_{t+k} = M_t$ for all $k \in \mathbb{Z}^d$. Consequently $\mathbf{f}(t)$ is $\mathbb{Z}^d$-periodic and $\mathbf{f}(0) = A$. Furthermore, A is invertible on $\ell^2(\mathbb{Z}^d)$ if and only if $\mathbf{f}(t)$ is invertible for all $t \in \mathbb{T}^d$.

It is natural to study the Fourier coefficients and the Fourier series of the matrix-valued function $\mathbf{f}(t)$. It has the following properties [3].

Lemma 8.6. —

(i) $\mathbf{f}(t)_{kl} = a_{kl}e^{2\pi i(k-l)\cdot t}$ *for* $k,l \in \mathbb{Z}^d$, $t \in \mathbb{T}^d$.

(ii) *The matrix-valued Fourier coefficients of* $\mathbf{f}(t)$ *are given by*

$$\widehat{\mathbf{f}}(n) = \int_{[0,1]^d} \mathbf{f}(t)e^{-2\pi i n\cdot t}\,dt = D_A(n) \tag{8.4}$$

(with the appropriate interpretation of the integral).

(iii) *Let* $\mathcal{A}(\mathbb{T}^d, \mathcal{B}(\ell^2))$ *be the space of all matrix-valued Fourier expansions* $\mathbf{g}$ *that are given by* $\mathbf{g}(t) = \sum_{n\in\mathbb{Z}^d} B_n\, e^{2\pi i n\cdot t}$ *with* $B_n \in \mathcal{B}(\ell^2)$ *and* $\big(\|B_n\|_{op}\big)_{n\in\mathbb{Z}^d} \in \mathcal{A}$. *Then*

$$A \in \mathcal{C}_\mathcal{A} \quad \Longleftrightarrow \quad \mathbf{f}(t) \in \mathcal{A}(\mathbb{T}^d, \mathcal{B}(\ell^2))\,. \tag{8.5}$$

Proof. — (i) follows from a simple calculation. For (ii) we interpret the integral entrywise and find that

$$\begin{aligned} \widehat{\mathbf{f}}(n)_{kl} \quad &= \quad \int_{[0,1]^d} \mathbf{f}(t)_{kl}\, e^{-2\pi i n\cdot t}\,dt \\ &= \quad a_{kl}\int_{[0,1]^d} e^{-2\pi i(n+l-k)\cdot t}\,dt = a_{k,l}\delta_{n+l-k}\,, \end{aligned}$$

and so $\widehat{\mathbf{f}}(n) = D_A(n)$ is the n-th side diagonal, as claimed.

(iii) follows from the definition of $\mathcal{C}_{\mathcal{A}}$. According to (ii), the n-th Fourier coefficient of $\mathbf{f}(t) = M_t A M_{-t}$ is just the n-th diagonal $D_A(n)$ of A and $\|D_A(n)\|_{op} = \mathbf{d}_A(n)$. So if $A \in \mathcal{C}_{\mathcal{A}}$, then $\mathbf{f}(t) = M_t A M_{-t}$ is in $\mathcal{A}(\mathbb{T}^d, \mathcal{B}(\ell^2))$. Conversely, if $\mathbf{f}(t) = M_t A M_{-t} \in \mathcal{A}(\mathbb{T}^d, \mathcal{B}(\ell^2))$, then $(\mathbf{d}_A(n))_{n\in\mathbb{Z}^d} = \big(\|\hat{A}(n)\|_{op}\big)_{n\in\mathbb{Z}^d} \in \mathcal{A}$ and thus $A \in \mathcal{C}_{\mathcal{A}}$. □

To prove the non-trivial implication of Theorem 3.5, we need Wiener's Lemma for matrix-valued functions.

Baskakov's proof makes use of the Bochner-Phillips version of Wiener's Lemma for absolutely convergent Fourier series with coefficient in a Banach algebra [6]. The proof exploits the relation between the ideal theory and the representation theory of a Banach algebra, and the description of invertibility by means of ideals.

Let $\mathcal{A}$ be a Banach algebra with an identity and $\mathcal{M} \subseteq \mathcal{A}$ be a closed left ideal, i.e., $\mathcal{A}\mathcal{M} \subseteq \mathcal{M}$. Then $\mathcal{A}$ acts on the Banach space $\mathcal{A}/\mathcal{M}$ by the left regular representation

$$\pi_{\mathcal{M}}(\mathbf{a})\tilde{x} = \widetilde{\mathbf{a}x} \qquad \text{for } \mathbf{a} \in \mathcal{A}, \tilde{x} \in \mathcal{A}/\mathcal{M}\,, \tag{8.6}$$

where $\tilde{x}$ is the equivalence class of x in $\mathcal{A}/\mathcal{M}$.

The following lemmata are standard and can be found in any textbook on Banach algebras, see e.g., [36, 7].

Lemma 8.7. — *If $\mathcal{M}$ is a* maximal *left ideal of $\mathcal{A}$, then $\pi_{\mathcal{M}}$ is algebraically irreducible. This means that the algebraically generated subspace $\{\pi(\mathbf{a})\tilde{x} : \mathbf{a} \in \mathcal{A}\}$ coincides with $\mathcal{A}/\mathcal{M}$ for* all $\tilde{x} \neq 0$.

Lemma 8.8. — *Let $\mathcal{A}$ be a Banach algebra with identity. An element $\mathbf{a} \in \mathcal{A}$ is left-invertible (right-invertible), if and only if $\pi_{\mathcal{M}}(\mathbf{a})$ is invertible for every maximal left (right) ideal $\mathcal{M} \subseteq \mathcal{A}$.*

Lemma 8.9 (Schur's Lemma for Banach Space Representations). — *Assume that $\pi : \mathcal{A} \to \mathcal{B}(X)$ is an algebraically irreducible representation of $\mathcal{A}$ on a Banach space X. If $T \in \mathcal{B}(X)$ and $T\pi(\mathbf{a}) = \pi(\mathbf{a})T$ for all $\mathbf{a} \in \mathcal{A}$, then T is a multiple of the identity operator I_X on X.*

Proof of Theorem 3.5 - Second Part. — **(iii) ⇒ (ii).** Assume that $A \in \mathcal{C}_{\mathcal{A}}$ is invertible in $\mathcal{B}(\ell^2)$. Then the associated function $\mathbf{f}$ defined by $\mathbf{f}(t) = M_t A M_{-t}$ possesses a $\mathcal{B}(\ell^2)$-valued Fourier series

$$\mathbf{f}(t) = \sum_{n\in\mathbb{Z}^d} D_A(n) e^{2\pi i n\cdot t}\,, \tag{8.7}$$

where $D_A(n)$ is the n-th side diagonal of A and the sequence $\mathbf{d}_A(n) = \|D_A(n)\|_{op}$ is in $\mathcal{A}$.

We identify the commutative algebra $\mathcal{A}$ with a subalgebra of $\mathcal{A}(\mathbb{T}^d, \mathcal{B}(\ell^2))$ via the embedding $j : \mathcal{A} \to \mathcal{A}(\mathbb{T}^d, \mathcal{B}(\ell^2))$

$$j(\mathbf{a})(t) = \sum_{n\in\mathbb{Z}^d} \mathbf{a}(n)\, e^{2\pi i n\cdot t}\mathrm{I} = \widehat{\mathbf{a}}(t)\mathrm{I}\,. \tag{8.8}$$

Clearly, since $j(\mathbf{a})$ is a multiple of the identity operator I, $j(\mathbf{a})$ commutes with every $T \in \mathcal{A}(\mathbb{T}^d, \mathcal{B}(\ell^2))$.

Now let $\mathcal{M}$ be a maximal left ideal of $\mathcal{A}(\mathbb{T}^d, \mathcal{B}(\ell^2))$ and $\pi_{\mathcal{M}}$ be the corresponding representation. Since every $j(\mathbf{a})$ commutes with every $T \in \mathcal{A}(\mathbb{T}^d, \mathcal{B}(\ell^2))$, we find that

$$\pi_{\mathcal{M}}(T)\pi_{\mathcal{M}}(j(\mathbf{a})) = \pi_{\mathcal{M}}(j(\mathbf{a}))\pi_{\mathcal{M}}(T) \qquad \forall T \in \mathcal{A}(\mathbb{T}^d, \mathcal{B}(\ell^2)), \mathbf{a} \in \mathcal{A}\,.$$

As a consequence of Lemma 8.9 on the algebraic irreducibility of $\pi_{\mathcal{M}}$, $\pi_{\mathcal{M}}(j(\mathbf{a}))$ must be a multiple of the identity, and since $\pi_{\mathcal{M}}$ is a homomorphism, there exists a multiplicative linear functional $\chi \in \widehat{\mathcal{A}}$, such that $\pi_{\mathcal{M}}(j(\mathbf{a})) = \chi(\mathbf{a})\,\mathrm{I}$.

Here comes in the crucial hypothesis that $\widehat{\mathcal{A}} \simeq \mathbb{T}^d$. This hypothesis says that there exists a $t_0 \in \mathbb{T}^d$ such that $\chi(\mathbf{a}) = \widehat{\mathbf{a}}(t_0)$. Consequently,

$$\pi_{\mathcal{M}}(j(\mathbf{a})) = \widehat{\mathbf{a}}(t_0)\,\mathrm{I} \qquad \text{for all } \mathbf{a} \in \mathcal{A}\,.$$

Let δ_n be the standard basis of $\ell^1(\mathbb{Z}^d)$. Since $\mathcal{A}$ is solid, $\delta_n \in \mathcal{A}$ and we have $j(\delta_n)(t) = e^{2\pi i n\cdot t}\mathrm{I}$. Thus, the function $\mathbf{f}$ given in (8.7) can be written as $\mathbf{f} = \sum_{n\in\mathbb{Z}^d} D_A(n) j(\delta_n)$. Consequently we have

$$\begin{aligned}
\pi_{\mathcal{M}}(\mathbf{f}) &= \pi_{\mathcal{M}}\Big(\sum_{n\in\mathbb{Z}^d} D_A(n) j(\delta_n)\Big) \\
&= \sum_{n\in\mathbb{Z}^d} \pi_{\mathcal{M}}(D_A(n))\pi_{\mathcal{M}}(j(\delta_n)) \\
&= \sum_{n\in\mathbb{Z}^d} \pi_{\mathcal{M}}(D_A(n)) e^{2\pi i n\cdot t_0}\mathrm{I} \\
&= \pi_{\mathcal{M}}\Big(\sum_{n\in\mathbb{Z}^d} D_A(n) e^{2\pi i n\cdot t_0}\Big) \\
&= \pi_{\mathcal{M}}(\mathbf{f}(t_0))\,,
\end{aligned} \tag{8.9}$$

If $A \in \mathcal{C}_{\mathcal{A}}$ is invertible in $\mathcal{B}(\ell^2)$, then clearly $\mathbf{f}(t) = M_t A M_{-t}$ is invertible in $\mathcal{B}(\ell^2)$ for every $t \in \mathbb{T}^d$ and consequently $\pi_{\mathcal{M}}(\mathbf{f}(t_0))$ is (left-) invertible for every maximal left ideal in $\mathcal{A}(\mathbb{T}^d, \mathcal{B}(\ell^2))$. By (8.9) we find that $\pi_{\mathcal{M}}(f)$ is invertible for every maximal left ideal in $\mathcal{A}(\mathbb{T}^d, \mathcal{B}(\ell^2))$. Thus, by Lemma 8.8,

$\mathbf{f}(t)$ is invertible in the algebra $\mathcal{A}(\mathbb{T}^d, \mathcal{B}(\ell^2))$. This means that $\mathbf{f}(t)^{-1}$ possesses a Fourier series

$$\mathbf{f}(t)^{-1} = M_t A^{-1} M_{-t} = \sum_{n\in\mathbb{Z}^d} B_n \, e^{2\pi i n\cdot t}$$

with $(\|B_n\|_{op})_{n\in\mathbb{Z}^d} \in \mathcal{A}$. Since by Lemma 8.6(b) the coefficients B_n are exactly the side diagonals of A^{-1}, Lemma 8.6(c) implies that $A^{-1} \in \mathcal{C}_{\mathcal{A}}$. This finishes the proof of Theorem 3.5. □

Remark 8.10. — Clearly Theorem 3.5 also works for arbitrary discrete abelian groups as index sets.

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