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MODULI SPACES OF PU(2)-INSTANTONS ON MINIMAL CLASS VII SURFACES WITH $b_2 = 1$

by Konrad SCHÖBEL

Abstract. — We describe explicitly the moduli spaces $\mathscr{M}_g^{\mathrm{pst}}(S, E)$ of polystable holomorphic structures $\mathcal{E}$ with $\det \mathcal{E} \cong \mathcal{K}$ on a rank two vector bundle E with $c_1(E) = c_1(K)$ and $c_2(E) = 0$ for all minimal class VII surfaces S with $b_2(S) = 1$ and with respect to all possible Gauduchon metrics g. These surfaces S are non-elliptic and non-Kähler complex surfaces and have recently been completely classified. When S is a half or parabolic Inoue surface, $\mathscr{M}_g^{\mathrm{pst}}(S, E)$ is always a compact one-dimensional complex disc. When S is an Enoki surface, one obtains a complex disc with finitely many transverse self-intersections whose number becomes arbitrarily large when g varies in the space of Gauduchon metrics. $\mathscr{M}_g^{\mathrm{pst}}(S, E)$ can be identified with a moduli space of PU(2)-instantons. The moduli spaces of simple bundles of the above type lead to interesting examples of non-Hausdorff singular one-dimensional complex spaces.

Résumé. — Nous décrirons explicitement les espaces de modules $\mathscr{M}_g^{\mathrm{pst}}(S, E)$ de structures holomorphes polystables $\mathcal{E}$ avec $\det \mathcal{E} \cong \mathcal{K}$ sur un fibré vectoriel E de rang deux avec $c_1(E) = c_1(K)$ et $c_2(E) = 0$ pour toutes les surfaces S minimales de la classe VII avec $b_2(S) = 1$ et par rapport à toutes les métriques de Gauduchon g. Ces surfaces S sont des surfaces complexes non-elliptiques et non-Kählériennes et ont récemment été complètement classifiées. Si S est une demi-surface d'Inoue ou une surface d'Inoue parabolique, $\mathscr{M}_g^{\mathrm{pst}}(S, E)$ est toujours un disque complexe compact de dimension un. Si S est une surface d'Enoki, on obtient un disque complexe avec un nombre fini d'auto-intersections transverses, arbitrairement grand quand g varie dans l'espace des métriques de Gauduchon. $\mathscr{M}_g^{\mathrm{pst}}(S, E)$ peut être identifié à un espace de modules de PU(2)-instantons. Les espaces de modules de fibrés simples du type considéré mènent à des exemples intéressants d'espaces complexes singuliers non-Hausdorff de dimension un.

1. Introduction

In gauge theory, moduli spaces of anti-self-dual connections have led to striking results in differential four-manifold geometry; they are the main

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tool in the construction of the Donaldson polynomial invariants. However, the explicit computation of these moduli spaces in concrete situations is in general very difficult. Nevertheless, when the base manifold is a complex surface, the Kobayashi-Hitchin correspondence establishes a real analytic isomorphism between the moduli spaces of (irreducible) anti-self-dual connections and polystable (stable) holomorphic structures on a fixed differentiable vector bundle and makes thus possible the application of complex geometric methods for the computation of gauge-theoretical moduli spaces. S. K. Donaldson gave the first complete proof of this relationship on algebraic surfaces and used it to explicitly compute moduli spaces and the corresponding invariants for Dolgachev surfaces. This led to the first example of pairs of homeomorphic but not diffeomorphic four-manifolds [10].

Subsequently this strategy was carried out for a large variety of algebraic surfaces [32, 5, 15, 26, 33, 17, 9, 16]. However, it becomes very hard for non-algebraic surfaces due to the presence of non-filtrable holomorphic bundles in the moduli space. A complete classification of such bundles is considered to be an extremely difficult problem on non-elliptic surfaces because of the lack of a general method of construction and parametrisation. It could only be solved for a number of non-Kählerian elliptic surfaces [3, 29, 35, 40, 30, 4]. Note that for elliptic fibrations one solves this problem by regarding the restrictions to the fibres which (generically) are elliptic curves on which the classification of holomorphic bundles is well understood. This strategy is called the graph method and was used by P. J. Braam and J. Hurtubise to obtain the first explicit example of an SU(2)-instanton moduli space on a non-Kähler surface, namely an elliptic Hopf surface [3]. In this article we now compute moduli spaces of holomorphic bundles on *all* minimal class VII surfaces with $b_2 = 1$, endowed with *all* possible Gauduchon metrics. Being the first example of moduli spaces on surfaces that are both non-Kähler and non-elliptic, this is the reason why one expects essentially new phenomena for the behaviour of moduli spaces in general.

Our method to overcome the main difficulty of controlling non-filtrable bundles is the following: We first classify filtrable bundles and then show, using gauge-theory, that only a particular non-filtrable bundle can exist. In [36] it was shown that the moduli space does not contain a compact component consisting of both filtrable and non-filtrable bundles. We then show that the moduli space does not contain any compact component at all. This is true on blown-up primary Hopf surfaces by a recent result of M. Toma [41] and we conclude using a deformation argument, since any minimal class VII surface surface containing a global spherical shell (see below) is the degeneration of a blown-up primary Hopf surface [22].

A first interesting property of our moduli spaces is that the filtrable bundles are generic. This is surprising, because on Kähler surfaces the filtrable locus is a countable union of Zariski-closed sets and also in all formerly known examples on non-Kähler surfaces it was found to be Zariski-closed.

Class VII surfaces with $b_2 = 1$ are of particular interest in the light of the classification problem of complex surfaces. In the early 1960ies, K. Kodaira classified connected compact complex surfaces (*surfaces* for short) into seven classes [24]. Six of them are quite well understood but the seventh [25] has resisted a complete classification until the present day. A surface S is said to be of class VII if it has Kodaira dimension $\mathrm{kod}(S) = -\infty$ and first Betti number $b_1(S) = 1$. It can be blown down to a unique minimal model, i. e. a unique class VII surface not being the blow-up of another one. We denote the subclass of minimal class VII surfaces by VII_0. Class VII_0 surfaces with second Betti number $b_2 = 0$ are classified: They are either Hopf or Inoue surfaces [2, 28, 34]. As to class VII_0 surfaces with $b_2 > 0$, all known examples admit a so-called global spherical shell and can thus be explicitly constructed by successive blow-ups of the unit ball in $\mathbb{C}^2$ and a subsequent holomorphic surgery [22]. On the other hand, every class VII_0 surface S with exactly $b_2(S)$ rational curves possesses a global spherical shell [8]. The global spherical shell conjecture now states that every class VII_0 surface has such a global spherical shell and would reduce the classification of class VII surfaces to finding sufficiently many curves.

This was recently done by A. Teleman for the subclass VII_0^1 of class VII_0 surfaces with $b_2 = 1$ [36]. Supposing there did not exist any complex curves on the surface, he constructed a contradiction for the moduli space of polystable holomorphic structures $\mathcal{E}$ with $\det \mathcal{E} \cong \mathcal{K}$ on a fixed complex vector bundle E with $c_1(E) = c_1(K)$ and $c_2(E) = 0$. By the above, this accomplishes the classification of class VII_0^1 surfaces: Each class VII_0^1 surface is biholomorphic to either the half Inoue surface [21], the parabolic Inoue surface [20] or an Enoki surface [13]. We in turn now compute explicitly this moduli space for each of these surfaces and describe its properties in detail. This is possible with respect to any Gauduchon metric, due to a recent result classifying the possible degree maps on non-Kähler surfaces [6, 37]. We finally remark that the methods used can be extended to show the existence of a curve in the case $b_2 = 2$ [39].

The expected complex dimension of the above moduli space is

$$(1.1) \qquad -\chi(\mathcal{E}nd_0\mathcal{E}) = \big(4c_2(E) - c_1(E)^2\big) - \tfrac{3}{2}\big(b_2^+(S) - b_1(S) + 1\big) = 1\,,$$

but there are two deeper reasons for this particular choice of the Chern classes of E. Firstly, it allows one to write filtrable holomorphic bundles $\mathcal{E}$ as extensions of certain holomorphic line bundles. Secondly, it assures that the moduli space of anti-self-dual connections on E is compact so that the moduli space of stable holomorphic structures on E, embedded via the Kobayashi-Hitchin correspondence, can be compactified by adding only the irreducible part. This compactification is crucial in the step determining possible non-filtrable bundles.

The moduli spaces we get are compact one-dimensional complex discs when the surface is a half or parabolic Inoue surface. In the generic case of an Enoki surface it is a compact one-dimensional complex disc too, but with finitely many transverse self-intersections. The number of these singularities is unbounded when the metric varies in the space of Gauduchon metrics. This shows that there are infinitely many homeomorphism types of moduli spaces although there are only finitely many topological splittings of the underlying vector bundle. Furthermore, having a boundary, these moduli spaces are not complex spaces. This is in contrast to algebraic surfaces, where the Uhlenbeck compactification is known to be an algebraic variety [27], and to all known examples on non-algebraic surfaces. It will be one of our next steps to study the behaviour of the natural Hermitian metric [29] near this boundary.

Let us finally point out that our results could only be obtained via a close interplay between complex geometry and gauge theory. Although nowadays Seiberg-Witten theory has widely replaced Donaldson theory, recent developments show that Donaldson theory on definite 4-manifolds with $b_1 \geqslant 1$ is still an interesting open subject [38].

The structure of this article is the following: In the next section we briefly review the necessary properties of class VII_0^1 surfaces and summarise their classification. Then we parametrise filtrable holomorphic bundles in the moduli space (section 3), examine its local structure (section 4) and the stability condition (section 5). In section 6 we give the boundary structure of the moduli spaces of polystable bundles. Finally we determine non-filtrable bundles (section 7) which leads to a complete description of the entire moduli space in the last section.

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2. Minimal class VII surfaces with $b_2 = 1$

Let S be a class VII surface, that is a compact complex surface with first Betti number $b_1(S) = 1$ and Kodaira dimension $\mathrm{kod}(S) = -\infty$. By definition, the condition on the Kodaira dimension means that tensor powers of the canonical holomorphic line bundle $\mathcal{K}$ do not admit any non-trivial holomorphic sections: $H^0(\mathcal{K}^{\otimes n}) = 0$ for $n \geqslant 1$. For such a surface the Chern classes are given by [24]

$$c_2(S) = -c_1(S)^2 = b_2(S)\,. \tag{2.1}$$

Suppose now that S is of class VII_0^1, i.e. minimal with second Betti number $b_2(S) = 1$. As mentioned in the introduction, Teleman proved that in this case there exists at least one complex curve on S [36]. But any class VII_0^1 surface containing a curve is biholomorphic to one of the following surfaces [31]:

- A *half Inoue surface* [21]. It contains only a single complex curve, namely a singular rational curve C with one node and self intersection -1. The canonical bundle is given by

$$\mathcal{K} \cong \mathcal{F} \otimes \mathcal{O}(-C) \tag{2.2}$$

  where $\mathcal{F}$ is the unique non-trivial square-root of the trivial holomorphic line bundle $\mathcal{O}$ (see below). We have

$$c_1(\mathcal{O}(C)) = -c_1(\mathcal{K})\,.$$

- A surface in the family studied by Enoki [13] containing only a single complex curve, namely a singular rational curve C with one node and self intersection 0. We have

$$c_1(\mathcal{O}(C)) = 0\,.$$

  There is no expression for the canonical bundle $\mathcal{K}$ as in the other two cases. We will refer to class VII_0^1 surfaces of this type as *Enoki surfaces.*

- A *parabolic Inoue surface* [20]. It contains precisely two complex curves, namely a singular rational curve C with one node and self intersection 0 and an elliptic curve E with self-intersection -1. Both curves are disjoint. The canonical bundle is given by

$$\mathcal{K} \cong \mathcal{O}(-C - E)\,. \tag{2.3}$$

  We have

$$c_1(\mathcal{O}(C)) = 0 \qquad c_1(\mathcal{O}(E)) = -c_1(\mathcal{K})\,.$$

The Chern classes above follow from the intersection numbers since $H^2(S, \mathbb{Z})$ is torsion free (see below).

Convention. — *The family of class* VII_0^1 *surfaces constructed and classified by Enoki* [13, 14] *is characterised by the existence of a divisor* $D > 0$ *with* $D^2 = 0$*. As such it includes the parabolic Inoue surface. Nonetheless, to simplify our exposition we agree that* ***in this article we do not consider the parabolic Inoue surface as an Enoki surface****.*

Unless otherwise stated, S *will always denote a class* VII_0^1 *surface, i. e. one of the three types above.*

Remark 2.1. — As a two parameter family Enoki surfaces represent the generic case of class VII_0^1 surfaces. The half and the parabolic Inoue surface appear as degenerations of them.

The existence of a rational curve on a class VII_0^1 surface implies the existence of a so-called global spherical shell [8]. Surfaces admitting a global spherical shell can be constructed by successive blow-ups of the unit ball in $\mathbb{C}^2$ and a subsequent holomorphic surgery [22, 7]. A consequence of this construction is that all such surfaces are degenerations of blown-up primary Hopf surfaces. In particular they are all diffeomorphic with fundamental group $\pi_1(S) \cong \mathbb{Z}$. Thus $H_1(S, \mathbb{Z}) \cong \mathbb{Z}$ is free and from the universal coefficient theorem we conclude $H^2(S, \mathbb{Z}) \cong \mathbb{Z}$ because $b_2(S) = 1$. Furthermore, from (2.1) we see that $c_1(\mathcal{K})^2 = -1$, showing that $c_1(\mathcal{K})$ is a generator of $H^2(S, \mathbb{Z})$.

In the following we will frequently use the correspondence between line bundle morphisms $\mathcal{M}_1 \to \mathcal{M}_2$ and the sections of $\mathcal{M}_1^\vee \otimes \mathcal{M}_2$ they define. In particular every such morphism is the zero morphism if the corresponding bundle does not admit non-trivial sections, i. e. if $H^0(\mathcal{M}_1^\vee \otimes \mathcal{M}_2) = 0$.

Remark 2.2. — Note that a line bundle admits non-trivial sections if and only if it is isomorphic to $\mathcal{O}(D)$ for a divisor $D \geqslant 0$ on S, i. e. if it is of the form $\mathcal{O}(rC)$ on the half Inoue or an Enoki surface and of the form

$\mathcal{O}(rC + sE)$ on the parabolic Inoue surface for some $r, s \in \mathbb{N}$. This shows in particular that line bundles $\mathcal{M}$ on class VII_0^1 surfaces with $c_1(\mathcal{M}) = c_1(\mathcal{K}^{\otimes n})$ do not admit non-trivial sections if $n \geqslant 1$, a fact we will use frequently below without further mention.

The divisor D is the zero divisor of a section in the line bundle and uniquely determined since class VII$_0^1$ surfaces do not admit non-constant meromorphic functions. In particular we have $\dim H^0(\mathcal{M}) \leqslant 1$ for line bundles $\mathcal{M}$ on class VII$_0^1$ surfaces and if $\mathcal{M}$ is non-trivial then either $\mathcal{M}$ or $\mathcal{M}^\vee$ does not admit non-trivial sections.

The exponential sequence $0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$ gives rise to the long exact cohomology sequence

$$\dots \longrightarrow H^1(S,\mathbb{Z}) \longrightarrow H^1(S,\mathcal{O}) \xrightarrow{\exp^1} H^1(S,\mathcal{O}^*) \xrightarrow{c_1} H^2(S,\mathbb{Z}) \longrightarrow \dots .$$

Here $\mathrm{Pic}(S) := H^1(S,\mathcal{O}^*)$ is the Picard group, the Abelian group of isomorphism classes of holomorphic line bundles on S with group multiplication induced by the tensor product. On the other hand $H^2(S,\mathbb{Z})$ classifies isomorphism classes of complex line bundles via the first Chern class. The connecting operator c_1 is just the group homomorphism that associates to a holomorphic line bundle the first Chern class of its underlying topological line bundle. Its kernel, the image of $\exp^1$, is the subgroup $\mathrm{Pic}^0(S)$ of holomorphic structures on the topologically trivial line bundle. The Picard group $\mathrm{Pic}(S)$ has the structure of a complex Lie group and $\exp^1$ is an étale morphism [29].

Since $H^1(S,\mathbb{Z})$ is torsion free and $b_1(S) = 1$ we have $H^1(S,\mathbb{Z}) \cong \mathbb{Z}$. Furthermore, on class VII surfaces the natural inclusion $\mathbb{C} \hookrightarrow \mathcal{O}$ induces an isomorphism $H^1(S,\mathbb{C}) \xrightarrow{\cong} H^1(S,\mathcal{O})$ [24] showing that for $b_1(S) = 1$ there is a group isomorphism

$$\mathrm{Pic}^0(S) \cong \mathbb{C}^* .$$

In particular every holomorphic line bundle in $\mathrm{Pic}^0(S)$ has exactly two roots in $\mathrm{Pic}^0(S)$ which differ by the non-trivial root of $\mathcal{O}$ which we will denote by $\mathcal{F}$:

$$\mathcal{F} \otimes \mathcal{F} = \mathcal{O} \qquad \mathcal{F} \ncong \mathcal{O} .$$

Remark that in contrast to Kähler surfaces $\mathrm{Pic}^0(S)$ is non-compact here.

3. Filtrable holomorphic bundles

On surfaces topological complex vector bundles are classified up to isomorphisms by their rank and their first two Chern classes. We fix once and for all a complex vector bundle E on S with

$$\text{(3.1a)} \qquad \operatorname{rank} E = 2 \qquad c_1(E) = c_1(K) \qquad c_2(E) = 0\,,$$

where K is the canonical complex line bundle. Since $c_1(\det E) = c_1(E)$ this implies $\det E \cong K$. In the following we will study the simple holomorphic structures $\mathcal{E}$ on E with determinant

$$\text{(3.1b)} \qquad \det \mathcal{E} \cong \mathcal{K}\,.$$

At first we investigate *filtrable* bundles of type (3.1), because they admit a relatively simple description as extensions of certain holomorphic line bundles. In general a rank two bundle is filtrable if it admits a rank one subsheaf, but the notion simplifies considerably for surfaces:

Definition 3.1. — *A holomorphic rank two vector bundle $\mathcal{E}$ on a complex surface S is* filtrable *if and only if one of the following equivalent conditions is satisfied:*

(1) *$\mathcal{E}$ has a rank one subsheaf $\mathscr{S}$.*
(2) *$\mathcal{E}$ has a locally free rank one subsheaf $\mathcal{L}$.*
(3) *There exist holomorphic line bundles $\mathcal{L}$ and $\mathcal{R}$ on S that fit into a short exact sequence of the form*

$$\text{(3.2)} \qquad 0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{R} \otimes \mathscr{I}_Z \longrightarrow 0,$$

where $\mathscr{I}_Z$ is the ideal sheaf of a dimension zero locally complete intersection $Z \subset S$.

The proof of the equivalence is standard, see for example [12]. The first reason for choosing E to satisfy (3.1a) is that in this case we get rid of the (possibly very complicated) ideal sheaf $\mathscr{I}_Z$ in (3.2):

Proposition 3.2. — *On a class VII_0^1 surface S we have $Z = \varnothing$ and either $c_1(\mathcal{L}) = 0$ or $c_1(\mathcal{R}) = 0$ in* (3.2) *under the assumption* (3.1a).

Proof. — Since $c_1(\mathcal{K})$ is a generator of $H^2(S, \mathbb{Z}) \cong \mathbb{Z}$ we set $c_1(\mathcal{L}) = n{\cdot}c_1(\mathcal{K})$ with $n \in \mathbb{Z}$. A computation of the Chern classes of $\mathcal{E} \cong (\mathcal{E} \otimes \mathcal{L}^\vee) \otimes \mathcal{L}$ yields, since $c_1(\mathcal{K})^2 = -1$,

$$c_1(\mathcal{K}) = c_1(\mathcal{E}) = c_1(\mathcal{E} \otimes \mathcal{L}^\vee) + 2c_1(\mathcal{L})$$

and

$$0 = c_2(\mathcal{E}) = c_2(\mathcal{E} \otimes \mathcal{L}^\vee) + c_1(\mathcal{E} \otimes \mathcal{L}^\vee)c_1(\mathcal{L}) + c_1(\mathcal{L})^2 = |Z| + n(n-1).$$

Here $|Z|$ denotes the number of points in Z, counted with multiplicities. But the last equality can only be satisfied if $|Z| = 0$, i. e. $Z = \varnothing$, and $n = 0$ or 1. □

Now note that the determinant of the central term in a line bundle extension is the tensor product of the two corresponding line bundles.

Corollary 3.3. — *Any filtrable holomorphic vector bundle $\mathcal{E}$ of type* (3.1) *on a class* VII_0^1 *surface is the central term of an extension of one of the following two types,*

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \to \mathcal{L}^\vee \otimes \mathcal{K} \to 0 \tag{3.3a}$$

$$0 \to \mathcal{R}^\vee \otimes \mathcal{K} \to \mathcal{E} \longrightarrow \mathcal{R} \longrightarrow 0, \tag{3.3b}$$

where $\mathcal{L}, \mathcal{R} \in \mathrm{Pic}^0(S)$.

Moreover, given a line bundle inclusion $\mathcal{M} \hookrightarrow \mathcal{E}$ into a bundle $\mathcal{E}$ of type (3.1)*, we have either $c_1(\mathcal{M}) = 0$ and the inclusion extends to* (3.3a) *with $\mathcal{L} \cong \mathcal{M}$ or we have $c_1(\mathcal{M}) = c_1(\mathcal{K})$ and it extends to* (3.3b) *with $\mathcal{R}^\vee \otimes \mathcal{K} \cong \mathcal{M}$.*

The following lemma shows that the existence of non-trivial extensions (3.3) and the uniqueness of their central terms is determined by the existence of sections in certain line bundles. Line bundle extensions $0 \to \mathcal{M}_1 \to \mathcal{E} \to \mathcal{M}_2 \to 0$ or equivalently $0 \to \mathcal{M}_1 \otimes \mathcal{M}_2^\vee \to \mathcal{E} \otimes \mathcal{M}_2^\vee \to \mathcal{O} \to 0$ are determined by the image of the constant 1 section in $\mathcal{O}$ under the connecting operator $H^0(\mathcal{O}) \to H^1(\mathcal{M}_1 \otimes \mathcal{M}_2^\vee)$ in the associated cohomology sequence and vice versa. In particular extensions which differ by a non-zero constant in the classifying space $\mathrm{Ext}^1(\mathcal{M}_2, \mathcal{M}_1) := H^1(\mathcal{M}_1 \otimes \mathcal{M}_2^\vee)$ have isomorphic central terms. To compute the dimension of these spaces we will use the Hirzebruch-Riemann-Roch theorem which, using (2.1) and combined with the Serre duality, takes the particular form

$$h^0(\mathcal{M}) - h^1(\mathcal{M}) + h^0(\mathcal{M}^\vee \otimes \mathcal{K}) = \tfrac{1}{2}\, c_1(\mathcal{M}) \big(c_1(\mathcal{M}) - c_1(\mathcal{K})\big) \tag{3.4}$$

for a holomorphic line bundle $\mathcal{M}$ on S, where $h^p(\mathcal{M}) := \dim H^p(\mathcal{M})$ [1]. To simplify the notation we write $\mathcal{L}^2$ and $\mathcal{L}^{-2}$ for $\mathcal{L} \otimes \mathcal{L}$ and $\mathcal{L}^\vee \otimes \mathcal{L}^\vee$ respectively.

Proposition 3.4. — (1) *For every holomorphic line bundle $\mathcal{L} \in \mathrm{Pic}^0(S) \setminus Q(S)$, where*

$$Q(S) := \{\mathcal{L} \in \mathrm{Pic}^0(S) \colon H^0(\mathcal{L}^2 \otimes \mathcal{K}^\vee) \neq 0\},$$

there is a non-trivial extension

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E}_\mathcal{L} \longrightarrow \mathcal{L}^\vee \otimes \mathcal{K} \longrightarrow 0 \tag{3.5a}$$

with an (up to isomorphisms) uniquely determined central term $\mathcal{E}_\mathcal{L}$. If $\mathcal{L} \in Q(S)$ then the isomorphism classes of central terms in non-trivial extensions of the form (3.3a) *are parametrised by* $\mathbb{C}\mathrm{P}^1$.

(2) *For every holomorphic line bundle* $\mathcal{R} \in R(S)$, *where*

$$R(S) := \{\mathcal{R} \in \mathrm{Pic}^0(S)\colon H^0(\mathcal{R}^2) \neq 0\}\,,$$

there is a non-trivial extension

$$0 \longrightarrow \mathcal{R}^\vee \otimes \mathcal{K} \longrightarrow \mathcal{A}_\mathcal{R} \longrightarrow \mathcal{R} \longrightarrow 0 \tag{3.5b}$$

with an (up to isomorphisms) uniquely determined central term $\mathcal{A}_\mathcal{R}$. *If* $\mathcal{R} \in \mathrm{Pic}^0(S) \setminus R(S)$ *there are no non-trivial extensions of the form* (3.3b).

Proof. — Extensions of type (3.5b) are classified by $\mathrm{Ext}^1(\mathcal{R}, \mathcal{R}^\vee \otimes \mathcal{K}) \cong H^1(\mathcal{R}^{-2} \otimes \mathcal{K})$. From formula (3.4) for $\mathcal{M} = \mathcal{R}^{-2} \otimes \mathcal{K}$ we obtain $\dim \mathrm{Ext}^1(\mathcal{R}, \mathcal{R}^\vee \otimes \mathcal{K}) = h^0(\mathcal{R}^2)$ since $H^0(\mathcal{R}^{-2} \otimes \mathcal{K}) = 0$. This proves (2) because $h^0(\mathcal{R}^2) = 0$ or 1. Likewise, extensions of type (3.5a) are classified by $\mathrm{Ext}^1(\mathcal{L}^\vee \otimes \mathcal{K}, \mathcal{L}) \cong H^1(\mathcal{L}^2 \otimes \mathcal{K}^\vee)$ and from formula (3.4) we obtain $\dim \mathrm{Ext}^1(\mathcal{L}^\vee \otimes \mathcal{K}, \mathcal{L}) = 1 + h^0(\mathcal{L}^2 \otimes \mathcal{K}^\vee)$ since $H^0(\mathcal{L}^{-2} \otimes \mathcal{K}^2) = 0$. This proves the first part of (1).

In the case $\mathcal{L} \in Q(S)$ we have $\dim \mathrm{Ext}^1(\mathcal{L}^\vee \otimes \mathcal{K}, \mathcal{L}) = 1 + h^0(\mathcal{L}^2 \otimes \mathcal{K}^\vee) = 2$ because $0 \neq h^0(\mathcal{L}^2 \otimes \mathcal{K}^\vee) \leqslant 1$. Let $\varphi\colon \mathcal{E}_1 \to \mathcal{E}_2$ be a bundle isomorphism between the central terms of two different extensions in the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L} \arrow[r, "\alpha_1"] \arrow[d, dashed] & \mathcal{E}_1 \arrow[r] \arrow[d, "\varphi"] & \mathcal{L}^\vee \otimes \mathcal{K} \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & \mathcal{L} \arrow[r] & \mathcal{E}_2 \arrow[r, "\beta_2"] & \mathcal{L}^\vee \otimes \mathcal{K} \arrow[r] & 0 \end{tikzcd} \tag{3.6}$$

The composition $\beta_2 \circ \varphi \circ \alpha_1$ must vanish since it defines a section of $\mathcal{L}^{-2} \otimes \mathcal{K}$. Thereby φ induces endomorphisms $\mathcal{L} \to \mathcal{L}$ and $\mathcal{L}^\vee \otimes \mathcal{K} \to \mathcal{L}^\vee \otimes \mathcal{K}$ (the vertical dashed morphisms) defining sections of $\mathcal{O}$. That φ is an isomorphism shows that both are non-trivial and thus non-zero multiples of the identity. But then the two extensions differ by a non-zero constant in $\mathrm{Ext}^1(\mathcal{L}^\vee \otimes \mathcal{K}, \mathcal{L})$. □

$R(S)$ is the set of those line bundles $\mathcal{R} \in \mathrm{Pic}^0(S)$ that define a (unique) bundle $\mathcal{A}_\mathcal{R}$ and $Q(S)$ is the set of those line bundles $\mathcal{L} \in \mathrm{Pic}^0(S)$ that do *not* define a unique bundle $\mathcal{E}_\mathcal{L}$. In the following we always imply $\mathcal{R} \in R(S)$ and $\mathcal{L} \in \mathrm{Pic}^0(S) \setminus Q(S)$ when we write $\mathcal{A}_\mathcal{R}$ and $\mathcal{E}_\mathcal{L}$ respectively.

Remark 3.5. — Using remark 2.2 and evaluating the first Chern class, it is not difficult to see that the above sets have the following form on the different class VII_0^1 surfaces:

- For S the half Inoue surface, $R(S) = \sqrt{\mathcal{O}} = \{\mathcal{O}, \mathcal{F}\}$ and $Q(S) = \sqrt{\mathcal{F}}$.
- For S an Enoki or the parabolic Inoue surface,

$$R(S) = \{\mathcal{M} \otimes \mathcal{O}(rC) \colon \mathcal{M} \in \sqrt{\mathcal{O}} \cup \sqrt{\mathcal{O}(C)},\ r \in \mathbb{N}\}\,.$$

- For S an Enoki surface, $Q(S) = \varnothing$.
- For S the parabolic Inoue surface, $Q(S) = R(S) \cup \sqrt{\mathcal{O}(-C)}$.

In particular, since every line bundle in $\mathrm{Pic}^0(S)$ has exactly two square roots, the above sets are finite or countable so that the bundles $\mathcal{E}_{\mathcal{L}}$ with $\mathcal{L} \in \mathrm{Pic}^0(S) \setminus Q(S)$ represent the generic case among the filtrable bundles of type (3.1).

We now restrict our attention to simple bundles. Simplicity assures that the resulting moduli space is a complex analytic space.

Definition 3.6. — *A holomorphic vector bundle $\mathcal{E}$ is called* simple *if the only holomorphic endomorphisms of $\mathcal{E}$ are multiples of the identity.*

Proposition 3.7. — (1) *The central terms of trivial extensions of type* (3.3) *are never simple.*
(2) *The bundles $\mathcal{E}_{\mathcal{L}}$, $\mathcal{L} \in \mathrm{Pic}^0(S) \setminus Q(S)$, are simple.*
(3) *For $\mathcal{L} \in Q(S)$ the central terms of non-trivial extensions* (3.3a) *are not simple.*
(4) *A bundle $\mathcal{A}_{\mathcal{R}}$ is simple if $\mathcal{R} \in R(S) \setminus Q(S)$.* (1)

Moreover, every simple filtrable holomorphic bundle of type (3.1) *is isomorphic to either a bundle $\mathcal{E}_{\mathcal{L}}$ for some $\mathcal{L} \in \mathrm{Pic}^0(S) \setminus Q(S)$ or to a bundle $\mathcal{A}_{\mathcal{R}}$ for some $\mathcal{R} \in R(S)$.*

Proof. — (1) is evident. To prove (2) and (3) regard diagram (3.6) for $\mathcal{E}_1 = \mathcal{E}_2 =: \mathcal{E}$ and an *endo*morphism $\varphi\colon \mathcal{E} \to \mathcal{E}$. As in the proof of proposition 3.4, φ induces endomorphisms $\mathcal{L} \to \mathcal{L}$ and $\mathcal{L}^\vee \otimes \mathcal{K} \to \mathcal{L}^\vee \otimes \mathcal{K}$ (the vertical dashed morphisms) which must be multiples of the identity since they define sections in $\mathcal{O}$. Let the latter one be $\zeta\,\mathrm{id}_{\mathcal{L}^\vee \otimes \mathcal{K}}$ with $\zeta \in \mathbb{C}$. Then we can substitute φ by $\varphi - \zeta\,\mathrm{id}_{\mathcal{E}}$ to obtain the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L} \arrow[r] \arrow[d, dashed] & \mathcal{E} \arrow[r, "\beta"] \arrow[d, "\varphi-\zeta\,\mathrm{id}_{\mathcal{E}}"] \arrow[dl, dotted] & \mathcal{L}^\vee \otimes \mathcal{K} \arrow[r] \arrow[d, dashed, "0"] & 0 \\
0 \arrow[r] & \mathcal{L} \arrow[r, "\alpha"] & \mathcal{E} \arrow[r] & \mathcal{L}^\vee \otimes \mathcal{K} \arrow[r] & 0
\end{tikzcd}$$

(1) Later on we will see that this is actually an "if and only if".

where this time the endomorphism $\mathcal{L}^\vee \otimes \mathcal{K} \to \mathcal{L}^\vee \otimes \mathcal{K}$ on the right is zero. Therefore $\varphi - \zeta\,\mathrm{id}_\mathcal{E}$ factorises through α. But now the endomorphism $\mathcal{L} \to \mathcal{L}$ on the left must be zero too. Indeed, if not, it would be an isomorphism and its inverse composed with the morphism $\mathcal{E} \to \mathcal{L}$ would define a splitting of the first extension. This induces yet another morphism $\sigma\colon \mathcal{L}^\vee \otimes \mathcal{K} \to \mathcal{L}$ from the bundle $\mathcal{L}^\vee \otimes \mathcal{K}$ in the upper extension to the bundle $\mathcal{L}$ in the lower extension (not indicated). This morphism defines an element of $H^0(\mathcal{L}^2 \otimes K^\vee)$ and is zero if and only if $\varphi - \zeta\,\mathrm{id}_\mathcal{E} = \alpha \circ \sigma \circ \beta$ is. This demonstrates (2) and (3).

The proof of (4) is analogous. In the corresponding diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{R}^\vee \otimes \mathcal{K} \arrow[r, "\alpha"] \arrow[d, dashed] & \mathcal{A}_\mathcal{R} \arrow[r] \arrow[d, "\varphi"] & \mathcal{R} \arrow[r] \arrow[d, dashed, "\zeta\,\mathrm{id}_\mathcal{R}"] & 0 \\
0 \arrow[r] & \mathcal{R}^\vee \otimes \mathcal{K} \arrow[r] & \mathcal{A}_\mathcal{R} \arrow[r, "\beta"] & \mathcal{R} \arrow[r] & 0
\end{tikzcd}$$

for a bundle endomorphism $\varphi\colon \mathcal{A}_\mathcal{R} \to \mathcal{A}_\mathcal{R}$ the composition $\beta \circ \varphi \circ \alpha$ is zero by hypothesis since it defines a section of $\mathcal{R}^2 \otimes \mathcal{K}^\vee$. As before we can substitute this diagram by

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{R}^\vee \otimes \mathcal{K} \arrow[r] \arrow[d, dashed, "0"'] & \mathcal{A}_\mathcal{R} \arrow[r] \arrow[d, "\varphi - \zeta\,\mathrm{id}"] \arrow[dl, dotted] & \mathcal{R} \arrow[r] \arrow[d, dashed, "0"] & 0 \\
0 \arrow[r] & \mathcal{R}^\vee \otimes \mathcal{K} \arrow[r] & \mathcal{A}_\mathcal{R} \arrow[r] & \mathcal{R} \arrow[r] & 0
\end{tikzcd}$$

Concluding as above we have $\varphi = \zeta\,\mathrm{id}$ since $H^0(\mathcal{R}^{-2} \otimes \mathcal{K}) = 0$.

The last statement is now a consequence of corollary 3.3 and proposition 3.4. □

To obtain a *bijective* parametrisation of simple filtrable bundles of type (3.1) we will have to determine possible isomorphisms of the forms

$$\mathcal{E}_{\mathcal{L}'} \cong \mathcal{E}_\mathcal{L} \qquad \mathcal{A}_{\mathcal{R}'} \cong \mathcal{A}_\mathcal{R} \qquad \mathcal{A}_\mathcal{R} \cong \mathcal{E}_\mathcal{L}\,. \tag{3.7}$$

Regarding the defining extensions (3.5) and corollary 3.3, these are given by holomorphic bundle embeddings $\mathcal{L}' \hookrightarrow \mathcal{E}_\mathcal{L}$, $\mathcal{R}' \hookrightarrow \mathcal{A}_\mathcal{R}$ and $\mathcal{L} \hookrightarrow \mathcal{A}_\mathcal{R}$.

A line bundle extension $0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{O} \to 0$ is determined by the image $\delta_h(1)$ of the constant 1 section in $\mathcal{O}$ under the connecting operator $\delta_h\colon H^0(\mathcal{O}) \to H^1(\mathcal{M})$ in the associated cohomology sequence. Given, in addition, a divisor $D > 0$ on S there is a second connecting operator $\delta_v : H^0(\mathcal{M}_D(D)) \to H^1(\mathcal{M})$ from the cohomology sequence associated to the short exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}(D) \longrightarrow \mathcal{M}_D(D) \longrightarrow 0\,. \tag{3.8}$$

This sequence is the defining sequence for $\mathcal{M}_D(D)$ where we write $\mathcal{M}(D)$ for $\mathcal{M}\otimes\mathcal{O}(D)$ and $\mathcal{M}_D$ for the restriction of $\mathcal{M}$ to D, i. e. $\mathcal{M}_D := \mathcal{M}\otimes\mathcal{O}_D$.

In [37] we find the following criterion:

Proposition 3.8. — *With the above notation, the natural map $\mathcal{O}(-D) \to \mathcal{O}$ can be lifted to a bundle embedding*

$$\begin{tikzcd} & & \mathcal{O}(-D) \arrow[d, dashed] \arrow[dr] & & \\ 0 \arrow[r] & \mathcal{M} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{O} \arrow[r] & 0 \end{tikzcd} \tag{3.9}$$

if and only if there exists a section $\sigma \in H^0(\mathcal{M}_D(D))$ defining a trivialisation $\mathcal{M}_D(D) \cong \mathcal{O}_D$ such that $\delta_h(1)=\delta_v(\sigma)$

Applying this criterion to the extensions (3.5) yields the following

Corollary 3.9. — (1) *$\mathcal{E}_{\mathcal{L}'} \cong \mathcal{E}_{\mathcal{L}}$ if and only if $\mathcal{L}' \cong \mathcal{L}$.*

(2) *Suppose $\mathcal{R}' \ncong \mathcal{R}$ and that $\mathcal{A}_{\mathcal{R}}$ and $\mathcal{A}_{\mathcal{R}'}$ are simple. Then $\mathcal{A}_{\mathcal{R}'} \cong \mathcal{A}_{\mathcal{R}}$ if and only if there exists a divisor $D > 0$ with $\mathcal{R}\otimes\mathcal{R}' \cong \mathcal{K}(D)$ and $\mathcal{R}'_D \cong \mathcal{R}_D$.*

(3) *Suppose $\mathcal{A}_{\mathcal{R}}$ is simple. Then $\mathcal{A}_{\mathcal{R}} \cong \mathcal{E}_{\mathcal{L}}$ if and only if there exists a divisor $D > 0$ with $\mathcal{L} \cong \mathcal{R}(-D)$ and $\mathcal{R}^2_D \cong \mathcal{K}_D(D)$.*

Proof. — We can show the first statement without using proposition 3.8. Take two non-isomorphic bundles $\mathcal{L}$ and $\mathcal{L}'$. Then either $\mathcal{L}^\vee \otimes \mathcal{L}'$ or $\mathcal{L} \otimes \mathcal{L}'^\vee$ has only trivial sections, cf. remark 2.2. We can assume the latter by possibly interchanging $\mathcal{L}$ and $\mathcal{L}'$. Let now $\varphi\colon \mathcal{E}_{\mathcal{L}} \to \mathcal{E}_{\mathcal{L}'}$ be an isomorphism between the corresponding bundles $\mathcal{E}_{\mathcal{L}}$ and $\mathcal{E}_{\mathcal{L}'}$ and regard the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L} \arrow[r, "\alpha_1"] & \mathcal{E}_{\mathcal{L}} \arrow[r] \arrow[d, "\varphi"] \arrow[dl, dotted] & \mathcal{L}^\vee \otimes \mathcal{K} \arrow[r] \arrow[d, dashed, "0"] & 0 \\ 0 \arrow[r] & \mathcal{L}' \arrow[r, "\alpha_2"] & \mathcal{E}_{\mathcal{L}'} \arrow[r, "\beta_2"] & \mathcal{L}'^\vee \otimes \mathcal{K} \arrow[r] & 0 \end{tikzcd}$$

The composition $\beta_2\circ\varphi\circ\alpha_1$ vanishes since it defines a section of $\mathcal{L}^\vee\otimes\mathcal{L}'^\vee\otimes\mathcal{K}$. Thus φ induces a morphism $\mathcal{L}^\vee \otimes \mathcal{K} \to \mathcal{L}'^\vee \otimes \mathcal{K}$ (the vertical dashed morphism). This defines a section of $\mathcal{L} \otimes \mathcal{L}'^\vee$ which is zero by the above choice of $\mathcal{L}$ and $\mathcal{L}'$. Consequently φ factorises through α_2, showing that it can not be an isomorphism. This proves the first statement.

To prove the second statement we can assume that $\mathcal{R}^\vee \otimes \mathcal{R}'$ does only admit trivial sections by possibly interchanging $\mathcal{R}$ and $\mathcal{R}'$, cf. remark 2.2.

Now observe that an isomorphism $\mathcal{A}_{\mathcal{R}'} \cong \mathcal{A}_{\mathcal{R}}$ gives, after tensorising the defining extensions for $\mathcal{A}_{\mathcal{R}}$ and $\mathcal{A}_{\mathcal{R}'}$ by $\mathcal{R}^\vee$, a bundle embedding α,

$$\begin{tikzcd}
 & \mathcal{M} \arrow[d, equal] & \mathcal{R}^\vee \otimes \mathcal{R}'^\vee \otimes \mathcal{K} \arrow[d, "\alpha"'] \arrow[dr] & & \\
0 \arrow[r] & \mathcal{R}^{-2} \otimes \mathcal{K} \arrow[r] & \mathcal{R}^\vee \otimes \mathcal{A}_{\mathcal{R}} \arrow[r] & \mathcal{O} \arrow[r] & 0\,,
\end{tikzcd} \tag{3.10}$$

and thus a bundle morphism $\mathcal{R}^\vee \otimes \mathcal{R}'^\vee \otimes \mathcal{K} \to \mathcal{O}$. If it was trivial, α would induce a morphism $\mathcal{R}^\vee \otimes \mathcal{R}'^\vee \otimes \mathcal{K} \to \mathcal{R}^{-2} \otimes \mathcal{K}$ defining a section of $\mathcal{R}^\vee \otimes \mathcal{R}'$ which is zero by assumption. This would contradict the fact that α is a bundle embedding. So the morphism $\mathcal{R}^\vee \otimes \mathcal{R}'^\vee \otimes \mathcal{K} \to \mathcal{O}$ is non-trivial, showing the existence of a divisor $D \geqslant 0$ with $\mathcal{R} \otimes \mathcal{R}' \cong \mathcal{K}(D)$. We have $D \neq 0$ because otherwise this would give a splitting of the extension defining $\mathcal{A}_{\mathcal{R}}$, but $\mathcal{A}_{\mathcal{R}}$ is simple by hypothesis. Proposition 3.8 applied to $\mathcal{M} = \mathcal{R}^{-2} \otimes \mathcal{K}$ now yields $\mathcal{R}_D^2 \cong \mathcal{K}_D(D)$ or equivalently $\mathcal{R}'_D \cong \mathcal{R}_D$.

Conversely, suppose $\mathcal{R} \otimes \mathcal{R}' \cong \mathcal{K}(D)$ and $\mathcal{R}'_D \cong \mathcal{R}_D$. Again we can assume that $\mathcal{R}^\vee \otimes \mathcal{R}'$ does only admit trivial sections by possibly interchanging $\mathcal{R}$ and $\mathcal{R}'$. Consider the short exact sequence (3.8) for $\mathcal{M} = \mathcal{R}^{-2} \otimes \mathcal{K}$ and regard the associated long exact sequence

$$\ldots \longrightarrow H^0(\mathcal{R}^\vee \otimes \mathcal{R}') \longrightarrow H^0(\mathcal{O}_D) \xrightarrow{\delta_v} H^1(\mathcal{R}^{-2} \otimes \mathcal{K}) \longrightarrow \ldots\,.$$

We have $h^0(\mathcal{R}^\vee \otimes \mathcal{R}') = 0$ by assumption, so the connecting operator δ_v is injective. As we saw in the proof of proposition 3.4, $h^1(\mathcal{R}^{-2} \otimes \mathcal{K}) = \dim \operatorname{Ext}^1(\mathcal{R}, \mathcal{R}^\vee \otimes \mathcal{K}) = 1$. Together with $h^0(\mathcal{O}_D) \geqslant 1$ this shows that δ_v is an isomorphism and $h^0(\mathcal{O}_D) = 1$. Thus the preimage σ of $\delta_h(1) \in H^1(\mathcal{R}^{-2} \otimes \mathcal{K})$ under δ_v is a non-zero constant section of $\mathcal{M}_D(D) \cong \mathcal{O}_D$ and therefore defines a trivialisation. Applying proposition 3.8 to $\mathcal{M} = \mathcal{R}^{-2} \otimes \mathcal{K}$ now gives a line bundle inclusion α in (3.10). By corollary 3.3 the resulting bundle embedding $\mathcal{R}'^\vee \otimes \mathcal{K} \to \mathcal{A}_{\mathcal{R}}$ extends to an extension $0 \to \mathcal{R}'^\vee \otimes \mathcal{K} \to \mathcal{A}_{\mathcal{R}} \to \mathcal{R}' \to 0$. It is non-trivial because $A_{\mathcal{R}}$ is simple. Then by the definition of $A_{\mathcal{R}'}$ we have $\mathcal{A}_{\mathcal{R}'} \cong \mathcal{A}_{\mathcal{R}}$.

The proof of the last statement is analogous because the corresponding diagram is

$$\begin{tikzcd}
 & \mathcal{M} \arrow[d, equal] & \mathcal{R}^\vee \otimes \mathcal{L} \arrow[d, "\alpha"] \arrow[dr, dashed] & & \\
0 \arrow[r] & \mathcal{R}^{-2} \otimes \mathcal{K} \arrow[r] & \mathcal{R}^\vee \otimes \mathcal{A}_{\mathcal{R}} \arrow[r] & \mathcal{O} \arrow[r] & 0
\end{tikzcd}$$

and the cohomology sequence of (3.8) for $\mathcal{M} = \mathcal{R}^{-2} \otimes \mathcal{K}$ reads

$$\ldots \longrightarrow H^0(\mathcal{R}^\vee \otimes \mathcal{L}^\vee \otimes \mathcal{K}) \longrightarrow H^0(\mathcal{O}_D) \xrightarrow{\delta_v} H^1(\mathcal{R}^{-2} \otimes \mathcal{K}) \longrightarrow \ldots .$$

But in this case $\mathcal{R}^\vee \otimes \mathcal{L}^\vee \otimes \mathcal{K}$ does not admit non-trivial sections. □

We will now examine the above criteria on each type of class VII_0^1 surfaces. For the half Inoue surface we first need the following fact.

Lemma 3.10. — *A singular rational curve C with one node on a complex surface satisfies $\mathcal{K}_C(C) \cong \mathcal{O}_C$.*

Proof. — Note that $\mathcal{K}_C(C)$ is the dualising bundle of C which is independent of the particular embedding of C [1] and we can embed C as a cubic in $\mathbb{C}\mathrm{P}^2$. But there $\mathcal{K} = \mathcal{O}(-3)$ and $\mathcal{O}(C) = \mathcal{O}(3)$ so that $\mathcal{K}(C)$ is already trivial. □

Theorem 3.11. — *For S the half Inoue surface, there is an isomorphism*

$$\mathcal{A}_\mathcal{O} \cong \mathcal{A}_\mathcal{F} \tag{3.11}$$

and the filtrable simple holomorphic bundles of type (3.1) *are bijectively parametrised by the disjoint union* $\big(\mathrm{Pic}^0(S) \setminus \sqrt{\mathcal{F}}\big) \cup \{0\}$, *mapping* $\mathcal{L} \mapsto \mathcal{E}_\mathcal{L}$ *and* $0 \mapsto \mathcal{A}_\mathcal{O} \cong \mathcal{A}_\mathcal{F}$.

Proof. — The bundles $\mathcal{E}_\mathcal{L}$ are simple by proposition 3.7 as well as is $\mathcal{A}_\mathcal{O}$ because $\mathcal{O} \notin Q(S) = \sqrt{\mathcal{F}}$. The isomorphism $\mathcal{A}_\mathcal{F} \cong \mathcal{A}_\mathcal{O}$ follows directly from corollary 3.9(2) and (2.2) together with the lemma. Notice that by remark 3.5 there are no further bundles of the form $\mathcal{A}_\mathcal{R}$. By corollary 3.9 the bundles $\mathcal{E}_\mathcal{L}$ are pairwise non-isomorphic and there can be no isomorphism $\mathcal{A}_\mathcal{O} \cong \mathcal{E}_\mathcal{L}$. Indeed, taking the first Chern class of $\mathcal{L} \cong \mathcal{R}(-D)$ shows $c_1(\mathcal{O}(D)) = 0$, contradicting $D \neq 0$. This shows injectivity. Surjectivity follows from proposition 3.7. □

Theorem 3.12. — *For S the parabolic Inoue surface there are isomorphisms*

$$\mathcal{A}_\mathcal{R} \cong \mathcal{R}(-E) \oplus \mathcal{R}^\vee(-C) \qquad \mathcal{R} \in R(S)\,, \tag{3.12}$$

so the bundles $\mathcal{A}_\mathcal{R}$ are not simple. The filtrable simple bundles of type (3.1) *are bijectively parametrised by* $\mathrm{Pic}^0(S) \setminus Q(S)$, *mapping* $\mathcal{L} \mapsto \mathcal{E}_\mathcal{L}$.

Proof. — To show (3.12), take a bundle $\mathcal{R} \in R(S)$ with $\mathcal{R}^2 \cong \mathcal{O}(rC)$ for some $r \in \mathbb{N}$. Using $\mathcal{K} \cong \mathcal{O}(-E-C)$ we get

$$(\mathcal{K} \otimes \mathcal{R}^\vee)^\vee \;\otimes\; \big(\mathcal{R}(-E) \oplus \mathcal{R}^\vee(-C)\big) \;=\; \mathcal{O}\big((r+1)C\big) \;\oplus\; \mathcal{O}(E)\,.$$

Since $C \cap E = \varnothing$, this bundle admits a non-vanishing section giving rise to a bundle embedding $\mathcal{K} \otimes \mathcal{R}^\vee \hookrightarrow \mathcal{R}(-E) \oplus \mathcal{R}^\vee(-C)$. But, as one easily checks, $\mathcal{R}(-E)\oplus\mathcal{R}^\vee(-C)$ is of type (3.1). So by corollary 3.3 this inclusion extends to

$$0 \longrightarrow \mathcal{K}\otimes\mathcal{R}^\vee \longrightarrow \mathcal{R}(-E)\oplus\mathcal{R}^\vee(-C) \longrightarrow \mathcal{R} \longrightarrow 0\,.$$

Assume this extension splits, i. e. $\mathcal{R}(-E)\oplus\mathcal{R}^\vee(-C)\cong(\mathcal{K}\otimes\mathcal{R}^\vee)\oplus\mathcal{R}$. Tensorising with $\mathcal{R}^\vee$ gives $\mathcal{O}(-E)\oplus\mathcal{O}(-(r+1)C)\cong(\mathcal{K}\otimes\mathcal{R}^{-2})\oplus\mathcal{O}$ which is impossible because the left hand side admits no non-trivial sections while the right hand side does. Therefore the above extension is non-trivial and determines the isomorphism (3.12) by the very definition of $\mathcal{A}_\mathcal{R}$. The rest follows from 3.7 and 3.9(1). □

To apply corollary 3.9 in the remaining case of an Enoki surface, we need the following generalisation of lemma 3.10 for Enoki surfaces.

Lemma 3.13. — *On an Enoki surface one has $\mathcal{K}_{rC}(C)\cong\mathcal{O}_{rC}$ for $r\in\mathbb{N}\setminus\{0\}$.*

Proof. — We prove by induction on r. For $r=1$ this is just lemma 3.10, so let us suppose $\mathcal{K}_{rC}(C)\cong\mathcal{O}_{rC}$ for some $r\geqslant 1$. The restriction ρ of the holomorphic line bundle $\mathcal{K}(C)$ from $(r+1)C$ to rC gives the following exact sequence [1]:

$$0 \longrightarrow \mathcal{K}_C\big(-(r-1)C\big) \longrightarrow \mathcal{K}_{(r+1)C}(C) \xrightarrow{\rho} \mathcal{K}_{rC}(C) \longrightarrow 0\,.$$

It suffices to show that the induced map ρ_* in the corresponding long exact cohomology sequence

$$\ldots \longrightarrow H^0\big(\mathcal{K}_{(r+1)C}(C)\big) \xrightarrow{\rho_*} H^0\big(\mathcal{K}_{rC}(C)\big) \longrightarrow H^1\big(\mathcal{K}_C(-(r-1)C)\big) \longrightarrow \ldots$$

is surjective, because then a trivialising section of $\mathcal{K}_{rC}(C)$ lifts to a section trivialising $\mathcal{K}_{(r+1)C}(C)$. But the surjectivity of ρ_* is equivalent to $H^1\big(\mathcal{K}_C(-(r-1)C)\big)=0$ or, after applying the Serre duality for embedded curves [1], to $H^0(\mathcal{O}_C(rC))=0$. The proof is therefore finished if we show that $\mathcal{O}_C(rC)$ has only trivial sections.

Recall that flat line bundles are given by representations of the fundamental group π_1 on $\mathbb{C}$ and have vanishing first Chern class. This gives the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(\pi_1(S),\mathbb{C}^*) \arrow[r] \arrow[d] & \mathrm{Pic}^0(S) \arrow[d] \\
\mathrm{Hom}(\pi_1(C),\mathbb{C}^*) \arrow[r] & \mathrm{Pic}^0(C)\,.
\end{tikzcd}$$

In our case both $\pi_1(S)$ and $\pi_1(C)$ are isomorphic to $\mathbb{Z}$. Moreover, on an Enoki surface the morphism $\pi_1(C)\to\pi_1(S)$ induced from the inclusion

$C \hookrightarrow S$ is in fact an isomorphism [31]. Thus the vertical map on the left-hand side is an isomorphism. The lower horizontal map is an isomorphism too, since it can be interpreted as follows: The normalisation $\hat{C} \to C$ of C maps two different points x_1 and x_2 to the singularity of C. Every line bundle in $\mathrm{Pic}^0(C)$ pulls back to a topologically trivial bundle on the projective line $\hat{C}$ and is thus determined by a complex number $\zeta \in \mathbb{C}^*$ identifying the fibres over x_1 and x_2. The above diagram is thus composed of isomorphisms between groups isomorphic to $\mathbb{C}^*$.

Now note that the bundle $\mathcal{O}(rC) \in \mathrm{Pic}^0(S)$ is non-trivial, cf. remark 2.2. Its restriction $\mathcal{O}_C(rC)$ therefore corresponds to a $\zeta \neq 1$. Under the normalising map a section $s \in H^0\big(\mathcal{O}_C(rC)\big)$ pulls back to a constant section $\hat{s}$ with $\hat{s}(x_1) = \zeta\hat{s}(x_2)$. This shows $\hat{s} = 0$ and thus $s = 0$. □

Theorem 3.14. — *On an Enoki surface S we have isomorphisms*

$$\mathcal{A}_{\mathcal{R}} \cong \mathcal{E}_{\mathcal{R}^\vee(-C)} \qquad \mathcal{R} \in R(S) \tag{3.13}$$

and the filtrable simple bundles of type (3.1) *are bijectively parametrised by* $\mathrm{Pic}^0(S)$, *mapping* $\mathcal{L} \mapsto \mathcal{E}_{\mathcal{L}}$.

Proof. — First recall that divisors on an Enoki surface are multiples of the curve C. Therefore $\mathcal{R}^2 \cong \mathcal{O}(rC)$ for some $r \in \mathbb{N}$ if $\mathcal{R} \in R(S)$ and the above lemma shows the existence of a divisor $D = (r+1)C$ with $\mathcal{K}_D(D) \cong \mathcal{O}_D(rC) \cong \mathcal{R}_D^2$. Corollary 3.9(3) thus gives an isomorphism $\mathcal{A}_{\mathcal{R}} \cong \mathcal{E}_{\mathcal{L}}$ with $\mathcal{L} \cong \mathcal{R}\big(-(r+1)C\big) = \mathcal{R}^\vee(-C)$. Remarking that $Q(S) = \varnothing$ for Enoki surfaces, the rest follows from 3.7 and 3.9(1). □

Resuming, we saw that with exception of the bundle $\mathcal{A}_{\mathcal{O}} \cong \mathcal{A}_{\mathcal{F}}$ on the half Inoue surface every filtrable simple holomorphic bundle of type (3.1) on a class VII_0^1 surface S is of the form $\mathcal{E}_{\mathcal{L}}$ with $\mathcal{L} \in \mathrm{Pic}^0(S) \setminus Q(S)$. Taking into account remark 3.5, a bundle of the form $\mathcal{A}_{\mathcal{R}}$ is simple if and only if $\mathcal{R} \in R(S) \setminus Q(S)$.

4. The local structure of the moduli space

Definition 4.1. — *We denote by*

$$\mathscr{M}^{\mathrm{s}}(S) := \{\mathcal{E} \textit{ simple hol. str. on } E\colon \det\mathcal{E} \cong \mathcal{K}\}/\Gamma\big(S, \mathrm{GL}(E)\big)$$

the moduli space of simple holomorphic bundles *of type* (3.1) *on* S.

This is a (possibly non-Hausdorff) complex space. The local structure of this moduli space is given by the following proposition whose proof is a straightforward generalisation of the case $R(S) = \sqrt{\mathcal{O}}$ in [36].

Proposition 4.2. — (1) *If* $\mathcal{L} \in \mathrm{Pic}^0(S) \setminus \big(R(S) \cup Q(S)\big)$ *then* $\mathscr{M}^{\mathrm{s}}(S)$ *is a smooth complex curve* $C_{\mathcal{L}}$ *in a neighbourhood of* $\mathcal{E}_{\mathcal{L}}$*, given by* $\mathcal{L}' \mapsto \mathcal{E}_{\mathcal{L}'}$.

(2) *If* $\mathcal{R} \in R(S) \setminus Q(S)$ *then* $\mathscr{M}^{\mathrm{s}}(S)$ *is the intersection of two complex curves* $C_{\mathcal{R}}$ *and* $C'_{\mathcal{R}}$ *in a neighbourhood of* $\mathcal{E}_{\mathcal{R}}$*, where* $C_{\mathcal{R}}$ *is given by* $\mathcal{L}' \mapsto \mathcal{E}_{\mathcal{L}'}$.

(3) *If* $\mathcal{R} \in R(S) \setminus Q(S)$ *then* $\mathscr{M}^{\mathrm{s}}(S)$ *is a smooth complex curve* $C''_{\mathcal{R}}$ *in a neighbourhood of* $\mathcal{A}_{\mathcal{R}}$.

(4) *The points* $\mathcal{E}_{\mathcal{R}}$ *and* $\mathcal{A}_{\mathcal{R}}$ *are not separable. More precisely, we find neighbourhoods* U'*,* U'' *of* $\mathcal{E}_{\mathcal{R}}$ *and* $\mathcal{A}_{\mathcal{R}}$ *respectively with* $\big(C'_{\mathcal{R}} \setminus \{\mathcal{E}_{\mathcal{R}}\}\big) \cap U' = \big(C''_{\mathcal{R}} \setminus \{\mathcal{A}_{\mathcal{R}}\}\big) \cap U''$.

(5) $\mathscr{M}^{\mathrm{s}}(S)$ *is a smooth complex curve in a neighbourhood of every non-filtrable bundle.*

Moreover, $\mathscr{M}^{\mathrm{s}}(S)$ *is regular in all smooth points.*

We have depicted this situation in figure 4.1 (dividing the real dimension by two). The vertical arrows symbolise identification of the corresponding curves with exception of the two points joined by the dotted line. We can regard the curves $C'_{\mathcal{R}}$ and $C''_{\mathcal{R}}$ as one single curve with a double point consisting of $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{A}_{\mathcal{R}}$. This "curve" is smooth at the point $\mathcal{A}_{\mathcal{R}}$ but is transversely crossed by the curve $C_{\mathcal{R}}$ at the point $\mathcal{E}_{\mathcal{R}}$.

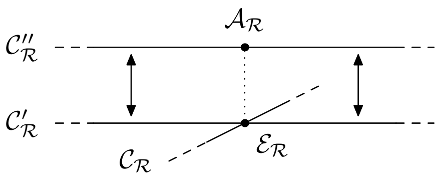

Figure 4.1. Local structure of the moduli space at $\mathcal{E}_{\mathcal{R}}$ *and* $\mathcal{A}_{\mathcal{R}}$

In the case of the parabolic Inoue surface and an Enoki surface the above theorem determines completely the structure of the moduli space in a neighbourhood of every filtrable bundle. Recall that for the parabolic Inoue surface $R(S) \setminus Q(S) = \varnothing$ so that the situation is particularly simple: Theorem 3.12 actually establishes an *isomorphism* between $\mathrm{Pic}^0(S) \setminus Q(S)$ and the filtrable part of the moduli space of simple bundles, given by $\mathcal{L} \mapsto \mathcal{E}_{\mathcal{L}}$. For an Enoki surface the isomorphisms $\mathcal{A}_{\mathcal{R}} \cong \mathcal{E}_{\mathcal{R}^\vee(-C)}$ immediately tell us that $C''_{\mathcal{R}} = C_{\mathcal{R}^\vee(-C)}$. In the remaining case of the half Inoue surface the situation is slightly more complicated. We can not yet identify the curves $C'_{\mathcal{R}}$ and $C''_{\mathcal{R}}$ and will do this in section 7.

5. Stability

The moduli space important for gauge theory is the moduli space of stable holomorphic bundles and is a *Hausdorff* complex space. Stability is defined with respect to a *Gauduchon metric* g on S which is a Hermitian metric whose associated $(1,1)$-form ω_g verifies $\partial\bar\partial\omega_g = 0$. Such a metric always exists [18] and allows one to define the *degree map* by

$$\begin{array}{rccl} \deg\colon \operatorname{Pic}(S) & \longrightarrow & \mathbb{R} & \\ \mathcal{L} & \longmapsto & \deg\mathcal{L} := \displaystyle\int_S c_1(\mathcal{L}, A_h) \wedge \omega_g \,, \end{array}$$

where $c_1(\mathcal{L}, A_h)$ is the first Chern form associated to the Chern connection A_h of a Hermitian metric h in $\mathcal{L}$ (i.e. locally $c_1(\mathcal{L}, A_h) = \partial\bar\partial \log h$). This map is a Lie group morphism and independent of the particular choice of h. Note that on non-Kähler surfaces the degree map is never a topological invariant and therefore non-constant on $\operatorname{Pic}^0(S)$ [29].

Example 5.1. —
1. $\deg\mathcal{O} = \deg\mathcal{F} = 0$ because the square roots of $\mathcal{O}$ are torsion elements in $\operatorname{Pic}^0(S)$.
2. $\deg\mathcal{O}(D) = \operatorname{vol} D > 0$ for any divisor $D > 0$ on S. This is a consequence of the Poincaré-Lelong formula [19].
3. On the half and the parabolic Inoue surface $\deg\mathcal{K} < 0$ for any Gauduchon metric. This follows from the previous examples together with (2.2) and (2.3) respectively. For Enoki surfaces $\deg\mathcal{K}$ attains every value in $\mathbb{R}$ when g varies in the space of Gauduchon metrics. This was shown in [37], based on results of [6].
4. $\deg\mathcal{R} \geqslant 0$ if $\mathcal{R} \in R(S)$, because $\mathcal{R}^2 \cong \mathcal{O}(D)$ for some divisor $D \geqslant 0$.
5. Likewise, $\deg\mathcal{L} \geqslant \frac{1}{2}\deg\mathcal{K}$ if $\mathcal{L} \in Q(S)$.

Now the (slope-)stability is defined using the *g-slope* of a coherent sheaf $\mathscr{S}$

$$\mu_g(\mathscr{S}) := \frac{\deg\det\mathscr{S}}{\operatorname{rank}\mathscr{S}}\,.$$

DEFINITION 5.2. — *A holomorphic rank two vector bundle $\mathcal{E}$ over a complex surface S is called g-*stable *if for every rank one subsheaf $\mathscr{S} \subset \mathcal{E}$ we have $\mu_g(\mathscr{S}) < \mu_g(\mathcal{E})$.*

This definition simplifies in our case to:

PROPOSITION 5.3. — *A holomorphic vector bundle $\mathcal{E}$ of type* (3.1) *on a class* VII_0^1 *surface S is g-stable if and only if for every holomorphic line-subbundle $\mathcal{L} \subset \mathcal{E}$ we have* $\deg\mathcal{L} < \frac{1}{2}\deg\mathcal{K}$.

The proof is standard, see for example [23]. Non-filtrable bundles are stable by definition and stable bundles are simple [23], so it remains to examine stability for simple filtrable bundles (cf. section 3).

Proposition 5.4. — (1) *On the half Inoue surface the bundle* $\mathcal{A}_{\mathcal{O}} \cong \mathcal{A}_{\mathcal{F}}$ *has exactly two holomorphic line subbundles, namely* $\mathcal{K}$ *and* $\mathcal{F} \otimes \mathcal{K}$.

(2) *On an Enoki surface the bundles* $\mathcal{A}_{\mathcal{R}} \cong \mathcal{E}_{\mathcal{R}^\vee(-C)}$ *have exactly two holomorphic line subbundles, namely* $\mathcal{R}^\vee \otimes \mathcal{K}$ *and* $\mathcal{R}^\vee(-C)$.

(3) *On an arbitrary class* VII_0^1 *surface a bundle* $\mathcal{E}_{\mathcal{L}}$ *has no holomorphic line subbundle other than* $\mathcal{L}$ *if it does not belong to case (2).*

Proof. — By definition the bundles $\mathcal{E}_{\mathcal{L}}$ and $\mathcal{A}_{\mathcal{R}}$ have as holomorphic line subbundles $\mathcal{L}$ and $\mathcal{R}^\vee \otimes \mathcal{K}$ respectively. By corollary 3.3 every other inclusion of a holomorphic line bundle into $\mathcal{E}_{\mathcal{L}}$ or $\mathcal{A}_{\mathcal{R}}$ extends to an extension of type (3.3). This extension is non-trivial if the bundle $\mathcal{E}_{\mathcal{L}}$ respectively $\mathcal{A}_{\mathcal{R}}$ is simple and thus determines an isomorphism (3.7) of the corresponding central terms. The proposition follows now from the classification (3.11)–(3.13) of all possible such isomorphisms. □

Corollary 5.5. — (1) *On the half Inoue surface the bundle* $\mathcal{A}_{\mathcal{O}} \cong \mathcal{A}_{\mathcal{F}}$ *is g-stable for any Gauduchon metric g.*

(2) *On an Enoki surface the bundle* $\mathcal{A}_{\mathcal{R}} \cong \mathcal{E}_{\mathcal{R}^\vee(-C)}$ *is g-stable if and only if*

$$\begin{cases} \deg \mathcal{R}^\vee(-C) < \frac{1}{2}\deg K & \text{in case} \quad \deg \mathcal{K} < 0 \\ \frac{1}{2}\deg K < \deg \mathcal{R} & \text{in case} \quad \deg \mathcal{K} \geqslant 0 \end{cases}.$$

In either case one inequality implies the other.

(3) *On an arbitrary class* VII_0^1 *surface a bundle* $\mathcal{E}_{\mathcal{L}}$ *not belonging to case (2) is g-stable if and only if* $\deg \mathcal{L} < \frac{1}{2}\deg \mathcal{K}$.

Proof. — Combine the previous two propositions with the examples 5.1. □

Remark 5.6. — We see that always at least one of the two non-separable bundles $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{A}_{\mathcal{R}}$ is unstable as it should be, for the moduli space of stable bundles is Hausdorff.

The degree homomorphism is non-constant on $\mathrm{Pic}^0(S) \cong \mathbb{C}^*$, so the degree corresponds to a non-zero multiple of the logarithm of the modulus in $\mathbb{C}$. Regarding 5.5(3) we fix an isomorphism $\mathrm{Pic}^0(S) \cong \mathbb{C}^*$ that identifies

$$\mathrm{Pic}^0_{<\varrho}(S) := \{\mathcal{L} \in \mathrm{Pic}^0(S) \colon \deg \mathcal{L} < \varrho\} \qquad \varrho := \tfrac{1}{2}\deg \mathcal{K}$$

to an open disc in $\mathbb{C}$ with radius ρ, punctured at the center 0, corresponding to $\deg\mathcal{L} \to -\infty$. In view of 5.5(2) we also define the set

$$U(S) := \{\mathcal{R}^\vee(-C) \in \mathrm{Pic}^0(S)\colon \mathcal{R} \in R(S),\ \deg\mathcal{R} \leqslant \tfrac{1}{2}\deg K\}\,. \tag{5.1}$$

From 5.1(4) we see that $U(S) = \varnothing$ if $\deg\mathcal{K} < 0$ — in particular if S is the half or the parabolic Inoue surface. If S is an Enoki surface then $U(S)$ is the finite set consisting of those line bundles $\mathcal{L} \in \mathrm{Pic}^0(S)$ with $\deg\mathcal{L} < \varrho$ that define an unstable bundle $\mathcal{E}_\mathcal{L}$. Note that under the map $\mathcal{R} \mapsto \mathcal{R}^\vee(-C)$ the set $U(S)$ is in bijection to the set $R_{\leqslant\varrho}(S) := R(S) \cap \mathrm{Pic}^0_{\leqslant\varrho}(S)$ defining singular stable points $\mathcal{E}_\mathcal{R}$ in the moduli space.

Corollary 5.7. — *The filtrable part of the moduli space of g-stable holomorphic bundles is bijectively parametrised by*

- $\mathrm{Pic}^0_{<\varrho}(S)$ *if S is the parabolic Inoue surface ($U(S) = \varnothing$),*
- $\mathrm{Pic}^0_{<\varrho}(S) \cup \{0\}$ *if S is the half Inoue surface ($U(S) = \varnothing$) and*
- $\mathrm{Pic}^0_{<\varrho}(S) \setminus U(S)$ *if S is an Enoki surface,*

mapping $\mathrm{Pic}^0_{<\varrho}(S) \ni \mathcal{L} \mapsto \mathcal{E}_\mathcal{L}$ *and* $0 \mapsto \mathcal{A}_\mathcal{O}$.

Proof. — This follows from the above corollary together with the theorems 3.11, 3.12, 3.14 and the observation from example 5.1(5) that $Q(S) \cap \mathrm{Pic}^0_{<\varrho}(S) = \varnothing$. □

6. The boundary of the moduli space of polystable bundles

We want to compute the moduli spaces of polystable holomorphic bundles of type (3.1) for any class VII_0^1 surface S. Throughout this section we fix S and omit it in our notations.

Definition 6.1. — *A holomorphic rank two vector bundle $\mathcal{E}$ is g-*polystable *if it is g-stable (definition 5.2) or if*

$$\mathcal{E} = \mathcal{L} \oplus \mathcal{M} \qquad \textit{with} \quad \deg\mathcal{M} = \deg\mathcal{L}\,. \tag{6.1}$$

In the latter case we call $\mathcal{E}$ a split g-polystable bundle. *We denote by*

$$\mathscr{M}^{(\mathrm{p})\mathrm{st}} := \{\mathcal{E} \text{ (poly)stable hol. str. on } E\colon \det\mathcal{E} \cong \mathcal{K}\}/\Gamma\big(S, \mathrm{GL}(E)\big)$$

the moduli space of (poly)stable holomorphic bundles of type (3.1).

In the previous sections we showed that there is an injection of $\mathrm{Pic}^0_{<\varrho} \setminus U$ into the filtrable part of the moduli space $\mathscr{M}^{\mathrm{st}}$ of stable bundles given by

$\mathcal{L} \mapsto \mathcal{E}_{\mathcal{L}}$ (corollary 5.7) and which is holomorphic on $\operatorname{Pic}^0_{<\varrho} \setminus \bigl(U \cup R_{\leqslant\varrho}\bigr)$ (proposition 4.2). Now define the closed punctured disc

$$\operatorname{Pic}^0_{\leqslant\varrho} := \{\mathcal{L} \in \operatorname{Pic}^0 \colon \deg \mathcal{L} \leqslant \varrho\} \subset \operatorname{Pic}^0 \cong \mathbb{C}^* \qquad \varrho = \tfrac{1}{2} \deg \mathcal{K}\,.$$

Its boundary is the circle $\operatorname{Pic}^0_{=\varrho}(S)$ of line bundles $\mathcal{L} \in \operatorname{Pic}^0(S)$ with $\deg \mathcal{L} = \frac{1}{2} \deg \mathcal{K}$ and can be mapped to the split polystable bundles by $\mathcal{L} \mapsto \mathcal{L} \oplus (\mathcal{L}^\vee \otimes \mathcal{K})$.

In the following we use results about the gauge theoretical counterpart of our complex geometric moduli space. Equip the bundle E with a Hermitian metric h and fix a $(\det h)$-unitary connection a in the determinant line bundle $\det E = K$. Denote by

$$\begin{aligned}&\mathscr{M}^{\mathrm{ASD}}\\ &:= \{A \text{ } h\text{-unitary connection on } E\colon (F^0_A)^+ = 0, \det A = a\}/\Gamma\bigl(S, \mathrm{SU}(E)\bigr)\end{aligned}$$

the moduli space of oriented projectively anti-self-dual connections. A connection A is called *reducible* if there is an A-parallel splitting of E into two line bundles, i. e. $E = L \oplus M$ and $A = A_L \oplus A_M$ where A_L and A_M are connections on the line bundles L and M respectively. We write $\bigl(\mathscr{M}^{\mathrm{ASD}}\bigr)^*$ for the irreducible part of $\mathscr{M}^{\mathrm{ASD}}$ which is naturally a real analytic space.

The relation between this gauge theoretical moduli space of anti-self-dual connections and the complex geometric moduli space of holomorphic bundles is given by the Kobayashi-Hitchin correspondence [29], a natural real analytic isomorphism

$$\mathrm{KH}\colon \bigl(\mathscr{M}^{\mathrm{ASD}}\bigr)^* \xrightarrow{\cong} \mathscr{M}^{\mathrm{st}} \tag{6.2}$$

given by mapping the gauge equivalence class $[A]$ of an anti-self-dual connection A to the holomorphic structure in E determined by the corresponding $\bar\partial$-operator $\bar\partial_A$.

Now the second reason for our particular choice of the Chern classes (3.1a) of E becomes apparent. The moduli space $\mathscr{M}^{\mathrm{ASD}}$ has a natural compactification — the Uhlenbeck compactification [11], constructed by attaching further strata involving moduli spaces $\mathscr{M}^{\mathrm{ASD}}(E_k)$ of oriented ASD connections on rank two bundles E_k with

$$c_1(E_k) = c_1(E) \quad \text{and} \quad c_2(E_k) = c_2(E) - k\,, \qquad k = 1, 2, \dots\,.$$

But in our case (3.1a) assures that $4c_2(E_k) - c_1(E_k)^2 < 0$, a condition under which the expected dimension (1.1) of $\mathscr{M}^{\mathrm{ASD}}(E_k)$ is negative and the attached strata in the Uhlenbeck compactification of $\mathscr{M}^{\mathrm{ASD}}$ are all empty. This means that $\mathscr{M}^{\mathrm{ASD}}$ *is already compact* and the irreducible part $\bigl(\mathscr{M}^{\mathrm{ASD}}\bigr)^*$ of $\mathscr{M}^{\mathrm{ASD}}$ can be compactified by adding only the reducible part.

The latter can be shown to be the circle $iH^1(S,\mathbb{R})/2\pi i H^1(S,\mathbb{Z})$. In fact, applying the Kobayashi-Hitchin-correspondence for line bundles separately to the line bundles in the splitting (6.1) of a split polystable bundle maps the circle of split polystable bundles to this circle of reducible connections.

Putting together the above, we get the following commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}^0_{<\varrho} \setminus U \arrow[r] \arrow[d, hook] & \mathscr{M}^{\mathrm{st}} \arrow[r, "\cong", "\mathrm{KH}"'] \arrow[d, hook] & \left(\mathscr{M}^{\mathrm{ASD}}\right)^* \arrow[d, hook] \\
\mathrm{Pic}^0_{\leqslant\varrho} \setminus U \arrow[r] & \mathscr{M}^{\mathrm{pst}} \arrow[r, "\cong"] & \mathscr{M}^{\mathrm{ASD}}
\end{tikzcd}$$

where the vertical arrows are natural inclusions. Remark that a priori there is no natural topology on the moduli space of polystable bundles and the bijection $\mathscr{M}^{\mathrm{pst}} \to \mathscr{M}^{\mathrm{ASD}}$ is only set theoretical. It is turned tautologically into a homeomorphism by equipping $\mathscr{M}^{\mathrm{pst}}$ with the induced topology.

Proposition 6.2. — *The above inclusion* $\mathrm{Pic}^0_{\leqslant\varrho} \setminus U \hookrightarrow \mathscr{M}^{\mathrm{pst}}$ *maps* $\mathrm{Pic}^0_{\leqslant\varrho} \setminus \left(U \cup R_{\leqslant\varrho}\right)$ *homeomorphically to an open subspace of* $\mathscr{M}^{\mathrm{pst}}$*. In particular, if there is no bundle* $\mathcal{R} \in R$ *with* $\deg \mathcal{R} = \varrho$*,* $\mathscr{M}^{\mathrm{pst}}$ *possesses the structure of a real two-dimensional manifold with boundary in the neighbourhood of the image of the circle* $\mathrm{Pic}^0_{=\varrho}$*.*

Proof. — Using the following lemma, we can apply the proof of [36, proposition 4.4]. Remark that $\mathrm{Pic}^0_{=\varrho} \cap U = \varnothing$. $\square$

Lemma 6.3. — *Let* $\mathcal{E}$ *be a stable holomorphic bundle of type* (3.1) *and* $\varepsilon > 0$ *be sufficiently small. Then a line bundle* $\mathcal{M} \in \mathrm{Pic}^0$ *with* $H^0(\mathcal{M}^\vee \otimes \mathcal{E}) \neq 0$ *and* $\varrho - \varepsilon \leqslant \deg \mathcal{M} \leqslant \varrho$ *is unique.*

Proof. — The existence of such a line bundle $\mathcal{M}$ implies that $\mathcal{E}$ is filtrable and as in the proof of the equivalence in definition 3.1 we can construct a non-trivial sheaf morphism $\mathcal{M} \to \mathcal{L}$ to a line subbundle $\mathcal{L}$ of $\mathcal{E}$. So $\mathcal{M} \cong \mathcal{L}(-D)$ for some divisor $D \geqslant 0$ on S. Since $\mathcal{E}$ is stable we have $\deg \mathcal{L} < \varrho$ and $\mathrm{vol}\, D = \deg \mathcal{L} - \deg \mathcal{M} < \varepsilon$. If we choose ε less than the volume of any curve on the surface then $D = 0$ and $\mathcal{M} \cong \mathcal{L}$. But proposition 5.4 shows that a line subbundle $\mathcal{L} \in \mathrm{Pic}^0$ of $\mathcal{E}$ is unique. $\square$

The proof of proposition 6.2 fails at points $\mathcal{R} \in R$ with $\deg \mathcal{R} = \varrho$, cf. [36, lemma 4.3]. This can only occur on Enoki surfaces for Gauduchon metrics with $\deg \mathcal{K} > 0$ and we will account for this situation in the last section when we discuss the structure of the entire moduli space.

7. Non-filtrable holomorphic bundles

The next proposition [36, proposition 4.5] says that the structure of the moduli space around the origin is the natural one given by the closure $\mathrm{Pic}^0_{\leqslant\varrho}\cup\{0\}$ of $\mathrm{Pic}^0_{\leqslant\varrho}$ in $\mathbb{C}$.

Proposition 7.1. — *The inclusion* $\mathrm{Pic}^0_{\leqslant\varrho}\setminus U \hookrightarrow \mathscr{M}^{\mathrm{pst}}$ *extends to an inclusion*

$$\left(\mathrm{Pic}^0_{\leqslant\varrho}\cup\{0\}\right)\setminus U \hookrightarrow \mathscr{M}^{\mathrm{pst}}\,,$$

holomorphic at the centre 0*. Moreover,* 0 *is mapped to a bundle* $\mathcal{E}$ *verifying*

$$\mathcal{E}\otimes\mathcal{F}\cong\mathcal{E}\,, \tag{7.1}$$

where $\mathcal{F}$ *is the (unique) non-trivial square-root of* $\mathcal{O}$*.*

The invariance property (7.1) follows from the following lemma in the limit $\mathcal{L}\to 0$, or $\deg\mathcal{L}\to-\infty$, since $\deg(\mathcal{L}\otimes\mathcal{F})=\deg\mathcal{L}$.

Lemma 7.2. — $\mathcal{E}_{\mathcal{L}}\otimes\mathcal{F}\cong\mathcal{E}_{\mathcal{L}\otimes\mathcal{F}}$ *and* $\mathcal{A}_{\mathcal{R}}\otimes\mathcal{F}\cong\mathcal{A}_{\mathcal{R}\otimes\mathcal{F}}$*.*

Proof. — First notice that $\mathcal{E}_{\mathcal{L}}\otimes\mathcal{F}$ and $\mathcal{A}_{\mathcal{R}}\otimes\mathcal{F}$ are of type (3.1). Tensorise the defining extensions for $\mathcal{E}_{\mathcal{L}}$ and $\mathcal{A}_{\mathcal{R}}$ by $\mathcal{F}$ and compare with the defining extensions for $\mathcal{E}_{\mathcal{L}\otimes\mathcal{F}}$ and $\mathcal{A}_{\mathcal{R}\otimes\mathcal{F}}$ respectively. ☐

This also makes explicit the $\mathbb{Z}_2$ symmetry of the moduli spaces of bundles of type (3.1) under tensorising with the square roots of $\mathcal{O}$. We see that (7.1) holds for $\mathcal{A}_{\mathcal{O}}\cong\mathcal{A}_{\mathcal{F}}$.

Corollary 7.3. — *On the half Inoue surface,* $\mathcal{E}$ *is the filtrable bundle* $\mathcal{A}_{\mathcal{O}}$*. On an Enoki or the parabolic Inoue surface* $\mathcal{E}$ *is a non-filtrable bundle.*

Proof. — Suppose $\mathcal{E}_{\mathcal{L}}\otimes\mathcal{F}\cong\mathcal{E}_{\mathcal{L}}$ for S an arbitrary class VII_0^1 surface. Then $\mathcal{E}_{\mathcal{L}\otimes\mathcal{F}}\cong\mathcal{E}_{\mathcal{L}}$ by lemma 7.2 and thus $\mathcal{L}\otimes\mathcal{F}\cong\mathcal{L}$ by corollary 3.9, contradicting the non-triviality of $\mathcal{F}$. Therefore either $\mathcal{E}$ is non-filtrable or S is the half Inoue surface and $\mathcal{E}\cong\mathcal{A}_{\mathcal{O}}$. $\mathcal{E}$ cannot be non-filtrable on the half Inoue surface because this would imply that $\mathcal{A}_{\mathcal{O}}$ lies on another component of the moduli space. But this is excluded by corollary 7.8 below. ☐

Remark 7.4. — One can show that (7.1) implies that the pull-back of $\mathcal{E}$ to a double cover of S splits into a sum of two line bundles.

For a complete description it only remains to show that our moduli spaces do not contain further connected components. Non-filtrable bundles are stable by definition and we saw that all unstable filtrable bundles lie on the component we already described. Thus another component would be contained in the moduli space of polystable bundles and therefore be compact.

But M. Toma showed that this is impossible on blown-up primary Hopf surfaces [41] and we know that every class VII_0 surface containing a global spherical shell — in particular every class VII_0^1 surface — is a degeneration of blown-up primary Hopf surfaces [22]. In the following we will prove that a compact component in the moduli space would be preserved under small deformations. We do this using a third guise of our moduli space, justifying at the same time, finally, why we speak of "PU(2)-instantons".

Let P be the principal $\mathrm{PU}(2)$-bundle obtained as the quotient of the principal $\mathrm{U}(2)$ frame bundle of E by the centre of $\mathrm{U}(2)$. Remark that the adjoint action Ad of $\mathrm{SU}(2)$ on itself descends to an action of $\mathrm{PU}(2) \cong \mathrm{SU}(2)/\{\pm 1\}$ on $\mathrm{SU}(2)$ so that we can define the gauge group $\mathscr{G} := \Gamma(P \times_{\mathrm{Ad}} \mathrm{SU}(2))$. This group acts naturally on the affine space $\mathscr{A}$ of connections on P. We call a connection irreducible if its stabiliser in $\mathscr{G}$ is minimal, i. e. the center $\{\pm 1\}$ of $\mathscr{G}$, and denote by $\mathscr{A}^*$ the space of irreducible connections. The moduli space of irreducible anti-self dual connections on P is now defined as the quotient

$$\mathscr{M}^{\mathrm{ASD}}(P)^* := \{A \in \mathscr{A}^* \colon F_A^+ = 0\}/\mathscr{G}$$

where F_A^+ denotes the self-dual part of the curvature F_A of A. There is a canonical isomorphism

$$\mathscr{M}^{\mathrm{ASD}}(P)^* \cong \mathscr{M}^{\mathrm{ASD}}(E)^*$$

with the moduli space of irreducible anti-self-dual connections on E from the previous section, independent of the fixed connection a on $\det E$. This independence will allow us to construct a parametrised moduli space for a deformation of our surface.

To do this we write this moduli space in a different way as follows. The space $\mathscr{A}^*$ is a principal $\mathscr{G}/\{\pm 1\}$-bundle over the corresponding orbit space $\mathscr{B}^* := \mathscr{A}^*/\mathscr{G}$. The map $F^+ \colon \mathscr{A} \to \Omega^2_+(\mathrm{ad}\, P)$ associating to a connection A the self-dual part F_A^+ of its curvature is $\mathscr{G}$-equivariant and therefore defines a section $F^+ \colon \mathscr{B}^* \to \mathscr{E}$ in the associated vector bundle $\mathscr{E} := \mathscr{A}^* \times_{\mathrm{ad}} \Omega^2_+(\mathrm{ad}\, P)$ over $\mathscr{B}^*$. The moduli space $\mathscr{M}^{\mathrm{ASD}}(P)^*$ is then simply the vanishing locus of this section. Using suitable Sobolev completions, F^+ is a Fredholm map between Banach manifolds. A set $\mathscr{C} \subset \mathscr{M}^{\mathrm{ASD}}(P)^*$ is said to be regular if F^+ is regular at every point of $\mathscr{C}$. The following proposition allows one to check regularity using the complex geometric framework. It results from comparing the local models of the moduli spaces $\left(\mathscr{M}^{\mathrm{ASD}}\right)^*$ and $\mathscr{M}^{\mathrm{st}}$ [29].

Proposition 7.5. — *A point in $\left(\mathscr{M}^{\mathrm{ASD}}\right)^*$ is regular if and only if its image in $\mathscr{M}^{\mathrm{st}}$ under the Kobayashi-Hitchin-correspondence* (6.2) *is regular.*

Corollary 7.6. — *For a class* VII_0^1 *surface* S *every compact component* $\mathscr{C} \subset \mathscr{M}^{\mathrm{ASD}}(S)^*$ *is regular.*

Proof. — By proposition 4.2, $\mathscr{M}^{\mathrm{st}}(S)$ is regular at every smooth point and we saw that all singular points lie on a non-compact component. $\square$

We show that in general a regular compact component of the moduli space of irreducible anti-self-dual connections is preserved under small deformations of the metric. For this we consider a parametrised version of the above construction of the moduli space $\mathscr{M}^{\mathrm{ASD}}(P)^*$. Let I be the interval $[-1,+1]$ and $(g_t)_{t\in I}$ a smooth one-parameter family of Riemannian metrics g_t on the base manifold. Again, $\underline{\mathscr{A}}^* := \mathscr{A}^* \times I$ is a principal $\mathscr{G}/\{\pm 1\}$-bundle over $\underline{\mathscr{B}}^* := \mathscr{B}^* \times I$. The map $\underline{F}^+ \colon \underline{\mathscr{A}} \to \Omega^2(\operatorname{ad} P)$, defined by mapping (A,t) to the self-dual part $F_A^{+_t}$ of the curvature F_A with respect to the metric g_t, is $\mathscr{G}$-equivariant and defines a section $\underline{\mathscr{B}}^* \to \underline{\mathscr{A}}^* \times_{\mathrm{ad}} \Omega^2(\operatorname{ad} P)$. This section actually takes values in the subbundle $\underline{\mathscr{E}}$ whose fibre over $([A],t)$ is the space $\Omega^2_{+_t}(\operatorname{ad} P)$ of $(\operatorname{ad} P)$-valued two-forms that are self-dual with respect to the metric g_t. This gives a section $\underline{F}^+ \colon \underline{\mathscr{B}}^* \to \underline{\mathscr{E}}$ whose vanishing locus is the parametrised moduli space

$$\big(\underline{\mathscr{M}}^{\mathrm{ASD}}\big)^* := \Big\{\big([A],t\big) \in \mathscr{B}^* \times I \colon F_A^{+_t} = 0\Big\}.$$

The restriction $\pi \colon \big(\underline{\mathscr{M}}^{\mathrm{ASD}}\big)^* \to I$ of the projection $\mathscr{B}^* \times I \to I$ gives a fibration

$$\big(\underline{\mathscr{M}}^{\mathrm{ASD}}\big)^* = \bigcup_{t\in I} \pi^{-1}(t) \qquad \text{with} \quad \pi^{-1}(t) = \mathscr{M}^{\mathrm{ASD}}(g_t)^* \times \{t\}\,.$$

Proposition 7.7. — *For t sufficiently small* $\mathscr{M}^{\mathrm{ASD}}(g_t)^*$ *contains a regular compact component if* $\mathscr{M}^{\mathrm{ASD}}(g_0)^*$ *does.*

Proof. — Let $\mathscr{C} \subset \mathscr{M}^{\mathrm{ASD}}(g_0)^*$ be such a regular compact component. The restriction of $\underline{F}^+$ to $\mathscr{M}^{\mathrm{ASD}}(g_0)^* = \pi^{-1}(0)$ is just the above map F^+ and thus regular on $\mathscr{C}$. Therefore $\underline{F}^+$ itself is regular on $\mathscr{C}$. Regularity is an open condition so $\underline{F}^+$ is regular on an open neighbourhood N of $\mathscr{C}$ in $\big(\underline{\mathscr{M}}^{\mathrm{ASD}}\big)^*$. It follows that N is a finite-dimensional smooth open manifold. Then, as $\mathscr{C}$ is compact, we can choose a compact neighbourhood K of $\mathscr{C}$ in N with $K \cap \pi^{-1}(0) = \mathscr{C} \subset \mathring{K}$. We have $\mathring{K} \cap \pi^{-1}(0) = K \cap \pi^{-1}(0)$. It suffices to show that $\mathring{K} \cap \pi^{-1}(t) = K \cap \pi^{-1}(t)$ for t sufficiently small. Suppose not. Then there exists a sequence of points $\big([A_n],t_n\big) \in (K \setminus \mathring{K}) \cap \pi^{-1}(t_n)$ with $t_n \to 0$. But K being compact, some subsequence of it converges to a point $([A],0) \in (K \setminus \mathring{K}) \cap \pi^{-1}(0) = \varnothing$ which is a contradiction. $\square$

Corollary 7.8. — *For a class* VII_0^1 *surface* S *all moduli spaces*

$$\mathscr{M}^{\mathrm{st}}(S) \cong \mathscr{M}^{\mathrm{ASD}}(S)^*, \quad \mathscr{M}^{\mathrm{pst}}(S) \cong \mathscr{M}^{\mathrm{ASD}}(S) \quad \text{and} \quad \mathscr{M}^{\mathrm{s}}(S)$$

are connected.

Proof. — We saw that another connected component in one of these moduli spaces, other than the one we already described, would belong to $\mathscr{M}^{\mathrm{ASD}}(S)^*$ and therefore be compact. By corollary 7.6 it would also be regular. Let now $(J_t)_{t\in I}$ be a family of complex structures on the real manifold underlying S, parametrising a degeneration $(S_t)_{t\in I}$ of blown-up primary Hopf surfaces S_t, $t \neq 0$, into $S_0 := S$. We can take $(g_t)_{t\in I}$ to be a corresponding smooth family of Gauduchon metrics g_t on S_t. Indeed, a Gauduchon metric $g_t' := \mathrm{e}^{\varphi_t}\, g_t$ can be obtained from an arbitrary metric g_t by finding a solution φ_t to $\partial\bar\partial(\mathrm{e}^{\varphi_t}\,\omega_t) = 0$. Such a solution is smooth by elliptic regularity and unique up to a constant. But since existence is established, the smooth dependence on t results from the implicit function theorem applied to the map

$$\begin{array}{ccc} \Omega^0_0(S) \times I & \to & \Omega^4(S) \\ (\ \varphi\ ,\ t\ ) & \mapsto & \partial\bar\partial(\mathrm{e}^{\varphi}\,\omega_t) \end{array}$$

in suitable Sobolev completions, where $\Omega^0_0(S)$ denotes the set of functions $\varphi \in \Omega^0(S)$ verifying $\int_S \varphi\omega_0 \wedge \omega_0 = 0$. Now the preceding proposition says that for small t the moduli space $\mathscr{M}^{\mathrm{ASD}}(S_t)^*$ would contain a compact component too, contradicting [41]. $\square$

8. The moduli spaces

We can finally assemble all our results to a complete description of the moduli spaces. By a *compact complex space with smooth boundary* we mean a compact real analytic space with a smooth boundary structure and a possibly singular complex structure on its interior. We write "$(\leqslant)$" for "$< (\leqslant)$".

Theorem 8.1. — *Let* S *be a minimal class* VII *surface with* $b_2(S) = 1$.

(1) *If* $\deg K < 0$ *— i. e. if* S *is the half or the parabolic Inoue surface or an Enoki surface with* $\deg K < 0$ *— then the entire moduli space* $\mathscr{M}^{\mathrm{(p)st}}(S)$ *of (poly)stable holomorphic bundles of type* (3.1) *is bijectively parametrised by the open (closed) complex one-dimensional disc* $\mathrm{Pic}^0_{(\leqslant)\varrho}(S) \cup \{0\}$.

(2) *If* $\deg K \geqslant 0$ *— i. e.* S *is an Enoki surface with* $\deg K \geqslant 0$ *— then* $\mathscr{M}^{(\mathrm{p})\mathrm{st}}(S)$ *is bijectively parametrised by* $\left(\mathrm{Pic}^0_{(\leqslant)\varrho}(S) \cup \{0\}\right) \setminus U(S)$ *where* $U(S)$ *is the finite set* (5.1).

The parametrisation is given by mapping

$$\mathrm{Pic}^0_{=\varrho}(S) \ni \mathcal{L} \mapsto \mathcal{L} \oplus (\mathcal{L}^\vee \otimes \mathcal{K}) \,, \qquad \mathrm{Pic}^0_{<\varrho}(S) \ni \mathcal{L} \mapsto \mathcal{E}_{\mathcal{L}} \qquad \textit{and} \qquad 0 \mapsto \mathcal{E} \,,$$

where:

(3) *On the half Inoue surface,* $\mathcal{E}$ *is the filtrable bundle* $\mathcal{A}_{\mathcal{O}}$ *and* $\mathscr{M}^{(\mathrm{p})\mathrm{st}}(S)$ *contains no non-filtrable bundles.*

(4) *On an Enoki or parabolic Inoue surface* $\mathcal{E}$ *is the only non-filtrable bundle in* $\mathscr{M}^{(\mathrm{p})\mathrm{st}}(S)$.

In case (1) this is a homeomorphism, holomorphic on the stable part. In case (2) this is a local homeomorphism except at points $\mathcal{R} \in R(S)$*, holomorphic on the stable part minus* $R(S)$*.* $\mathscr{M}^{\mathrm{st}}(S)$ *is a one-dimensional complex space whose singularities are simple normal crossings at the points* $\mathcal{E}_{\mathcal{R}}$ *characterised by*

$$\lim_{\mathcal{L} \to \mathcal{R}^\vee(-C)} \mathcal{E}_{\mathcal{L}} = \mathcal{E}_{\mathcal{R}} = \lim_{\mathcal{L} \to \mathcal{R}} \mathcal{E}_{\mathcal{L}} \qquad \textit{for} \quad \mathcal{R}^\vee(-C) \in U(S) \,.$$

Their number $|U(S)|$ *is finite but unbounded if the metric varies in the space of Gauduchon metrics.*

Therefore, except for the case $\mathrm{Pic}^0_{=\varrho}(S) \cap R(S) \neq \varnothing$ *on an Enoki surface,* $\mathscr{M}^{\mathrm{pst}}(S)$ *is a one-dimensional compact complex space with smooth boundary a circle and interior* $\mathscr{M}^{\mathrm{st}}(S)$*, smooth in case (1) and in general singular in case (2).*

For an Enoki surface S the moduli space $\mathscr{M}^{\mathrm{pst}}(S)$ can be viewed as a closed complex disc with finitely many self intersections as in figure 8.1 (where we divided the real dimension by two). Notice that the degree corresponds to (the logarithm of) the “distance” from the center of the disc. In the limit case where a line bundle $\mathcal{R} \in R(S)$ happens to lie on the boundary circle of this disc, the self intersection is merely a “touch” of a point on the boundary circle with an interior point, but both points do not belong to the moduli space since they correspond to the unstable bundles $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}^\vee(-C)}$.

Since non-filtrable bundles are stable by definition, the above also completes our description of the moduli space $\mathscr{M}^{\mathrm{s}}(S)$ of simple holomorphic bundles of type (3.1). If S is the parabolic Inoue surface then $\mathscr{M}^{\mathrm{s}}(S)$ is simply isomorphic to $(\mathrm{Pic}^0(S) \cup \{0\}) \setminus Q(S)$, i. e. to the complex line $\mathbb{C}$ minus a discrete set of points.

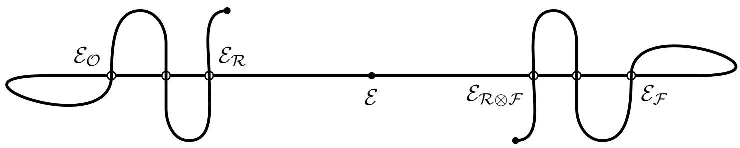

Figure 8.1. The moduli space of polystable bundles for an Enoki surface

If S is the half Inoue surface then, due to the isomorphism $\mathcal{A}_{\mathcal{F}} \cong \mathcal{A}_{\mathcal{O}}$, the smooth branches in the two local pictures in figure 4.1 for $\mathcal{R} = \mathcal{O}$ and $\mathcal{R} = \mathcal{F}$ coincide. With notations as in proposition 4.2, we can regard the curves $C''_{\mathcal{O}} = C''_{\mathcal{F}}$, $C'_{\mathcal{O}}$ and $C'_{\mathcal{F}}$ as one single "curve" with a triple point consisting of the three non-separable points $\mathcal{A}_{\mathcal{O}}$, $\mathcal{E}_{\mathcal{O}}$ and $\mathcal{E}_{\mathcal{F}}$. This curve is smooth at $\mathcal{A}_{\mathcal{O}}$ but transversely crossed by $C_{\mathcal{O}}$ at $\mathcal{E}_{\mathcal{O}}$ and by $C_{\mathcal{F}}$ at $\mathcal{E}_{\mathcal{F}}$. The resulting moduli space $\mathscr{M}^{\mathrm{s}}(S)$ is depicted in figure 8.2 (where the stable part is marked in bold and we omitted indicating the punctures corresponding to bundles in $Q(S)$).

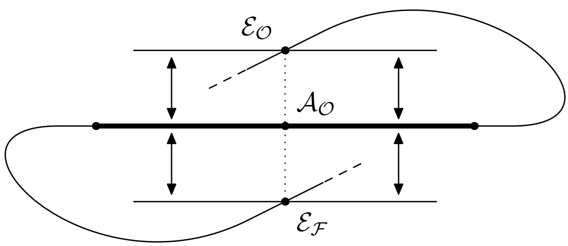

Figure 8.2. The moduli space for the half Inoue surface

If S is an Enoki surface then $\mathscr{M}^{\mathrm{s}}(S)$ contains no such triple points but countably infinitely many pairs of inseparable points $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}^\vee(-C)}$ corresponding to line bundles $\mathcal{R} \in R(S)$, the first of them being singular and the second smooth as in figure 4.1. This is shown in figure 8.3.

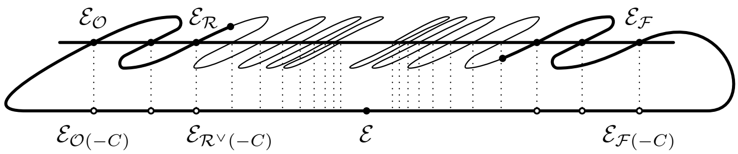

Figure 8.3. The moduli space for an Enoki surface

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