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Mladen BOŽIČEVIĆ

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LIMIT FORMULAS FOR GROUPS WITH ONE CONJUGACY CLASS OF CARTAN SUBGROUPS

by Mladen BOŽIČEVIĆ

Abstract. — Limit formulas for the computation of the canonical measure on a nilpotent coadjoint orbit in terms of the canonical measures on regular semisimple coadjoint orbits arise naturally in the study of invariant eigendistributions on a reductive Lie algebra. In the present paper we consider a particular type of the limit formula for canonical measures which was proposed by Rossmann. The main technical tool in our analysis are the results of Schmid and Vilonen on the equivariant sheaves on the flag variety and their characteristic cycles. We combine the theory of Schmid and Vilonen, and the work of Rossmann to compute canonical measures on nilpotent orbits for the real semisimple Lie groups with one conjugacy class of Cartan subgroups.

Résumé. — Les formules limites qui relient la mesure canonique sur une orbite coadjointe nilpotente aux mesures canoniques sur les orbites semi-simples régulières jouent un rôle important dans les études des distributions invariantes sur les groupes de Lie réels réductifs. Le but de cet article est d'étudier un type particulier de la formule limite proposée par Rossmann. En utilisant les résultats de Schmid et Vilonen concernant les faisceaux équivariants sur la variété de drapeaux d'une algèbre de Lie réductifs, nous calculons les mesures invariantes associées aux orbites nilpotentes pour les groupes de Lie semi-simples ayant l'unique classe de conjugaison de sous-groupes de Cartan.

Introduction

Let $G_{\mathbb{R}}$ be a semisimple Lie group, $\mathfrak{g}_{\mathbb{R}}$ the Lie algebra of $G_{\mathbb{R}}$, $\mathfrak{g}$ the complexification of $\mathfrak{g}_{\mathbb{R}}$, and X the flag variety of $\mathfrak{g}$. In case $\mathfrak{g}_{\mathbb{R}}$ has a complex structure it was observed first by Rossmann [16] that the invariant eigendistributions on $\mathfrak{g}_{\mathbb{R}}$ can be expressed as integrals of certain equivariant forms over homology classes on the conormal variety of $G_{\mathbb{R}}$-action on X. These ideas were later refined and generalized to arbitrary semisimple groups by

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Schmid and Vilonen [19]. The formulas that relate invariant eigendistributions and homology classes are usually called Rossmann integral formulas. They have proved to be important in studying asymptotic properties of invariant eigendistributions, and in particular, for computing the Liouville measure on a coadjoint nilpotent orbit in terms of Liouville measures on regular semisimple orbits. The corresponding formulas are known as limit formulas. They already appear in the classical work of Harish-Chandra on the harmonic analysis on semisimple groups. Namely, the simplest example of limit formulas is the Harish-Chandra's formula for delta function at zero.

Liouville measures on nilpotent orbits for complex groups were computed independently by Rossmann [16] and Hotta and Kashiwara [12], and for special orbits by Barbasch and Vogan [1] [2]. Rossmann proposed in [15] a method for computing nilpotent Liouville measures for arbitrary semisimple groups, which was based on his theory of Weyl group representations on homology classes of conormal varieties, and on the notion of character contours. Subsequent work of Schmid and Vilonen on the characteristic cycles of equivariant sheaves provides the tools for the analysis of cycles that enter the integral formulas. The main goal of the present paper is to combine and relate the methods of Schmid and Vilonen [18] [20] to those of Rossmann [16] [17], and to use them to compute Liouville measures for semisimple groups with one conjugacy class of Cartan subgroups. We should point out that our hypothesis on a group is quite restrictive. If $G_{\mathbb{R}}$ is a simple Lie group with one conjugacy class of Cartan subgroups, which is neither complex nor compact, then $\mathfrak{g}_{\mathbb{R}}$ is one of the following three types: AII, DII, EIV [10], Ch.IX, 6.1, Ch.X, F.1-9. For such groups the structure of the real nilpotent cone is relatively simple: distinct real nilpotent orbits are non-conjugate under the action of the complex group. The fact that this is not true in general represents the major difficulty in extending the results of the present paper to an arbitrary semisimple group. The main ingredients in our analysis, Proposition 2.4 and Theorem 3.1, which relate the work of Schmid and Vilonen to the work of Rossmann, appropriately generalize to the setting of arbitrary semisimple groups, and perhaps could be considered even more interesting than the main result Theorem 3.4. In view of these facts, we expect some of the ideas introduced in the present paper will be useful in pursuing the problem of limit formulas in a more general context.

1. Preliminaries

Suppose $G_{\mathbb{R}}$ is a real, connected, linear, semisimple Lie group. We assume that $G_{\mathbb{R}}$ has a unique conjugacy class of Cartan subgroups. We embed $G_{\mathbb{R}}$ into a complexification G and denote by

$$\tau : G \longrightarrow G$$

the involution on G having $G_{\mathbb{R}}$ as the connected component of the set of fixed points. Next we choose a Cartan involution

$$\theta : G_{\mathbb{R}} \longrightarrow G_{\mathbb{R}},$$

and extend it to G. Denote by $K_{\mathbb{R}}$ resp. K the set of fixed points of θ on $G_{\mathbb{R}}$ resp. G. Observe that $\theta\tau$ is a Cartan involution on G. We denote by $U_{\mathbb{R}}$ the set of fixed points. Write $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{g}_{\mathbb{R}}$, $\mathfrak{k}_{\mathbb{R}}$, $\mathfrak{u}_{\mathbb{R}}$ for the Lie algebras of G, K, $G_{\mathbb{R}}$, $K_{\mathbb{R}}$, $U_{\mathbb{R}}$ respectively. Denote the involutions on $\mathfrak{g}$ induced by θ, τ by the same letters. In addition, let

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}\,,\ \ \mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be the eigenspace decompositions defined by θ. Let $(\, , \,)$ be the Killing form on $\mathfrak{g}$. We will use it whenever convenient to identify $\mathfrak{g}$ and the dual space $\mathfrak{g}^*$.

Now we fix a θ-stable Cartan subalgebra $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$. Let

$$\mathfrak{h}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}},\ \ \mathfrak{t}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}} \cap \mathfrak{k}_{\mathbb{R}},\ \mathfrak{a}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}} \cap \mathfrak{p}_{\mathbb{R}}$$

be the Cartan decomposition, and $\mathfrak{h}$ the complexification of $\mathfrak{h}_{\mathbb{R}}$. Denote by

$$\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$$

the root system of the pair $(\mathfrak{g}, \mathfrak{h})$. By our assumption on the group, $\mathfrak{h}_{\mathbb{R}}$ is both a fundamental and maximally split Cartan subalgebra, so there are no real and noncompact imaginary roots in Δ. Denote by Δ_c the set of compact imaginary and by Δ_{cx} the set of complex roots in Δ. Then we have

$$\Delta = \Delta_c \cup \Delta_{cx}.$$

We fix a positive subsystem $\Delta^+ \subset \Delta$ such that

$$\theta\Delta^+ = \Delta^+. \tag{1.1}$$

Let

$$\Pi = \{\alpha_1, \cdots, \alpha_k, \alpha_{k+1}, \cdots \alpha_l\}\,,\ \ \alpha_i \in \Delta_c,\ i \leqslant k;\ \alpha_j \in \Delta_{cx},\ j \geqslant k+1,$$

be the corresponding set of simple roots. Next we recall some facts about real Weyl groups following [23]. Write $Z_{G_{\mathbb{R}}}(A)$ (resp. $N_{G_{\mathbb{R}}}(A)$) for the centralizer (resp. normalizer) of $A \subset \mathfrak{g}$. Let

$$H_{\mathbb{R}} = Z_{G_{\mathbb{R}}}(\mathfrak{h}_{\mathbb{R}})$$

be the Cartan subgroup defined by $\mathfrak{h}_{\mathbb{R}}$. Set

$$W(G_{\mathbb{R}}, H_{\mathbb{R}}) = N_{G_{\mathbb{R}}}(\mathfrak{h}_{\mathbb{R}})/H_{\mathbb{R}}.$$

Given a root system R we denote by $W(R)$ the Weyl group of R. Recall that $W(R)$ is generated by the reflections s_α, $\alpha \in R$. In patricular, we write $W = W(\Delta)$. We will consider W also as a group of linear endomorphisms of $\mathfrak{h}$ and $\mathfrak{h}^*$. It is then not difficult to deduce

$$W(G_{\mathbb{R}}, H_{\mathbb{R}}) \subset W. \tag{1.2}$$

Observe that Δ_c is a root system. Set

$$W_c = W(\Delta_c)\,, \;\; 2\rho_c = \sum_{\alpha \in \Delta_c \cap \Delta^+} \alpha.$$

Then the condition

$$\Delta_{\mathbb{C}} = \{\alpha \in \Delta : (\alpha, \rho_c) = 0\}$$

defines a root system and $\theta\Delta_{\mathbb{C}} = \Delta_{\mathbb{C}}$. Observe that $\alpha \in \Delta_c$ implies $(\alpha, \rho_c) \neq 0$, hence

$$\Delta_{\mathbb{C}} \subset \Delta_{cx}\,.$$

Finally we set

$$W_{\mathbb{C}} = W(\Delta_{\mathbb{C}}), \;\; W^\theta = \{w \in W : w\theta = \theta w\}\,, \;\; W_{\mathbb{C}}^\theta = W_{\mathbb{C}} \cap W^\theta.$$

The next proposition is a special case of [23], 3.12, 4.16.

Proposition 1.1.

1. *$W_{\mathbb{C}}^\theta$ is generated by $s_\alpha s_{\theta\alpha}$, where $\alpha \in \Pi \cap \Delta_{\mathbb{C}}$.*
2. *W_c is a normal subgroup of W^θ and*

$$W^\theta = W_{\mathbb{C}}^\theta \ltimes W_c$$

3. *The embedding 1.2 induces an isomorphism*

$$W(G_{\mathbb{R}}, H_{\mathbb{R}}) \cong W^\theta.$$

Denote by $\mathcal{N}$ the set of nilpotent elements in $\mathfrak{g}$. There exists a natural bijection between the sets of $G_\mathbb{R}$-orbits in $\mathcal{N} \cap i\mathfrak{g}_\mathbb{R}$ and K-orbits in $\mathcal{N} \cap \mathfrak{p}$, called the Kostant-Sekiguchi correspondence [21]. We recall the construction. We say that elements (h, e, f) from $\mathfrak{g}$ form an SL_2-triple if the following commutation relations hold

$$[h, e] = 2e,\ [h, f] = -2f,\ [e, f] = h.$$

We choose an SL_2-triple (h, e, f) such that

$$e, f \in \mathfrak{p},\ \tau e = f\,, \tag{1.3}$$

and set

$$h' = e + f,\ e' = \frac{1}{2}(e - f - h),\ f' = \frac{1}{2}(f - e - h).$$

Then (h', e', f') is also an SL_2-triple in $\mathfrak{g}$, and it is not difficult to show that

$$h' \in \mathfrak{p}_\mathbb{R},\ e', f' \in i\mathfrak{g}_\mathbb{R},\ \theta e' = f'.$$

Put $\mathcal{V} = K \cdot e$ and $\mathcal{O} = G_\mathbb{R} \cdot e'$. Then the association

$$\mathcal{V} \mapsto \mathcal{O}$$

defines a bijection between finite sets $\mathcal{N} \cap \mathfrak{p}/K$ and $\mathcal{N} \cap i\mathfrak{g}_\mathbb{R}/G_\mathbb{R}$. This bijection has an additional important property. Let $\mathcal{O}_\mathbb{C}$ be a nilpotent G-orbit. Then

$$\mathcal{O}_\mathbb{C} \cap i\mathfrak{g}_\mathbb{R} \neq \emptyset \Leftrightarrow \mathcal{O}_\mathbb{C} \cap \mathfrak{p} \neq \emptyset,$$

and the Sekiguchi correspondence induces a bijection between finite sets of orbits

$$\mathcal{O}_\mathbb{C} \cap i\mathfrak{g}_\mathbb{R}/G_\mathbb{R} \longleftrightarrow \mathcal{O}_\mathbb{C} \cap \mathfrak{p}/K. \tag{1.4}$$

Our goal is to show that for groups with one conjugacy class of Cartan subgroups, if $\mathcal{O}_\mathbb{C} \cap i\mathfrak{g}_\mathbb{R} \neq \emptyset$, then it is a single $G_\mathbb{R}$-orbit. The proof of this fact is sketched in [22], Prop. 13. Here we present an alternative argument. First, we choose a set $A \subset \Delta_{cx}$ such that

$$\Delta_{cx} = A \cup \theta A$$

is a disjoint union. For $\alpha \in \Delta$ let $\mathfrak{g}_\alpha$ be the corresponding root space, and $X_\alpha \in \mathfrak{g}_\alpha$. We write the root space decomposition in the form

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta_c} \mathfrak{g}_\alpha + \sum_{\alpha \in A} \mathbb{C} \cdot (X_\alpha + \theta X_\alpha) + \sum_{\alpha \in A} \mathbb{C} \cdot (X_\alpha - \theta X_\alpha).$$

If $\alpha \in A$, then $\alpha|\mathfrak{t} = \theta\alpha|\mathfrak{t}$, hence the above decomposition implies

$$\Delta(\mathfrak{k}, \mathfrak{t}) = \Delta(\mathfrak{g}, \mathfrak{h})|\mathfrak{t}.$$

In particular, a positive root system in $\Delta(\mathfrak{k}, \mathfrak{t})$ is determined by

$$\Delta(\mathfrak{k}, \mathfrak{t})^+ = \Delta(\mathfrak{g}, \mathfrak{h})^+|\mathfrak{t}.$$

The corresponding closed chambers in $\mathfrak{h}$ and $\mathfrak{t}$ are given by

$$\overline{C}_{\mathfrak{g}} = \left\{x \in i\mathfrak{t}_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}} : \alpha(x) \geqslant 0,\, \alpha \in \Delta^+\right\},$$
$$\overline{C}_{\mathfrak{k}} = \left\{x \in i\mathfrak{t}_{\mathbb{R}} : \alpha(x) \geqslant 0, \alpha \in \Delta(\mathfrak{k}, \mathfrak{t})^+\right\}.$$

Proposition 1.2. — *Let $\mathcal{O}_{\mathbb{C}}$ be a nilpotent G-orbit. If $\mathcal{O}_{\mathbb{C}} \cap i\mathfrak{g}_{\mathbb{R}} \neq \emptyset$, then it is a single $G_{\mathbb{R}}$-orbit.*

Proof. — By the remark (1.4), it will suffice to prove that $\mathcal{O}_{\mathbb{C}} \cap \mathfrak{p}$ is a single K-orbit. Let $\mathcal{O}$ and $\mathcal{O}_1$ be K-orbits in $\mathcal{O}_{\mathbb{C}} \cap \mathfrak{p}$. Let (h, e, f) and (h_1, e_1, f_1) be SL_2-triples associated with orbits $\mathcal{O}$ and $\mathcal{O}_1$ as in (1.3). Then h, $h_1 \in i\mathfrak{k}_{\mathbb{R}}$, hence conjugating by $K_{\mathbb{R}}$, if necessary, we may assume h, $h_1 \in \overline{C}_{\mathfrak{k}}$. The definition of the positive root system $\Delta(\mathfrak{k}, \mathfrak{t})^+$ implies $\overline{C}_{\mathfrak{k}} \subset \overline{C}_{\mathfrak{g}}$, hence we also have h, $h_1 \in \overline{C}_{\mathfrak{g}}$. On the other hand by [9], Th. 2.2.4 $G \cdot h \cap \overline{C}_{\mathfrak{g}}$ is a single element, thus we obtain $h = h_1$. Finally, by [9], Th. 9.4.4 the triples (h, e, f) and (h, e_1, f_1) are K-conjugate. In particular $\mathcal{O} = \mathcal{O}_1$, as desired. □

Next we recall some facts on the $G_{\mathbb{R}}$-orbit structure of the flag variety. Denote by X the flag variety of Borel subalgebras of $\mathfrak{g}$. We view X as a homogeneous space for G. Matsuki [14] shows that the number of $G_{\mathbb{R}}$-orbits on X is finite. In the present setting these orbits can be described as follows. Given $w \in W$ write $\mathfrak{b}_w$ for the Borel subalgebra defined by the pair $(\mathfrak{h}, w\Delta^+)$ and $x_w \in X$ for the corresponding point. Set

$$S_w = G_{\mathbb{R}} \cdot x_w.$$

Then the map $w \mapsto S_w$ induces a bijection

$$W/W^\theta \longleftrightarrow X/G_{\mathbb{R}}.$$

Recall that

$$S_w \subset X \text{ open } \iff \theta(w\Delta^+) = w\Delta^+ \iff w \in W^\theta.$$

2. Intertwining functors

The goal of this section is to describe the K-group of $G_{\mathbb{R}}$-equivariant sheaves on X as a module for the Weyl group. Similar results, in the setting of $\mathcal{D}$-modules, appear in [22]. In view of our applications, it will be convenient to work in the setting of semi-algebraic sets and semialgebraic maps, as in [18], § 6, for example.

Given a real algebraic manifold Y we denote by $Sh_c(Y)$ the category of sheaves of (complex) vector spaces constructible for semi-algebraic stratifications on Y [18], § 6, and by $D(Y)$ the corresponding bounded derived category. Let $f : Y \longrightarrow Z$ be a semi-algebraic map of (locally compact) semi-algebraic sets. Then the notation for functors

$$Rf_* : D(Y) \longrightarrow D(Z), \ \ Rf_! : D(Y) \longrightarrow D(Z),$$
$$f^{-1} : D(Z) \longrightarrow D(Y), \ \ f^! : D(Z) \longrightarrow D(Y)$$

is the same as in [13], Ch. II, Ch. III. Suppose that A is a real algebraic group acting on Y. Then we denote by $Sh_{A,c}(Y)$ the full subcategory of A-equivariant sheaves in $Sh_c(Y)$ [3], 0.2, 1.10. We remark that the notion of equivariant derived category from [3] will not be used in this paper. We return now to the setting of flag variety X.

Following [18], § 7 we will define intertwining functors on $D(X)$. If $w \in W$ write $l(w)$ for the length function. Let

$$Y_w \subset X \times X$$

be the variety of pairs of Borel subalgebras in the relative position w, and

$$p_1, p_2 : Y_w \longrightarrow X$$

projections onto the first and second factor in $X \times X$. Then we define the intertwining functor attached to $w \in W$ by the formula:

$$I_w = Rp_{1*}p_2^{-1}[l(w)] : D(X) \longrightarrow D(X),$$

One can show that I_w is an equivalence of categories. Moreover, the equivalences I_w induce an action of the Weyl group W on the K-group $K(D(X))$. We write $[\mathcal{F}] \in K(D(X))$ for the image of an object $\mathcal{F}$ from $D(X)$. The action of W on $K(D(X))$ will be denoted by

$$w \cdot [\mathcal{F}] = [I_w(\mathcal{F})].$$

Observe that the $G_{\mathbb{R}}$-orbit stratification on X is semi-algebraic, and any $\mathcal{F} \in Sh_{G_{\mathbb{R}},c}(X)$ is constructible for the orbit stratification. We know that the category $Sh_{G_{\mathbb{R}},c}(X)$ is abelian, hence we can also define the K-group $K(Sh_{G_{\mathbb{R}},c}(X))$. It is known [19], 6.2 that $K(Sh_{G_{\mathbb{R}},c}(X))$ is generated by standard sheaves. We recall the definition. Let $S \subset X$ be a $G_{\mathbb{R}}$-orbit and τ an irreducible $G_{\mathbb{R}}$-equivariant local system on S. To the pair (S, τ) we associate the standard sheaf

$$\mathcal{I}(S, \tau) = i_{S*}(\tau).$$

Here $i_S : S \longrightarrow X$ denotes the inclusion map. We describe in more details $G_{\mathbb{R}}$-equivariant local systems on the orbit S. Recall that irreducible

$G_{\mathbb{R}}$-equivariant local systems on S are parametrized by irreducible representations of $Z_{G_{\mathbb{R}}}(x)/Z_{G_{\mathbb{R}}}(x)^{\circ}$, the group of connected components of the centralizer of $x \in S$ in $G_{\mathbb{R}}$. If x is fixed by a θ and τ-stable Cartan subgroup $H \subset G$ then

$$Z_{G_{\mathbb{R}}}(x)/Z_{G_{\mathbb{R}}}(x)^{\circ} \cong H \cap G_{\mathbb{R}}/(H \cap G_{\mathbb{R}})^{\circ}.$$

In particular, in our case $H \cap G_{\mathbb{R}}$ is connected, so it follows that the constant sheaf $\mathbb{C}_S$ is up to isomorphism the only $G_{\mathbb{R}}$-equivariant local system on S. Hence, we deduce

$$\dim_{\mathbb{C}} K(Sh_{G_{\mathbb{R}},c}(X))_{\mathbb{C}} = \#(W/W^{\theta}), \tag{2.1}$$

where the subscript $\mathbb{C}$ stands for the complexification of the K-group.

Following [19], § 10 we will recall the formulas for the action of simple reflections on standard modules. To simplify the notation we write $\mathcal{I}(S_w) = \mathcal{I}(S_w, \mathbb{C}_S)$, $w \in W$.

Lemma 2.1. — *Let $\alpha \in \Delta^+$ be a simple root, and $w \in W$.*

1. *If $w\alpha \in \Delta_c$ then $I_{s_\alpha}\mathcal{I}(S_w) = \mathcal{I}(S_w)[1]$.*
2. *If $w\alpha \in \Delta_{cx}$, and $\theta(w\alpha) \in -w\Delta^+$, then $I_{s_\alpha}\mathcal{I}(S_w) = \mathcal{I}(S_{ws_\alpha})[1]$.*

Proof. — We can argue similarly as in [19], 10.17 to prove both formulas. Actually, Schmid and Vilonen work with twisted equivariant sheaves, so in our case the argument is even simpler. □

Lemma 2.2. — *Let $w \in W$, $w \notin W^{\theta}$, and let S_w be the corresponding $G_{\mathbb{R}}$-orbit. There exist simple roots $\alpha_1, \cdots, \alpha_m \in \Delta^+$ such that*

$$I_{s_{\alpha_m}} \circ \cdots \circ I_{s_{\alpha_1}}(\mathcal{I}(S_w)) = \mathcal{I}(S_e)[m].$$

Proof. — For $v \in W$ set

$$D(S_v) = \left\{\alpha \in \Delta^+ :\ w\alpha \in \Delta^+_{cx},\ \theta w\alpha \in -w\Delta^+\right\}, \quad d(S_v) = \#(D(S_v)).$$

Then we have $d(S_v) = 0 \Leftrightarrow v \in W^{\theta}$. By the assumption $d(S_w) > 0$, hence we can find a simple root $\alpha_1 \in D(S_w)$ (compare [19], 9.1). It is not difficult to show $d(S_{ws_{\alpha_1}}) = d(S_w) - 1$. Now we use Lemma 2.1, and induction on $d(S_w)$ to complete the proof. □

Observe that $K(Sh_{G_{\mathbb{R}},c}(X))_{\mathbb{C}}$ is a subspace of $K(D(X))_{\mathbb{C}}$. We already remarked that standard sheaves generate $K(Sh_{G_{\mathbb{R}},c}(X))_{\mathbb{C}}$, hence the above lemmas impply that $K(Sh_{G_{\mathbb{R}},c}(X))_{\mathbb{C}}$ is W-invariant. In the following proposition we describe the W-module structure on $K(Sh_{G_{\mathbb{R}},c}(X))_{\mathbb{C}}$ more explicitly.

Proposition 2.3. — *Write $\epsilon_\theta(w) = (-1)^{l(w)}$ for $w \in W^\theta$. As a W-module $K(Sh_{G_{\mathbb{R}},c}(X))_{\mathbb{C}}$ is generated by $[\mathcal{I}(S_e)]$ and*

$$\mathrm{Ind}_{W^\theta}^{W}(\epsilon_\theta) \cong K(Sh_{G_{\mathbb{R}},c}(X))_{\mathbb{C}}.$$

Proof. — First we show

$$w \cdot [\mathcal{I}(S_e)] = \epsilon_\theta(w)[\mathcal{I}(S_e)],\ w \in W^\theta.$$

We use the result 1.1 on the structure of W^θ. It will suffice to check

$$s_{\theta\alpha}s_\alpha[\mathcal{I}(S_e)] = [\mathcal{I}(S_e)],$$

if α is a simple complex root. Observe that $\alpha \pm \theta\alpha$ are not roots, hence

$$s_{\theta\alpha}s_\alpha = s_\alpha s_{\theta\alpha}.$$

It follows that $s_{\theta\alpha}s_\alpha \in W^\theta$. By 2.1 we have

$$I_{s_\alpha}\mathcal{I}(S_e) = \mathcal{I}(S_{s_\alpha})[1] \ \text{ and } \ I_{s_{\theta\alpha}}\mathcal{I}(S_e) = \mathcal{I}(S_{s_{\theta\alpha}})[1].$$

Since $s_{\theta\alpha}s_\alpha \in W^\theta$ we have $S_{s_\alpha} = S_{s_{\theta\alpha}}$. Finally, we conclude $I_{s_{\theta\alpha}s_\alpha}\mathcal{I}(S_e) = \mathcal{I}(S_e)$, as desired. To complete the proof, observe that we have a natural map

$$\mathrm{Ind}_{W^\theta}^{W}(\epsilon_\theta) \longrightarrow K(Sh_{G_{\mathbb{R}},c}(X))_{\mathbb{C}}.$$

By 2.2 this map is necessarily surjective, hence by (2.1) it is also an isomorphism. □

Finally, we relate the K-group of the $G_{\mathbb{R}}$-equivariant sheaves to the characteristic cycle construction. In order to explain this, we need some additional notation. If Y is a locally compact space, we denote by $H_i(Y, \mathbb{Z})$ resp. $H_i(Y, \mathbb{C})$, $i \in \mathbb{Z}$, the Borel-Moore homology groups with integral resp. complex coefficients. Suppose that Y is a real algebraic manifold. The characteristic cycle $CC(\mathcal{F})$ of a constructible sheaf $\mathcal{F}$ from $D(Y)$ was defined by Kashiwara [13], Ch.IX, [19]. Recall that $CC(\mathcal{F})$ is defined as a Lagrangian cycle in the real cotangent bundle T^*Y. In fact, let $\mathcal{S}$ be a semi-algebraic Whitney stratification on Y, and $\mathcal{F}$ a complex of sheaves on Y constructible for $\mathcal{S}$. Denote by $T^*_{\mathcal{S}}Y$ the union of conormal bundles to the strata. Then

$$CC(\mathcal{F}) \in H_m(T^*_{\mathcal{S}}Y, \mathbb{Z}), \qquad m = \dim_{\mathbb{R}} Y.$$

Returning to the flag variety X, denote by $T^*_{G_{\mathbb{R}}}X$ the union of the conormal bundles to the $G_{\mathbb{R}}$-orbits. Recall that CC is additive on exact sequences in $Sh_{G_{\mathbb{R}},c}(X)$, and for any $\mathcal{F}$ from $Sh_{G_{\mathbb{R}},c}(X)$, $CC(\mathcal{F})$ is supported in $T^*_{G_{\mathbb{R}}}X$. We conclude that the characteristic cycle map determines a homomorphism of abelian groups

$$CC : K(Sh_{G_{\mathbb{R}},c}(X)) \longrightarrow H_{2n}(T^*_{G_{\mathbb{R}}}X, \mathbb{Z}).$$

We will denote by the same symbol the complexified homomorphism

$$CC : K(Sh_{G_{\mathbb{R}},c}(X))_{\mathbb{C}} \longrightarrow H_{2n}(T^*_{G_{\mathbb{R}}}X, \mathbb{C}). \tag{2.2}$$

We know already that the structure of W-module on $K(Sh_{G_{\mathbb{R}},c}(X))_{\mathbb{C}}$ is defined by the intertwining functors. On the other hand, the structure of W-module on $H_{2n}(T^*_{G_{\mathbb{R}}}X, \mathbb{C})$ was defined by Rossmann. We refer to [18], § 8 for the details of Rossmann's construction. Then [18], 9.1 implies that 2.2 is a homomorphism of W-modules. By [7], 2.5, the characteristic cycles of standard sheaves generate (even over $\mathbb{Z}$) $H_{2n}(T^*_{G_{\mathbb{R}}}X, \mathbb{C})$. The next proposition will be the main ingredient in the proof of the limit formula. It follows immediately from the above discussion and equation (2.1).

Proposition 2.4. — *The homomorphism (2.2) is an isomorphism of W-modules.*

3. Limit formula

We begin by introducing two maps, the moment map and the twisted moment map, that are used to transfer geometric information from the cotangent bundle of the flag variety to the Lie algebra. Denote by T^*X the cotangent bundle of X. Given $x \in X$ denote by $\mathfrak{b}_x$ the Lie algebra of the Borel subgroup of G which normalizes x, and by $\mathfrak{b}_x^{\perp} \subset \mathfrak{g}^*$ the space of liner forms vanishing on $\mathfrak{b}_x$. We use the identification

$$T^*X \cong \left\{(x, \xi) : x \in X,\ \xi \in \mathfrak{b}_x^{\perp}\right\},$$

to consider T^*X as a submanifold of $X \times \mathfrak{g}^*$ The moment map of X is then defined by

$$\mu : T^*X \longrightarrow \mathfrak{g}^*, \qquad \mu(x, \xi) = \xi.$$

The definition of the twisted moment map is due to Rossmann [17], 2.3(5). We can use the decomposition

$$\mathfrak{g} = \mathfrak{h} + [\mathfrak{h}, \mathfrak{g}]$$

to view $\mathfrak{h}^*$ as a subspace of $\mathfrak{g}^*$. The twisted moment map depends on the parameter $\lambda \in \mathfrak{h}^*$. Observe that X is a homogeneous space for $U_{\mathbb{R}}$: $X = U_{\mathbb{R}} \cdot x_e$. Then we define the twisted moment map $\mu_\lambda : T^*X \longrightarrow G \cdot \lambda$ by the formula

$$\mu_\lambda(u \cdot x_e, \xi) = u \cdot \lambda + \mu(u \cdot x_e, \xi), \quad u \in U_{\mathbb{R}},\ \xi \in \mathfrak{b}^{\perp}_{u \cdot x_e}.$$

One can show that μ_λ is well-defined, and moreover, it is a $U_{\mathbb{R}}$-equivariant, real algebraic isomorphism if λ is regular.

Next we recall some facts on Weyl group representations. When

$$S \subset \mathfrak{g}^*$$

satisfies certain natural assumptions [16], II, § 2 Rossmann defines W-module structure on homology groups

$$H_*(\mu^{-1}(S), \mathbb{C}).$$

In particular, we obtain W-modules in the following cases:

$$S = i\mathfrak{g}_{\mathbb{R}}^* \cap \mathcal{N}^*,\ S = \overline{\mathcal{O}},\ S = \mathcal{O}, S = \{\nu\}\,.$$

Here $\mathcal{O}$ is a $G_{\mathbb{R}}$-orbit and $\nu \in \mathcal{N}^*$. In the first case we have

$$\mu^{-1}(i\mathfrak{g}_{\mathbb{R}}^* \cap \mathcal{N}^*) = T^*_{G_{\mathbb{R}}}X,$$

and the corresponding W-module structure was already considered in section 2. Rossmann shows [17], 4.4.1 that inclusions of the orbit closures are compatible with W-module structure on homology groups. In fact,

(3.1)

$$0 \longrightarrow H_{2n}(\mu^{-1}(\overline{\mathcal{O}} \setminus \mathcal{O}), \mathbb{C}) \longrightarrow H_{2n}(\mu^{-1}(\overline{\mathcal{O}}), \mathbb{C}) \longrightarrow H_{2n}(\mu^{-1}(\mathcal{O}), \mathbb{C}) \longrightarrow 0$$

is an exact sequence of W-modules. Denote by

$$C_G(\nu) \ \text{ resp } \ C_{G_{\mathbb{R}}}(\nu)$$

the group of connected components of the centralizer of ν in G resp. $G_{\mathbb{R}}$. Let

$$d = d(\nu) = \dim_{\mathbb{C}} \mu^{-1}(\nu).$$

Then $C_G(\nu)$ acts on $H_{2d}(\mu^{-1}(\nu), \mathbb{C})$ by permuting the irreducible components, and this action commutes with W-action. Hence

$$H_{2d}(\mu^{-1}(\nu), \mathbb{C})^{C_G(\nu)} \subset H_{2d}(\mu^{-1}(\nu), \mathbb{C})$$

and

$$H_{2d}(\mu^{-1}(\nu), \mathbb{C})^{C_{G_{\mathbb{R}}}(\nu)} \subset H_{2d}(\mu^{-1}(\nu), \mathbb{C})$$

are W-submodules, and the natural projection

$$H_{2d}(\mu^{-1}(\nu), \mathbb{C})^{C_{G_{\mathbb{R}}}(\nu)} \longrightarrow H_{2d}(\mu^{-1}(\nu), \mathbb{C})^{C_G(\nu)}$$

is a map of W-modules. Recall that the W-module $H_{2d}(\mu^{-1}(\nu), \mathbb{C})^{C_G(\nu)}$ is irreducible [17], Th. 4.5. This is the Springer representation associated to the orbit $G \cdot \nu$, and we denote the corresponding character by χ_ν. We will also need the following isomorphism of W-modules [17], 4.4.1:

$$H_{2n}(\mu^{-1}(\mathcal{O}), \mathbb{C}) \cong H_{2d}(\mu^{-1}(\nu), \mathbb{C})^{C_{G_{\mathbb{R}}}(\nu)}. \tag{3.2}$$

Next we introduce differential forms that will be used to define invariant distributions on the Lie algebra. Suppose $\mathcal{V}$ is a coadjoint G-orbit in $\mathfrak{g}^*$

or a coadjoint $G_{\mathbb{R}}$-orbit in $i\mathfrak{g}_{\mathbb{R}}^*$. To treat both cases simultaneously write $M = G$ or $M = G_{\mathbb{R}}$, and denote by $\mathfrak{m}$ the Lie algebra of M. The space

$$\mathfrak{m} \cdot \xi = \{\mathrm{ad}^*(x)(\xi) : x \in \mathfrak{m}\}$$

identifies with tangent space $T_\xi \mathcal{V}$ of $\mathcal{V}$ at ξ, and we define a M-equivariant 2-form $\sigma_{\mathcal{V}}$ on $\mathcal{V}$ by the formula

$$\sigma_{\mathcal{V},\xi}(x \cdot \xi, y \cdot \xi) = \xi[x, y]\,, \;\; x, y \in \mathfrak{m}.$$

In case $M = G_{\mathbb{R}}$ the form $-i\sigma_{\mathcal{V}}$ is real valued and we use the form

$$(-i\sigma_{\mathcal{V}})^k\,, \;\; 2k = \dim_{\mathbb{R}} \mathcal{V}$$

to orient $\mathcal{V}$. In this case we define the measure $m_{\mathcal{V}}$ by the formula

$$m_{\mathcal{V}} = \frac{1}{(2\pi i)^k k!}\sigma_{\mathcal{V}}^k, \tag{3.3}$$

and call it the Lioville measure. When $\mathcal{V} = M \cdot \lambda$, $\lambda \in \mathfrak{h}^*$, we will use the following notation

$$\sigma_{\mathcal{V}} = \sigma_\lambda.$$

Let $\lambda \in \mathfrak{h}^*$. Then a $U_{\mathbb{R}}$-equivariant 2-form τ_λ on X is defined at $x_e \in X$ by

$$\tau_\lambda(a_{x_e}, b_{x_e}) = \lambda([a, b]).$$

Here a_{x_e} and b_{x_e} denote the tangent vectors at $x_e \in X$, which a, $b \in \mathfrak{u}_{\mathbb{R}}$ induce by the differentiation of $U_{\mathbb{R}}$-action. Denote by

$$\pi : T^*X \longrightarrow X$$

the natural projection, and by σ the canonical symplectic form on T^*X. For $\lambda \in \mathfrak{h}^*_{\mathrm{reg}}$ the following formula holds [19], Prop. 3.3:

$$\mu_\lambda^*(\sigma_\lambda) = -\sigma + \pi^*(\tau_\lambda). \tag{3.4}$$

Next we recall, following [19], § 3, the definition of invariant distributions on the Lie algebra as integrals of certain differential forms over the semi-algebraic cycles in T^*X. The Fourier transform of a test function $\phi \in C_c^\infty(\mathfrak{g}_{\mathbb{R}})$ will be defined by

$$\hat{\phi}(\xi) = \int_{\mathfrak{g}_{\mathbb{R}}} e^{\xi(x)}\phi(x)dx\,, \;\; \xi \in \mathfrak{g}^*,$$

without the usual i in the exponential. Here dx denotes a suitably normalized Lebesgue measure on $\mathfrak{g}_{\mathbb{R}}$. Let Γ be a semi-algebraic chain in T^*X. We say that Γ is $\mathbb{R}$-bounded if

$$\mathrm{Re}\,\mu(\mathrm{supp}(\Gamma)) \subset \mathfrak{g}^*$$

is bounded. Here Re is defined with respect to $\mathfrak{g}_{\mathbb{R}}^*$. If Γ is a semi-algebraic, $\mathbb{R}$-bounded, $2n$-chain in T^*X one can prove that for a test function $\phi \in C_c^\infty(\mathfrak{g}_{\mathbb{R}})$ and $\lambda \in \mathfrak{h}^*$ the integral

$$\Theta(\Gamma,\lambda)(\phi) = \int_\Gamma \mu_\lambda^*(\hat{\phi})(-\sigma + \pi^*\tau_\lambda)^n \tag{3.5}$$

converges and depends holomorphically on λ. In particular, this is true for a cycle $\Gamma \in H_{2n}(T^*_{G_{\mathbb{R}}}X, \mathbb{C})$. In this case $\Theta(\Gamma,\lambda)$ is a $G_{\mathbb{R}}$-invariant distribution on $\mathfrak{g}_{\mathbb{R}}$. We mention that this facts depend essentially on the rapid decay of $\hat{\phi}$ in imaginary directions. Moreover, Rossmann's definition of W-action on $H_{2n}(T^*_{G_{\mathbb{R}}}X, \mathbb{C})$ implies the following W-equivariance formula for distributions $\Theta(\Gamma,\lambda)$ [16], 3.1:

$$\Theta(w\Gamma,\lambda) = \Theta(\Gamma, w^{-1}\lambda)\,, \;\; w \in W,\; \lambda \in \mathfrak{h}^*_{\mathrm{reg}}. \tag{3.6}$$

We say that two semi-algebraic, $\mathbb{R}$-bounded, $2n$-cycles Γ_1 and Γ_2 in T^*X are $\mathbb{R}$-homologous if

$$\Gamma_1 - \Gamma_2 = \partial\Gamma$$

for a semi-algebraic, $\mathbb{R}$-bounded, $(2n+1)$-chain Γ in T^*X. In this case we have [19], 3.19

$$\int_{\Gamma_1} \mu_\lambda^*(\hat{\phi})(-\sigma + \pi^*\tau_\lambda)^n = \int_{\Gamma_2} \mu_\lambda^*(\hat{\phi})(-\sigma + \pi^*\tau_\lambda)^n.$$

Now we can state Rossmann's integral formula in the form convenient for applications we have in mind.

Theorem 3.1. — *Let $\phi \in C_c^\infty(\mathfrak{g}_{\mathbb{R}})$ and let $C \subset i\mathfrak{t}_{\mathbb{R}}^*$ be the positive chamber defined by $\mathfrak{k}/\mathfrak{b}_e \cap \mathfrak{k}$. Write $s = \dim_{\mathbb{C}} [\mathfrak{b}_e, \mathfrak{b}_e] \cap \mathfrak{k}$. Then for $\lambda \in C + i\mathfrak{a}_{\mathbb{R}}^*$ we have*

$$\int_{CC(Ri_{e*}\mathbb{C}_{S_e})} \mu_\lambda^*(\hat{\phi}\sigma_\lambda^n) = (-1)^s \int_{G_{\mathbb{R}}\cdot\lambda} \hat{\phi}\sigma_\lambda^n.$$

Proof. — It was proved in [6], Th. 1, [8], 3.4 that for $\lambda \in C$ the cycles $CC(Ri_{e*}\mathbb{C}_{S_e})$ and $(-1)^s\mu_\lambda^{-1}(G_{\mathbb{R}}\cdot\lambda)$ are $\mathbb{R}$-homologous. We will use a similar argument to extend the formula to the case $\lambda \in C + i\mathfrak{a}_{\mathbb{R}}^*$. Write

$$\lambda = \lambda_1 + \lambda_2\,, \;\; \lambda_1 \in C,\; \lambda_2 \in i\mathfrak{a}_{\mathbb{R}}^*.$$

Set $\lambda(t) = \lambda_1 + t\lambda_2$, $t \in [0,1]$. It is not difficult to show that that for $\lambda \in C + i\mathfrak{a}_{\mathbb{R}}^*$

$$\mu_\lambda^{-1}(\mathrm{Ad}(g)\lambda) = (g.x_e, \mathrm{Ad}(g)\lambda - \mathrm{Ad}(u)\lambda),\;\; g \in G_{\mathbb{R}},\, u \in U_{\mathbb{R}},\; g.x_e = u.x_e. \tag{3.7}$$

Consider the following map

$$\Phi : [0,1] \times G_{\mathbb{R}}/H_{\mathbb{R}} \longrightarrow T^*X\,,\;\; \Phi(t, gH_{\mathbb{R}}) = \mu_{\lambda(t)}^{-1}(\mathrm{Ad}(g)\lambda(t)).$$

Then Φ is a homotopy, and (3.7) implies that $\operatorname{Re}\mu(\Phi([0,1]\times G_{\mathbb{R}}/H_{\mathbb{R}}))$ is bounded. It follows that the cycles $\mu^{-1}_{\lambda(0)}(G_{\mathbb{R}}\cdot\lambda(0))$ and $\mu^{-1}_{\lambda(1)}(G_{\mathbb{R}}\cdot\lambda(1))$ are $\mathbb{R}$-homologous. Hence, for $\lambda\in C+i\mathfrak{a}^*_{\mathbb{R}}$, we have

$$\begin{aligned}\int_{CC(Ri_{e*}\mathbb{C}_S)}\mu^*_\lambda(\hat\phi\sigma^n_\lambda) &= (-1)^s\int_{\mu^{-1}_{\lambda(0)}(G_{\mathbb{R}}\cdot\lambda(0))}\mu^*_\lambda(\hat\phi\sigma^n_\lambda) = \\ (-1)^s\int_{\mu^{-1}_{\lambda(1)}(G_{\mathbb{R}}\cdot\lambda(1))}\mu^*_\lambda(\hat\phi\sigma^n_\lambda) &= (-1)^s\int_{G_{\mathbb{R}}\cdot\lambda}\hat\phi\sigma^n_\lambda,\end{aligned}$$

as desired. □

Our goal is to study the asymptotic behaviour of distributions $\Theta(\Gamma,\lambda)$ when $\lambda\in\mathfrak{h}^*_{\mathrm{reg}}$ approaches zero. Some additional results are needed for this analysis.

Denote by $\mathcal{H}_d(\mathfrak{h}^*)$ ($\mathcal{H}_d(\mathfrak{h})$) the space of harmonic polynomials on $\mathfrak{h}^*$ ($\mathfrak{h}$) of degree d. The map

$$H_{2d}(X,\mathbb{C})\longrightarrow\mathcal{H}_d(\mathfrak{h}^*)\,,\ \ \gamma\mapsto b(\gamma)=\frac{1}{(2\pi i)^d d!}\int_\gamma\tau^d_\lambda \tag{3.8}$$

is an isomorphism of W-modules, usually called the Borel isomorphism [4]. Here, we consider the W-action on $H_{2d}(X,\mathbb{C})$ induced by the natural W-action on X. On the other hand, we have a natural homomorphism

$$H_{2d}(\mu^{-1}(\nu),\mathbb{C})\longrightarrow H_{2d}(X,\mathbb{C}), \tag{3.9}$$

defined by the inclusion $\mu^{-1}(\nu)\longrightarrow X\times\{\nu\}$. Rossmann shows this is a nonzero W-module homomorphism [16], Cor. 3.2, which factors through the projection

$$H_{2d}(\mu^{-1}(\nu),\mathbb{C})\longrightarrow H_{2d}(\mu^{-1}(\nu),\mathbb{C})^{C_G(\nu)}. \tag{3.10}$$

It is known that χ_ν appears exactly once in $\mathcal{H}_d(\mathfrak{h}^*)$ [5], Cor. 4. We denote the corresponding subspace by $\mathcal{H}_d(\mathfrak{h}^*)_\nu$. Now taking into account (3.1), (3.2), (3.8), (3.9), (3.10) we obtain a surjective homomorphism of W-modules

$$H_{2n}(\mu^{-1}(\overline{\mathcal{O}}),\mathbb{C})\longrightarrow\mathcal{H}_d(\mathfrak{h}^*)_\nu\,,\ \ \Gamma\mapsto p_\Gamma. \tag{3.11}$$

Denote by $\Theta_{\mathcal{O}}$ the Fourier transform of the Liouville measure $m_{\mathcal{O}}$. In more details,

$$\Theta_{\mathcal{O}}(\phi)=\frac{1}{(2\pi i)^k k!}\int_{\mathcal{O}}\hat\phi\sigma^k_{\mathcal{O}}\,,\ \ 2k=\dim_{\mathbb{R}}\mathcal{O},\ \phi\in C^\infty_c(\mathfrak{g}_{\mathbb{R}}).$$

Let $\Gamma\in H_{2n}(\mu^{-1}(\overline{\mathcal{O}}),\mathbb{C})$. Applying Fubini's theorem to the fibration $\mu^{-1}(\mathcal{O})\longrightarrow\mathcal{O}$, using (3.11), and $\mu^*\sigma_{\mathcal{O}}=-\sigma|\mu^{-1}(\mathcal{O})$ (at the smooth points) [18], Lem. 8.19, Rossmann proves the following formula relating distributions $\Theta(\Gamma,\lambda)$ and $\Theta_{\mathcal{O}}$

$$\Theta(\Gamma,\lambda) = p_\Gamma(\lambda)\Theta_{\mathcal{O}} + o(\lambda^d). \tag{3.12}$$

The term $o(\lambda^d)$ can be described as follows. For any $\phi \in C_c^\infty(\mathfrak{g}_\mathbb{R})$, $o(\lambda^d)(\phi)$ is a holomorphic function of λ and

$$\lim_{t\longrightarrow 0} \frac{o((t\lambda)^d)(\phi)}{t^d} = 0.$$

Denote by $\mathbb{C}[\mathfrak{h}]$ resp. $\mathbb{C}[\mathfrak{h}^*]$ the algebra of polynomial functions on $\mathfrak{h}$ resp. $\mathfrak{h}^*$. Write $S(\mathfrak{h})$ resp. $S(\mathfrak{h}^*)$ for the symmetric algebra of $\mathfrak{h}$ resp. $\mathfrak{h}^*$. Recall that we have canonical isomorphisms

$$\mathbb{C}[\mathfrak{h}] \cong S(\mathfrak{h}^*) \text{ and } \mathbb{C}[\mathfrak{h}^*] \cong S(\mathfrak{h}).$$

On the other hand the map

$$v \mapsto \partial(v), \quad \partial(v)f(\lambda) = \lim_{t\longrightarrow 0}(f(\lambda+tv)-f(\lambda))/t, \quad \lambda,\, v\in\mathfrak{h}^*,\ f\in C^\infty(\mathfrak{h}^*)$$

extends to an isomorphism of $S(\mathfrak{h}^*)$ and the algebra $D(\mathfrak{h}^*)$ of differential operators on $\mathfrak{h}^*$ with constant ceofficients. Thus we obtain the isomorphism of algebras

$$\mathbb{C}[\mathfrak{h}] \cong D(\mathfrak{h}^*),\ \ p\mapsto p(\partial),\, p\in\mathbb{C}[\mathfrak{h}].$$

Let $\delta : \mathfrak{h} \longrightarrow \mathfrak{h}^*$ be the isomorphism defined by the Killing form and

$$\delta : S(\mathfrak{h}) \longrightarrow S(\mathfrak{h}^*)$$

the induced isomorphism of algebras. We write

$$\delta^{-1}(\lambda) = h_\lambda\,,\ \lambda\in\mathfrak{h}^*.$$

Put

$$\mathfrak{h}_0^* = \sum_{\alpha\in\Delta^+} \mathbb{R}\cdot\alpha,$$

and denote by

$$^{-}\colon \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$$

the conjugation with respect to $\mathfrak{h}_0^*$. Let

$$^{-}\colon S(\mathfrak{h}^*) \longrightarrow S(\mathfrak{h}^*)$$

be the induced conjugation of $S(\mathfrak{h}^*)$.

Lemma 3.2. — *Let $(r_1,\cdots,r_s)$ be a basis in $\mathcal{H}_d(\mathfrak{h}^*)_\nu \subset S(\mathfrak{h})$. Put $p_i=\delta(r_i)$, $i=1,\cdots,s$, and let*

$$V_d = \sum_{i=1}^{s} \mathbb{C}\cdot\bar{p_i}\,.$$

Then V_d is a W-module isomorphic to $\mathcal{H}_d(\mathfrak{h}^*)_\nu$.

Proof. — Since δ is an isomorphism of W-modules, $\sum_{i=1}^s \mathbb{C}\cdot p_i$ is isomorphic to $\mathcal{H}_d(\mathfrak{h}^*)_\nu$. Observe that $\mathfrak{h}_0^*$ is invariant for W, hence

$$w\bar{p}_i = \overline{wp}_i\,, \quad w\in W.$$

It follows that V_d is a W-module. Moreover the corresponding character $\chi(V_d)$ satisfies

$$\chi(V_d) = \overline{\chi}_\nu.$$

On the other hand by the Springer theory of Weyl group representations χ_ν is defined over $\mathbb{Q}$ [5], Th. 3. Hence $\chi_\nu = \overline{\chi}_\nu$, which implies the statement. □

Let $\mathfrak{h}_0 = \sum_{\alpha\in\Delta^+}\mathbb{R}\cdot h_\alpha$. Observe that $(.,.)$ is positive definite on $\mathfrak{h}_0$, hence we can choose an orthonormal basis

$$(e_1,\cdots,e_l)$$

in $\mathfrak{h}_0$. Then

$$(\epsilon_1=\delta(e_1),\cdots,\epsilon_l=\delta(e_l))$$

is the dual basis in $\mathfrak{h}_0^*$.

Lemma 3.3. — *Let $\Gamma\in H_{2n}(T^*_{\mathbb{R}}X,\mathbb{C})$, $\lambda\in\mathfrak{h}^*$, $p\in\mathbb{C}[\mathfrak{h}]$ and $w\in W$.*

1. $\lim_{\lambda\to 0} p(\partial)\Theta(\Gamma,\lambda)$ *exists as a distribution on* $\mathfrak{g}_{\mathbb{R}}$.
2. $\lim_{\lambda\to 0} w^{-1}p(\partial)\Theta(\Gamma,\lambda) = \lim_{\lambda\to 0} p(\partial)\Theta(w\Gamma,\lambda)$.

Proof. — Let $i_1,\cdots,i_m\in\{1,\cdots,l\}$ and $\phi\in C_c^\infty(\mathfrak{g}_{\mathbb{R}})$. We know that $\Theta(\Gamma,\lambda)(\phi)$ depends holomorphically on λ, hence using repeatedly [11], Th. 2.1.8 we deduce that

$$\phi\mapsto \partial(\epsilon_{i_1})\cdots\partial(\epsilon_{i_m})\Theta(\Gamma,\lambda)(\phi)$$

is a distribution on $\mathfrak{g}_{\mathbb{R}}$. The first claim now follows. To prove the second statement consider the Taylor series expansion

$$\Theta(\Gamma,\lambda)(\phi) = \sum_{n_1,\cdots,n_l\in\mathbb{Z}_+} a_{n_1\cdots n_l}(\Gamma)(\phi)\lambda(e_1)^{n_1}\cdots\lambda(e_l)^{n_l}.$$

Then we have

$$p(\partial)\Theta(\Gamma,\lambda)(\phi) = \sum_{n_1,\cdots,n_l\in\mathbb{Z}_+} a_{n_1\cdots n_l}(\Gamma)(\phi)p(\partial)(\lambda(e_1)^{n_1}\cdots\lambda(e_l)^{n_l}),$$

and by (3.6)

$$\Theta(w\Gamma,\lambda)(\phi) = \sum_{n_1,\cdots,n_l\in\mathbb{Z}_+} a_{n_1\cdots n_l}(\Gamma)(\phi)\lambda(we_1)^{n_1}\cdots\lambda(we_l)^{n_l}.$$

We conclude it will suffice to prove

$$\lim_{\lambda\to 0} \partial^{m_1}(w^{-1}\epsilon_1)\cdots\partial^{m_l}(w^{-1}\epsilon_l)(\lambda(e_1)^{n_1}\cdots\lambda(e_l)^{n_l}) = \lim_{\lambda\to 0} \partial^{m_1}(\epsilon_1)\cdots\partial^{m_l}(\epsilon_l)(\lambda(we_1)^{n_1}\cdots\lambda(we_l)^{n_l}),$$

for any $m_1, \cdots, m_l \in \mathbb{Z}_+$. To prove the last formula we use induction on $m_1 + \cdots + m_l$. Assume

$$\partial^{m_1}(w^{-1}\epsilon_1)\cdots\partial^{m_l}(w^{-1}\epsilon_l)(\lambda(e_1)^{n_1}\cdots\lambda(e_l)^{n_l}) = \sum_{k_i\leqslant n_i} a_{k_1\cdots k_l}\lambda(e_1)^{k_1}\cdots\lambda(e_l)^{k_l},$$

$$\partial^{m_1}(\epsilon_1)\cdots\partial^{m_l}(\epsilon_l)(\lambda(we_1)^{n_1}\cdots\lambda(we_l)^{n_l}) = \sum_{k_i\leqslant n_i} a_{k_1\cdots k_l}\lambda(w^{-1}e_1)^{k_1}\cdots\lambda(w^{-1}e_l)^{k_l}.$$

We use the formulas

$$\partial(w^{-1}\epsilon_j)(\lambda(e_1)^{k_1}\cdots\lambda(e_l)^{k_l}) = \sum_{i=1}^{l} w^{-1}\epsilon_j(e_i)\lambda(e_1)^{k_1}\cdots\lambda(e_i)^{k_i-1}\cdots\lambda(e_l)^{k_l},$$

$$\partial(\epsilon_j)(\lambda(we_1)^{k_1}\cdots\lambda(we_l)^{k_l}) = \sum_{i=1}^{l} w^{-1}\epsilon_j(e_i)\lambda(we_1)^{k_1}\cdots\lambda(we_i)^{k_i-1}\cdots\lambda(we_l)^{k_l}$$

to complete the inductive proof. □

Now we can state and prove the main result of the paper.

Theorem 3.4. — *Suppose $G_{\mathbb{R}}$ is a connected linear semisimple Lie group with one conjugacy class of Cartan subgroups. Let $\mathcal{O} \subset i\mathfrak{g}_{\mathbb{R}}^*$ be a nilpotent coadjoint $G_{\mathbb{R}}$-orbit. Let $m_{\mathcal{O}}$ and m_λ, $\lambda \in i\mathfrak{h}_{\mathbb{R}}^*$ be the Liouville measures on $\mathcal{O}$ and $G_{\mathbb{R}} \cdot \lambda$ defined in (3.3). Then there exists up to a constant unique harmonic polynomial $p \in \mathbb{C}[\mathfrak{h}]$ corresponding to the W-character χ_ν, $\nu \in \mathcal{O}$, and transforming by ϵ_θ under W^θ, such that the following limit formula for the orbital measures holds*

$$\lim_{\lambda\to 0(C+i\mathfrak{a}_{\mathbb{R}}^*)} p(\partial)m_\lambda = \kappa m_{\mathcal{O}}.$$

Here κ is a nonzero constant and C is as in 3.1.

Proof. — To simplify notation we write

$$V = H_{2n}(T_{G_{\mathbb{R}}}X, \mathbb{C}).$$

First we remark that as a W-module [17], 4.4.1

$$V \cong \sum_{\mathcal{V} \in i\mathcal{N}_{\mathbb{R}}^{*}/G_{\mathbb{R}}} H_{2n}(\mu^{-1}(\mathcal{V}), \mathbb{C}).$$

By 1.2 distinct real orbits belong to distinct complex nilpotent orbits, hence we conclude that

$$[V : \chi_\nu] = 1.$$

Let P_{χ_ν} be the projection to the isotypical component of type χ_ν. Explicitly

$$P_{\chi_\nu} : V \longrightarrow V, \ \ P_{\chi_\nu}(\Gamma) = \frac{\deg \chi_\nu}{|W|} \sum_{w \in W} \chi_\nu(w^{-1}) w\Gamma.$$

The multiplicity one property implies that

$$P_{\chi_\nu} V \subset H_{2n}(\mu^{-1}(\overline{\mathcal{O}}), \mathbb{C}).$$

Let

$$\Gamma_0 = CC(Ri_{e*}(\mathbb{C}_{S_e})).$$

By 2.3 Γ_0 generates V as W-module, hence $P_{\chi_\nu}\Gamma_0 \neq 0$. Then $r = b(P_{\chi_\nu}\Gamma_0) \neq 0$ and applying (3.12) we obtain

$$\Theta(P_{\chi_\nu}\Gamma_0, \lambda) = r(\lambda)\Theta_{\mathcal{O}} + o(\lambda^d).$$

Set $p = \overline{\delta(r)}$. Then by 3.2 p is a harmonic polynomial on $\mathfrak{h}$ corresponding to the W-character χ_ν. Moreover, the definition of p implies

$$p(\partial)r(\lambda) = p(\partial)r(0) \neq 0.$$

Now we apply 3.3 to conclude

$$\lim_{\lambda \to 0} p(\partial)\Theta(\Gamma_0, \lambda) = p(\partial)r(0)\Theta_{\mathcal{O}}.$$

Here we used the formula $\chi_\nu(w^{-1}) = \chi_\nu(w)$, $w \in W$, which is a consequence of the fact that χ_ν is defined over $\mathbb{Q}$ [5], Th. 3. To complete the proof it will suffice to use 3.1, and take the inverse Fourier transform. $\square$

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Mladen BOŽIČEVIĆ
University of Zagreb
Department of Geotechnical Engineering
Hallerova 7
42000 Varaždin (Croatia)
mbozicev@gfv.hr