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SYSTOLIC INVARIANTS OF GROUPS AND 2-COMPLEXES VIA GRUSHKO DECOMPOSITION

by Yuli B. RUDYAK & Stéphane SABOURAU (*)

Abstract. — We prove a finiteness result for the systolic area of groups. Namely, we show that there are only finitely many possible unfree factors of fundamental groups of 2-complexes whose systolic area is uniformly bounded. We also show that the number of freely indecomposable such groups grows at least exponentially with the bound on the systolic area. Furthermore, we prove a uniform systolic inequality for all 2-complexes with unfree fundamental group that improves the previously known bounds in this dimension.

Résumé. — Nous prouvons un résultat de finitude pour l'aire systolique des groupes. Précisément, nous montrons qu'il n'existe qu'un nombre fini de facteurs non-libres dans les groupes fondamentaux des 2-complexes d'aire systolique uniformément bornée. Nous montrons aussi que le nombre de tels groupes librement indécomposables croît au moins exponentiellement avec la borne sur l'aire systolique. De plus, nous prouvons une inégalité systolique uniforme pour tous les 2-complexes de groupe fondamental non-libre qui améliore les bornes précédemment connues dans cette dimension.

1. Introduction

Throughout the article the word "complex" means "finite simplicial complex", unless something else is said explicitly.

Consider a piecewise smooth metric $\mathcal{G}$ on a 2-complex X. The systole of $\mathcal{G}$, denoted $\mathrm{sys}\pi_1(\mathcal{G})$, is defined as the least length of a noncontractible loop in X. We define the systolic ratio of $\mathcal{G}$ as

$$\mathrm{SR}(\mathcal{G}) = \frac{\mathrm{sys}\pi_1(\mathcal{G})^2}{\mathrm{area}(\mathcal{G})}, \tag{1.1}$$

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and the systolic ratio of X as

$$\mathrm{SR}(X) = \sup_{\mathcal{G}} \mathrm{SR}(\mathcal{G}), \tag{1.2}$$

where the supremum is taken over the space of all the piecewise flat metrics $\mathcal{G}$ on X. Note that taking the supremum over the space of all piecewise smooth metrics on X would yield the same value, *cf.* [1], [6, §3].

We also define the *systolic ratio* of a finitely presentable group G as

$$\mathrm{SR}(G) = \sup_{X} \mathrm{SR}(X), \tag{1.3}$$

where the supremum is taken over all finite 2-complexes X with fundamental group isomorphic to G. It is also convenient to introduce the *systolic area* $\sigma(G)$ of G, *cf.* [12, p. 337], by setting

$$\sigma(G) = \mathrm{SR}(G)^{-1}.$$

Similarly, we define the systolic area of a 2-complex X and of a piecewise flat metric $\mathcal{G}$ on X as $\sigma(X) = \mathrm{SR}(X)^{-1}$ and $\sigma(\mathcal{G}) = \mathrm{SR}(\mathcal{G})^{-1}$, respectively. For instance, the systolic area of the wedge of p circles is zero. Thus, $\sigma(F_p) = 0$ where F_p is the free group of rank p.

A group is said to be *unfree* if it is not free.

In this article, we study the systolic ratio of groups, or equivalently the systolic ratio of 2-complexes. Before stating our results, let us review what was previously known on the subject.

M. Gromov [11, 6.7.A] (note a misprint in the exponent) showed that every 2-complex X with unfree fundamental group satisfies the systolic inequality

$$\mathrm{SR}(X) \leqslant 10^4. \tag{1.4}$$

Contrary to the case of surfaces, where a (better) systolic inequality can be derived by simple techniques [13, $4.5\frac{3}{4}{}_+$], the proof of inequality (1.4) depends on the advanced filling techniques of [11].

Recently, in collaboration with M. Katz, we improved the bound (1.4) using "elementary" techniques and characterized the 2-complexes satisfying a systolic inequality, *cf.* [17]. Specifically, we showed that every 2-complex X with unfree fundamental group satisfies

$$\mathrm{SR}(X) \leqslant 12. \tag{1.5}$$

Furthermore, we proved that 2-complexes with unfree fundamental groups are the only 2-complexes satisfying a systolic inequality, *i.e.*, for which the systolic ratio is finite, *cf.* [17].

If one restricts oneself to surfaces, numerous systolic inequalities are available. These inequalities fall into two categories. The best estimates for surfaces of low Euler characteristic can be found in [23, 11, 4, 13, 19, 20, 3]. Near-optimal asymptotic bounds for the systolic ratio of surfaces of large genus have been established in [11, 2, 18, 24] and [7, 21].

We refer to the expository texts [12, 13, 8, 16] and the reference therein for an account on higher-dimensional systolic inequalities and other related curvature-free inequalities.

In order to state our main results, we need to recall Grushko decomposition in group theory. By Grushko's theorem [25, 22], every finitely generated group G has a decomposition as a free product of subgroups

$$G = F_p * H_1 * \cdots * H_q \tag{1.6}$$

such that F_p is free of rank p, while every H_i is nontrivial, non isomorphic to $\mathbb{Z}$ and freely indecomposable. Furthermore, given another decomposition of this sort, say $G = F_r * H'_1 * \cdots * H'_s$, one necessarily has $r = p$, $s = q$ and, after reordering, H'_i is conjugate to H_i. We will refer to the number p in decomposition (1.6) as the *Grushko free index* of G.

Thus, every finitely generated group G of Grushko free index p can be decomposed as

$$G = F_p * H_G, \tag{1.7}$$

where F_p is free of rank p and H_G is of zero Grushko free index. The subgroup H_G is unique up to isomorphism. Its isomorphism class is called the *unfree factor* of (the isomorphism class of) G.

The Grushko free index of a complex is defined as the Grushko free index of its fundamental group.

In [12, p. 337], M. Gromov asks how large is the set of isomorphism classes of groups with systolic area bounded by a given constant.

Our main result deals with this question. Specifically, we obtain the following finiteness result.

Theorem 1.1. — *Let $C > 0$. The isomorphism classes of the unfree factors H_i of the finitely presentable groups G with $\sigma(G) < C$ lie in a finite set with at most*

$$A^{C^3}$$

elements, where A is an explicit universal constant.

Clearly, we have $\sigma(G_1 * G_2) \leqslant \sigma(G_1) + \sigma(G_2)$ for every finitely presentable groups G_1 and G_2 (by taking the wedge of the corresponding complexes). In particular, the inequality $\sigma(F_p * G) \leqslant \sigma(G)$ holds for every p. Thus, the

Grushko free index of a group with bounded systolic area can be arbitrarily large, which explains why we considered only the unfree factors in the previous theorem.

QUESTION 1.2. — *Given an unfree finitely presentable group G, does the relation $\sigma(G * \mathbf{Z}) = \sigma(G)$ hold?*

On the other hand, using a result of I. Kapovich and P. Schupp [15], it is not difficult to show that the number of isomorphism classes of freely indecomposable groups G with $\sigma(G) < C$ grows at least exponentially with C. Specifically, we have the following.

THEOREM 1.3. — *Let $C > 0$. The number of isomorphism classes of freely indecomposable finitely presentable groups G with $\sigma(G) < C$ is at least 2^C for C large enough.*

Thus, by providing a lower and an upper bound on the number of isomorphism classes of groups with systolic area bounded by a given constant, Theorem 1.1 and Theorem 1.3 address to some extend M. Gromov's question [12, p. 337].

While proving Theorem 1.1, we improve the systolic inequalities (1.4) and (1.5).

THEOREM 1.4. — *Every unfree finitely presentable group satisfies the inequality*

$$\mathrm{SR}(G) \leqslant \frac{16}{\pi}. \tag{1.8}$$

QUESTION 1.5. — *Does every 2-complex with unfree fundamental group satisfy Pu's inequality for $\mathbb{RP}^2$? Equivalently, is the optimal constant in* (1.8) *equal to $\frac{\pi}{2}$? This is known to be true for nonsimply connected closed surfaces from the combination of the systolic inequalities in* [23] *and* [11, 5.2.B].

The article is organized as follows. In Section 2, we recall some topological preliminaries. In Section 3, we investigate the geometry of pointed systoles and establish a lower bound on the area of "small" balls on 2-complexes with zero Grushko free index. This yields a systolic inequality. The existence of "almost extremal regular" metrics is established in Section 4. Section 5 contains some combinatorial results: we count the number of fundamental groups of complexes with some prescribed properties. Using these results, we derive two finiteness results about the fundamental groups of certain 2-complexes in Section 6. In Section 7, we relate the systolic ratio of a group to the systolic ratio of the free product of this group with $\mathbb{Z}$. In Section 8, we combine all the results from the previous sections to prove our main theorems. In the last section, we obtain an exponential lower bound on the number of freely indecomposable groups with bounded systolic area.

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2. Topological preliminaries

A proof of the following result, derived easily from the Seifert–van Kampen Theorem, can be found in [17].

LEMMA 2.1. — *Let (X, A) be a CW-pair with X and A connected. If the inclusion $j : A \to X$ induces the zero homomorphism $j_* : \pi_1(A) \to \pi_1(X)$ of fundamental groups, then the quotient map $q : X \to X/A$ induces an isomorphism of fundamental groups.*

Let X be a finite connected complex and let $f : X \to \mathbb{R}$ be a function on X. Let

$$[f \leqslant r] := \{x \in X \mid f(x) \leqslant r\} \text{ and } [f \geqslant r] := \{x \in X \mid f(x) \geqslant r\}$$

denote the sublevel and superlevel sets of f, respectively.

DEFINITION 2.2. — *Suppose that a single path-connected component of the superlevel set $[f \geqslant r]$ contains k path-connected components of the level set $f^{-1}(r)$. Then we will say that the k path-connected components* coalesce forward.

We will need the following result (we refer to [17] for a more complete statement and a more detailed proof).

LEMMA 2.3. — *Assume that the pairs $([f \geqslant r], f^{-1}(r))$ and $(X, [f \leqslant r])$ are homeomorphic to CW-pair. Suppose that the set $[f \leqslant r]$ is connected and that two connected components of $f^{-1}(r)$ coalesce forward. If the inclusion*

$$[f \leqslant r] \subset X$$

of the sublevel set $[f \leqslant r]$ induces the zero homomorphism of fundamental groups, then the Grushko free index of X is positive.

Proof. — Let $Y = [f \geqslant r]/\sim$ where $x \sim y$ if and only if x, y belong to the same component of $f^{-1}(r)$. The images a_i of the components of $f^{-1}(r)$

under the quotient map $[f \geqslant r] \to Y$ form a finite set $A \subset Y$. By assumption, two points of A are joined by an arc in $[f \geqslant r]$. Therefore, the space $Y \cup CA$, obtained by gluing an abstract cone over A to Y, is homotopy equivalent to the wedge of S^1 with another space Z. Hence,

$$X/[f \leqslant r] = Y/A \simeq Y \cup CA \simeq S^1 \vee Z.$$

Thus, by the Seifert–van Kampen Theorem, the Grushko free index of $\pi_1(X/[f \leqslant r])$ is positive. Since the inclusion $[f \leqslant r] \subset X$ induces the zero homomorphism of fundamental groups, we conclude that the group $\pi_1(X/[f \leqslant r])$ is isomorphic to $\pi_1(X)$ by Lemma 2.1. □

We will also need the following technical result.

Proposition 2.4. — *A level set of the distance function f from a point in a piecewise flat 2-complex X is a finite graph. In particular, the triangulation of X can be refined in such a way that the sets $[f \leqslant r]$, $f^{-1}(r)$ and $[f \geqslant r]$ become CW-subspaces of X.*

Furthermore, the function $\ell(r) = \text{length}\, f^{-1}(r)$ is piecewise continuous.

Proposition 2.4 is a consequence of standard results in real algebraic geometry, *cf.* [5]. Indeed, note that X can be embedded into some $\mathbb{R}^N$ as a semialgebraic set and that the distance function f is a continuous semialgebraic function on X. Thus, the level curve $f^{-1}(r)$ is a semialgebraic subset of X and, therefore, a finite graph, *cf.* [5, §9.2]. A more precise description of the level curves of f appears in [17].

The second part of the proposition also follows from [5, §9.3].

3. Complexes of zero Grushko free index

The results of this section will be used repeatedly in the sequel. These results also appear in [17]. We duplicate them here for the reader's convenience.

Definition 3.1. — *Let X be a complex equipped with a piecewise smooth metric. A shortest noncontractible loop of X based at $x \in X$ is called a pointed systolic loop at x. Its length, denoted by $\text{sys}\pi_1(X, x)$, is called the pointed systole at x.*

As usual, given $x \in X$ and $r \in \mathbb{R}$, we denote by $B(x, r)$ the ball of radius r centered at x, $B(x, r) = \{a \in X \mid \text{dist}(x, a) \leqslant r\}$.

Proposition 3.2. — *If $r < \frac{1}{2}\,\text{sys}\pi_1(X, x)$ then the inclusion $B(x, r) \subset X$ induces the zero homomorphism of fundamental groups.*

Proof. — Suppose the contrary. Consider all the loops of $B(x,r)$ based at x that are noncontractible in X. Let $\gamma \subset B(x,r)$ be the shortest of these loops. We have $L = \text{length}(\gamma) \geqslant \text{sys}\pi_1(X,x)$. Let a be the point of γ that divides γ into two arcs γ_1 and γ_2 of the same length $L/2$. Consider a shortest geodesic path c, of length $d = d(x,a) < r$, that joins x to a. Since at least one of the curves $\gamma_1 \cup c_-$ or $c \cup \gamma_2$ is noncontractible, we conclude that $d + L/2 \geqslant L$, *i.e.*, $d \geqslant L/2$ (here c_- denotes the path c with the opposite orientation). Thus

$$\text{sys}\pi_1(X,x) > 2r \geqslant 2d \geqslant L \geqslant \text{sys}\pi_1(X,x).$$

That is a contradiction. □

The following lemma describes the structure of a pointed systolic loop.

Lemma 3.3. — *Let X be a complex equipped with a piecewise flat metric. Let γ be a pointed systolic loop at $x \in X$ of length $L = \text{sys}\pi_1(X,x)$.*

(i) *The loop γ is formed of two distance-minimizing arcs, starting at p and ending at a common endpoint, of length $L/2$.*
(ii) *Any point of self-intersection of the loop γ is no further than $\frac{1}{2}$ $(\text{sys}\pi_1(X,x) - \text{sys}\pi_1(X))$ from x.*

Proof. — Consider the arc length parameterization $\gamma(s)$ of the loop γ with $\gamma(0) = \gamma(L) = x$. Let $y = \gamma(L/2) \in X$ be the "midpoint" of γ. Then y splits γ into a pair of paths of the same length $L/2$, joining x to y. By Proposition 3.2, if y were contained in the open ball $B(x, L/2)$, the loop γ would be contractible. This proves item (i).

If x' is a self-intersection point of γ, the loop γ decomposes into two loops γ_1 and γ_2 based at x', with $x \in \gamma_1$. Since the loop γ_1 is shorter than the pointed systolic loop γ at x, it must be contractible. Hence γ_2 is noncontractible, so that

$$\text{length}(\gamma_2) \geqslant \text{sys}\pi_1(X).$$

Therefore,

$$\text{length}(\gamma_1) = L - \text{length}(\gamma_2) \leqslant \text{sys}\pi_1(X,x) - \text{sys}\pi_1(X),$$

proving item (ii). □

The following proposition provides a lower bound for the length of level curves in a 2-complex.

Proposition 3.4. — *Let X be a piecewise flat 2-complex. Fix $x \in X$. Let r be a real number satisfying*

$$\text{sys}\pi_1(X,x) - \text{sys}\pi_1(X) < 2r < \text{sys}\pi_1(X,x).$$

Consider the curve $S = \{a \in X \mid \operatorname{dist}(x,a) = r\}$. *Let* γ *be a pointed systolic loop at* x. *If* γ *intersects exactly one connected component of* S, *then*

$$\operatorname{length} S \geqslant 2r - (\operatorname{sys}\pi_1(X,x) - \operatorname{sys}\pi_1(X)) . \tag{3.1}$$

Proof. — By Lemma 3.3, the loop γ is formed of two distance-minimizing arcs which do not meet at distance r from x. Thus, the loop γ intersects S at exactly two points. Let $\gamma' = \gamma \cap B$ be the subarc of γ lying in B.

If γ meets exactly one connected component of S, there exists an embedded arc $\alpha \subset S$ connecting the endpoints of γ'. By Proposition 3.2, every loop based at x and lying in $B(x,r)$ is contractible in X. Hence γ' and α are homotopic, and the loop $\alpha \cup (\gamma \setminus \gamma')$ is homotopic to γ. Hence,

$$\operatorname{length}(\alpha) + \operatorname{length}(\gamma) - \operatorname{length}(\gamma') \geqslant \operatorname{sys}\pi_1(X). \tag{3.2}$$

Meanwhile, $\operatorname{length}(\gamma) = \operatorname{sys}\pi_1(X,x)$ and $\operatorname{length}(\gamma') = 2r$, proving the lower bound (3.1), since $\operatorname{length}(S) \geqslant \operatorname{length}(\alpha)$. □

The following result provides a lower bound on the area of "small" balls of 2-complexes with zero Grushko free index, *cf.* Section 1.

Theorem 3.5. — *Let* X *be a piecewise flat* 2*-complex with zero Grushko free index. Fix* $x \in X$. *For every real number* R *such that*

$$\operatorname{sys}\pi_1(X,x) - \operatorname{sys}\pi_1(X) \leqslant 2R \leqslant \operatorname{sys}\pi_1(X,x), \tag{3.3}$$

the area of the ball $B(x,R)$ *of radius* R *centered at* x *satisfies*

$$\operatorname{area} B(x,R) \geqslant \left(R - \tfrac{1}{2}(\operatorname{sys}\pi_1(X,x) - \operatorname{sys}\pi_1(X))\right)^2 . \tag{3.4}$$

In particular, we have

$$\operatorname{SR}(X) \leqslant 4.$$

Remark 3.6. — The example of a piecewise flat 2-complex X with a circle of length the systole of X attached to it shows that the assumption on the fundamental group of the complex cannot be dropped.

Proof of Theorem 3.5. — Let $L = \operatorname{sys}\pi_1(X,x)$. Let r be a real number satisfying $L - \operatorname{sys}\pi_1(X) \leqslant 2r \leqslant L$. Denote by $S = S(x,r)$ and $B = B(x,r)$, respectively, the level curve and the ball of radius r centered at x. Let γ be a pointed systolic loop at x.

If γ intersects two connected components of S, then by Lemma 3.3, there exists an arc of γ lying in $X \setminus \operatorname{Int}(B)$, which joins these two components of S. That is, the components coalesce forward. Thus, by Lemma 2.3 and Proposition 3.2, the complex X has a positive Grushko free index, which is excluded.

Therefore, the loop γ meets a single connected component of S. Now, Proposition 3.4 implies that

$$\text{length}\, S(r) \geqslant 2r - (\text{sys}\pi_1(X,x) - \text{sys}\pi_1(X))\,. \tag{3.5}$$

Let $\varepsilon = \text{sys}\pi_1(X,x) - \text{sys}\pi_1(X)$. Using the coarea formula, *cf.* [10, 3.2.11], [6, 13.4], as in [6, Theorem 5.3.1], [14] and [11, 5.1.B], we obtain

$$\begin{aligned} \text{area}\, B(x,R) &\geqslant \int_{\frac{\varepsilon}{2}}^{R} \text{length}\, S(r)\, dr \\ &\geqslant \int_{\frac{\varepsilon}{2}}^{R} (2r - \varepsilon)\, dr \\ &\geqslant \left(R - \frac{\varepsilon}{2}\right)^2 \end{aligned}$$

for every R satisfying (3.3).

Now, if we choose $x \in X$ such that a systolic loop passes through x, then $\text{sys}\pi_1(X,x) = \text{sys}\pi_1(X)$. In this case, setting $R = \frac{1}{2}\,\text{sys}\pi_1(X,x)$, we obtain $\text{area}(X) \geqslant \frac{1}{4}\,\text{sys}\pi_1(X)^2$, as required. □

4. Existence of ε-regular metrics

DEFINITION 4.1. — *A metric on a complex X is said to be ε-regular if $\text{sys}\pi_1(X,x) < (1+\varepsilon)\,\text{sys}\pi_1(X)$ for every x in X.*

LEMMA 4.2. — *Let X be a 2-complex with unfree fundamental group. Given a metric $\mathcal{G}$ on X and $\varepsilon > 0$, there exists an ε-regular piecewise flat metric $\mathcal{G}_\varepsilon$ on X with a systolic ratio as good as for $\mathcal{G}$, i.e., $\text{SR}(\mathcal{G}_\varepsilon) \geqslant \text{SR}(\mathcal{G})$.*

Proof. — We argue as in [11, 5.6.C'']. Choose $\varepsilon' > 0$ such that $\varepsilon' < \min\{\varepsilon, 1\}$. Fix $r' = \frac{1}{2}\varepsilon'\,\text{sys}\pi_1(\mathcal{G})$ and $r > 0$, with $r < r'$. Subdividing X if necessary, we can assume that the diameter of the simplices of X is less than $r' - r$. The *approximating ball* $B'(x,r)$ is defined as the union of all simplices of X intersecting $B(x,r)$. By construction, $B'(x,r)$ is a path connected subcomplex of X which contains $B(x,r)$ and is contained in $B(x,r')$. In particular, the inclusion $B' \subset X$ induces the trivial homomorphism of fundamental groups.

Assume that the metric $\mathcal{G}_0 = \mathcal{G}$ on $X_0 = X$ is not already ε'-regular. There exists a point x_0 of X_0 such that

$$\text{sys}\pi_1(X_0,x_0) > (1+\varepsilon')\,\text{sys}\pi_1(X_0). \tag{4.1}$$

Consider the space

$$X_1 = X_0/B'_0$$

obtained by collapsing the approximating ball $B'_0 := B'(x_0, r)$. Denote by $\mathcal{G}_1$ the length structure induced by $\mathcal{G}_0$ on X_1. Let $p_0 : X_0 \longrightarrow X_1$ be the (non-expanding) canonical projection. By Lemma 2.1, the projection p_0 induces an isomorphism of fundamental groups. Consider a systolic loop γ of $\mathcal{G}_1$. Clearly, $\text{length}_{\mathcal{G}_1}(\gamma) \leqslant \text{sys}\pi_1(\mathcal{G}_0)$.

If γ does not pass through the point $p_0(B'_0)$, then the preimage of γ under p_0 is a noncontractible loop of the same length as γ. Therefore, $\text{sys}\pi_1(\mathcal{G}_1) = \text{sys}\pi_1(\mathcal{G}_0)$.

Otherwise, γ is a loop based at the point $p_0(B'_0)$. It is possible to construct a (noncontractible) loop $\overline{\gamma}$ on X_0 passing through x_0 with

$$\text{length}_{\mathcal{G}_0}(\overline{\gamma}) \leqslant \text{length}_{\mathcal{G}_1}(\gamma) + 2r',$$

whose image under p_0 agrees with γ. From (4.1), we deduce that

$$\text{length}_{\mathcal{G}_1}(\gamma) \geqslant \text{length}_{\mathcal{G}_0}(\overline{\gamma}) - 2r' \geqslant (1+\varepsilon')\,\text{sys}\pi_1(\mathcal{G}_0) - 2r' = \text{sys}\pi_1(\mathcal{G}_0).$$

Thus, the systole of $\mathcal{G}_1$ is the same as the systole of $\mathcal{G}_0$ and its area (or Hausdorff measure) is at most the area of $\mathcal{G}_0$. Hence, $\text{SR}(\mathcal{G}_1) \geqslant \text{SR}(\mathcal{G}_0)$.

If $\mathcal{G}_1$ is not ε'-regular, we apply the same process to $\mathcal{G}_1$. By induction, we construct:

- a sequence of points $x_i \in X_i$ with
$$\text{sys}\pi_1(X_i, x_i) > (1+\varepsilon')\,\text{sys}\pi_1(X_i),$$
- a sequence of approximating balls $B'_i := B'(x_i, r)$ in X_i,
- a sequence of spaces X_{i+1} obtained from X_i by collapsing B'_i into a point (with $\pi_1(X_{i+1}) \simeq \pi_1(X_i)$),
- a sequence of metrics $\mathcal{G}_{i+1}$ induced by $\mathcal{G}_i$ on X_{i+1},
- a sequence of canonical projections $p_i : X_i \longrightarrow X_{i+1}$.

This process stops when we obtain an ε'-regular metric (with a systolic ratio as good as the one of $\mathcal{G}$).

Now we show that this process really stops. Let $B^i_1, \ldots, B^i_{N_i}$ be a maximal system of disjoint balls of radius $r/3$ in X_i. Since p_{i-1} is non-expanding, the preimage $p^{-1}_{i-1}(B^i_k)$ of B^i_k contains a ball of radius $r/3$ in X_{i-1}. Furthermore, the preimage $p^{-1}_{i-1}(x_i)$ of x_i contains a ball B_{i-1} of radius r in X_{i-1}. Thus, two balls of radius $r/3$ lie in the preimage of x_i under p_{i-1}. It is then possible to construct a system of N_i+1 disjoint disks of radius $r/3$ in X_{i-1}. Thus, $N_{i-1} \geqslant N_i + 1$ where N_i is the maximal number of disjoint balls of radius $r/3$ in X_i. Therefore, the process stops after N steps with $N \leqslant N_0$.

Let $\pi = p_{N-1} \circ \cdots \circ p_0$ be the projection from X to X_N. Denote by Δ the set formed of the points of X_N whose preimage under π is a singleton, *i.e.*,

$$\Delta = \{y \in X_N \mid \text{card}\,\pi^{-1}(y) = 1\}.$$

By construction, the set $X_N \setminus \Delta$ has at most N points, which will be called the singularities of X_N.

Let $\mathcal{G}_t$ be the length structure on X induced by $e^{-t\varphi}\mathcal{G}$, where $t > 0$ and $\varphi(x) = \mathrm{dist}_{\mathcal{G}}(\pi^{-1}(\Delta), x)$ for $x \in X$. Clearly, $\mathrm{area}(\mathcal{G}_t) \leqslant \mathrm{area}(\mathcal{G})$ and $\mathrm{sys}\pi_1(\mathcal{G}_t) \geqslant \mathrm{sys}\pi_1(X_N) = \mathrm{sys}\pi_1(\mathcal{G})$. Therefore, $\mathrm{SR}(\mathcal{G}_t) \geqslant \mathrm{SR}(\mathcal{G})$.

It suffices to prove that $\mathcal{G}_t$ is ε-regular for t large enough. Since X_N is ε'-regular, this follows, in turn, from the Claim 4.3 below.

Strictly speaking, the metrics $\mathcal{G}_t$ are not piecewise flat but we can approximate them by piecewise flat metrics as in [1] (see also [6, §3]) to obtain the desired conclusion.

Claim 4.3. — *The family* $\{\mathrm{sys}\pi_1(\mathcal{G}_t, x)\}$ *converges to* $\mathrm{sys}\pi_1(X_N, \pi(x))$ *uniformly in x as t goes to infinity.*

Clearly, for every x in X and $t > 0$, we have

$$\mathrm{sys}\pi_1(X_N, \pi(x)) \leqslant \mathrm{sys}\pi_1(\mathcal{G}_t, x). \tag{4.2}$$

Fix $\delta > 0$. Take a pointed systolic loop $\gamma \subset X_N$ at some fixed point y of X_N and let γ pass through $k(y)$ singularities of X_N. Given $z \in X_N$ at distance at most $R = \delta/5$ from y, the loop $[z, y] \cup \gamma \cup [y, z]$ based at z, where $[a, b]$ represents a segment joining a to b, is freely homotopic to γ and passes through at most $k(y) + 2N$ singularities. Moreover, its length is at most $\mathrm{sys}\pi_1(X_N, y) + 2R \leqslant \mathrm{sys}\pi_1(X_N, z) + 4R$ since $\mathrm{sys}\pi_1(X_N, \cdot)$ is 2-Lipschitz.

Let $k = \max_i k(y_i) + 2N$ where the y_i's are the centers of a maximal system of disjoint balls of radius $R/2$ in X_N. It is possible to construct for every z in X_N a noncontractible loop γ_z based at z passing through at most k singularities of length at most $\mathrm{sys}\pi_1(X_N, z) + 4R$.

The preimages U_i under $\pi : X \to X_N$ of the singularities of X_N are path-connected. Choose t large enough so that every pair of points in U_i can be joined by an arc of U_i of $\mathcal{G}_t$-length less than some fixed $\eta > 0$ with $\eta < R/k$. Fix $x \in X$. Consider the loop $\gamma = \gamma_z$ of X_N based at $z = \pi(x)$ previously defined. There exists a noncontractible loop $\overline{\gamma} \subset X$ based at x of length

$$\mathrm{length}_{\mathcal{G}_t}(\overline{\gamma}) \leqslant \mathrm{length}_{X_N}(\gamma) + k\eta$$

whose image under π agrees with γ. Therefore,

$$\mathrm{sys}\pi_1(\mathcal{G}_t, x) \leqslant \mathrm{sys}\pi_1(X_N, \pi(x)) + 4R + R.$$

Hence,

$$\mathrm{sys}\pi_1(\mathcal{G}_t, x) \leqslant \mathrm{sys}\pi_1(X_N, \pi(x)) + \delta. \tag{4.3}$$

Since $\mathrm{sys}\pi_1(\mathcal{G}_{t'}, x) \leqslant \mathrm{sys}\pi_1(\mathcal{G}_t, x)$ for every $t' \geqslant t$, the inequalities (4.2) and (4.3) lead to the desired claim.

This conclude the proof of the lemma. □

5. Counting fundamental groups

Let X be a complex endowed with a piecewise flat metric, Consider a finite covering $\{B_i\}$ of X by open balls of radius $R = \frac{1}{6}\,\mathrm{sys}\pi_1(X)$. Denote by $\mathcal{N}$ the nerve of this covering.

Lemma 5.1. — *The fundamental groups of X and $\mathcal{N}$ are isomorphic.*

Proof. — Recall that, by definition, the vertices p_i of $\mathcal{N}$ are identified with the balls B_i. Furthermore, $k+1$ vertices $p_{i_0}, \dots, p_{i_k}$ form a k-simplex of $\mathcal{N}$ if and only if $B_{i_0} \cap \cdots \cap B_{i_k} \neq \emptyset$. Given x and y in X, fix a minimizing path (not necessarily unique), denoted by $[x, y]$, from x to y.

We denote by $\mathcal{N}_i$ the i-skeleton of $\mathcal{N}$. Define a map $f : \mathcal{N}_1 \longrightarrow X$ as follows. The map f sends the vertices p_i to the centers x_i of the balls B_i and the edges $[p_i, p_j]$ to the segments $[x_i, x_j]$ (previously chosen). By construction, the distance between two centers x_i and x_j corresponding to a pair of adjacent vertices is less than $2R$. Thus, the map f sends the boundary of each 2-simplex of $\mathcal{N}$ to loops of length less than $6R = \mathrm{sys}\pi_1(X)$. By definition of the systole, these loops are contractible in X. Therefore, the map f extends to a map $F : \mathcal{N}_2 \longrightarrow X$.

Choose a center $x_{\alpha(0)}$ of some of the balls B_i. We claim that the homomorphism $F_* : \pi_1(\mathcal{N}_2, p_{\alpha(0)}) \longrightarrow \pi_1(X, x_{\alpha(0)})$ is an isomorphism. Since the nerve $\mathcal{N}$ and its 2-skeleton $\mathcal{N}_2$ have the same fundamental group, we conclude that $\pi_1(X)$ and $\pi_1(\mathcal{N})$ are isomorphic.

We prove the surjectivity of F_* only. The injectivity can be proved in a similar way, we leave it to the reader.

Given a piecewise smooth path $\gamma : I \longrightarrow X$, $\gamma(0) = \gamma(1) = x_{\alpha(0)}$, we construct the following path $\overline{\gamma} : I \longrightarrow \mathcal{N}_1, \overline{\gamma}(0) = \overline{\gamma}(1) = p_{\alpha(0)}$ such that the loop $F(\overline{\gamma})$ is homotopic to γ. Fix a subdivision $t_0 = 0 < t_1 < \cdots < t_m < t_{m+1} = 1$ of I such that $\gamma([t_k, t_{k+1}])$ is contained in some $B_{\alpha(k)}$ and the length of $\gamma_{|[t_k, t_{k+1}]}$ is less than $\frac{1}{3}$ for $k = 0, \dots, m$. The map $\overline{\gamma}$ takes the segment $[t_k, t_{k+1}]$ to the edge $[p_{\alpha(k)}, p_{\alpha(k+1)}]$ of $\mathcal{N}$. By construction, we have $\overline{\gamma}(t_k) = p_{\alpha(k)}$ and $F(\overline{\gamma}(t_k)) = x_{\alpha(k)}$. Therefore, the image of $\overline{\gamma}$ under F is a piecewise linear loop which agrees with the union

$$\bigcup_{k=0}^{m} [x_{\alpha(k)}, x_{\alpha(k+1)}]$$

where the segments $[x_{\alpha(k)}, x_{\alpha(k+1)}]$ are previously fixed.

Consider the following loops of X

$$c_k = \gamma([t_k, t_{k+1}]) \cup [\gamma(t_{k+1}), x_{\alpha(k+1)}] \cup [x_{\alpha(k+1)}, x_{\alpha(k)}] \cup [x_{\alpha(k)}, \gamma(t_k)]$$

where $[x_{\alpha(k+1)}, x_{\alpha(k)}]$ agrees with $F \circ \overline{\gamma}([t_k, t_{k+1}])$. The length of c_k is

$$\text{length}(c_k) < \frac{1}{3}\,\text{sys}\pi_1(X) + R + 2R + R = \text{sys}\pi_1(X).$$

Hence, the loop c_k is contractible. Therefore, the loops γ and $F \circ \overline{\gamma}$ are homotopic, and thus the homomorphism F_* is surjective. □

Definition 5.2. — *The isomorphism classes of the fundamental groups of the finite* 2*-complexes with at most* n *vertices form a finite set* $\Gamma(n)$*. We define* $\Gamma'(n)$ *as the union of* $\Gamma(n)$ *and the set formed of the unfree factors of the elements of* $\Gamma(n)$*.*

Corollary 5.3. — *Suppose that the covering* $\{B_i\}$ *of* X *in Lemma 5.1 consists of* m *elements. Then* $\pi_1(X) \in \Gamma(m)$*.*

Proof. — This follows from Lemma 5.1, since the nerve of the covering has m vertices. □

Now we estimate the numbers $\Gamma(n)$ and $\Gamma'(n)$.

Lemma 5.4. — *Up to isomorphism, the number of* 2*-dimensional simplicial complexes having* n *vertices is at most*

$$2^{\frac{(n-1)n(n+1)}{6}} < 2^{n^3}. \tag{5.1}$$

In particular, the sets $\Gamma(n)$ *and* $\Gamma'(n)$ *contain less than* 2^{n^3} *elements.*

Proof. — The maximal number of edges in a simplicial complex with n vertices is equal to the cardinality of $\{(i,j) \mid 1 \leqslant i < j \leqslant n\}$, which is $\frac{n(n-1)}{2}$. Similarly, the maximal number of triangles in a simplicial complex with n vertices is equal to the cardinality of $\{(i,j,k) \mid 1 \leqslant i < j < k \leqslant n\}$, which is $\frac{n(n-1)(n-2)}{6}$. Thus, the number of isomorphism classes of 1-dimensional simplicial complexes having n vertices is at most

$$2^{\frac{n(n-1)}{2}}. \tag{5.2}$$

Therefore, the number of 2-dimensional simplicial complexes whose 1-skeleton agrees with one of these 1-dimensional simplicial complexes is at most

$$2^{\frac{n(n-1)(n-2)}{6}}. \tag{5.3}$$

The product of (5.2) and (5.3) yields an upper bound on the number of isomorphism classes of 2-dimensional simplicial complexes having n vertices.

Note that $\Gamma'(n)$ has at most twice as many elements as $\Gamma(n)$. The second part of the lemma follows then from the first part. □

6. Two systolic finiteness results

PROPOSITION 6.1. — *Let X be a 2-complex equipped with a piecewise flat metric. Suppose that the area of every ball $B(R)$ of radius $R = \frac{1}{12}\operatorname{sys}\pi_1(X)$ in X is at least $\alpha \operatorname{sys}\pi_1(X)^2$, i.e.,*

$$\operatorname{area} B(R) \geqslant \alpha \operatorname{sys}\pi_1(X)^2. \tag{6.1}$$

If $\sigma(X) < C$, then the isomorphism class of the fundamental group of X lies in the finite set $\Gamma(C/\alpha)$.

Proof. — Consider a maximal system of disjoint open balls $B(x_i, R)$ in X of radius $R = \frac{1}{12}\operatorname{sys}\pi_1(X)$ with centers x_i, $i = 1, \dots, m$. By the assumption,

$$\operatorname{area} B(x_i, R) \geqslant \alpha \operatorname{sys}\pi_1(X)^2. \tag{6.2}$$

Therefore, this system admits at most $\frac{\operatorname{area}(X)}{\alpha \operatorname{sys}\pi_1(X)^2}$ balls. Thus,

$$m \leqslant C/\alpha. \tag{6.3}$$

The open balls B_i of radius $2R = \frac{1}{6}\operatorname{sys}\pi_1(X)$ centered at x_i form a covering of X. From Corollary 5.3, the fundamental group of X lies in $\Gamma(m) \subset \Gamma(C/\alpha)$. □

THEOREM 6.2. — *Given $C > 0$, there are finitely many isomorphism classes of finitely presented groups G of zero Grushko free index such that $\sigma(G) < C$.*

More precisely, the isomorphism class of every finitely presented group G with zero Grushko free index and $\sigma(G) < C$ lies in the finite set $\Gamma(144\,C)$, which has at most

$$K^{C^3},$$

elements. Here, K is an explicit universal constant.

Remark 6.3. — The remark following Theorem 1.1 also shows that the assumption that G has zero Grushko free index cannot be dropped in Theorem 6.2.

Proof of Theorem 6.2. — Consider a finitely presentable group G of zero Grushko free index and such that $\sigma(G) < C$. There exist a 2-complex X with fundamental group isomorphic to G and a piecewise flat metric $\mathcal{G}$ on X such that $\sigma(\mathcal{G}) < C$. Let $0 < \varepsilon < \frac{1}{12}$. Fix a 2ε-regular piecewise flat metric on X with a better systolic ratio than the one of $\mathcal{G}$, *cf.* Lemma 4.2. By Theorem 3.5,

$$\text{area}\, B(R) \geqslant \left(\frac{1}{12} - \varepsilon\right)^2 \text{sys}\pi_1(X)^2 \tag{6.4}$$

for all balls $B(R)$ of radius $R = \frac{1}{12}\,\text{sys}\pi_1(X)$. Since $\sigma(X) < C$, we deduce from Proposition 6.1 that the isomorphism class of the fundamental group of X lies in the finite set

$$\Gamma\left(\frac{C}{(\frac{1}{12} - \varepsilon)^2}\right) = \Gamma\left(\frac{144\,C}{(1-12\varepsilon)^2}\right)$$

for every $\varepsilon > 0$ small enough. Thus, the isomorphism class of G lies in $\Gamma(144\,C)$.

By Lemma 5.4, this set has at most

$$(2^{12^6})^{C^3}$$

elements. Hence the result. □

Example 6.4. — It follows from Theorem 6.2 that the systolic ratio of the cyclic groups $\mathbb{Z}/n\mathbb{Z}$ of order n goes to zero as $n \to \infty$, *i.e.*,

$$\lim_{n\to\infty} \text{SR}(\mathbb{Z}/n\mathbb{Z}) = 0.$$

It would be interesting, however, to evaluate the value $\text{SR}(\mathbb{Z}/n\mathbb{Z})$.

7. Systolic area comparison

Let G be an unfree finitely presentable group with $G = F_p * H$ where F_p is free of rank p and H is of zero Grushko free index. Fix $\delta \in (0, \frac{1}{12})$ (close to zero) and $\lambda > \frac{1}{\pi}$ (close to $\frac{1}{\pi}$). Choose $\varepsilon < \delta$ (close to zero) such that $0 < \varepsilon < 4(\lambda - \frac{1}{\pi})(\delta - \varepsilon)^2$. From Lemma 4.2, there exists a 2-complex X with fundamental group isomorphic to G and a 2ε-regular piecewise flat metric $\mathcal{G}$ on X such that

$$\sigma(\mathcal{G}) \leqslant \sigma(G) + \varepsilon. \tag{7.1}$$

We normalize the metric $\mathcal{G}$ on X so that its systole is equal to 1.

Denote by $B(x,r)$ and $S(x,r)$ the ball and the sphere of radius $r < \frac{1}{2}$ centered at some point x of X. Note that

$$\delta > \varepsilon > \frac{1}{2}\left(\text{sys}\pi_1(X,x) - \text{sys}\pi_1(X)\right) \tag{7.2}$$

for every $x \in X$.

Lemma 7.1. — *Suppose that there exist $x_0 \in X$ and $r_0 \in (\delta, \frac{1}{2})$ such that*

$$\operatorname{area} B > \lambda\, (\operatorname{length} S)^2 \tag{7.3}$$

where $B = B(x_0, r_0)$ and $S = S(x_0, r_0)$. Then, the Grushko free index p of G is positive, and

$$\sigma(G) \geqslant \sigma(F_{p-1} * H) - \varepsilon. \tag{7.4}$$

Proof. — First, we prove that $p > 0$. We let $f(x) = \operatorname{dist}(x_0, x)$ and show that two path-connected components of $S = f^{-1}(r_0)$ coalesce forward, *cf.* Definition 2.2 and Lemma 2.3. Denote by $\overline{X}$ the 2-complex obtained from X by attaching cones C_i over each connected component S_i of S, $1 \leqslant i \leqslant m$. By Proposition 3.2, the connected components S_i are contractible in X. Therefore, the fundamental groups of X and $\overline{X}$ are isomorphic, *i.e.*,

$$\pi_1(X) \simeq \pi_1(\overline{X}). \tag{7.5}$$

Fix a segment $[x_0, x_i]$ joining x_0 to S_i in B. There exists a tree T in the union of the $[x_0, x_i]$ containing x_0 with endpoints x_i.

Let $\widehat{X} := (\overline{X} \setminus \operatorname{Int} B) \cup T$ and $\widehat{B} := B \cup (\cup_i C_i)$. Notice that $\widehat{X}$ is (path) connected. Indeed, every point $x \in X \setminus \operatorname{Int} B$ can be connected to some S_i by a path in $X \setminus \operatorname{Int} B$ (every path from x_0 to x intersects S), while every point of each component S_i can be connected to x_0 by a path in $S_i \cup T \subset \widehat{X}$. By the results of Section 2, the triad $(\overline{X}; \widehat{X}, \widehat{B})$ is a CW-triad. Since every loop in $\widehat{B}$ can be deformed into a loop in B, the inclusion $\widehat{B} \subset \overline{X}$ induces a trivial homomorphism of fundamental groups because of Proposition 3.2. Furthermore, the space $\widehat{X} \cap \widehat{B} = T \cup (\cup C_i)$ is simply connected. Since $\overline{X} = \widehat{X} \cup \widehat{B}$, we deduce from Seifert–van Kampen theorem that the inclusion $\widehat{X} \subset \overline{X}$ induces an isomorphism of fundamental groups. Thus, the relation (7.5) leads to

$$\pi_1(\widehat{X}) \simeq \pi_1(X) \simeq G. \tag{7.6}$$

We endow each cone C_i over S_i with the round metric described in the appendix. By Appendix A, the area of C_i is equal to $\frac{1}{\pi}(\operatorname{length} S_i)^2$. Since the sum of the lengths of the S_i's is equal to the length of S, the total area of $\cup_i C_i$ is at most $\frac{1}{\pi}(\operatorname{length} S)^2$. The tree T is endowed with its standard metric, *i.e.*, the length of each of its edges is equal to 1. The metrics on $X \setminus B$, $\cup_i C_i$ and T induce a metric, noted $\widehat{\mathcal{G}}$, on the union $\widehat{X} = (X \setminus B) \cup (\cup_i C_i) \cup T$.

By construction, one has

$$\operatorname{sys}\pi_1(\widehat{X}) \geqslant \operatorname{sys}\pi_1(X) = 1. \tag{7.7}$$

Furthermore, we have

$$\text{area}\,\widehat{X} \leqslant \text{area}\,X - \text{area}\,B + \frac{1}{\pi}(\text{length}\,S)^2. \tag{7.8}$$

The inequality (7.3) leads to

$$\text{area}\,\widehat{X} \leqslant \text{area}\,X - \left(\lambda - \frac{1}{\pi}\right)(\text{length}\,S)^2. \tag{7.9}$$

Hence, $\sigma(\widehat{\mathcal{G}}) \leqslant \sigma(\mathcal{G}) \leqslant \sigma(G)+\varepsilon$. Here, the first inequality holds since $\lambda > \frac{1}{\pi}$ while the second one follows from (7.1).

Since $\sigma(G) \leqslant \text{area}(\widehat{X})$ and $\text{area}(X) \leqslant \sigma(G) + \varepsilon$, we also obtain

$$\left(\lambda - \frac{1}{\pi}\right)(\text{length}\,S)^2 < \varepsilon. \tag{7.10}$$

Since $\varepsilon < 4(\lambda - \frac{1}{\pi})(\delta - \varepsilon)^2$ and $\delta \leqslant r_0$, we deduce that

$$\text{length}\,S < 2(\delta - \varepsilon) \leqslant 2r_0 - 2\varepsilon. \tag{7.11}$$

Now, by Lemma 3.3 and Proposition 3.4, every pointed systolic loop $\gamma \subset X$ at x_0 intersects exactly two path-connected components of S, say S_1 and S_2 (recall that $r_0 \geqslant \delta > \frac{1}{2}\,(\text{sys}\pi_1(X,x) - \text{sys}\pi_1(X))$, *cf.* (7.2)). Since γ contains an arc of $X \setminus \text{Int}(B)$ joining S_1 to S_2, *cf.* Lemma 3.3, we conclude that two path-connected components of S coalesce forward. Thus, by Proposition 3.2 and Lemma 2.3, G has a positive Grushko free index.

Now, the points x_1 and x_2, which are joined by a path in $X \setminus \text{Int}\,B$, are also joined to x_0 by a unique geodesic arc in the tree T. Identify the unique edge of the tree T which contains x_1 with the segment $[0,1]$. Set $Y := \widehat{X} \setminus I \subset \widehat{X}$, where $I = (\frac{1}{3}, \frac{2}{3})$. Since $\widehat{X}$ is connected and the endpoints of I are joined by a path in $\widehat{X} \setminus I$, we conclude that Y is connected. Furthermore, the space $\widehat{X}$, obtained by gluing back the interval I to Y, is homotopy equivalent to $Y \vee S^1$. In particular,

$$G \simeq \pi_1(\widehat{X}) \simeq \pi_1(Y) * \mathbb{Z}. \tag{7.12}$$

By uniqueness (up to isomorphism) of the Grushko decomposition $G \simeq F_p * H$, we obtain

$$\pi_1(Y) \simeq F_{p-1} * H. \tag{7.13}$$

Furthermore, $\sigma(Y) \leqslant \sigma(\widehat{\mathcal{G}}) \leqslant \sigma(G) + \varepsilon$. In particular, we deduce that $\sigma(F_{p-1} * H) \leqslant \sigma(G) + \varepsilon$, which concludes the proof of Lemma 7.1. □

Proposition 7.2. — *With the previous notation,*

(i) *either the Grushko free index p of G is positive, and*

$$\sigma(G) \geqslant \sigma(F_{p-1} * H) - \varepsilon, \tag{7.14}$$

(ii) *or*

$$\operatorname{area} B(x,r) \geqslant \frac{1}{4\lambda}(r-\delta)^2 \tag{7.15}$$

for every $x \in X$ *and every* $r \in (\delta, \frac{1}{2})$*.*

Proof. — By Lemma 7.1, we can assume that

$$\operatorname{area} B(x,r) \leqslant \lambda\, (\operatorname{length} S(x,r))^2 \tag{7.16}$$

for every $x \in X$ and $r \in (\delta, \frac{1}{2})$, otherwise the claim (i) holds. Now, if $a(r)$ and $\ell(r)$ represent the area of $B(x,r)$ and the length of $S(x,r)$, respectively, then the claim (ii) follows from Lemma 7.3 below along with the coarea formula. □

Lemma 7.3. — *Assume that, for all* $r \in (\delta, \frac{1}{2})$*, we have*

$$a(r) := \int_0^r \ell(s)\, ds \leqslant \lambda\, \ell(r)^2. \tag{7.17}$$

Then, for every $r \in (\delta, \frac{1}{2})$*, we have*

$$a(r) \geqslant \frac{1}{4\lambda}(r-\delta)^2. \tag{7.18}$$

Proof. — The function $\ell(r)$ is a piecewise continuous positive function by Proposition 2.4. So, the function $a(r)$ is continuously differentiable for all but finitely many r in $(\delta, \frac{1}{2})$. Furthermore, $a'(r) = \ell(r)$ for all but finitely many r in $(\delta, \frac{1}{2})$. By assumption, we have

$$a(r) \leqslant \lambda\, a'(r)^2$$

for all but finitely many $r \in (\delta, \frac{1}{2})$. That is,

$$\left(\sqrt{a(r)}\right)' = \frac{a'(r)}{2\sqrt{a(r)}} \geqslant \frac{1}{2\sqrt{\lambda}}.$$

Integrating this inequality from δ to r, we get

$$\sqrt{a(r)} \geqslant \frac{1}{2\sqrt{\lambda}}(r-\delta).$$

Hence, for every $r \in (\delta, \frac{1}{2})$, we obtain

$$a(r) \geqslant \frac{1}{4\lambda}(r-\delta)^2.$$

□

8. Main results

In this section, we extend previous results for groups of zero Grushko free index to arbitrary finitely presentable groups. More precisely, we establish a uniform bound on the systolic ratio of unfree finitely presented groups and a finiteness result for the unfree part of a group with systolic ratio bounded away from zero.

THEOREM 8.1. — *Every unfree finitely presentable group G satisfies*

$$\mathrm{SR}(G) \leqslant \frac{16}{\pi}. \tag{8.1}$$

Remark 8.2. — The upper bound by $\frac{16}{\pi}$ on the systolic ratio in (8.1) is not as good as the upper bound by 4 obtained in Theorem 3.5 in the zero Grushko free index case.

Proof of Theorem 8.1. — Let us prove the inequality (8.1) by induction on the Grushko free index of G. To start the induction, consider a finitely presentable group G of zero Grushko free index. Then, by Theorem 3.5,

$$\sigma(G) \geqslant \frac{1}{4} > \frac{\pi}{16}.$$

Now, assume that the inequality (8.1) holds for all finitely presented groups whose Grushko free index is less than p. Consider a finitely presentable group G with positive Grushko free index p. The group G decomposes as $G = F_p * H$ where F_p is free of rank p and H is of zero Grushko free index. We will use the notation of Section 7.

If the inequality (7.15) holds for all $x \in X$ and $r \in (\delta, \frac{1}{2})$, then

$$\sigma(\mathcal{G}) = \operatorname{area} X \geqslant \frac{1}{4\lambda} \left(\frac{1}{2} - \delta \right)^2. \tag{8.2}$$

That is,

$$\sigma(G) \geqslant \frac{1}{4\lambda} \left(\frac{1}{2} - \delta \right)^2 - \varepsilon. \tag{8.3}$$

Note that the right-hand term goes to $\frac{\pi}{16}$ as $\delta \to 0$, $\lambda \to \frac{1}{\pi}$ and $\varepsilon \to 0$. Thus, $\sigma(G) \geqslant \frac{\pi}{16}$, *i.e.*, the inequality (8.1) holds.

Therefore, we can assume that the inequality (7.14) holds, *i.e.*,

$$\sigma(G) \geqslant \sigma(F_{p-1} * H) - \varepsilon.$$

By induction on p, we obtain

$$\sigma(G) \geqslant \frac{\pi}{16} - \varepsilon. \tag{8.4}$$

This implies the inequality (8.1) as $\varepsilon \to 0$. □

THEOREM 8.3. — *Let G be a finitely presentable group. If $\sigma(G) < C$ for some $C > 0$, then the isomorphism class of the unfree factor of G lies in the finite set $\Gamma'\left(\frac{576\,C}{\pi}\right)$.*

In particular, the number of isomorphism classes of the unfree factors of the finitely presentable groups G such that $\sigma(G) < C$ is at most

$$A^{C^3}, \tag{8.5}$$

where A is an explicit universal constant.

Proof of Theorem 8.3. — We prove the result by induction on the Grushko free index of G. Theorem 6.2 shows that the isomorphism class of every finitely presented group H of zero Grushko free index with $\sigma(H) < C$ lies in $\Gamma(144\,C) \subset \Gamma'(\frac{576\,C}{\pi})$.

Now, let G be a finitely presentable group of positive Grushko free index p, that is $G = F_p * H$ where H has zero Grushko free index. Suppose that $\sigma(G) < C$. We will use the notation of Section 7. Note that we can always assume that $\sigma(\mathcal{G}) < C$ for $\mathcal{G}$ as in (7.1).

If the inequality (7.15) holds for all $x \in X$ and $r \in (\delta, \frac{1}{2})$, then the inequality (6.1) holds for $\alpha = \frac{1}{4\lambda}(\frac{1}{12} - \delta)^2$. Hence, by Proposition 6.1, the isomorphism class of the group G lies in the finite set $\Gamma(C/\alpha)$, which is contained in

$$\Gamma\left(\frac{576\,C}{\pi}\right) \subset \Gamma'\left(\frac{576\,C}{\pi}\right)$$

if δ is close enough to 0 and λ is close enough to $\frac{1}{\pi}$. In particular, this shows that the isomorphism class of the group H lies in $\Gamma'\left(\frac{576\,C}{\pi}\right)$.

So, we can assume that the inequality (7.14) holds. Since $\sigma(G) < C$, we obtain

$$\sigma(F_{p-1} * H) < C + \varepsilon.$$

By induction on p, we derive that the isomorphism class of H lies in $\Gamma'\left(\frac{576\,(C+\varepsilon)}{\pi}\right)$ for all $\varepsilon > 0$. Thus, the isomorphism class of H lies in $\Gamma'\left(\frac{576\,C}{\pi}\right)$.

Finally, by Lemma 5.4, we can take $A = 2^{\left(\frac{576}{\pi}\right)^3}$ in (8.5). □

We have the following Corollary that generalizes Example 6.4.

COROLLARY 8.4. — *Let $G_1, \dots, G_n, \dots$ be a sequence of pairwise non-isomorphic groups of bounded Grushko free index. Then*

$$\lim_{n \to \infty} \mathrm{SR}(G_n) = 0.$$

Proof. — This follows from Theorem 8.3 since, given $\varepsilon > 0$, there are only finite number of n's with $\mathrm{SR}(G_n) < \varepsilon$. □

Example 8.5. — Let G_n be the free product of n unfree finitely presentable groups. As in Corollary 8.4, we obtain from Theorem 8.3 that the systolic ratio of the sequence $\{G_n\}$ tends to zero as $n \to \infty$, *cf.* [12, p. 337].

9. Counting freely indecomposable groups

In this section, using a result of I. Kapovich and P. Schupp [15], we show that the number of isomorphism classes of freely indecomposable groups grows at least exponentially with a bound on their systolic area.

THEOREM 9.1. — *There exists $C_0 > 0$ such that for every $C \geqslant C_0$ the number of isomorphism classes of freely indecomposable finitely presentable groups G with $\sigma(G) < C$ is at least 2^C.*

Proof. — A relation R on two letters a and b defines a two-generator one-relator group $G = \langle a, b \mid R \rangle$. Denote by k the length of the relation R with respect to the alphabet formed of the two letters a and b. The group G can be described as the fundamental group of a 2-complex $X = X_R$, where X is obtained from a polygon (disk) D with k edges that are identified according to the word R.

We can equip X with a round metric $\mathcal{G}$ for which D is a metric disk of constant curvature $\frac{1}{4}$ with radius π, perimeter $2\pi k$ and a conical singularity of angle $k\pi$ at its center. This metric $\mathcal{G}$ can be written in polar coordinate (r, θ) on D as

$$\mathcal{G} = dr^2 + \sin^2(r/2)\, d\theta^2, \quad 0 < r \leqslant \pi,\ \theta \in \mathbb{R}/2\pi k\mathbb{Z}.$$

By construction, the area of $(X, \mathcal{G})$ is equal to $8\pi k$ and its systole is equal to 2π. Hence,

$$\sigma(G) \leqslant \sigma(X, \mathcal{G}) = \frac{2k}{\pi}.$$

Since there are at least 2^k isomorphism classes of two-generator one-relator groups with defining relation of length k for k large enough, *cf.* [15, Theorem B], we obtain the desired bound after showing that most of these groups are freely indecomposable.

Let H be a freely decomposable two-generator one-relator group. By Grushko's theorem, the group H decomposes as $H = H_1 * H_2$, where $H_i \simeq \mathbb{Z}/p_i\mathbb{Z}$. From [9], we have $p_1 + p_2 \leqslant k + 2$, where k is the length of the defining relation of H. Thus, the number of isomorphism classes of such groups H is at most polynomial in k. Hence the conclusion. □

Appendix A. A round metric

Consider the upper hemisphere H of the radius r,

$$H := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = r^2, z \geqslant 0\}.$$

We equip H with the sphere metric dist_H. Let $K = \{(x, y, z) \in H \mid z = 0\}$ and $p = (0, 0, r) \in H$. Given a point $q \in H, q \neq p$, consider the geodesic arc of length $\pi r/2$ that starts at p, passes through q and ends at some point $x = x(q) \in K$. We define $t = t(q)$ as the length of the geodesic segment joining p and q. Clearly, q determines and is uniquely determined by x and t. Thus, every point of H can be described as a pair (x, t) where $x \in K$ and $t \in [0, \pi\, r/2]$. Here, $(x, 0) = p$ for all x.

We define a function $f : [0, \pi r] \times [0, \pi r/2]^2 \to \mathbb{R}$ by setting

$$f(R, t_1, t_2) = \mathrm{dist}_H((x_1, t_1), (x_2, t_2)) \tag{A.1}$$

where $(x_i, t_i) \in H, i = 1, 2$ are such that $\mathrm{dist}_K(x_1, x_2) = R$. Clearly, the function f is well-defined.

Now, let S be a finite metric graph of total length L and set $r = L/\pi$. Consider the cone $C = (S \times [0, \pi\, r/2])/(S \times \{0\})$. Every point of C can be written as (x, t) where $x \in S$ and $t \in [0, \pi r/2]$. We denote by v the vertex $(x, 0)$ of the cone. We equip C with a piecewise smooth metric by setting

$$\mathrm{dist}_C((x_1, t_1), (x_2, t_2)) = f(\mathrm{dist}_S(x_1, x_2), t_1, t_2)$$

where f is the function defined in (A.1). It is clear that dist_C is a metric since dist_H is, and it is piecewise smooth since dist_S is. Clearly, the inclusion $S \subset C$ is an isometry.

Furthermore, the region $(e \times [0, \pi r/2])/(e \times \{0\})$ of C, where e is an edge of S, is isometric to a sector of the hemisphere H of angle $\frac{1}{r}\,\mathrm{length}(e)$. Thus, the area of this region is equal to $r\,\mathrm{length}(e)$. We immediately deduce the following result.

Proposition A.1. — *The area of the cone C is given by*

$$\mathrm{area}(C) = rL = L^2/\pi.$$

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