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RATIONAL POINTS AND COXETER GROUP ACTIONS ON THE COHOMOLOGY OF TORIC VARIETIES

by Gustav I. LEHRER

To Alex and Claire

ABSTRACT. — We derive a simple formula for the action of a finite crystallographic Coxeter group on the cohomology of its associated complex toric variety, using the method of counting rational points over finite fields, and the Hodge structure of the cohomology. Various applications are given, including the determination of the graded multiplicity of the reflection representation.

RéSUMÉ. — On donne une formule simple pour l’action d’un groupe de Coxeter fini crystallographique sur la cohomologie de la variété torique complexe associée. La méthode utilise la structure de Hodge sur la cohomologie pour relier le nombre des points rationnels sur un corps fini à cette action. On utilise la formule pour quelques applications, telles que la détermination de la multiplicité graduée de la représentation par réflexions dans la cohomologie.

1. Introduction and statement of main result

Let $V$ be an vector space of finite dimension $n$ over $\mathbb{R}$. Let $\Phi$ be a root system in $V$, and let $W$ be the associated Coxeter group, which is generated by the reflections in hyperplanes orthogonal to the roots; we take $W$ to be finite and crystallographic, and write $\langle -, - \rangle$ for a $W$-invariant positive definite bilinear form on $V$. Assume chosen a simple system $\Pi \subseteq \Phi$, which forms a basis of $V$. Let $L := \mathbb{Z}\Phi$ be the root lattice, and $M := \{v \in V \mid \langle v, \alpha \rangle \in \mathbb{Z}, \ \forall \alpha \in L\}$ be the corresponding weight lattice.

As explained in [8], there is a fan $\Delta = \Delta_W$ of convex polyhedral cones in $M$, and hence a “toric variety” associated with this data. This is a smooth complex projective variety, which we shall denote by $T_W$. This variety, and

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hence its cohomology, carries a natural action of the group $W$. In this work we shall determine this action, in the sense that we shall give an explicit formula for the equivariant Poincaré polynomial

$$P_W(t, w) := \sum_{i \geq 0} \text{Trace} \left( w, H^i(T_W, \mathbb{C}) \right) t^i \in \mathbb{C}[t],$$

for each element $w \in W$.

Equivalently, if $R(W)$ denotes the complex character ring of $W$, we shall determine the element

$$P_W(t) := \sum_i H^i(T_W, \mathbb{C}) t^i \in R(W)[t]$$

by means of its value on elements $w \in W$. This question arises, among other places, in the study (cf. [5, 4]) of compactifications of reductive groups and the cohomology of complete symmetric varieties.

This problem was addressed by different methods by Procesi in [14]. He made use of the fact that $T_W$ may be described in terms of repeated blowups, and the cohomology of the blowup of a space along a subspace is straightforward to compute. Procesi’s result is well suited to the recursive determination of $P_W(t)$.

Stembridge, in [15], studied the same problem indirectly, using a result of Danilov [3] to identify the cohomology ring with a certain commutative algebra. His result [15, Corollary 1.6] is similar to our Theorem 1.1 below, but retains a recursive flavour. The main thrust of [15] is the identification of the total cohomology with a permutation representation of $W$. In their work [7, Theorem 2.1] Dolgachev and Lunts also prove the same formula as Stembridge, using the $T$-equivariant cohomology of the toric variety $T_W$. In the Appendix below, we prove that the Dolgachev-Lunts-Stembridge formula is equivalent to ours.

In this work we use the method of counting rational points over finite fields, combined with the Hodge structure of the cohomology, developed in [11, 6, 9, 10]. This results in a quick and direct derivation of a closed formula for $P_W(w, t)$, which is quite easy to evaluate in many cases. Some consequences of the general formula are discussed in §§3,4; included among these are the fact that the alternating representation of $W$ does not occur in $H^*(T_W, \mathbb{C})$, and a formula giving the graded multiplicity of the reflection representation of $W$ in the cohomology ring (see §4.1).

The case $w = 1$ of our formula (1) below was proved by Fulton in [8, §4.5], for smooth complete toric varieties $T$ which includes our $T_W$; the weight filtration of the cohomology $H^*(T, \mathbb{C})$ is also determined for these $T$. Our
work may be regarded as an equivariant generalisation of the results in [8, loc. cit.] for the particular varieties $T_W$.

We proceed now to state our basic formula.

For each subset $J \subseteq \Pi$, let $W_J$ be the corresponding parabolic subgroup generated by the reflections in the hyperplanes orthogonal to the roots in $J$, and $V_J$ the linear span of $J$.

**Theorem 1.1.** — Let $\Phi, \Pi, W$ and $T_W$ be as above. Then $H^i(T_W, \mathbb{C}) = 0$ if $i$ is odd. The even dimensional cohomology is described as follows. For each $J \subseteq \Pi$, let $\gamma_J(t)$ be the $\mathbb{C}[t]$-valued class function on $W_J$ given by $\gamma_J(t)(w) := \det_{V_J}(t^2 - w)$, where this is interpreted as 1 if $J = \emptyset$. Then

$$P_W(t) = \sum_{J \subseteq \Pi} \text{Ind}_W^W(\gamma_J(t)).$$

This may be reformulated as follows.

**Corollary 1.2.** — Maintain the above notation. For each subset $J \subseteq \Pi$, let $\rho_{J,i}$ be the $i$th exterior power of the (reflection) representation of $W_J$ on $V_J$ $(i = 1, \ldots, |J|)$. Then

$$P_W(t) = \sum_{i=0}^n (-t^2)^i \sum_{J \subseteq \Pi, |J| \geq i} \left(-1\right)^{|J|} \text{Ind}_W^W \rho_{J,|J|-i}.$$ 

Proof. — If $w \in W_J$ has eigenvalues $\lambda_1, \ldots, \lambda_{|J|}$ on $V_J$, then $\det_{V_J}(t^2 - w) = \prod_{j=1}^{|J|} (t^2 - \lambda_j)$. It follows that

$$\gamma_J(t)(w) = \sum_{i=0}^{|J|} t^{2(|J|-i)} (-1)^i \rho_{J,i}(w).$$

The assertion is now immediate from Theorem 1.1. \qed

Theorem 1.1 may be restated as the assertion that $H^{2i+1}(T_W, \mathbb{C}) = 0$ for all $i$, which of course is well known (cf. [8, Prop., p. 92], or (2.1)(iii) below), while as $W$-module,

$$H^{2i}(T_W, \mathbb{C}) \cong (-1)^i \sum_{J \subseteq \Pi, |J| \geq i} \left(-1\right)^{|J|} \text{Ind}_W^W \rho_{J,|J|-i}.$$ 

**2. Proof of the main theorem**

Our basic tools will be the Hodge structure of $H^*(T_W)$, and the counting of rational points over finite fields (cf. [8, p. 94] and [9, 10, 12]). The following result is well known.
Lemma 2.1.

(i) Let $Z = Z(\Delta)$ be the toric variety associated with a fan $\Delta$. If $d_k$ is the number of $k$-dimensional polyhedral cones in $\Delta$ ($k = 1, \ldots, n = \dim Z$), then the (non-equivariant) compactly supported weight polynomial (for the definition see [6, (1.5)]) is given by

$$W_c(Z, t) = \sum_{k=0}^{n} d_k (t^2 - 1)^{n-k}.$$ 

(ii) [8, p. 94] The number of points of the $\mathbb{Z}$-scheme $Z$ over $\mathbb{F}_q$ is

$$|Z(\mathbb{F}_q)| = \sum_{k=0}^{n} d_k (q - 1)^{n-k} := S(q).$$

(iii) (1) If $\Delta$ is simplicial and complete, in particular if $Z$ is non-singular and projective, then $Z$ has only even cohomology. Moreover $H^{2j}(Z, \mathbb{C})$ is a pure Hodge structure of type $(j, j)$. Thus $Z$ is mixed Tate in the sense of [10].

Proof. — The statements (i) and (ii) may be found in [8, p. 94, 104] and in [6, (2.8), (3.3), §5].

If $\Delta$ is simplicial and complete, then the compact supports weight polynomial $W_c(Z, 1) = \dim H^*_c(Z, \mathbb{C}) = \sum_j \dim H^j_c(Z, \mathbb{C})$, (see [8, p. 93–94]). Thus, writing $S(q)$ for the polynomial which gives the number of $\mathbb{F}_q$-points of $Z$, we have $S(1) = d_0 = \dim H^*_c(Z, \mathbb{C})$. All the assertions of (iii) now follow immediately from [10, Proposition 3.3(2)].

For any variety (i.e. reduced scheme of finite type) $X$ defined over the finite field $\mathbb{F}_q$, denote by $F$ the endomorphism of $X \otimes \mathbb{F}_q := X(q)$ obtained by raising local coordinates to the $q^{th}$ power. The action induced by $F$ on $\ell$-adic cohomology is defined as follows. There is a natural action of $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on the $\ell$-adic cohomology spaces $H^j_c(X \otimes \overline{\mathbb{F}}_q, \overline{\mathbb{Q}}_\ell)$. The action induced on $H^j_c(X \otimes \mathbb{F}_q, \overline{\mathbb{Q}}_\ell)$ by the inverse of the arithmetic ($q$-power) Frobenius automorphism in $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ will also be denoted by $F$. With this convention, we have the well known fixed point formula of Grothendieck:

$$|X^F| = \sum_{j=0}^{2 \dim X} (-1)^j \text{Trace}(F, H^j_c(X(q), \overline{\mathbb{Q}}_\ell)).$$

(4)

Proposition 2.2. — For any element $w \in W$, the cardinality $|T_w^F|$ is a polynomial $S(q, w)$ in $q$, and we have

$$P_W(t, w) = S(t^2, w).$$

(1) See Remark 2.4 below
Proof. — The automorphism $w$ of $T_W$ clearly commutes with the geometric Frobenius endomorphism described above. It follows that $w$ and $F$ induce commuting endomorphisms on $H^i_c(X(q), \overline{Q}_\ell)$. Hence from Grothendieck’s fixed point formula (4) we have

$$|T_W wF| = \sum_{j=0}^{2n} (-1)^j \text{Trace}(wF, H^j_c(T_W(q), \overline{Q}_\ell))$$

$$= \sum_{j=0}^{2n} \text{Trace}(w, H^{2j}(T_W(q), \overline{Q}_\ell))q^j$$

by Poincaré duality and 2.1(iii)

$$= \sum_{j=0}^{2n} \text{Trace}(w, H^{2j}(T_W, \mathbb{C}))q^j$$

for almost all $q$, by [9, (1.2)].

The proposition is now immediate. □

Rather than applying Proposition 2.2 directly, we shall make use of the fact that there is an action of the torus $T \cong (\mathbb{C}^\times)^n$ on $T_W$ which partitions $T_W$ into the (finite) union of its orbits, which are locally closed subvarieties, each isomorphic to a torus.

The following result, (see [12, Theorem 2.5]), is designed to handle this type of situation. For any complex algebraic variety with a $G$-action, where $G$ is a finite group, $W_{c,X}^G(t)$ denotes the compactly supported equivariant weight polynomial

$$W_{c,X}^G(t) = \sum_m \sum_j (-1)^j \text{Gr}_m^W H^j_c(X, \mathbb{C}) t^m,$$

regarded as an element of $R(G)[t]$, where $R(G)$ is the Grothendieck ring of complex representations of $G$ and $\text{Gr}_m^W$ denotes the $m^{th}$ graded component of the weight filtration of $H^j_c$.

**Proposition 2.3.** — (cf. [6, 12, 13]) Let $X$ be a complex algebraic variety with a $G$-action, where $G$ is a finite group. Suppose $X$ is a finite disjoint union $X = \bigsqcup_{i \in I} X_i$ of locally closed subvarieties $X_i$ which are permuted by $G$. Then

$$W_{c,X}^G(t) = \sum_{i \in \mathcal{I}/G} \text{Ind}_{G_i}^G W_{c,X_i}^G(t),$$

where the sum is over the $G$-orbits $i$ in $\mathcal{I}$, $i$ is any element of $i$, and $G_i$ is the isotropy group of $i$ in $G$.

We are now in a position to give the
Proof of Theorem 1.1. — In case $X = T_W$ and $G = W$, let $Γ$ be the set of polyhedral cones of the fan defined by the root system $Φ$. This is also described as the set of closures of the regions into which $V$ is partitioned by the reflecting hyperplanes of $W$. As explained in [8, Chapter 3], the torus $T = T_Λ ≅ (\mathbb{C}^*)^n$ acts on $T_W$. For each cone $τ ∈ Γ$, there is a distinguished point $x_τ ∈ T_W$, and the orbit $Z(τ) := T · x_τ$ is isomorphic to a torus of dimension equal to $n − \dim τ$. Moreover $T_W$ is the disjoint union of the tori $Z(τ)$. To describe the $W$-action, we require the following details.

The cones $τ ∈ Γ$ are in bijection with the cosets $wW_J$ ($w ∈ W, J ⊆ Π$) of the standard parabolic subgroups $W_J$ of $W$. We have $\dim Z(τ) = n − \dim τ$, and $wW_J$ is a face of $w′W_{J′}$ if $wW_J ⊇ w′W_{J′}$. If $τ(wW_J)$ denotes the cone corresponding to $wW_J$ and $Z(wW_J)$ denotes the corresponding $T$-orbit in $T_W$, then $\dim (wW_J) = n − |J|$, so $\dim Z(wW_J) = |J|$, and the character group of $Z(wW_J)$ is the lattice $\mathbb{Z}Φ_{w(J)}$, where $Φ_K$ is the sub-root system of $Φ$ spanned by $K$. Thus the cone $τ = \{0\}$ corresponds to $W$, and $Z(\{0\}) = Z(W)$ is the dense orbit $T = (\mathbb{C}^*)^n$ in $T_W$. Similarly the $|W|$ chambers of $V$ each correspond to a torus of dimension 0, i.e. a point in $T_W$.

The action of $W$ is described as follows. The element $g ∈ W$ takes $Z(τ)$ to $Z(gτ)$, i.e. $Z(wW_J)$ to $Z(gwW_J)$. The set $Γ/W$ of orbits of $W$ on $Γ$ is therefore in bijection with the subsets $J$ of $Π$. If $O_J$ is the orbit corresponding to $J$, then we may (and do) select $τ(W_J) ∈ O_J$ as the representative element of the orbit. Note that the set of representatives $\{τ(W_J) \mid J ⊆ Π\}$ is precisely the set of facets of the fundamental chamber of the $W$-action on $V$ which corresponds to the simple system $Π$. Since the isotropy group of $τ(W_J)$ is $W_J$, we have the following immediate consequence of Proposition 2.3.

\begin{equation}
W_{c,T_W}^W(t) = \sum_{J ⊆ Π} \text{Ind}_{W_J}^{W} W_{c,Z(W_J)}^W(t).
\end{equation}

We are therefore reduced to computing

\begin{equation}
W_{c,Z(W_J)}^W(t, w) = \sum_m \sum_j (-1)^j \text{Trace} \left( w, \text{Gr}_m H_c^W(Z(W_J), \mathbb{C}) \right) t^m
\end{equation}

for $w ∈ W_J$. For this, observe first that $Z(W_J)$ is a torus of dimension $|J|$, and therefore is minimally pure [6, §3]. Thus $H_c^W(Z(W_J), \mathbb{C})$ is pure of weight $2j − 2|J|$. Hence by [12, (2.6)], we have

\begin{equation}
W_{c,Z(W_J)}^W(t, w) = \left| Z(W_J)^w F \right|_{q \to t^2} = S_{Z(W_J)}(t^2, w),
\end{equation}

where $S_{Z(W_J)}(q, w)$ is the polynomial in $q$ which gives the number of points of $Z(W_J)$ fixed by $wF$ for almost all $q$.  

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But $wF$ acts on the character group $\mathbb{Z} \Phi_J$ of $Z(W_J)$ as $qw$. It follows (see, e.g. [2, 3.2.3]) that

$$S_{Z(W_J)}(q, w) = |\det_{V_J}(qw - 1)| = |\det_{V_J}(q - w)|.$$  

Moreover since those eigenvalues of $w$ which are not $\pm 1$ come in conjugate pairs $e^{\pm i\theta}$ and $(q - e^{i\theta})(q - e^{-i\theta}) = q^2 - 2q \cos \theta + 1 \geq (q - 1)^2 \geq 0$, we see that $S_{Z(W_J)}(q, w) = \det_{V_J}(q - w)$. Combining this with (7) and (6) we obtain

$$W_{c,T_W}(t) = \sum_{J \subseteq \Pi} \text{Ind}_{W_J}^{W} \gamma_J(t),$$

where, as in §1 above, $\gamma_J$ is the class function on $W_J$ which takes the value $\det_{V_J}(q - w)$ on $w \in W_J$.

But by Lemma 2.1 (iii) and Poincaré duality, $H^{2j}_c(T_W, \mathbb{C})$ is a pure Hodge structure of weight $2j$, while $H^{2j+1}_c(T_W, \mathbb{C}) = 0$ for all $j$. It follows that

$$W_{c,T_W}^W(t) = \sum_m \sum_j (-1)^j \text{Gr}_m^{W} H^{2j}_c(T_W, \mathbb{C}) t^m = \sum_j H^{2j}_c(T_W, \mathbb{C}) t^{2j} \text{ by Lemma 2.1 (iii) and Poincaré duality} = P_{T_W}(t).$$

This completes the proof of Theorem 1.1. □

**Remark 2.4.** — The proof of Theorem 1.1 above amounts to the computation of the polynomial $S(q, w)$ of Proposition 2.2, with the induced character formula being a convenient way to organise the computation. Explicitly, we have proved that

$$S(q, w) = \sum_{J \subseteq \Pi} \text{Ind}_{W_J}^{W}(\det_{V_J}(q - w)),$$

where $w \mapsto \det_{V_J}(q - w)$ is to be thought of as a class function on $W_J$ (when $J = \emptyset$, this function is identically 1). In this sense, our main result is a generalisation, of course applicable only to the varieties $T_W$, of the formula in [8, p. 94], which is the case $w = 1$ of our formula.

It follows from this formula (which is proved independently of any assertions concerning the cohomology) that $S(1, 1) = |W|$. Moreover by (7), the weight polynomial $W_{c,T_W}(t, 1) = S(q, 1)_{q \mapsto t^2} = S(t^2, 1)$. But since $T_W$ is smooth and projective, it follows from the results of [6] or from [8, (1), p. 92] that $W_{c,T_W}(t, 1)$ coincides with the Poincaré polynomial of $T_W$. This shows immediately (as is pointed out in [8, p. 92]) that its odd cohomology vanishes and that $\sum_j \dim H^j(T, \mathbb{C}) = |W| = S(1, 1)$. Moreover it follows from
[10, Proposition 3.3(2)] that \( T_W \) is mixed Tate. See [6] for a general discussion of weight polynomials along the lines of [8, p. 92–95].

Alternatively, it follows from the non-singular projective nature of \( T_W \) that [10, (3.7.1)] holds, i.e. that the eigenvalues of Frobenius on \( H^i(T_W, \mathbb{Q}_\ell) \) all have absolute value \( q^{\frac{i}{2}} \). The arguments on [10, p. 212] then show that all the above facts for \( T_W \) follow from the polynomial nature of \( |T_W(F_q)| \).

Thus the case \( w = 1 \) of our formula (which is due to Fulton [8]) suffices to determine the Hodge structure of the cohomology, and this in turn permits the application of our counting argument to the determination of the graded character.

3. Some applications

In this section we point out some consequences of the results above. We begin by noting that it suffices to consider irreducible root systems.

**Proposition 3.1.** — Suppose \( \Phi \) is reducible. Then \( \Phi = \Phi_1 \sqcup \Phi_2 \), where the \( \Phi_i \) are mutually orthogonal, and if \( V_i = \mathbb{C}\Phi_i \) (\( i = 1, 2 \)) then \( V = V_1 \oplus V_2 \), and \( W = W_1 \times W_2 \), where \( W_i \) is the Coxeter group with root system \( \Phi_i \) in \( V_i \).

With notation as in Theorem 1.1, we have for \( w = (w_1, w_2) \in W \)
\[
P_W(t, w) = P_{W_1}(t, w_1)P_{W_2}(t, w_2).
\]
Equivalently, if \( p_1, p_2 \) are functions on \( W_1 \) and \( W_2 \) respectively, define \( p = p_1p_2 \) to be the function on \( W = W_1 \times W_2 \) given by \( p(w_1, w_2) = p_1(w_1)p_2(w_2) \) (where \( w_i \in W_i \)). Then we have the following equation in \( R(W)[t] \).
\[
P_W(t) = P_{W_1}(t)P_{W_2}(t).
\]

**Proof.** — This is a simple consequence of the character formula provided by Theorem 1.1. \( \square \)

**Remark 3.2.** — Proposition 3.1 may also be deduced using the Künneth theorem from the following general fact.

**Proposition 3.3** (cf. [8], p. 19–20). — Let \( N_1 \) and \( N_2 \) be lattices in the real vector spaces \( V_1 \) and \( V_2 \). Let \( \Delta_1 \) and \( \Delta_2 \) be fans of rational convex polyhedral cones in \( V_1, V_2 \) respectively, and let \( T_{\Delta_1}, T_{\Delta_2} \) be the corresponding toric varieties. Define the fan \( \Delta_1 \oplus \Delta_2 \) in \( V_1 \oplus V_2 \) as that which contains the cones \( \sigma_1 \oplus \sigma_2 \), where \( \sigma_i \in \Delta_i \). Let \( T_{\Delta_1 \oplus \Delta_2} \) be the corresponding toric variety.

Then \( T_{\Delta_1 \oplus \Delta_2} \simeq T_1 \times T_2 \).
The proof of Proposition 3.3 reduces easily to the affine case, where it is straightforward. As an easy consequence, we have

**Corollary 3.4.** — With notation as in the statement of Proposition 3.1, we have $T_W \cong T_{W_1} \times T_{W_2}$.

Applying the Künneth theorem to compute the cohomology of $T_W$ using Corollary 3.4, we obtain Proposition 3.1.

**Theorem 3.5.** — Let $W$ be a finite crystallographic Coxeter group, and $T_W$ be the corresponding toric variety. Then in the notation above:

(i) ([8, p. 94]) The Poincaré polynomial of $T_W$ is given by

$$P_W(1, t) = \sum_j \dim H^j(T_W, \mathbb{C}) t^j = \sum_{J \subseteq \Pi} |W : W_J|(t^2 - 1)^{|J|}.$$

(ii) We have $(P_W(t), 1)_W = (1 + t^2)^n$, where $(-, -)_W$ denotes inner product of class functions, and $P_W(t)$ is the class function given by $P_W(t)(w) = \sum_i \text{Trace}(w, H^i(T_W, \mathbb{C})) t^i$.

(iii) The alternating character of $W$ does not occur in $H^i(T_W, \mathbb{C})$ for any $i$.

**Proof.** — The statement (i) is simply the case $w = 1$ of Theorem 1.1.

To see (ii), observe that by Frobenius reciprocity, it follows from Corollary 1.2 that

$$(P_W(t), 1)_W = \sum_{i=0}^n t^{2i} \sum_{J \subseteq \Pi, |J| \geq i} (-1)^{|J| - i} (\rho_{J, |J| - i}(1)_{W_J}).$$

But by [1, Exercice 3(a), p. 127], $(\rho_{J, |J| - i}, 1)_{W_J} = 0$ unless $i = |J|$, in which case it is 1. Hence

$$(P_W(t), 1)_W = \sum_{i=0}^n t^{2i} \sum_{J \subseteq \Pi, |J| = i} 1 = (1 + t^2)^n,$$

which is the statement (ii).

Finally, in order to compute $(P_W(t), \varepsilon_W)_W$, note that the computation above shows that we need to know $(\rho_{J, k}, \varepsilon_J)_{W_J}$ for each $J \subseteq \Pi$ and $k = 0, 1, \ldots, |J|$, where $\varepsilon_J$ is the alternating character of $W_J$. For this, we note that for any $k$, $\rho_{J, |J| - k} \cong \varepsilon_J \rho_{J, k}$. Hence by the argument above, $(\rho_{J, k}, \varepsilon_J)_{W_J} = 0$ unless $k = |J|$, and is 1 in that case. Hence again applying Corollary 1.2, it follows that

$$(P_W(t), \varepsilon)_W = \sum_{i=0}^n t^{2i} \sum_{J \subseteq \Pi, |J| \geq i} (-1)^{|J| - i} (\rho_{J, |J| - i}(\varepsilon_J)_{W_J}) = \sum_{J \subseteq \Pi} (-1)^{|J|} = 0,$$
as asserted in (iii).

Remark 3.6 — Note that in view of Theorem 3.5(i), the polynomial
\[ \sum_{J \subseteq \Pi} |W : W_J|(t^2 - 1)^{|J|} \]
has positive coefficients, a fact which is not entirely obvious.

Proposition 3.7. — We have
(i) The character of \( W \) on the total cohomology ring is given by

\[ P_W(1) = \sum_{J \subseteq \Pi} \text{Ind}_{W_J}^W(\gamma_{J,1}), \]

where \( \gamma_{J,1}(w) = \det_{V_J}(1 - w) \) for \( w \in W_J \). It is a non-negative integer for any \( w \in W \) (see [15, Proposition 1.7]).
(ii) If \( w \) is a Coxeter element of \( W \) then

\[ P_W(t, w) = \prod_{j=1}^{n} (t^2 - \exp(\frac{2\pi i m_j}{h})) \]

where \( h \) is the Coxeter number of \( W \) and \( m_1, \ldots, m_n \) are its exponents.
(iii) If \( w \) is any elliptic element of \( W \), \( P_W(t, w) = \prod_{j=1}^{n} (t^2 - \lambda_j) \), where the \( \lambda_j \) are the eigenvalues of \( w \) on \( V \).

Proof. — The first part of (i) follows immediately by putting \( t = 1 \) in Theorem 1.1. Further, the argument in the proof of Theorem 1.1 above shows that \( \det_{V_J}(q - w) \geq 0 \) for any real number \( q \geq 1 \), whence the positivity assertion (which is due to Stembridge).

Since \( w \) has no non-zero fixed points in \( V \), \( w \) has no conjugates in \( W_J \) for \( J \neq \Pi \). Thus by (1.1), \( P_{T_W}(t, w) = \det_{V}(t - w) \). But the eigenvalues of \( w \) on \( V \) are precisely \( \left\{ \exp(\frac{2\pi i m_j}{h}) \mid j = 1, \ldots, n \right\} \), and the statement (ii) is immediate. The proof of (iii) is the same. \( \square \)

In the special case when \( \Phi \) is of type \( A_n \), so that \( W \cong \text{Sym}_{n+1} \), we can be more explicit about the polynomials \( P_W(t, w) \).

Proposition 3.8. — Let \( W \) be the Coxeter group of type \( A_n \), so that \( W \cong \text{Sym}_{n+1} \). Then
(i) If \( w \) is a Coxeter element of \( W \), then \( P_W(t, w) = 1 + t^2 + t^4 + \cdots + t^{2n} \).
(ii) The character of \( W \) on the total cohomology ring is given by

\[ P_W(1, w) = (\sum_i m_i)! \prod_i i^{m_i} \] if \( w \) has cycle type \((i^{m_i})\), i.e. \( m_i \) cycles of length \( i \) for \( i = 1, 2, \ldots \).
Proof. — The first statement is a special case of (3.7)(ii).
For the second, we apply (3.7)(i), noting that $\gamma_{J,1}$ is supported on the Coxeter class of $W_J$. Thus in order to apply Frobenius' formula for evaluation of induced characters, we note that to evaluate the right side of (3.7)(i) at $w$, only those $J$ with associated partition $(i^{m_i})$ contribute. The actual evaluation is easy □

Remark 3.9. — Combining the statements (3.5(iii)) and (3.8(ii)), we obtain
$$\sum_{\lambda=(i^{m_i})} \frac{(\sum_i m_i)!}{\prod_i m_i!} = 2^{n-1},$$
where the sum is over the partitions $\lambda$ of $n$.

Remark 3.10. — The varieties $T_W$ are clearly defined over $\mathbb{R}$, and one may therefore speak of the space $T_W(\mathbb{R})$ of real points of $T_W$. The methods of [10, §5] may be used to investigate these spaces. As a very simple example we cite type $A_1$, where $T_W = \mathbb{P}^1(\mathbb{C})$ and $T_W(\mathbb{R}) = \mathbb{P}^1(\mathbb{R})$. In this case we have in the above notation (with $P_Y(t)$ denoting the usual Poincaré polynomial of a topological space),
$$P_{T_W(\mathbb{R})}(t) = 1 + t = |T_W(\mathbb{F}_q)^F|_{q \to t}.$$ This example leads naturally to the question of how the Poincaré polynomials of the real varieties $T_W(\mathbb{R})$ (both equivariant and otherwise) are related to the corresponding polynomials for the complex or finite field cases.

We conclude this section by giving the values of the polynomials $P_W(t, w)$ when $\Phi$ is the root system of type $B_3$. This is quickly calculated by hand using the results above. Recall that the conjugacy classes of the Weyl groups of type $B_n$ are characterised by their “cycle type” $\lambda^\pm_1, \ldots, \lambda^\pm_p$, where $\sum_j \lambda_j = n$. For example $-\text{Id}$ is of type $1^-, \ldots, 1^-$. There are ten conjugacy classes in $W(B_3)$, and the values of $P_W(t, w)$ are given in the table below.

<table>
<thead>
<tr>
<th>Conjugacy class $(w)$</th>
<th>$P_W(t, w)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1,1,1)$</td>
<td>$t^6 + 23t^4 + 23t^2 + 1$</td>
</tr>
<tr>
<td>$(1,2)$</td>
<td>$t^6 + 7t^4 + 7t^2 + 1$</td>
</tr>
<tr>
<td>$(1^-,1,1)$</td>
<td>$t^6 + 7t^4 + 7t^2 + 1$</td>
</tr>
<tr>
<td>$(3)$</td>
<td>$t^6 + 2t^4 + 2t^2 + 1$</td>
</tr>
<tr>
<td>$1^-,2)$</td>
<td>$t^6 + 3t^4 + 3t^2 + 1$</td>
</tr>
<tr>
<td>$(1,2^-)$</td>
<td>$t^6 + t^4 + t^2 + 1$</td>
</tr>
<tr>
<td>$(1^-,1^-,1)$</td>
<td>$t^6 + 3t^4 + 3t^2 + 1$</td>
</tr>
<tr>
<td>$(3^-)$</td>
<td>$t^6 + t^4 + t^2 + 1$</td>
</tr>
<tr>
<td>$(1^-,2^-)$</td>
<td>$t^6 + 3t^4 + 3t^2 + 1$</td>
</tr>
<tr>
<td>$1^-,1^-,1^-$</td>
<td>$t^6 + 3t^4 + 3t^2 + 1$</td>
</tr>
</tbody>
</table>
4. The reflection representation

In this section we shall apply our main theorem to determine the multiplicity of the reflection representation $\rho = \rho_W$ of $W$ in each cohomology space $H^2(T_W, \mathbb{C})$. We start with some basic facts concerning the reflection representation.

4.1. The reflection representation

Let $K$ be a simple system for a reflection group $H$ in $V = \mathbb{R}^n$. Suppose $K = \bigoplus_{i=1}^{c} K_i$ is the decomposition of $K$ into irreducible components. Then correspondingly, $H = H_1 \times \cdots \times H_c$, and $H_i = H_{K_i}$ acts irreducibly on $V_{K_i}$, the linear span of $K_i$, through its reflection representation $\rho_i$. Moreover if $\rho_K$ is the reflection representation of $H$, its decomposition into irreducible components is given by

$$\rho_K = \bigoplus_{i=1}^{c} 1_{H_1} \otimes \cdots \otimes 1_{H_{i-1}} \otimes \rho_i \otimes 1_{H_{i+1}} \otimes \cdots \otimes 1_{H_c}. \quad (10)$$

Suppose $\Pi$ is as in Theorem 1.1 and let $J \subseteq \Pi$. Then the restriction to $W_J$ of the reflection representation $\rho$ of $W$ is given by

$$\text{Res}_{W_J}^W \rho = \rho_J \oplus |\Pi \setminus J| \Lambda W_J,$$

where $\rho_J$ is the reflection representation of $W_J$ (on $V_J$).

Next, recall that if $V_1$ and $V_2$ are vector spaces, there is a canonical isomorphism of graded vector spaces $\Lambda(V_1 \oplus V_2) \xrightarrow{\cong} \Lambda(V_1) \otimes \Lambda(V_2)$; i.e. for each index $k$, we have $\Lambda^k(V_1 \oplus V_2) \cong \oplus_{i+j=k} \Lambda^i(V_1) \otimes \Lambda^j(V_2)$. It follows from (10) that the decomposition of $\Lambda^k \rho_K$ into irreducibles is given by

$$\Lambda^k \rho_K = \oplus_{i_1+\cdots+i_c=k} \Lambda^i_1 \rho_1 \otimes \Lambda^i_2 \rho_2 \otimes \cdots \otimes \Lambda^i_c \rho_c. \quad (12)$$

Note that since the representations $\Lambda^i \rho_j$ are irreducible, this implies that $\Lambda^k \rho_K$ is multiplicity free.

4.2. A combinatorial result about trees

Our multiplicity formula will involve the Dynkin diagram of $\Phi$, and to evaluate it explicitly, the following discussion will be useful. The author thanks Anthony Henderson for pointing out the degree of generality in which Proposition 4.1 below holds.
Let $\Theta$ be a tree, that is, a finite connected undirected graph with no circuits. Write $n = |\Theta|$, and for $0 \leq k \leq n$ define $c(\Theta, k)$ by

\begin{equation}
(13) \quad c(\Theta, k) = \sum_{\substack{J \subseteq \Theta \\
|J| = k}} c(J),
\end{equation}

where $c(J)$ is the number of connected components of the subgraph (forest) spanned by $J$. Putting the $c(\Theta, k)$ into a generating polynomial, we define

\begin{equation}
(14) \quad c_\Theta(t) := \sum_{k=0}^{n} c(\Theta, k)t^{n-k} \in \mathbb{Z}[t].
\end{equation}

We shall prove

**Proposition 4.1.** Let $\Theta$ be any tree with $n$ vertices. Then

\[ c_\Theta(t) = (1 + t)^{n-2}(1 + nt). \]

**Proof.** Note first that for any tree, the number of vertices is one more than the number of edges. Since any subset $J$ of $\Theta$ spans a forest (disjoint union of trees), it follows that $c(J)$ is the difference between $k = |J|$ and the number $e(J)$ of edges of $J$. Further, each edge of $\Theta$ occurs in precisely $\binom{n-2}{k-2}$ subsets $J$. It follows that

\[ c(\Theta, k) = \sum_{\substack{J \subseteq \Theta \\
|J| = k}} c(J) = \sum_{\substack{J \subseteq \Theta \\
|J| = k}} (k - e(J)) = k \binom{n}{k} - (n - 1) \binom{n-2}{k-2} = (n - k + 1) \binom{n-1}{k-1}. \]

Hence

\[ c_\Theta(t) = \sum_{k=0}^{n} c(\Theta, k)t^{n-k} = \sum_{k=1}^{n} (n - k + 1) \binom{n-1}{k-1} t^{n-k} \]

\[ = \frac{d}{dt} \sum_{k=1}^{n} \binom{n-1}{k-1} t^{n-k+1} = \frac{d}{dt} (t(1 + t)^{n-1}) = (1 + t)^{n-2}(1 + nt), \]

as stated. \qed

**Definition 4.2.** Define the polynomial $u_n(t)$ as the value of $c_\Theta(t)$ for any tree $\Theta$ with $n$ vertices. That is,

\[ u_n(t) := (1 + t)^{n-2}(1 + nt). \]
4.3. The multiplicity theorem

In order to discuss our result, it is convenient to define the polynomial $N_{\Phi}(t)$ which is associated with the root system $\Phi$.

**Definition 4.3.** — Let $\Phi$ be a root system and let $\Pi \subset \Phi$ be a simple system in $\Phi$. For each subset $J \subseteq \Pi$ denote by $c(J)$ the number of connected components of $J$ (or of the root system $\Phi_J$ spanned by $J$). For each integer $i \geq 0$ write

$$\nu_{\Phi}(i) = \sum_{\substack{J \subseteq \Pi \mid |J| = i + 1}} c(J).$$

Then $N_{\Phi}(t) := \sum_{i \geq 0} \nu_{\Phi}(i)t^i$.

**Lemma 4.4.** — If $\Phi$ is an irreducible root system of rank $n$, then $N_{\Phi}(t) = (1 + t)^{n-2}(n + t)$.

**Proof.** — Let $\Theta$ be the Dynkin diagram of $\Phi$. Then evidently for $i = 0, 1, \ldots, n - 1$, $\nu_{\Phi}(i) = c(\Theta, i + 1)$. It follows easily that

$$N_{\Phi}(t) = t^{n-1}u_n(t^{-1}) = (1 + t)^{n-2}(n + t).$$

□

**Theorem 4.5.** — Let $\Phi$ be any irreducible root system of rank $n$ ($n \geq 2$). Then $\sum_{i=0}^{n} (H^{2i}(T_W, \mathbb{C}), \rho_W) t^i = (n - 1)t(1 + t)^{n-2}$.

A straightforward consequence of Theorem 4.5 is

**Corollary 4.6.** — Let $\Phi$ be any root system of rank $n$, and denote by $W$ and $c(\Phi)$ respectively, the corresponding Weyl group and the number of irreducible components of $\Phi$. Then

$$\sum_{i=0}^{n} \left( H^{2i}(T_W, \mathbb{C}), \rho_W \right) t^i = (n - c(\Phi)) t(1 + t)^{n-2}. \tag{15}$$

**Proof of Corollary 4.6.** — Writing $c = c(\Phi)$, and using notation analogous to that at the beginning of this section, we have

$$\rho_W = \bigoplus_{i=1}^{c} 1_{W_1} \otimes \cdots \otimes \rho_i \otimes \cdots \otimes 1_{W_c}.$$ 

Since $H^*(T_W, \mathbb{C}) \cong \bigotimes_{i=1}^{c} H^*(T_{W_i}, \mathbb{C})$, it follows from Theorem 3.5(iii) and Theorem 4.5 above that

$$\sum_{i=0}^{n} \left( H^{2i}(T_W, \mathbb{C}), \rho_W \right) t^i = \sum_{j=1}^{c} \prod_{i=1}^{c} (1 + t)^{n_i}(n_j - 1)t(1 + t)^{n_j - 2},$$
where \( n_i \) is the rank of the irreducible component \( \Phi_i \) of \( \Phi \). The required statement follows easily. \( \square \)

Proof of Theorem 4.5. — Our starting point is the formula (3) which describes \( H^{2i}(T_W, \mathbb{C}) \) as a \( W \)-module.

\[
H^{2i}(T_W, \mathbb{C}) \cong \sum_{J \subseteq \Pi \atop |J| \geq i} (-1)^{|J| - i} \text{Ind}_{W_J}^W (\Lambda^{|J| - i} \rho_J),
\]

where \( \rho_J \) is the reflection representation of \( W_J \).

By Frobenius reciprocity, it follows that

\[
\kappa_i := (H^{2i}(T_W, \mathbb{C}), \rho)_W = \sum_{J \subseteq \Pi \atop |J| \geq i} (-1)^{|J| - i} \left( \text{Res}^W_{W_J} \rho, \Lambda^{|J| - i} \rho_J \right)_{W_J}.
\]

We therefore turn our attention to the computation of the \( \kappa_i \).

Now \( \kappa_i = \sum_{J \subseteq \Pi \atop |J| \geq i} (-1)^{|J| - i} \kappa_i(J) \), where \( \kappa_i(J) = (\text{Res}^W_{W_J} \rho, \Lambda^{|J| - i} \rho_J)_{W_J} \).

Further, by (11), we have

\[
\kappa_i(J) = (\rho_J, \Lambda^{|J| - i} \rho_J)_{W_J} + |\Pi \smallsetminus J|(1_{W_J}, \Lambda^{|J| - i} \rho_J)_{W_J}.
\]

We have seen that \((1_{W_J}, \Lambda^{|J| - i} \rho_J)_{W_J} = 0\) unless \(|J| = i\), in which case the multiplicity is 1. To compute \((\rho_J, \Lambda^{|J| - i} \rho_J)_{W_J}\), write \( J = J_1 \Pi \cdots \Pi J_{c(J)} \) for the decomposition of \( J \) into connected components (cf. 10), and let \( k = |J| - i \). Then from (10) and (12) we see that \((\rho_J, \Lambda^k \rho_J)_{W_J} = 0\) unless \( k = 1 \), and when \( k = 1 \), \((\rho_J, \rho_J)_{W_J} = c(J)\).

Hence

\[
\kappa_i(J) = \begin{cases} |\Pi \smallsetminus J| & \text{if } |J| = i \\ c(J) & \text{if } |J| = i + 1 \\ 0 & \text{otherwise} \end{cases}
\]

It follows that

\[
\sum_{i=0}^{n}(H^{2i}(T_W, \mathbb{C}), \rho_W) t^i = \sum_{i=0}^{n} \kappa_i t^i = \sum_{i=0}^{n} \sum_{J \subseteq \Pi \atop |J| \geq i} (-1)^{|J| - i} \kappa_i(J) t^i
\]

\[
= \sum_{i=0}^{n} \left( \sum_{J \subseteq \Pi \atop |J| = i} \binom{n - i}{i} - \sum_{J \subseteq \Pi \atop |J| = i + 1} c(J) \right) t^i
\]

\[
= \sum_{i=0}^{n} \left( \binom{n}{i} (n - i) - \nu_{\Phi}(i) \right) t^i
\]

\[
= n(1 + t)^{n-1} - N_{\Phi}(t).
\]
Finally, it follows from Lemma 4.4 that $N_{\Phi}(t) = (1 + t)^{n-2}(n + t)$. Substituting into the expression above, we obtain the theorem. \hfill \Box

Appendix A. Equivalence to the Dolgachev-Lunts-Stembridge formula

In this appendix we shall show how the character formula of Dogachev, Lunts and Stembridge can be derived from our Theorem 1.1 and vice versa. To do this, we shall evaluate our formula (1) at an element $w \in W$, and compare with the formula in [15, Cor. 1.6].

We start by noting that given an element $w \in W$, we may apply Frobenius' formula for induced characters to the formula (1) to obtain the following expression for $P_W(t, w) := \sum_{i \geq 0} \text{Trace}(w, H_i(T_W, \mathbb{C})) t^i \in \mathbb{C}[t]$.

\begin{equation}
P_W(t, w) = \sum_{xW_j \subset W_J} \det_{V_J}(t^2 - x^{-1}wx),
\end{equation}

where the sum is over all cosets $xW_j$ of parabolic subgroups $W_J$ ($J \subseteq \Pi$) which are fixed by $w$, and $V_J$ is the span of the simple roots in $J$.

Next we translate the formula in [15, Cor. 1.6] into the notation of the current work. Let $\Delta = \Delta_W$ be the fan in $V$ which corresponds to the root system $\Phi$. Then in the language of the proof of Theorem 1.1 above, $\Delta$ is the union of the cones $\tau(xW_J)$ over all cosets $xW_J$. Fix $w \in W$ and define

\begin{equation}
Q_W(t, w) = P_{\Delta^w}(t)(1 - t^2)^{- \dim V^w} \det V(1 - wt^2),
\end{equation}

where $V^w = \ker(w - 1)$ is the fixed point subspace of $w$, and $P_{\Delta^w}(t)$ is the Poincaré polynomial of the toric variety $T(\Delta^w)$ corresponding to the fan $\Delta^w$ obtained by intersecting the cones of $\Delta$ with $V^w$.

Then [15, Cor. 1.6] asserts that $Q_W(t, w) = P_W(t, w)$. The equivalence of this statement to our Theorem 1.1 will follow from

**Proposition A.1.** — Let $w \in W$, and define $R_W(t, w)$ to be the right side of the equation (16). Then $R_W(t, w) = Q_W(t, w)$.

**Proof.** — First, note that the cones of $\Delta^w$ are precisely those cones of $\Delta$ which are fixed by $w$; this is because $w$ fixes a cone $\tau$ (setwise) if and only $w$ fixes $\tau$ pointwise. That is, in the language above, $\Delta^w = \{ \tau(xW_J) \in \Delta \mid x^{-1}wx \in W_J \}$. 

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It follows, using [8, p. 94] and the fact that $\dim \tau(xW_J) = n - |J|$, that
\begin{equation}
P_{\Delta^w}(t) = \sum_{\tau \in \Delta^w} (t^2 - 1)^{\dim V^w - \dim \tau}
\end{equation}

\begin{equation}
= \sum_{xW_J \in \Delta^w} (t^2 - 1)^{\dim V^w - n + |J|}.
\end{equation}

Substituting the expression (18) into (17) and simplifying, we obtain
\begin{equation}
Q(t, w) = (-1)^{\dim V^w} \det V(1 - wt^2) \sum_{xW_J \in \Delta^w} (t^2 - 1)^{n - |J|}.
\end{equation}

Now if $x^{-1}wx \in W_J$, then $x^{-1}wx$ fixes $V^w_J$ pointwise, so that $V^w \supseteq V^w_J$. Hence for such $xW_J$, we have
\begin{equation}
\det V(1 - wt^2) = \det V(1 - x^{-1}wx t^2)
\end{equation}
\begin{equation}
= \det V_J(1 - x^{-1}wx t^2) \det V^w_J(1 - x^{-1}wx t^2)
\end{equation}
\begin{equation}
= \det V_J(1 - x^{-1}wx t^2)(1 - t^2)^{n - |J|}.
\end{equation}

Now substitute this last expression into (19), to obtain
\begin{equation}
Q(t, w) = (-1)^{\dim V^w} \sum_{xW_J \in \Delta^w} (t^2 - 1)^{n - |J|} \det V_J(1 - x^{-1}wx t^2)(1 - t^2)^{n - |J|}
\end{equation}
\begin{equation}
= (-1)^{\dim V^w} \sum_{xW_J \in \Delta^w} (-1)^{n - |J|} \det V_J(1 - x^{-1}wx t^2)
\end{equation}
\begin{equation}
= \sum_{xW_J \in \Delta^w} (-1)^{\dim V^w + n} \det V_J(x^{-1}wx t^2 - 1).
\end{equation}

Finally, since $x^{-1}wx$ has eigenvalues (on $V$, and therefore $V_J$) which come in complex conjugate pairs or are equal to $\pm 1$, it follows that
\begin{equation}
\det V(x^{-1}wx) = (-1)^{n + \dim V^w} = \det V_J(x^{-1}wx),
\end{equation}
since $x^{-1}wx$ acts trivially on $V^w_J$. It follows from the last line of (21) that
\begin{equation}
Q(t, w) = \sum_{xW_J \in \Delta^w} \det V_J(t^2 - x^{-1}wx) = R(t, w),
\end{equation}
and the proof of the proposition is complete.
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