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# ALEXANDROV'S THEOREM, WEIGHTED DELAUNAY TRIANGULATIONS, AND MIXED VOLUMES

by Alexander I. BOBENKO & Ivan IZMESTIEV (*)

ABSTRACT. — We present a constructive proof of Alexandrov's theorem on the existence of a convex polytope with a given metric on the boundary. The polytope is obtained by deforming certain generalized convex polytopes with the given boundary. We study the space of generalized convex polytopes and discover a connection with weighted Delaunay triangulations of polyhedral surfaces. The existence of the deformation follows from the non-degeneracy of the Hessian of the total scalar curvature of generalized convex polytopes with positive singular curvature. This Hessian is shown to be equal to the Hessian of the volume of the dual generalized polyhedron. We prove the non-degeneracy by applying the technique used in the proof of Alexandrov-Fenchel inequality. Our construction of a convex polytope from a given metric is implemented in a computer program.

RÉSUMÉ. — Nous présentons, dans cet article, une démonstration constructive du théorème d'Alexandrov sur l'existence d'un polytope convexe ayant une métrique donnée sur son bord. Le polytope est obtenu en déformant des polytopes convexes généralisés dont le bord est donné. Nous étudions l'espace des polytopes convexes généralisés et mettons en évidence une relation avec les triangulations de Delaunay pondérées des surfaces polyédrales. L'existence de la déformation est une conséquence de la non-dégénérescence du hessien de la courbure scalaire totale des polytopes convexes généralisés ayant leurs courbures singulières positives. Ce hessien se révèle être égal au hessien du volume du polyèdre généralisé dual. Nous démontrons la non-dégénérescence en appliquant la technique utilisée dans la preuve de l'inégalité d'Alexandrov-Fenchel. Notre construction d'un polytope convexe à partir d'une métrique donnée est mise en œuvre dans un programme informatique.

## 1. Introduction

In 1942 A.D.Alexandrov [2] proved the following remarkable theorem:

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Theorem 1.1. — *Let $M$ be a sphere with a convex polyhedral metric. Then there exists a convex polytope $P \subset \mathbb{R}^3$ such that the boundary of $P$ is isometric to $M$. Moreover, $P$ is unique up to a rigid motion.*

The following definition explains the term *convex polyhedral metric*.

Definition 1.2. — *A* polyhedral metric *on a surface $M$ is a metric on $M$ such that any point $x \in M$ posesses an open neighborhood $U$ with one of the two following properties. Either $U$ is isometric to a subset of $\mathbb{R}^2$, or there is an isometry between $U$ and an open subset of a cone with angle $\alpha \neq 2\pi$ such that $x$ is mapped to the apex. In the first case $x$ is called a regular point, in the second case it is called a singular point of $M$. The set of singular points is denoted by $\Sigma$ and is required to be finite.*

*If for all $x \in \Sigma$ the angle at $x$ is less than $2\pi$, then the polyhedral metric is said to be convex.*

Other names for a polyhedral metric are *Euclidean cone metric* and *singular Euclidean metric*.

The polytope $P$ in Theorem 1.1 may degenerate to a two-sided planar convex polygon. In this case the boundary of $P$ consists of two copies of this polygon identified along the boundary.

The metric on the boundary of a convex polytope is an example of a convex polyhedral metric on the sphere. The only singular points are the vertices of the polytope, because any point in the interior of an edge posesses a Euclidean neighborhood. The intrinsic metric of the boundary of a polytope does not a priori distinguish the edges.

In practice, one specifies a polyhedral metric on the sphere by taking a collection of polygons and glueing them together along some pairs of edges. Alexandrov calls this representation a development, in analogy with developments of polytopes. By Theorem 1.1, from any development with cone angles less than $2\pi$ a convex polytope can be built. However, the edges of this polytope do in general differ from the edges of the development. To see this, take a convex polytope and cut it into planar pieces by geodesics different from the edges of the polytope. For a given development it is a hard problem to determine the edges of the corresponding polytope.

If a development and the true edges of the polytope are known, then the problem of constructing the polytope is still non-trivial, and is studied in [11] using polynomial invariants.

The uniqueness part of Theorem 1.1 can be proved by a modification of Cauchy's proof of global rigidity of convex polytopes. The original existence proof by Alexandrov [2], [3] is more involved and non-constructive. Alexandrov [3, pp. 320–321] discusses a possibility for a constructive proof.

Alexandrov's theorem is closely related to the following problem posed by Hermann Weyl in [31]: *Prove that any Riemannian metric of positive Gaussian curvature on the sphere can be realized as the metric of the boundary of a unique convex body in* $\mathbb{R}^3$. Alexandrov in [2] applied Theorem 1.1 to solve Weyl's problem for the metric of class $C^2$ through approximation by convex polyhedral metrics. The usual approach to Weyl's problem via partial differential equations was realized by Nirenberg [21].

An interesting remark regarding Weyl's problem was made by Blaschke and Herglotz in [6]. Consider all possible extensions of the given Riemannian metric on the sphere to a Riemannian metric inside the ball. On the space of such extensions take the total scalar curvature functional, also known as the Einstein-Hilbert action. It turns out that its critical points correspond to Euclidean metrics in the interior of the ball. It is not clear how to use this observation to solve Weyl's problem since the functional is not convex.

Since Alexandrov's theorem is a discrete version of Weyl's problem, one can try to discretize the Blaschke-Herglotz approach. The discrete analog of the Einstein-Hilbert action is known as the Regge action. If simplices of a triangulated closed 3-manifold are equipped with Euclidean structures, then the Regge action [23] is defined as

$$\sum_e \ell_e \kappa_e. \tag{1.1}$$

Here the sum ranges over all edges of the triangulation, $\ell_e$ is the length of the edge $e$, and $\kappa_e$ is the angle deficit, also called curvature.

This way to discretize the total scalar curvature was also known to Volkov, a student of Alexandrov. In [29] (translated and reprinted as [3, Section 12.1]), he applied it to study the dependence of the extrinsic metric of a convex polytope on the intrinsic metric on its boundary. Volkov also gave a new proof of Alexandrov's theorem in his PhD of 1955. It proceeds by extending the metric to a polyhedral metric in the ball in a special way and minimizing the sum of edge lengths over all such extensions. The discrete total scalar curvature functional is not used in the proof. Volkov's proof can be found in [30]. See also Zalgaller's remark in [3, pp. 504–505].

In this paper we investigate the total scalar curvature functional (1.1) and give a new proof of Alexandrov's theorem based on the properties of this functional. Let us sketch our proof.

A *generalized polytope* is a simplicial complex glued from pyramids over triangles of some geodesic triangulation $T$ of $M$. Vertex set of $T$ is the set $\Sigma$ of the cone singularities of $M$; edges are geodesic arcs. A generalized polytope can be described by a pair $(T, r)$, where $r = (r_i)_{i\in\Sigma}$ are lengths of the side edges of the pyramids. A *generalized convex polytope* is a generalized polytope with all of the dihedral angles $\theta_{ij}$ at the boundary edges less or equal to $\pi$. In Section 2 we show that for any $r$ there is at most one generalized convex polytope and describe the space of all generalized convex polytopes with boundary $M$.

In Section 3 we study the total scalar curvature of generalized convex polytopes:

$$H(P(T,r)) = \sum_{i\in\mathcal{V}(T)} r_i\kappa_i + \sum_{ij\in\mathcal{E}(T)} \ell_{ij}(\pi - \theta_{ij}),$$

where $\mathcal{V}(T)$ and $\mathcal{E}(T)$ denote the vertex set and the edge set of $T$, $\kappa_i$ is the curvature at the $i$-th radial edge, and $\ell_{ij}$ is the length of the boundary edge joining the vertices $i$ and $j$. The main result of Section 3 is the non-degeneracy of the Hessian of $H$ if

$$0 < \kappa_i < \delta_i \quad \text{for every } i, \tag{1.2}$$

where $\delta_i$ is the angle deficit at the $i$-th singularity of the metric of $M$.

Section 4 contains the proof of Alexandrov's theorem. The idea is to start with a certain generalized convex polytope over $M$ and deform it into a convex polytope. As the starting point we take the generalized polytope $(T_D, r)$, where $T_D$ is the Delaunay triangulation of $M$, and $r_i = R$ for every $i$, with a sufficiently large $R$. We show that this generalized polytope is convex and satisfies condition (1.2). In order to show that any small deformation of the curvatures $\kappa_i$ can be achieved by a deformation of radii $r_i$ it suffices to prove the non-degeneracy of the Jacobian $\left(\frac{\partial\kappa_i}{\partial r_j}\right)$. Here the functional $H$ comes into play since

$$\frac{\partial\kappa_i}{\partial r_j} = \frac{\partial^2 H}{\partial r_i \partial r_j}.$$

We deform $\kappa_i$ by the rule

$$\kappa_i(t) = t\cdot\kappa_i, \tag{1.3}$$

where $t$ goes from 1 to 0. In particular, condition (1.2) remains valid during the deformation. We show that when $r$ is changed according to (1.3), the generalized convex polytope does not degenerate. In the limit as $t \to 0$ we get a convex polytope with boundary $M$.

A new proof of Alexandrov's theorem is not the only result of this paper. Coming across *weighted Delaunay triangulations* and *mixed volumes* while studying the total scalar curvature functional was a big surprise for us. In Section 2 we give a new geometric interpretation of weighted Delaunay triangulations in terms of generalized convex polytopes. Also, for a given polyhedral surface with marked points, we explicitely describe the space of admissible weights of weighted Delaunay triangulations. Further, we show that a weighted Delaunay triangulation with given weights can be obtained via a flip algorithm.

In Section 3 we define for any generalized convex polytope $P$ its dual generalized convex polyhedron $P^*$. We find a surprising relation between the total scalar curvature of $P$ and the volume of $P^*$:

$$\frac{\partial^2 H}{\partial r_i \partial r_j}(P) = \frac{\partial^2 \mathrm{vol}}{\partial h_i \partial h_j}(P^*). \tag{1.4}$$

Here the $h_i$ are the altitudes of $P^*$, see Subsection 3.2. We prove that the Hessian of $\mathrm{vol}(P^*)$ is non-degenerate if the curvatures satisfy (1.2). The proof uses the theory of mixed volumes: we partly generalize the classical Alexandrov-Fenchel inequalities [27]. Note that mixed volumes play similar roles in our proof and in the variational proof of Minkowski's theorem, see [3, Section 7.2].

The proof of Alexandrov's theorem presented in this paper provides a numerical algorithm to construct a convex polytope from a given development. This algorithm was implemented by Stefan Sechelmann, the program is available at `http://www.math.tu-berlin.de/geometrie/ps/`. Examples of surfaces constructed with the help of this program are shown in Figure 1.1. Two convex polygons $A$ and $B$ of the same perimeter identified isometrically along the boundary produce a convex polyhedral metric on the sphere. In the limit, $A$ and $B$ can be any two convex plane figures. The resulting convex body in $\mathbb{R}^3$ is the convex hull of a space curve that splits its boundary into two regions with Euclidean metric. This construction was communicated to us by Johannes Wallner.

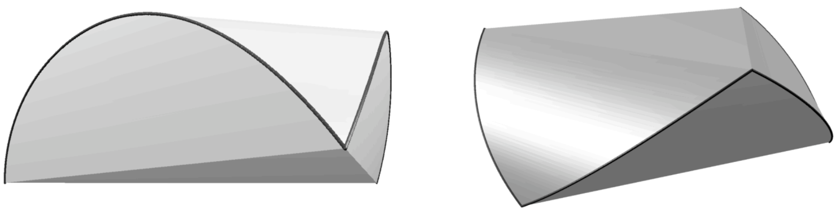

*Figure 1.1. Convex surfaces glued from two Euclidean pieces identified along the boundary.* Left: *Disc and equilateral triangle.* Right: *Two Reuleaux triangles (triangles of constant width); the vertices of one are identified with the midpoints of the sides of the other.*

There exist numerous generalizations of Alexandrov's theorem: to surfaces of arbitrary genus, to hyperbolic and spherical polyhedral metrics, to singularities of negative curvature (see [12] and references therein). Perhaps our approach can be generalized to these cases.

## 2. Generalized convex polytopes and weighted Delaunay triangulations

For a fixed polyhedral metric $M$ on the sphere, we consider the space $\mathcal{P}(M)$ of all generalized convex polytopes with boundary $M$, see Definition 2.1. Our goal here is to introduce coordinates on $\mathcal{P}(M)$ and to describe its boundary. We also discover a connection between generalized convex polytopes and weighted Delaunay triangulations of a sphere with a Euclidean polyhedral metric. This leads to an explicit description of the space of weighted Delaunay triangulations.

### 2.1. Geodesic triangulations

Let $M$ be a closed surface with a Euclidean polyhedral metric with a non-empty singular set $\Sigma$. A *geodesic triangulation* $T$ of $M$ is a decomposition of $M$ into Euclidean triangles by (not necessarily shortest) geodesics with endpoints in $\Sigma$. By $\mathcal{V}(T)$, $\mathcal{E}(T)$, $\mathcal{F}(T)$ we denote the sets of vertices, edges, and faces of $T$, respectively. By definition, $\mathcal{V}(T) = \Sigma$. Note that in $\mathcal{E}(T)$ there may be multiple edges as well as loops. Triangles of $\mathcal{F}(T)$ may have self-identifications on the boundary.

Singularities of $M$ are denoted by letters $i, j, k, \ldots$. An edge of $T$ with endpoints $i$ and $j$ is denoted $ij$, a triangle with vertices $i$, $j$, $k$ is denoted $ijk$. Note that different edges or triangles can have the same name and letters in a name may repeat.

It is not hard to show that for any $M$ there is a geodesic triangulation $T$. In the case when $M$ is a sphere with all singularities of positive curvature, this is proved in [3, Section 4.1, Lemma 2]. The following examples of geodesic triangulations should illustrate the concept.

Take two copies of a euclidean triangle and identify them along the boundary. The space obtained is a geodesically triangulated sphere with a Euclidean polyhedral metric. There are other geodesic triangulations of the same surface. Namely, remove from the triangulation a side of the triangle that forms acute angles with two other sides and draw instead a geodesic loop based at the opposite vertex (the loop is formed by two copies of the corresponding altitude of the triangle).

Let $M$ be the boundary of a cube. Take two opposite faces and subdivide each of them by a diagonal. The remaining four faces form a cylinder, both ends of which contain four points from $\Sigma$. There are infinitely many geodesic triangulations of the cylinder with vertices in $\Sigma$.

### 2.2. Generalized convex polytopes

Let $T$ be a geodesic triangulation of $M$. Let $r = (r_1, \dots, r_n)$ be a collection of positive numbers, called radii, subject to the condition that for every triangle $ijk \in \mathcal{F}(T)$ there exists a pyramid with base $ijk$ and lengths $r_i, r_j, r_k$ of side edges. The generalized polytope defined by the pair $(T, r)$ is a polyhedral complex glued from these pyramids following the gluing rules in the triangulation. As a consequence, the apices of all pyramids are glued to one point $a$ which we call the apex of the generalized polytope.

In a generalized polytope consider the interior edge that joins the apex to the vertex $i$. We denote by $\omega_i$ the sum of all dihedral angles of pyramids at this edge. If for some $i$ we have $\omega_i \neq 2\pi$, then the generalized polytope cannot be embedded into $\mathbb{R}^3$ and has to be considered as an abstract polyhedral complex. The quantity

$$\kappa_i = 2\pi - \omega_i$$

is called the *curvature* at the corresponding edge. Further, for every boundary edge $ij \in \mathcal{E}(T)$ of the generalized polytope we denote by $\theta_{ij}$ the sum of the dihedral angles of pyramids at this edge.

Definition 2.1. — *A generalized polytope with boundary $M$ is an equivalence class of pairs $(T, r)$ as above, where two pairs are equivalent if and only if there is an isometry between the resulting polyhedra which is identical on the boundary and maps the apex to the apex. A generalized convex polytope is a generalized polytope such that $\theta_{ij} \leqslant \pi$ for any edge $ij \in \mathcal{E}(T)$ in some (and hence in any) associated triangulation $T$.*

For brevity, we sometimes use the word polytope when talking about generalized convex polytopes. Here are two important examples.

*Example 1.* It is not hard to show that a generalized convex polytope with $\kappa_i = 0$ for all $i$ embeds into $\mathbb{R}^3$ and thus produces a classical convex polytope. Conversely, take a convex polytope in $\mathbb{R}^3$, triangulate its non-triangular faces without adding new vertices, and choose a point in its interior. The triangulation of the boundary and the distances from the chosen point to the vertices provide us with a pair $(T, r)$. Another pair $(T', r)$ representing the same generalized convex polytope can be obtained by choosing some other triangulation of non-triangular faces. But a different choice of an interior point as the apex leads to a different generalized polytope.

*Example 2.* Let $T$ be any triangulation, and $R > 0$. Take $r_i = R$ for every $i \in \mathcal{V}(T)$. If $R$ is large enough, then over every triangle $ijk$ there exists an isosceles pyramid with all side edges of length $R$. Thus $(T, r)$ defines a generalized polytope. It is convex if and only if $T$ is a Delaunay triangulation of $M$, see Definition 2.8 or [7]. This is a special case of Theorem 2.10 or can be shown directly by a geometric argument similar to the proof of Lemma 2.4.

In our proof of Alexandrov's theorem we take a polytope from the second example and deform it by changing the radii and keeping the boundary $M$ fixed to a polytope from the first example. The explanation why this is possible is rather involved and is based on the analysis of some unexpected connections to the theory of mixed volumes.

### 2.3. A generalized convex polytope is determined by its radii

Consider the space

$$\mathcal{P}(M) := \{\text{generalized convex polytopes with boundary } M\}, \tag{2.1}$$

where $M$ is a sphere with convex polyhedral metric. We have

$$\mathcal{P}(M) = \bigcup_T \mathcal{P}^T(M),$$

where $\mathcal{P}^T(M)$ consists of those generalized convex polytopes that have a representative of the form $(T, r)$ for the given $T$. The map $(T, r) \mapsto r$ defines an embedding of $\mathcal{P}^T(M)$ into $\mathbb{R}^n$, where $n = |\Sigma|$. In this subsection we show that this map is injective on $\mathcal{P}(M)$ that is, for a given assignment of radii there is at most one generalized convex polytope.

Proposition 2.2. — *Suppose that $P, P' \in \mathcal{P}(M)$ are two generalized convex polytopes represented by the pairs $(T, r)$ and $(T', r')$, respectively. Then $r = r'$ implies $P = P'$.*

We introduce some technical notions that are used in the proof of this proposition.

Let us call a *Q-function* any function of the form

$$x \mapsto \|x - a\|^2 + b$$

defined on some subset of a Euclidean space $\mathbb{R}^m$. Here $a \in \mathbb{R}^m$ and $b \in \mathbb{R}$. A *PQ-function* on $M$ is a continuous function which is a $Q$-function on each triangle of some geodesic triangulation. (To define a $Q$-function on a triangle of $T$, develop this triangle into the plane.) The restriction of a $Q$-function to a Euclidean subspace is a $Q$-function on this subspace. Similarly, the restriction of a $PQ$-function on $M$ to a geodesic arc is a $PQ$-function of the arc length. We call a $PQ$-function *Q-concave* iff on each geodesic arc it becomes concave after subtracting the squared arc length. The difference of any two $Q$-functions is a linear function, so after subtracting the squared arc length we always obtain a piecewise linear function on a geodesic.

The terms $PQ$-function and $Q$-concave are abbreviations for "piecewise quadratic function" and "quasiconcave".

Definition 2.3. — *Let $T$ be a geodesic triangulation of $M$, and let $r$ be an assignment of positive numbers to the vertices of $T$. Define the function*

$$q_{T,r} : M \to \mathbb{R}$$

*as the PQ-function that takes the value $r_i^2$ at the singularity $i$ of $M$ and that is a Q-function on every triangle of $T$.*

Lemma 2.4. — *A pair $(T, r)$ represents a generalized convex polytope if and only if $q_{T,r}$ is a positive Q-concave PQ-function.*

*Proof.* — Let $(T, r)$ represent a generalized convex polytope. It is immediate that the function $q_{T,r}$ is the squared distance from the apex. Therefore it is a positive $PQ$-function. To show that it is $Q$-concave, it suffices to consider a short geodesic arc crossing an edge $ij \in \mathcal{E}(T)$. Take the adjacent triangles $ijk$ and $ijl$ and develop them nto the plane. We get a quadrilateral $ikjl$. Choose a point $a \in \mathbb{R}^3$ such that $\|a - i\| = r_i$, $\|a - j\| = r_j$, and $\|a - k\| = r_k$. We claim that $\|a - l\| \geqslant r_l$. To see this, compare the pyramid over $ikjl$ with the union of two corresponding pyramids in the polytope, and use the fact that $r_l$ is a monotone increasing function of $\theta_{ij}$, for constant $r_i$, $r_j$, $r_k$. Consider the function $q_{T,r}(x) - \|x - a\|^2$ on the quadrilateral. It is a piecewise linear function that is identically 0 on the triangle $ijk$ and non-positive at $l$. Therefore it is concave, which implies $Q$-concavity of $q_{T,r}$ on any short geodesic arc across $ij$. The left to right implication is proved. (Note that any of the vertices $i, j, k, l$ may coincide, and even triangles $ijk$ and $ijl$ may be two copies of the same triangle.)

Now let $(T, r)$ be any pair such that $q_{T,r}$ is positive and $Q$-concave. Let us show that for every triangle $ijk \in \mathcal{F}(T)$ there exists a pyramid over it with side edge lengths $r_i$, $r_j$, $r_k$. Develop the triangle $ijk$ and extend the function $q_{T,r}$ from it to a $Q$-function $\mathrm{ext}_{ijk}$ on the whole plane. The desired pyramid exists if and only if $\mathrm{ext}_{ijk}$ is positive everywhere on $\mathbb{R}^2$: the minimum value of the function is the square of the pyramids altitude, the point where the minimum is attained is the projection of the apex. So assume that there is a point $y \in \mathbb{R}^2$ such that $\mathrm{ext}_{ijk}(y) \leqslant 0$. Join $y$ with some point $x$ inside the triangle $ijk$ by a straight line segment. Now take the corresponding geodesic arc on $M$, which is possible for $x$ and $y$ in general position. Inside the triangle $ijk$ we have $q_{T,r} = \mathrm{ext}_{ijk}$. It follows from the $Q$-concavity of $q_{T,r}$ that $q_{T,r}(y) \leqslant 0$. This contradiction shows that there is a generalized polytope corresponding to the pair $(T, r)$. That it is convex follows by reversing the argument in the first part of the proof. □

*Proof. — of Proposition 2.2.* Let us show that a generalized convex polytope is determined by the function $q_{T,r}$:

$$(T, r) \sim (T', r') \Leftrightarrow q_{T,r} = q_{T',r'}. \tag{2.2}$$

The implication from the left to the right is obvious. To prove the reverse implication, consider a common geodesic subdivision $T''$ of $T$ and $T'$, where $T''$ is allowed to have vertices outside $\Sigma$. This yields decompositions of the both generalized polytopes $(T, r)$ and $(T', r')$ into smaller pyramids. The equality $q_{T,r} = q_{T',r'}$ implies that the pyramids over same triangle of $T''$ in both decompositions are equal.

Now it suffices to show that if for some $r$ there are two triangulations $T$ and $T'$ such that both functions $q_{T,r}$ and $q_{T',r}$ are $Q$-concave, then $q_{T,r} = q_{T',r}$.

Take any point $x \in M$. Let $ijk \in \mathcal{F}(T)$ be a triangle that contains $x$. Since $q_{T,r}$ is a $Q$-function on $ijk$, the function $q_{T',r} - q_{T,r}$ is piecewise linear and concave on $ijk$. Since it vanishes at the vertices, it is non-negative everywhere in $ijk$. Thus we have $q_{T',r}(x) \geqslant q_{T,r}(x)$ for any $x \in M$. Similarly, $q_{T,r}(x) \geqslant q_{T',r}(x)$. Therefore the functions are equal. □

### 2.4. The space of generalized convex polytopes

By Proposition 2.2, the radii $r_i$ can be considered as coordinates on the space $\mathcal{P}(M)$ which thus becomes a subset of $\mathbb{R}^n$. It is more convenient to take squares of radii as coordinates; so in the future we identify $\mathcal{P}(M)$ with its image under the embedding

$$
\begin{array}{rcl}
\mathcal{P}(M) & \to & \mathbb{R}^n, \\
(T, r) & \mapsto & (r_1^2, \dots, r_n^2).
\end{array}
\tag{2.3}
$$

In this subsection we derive a system of inequalities which describes $\mathcal{P}(M)$. By Lemma 2.4, we have

$$\mathcal{P}(M) = \{\text{positive } Q\text{-concave } PQ\text{-functions on } M\}.$$

First, we look at a larger space:

$$\mathcal{D}(M) = \{Q\text{-concave } PQ\text{-functions on } M\}.$$

For any map $q : \Sigma \to \mathbb{R}$ and any geodesic triangulation $T$ of $M$ denote by

$$\widetilde{q_T} : M \to \mathbb{R}$$

the $PQ$-extension of $q$ with respect to $T$. (Note that $q$ and $\widetilde{q_T}$ are just a different notation for $r^2$ and $q_{T,r}$, respectively. ) The space $\mathcal{D}(M)$ can be identified with those $q = (q_i)_{i \in \Sigma}$ for which there exists $T$ such that $\widetilde{q_T}$ is $Q$-concave. By $q_i$ we denote the value $q(i)$.

A Euclidean quadrilateral $ikjl$ in $M$ is an open subset of $M$ bounded by four simple geodesic arcs $ik, kj, jl, li$ and without singular points. Vertices and sides of the quadrilateral need not be distinct. Since a Euclidean quadrilateral contains no singularities, it can be developed into the plane. A Euclidean triangle $iji$ is an open subset of $M$ bounded by a geodesic arc $ij$ and a closed geodesic arc based at $i$ that separates $j$ from the other singularities. (Clearly, not for any choice of $i, j, k, l$ or of $i, j$ the surface $M$ contains a quadrilateral $ikjl$ or a triangle $iji$.)

Define the function $\mathrm{ext}_{ikl}$ on the quadrilateral $ikjl$ as the $Q$-function that takes values $q_i$, $q_k$, $q_l$ at the respective vertices.

Now we describe the *flip algorithm*. Let $ij$ be an edge in a geodesic triangulation $T$. If $ij$ is adjacent to two different triangles $ijk$ and $ijl$, then we have a Euclidean quadrilateral $ikjl$. If $ikjl$ is strictly convex, then we can transform the triangulation $T$ by replacing the diagonal $ij$ through the diagonal $kl$ in $ikjl$. This transformation is called a *flip*. The flip algorithm in a general setting works as follows. Assume that we have a rule to say for any edge $ij \in \mathcal{E}(T)$ whether $ij$ is "bad" or "good". The goal is to find a triangulation where all edges are good. Start with an arbitrary triangulation. If there is a bad edge that forms a diagonal of a strictly convex quadrilateral, then flip it. Look for a bad edge in the new triangulation, flip it if possible and so on. If any bad edge can be flipped and flipping bad edges cannot continue infinitely, then the flip algorithm yields a triangulation with only good edges.

Proposition 2.5. — *The set $\mathcal{D}(M)$ is a convex polyhedron which is the solution set of a system of linear inequalities of the form:*

$$q_j \geqslant \mathrm{ext}_{ikl}(j), \tag{2.4}$$

$$q_j \geqslant q_i - \ell_{ij}^2. \tag{2.5}$$

*There is one equation of the form (2.4) for each Euclidean quadrilateral $ikjl$ with the angle at $j$ greater or equal $\pi$, and one equation of the form (2.5) for each Euclidean triangle $iji$. By $\ell_{ij}$ we denote the length of the edge $ij$.*

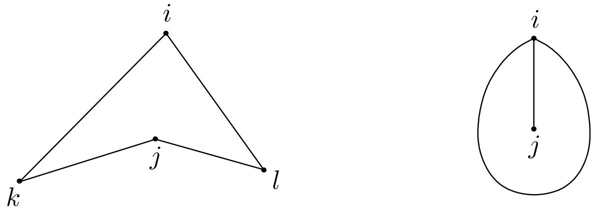


*Figure 2.1. Every non-strongly convex quadrilateral $ikjl$ gives rise to an inequality (2.4), every triangle $iji$ gives rise to an inequality (2.5).*

*Proof.* — Let us show that the right hand side of (2.4) is a linear function of $q_i$, $q_k$, $q_l$. Develop $ikjl$ onto the plane. When we vary $q_i$, the function $\mathrm{ext}_{ikl}$ changes by a linear function which vanishes along the line $kl$. Thus $\mathrm{ext}_{ikl}(j)$ depends linearly on $q_i$, as well as on $q_k$ and $q_l$. Moreover, the inequality (2.4) has the form

$$q_i \geqslant \lambda q_j + \mu q_k + \nu q_l + c,$$

where $\lambda + \mu + \nu = 1$, $\lambda$, $\mu$, $\nu \geqslant 0$ are barycentric coordinates of $j$ with respect to $i$, $k$, and $l$.

Let us show that conditions (2.4) and (2.5) are fulfilled for any assignment $q = (q_i)_{i\in\Sigma}$ that posesses a $Q$-concave extension $\widetilde{q}$. For (2.4), consider the function $\widetilde{q} - \mathrm{ext}_{ikl}$ on the quadrilateral $ikjl$. It is concave and vanishes at $i$, $k$, and $l$. Hence it is non-negative on the triangle $ikl$, in particular $q_j - \mathrm{ext}_{ikl}(j) \geqslant 0$. For (2.5), consider the function $f(x) = q(x) - \|x - j\|^2$ on the triangle $iji$. It is geodesically concave and takes values $q_j$ at $j$ and $q_i - \ell_{ij}^2$ at $i$. Hence $f(x) \geqslant q_i - \ell_{ij}^2$ for any $x$ on the edge $ii$. Due to concavity,

$f$ is also greater or equal to $q_i - \ell_{ij}^2$ at any regular point of the triangle $iji$. By continuity, $f(j) \geqslant q_i - \ell_{ij}^2$.

Now we prove sufficiency of the conditions (2.4) and (2.5). Let $T$ be any geodesic triangulation. Call an edge of $T$ *good*, if the function $\widetilde{q_T}$ is $Q$-concave across this edge, otherwise call an edge *bad*. Our aim is to find a triangulation $T$ with only good edges. Let us show that any bad edge can be flipped. Indeed, if $ij$ is adjacent to only one triangle, then the inequality (2.5) implies that $ij$ is good. If it is adjacent to two triangles that form a concave or non-strictly convex quadrilateral, then the inequality (2.4) implies that $ij$ is good. Thus any bad edge is a diagonal of a strictly convex quadrilateral and therefore can be flipped.

Let us show that the flip algorithm terminates. Note that when a bad edge is flipped, the function $\widetilde{q_T}$ increases pointwise. If there are only finitely many geodesic triangulations of $M$, this suffices. If not, we use the fact that for any $C$ there are only finitely many triangulations of $M$ with all geodesics of length less than $C$, [17]. Thus it suffices to show that no long edges can appear when performing the algorithm. Let $T'$ be a triangulation with an edge $ij$. Function $\widetilde{q_{T'}}$ restricted to $ij$ has the form $x^2 + ax + b$ and takes values $q_i$ and $q_j$ at the endpoints. There exists a constant $C$ such that if the length of $ij$ is greater than $C$, the minimum value of $\widetilde{q_{T'}}$ on $ij$ is smaller than $\min_{x \in M} \widetilde{q_T}(x)$. This means that no edge of length greater than $C$ can appear between the vertices $i$ and $j$ if we start the flip algorithm from the triangulation $T$. □

Let $\mathcal{D}^T(M)$ be the set of those functions $q \in \mathcal{D}(M)$ which are $PQ$ with respect to the triangulation $T$. Then we have a decomposition

$$\mathcal{D}(M) = \cup_T \mathcal{D}^T(M). \tag{2.6}$$

Proposition 2.6. — *For any surface $M$ with a polyhedral metric the polyhedron $\mathcal{D}(M)$ of $Q$-concave $PQ$-functions on $M$ has the following properties.*

(1) *The origin $0 \in \mathbb{R}^n$ is an interior point of $\mathcal{D}(M)$.*
(2) *Put $\mathcal{D}_0(M) = \{q \in \mathcal{D}(M) | \sum_i q_i = 0\}$. Then the polyhedron $\mathcal{D}(M)$ is the direct sum*

$$\mathcal{D}(M) = \mathcal{D}_0(M) \oplus L, \tag{2.7}$$

*where $L$ is the one-dimensional subspace of $\mathbb{R}^n$ spanned by the vector $(1, 1, \ldots, 1)$. Every $\mathcal{D}^T(M)$ is decomposed in the same way: if $\mathcal{D}_0^T(M) = \{q \in \mathcal{D}^T(M) | \sum_i q_i = 0\}$, then $\mathcal{D}^T(M) = \mathcal{D}_0^T(M) \oplus L$.*
(3) *Every $\mathcal{D}^T(M)$ is a convex polyhedron.*
(4) *The decomposition (2.6) is locally finite.*

(5) *For $q \in \mathcal{D}(M)$, let $\widetilde{q} : M \to \mathbb{R}$ denote the $Q$-concave $PQ$-extension of $q$. Then $\widetilde{q}$ depends continuously on $q$ in the $L^\infty$-metric.*

*Proof.* — If we put $q_i = 0$ for every $i$, then all of the inequalities (2.4) and (2.5) are fulfilled and are strict. This implies property 1.

Fix a triangulation $T$. If $q$ is a $PQ$-function with respect to $T$, then it is $Q$-concave across an edge $ij$ if and only if

$$q_l \leqslant \mathrm{ext}_{ijk}(l), \tag{2.8}$$

where $ijk$ and $ijl$ are triangles of $T$. The inequality (2.8) remains valid if we change all of the $q_i$'s by the same amount. Thus $D^T(M)$, and also the whole $D(M)$, is invariant under translations along $L$. This proves property 2.

The set $D^T(M)$ is a convex polyhedron because it is the set of solutions of a linear system (2.8). This shows 3.

Property 4 means that any bounded subset $U$ in $\mathbb{R}^n$ has a non-empty intersection with only finitely many of the $\mathcal{D}^T(M)$. For this, it suffices to show that the edge lenghts in all of the triangulations $T$ such that $\mathcal{D}^T(M) \cap U \neq \emptyset$ are uniformly bounded from above. Fix any triangulation $T_0$. For any $q \in U$ consider the functions $\widetilde{q_{T_0}}$ and $\widetilde{q_T}$, where $T$ is such that $\widetilde{q_T}$ is $Q$-concave. We have $\widetilde{q_{T_0}} \leqslant \widetilde{q_T}$. Since $U$ is bounded, functions $\widetilde{q_{T_0}}$ are uniformly bounded from below. Take any edge in $T$. The values of $q$ at the endpoints of the edge are $\leqslant \max_{q \in U,\, i \in \Sigma} q_i$. This, together with the uniform lower bound on $q$, implies that the edge cannot be too long.

In every $\mathcal{D}^T(M)$, $\widetilde{q}$ depends continuously on $q$. The continuity of $q \mapsto \widetilde{q}$ on the whole $\mathcal{D}(M)$ follows from property 4. □

Proposition 2.7. — *The space $\mathcal{P}(M)$ of convex generalized polytopes with boundary $M$ has the following properties.*

(1) *The point $(C, C, \ldots, C)$ is an interior point of $\mathcal{P}(M)$ for a sufficiently large $C$.*
(2) *If $q \in \mathcal{P}(M)$ is an interior point of $\mathcal{D}(M)$, then $q$ is an interior point of $\mathcal{P}(M)$ as well.*
(3) $\mathcal{P}^T(M) = \mathcal{D}^T(M) \cap \{q_i > 0\, \forall i \in \mathcal{V}(T)\} \cap \{CM_{ijk} > 0\, \forall ijk \in \mathcal{F}(T)\}$, *where*

$$CM_{ijk} = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & q_i & q_j & q_k \\ 1 & q_i & 0 & \ell_{ij}^2 & \ell_{ik}^2 \\ 1 & q_j & \ell_{ij}^2 & 0 & \ell_{jk}^2 \\ 1 & q_k & \ell_{ik}^2 & \ell_{jk}^2 & 0 \end{vmatrix}$$

*is the Cayley-Menger determinant.*

*Proof.* — Recall that

$$\mathcal{P}(M) = \{q \in \mathcal{D}(M) | \, \widetilde{q} > 0\}.$$

By Proposition 2.6, the point $(q_i = C)$ belongs to $\mathcal{D}(M)$ for any $C$. Since adding a constant to all of $q_i$ results in adding a constant function to $\widetilde{q}$, for a large $C$ the $PQ$-extension of $(q_i = C)$ is everywhere positive. That is, $(q_i = C) \in \mathcal{P}(M)$.

By property 5 of Proposition 2.6, the function $\widetilde{q}$ depends continuously on $q$. Therefore the condition $\widetilde{q} > 0$ is an open condition on $q$. This implies property 2. Also it completes the proof of property 1, since $q_i = C$ is an interior point of $\mathcal{D}(M)$.

Positivity of the Cayley-Menger determinant $CM_{ijk}$ is equivalent to the existence of a pyramid over the triangle $ijk$ with side edges of lengths $\sqrt{q_i}, \sqrt{q_j}, \sqrt{q_k}$. This implies property 3. □

Note that $q$ is a boundary point of $\mathcal{D}(M)$ if and only if one of the inequalities (2.4) and (2.5) becomes an equality. Geometrically this means that a generalized convex polytope $P$ is on the boundary of $\mathcal{P}(M)$ if and only if $P$ has a non-strongly convex face or an isolated vertex. Here by faces of $P$ we mean connected regions in $M$ after erasing flat edges of $P$.

## 2.5. Weighted Delaunay triangulations

We will now explain how generalized convex polytopes are related to weighted Delaunay triangulations of polyhedral surfaces.

Definition 2.8. — *Let $M$ be a polyhedral surface with singular set $\Sigma$. Let $V \subset M$ be a finite non-empty subset which contains $\Sigma$ and has at least one point in every boundary component of $M$. For a function $q : V \to \mathbb{R}$ and a geodesic triangulation $T$ with vertex set $V$ denote by $\widetilde{q_T}$ the PQ-extension of $q$ to $M$ with respect to $T$. A pair $(T, q)$ is called a* weighted Delaunay triangulation *of $(M, V)$ with weights $q$ iff the function $\widetilde{q_T}$ is Q-concave.*

*A Delaunay triangulation is a weighted Delaunay triangulation with all weights equal.*

Definition 2.9. — *A weighted Delaunay tesselation $(\overline{T}, q)$ is obtained from a weighted Delaunay triangulation $(T, q)$ by removing from $T$ all inessential edges. An edge $ij \in \mathcal{E}(T)$ is called inessential if the corresponding function $\widetilde{q_T}$ is a Q-function in the neighborhood of any interior point of $ij$.*

Theorem 2.10. — *If $M$ is the sphere and $V = \Sigma$, then weighted Delaunay tesselations of $(M, V)$ with positive associated functions $\widetilde{q_T}$ are in one-to-one correspondence with generalized convex polytopes with boundary $M$.*

*Proof.* — This is essentially a reformulation of Lemma 2.4. By (2.2), generalized convex polytopes with boundary $M$ are in one-to-one correspondence with positive $Q$-concave $PQ$-functions on $M$. To any such function $f$ there canonically corresponds a tesselation of $M$ into regions where $f$ is quadratic. Since the function $f$ is $Q$-concave, the tesselation together with values of $f$ at $\Sigma$ as weights is a weighted Delaunay tesselation. $\square$

Definition 2.11. — *Let $M$ and $V$ be as in Definition 2.8. Define the space of* admissible weights *$\mathcal{D}(M, V)$ as the set of maps $q : V \to \mathbb{R}$ such that there exists a weighted Delaunay triangulation of $M$ with weights $q$.*

Theorem 2.12. —

(1) *For any $(M, V)$ the space of admissible weights $\mathcal{D}(M, V)$ is a convex polyhedron in $\mathbb{R}^V$ defined by inequalities (2.4) and (2.5).*
(2) *For any weight $q \in \mathcal{D}(M, V)$ the weighted Delaunay tesselation with weights $q$ is unique.*
(3) *A weighted Delaunay triangulation can be found via the flip algorithm starting from any triangulation of $(M, V)$.*
(4) *Properties from Proposition 2.6 hold.*

*Proof.* — Statement 1 generalizes Proposition 2.5. In Proposition 2.5 we assumed that $M$ is a closed surface and $V = \Sigma$. But it is easy to see that the proof works equally well without these assumptions. Statement 3 is contained in the proof of Proposition 2.5.

Statement 2 is essentially Proposition 2.2.

Finally, for statement 4 nothing changes in the proof of Proposition 2.6. $\square$

The weighted Delaunay tesselation with all equal weights is called the *Delaunay tesselation.* Theorem 2.12 implies that for any $(M, V)$ the Delaunay tesselation is unique and a Delaunay triangulation can be obtained via the flip algorithm.

Delaunay triangulations of polyhedral surfaces were used as a technical tool in the study of moduli spaces of Euclidean polyhedral metrics on surfaces [8], [24]. Recently they were applied to define the discrete Laplace-Beltrami operator on polyhedral surfaces [7]. Further development was done in [16].

Originally weighted Delaunay triangulations are defined for $V \subset \mathbb{R}^2$, and $M$ the convex hull of $V$, see [4], [10], [14]. It can be shown that in this case Definition 2.8 is equivalent to the classical one. The terminology in the subject is not unified. Sometimes a triangulation is called *regular* if it is a weighted Delaunay triangulation for a suitable choice of weights. In [15], triangulations of this kind are called *coherent*. The definition in [15] is more close to our Definition 2.8: a triangulation $T$ of a point set $V \subset \mathbb{R}^n$ is coherent iff there is a piecewise linear with respect to $T$ function which is (strictly) concave. The connection with our definition is established through the fact that in $\mathbb{R}^n$ there exist globally quadratic functions, $\|x\|^2$ for example. If one subtracts from a $PQ$-function a globally quadratic function, then one obtains a piecewise linear function; the original $PQ$-function is $Q$-concave iff the difference is concave.

Another definition of weighted Delaunay triangulations is in the language of circle patterns. Let a set of circles whose centers form the set $V$ is given. There is the Voronoi diagram defined through the radical axes of the given circles. The dual triangulation is called the Delaunay triangulation with the circles as sites. It can be shown that this is the weighted Delaunay triangulation in the sense of Definition 2.8, where the weights are the squares of the circles radii. Thus a generalized convex polytope with boundary $M$ corresponds to a circle pattern on $M$ with the circles centered at the singular points. The positivity of the associated function $\widetilde{q_T}$ means that the pattern is *hypoideal*, that is the intersection of the circles centered at the vertices of a triangle of the triangulation has a non-empty interior.

Circle patterns on surfaces with conical singularities are considered in [25], [28]. These works deal with the question of existence and uniqueness of a *hyperideal* circle pattern with a prescribed combinatorics and intersection angles. They provide systems of linear inequalities for the intersection angles of circle patterns and the curvatures at the circles centers. The polyhedral metric that carries a circle pattern with given curvatures and intersection angles is shown to be unique. Note that the viewpoint of our paper is quite different, since we have a fixed polyhedral metric, changing triangulations, and no control over the intesection angles. However, we characterize in a certain way the collection $\mathcal{D}^T(M)$ of weights that produce the given triangulation $T$, see Proposition 2.6. It would be interesting to develop the connections further.

## 3. Total scalar curvature and volume of the dual

In Section 2 we have shown that the radii $(r_i)_{i\in\Sigma}$ can serve as coordinates on the space $\mathcal{P}(M)$ of generalized convex polytopes with boundary $M$.

Hence the curvatures $(\kappa_i)_{i\in\Sigma}$ become functions of the radii. In our proof of Alexandrov's theorem we will deform the radii so that at the end of the deformation all of the curvatures vanish. The goal of this section is to prove non-degeneracy of the Jacobian $\left(\frac{\partial\kappa_i}{\partial r_j}\right)$ under certain restrictions on the polytope (Corollary 3.18). By the inverse function theorem, this allows us to realize any infinitesimal deformation of curvatures by an infinitesimal deformation of radii.

### 3.1. Total scalar curvature of a generalized polytope

Definition 3.1. — *Let $P = (T, r)$ be a convex generalized polytope. The total scalar curvature of $P$ is*

$$H(P) = \sum_{i\in\mathcal{V}(T)} r_i\kappa_i + \sum_{ij\in\mathcal{E}(T)} \ell_{ij}(\pi - \theta_{ij}),$$

*where $\kappa_i = 2\pi - \omega_i$ and $\omega_i$ is the sum of all dihedral angles of pyramids at the $i$-th radial edge, $\ell_{ij}$ is the length of the edge $ij \in \mathcal{E}(T)$, and $\theta_{ij}$ is the dihedral angle of the polytope $P$ at the edge $ij$.*

In the case of a classical convex polytope the first sum in the definition of $H(P)$ vanishes. The second sum equals twice the *edge curvature* introduced by Steiner. It was proved by Minkowski that the edge curvature converges to the total mean curvature as the boundary of the polytope converges to a smooth surface.

If the triangulation $T$ is not unique, the value $H(P)$ does not depend on it, because $\pi - \theta_{ij} = 0$ for a flat edge $ij$.

Proposition 3.2. — *For a fixed Euclidean polyhedral metric $M$, function $H$ is of class $C^2$ on $\mathcal{P}(M)$. Its partial derivatives are:*

$$\frac{\partial H}{\partial r_i} = \kappa_i, \tag{3.1}$$

$$\frac{\partial\kappa_i}{\partial r_j} = \frac{\cot\alpha_{ij} + \cot\alpha_{ji}}{\ell_{ij}\sin\rho_{ij}\sin\rho_{ji}}, \tag{3.2}$$

$$\frac{\partial\kappa_i}{\partial r_i} = -\sum_{j\neq i}\cos\varphi_{ij}\frac{\partial\kappa_i}{\partial r_j}. \tag{3.3}$$

*Here $\alpha_{ij}$ and $\alpha_{ji}$ are the dihedral angles of the pyramids at the edge $ij$, thus $\alpha_{ij} + \alpha_{ji} = \theta_{ij}$; $\rho_{ij}$ is the angle at the vertex $i$ in the triangle $aij$; $\varphi_{ij}$ is the angle at $a$ in the same triangle, see Figure 3.1. If $ij$ is not an edge of $T$, then $\frac{\partial\kappa_i}{\partial r_j} = 0$.*

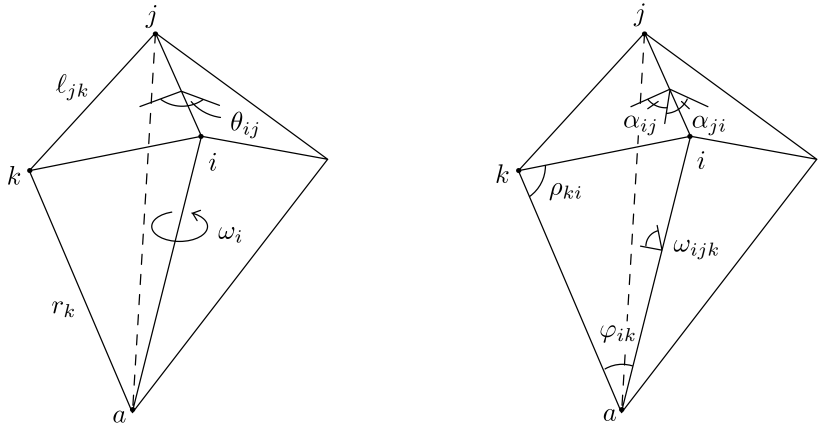


*Figure 3.1. Angles and lengths in a generalized convex polytope.*

*Proof.* — Let $P$ be strongly convex, that is, let $\theta_{ij} < \pi$ for all $ij \in T$. In this case all polytopes in some neighborhood of $P$ have the same triangulation $T$ on the boundary.

For $ijk \in \mathcal{F}(T)$ consider the corresponding pyramid. Denote by $\omega_{ijk}$ its dihedral angle at the edge $ai$. By Schläfli's formula [19],

$$r_i d\omega_{ijk} + r_j d\omega_{jik} + r_k d\omega_{kij} + \ell_{ij} d\alpha_{ij} + \ell_{jk} d\alpha_{jk} + \ell_{ki} d\alpha_{ki} = 0.$$

Summing this up over all $ijk \in \mathcal{F}(T)$ yields

$$\sum_{i\in\mathcal{V}(T)} r_i d\omega_i + \sum_{ij\in\mathcal{E}(T)} \ell_{ij} d\theta_{ij} = 0$$

which implies formula (3.1).

For $i \in \mathcal{V}(T)$ let

$$\text{lk } i = \{j \in \mathcal{V}(T) |\, ij \in \mathcal{E}(T)\}. \tag{3.4}$$

The angle $\omega_i$ and its parts $\omega_{ijk}$ can be viewed as functions of the angles $(\rho_{ij})_{j\in\text{lk } i}$. By Leibniz's rule we have

$$\frac{\partial\omega_{ijk}}{\partial r_j} = \frac{\partial\omega_{ijk}}{\partial\rho_{ij}}\frac{\partial\rho_{ij}}{\partial r_j},$$
$$\frac{\partial\omega_{ijk}}{\partial r_i} = \frac{\partial\omega_{ijk}}{\partial\rho_{ij}}\frac{\partial\rho_{ij}}{\partial r_i} + \frac{\partial\omega_{ijk}}{\partial\rho_{ik}}\frac{\partial\rho_{ik}}{\partial r_i}.$$

All partial derivatives on the right hand side are derivatives of the angles with respect to the side lengths in spherical and Euclidean triangles. Substituting the formulas of Lemma 5.1, we get the formulas (3.2) and (3.3).

Let $P$ be not strongly convex. Recall that $\mathcal{P}^T(M)$ denotes the subset of $\mathcal{P}(M)$ formed by the polytopes with an associated triangulation $T$. By property 3 from Proposition 2.6, $\mathcal{P}^T(M)$ are separated by affine hyperplanes in coordinates $(q_i) = (r_i^2)$. By the argument for a strongly convex $P$, there are directional derivatives of $H$ at $P$ in all directions. Since $\kappa_i$ is well-defined and continuous on $\mathcal{P}(M)$, it follows that the function $H$ is of class $C^1$. The same argument applies to the functions $\kappa_i$. The right hand sides of formulas (3.2) and (3.3) are well-defined and continuous because all of the summands that correspond to flat edges vanish. □

To be precise, the formulas of Proposition 3.2 are valid only when the triangulation $T$ contains neither loops nor multiple edges. Otherwise the formulation is somewhat cumbersome and looks as follows. For an oriented edge $e \in \mathcal{E}(T)$ let $a(e), b(e) \in \mathcal{V}(T)$ be its initial and terminal vertex, respectively. Then we have

$$d\kappa_i = \sum_{a(e)=i} \frac{\cot \alpha_e + \cot \alpha_{-e}}{\sin \rho_e} d\rho_e,$$

$$\frac{\partial \rho_e}{\partial r_j} = \begin{cases} \frac{1}{\ell_e \sin \rho_e} & \text{for } a(e) \neq j = b(e) \\ -\frac{\cos \varphi_e}{\ell_e \sin \rho_{-e}} & \text{for } a(e) = j \neq b(e) \\ \frac{\ell_e}{2r_i^2 \sin \rho_e} & \text{for } a(e) = j = b(e). \end{cases} \tag{3.5}$$

## 3.2. Generalized polyhedra

Convex polytope and convex polyhedron are two basic notions of the classical theory of polytopes. The former is defined as the convex hull of a finite number of points, and the latter as the intersection of a finite number of half-spaces. If we take a sphere with the center inside a convex polytope $P$, then the corresponding polar dual $P^*$ is a bounded convex polyhedron, and vice versa. On the other hand, there is a fundamental theorem saying that the class of bounded convex polyhedra coincides with the class of convex polytopes. For more information see [32].

In this section we will dualize the notion of generalized convex polytope from Section 2 by introducing generalized convex polyhedra. Informally speaking, a generalized convex polytope has curvatures concentrated along the segments joining the apex to the vertices. A generalized convex polyhedron has curvatures concentrated along the altitudes drawn from the apex to the faces. A common extension of these two notions is a polyhedral complex that has curvatures both along the radial edges and along the altitudes. This class of complexes is self-dual; however, we have no need for such a generalization.

Generalized convex polyhedra are introduced in Definitions 3.3 and 3.4. In order to motivate the abstract constructions let us look closer at the classical convex polyhedra.

Let us start by recalling the notion of the normal fan. Let $Q$ be a bounded 3-dimensional convex polyhedron. To a face $Q_i$ of $Q$ associate the ray in $\mathbb{R}^3$ based at the origin and spanned by the outer normal to $Q_i$. If two faces $Q_i$ and $Q_j$ have an edge in common, then denote this edge by $Q_{ij}$ and associate to it the positive span of the rays associated with $Q_i$ and $Q_j$. The flat angles thus obtained subdivide $\mathbb{R}^3$ into 3-dimensional cones which are the normal cones at the vertices of $Q$. The resulting complex of cones is called the *normal fan* of $Q$. If exactly three edges meet at every vertex of $Q$, then all vertex cones are trihedral angles, and the normal fan is called *simplicial*. Every normal fan can be subdivided to become simplicial, which corresponds to formally introducing in $Q$ additional edges of zero length. The spherical section of a simplicial fan is a geodesic triangulation of $\mathbb{S}^2$. Clearly, it completely determines the fan.

Definition 3.3. — *A generalized simplicial fan is a pair $(T, \varphi)$, where $T$ is a triangulation of the sphere, and*

$$\varphi : \mathcal{E}(T) \to (0, \pi)$$

*is a map such that for each triangle $ijk \in \mathcal{F}(T)$ there exists a spherical triangle with side lengths $\varphi_{ij}$, $\varphi_{jk}$, $\varphi_{ik}$.*

A simplex $OABC$ in $\mathbb{R}^3$ is called an *orthoscheme*, if the vectors $OA$, $AB$, $BC$ are pairwise orthogonal. Let $Q \subset \mathbb{R}^3$ be a convex polyhedron. Draw perpendiculars from a point $O \in \mathbb{R}^3$ to the planes of the faces of $Q$. Let $A_i$ be the foot of the perpendicular to the face $Q_i$. In the plane of $Q_i$ draw perpendiculars $A_iA_{ij}$ to the edges $Q_{ij}$. By $A_{ijk}$ denote the vertex of $Q$ common to the faces $Q_i, Q_j$ and $Q_k$. The collection of orthoschemes $OA_iA_{ij}A_{ijk}$ is called the *orthoscheme decomposition* of $Q$. Combinatorially the orthoscheme decomposition is the barycentric decomposition of $Q$. Note however that the feet of perpendiculars can lie outside the faces or edges, which makes the geometric picture complicated. Let $h_i$, $h_{ij}$, $h_{ijk}$ be the signed lengths of the segments $OA_i$, $A_iA_{ij}$, $A_{ij}A_{ijk}$, respectively. The length $h_i$ is negative if the point $O$ and the polytope $Q$ are on opposite sides of the face $Q_i$; similarly for $h_{ij}$ and $h_{ijk}$. We call the numbers $h_i$, $h_{ij}$, $h_{ijk}$ the *parameters of the orthoscheme $OA_iA_{ij}A_{ijk}$*, see Figure 3.2.

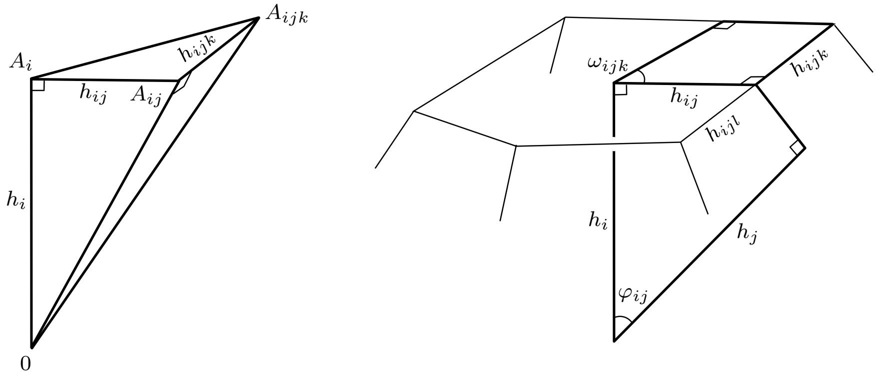


*Figure 3.2. The orthoscheme decomposition of a convex polyhedron.*

Clearly, the normal fan and the altitudes $h_i$ completely determine the polyhedron $Q$. In particular, the parameters $h_{ij}$ and $h_{ijk}$ can be expressed as functions of $(h_i)$ and the angles of the fan. In fact, one obtains

$$h_{ij} = \frac{h_j - h_i \cos\varphi_{ij}}{\sin\varphi_{ij}}, \tag{3.6}$$

where $\varphi_{ij}$ is the angle between the outer normals to faces $Q_i$ and $Q_j$. Similarly,

$$h_{ijk} = \frac{h_{ik} - h_{ij}\cos\omega_{ijk}}{\sin\omega_{ijk}}, \tag{3.7}$$

where $\omega_{ijk}$ is the exterior angle of the face $Q_i$ at the vertex $A_{ijk}$ or, equivalently, the angle at the vertex $i$ of the triangle $ijk$ in the spherical section of the normal fan. The length $\ell^*_{ij}$ of the edge $Q_{ij}$ is

$$\ell^*_{ij} = h_{ijk} + h_{ijl}, \tag{3.8}$$

where $Q_k$ and $Q_l$ are the faces that bound the edge $Q_{ij}$. Therefore the sum $h_{ijk} + h_{ijl}$ has to be non-negative, although the parameters $h_{ijk}$ are allowed to be negative. Note also that

$$h_{ijk} = h_{jik} \tag{3.9}$$

which is not obvious from (3.7), but is clear from the geometric meaning of $h_{ijk}$.

A generalized convex polyhedron is defined as a generalized simplicial fan together with the set of altitudes $h_i$. More intuitively, it is a complex of signed orthoschemes with parameters computed by formulas (3.6) and (3.7). Moreover, we require the edge lengths (3.8) to be non-negative.

Definition 3.4. — *Let $(T, \varphi)$ be a generalized simplicial fan, and let $h$ be a map $\mathcal{V}(T) \to \mathbb{R}$. A triple $Q = (T, \varphi, h)$ is called* a generalized convex polyhedron, *if $\ell^*_{ij} \geqslant 0$ for every edge $ij \in \mathcal{E}(T)$, where $\ell^*_{ij}$ is defined by (3.6), (3.7), and (3.8). Here $\omega_{ijk}$ is the angle at the vertex $i$ in the spherical triangle $ijk$ of the fan $(T, \varphi)$.*

*For every $i \in \mathcal{V}(T)$ define the area of the $i$-th face of $Q$ by*

$$F_i = \frac{1}{2} \sum_j h_{ij} \ell^*_{ij}. \tag{3.10}$$

*The volume of the polyhedron $Q$ is defined by*

$$\mathrm{vol} = \frac{1}{3} \sum_i h_i F_i. \tag{3.11}$$

Note that the right hand side of the equation (3.11) is nothing else than the sum of the signed volumes of the orthoschemes.

The space $\mathcal{Q}(T, \varphi)$ of all generalized convex polyhedra with a given normal fan $(T, \varphi)$ is a convex polyhedron in $\mathbb{R}^{\mathcal{V}(T)}$ with coordinates $(h_i)$. The interior of $\mathcal{Q}(T, \varphi)$ corresponds to generalized polyhedra with all edges $\ell^*_{ij}$ of positive length. The function vol is a homogeneous polynomial of degree 3 in $(h_i)$, hence differentiable.

Proposition 3.5. — *The volume of $Q \in \mathcal{Q}(T, \varphi)$ is a smooth function of $h$ with the following partial derivatives:*

$$\frac{\partial \mathrm{vol}}{\partial h_i} = F_i, \tag{3.12}$$

$$\frac{\partial F_i}{\partial h_j} = \frac{\ell^*_{ij}}{\sin \varphi_{ij}}, \tag{3.13}$$

$$\frac{\partial F_i}{\partial h_i} = -\sum_{j \neq i} \cos \varphi_{ij} \frac{\partial F_i}{\partial h_j}. \tag{3.14}$$

We need the following lemma.

Lemma 3.6. — *The face area $F_i$ as a function of the variables $h_{ij}$ has the following partial derivatives:*

$$\frac{\partial F_i}{\partial h_{ij}} = \ell^*_{ij}. \tag{3.15}$$

*Proof.* — It is not hard to see that

$$\frac{\partial F_i}{\partial h_{ij}} = \frac{1}{2} \left( \frac{\partial}{\partial h_{ij}} (h_{ij} h_{ijk} + h_{ik} h_{ikj}) + \frac{\partial}{\partial h_{ij}} (h_{ij} h_{ijl} + h_{ik} h_{ilj}) \right).$$

Now the Lemma follows from

$$\begin{aligned}\frac{\partial}{\partial h_{ij}}(h_{ij}h_{ijk}+h_{ik}h_{ikj}) &= h_{ijk}+h_{ij}\frac{\partial h_{ijk}}{\partial h_{ij}}+h_{ik}\frac{\partial h_{ikj}}{\partial h_{ij}}\\ &= h_{ijk}-h_{ij}\cot\omega_{ijk}+\frac{h_{ik}}{\sin\omega_{ikj}}\\ &= h_{ijk}+\frac{h_{ik}-h_{ij}\cos\omega ijk}{\sin\omega_{ijk}}\\ &= 2h_{ijk}\ .\end{aligned}$$

□

*Proof. — of Proposition 3.5.* Equations (3.13) and (3.14) follow from

$$\frac{\partial F_i}{\partial h_j}=\frac{\partial F_i}{\partial h_{ij}}\frac{\partial h_{ij}}{\partial h_j}$$

and

$$\frac{\partial F_i}{\partial h_i}=\sum_j\frac{\partial F_i}{\partial h_{ij}}\frac{\partial h_{ij}}{\partial h_i}\ .$$

To prove (3.12), differentiate (3.11) by Leibniz's rule and use equations (3.13) and (3.14). □

As a byproduct we obtain the following analogue of Schläfli's formula:

Corollary 3.7. —

$$\sum_i h_i dF_i = 2\,d\mathrm{vol}\ .$$

Note that for classical convex polyhedra equations (3.12)–(3.15) have a clear geometric interpretation.

The triangulation $T$ may have loops and multiple edges. Equations (3.13) and (3.14) can be generalized to this case in a similar way as (3.5).

### 3.3. Duality

The equations of Propositions 3.2 and 3.5 are closely related.

Definition 3.8. — *Let $P=(T,r)$ be a generalized convex polytope. Its spherical section centered at the apex produces a generalized simplicial fan $(T,\varphi)$, where $\varphi_{ij}$ is the angle of the triangle $aij$ at the vertex $a$. The triple $P^*=(T,\varphi,h)$ with $h_i=\frac{1}{r_i}$ is called the generalized convex polyhedron dual to $P$.*

We justify this definition by showing that $\ell^*_{ij}\geqslant 0$ for any $ij\in\mathcal{E}(T)$.

Lemma 3.9. — *If $OABC$ is an orthoscheme, then $OC^*B^*A^*$ is also an orthoscheme. Here $X^*$ is the point on the ray $OX$ such that $OX^* \cdot OX = 1$.*

*Proof.* — $OABC$ is an orthoscheme if and only if $OAB, OAC$, and $OBC$ are two-dimensional orthoschemes, that is, right triangles. Since triangles $OXY$ and $OY^*X^*$ are similar, $OC^*B^*, OC^*A^*$, and $OB^*A^*$ are also orthoschemes. This implies that $OC^*B^*A^*$ is an orthoscheme. □

We call $OC^*B^*A^*$ the orthoscheme dual to $OABC$.

Lemma 3.10. — *Decompose each of the constituting pyramids of $P$ into 6 orthoschemes. Then the duals to these orthoschemes are exactly those that form the orthoscheme decomposition of $P^*$. The following formulas hold:*

$$h_{ij} = \frac{\cot \rho_{ij}}{r_i}, \tag{3.16}$$

$$h_{ijk} = \frac{\cot \alpha_{ij}}{r_i \sin \rho_{ij}}. \tag{3.17}$$

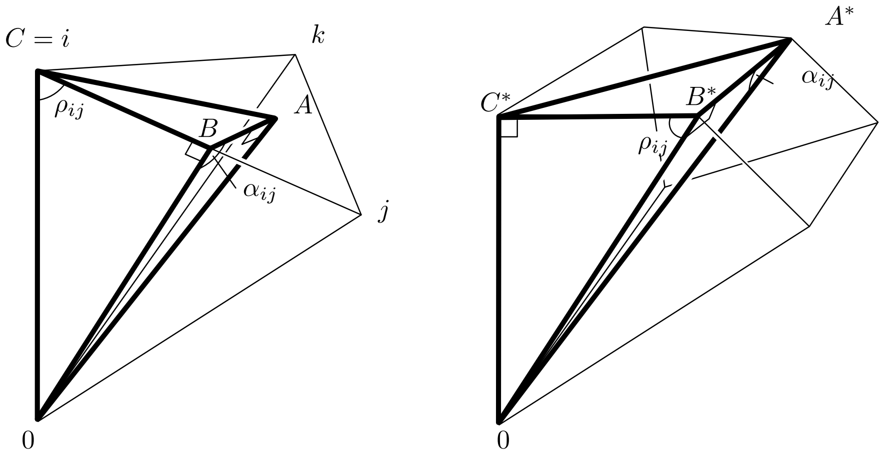


*Figure 3.3. Building blocks of a generalized convex polytope and of its dual polyhedron.*

*Proof.* — For a pyramid $ijk$, the six dual orthoschemes can be identified along common edges and faces. Together they form an object that has combinatorics of the cube, see Figure 3.3. The three faces of the cube that do not contain the apex are orthogonal to the side edges of the pyramid and have the distances $h_i, h_j, h_k$ from the apex. But this is exactly the construction in Definition 3.4. Therefore each of the dual orthoschemes has parameters $h_i, h_{ij}, h_{ijk}$ with indices appropriately permuted.

In the pyramid $ijk$ let $A$ be the foot of the perpendicular to the base, let $B$ be the foot of the perpendicular from $A$ to $ij$, and let $C$ be the vertex $i$. In the dual orthoscheme $C^*B^*A^*$ we have $A^*B^* = h_{ijk}$ and $B^*C^* = h_{ij}$. To prove the equations (3.16) and (3.17), note that $\angle OA^*B^* = \angle OBA = \alpha_{ij}$ and $\angle OB^*C^* = \angle OCB = \rho_{ij}$. □

Substitution of (3.17) into (3.8) yields

$$\ell^*_{ij} = \frac{\cot\alpha_{ij} + \cot\alpha_{ji}}{r_i \sin\rho_{ij}}, \tag{3.18}$$

which is non-negative because $\alpha_{ij} + \alpha_{ji} = \theta_{ij} \leqslant \pi$. This shows that $P^*$ is indeed a generalized convex polyhedron and justifies Definition 3.8.

*Remark.* An orthoscheme can degenerate so that some of the points $O$, $A$, $B$, $C$ coincide. A formal approach that includes this possibility is to define an orthoscheme as an orthonormal basis and a triple of real parameters. Then degenerations correspond to the vanishing of some of the parameters. If the first parameter vanishes, then so do the others, and the dual orthoscheme is infinitely large. This does not happen when we dualize the orthoschemes in Lemma 3.10, because there the first parameter is the altitude of a pyramid which does not vanish ($O \neq A$).

THEOREM 3.11. — *The Hessian of the total scalar curvature at $P$ is equal to the Hessian of the volume function at $P^*$:*

$$\frac{\partial^2 H}{\partial r_i \partial r_j}(P) = \frac{\partial^2 \mathrm{vol}}{\partial h_i \partial h_j}(P^*) \quad \textit{for all } i, j.$$

*Equivalently, the Jacobian of the curvatures of $P$ with respect to the radii equals the Jacobian of the face areas of $P^*$ with respect to the altitudes:*

$$\frac{\partial \kappa_i}{\partial r_j}(P) = \frac{\partial F_i}{\partial h_j}(P^*) \quad \textit{for all } i, j. \tag{3.19}$$

*Proof.* — Substitute (3.18) into (3.13) and (3.14). Apply the sine theorem to triangle $aij$, to see that the right hand sides of (3.13) and (3.14) are identical to those of (3.2) and (3.3). □

*Remarks.* A generalized simplicial cone $(T, \varphi)$ determines a spherical polyhedral metric $S$ on the sphere. As in Section 2, one may consider the set $\mathcal{Q}(S)$ of all generalized convex polyhedra with the spherical section $S$. Then a collection of altitudes $h_i$ defines the triangulation $T$ uniquely, up to the edges with zero dual edge length, $\ell^*_{ij} = 0$. The coordinates $(h_i)$ turn $\mathcal{Q}(S)$ into a polyhedron in $\mathbb{R}^n$. The cell $\mathcal{Q}^T(S)$ that corresponds to a triangulation $T$ is also a polyhedron and was denoted by $\mathcal{Q}(T, \varphi)$ in Subsection 3.2.

The polytope $P$ in Theorem 3.11 has a fixed boundary $M$ and depends only on $r$; similarly, the polyhedron $P^*$ has a fixed spherical section $S$ and depends on $h$. Note that the duality map $P \mapsto P^*$ maps the polytopes with the same boundary to polyhedra with different spherical sections.

The function vol is of class $C^2$ on $\mathcal{Q}(S)$, because the formulas of Proposition 3.5 agree at the common boundary points of chambers $\mathcal{Q}^{T_1}(S)$ and $\mathcal{Q}^{T_2}(S)$.

In the classical situation we have the space of polyhedra with given outer normals $v_i$ to the faces. Minkowski's theorem [20] says that the condition $\sum F_i v_i = 0$ is necessary and sufficient for the existence of a polyhedron with the given face normals and face areas $F_i$. The classical proof of Minkowski's theorem [27, Section 7.1] is based on the properties of the Hessian of the volume. Because our proof of Alexandrov's theorem uses the non-degeneracy of the Hessian of the total scalar curvature, Theorem 3.11 provides a connection between Minkowski's and Alexandrov's theorems.

### 3.4. Mixed volumes of generalized convex polyhedra

Theorem 3.11 implies that in order to prove the non-degeneracy of $\left(\frac{\partial \kappa_i}{\partial r_j}\right)$, it suffices to prove the non-degeneracy of $\left(\frac{\partial^2 \mathrm{vol}}{\partial h_i \partial h_j}\right)$. For classical convex polyhedra the signature of this Hessian is known. This information is contained in the Alexandrov-Fenchel inequalities [1], and more explicitely in the lemmas used to prove them, see [27, Section 6.3, Propositions 3 and 4].

Any generalized convex polyhedron has curvatures $\kappa_i$, defined as the curvatures of its spherical section.

Theorem 3.12. — *Let $Q$ be a generalized convex polyhedron such that*

$$\kappa_i > 0,\ h_i > 0,\ F_i > 0 \quad \textit{for all } i,$$

*where $\kappa_i$ is the curvature, $h_i$ is the altitude, and $F_i$ is the area of the face $i$. Then the Hessian of the volume of $Q$ is non-degenerate:*

$$\det \left( \frac{\partial^2 \mathrm{vol}}{\partial h_i \partial h_j} \right) \neq 0.$$

We will prove this theorem using the notion of mixed volumes. Let us introduce some notation. For any $x \in \mathbb{R}^{\mathcal{V}(T)}$ we denote by $x_{ij}$ and $x_{ijk}$ the linear functions of $x$ obtained by substituting $x$ for $h$ in formulas (3.6) and (3.7), respectively. In the same way we define functions $\ell^*_{ij}(x)$, $F_i(x)$, and $\mathrm{vol}(x)$. Furthermore, let $\overline{F_i}$ be the following function on $\mathbb{R}^{\mathrm{lk}\ i}$. The point with coordinates $(x_{ij})_{j\in \mathrm{lk}\ i}$ in $\mathbb{R}^{\mathrm{lk}\ i}$ is mapped to $\frac{1}{2}\sum_j x_{ij}\ell^*_{ij}$, where $\ell^*_{ij}$ is again computed via (3.8). In other words,

$$F_i = \overline{F_i} \circ \pi_i,$$

where $\pi_i : \mathbb{R}^{\mathcal{V}(T)} \to \mathbb{R}^{\mathrm{lk}\ i}$ maps $(x_i)$ to $(x_{ij})$.

It is immediate that all $F_i$ and $\overline{F_i}$ are quadratic forms and vol is a cubic form.

Definition 3.13. — *The* mixed areas $F_i(\cdot,\cdot)$ *and* $\overline{F_i}(\cdot,\cdot)$ *are the symmetric bilinear forms associated to the quadratic forms* $F_i$ *and* $\overline{F_i}$*. The* mixed volume $\mathrm{vol}(\cdot,\cdot,\cdot)$ *is the symmetric trilinear form associated to the cubic form* vol.

Proposition 3.14. — *The mixed areas and the mixed volume can be computed by the formulas*

$$F_i(x,y) = \frac{1}{2}\sum_j x_{ij}\ell^*_{ij}(y), \tag{3.20}$$

$$\mathrm{vol}(x,y,z) = \frac{1}{3}\sum_i x_i F_i(y,z). \tag{3.21}$$

*Proof.* — It suffices to prove (3.20) for the bilinear form $\overline{F_i}$. For any symmetric bilinear form $B$ we have:

$$B(x,y) = \frac{1}{2}\sum_i x_i \frac{\partial B(y,y)}{\partial y_i}.$$

Thus (3.20) follows from Lemma 3.6.

A similar argument shows that

$$\mathrm{vol}(x,y,y) = \frac{1}{3}\sum_i x_i F_i(y,y).$$

Using $2\,\mathrm{vol}(x,y,z) = \mathrm{vol}(x,y+z,y+z) - \mathrm{vol}(x,y,y) - \mathrm{vol}(x,z,z)$ we get (3.21). □

Lemma 3.15. — *If* $\kappa_i > 0$*, then the form* $\overline{F_i}$ *is non-degenerate and has exactly one positive eigenvalue, which is simple.*

*Proof.* — First let us prove the non-degeneracy. Assume that for a vector $y \in \mathbb{R}^{\mathrm{lk}\ i}$ we have $\overline{F_i}(x,y) = 0$ for any $x$. Then from (3.20) we have $\ell^*_{ij}(y) = 0$ for every $j$. We need to show that this implies $y = 0$.

Let us consider the situation geometrically. We have a generalized polygon with angles $\omega_{ijk}$ between the normals to consecutive sides. The numbers $y_{ij}$ are the altitudes to the sides. We can develop the boundary of the polygon into the plane so that it becomes a polygonal line. The rotation by the angle $\omega_i = \sum_{jk} \omega_{ijk}$ sends the starting point of the line to its endpoint. The equalities $\ell^*_{ij}(y) = 0$ mean that all of the segments of the line have zero length. Thus the line degenerates to a point invariant under rotation. Because $\omega_i = 2\pi - \kappa_i < 2\pi$, this point can be only the origin. Thus $y_{ij} = 0$ for every $j$.

To determine the signature, deform the collection of angles $(\omega_{ijk})_{jk}$ continuously so that they become all equal. If their sum remains less than $2\pi$ during the process of deformation, then the form $\overline{F_i}$ remains non-degenerate and it suffices to compute its signature at the final point. This can be done explicitely by extracting the coefficients from (3.7) and by computing the eigenvalues.

Note that in the classical situation $\omega_i = 2\pi$, and the polygon can degenerate to any point. This means that the form $\overline{F_i}$ has a 2-dimensional nullspace. The form $\overline{F_i}$ is non-degenerate if and only if $\omega_i$ is not an integer multiple of $2\pi$. The dimension of the positive subspace is $2k-1$, for $\omega_i \in (2(k-1)\pi, 2k\pi)$. $\square$

*Proof. — of Theorem 3.12.* We have

$$\mathrm{vol}(h+x) = \mathrm{vol}(h) + 3\mathrm{vol}(h,h,x) + 3\mathrm{vol}(h,x,x) + \mathrm{vol}(x).$$

We have to show that under the assumption that $\kappa_i > 0$ for all $i$, the quadratic form $\mathrm{vol}(h,\cdot,\cdot)$ is non-degenerate for any $h$ that satisfies $h_i > 0$, $F_i(h) > 0$ for all $i$.

Suppose this is not the case. Then there exists a non-zero vector $x$ such that $\mathrm{vol}(h,x,y) = 0$ for any $y$. Since $\mathrm{vol}(h,x,y) = \mathrm{vol}(y,h,x) = \frac{1}{3}\sum_i y_i F_i(h,x)$, this implies $F_i(h,x) = 0$ for every $i$. Thus $x$ is orthogonal to $h$ with respect to the form $F_i$. Because $F_i(h,h) > 0$ by assumption, Lemma 3.15 implies that $F_i(x,x) \leqslant 0$. Besides, if $F_i(x,x) = 0$, then $x_{ij} = 0$ for every $j \in \mathrm{lk}\ i$.

Consider the mixed volume $\mathrm{vol}(h,x,x)$. On the one hand we have

$$\mathrm{vol}(h,x,x) = \frac{1}{3}\sum_i x_i F_i(h,x) = 0.$$

On the other hand

$$\mathrm{vol}(h,x,x) = \frac{1}{3}\sum_i h_i F_i(x,x) \leqslant 0,$$

because $h_i > 0$ for every $i$. Hence $F_i(x,x) = 0$ for every $i$, and $x_{ij} = 0$ for every edge $ij$. Note that for every $i \in \mathcal{V}(T)$ we have $\mathrm{lk}\ i \neq \emptyset$, otherwise $F_i(h) = 0$. By equation (3.6), $x_{ij} = 0 = x_{ji}$ together with $\varphi_{ij} \in (0,\pi)$ implies $x_i = 0$. This contradicts the assumption $x \neq 0$. $\square$

Let $Q = P^*$ be the dual to a generalized convex polytope $P$. Then $h_i = \frac{1}{r_i} > 0$ for every $i$. Also, $P$ and $P^*$ have identical spherical sections, thus their curvatures are equal. However, there are examples of positively curved polytopes $P$ where $F_i(P^*) < 0$ for some $i$. The following proposition describes a class of polytopes with duals that satisfy the assumptions of Theorem 3.12.

Proposition 3.16. — *Let $P$ be a generalized convex polytope with boundary $M$ and suppose that*

$$0 < \kappa_i < \delta_i \quad \textit{for all } i,$$

*where $\delta_i$ is the angle defect of the $i$-th singularity on $M$. Then we have*

$$F_i > 0 \quad \textit{for all } i$$

*in the dual polyhedron $P^*$.*

*Proof.* — The $i$-th face $P_i^*$ of $P^*$ is a generalized convex polyhedron of dimension 2. It is defined by the angles $(\omega_{ijk})_{jk}$ between the normals to the sides and the altitudes $(h_{ij})_j$ of the sides. Formulas (3.7), (3.8), (3.10) allow us to compute the side lengths and the area of $P_i^*$. Alternatively, one could construct 2-dimensional orthoschemes (right triangles) with legs $h_{ij}, h_{ijk}$, and glue them along common sides.

The Euclidean generalized polyhedron $P_i^*$ has a spherical counterpart $SP_i^*$. Take a sphere of radius $\frac{1}{r_i}$ and place one of the orthoschemes that form $P_i^*$ in the plane tangent to the north pole so that the base point of the orthoscheme is the north pole. Project the orthoscheme to the northern hemisphere using the sphere center as the center of projection. The image is a spherical right triangle which we also call an orthoscheme, see Figure 3.4. The polyhedron $SP_i^*$ is defined as the complex of thus constructed spherical orthoschemes. Because of (3.16), the first parameter of the spherical orthoscheme equals $\frac{\pi}{2} - \rho_{ij}$. Therefore $SP_i^*$ is defined by the angles $\omega_{ijk}$ between the normals to the sides and the altitudes $\frac{\pi}{2} - \rho_{ij}$.

We claim that

$$\operatorname{area}(SP_i^*) = \delta_i - \kappa_i. \tag{3.22}$$

By Lemma 3.17 together with the assumption $\delta_i > \kappa_i$ this implies $F_i > 0$.

Let $SP_i$ be the spherical section of $P$ at the vertex $i$. It consists of the spherical triangles with two sides of length $\rho_{ij}, \rho_{ik}$ and angle $\omega_{ijk}$ between them. Let $\gamma_{ijk}$ be the length of the third side of this triangle. The spherical generalized 2-polytope $SP_i$ has $SP_i^*$ as its spherical dual, see Figure 3.5. Consider the pair of orthoschemes $(ij, ijk)$ and $(ik, ikj)$ in $SP_i^*$. Together

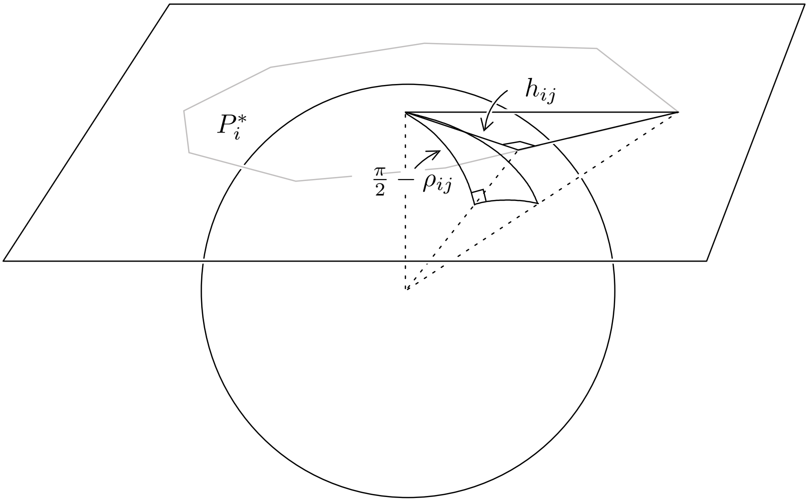


*Figure 3.4. To the definition of $SP_i^*$.*

they form a quadrilateral with two right opposite angles. The angle of this quadrilateral at the center of the polyhedron equals $\omega_{ijk}$, whereas the fourth angle is $\pi - \gamma_{ijk}$. Therefore the area of the quadrilateral is $\omega_{ijk} - \gamma_{ijk}$. This remains true when some parameters of the orthoschemes are negative, if one considers signed angles and signed areas. As a result, we have

$$\operatorname{area}(SP_i^*) = \sum_{jk} (\omega_{ijk} - \gamma_{ijk}) = \omega_i - \operatorname{per}(SP_i).$$

Since $\kappa_i = 2\pi - \omega_i$ and $\delta_i = 2\pi - \operatorname{per}(SP_i)$, this implies the equality (3.22). $\square$

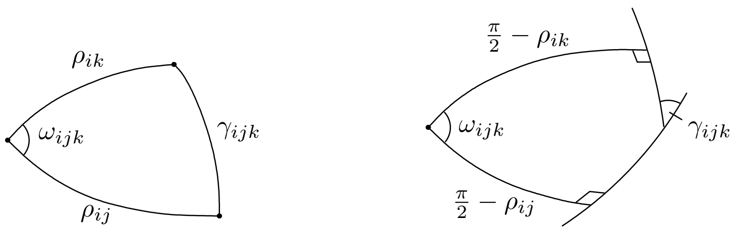


*Figure 3.5. The spherical generalized polygons $SP_i$ and $SP_i^*$.*

For a 2-dimensional generalized convex polyhedron $Q$ define its *spherical image* $SQ$ exactly as we defined $SP_i^*$ in the proof of Proposition 3.16, but taking a unit sphere instead, see Figure 3.4.

LEMMA 3.17. — *Let $C$ be a positively curved generalized convex polyhedron of dimension 2, and let $SC$ be its spherical image. Suppose that* $\operatorname{area}(SC) > 0$*. Then also* $\operatorname{area}(C) > 0$.

*Proof.* — If all of the altitudes of $C$ are negative, then so are the altitudes of $SC$, which leads to $\operatorname{area}(SC) < 0$.

Let $h_i$ be a positive altitude of $C$. We have $h_{ij} + h_{ik} = \ell_i^* \geqslant 0$, where $\ell_i^*$ is the length of the $i$-th side of $C$, and $h_{ij}, h_{ik}$ are the signed lengths of the segments into which the side is divided by the foot of the altitude. Without loss of generality $h_{ij} \geqslant 0$. Cut $C$ along the segment joining the vertex $ij$ to the apex. Develop the result into the plane. The two sides of the cut become the equal sides of an isosceles triangle $\Delta$, and the boundary of $C$ developes to a convex polygonal line joining the two base vertices of $\Delta$. It can be shown that, when extended by the base of $\Delta$, the line remains convex. Let $D$ be the convex polygon bounded by it. If $\kappa \geqslant \pi$, then $\Delta$ and $D$ lie on different sides from their common edge, and one easily sees that all of the altitudes of $C$ are positive. This implies $\operatorname{area}(C) > 0$. If $\kappa < \pi$, then we have

$$\operatorname{area}(C) = \operatorname{area}(D) - \operatorname{area}(\Delta),$$

and, if $J$ denotes the Jacobian of the projection to the sphere,

$$\operatorname{area}(SC) = \int_D J d\text{area} - \int_\Delta J d\text{area}.$$

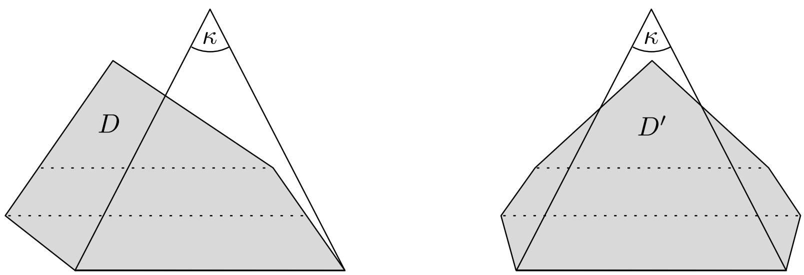


*Figure 3.6. Triangle $\Delta$, polygon $D$ and its Steiner symmetral $D'$.*

To show that $\operatorname{area}(SC) > 0$ implies $\operatorname{area}(C) > 0$, replace $D$ by its Steiner symmetral $D'$ with respect to the line perpendicular to the base of $\Delta$. This means that $D$ is decomposed into segments parallel to the base and every segment is translated so that its midpoint lies on the altitude of $\Delta$. The polygon $D'$ is symmetric with respect to the altitude of $\Delta$ and has the same area as $D$. On the other hand, it is not hard to see that $\int_{D'} J d\text{area} \geqslant \int_D J d\text{area}$. This implies

$$\int_{D'} J d\text{area} > \int_{\Delta} J d\text{area}.$$

It is easy to see that for any $x \in D' \setminus \Delta$ and for any $y \in \Delta \setminus D'$ we have $J(x) \leqslant J(y)$. It follows that $\text{area}(D') > \text{area}(\Delta)$ and thus $\text{area}(C) > 0$. $\square$

Corollary 3.18. — *Let $P$ be a generalized convex polytope with radii $(r_i)$ and curvatures $(\kappa_i)$. Suppose that*

$$0 < \kappa_i < \delta_i \quad \textit{for all } i,$$

*where $\delta_i$ is the angle defect of the $i$-th singularity on the boundary of $P$. Then the matrix $\left(\frac{\partial \kappa_i}{\partial r_j}\right)$ is non-degenerate.*

*Proof.* — This follows from Proposition 3.16, Theorem 3.12, and from the fact that the Jacobian of the map $r \mapsto \kappa$ equals the Hessian of $H$, see (3.1). $\square$

Now we will study the matrix $\left(\frac{\partial \kappa_i}{\partial r_j}\right)$ at $\kappa_i = 0$ for all $i$, that is, in the case when the generalized convex polytope is an ordinary convex polytope. As a corollary we will get a proof of the infinitesimal rigidity of polytopes.

Proposition 3.19. — *Let $P$ be a generalized convex polytope with all curvatures $0$. Then the Jacobian $\left(\frac{\partial \kappa_i}{\partial r_j}\right)$ has corank $3$. A vector $dr = (dr_i)$ lies in the kernel of $\left(\frac{\partial \kappa_i}{\partial r_j}\right)$ if and only if there exists an $x \in \mathbb{R}^3$ such that*

$$dr_i = \left\langle \frac{p_i - a}{\|p_i - a\|}, x \right\rangle \quad \textit{for all } i. \tag{3.23}$$

*Here $p_i$ are the vertices of $P$, and $a$ is the apex, for some realization of $P$ as a convex polytope in $\mathbb{R}^3$.*

*Proof.* — Use the equality (3.19). Here $P^*$ is the convex polytope in $\mathbb{R}^3$ polar to $P$. It is known that the matrix $\left(\frac{\partial F_i}{\partial h_j}\right)$ has corank 3, see [27, Section 6.3, Proposition 3]. Hence the Jacobian $\left(\frac{\partial \kappa_i}{\partial r_j}\right)$ also has corank 3.

Now let us find vectors in the kernel. Let $x \in \mathbb{R}^3$ be any vector. Consider the family $\{P_x(t) | t \in (-\varepsilon, \varepsilon)\}$ of generalized polytopes with zero curvatures, where $P_x(t) \subset \mathbb{R}^3$ has vertices $p_i$ and the apex $a - tx$. From $r_i = \|p_i - a\|$ it is easy to get

$$\left.\frac{dr_i}{dt}\right|_{t=0} = \left\langle \frac{p_i - a}{\|p_i - a\|}, x \right\rangle.$$

Therefore any infinitesimal deformation given by (3.23) results in $d\kappa_i = 0$ for all $i$. ☐

Corollary 3.20. — *Convex polytopes are infinitesimally rigid.*

*Proof.* — Let $P \subset \mathbb{R}^3$ be a convex polytope with vertices $(p_i)$. Triangulate its non-triangular faces by diagonals. Let $(dp_i)$ be infinitesimal motions of the vertices that result in a zero first order variation of the edge lengths. Take any point $a$ in the interior of $P$ and cut $P$ into pyramids with the apex $a$. The deformation changes the dihedral angles of the pyramids, but the total angles $\omega_i$ around the radial edges remain equal to $2\pi$. This means that the vector $(dr_i = d\|p_i - a\|)$ belongs to the kernel of the Jacobian $\left(\frac{\partial \omega_i}{\partial r_j}\right)$. Recall that $\kappa_i = 2\pi - \omega_i$. By Proposition 3.19 we have (3.23). It follows that $(dp_i)$ is the restriction of the differential of a rigid motion of $P$. ☐

*Remarks.* The infinitesimal rigidity of convex polytopes was proved by Dehn [9]. Recently different new proofs were suggested by Filliman [13], Schlenker [26] and Pak [22]. The proof of Filliman [13] is very similar to ours and is formulated in the language of stresses on frameworks. Schlenker [26] studies infinitesimal isometric deformations of spherical and Euclidean polygons and the corresponding first-order variations of their angles. He introduces a quadratic form on the space of infinitesimal deformations and proves a remarkable positivity property for it.

The infinitesimal rigidity of smooth convex surfaces is an older result due to Liebmann [18]. A proof that uses theory of mixed volumes was suggested by Blaschke [5].

It is natural to ask what is the signature of the Hessian of the total scalar curvature (equivalently, of the volume of the dual) for $0 < \kappa_i < \delta_i$. In the case $\kappa = 0$ it is known [27, Section 6.3] that the corank is 3 and there is a unique positive eigenvalue. For generalized polyhedra the argument from [27] cannot be applied since it uses Minkowski's inequalities for mixed volumes. However, we are able to determine the signature in an important special case. Let $P(R)$ be the generalized polytope constructed from isosceles pyramids with edge lengts $R$ over the Delaunay triangulation of $M$, see Example 2 in Subsection 2.2. An explicit computation shows that

$$\left(\frac{\partial^2 \mathrm{vol}}{\partial h_i \partial h_j}\right)(P(R)^*) = -R^{-1}\Delta_M + o(R^{-1}),$$

where $\Delta_M$ is the discrete Laplace-Beltrami operator for the metric $M$ [7]. The operator $\Delta_M$ is known to be positively semidefinite with the one-dimensional kernel. Thus the Hessian of the volume function at $P(R)$ has at least $(n-1)$-dimensional negative subspace. On the other hand, we have

$$\sum_{i,j=1}^{n} \frac{\partial^2 \mathrm{vol}}{\partial h_i \partial h_j} h_i h_j = \sum_i h_i \sum_j \frac{\partial F_i}{\partial h_j} h_j = 2 \sum_i h_i F_i = 6\mathrm{vol},$$

where the middle equation follows from the Euler formula for the homogeneous functions $F_i$. The generalized polytope $P(R)$ satisfies $0 < \kappa_i < \delta_i$ (this will be proved later). Hence by Proposition 3.16 its dual has $F_i > 0$. Therefore $\mathrm{vol}(P(R)^*) > 0$ and the Hessian has at least one positive eigenvalue. Thus the signature of the Hessian of the total scalar curvature for the polytope $P(R)$ is $(1, n-1)$.

## 4. Proof of Alexandrov's theorem

In this section we prove the existence part of Theorem 1.1. The uniqueness can be proved by extending the classical Cauchy's argument [3, Section 3.3]. In order to establish existence, we construct a family $\{P(t)\}_{t\in[0,1]}$ of generalized convex polytopes with the boundary $M$ such that the curvatures of $P(0)$ vanish. Then $P(0)$ is the desired convex polytope.

Theorem 4.1. — *For any convex Euclidean polyhedral metric $M$ on the sphere there exists a family $\{P(t) = (T(t), r(t)) \mid t \in (0,1]\}$ of generalized convex polytopes with boundary $M$ such that*

1. *$T(1)$ is the Delaunay triangulation of $M$, and $r_i(1) = R$ for all $i$ and a sufficiently large $R$;*
2. *the curvatures $\kappa_i(t)$ of the polytope $P(t)$ are proportional to $t$:*

$$\kappa_i(t) = t \cdot \kappa_i(1) \quad \text{for all } i. \tag{4.1}$$

3. *$r(t)$ is of class $C^1$ on $(0,1]$;*
4. *there exists the limit $r(0) = \lim_{t\to 0} r(t)$;*
5. *there exists a convex polytope $P \subset \mathbb{R}^3$ with boundary isometric to $M$ and vertices $p_i$ such that $r_i(0) = \|p_i - a\|$, where $a \in P$ is the unique point that satisfies the condition*

$$\sum_i \kappa_i(1) \frac{p_i - a}{\|p_i - a\|} = 0. \tag{4.2}$$

Theorem 4.1 provides us with the following recipe for constructing a polytope with the boundary $M$. The path $P(t)_{t\in(0,1]}$ in the space $\mathcal{P}(M)$ is the preimage of the path $\kappa(t)_{t\in(0,1]}$ given by (4.1) under the map $r \mapsto \kappa$. Since a generalized convex polytope is uniquely determined by its radii, we have a system of differential equations:

$$\frac{dr}{dt} = \left(\frac{\partial \kappa_i}{\partial r_j}\right)^{-1} \kappa(1). \tag{4.3}$$

Theorem 4.1 states that the system has a solution on $(0,1]$ with the initial condition $r(1) = (R, \dots, R)$ for $R$ sufficiently large. Furthermore, $r(t)$ converges for $t \to 0$. At this point there are two possibilities: either the pyramids with edge lengths $r_i(0)$ over some triangulation $T(0)$ form a convex polytope, or all of the pyramids degenerate and their bases form a doubly-covered polygon. When solving the system (4.3) numerically, one should not forget that the formulas (3.2), (3.3) for the partial derivatives $\frac{\partial \kappa_i}{\partial r_j}$ depend on the combinatorics of the triangulation $T(t)$. During the deformation of the generalized convex polytope $P(t)$ the triangulation $T(t)$ may change. Generically, the flips happen at different moments.

### 4.1. Proof of Theorem 4.1

The proof uses the description of the space $\mathcal{P}(M)$ of generalized convex polytopes and the non-degeneracy of the Hessian of the functional $H$. In particular we refer to Proposition 2.7 and Corollary 3.18. Several lemmas needed to deal with degenerating generalized polytopes are postponed to Section 5.

In Section 2 we used an embedding of $\mathcal{P}(M)$ into $\mathbb{R}^n$ via the squares of the radii. Here we consider the radii as coordinates on $\mathcal{P}(M)$. Thus a generalized convex polytope is identified with a point $r = (r_1, \dots, r_n) \in \mathbb{R}^n$. The map

$$\begin{array}{rcl} \mathcal{P}(M) & \to & \mathbb{R}^n, \\ r & \mapsto & \kappa, \end{array} \tag{4.4}$$

is of class $C^1$ in the interior of $\mathcal{P}(M)$ by Proposition 3.2. We have to show that the path $\kappa : (0,1] \to \mathbb{R}^n$ given by (4.1) can be lifted through the map (4.4) to a path $r : (0,1] \to \mathcal{P}(M)$. The question of convergence of $r$ as $t \to 0$ will be treated later.

Lemma 4.2. — *For $R$ large enough, there is a lift of the path (4.1) on an interval $(t_0, 1]$ for some $t_0$.*

*Proof.* — By property 1 of Proposition 2.7, there is a generalized convex polytope $P(1)$ with large equal radii. Moreover, $P(1)$ lies in the interior of the space $\mathcal{P}(M)$. The corresponding triangulation $T(1)$ of $M$ is the weighted Delaunay triangulation with equal weights, that is the Delaunay triangulation.

The spherical section of $P(1)$ at the vertex $i$ is a convex star polygon, see Figure 5.1. We have $\rho_j < \frac{\pi}{2}$ for every $j$, because $\rho_j$ is an angle at the base of an isosceles triangle. Therefore by Lemma 5.3 we have $\kappa_i(1) < \delta_i$. Furthermore, as $R$ tends to infinity, every $\rho_j$ tends to $\frac{\pi}{2}$. This implies that $\kappa_i(1)$ is close to $\delta_i$ for $R$ sufficiently large; in particular $\kappa_i(1) > 0$. Thus we have

$$0 < \kappa_i(1) < \delta_i \quad \text{for all } i. \tag{4.5}$$

By Corollary 3.18, this implies that the Jacobian of (4.4) at $r(1)$ is nondegenerate. The Lemma follows from the inverse function theorem. □

LEMMA 4.3. — *If $r : (t_0, 1] \to \mathcal{P}(M)$ and $r' : (t'_0, 1] \to \mathcal{P}(M)$ are two lifts of the path (4.1), then they coincide on* $(t_0, 1] \cap (t'_0, 1]$. *The maximal interval where the lift exists is open in* $(0, 1]$.

*Proof.* — Suppose that the first statement of the lemma is not true. Let

$$t_1 = \inf\{t |\, r(t) = r'(t)\}.$$

Then we have $r(t_1) = r'(t_1)$. By (4.1) and (4.5), we have

$$0 < \kappa_i(t_1) < \delta_i \quad \text{for all } i.$$

Let us show that $r(t_1)$ is an interior point of $\mathcal{P}(M)$. If $r(t_1)$ is a boundary point of $\mathcal{P}(M)$, then by Proposition 2.7, $r(t_1)$ is also a boundary point of $\mathcal{D}(M)$. This means that one of the inequalities (2.4) or (2.5) becomes an equality after the substitution of $r^2(t_1)$ for $q$. If we have equality in (2.4), then the spherical section at the vertex $j$ contains a boundary geodesic arc of length at least $\pi$. This contradicts Lemma 5.4. If an equality in (2.5) is attained, then in the pyramid over the triangle $iji$ the vertex $j$ is the foot of the altitude. This implies $\kappa_j(t_1) = \delta_j$ which contradicts (4.5).

By Corollary 3.18 and inverse function theorem, the lift is unique in a neighborhood of $t_1$. This implies the first statement of the lemma.

The same argument shows that a lift on any closed interval $[t_1, 1] \subset (0, 1]$ can be extended to a neighborhood of $t_1$. This implies the second statement of the lemma. □

Let $(t_0, 1] \subset (0, 1]$ be the maximum interval where the lift of (4.1) exists. We have to prove $t_0 = 0$.

Lemma 4.4. — *Let* $(t_0, 1] \subset (0, 1]$ *be the maximum interval where the lift* $r : (t_0, 1] \to \mathcal{P}(M)$ *exists, and suppose* $t_0 > 0$. *Assume that there is a sequence* $(t_n)$ *in* $(t_0, 1]$ *such that* $t_n$ *converges to* $t_0$ *and* $r(t_n)$ *converges to a point* $r$ *in* $\mathcal{P}(M)$. *Then the path* $r : (t_0, 1] \to \mathcal{P}(M)$ *can be extended to* $t_0$ *in a* $C^1$ *way by letting* $r(t_0) = r$.

*Proof.* — Let $\kappa_i$ be the curvatures at the point $r = \lim_{n\to\infty} r(t_n)$. Since $\kappa_i(t_n) = t_n \cdot \kappa_i(1)$ and due to the continuity of the map (4.4), we have $\kappa_i = t_0 \cdot \kappa_i(1)$. Therefore $0 < \kappa_i < \delta_i$. As in the proof of Lemma 4.3, this implies that $r$ is an interior point of $\mathcal{P}(M)$. Furthermore, by Corollary 3.18, the Jacobian of the map (4.4) at $r$ is non-degenerate. Thus (4.4) maps a neigborhood $U$ of $r$ diffeomorphically to a neighborhood $V$ of $\kappa$. For $n$ large enough, $r(t_n)$ lies in $U$. Since the path $\kappa(t)$ is regular at $t = t_0$, the path $r(t)$ extends smoothly to a neighborhood of $t_0$. □

In order to show $t_0 = 0$ it suffices to find a sequence $(t_n)$ as in Lemma 4.4. It is not hard to satisfy the condition that $r(t_n)$ converges: one needs only boundedness of $r$ on $(t_0, 1]$ which is proved in Lemma 4.6 below. A bigger problem is to show that the limit lies in $\mathcal{P}(M)$. This is the purpose of Lemmas 4.7–4.10.

Definition 4.5. — *The spherical section of a generalized convex polytope* $P$ *is a sphere with a spherical polyhedral metric that is obtained by gluing together the sections of the pyramids that form* $P$ *with the center at the apex.*

Clearly, the curvatures of the spherical section equal the curvatures of the polytope. The area of the spherical section is

$$\operatorname{area}(S) = 4\pi - \sum_i \kappa_i. \tag{4.6}$$

For the generalized convex polytope $P(t)$ we denote its spherical section by $S(t)$. Instead of $P(t_n)$ and $S(t_n)$ we write briefly $P_n$ and $S_n$.

Lemma 4.6. — *Let* $r : (t_0, 1] \to \mathcal{P}(M)$ *be a lift of the path (4.1) through the map (4.4) for some* $t_0 \geqslant 0$. *Then* $r$ *is bounded on* $(t_0, 1]$.

*Proof.* — Assume that $r_i(t) > C$ for some $t \in (t_0, 1]$, some $i$, and some large $C$. Then the triangle inequality in the metric space $P(t)$ implies that $\operatorname{dist}(a, x) > C - \operatorname{diam}(M)$ for any $x \in M$, where $a$ is the apex of $P(t)$. We claim that

$$\operatorname{area}(S(t)) < \frac{\operatorname{area}(M)}{(C - \operatorname{diam}(M))^2}. \tag{4.7}$$

For $C$ large enough this contradicts

$$\operatorname{area}(S(t)) = 4\pi - \sum_i \kappa_i(t) > 4\pi - \sum_i \kappa_i(1) = \operatorname{area}(S(1)).$$

In order to prove (4.7), consider one of the pyramids forming $P(t)$. Denote by $\alpha$ its solid angle at the apex, and by $A$ the area of its base. By our assumption, the distance from the apex to any point of the base is larger than $C - \operatorname{diam}(M)$. It follows that the Jacobian of the central projection of the base to the unit sphere centered at the apex is less than $(C - \operatorname{diam}(M))^{-2}$. Thus we have $\alpha < A(C - \operatorname{diam}(M))^{-2}$. This implies the inequality (4.7), and the lemma is proved. $\square$

It follows that there is a sequence $(t_n)$ such that $r(t_n)$ converges to some point $r \in \mathbb{R}^n$. Let $\widetilde{q_n}$ be the $Q$-concave $PQ$-function on $M$ that takes the value $r_i^2$ at the singularity $i$, for all $i$. Recall that $\widetilde{q_n}(x)$ is the squared distance from the apex of the polytope $P_n$ to the point $x \in M = \partial P_n$, and that the space $\mathcal{P}(M)$ is identified with the space of positive $Q$-concave $PQ$-functions on $M$, see Lemma 2.4. By property 5 of Proposition 2.6, $\widetilde{q_n}$ depends continuously on $r(t_n)$ in the $L^\infty$-metric. Since the space $\mathcal{D}(M)$ of all $Q$-concave $PQ$-functions is closed, this implies that the sequence $(\widetilde{q_n})$ converges uniformly to a $Q$-concave $PQ$-function $\widetilde{q}$. Function $\widetilde{q}$ takes value $r_i$ at the singularity $i$, for all $i$, and is non-negative. If $\widetilde{q}$ is positive, then the point $r$ lies in $\mathcal{P}(M)$, and we are done. Otherwise there is a point $x \in M$ such that $\widetilde{q}(x) = 0$. Geometrically this means that in the polytopes $P_n$ the apex tends to the point $x \in M = \partial P_n$. Since the decomposition $\mathcal{P}(M) = \cup_T \mathcal{P}^T(M)$ is locally finite by Proposition 2.7, we can assume, after replacing $(t_n)$ by a subsequence if necessary, that all of the polytopes $P_n$ have the same triangulation $T(t_n) = T$. Now there are three possibilities: $x$ lies in the interior of a face of $T$, or $x$ lies in the interior of an edge of $T$, or $x$ is a singularity $i$ of $M$.

Lemma 4.7. — *The point $x$ cannot be an interior point of a face of $T$.*

*Proof.* — Suppose the opposite is true. Let $\Delta \in \mathcal{F}(T)$ be the triangle that contains $x$ in the interior, and let $S\Delta_n$ be the corresponding spherical triangle in $S_n$. As $n \to \infty$, the spherical triangle $S\Delta_n$ tends to a hemisphere, and its angles tend to $\pi$. There are no self-identifications on the boundary of $S\Delta_n$, because $\kappa_i(t_n) \to \kappa_i(t_0) > 0$. Therefore the complement $S\Delta'_n$ to $S\Delta_n$ in $S_n$ is a singular spherical triangle in the sense of Definition 5.2. If $i$ is a vertex of $\Delta$, then the angle of $S\Delta'_n$ at $i$ tends to $\pi - \kappa_i(t_0)$. The side lengths of $S\Delta'_n$ also have limits as $n \to \infty$, these are the angles under which the sides of the triangle $\Delta$ are seen from the point $x$. On the other hand, the perimeter of $S\Delta'_n$ equals the perimeter of $S\Delta_n$ which tends to $2\pi$. Therefore, for sufficiently large $n$ we get a contradiction to Lemma 5.5. $\square$

Lemma 4.8. — *The point $x$ cannot be an interior point of an edge of $T$.*

*Proof.* — Assume that $x$ is an interior point of an edge $ij \in \mathcal{E}(T)$. Let $\Delta^1, \Delta^2 \in \mathcal{F}(T)$ be the triangles incident to $ij$, and let $S\Delta^1_n$, $S\Delta^2_n$ be their spherical images in the spherical section of the polytope $P_n$. Consider the spherical quadrilateral $S\Diamond_n = S\Delta^1_n \cup S\Delta^2_n$. As $n$ goes to $\infty$, the length of the diagonal $ij$ in $S\Diamond_n$ tends to $\pi$. At the same time, the side lengths of $S\Diamond_n$ have limits different from 0 and $\pi$, see Figure 4.1. The angles of $S\Diamond_n$ opposite to the diagonal $ij$ tend to $\pi$. Let $\alpha_n$ and $\beta_n$ be the angles at the vertices $i$ and $j$ in $S\Diamond_n$. We have

$$\lim(\alpha_n + \theta_{ij}(n)) = \lim(\beta_n + \theta_{ij}(n)) = 2\pi,$$

but the angle $\theta_{ij}(n)$ need not have the limit. However, $\theta_{ij}(n) \leqslant \pi$ implies

$$\liminf \alpha_n \geqslant \pi \quad \text{and} \quad \liminf \beta_n \geqslant \pi.$$

It follows that there are no self-identifications on the boundary of $S\Diamond_n$, thus its complement $S\Diamond'_n$ is a singular spherical quadrilateral. The perimeter of $S\Diamond'_n$ tends to $2\pi$, whereas its side lengths have limits different from 0 and $\pi$. Also, due to

$$\lim_{n\to\infty} \kappa_k(t_n) = t_0 \cdot \kappa_k(1) > 0 \quad \text{for all } k,$$

the angles of $S\Diamond'_n$ have upper limits less than $\pi$. This contradicts Lemma 5.5. $\square$

The most delicate case is when $x$ is a singularity of $M$. To deal with it we need the hard Lemma 5.6.

Lemma 4.9. — *Function $\widetilde{q}$ cannot vanish at more than one vertex of $T$.*

*Proof.* — Assume $\widetilde{q}(i) = 0$, that is $r_i = 0$. Consider the star $\mathrm{st}i$ of the vertex $i$ in the triangulation $T$ and its spherical image $S(\mathrm{st}i)_n$ in $S_n$, see Figure 4.2. Here all of the angles and lengths depend on $n$. Let

$$\eta_j(n) = \omega_{jki}(n) + \omega_{jli}(n)$$

be the angle at the vertex $j$ in $S(\mathrm{st}i)_n$. We have

$$\mathrm{area}(S(\mathrm{st}i)_n) = 2\pi - \kappa_i(t_n) - \sum_{j\in\mathrm{lk}\ i} (\pi - \eta_j(n)).$$

By convexity of $P_n$,

$$\liminf_{n\to\infty} \eta_j(n) \geqslant \pi. \tag{4.8}$$

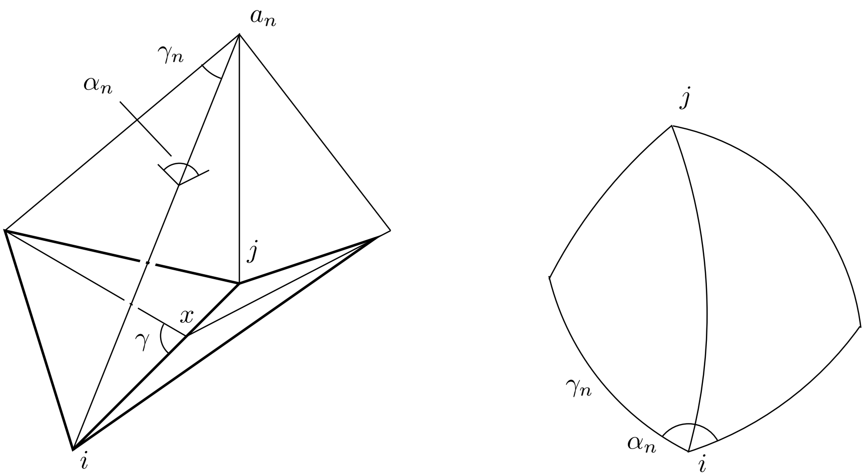


*Figure 4.1. As the apex $a_n$ tends to the point $x$, the angle $\gamma_n$ tends to $\gamma$. The angles opposite to the diagonal $ij$ in the spherical quadrilateral $S\Diamond_n$ tend to $\pi$. The angle $\alpha_n$ may have no limit, since we have no control over the dihedral angles $\theta_{ij}(n)$.*

Therefore we have

$$\liminf_{n\to\infty} \operatorname{area}(S(\mathrm{st}i)_n) \geqslant 2\pi - t_0 \cdot \kappa_i(1).$$

On the other hand, by (4.6)

$$\lim_{n\to\infty} \operatorname{area}(S_n) = 4\pi - t_0 \cdot \sum_i \kappa_i(1).$$

Thus by the first part of Lemma 5.6 we have

$$\liminf_{n\to\infty} \operatorname{area}(S(\mathrm{st}i)_n) > \frac{1}{2} \lim_{n\to\infty} \operatorname{area}(S_n). \tag{4.9}$$

Now, if there is another singularity $m$ such that $r_m \to 0$, then the stars of $i$ and $m$ have disjoint interiors. On the other hand, by (4.9) both $S(\mathrm{st}i)_n$ and $S(\mathrm{st}m)_n$ make at least a half of $S_n$, for $n$ large enough. This contradiction proves the lemma. □

LEMMA 4.10. — *Function $\widetilde{q}$ cannot vanish at a vertex of $T$.*

*Proof.* — It is easy to see that

$$\lim_{n\to\infty} \varphi_{jk}(n) = \gamma_{ijk}, \tag{4.10}$$

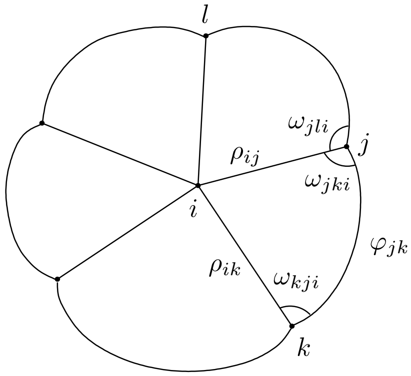


*Figure 4.2. The spherical image of the star of vertex $i$ as $r_i$ tends to* 0.

where $\gamma_{ijk}$ is the angle at the vertex $i$ in the Euclidean triangle $ijk \in \mathcal{F}(T)$. Consider the angles $\omega_{kji}(n), \omega_{jki}(n), \omega_{jli}(n)$ and so on, and call them the *base angles* of the star. The base angles can behave badly. After choosing a subsequence if necessary, we may assume that there exist limits

$$\lim_{n\to\infty} \omega_{jki}(n) = \omega_{jki} \in [0, \pi] \quad \text{etc.}$$

*Case 1.* The star of $i$ does not degenerate, that is, the limits of all of the base angles are in $(0, \pi)$. In this case the sequence of spherical sections $(S_n)$ converges to a spherical polyhedral metric that satisfies all of the assumptions of Lemma 5.6. Here the distinguished singularity 0 is the vertex $i$, and the star polygon $C$ is the limit of the stars $S(\mathrm{st}i)_n$. Angles of $C$ are greater or equal $\pi$ because of the convexity of the polytopes $P_n$. Note that the curvatures of the resulting polyhedral metric are equal to $t_0 \cdot \kappa_i(1)$ for all $i \in \mathcal{V}(T)$. Thus by (5.4) we have

$$\kappa_i(1) \geqslant \left(1 - \frac{\mathrm{per}(C)}{2\pi}\right) \sum_{j\neq i} \kappa_j(1). \tag{4.11}$$

Because of (4.10) and (4.5) we have

$$\mathrm{per}(C) = 2\pi - \delta_i < 2\pi - \kappa_i(1).$$

Therefore (4.11) implies

$$\sum_{j\neq i} \kappa_j(1) < 2\pi. \tag{4.12}$$

On the other hand, we have

$$\sum_{j\neq i} \delta_j > 2\pi,$$

since $\sum_{j\in\Sigma} \delta_j = 4\pi$ and $\delta_i < 2\pi$. But if the number $R$ was chosen sufficiently large, then $\kappa_j(1)$ are close to $\delta_j$ (see the proof of Lemma 4.2) and also satisfy $\sum_{j\neq i} \kappa_j(1) > 2\pi$. This contradicts equation (4.12), and Case 1 is ruled out.

Let us show that if one of the base angles tends to 0 or $\pi$, then we may assume without loss of generality that

$$\omega_{kji} = \pi, \quad \omega_{jki} > 0. \tag{4.13}$$

Suppose one of the base angles tends to $\pi$, say $\omega_{kji} = \pi$. If (4.13) fails, then $\omega_{jki} = 0$. Then (4.8) implies $\omega_{jli} = \pi$. If $\omega_{lji} > 0$, then after relabeling we have (4.13). So again $\omega_{lji} = 0$. Proceeding in this manner we see that all of the base angles tend to 0 or $\pi$ depending on their orientation. But this implies

$$\rho_{ik}(n) < \rho_{ij}(n) < \rho_{il}(n) < \cdots < \rho_{ik}(n)$$

for a sufficiently large $n$. We obtain a similar chain of inequalities if we assume that one of the base angles tends to 0, and (4.13) fails for any edge $jk \in \mathrm{st}\ i$. This contradiction shows that we may assume (4.13).

*Case 2.* We have $\omega_{kji} = \pi, \omega_{jki} \in (0, \pi)$. This implies

$$\lim_{n\to\infty} \rho_{ij}(n) = \pi, \quad \lim_{n\to\infty} \omega_{lji} = \pi.$$

To the spherical quadrilateral $ikjl$ the same argument as in the proof of Lemma 4.8 can be applied. This leads to a contradiction.

*Case 3.* We have $\omega_{kji} = \omega_{jki} = \pi$. We may assume that the limits

$$\lim_{n\to\infty} \rho_{ij}(n) = \rho_{ij}, \quad \lim_{n\to\infty} \rho_{ik}(n) = \rho_{ik}$$

exist. If both are less than $\pi$, then we apply the argument from Lemma 4.7 to the spherical triangle $ijk$. If $\rho_{ij} = \pi$, then both $\rho_{ik}$ and $\rho_{il}$ are less than $\pi$. In this case we consider the spherical quadrilateral $ikjl$ and apply the argument of Lemma 4.8 to it. □

We have proved that $t_0 = 0$ for the maximum lift $r : (t_0, 1] \to \mathcal{P}(M)$. Thus the existence of a family of generalized convex polytopes $\{P(t)|\ t \in (0, 1]\}$ that satisfies the properties 1.–3. from Theorem 4.1 is established. It remains to show that $P(t)$ converges to a convex polytope $P$ with a marked point $a$ (property 5 in Theorem 4.1).

Lemma 4.11. — *Let $(t_n)$ be a sequence in $(0,1]$ with $\lim_{n\to\infty} t_n = 0$ such that there exist the limits*

$$r_i(0) = \lim_{n\to\infty} r_i(t_n) \quad \text{for all } i. \tag{4.14}$$

*Then there exists a convex polytope $P \subset \mathbb{R}^3$ with boundary isometric to $M$ and a point $a \in P$ such that $r_i(0) = \|p_i - a\|$, where $p_i$ are the vertices of $P$.*

*Proof.* — The lemma is obvious if $r(0) \in \mathcal{P}(M)$. Then $r(0)$ defines a generalized polytope $P(0)$ with curvatures

$$\kappa_i(0) = \lim_{n\to\infty} \kappa_i(t_n) = 0.$$

Therefore $P(0)$ is isometric to a convex polytope $P \subset \mathbb{R}^3$ with a marked interior point $a$.

In the general case the following argument works. We may assume that all triangulations $T(t_n)$ are equal to some triangulation $T$. For every edge $ij \in \mathcal{E}(T)$ consider the dihedral angle $\theta_{ij}(t)$ of the polytope $P(t)$ at $ij$. The angles $\theta_{ij}(t)$ vary in $(0,\pi]$, so after choosing a subsequence if necessary, we have the limits

$$\lim_{n\to\infty} \theta_{ij}(t_n) = \theta_{ij} \in [0,\pi].$$

Now pick a face $F \in \mathcal{F}(T)$ and place it in $\mathbb{R}^3$. The radii $r_i(0)$ for $i \in F$ determine the position of the point $a$, up to the choice of an orientation. The dihedral angles at the edges of $F$ determine the positions of the faces adjacent to $F$. Furthermore, the condition $r_i(0) = \|p_i - a\|$ is satisfied for the new vertices. The new faces determine the positions of their neighbors, and so on. The faces fit nicely together and bound a convex polytope if they fit together around every vertex. Consider the sequence of spherical sections $S_n^i$ at the vertex $i$ in the generalized polytopes $P(t_n)$. This is a sequence of singular star polygons with the angles $\theta_{ij}(t_n)$, curvature $\kappa_i(t_n)$, and fixed edge lengths. It is not hard to prove that $S_n^i$ converges to a convex spherical polygon with angles $\theta_{ij}$, possibly degenerated to a doubly-covered spherical arc. This means that the faces around the vertex $i$ can be put together. □

Due to Lemma 4.6, the radii $r_i(t)$ are bounded on $(0,1]$. Therefore there is a sequence $t_n \to 0$ such that the limits (4.14) exist. By Lemma 4.11, this gives a convex polytope $P$ and a point $a \in P$. This proves the existence part of Alexandrov's theorem. However, properties 4 and 5 are still to be proved.

By the uniqueness part of Alexandrov's theorem, the polytope $P$ does not depend on the choice of a sequence $(t_n)$. It remains to prove that the point $a$ is also independent of this choice. We will show that the point $a$

satisfies the condition (4.2) and that the condition (4.2) defines the point uniquely. This will imply the properties 4 and 5 and thus will complete the proof of Theorem 4.1.

Let $S_n$ be the spherical section of the generalized polytope $P(t_n)$, see Definition 4.5.

LEMMA 4.12. — *Suppose that $S_n$ converges to a spherical section $S$ with the vertices $v_i$. Then the following equality holds:*

$$\sum_i \kappa_i(1)v_i = 0. \tag{4.15}$$

Some explanations are necessary. The spherical section $S_n$ inherits from the generalized polytope $P(t_n)$ the triangulation $T(t_n)$. We say that the sequence of spherical sections $S_n$ converges if $T(t_n) = T$ for almost all $n$, and for every triangle $\Delta \in \mathcal{F}(T)$ the angles and the side lengths of its spherical image $S\Delta_n$ converge. If $(t_n)$ is as in Lemma 4.11, and the point $a$ is in the interior of $P$, then the sequence $S_n$ converges to the spherical section $S$ of the polytope $P$ viewed as a generalized polytope with the apex $a$. If $a$ is an interior point of a face of $P$, then $S_n$ converges to a spherical section $S$ that contains one degenerate triangle with all angles equal to $\pi$. If $a$ is an interior point of an edge, then $S_n$ does not necessarily converge, but can be replaced by a converging subsequence. The limit section $S$ contains two degenerate triangles. In each of these cases the vertices of the section $S$ are the unit vectors directed from $a$ to $p_i$:

$$v_i = \frac{p_i - a}{\|p_i - a\|}.$$

Finally, suppose $a = p_i$. Then $S_n$ contains a converging subsequence. Its limit $S$ may contain degenerate triangles in the star of $i$. However, $S$ has all curvatures zero and defines a unit vector $v_i$, "the direction from $a$ to $p_i$".

*Proof.* — In the spherical section $S_n$ mark one of the singularities by 1 and join it by geodesic arcs to all of the other singularities. The geodesics will not pass through singularities since they have positive curvature. Mark the other singularities by $2, \dots, m$ in the clockwise order around 1. Cut $S_n$ along the geodesic arcs and develop the result onto the unit sphere $\mathbb{S}^2$. Denote by $v_i(n)$ the images of the singularities $i, 2 \leqslant i \leqslant m$, and by $v_1(n)$ the image of 1 that lies between $m$ and 2. See Fig. 4.3, where we have omitted the index $n$. Clearly we can achieve $\lim_{n\to\infty} v_i(n) = v_i$ by suitable rotations of $\mathbb{S}^2$.

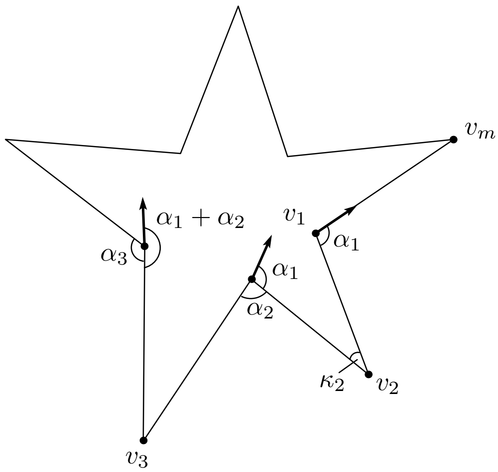


*Figure 4.3. Spherical section $S_n$ developed onto the unit sphere. The development is the exterior of the star.*

Let $R_i \in SO(3)$ be the rotation around the vector $v_i(n)$ by the angle $\kappa_i(t_n)$. We claim that

$$R_m \circ \cdots \circ R_2 \circ R_1 = \mathrm{id}. \tag{4.16}$$

Indeed, from Fig. 4.3 it is clear that $R_m \circ \cdots \circ R_2(v_1(n)) = v_1(n)$. In the same figure the trajectory of a tangent vector at the point $v_1(n)$ is shown. The vector gets rotated by the angle $\alpha_1(n)+\cdots+\alpha_{m-1}(n) = \omega_1(n) = 2\pi - \kappa_1(n)$. Equation (4.16) follows.

Since $\kappa_i(t_n) = t_n \cdot \kappa_i(1)$, we have for any vector $x \in \mathbb{R}^3$

$$R_i(x) = x + t_n \cdot \kappa_i(1)[v_i(n), x] + o(t_n),$$

where $[\cdot,\cdot]$ denotes the cross product. Then (4.16) implies

$$\begin{aligned} x = R_m \circ \cdots \circ R_2 \circ R_1(x) &= x + \sum_{i=1}^m t_n \cdot \kappa_i(1)[v_i(n), x] + o(t_n) \\ &= x + t_n \cdot \left[\sum_{i=1}^m \kappa_i(1) v_i(n), x\right] + o(t_n). \end{aligned}$$

It follows that

$$\sum_{i=1}^m \kappa_i(1) v_i(n) = o(1),$$

and since $v_i(n) \to v_i$, we have (4.15). $\square$

*Remark.* When $a$ is an interior point of the polytope $P$, Lemma 4.12 can be proved using Proposition 3.19 and the fact that the matrix $(\frac{\partial \kappa_i}{\partial r_j})$ is symmetric.

LEMMA 4.13. — *Let $P \subset \mathbb{R}^3$ be a convex polytope with the vertices $p_1, \ldots, p_m$. Let $\delta_i$ be the angular defect at the vertex $p_i$. Then there exists a unique point $a = a(P)$ in the interior of $P$ (in the relative interior, if $P$ is 2-dimensional) such that*

$$\sum_{i=1}^{m} \delta_i \frac{p_i - a}{\|p_i - a\|} = 0. \tag{4.17}$$

*Proof.* — Consider the function $f : P \to \mathbb{R}$,

$$f(x) = \sum_{i=1}^{m} \delta_i \cdot \|p_i - x\|.$$

It is differentiable everywhere except the vertices of $P$, and

$$\mathrm{grad} f(x) = -\sum_{i=1}^{m} \delta_i \frac{p_i - x}{\|p_i - x\|}.$$

Thus we are looking for the critical points of the function $f$. Let $a$ be the point of the global minimum of $f$:

$$f(a) = \min_{x \in P} f(x).$$

Let us show that $a$ is a (relative) interior point of $P$. Indeed, if $a$ is a boundary point of $P$ and not a vertex, then the vector $-\mathrm{grad} f$ at $a$ is directed inside $P$ (which should by understood appropriately if $P$ is 2-dimensional). Thus $a$ cannot even be a local minimum. Assume now $a = p_i$. Then it is easy to see that

$$\delta_i \geqslant \left| \sum_{j \neq i} \delta_j \frac{p_j - p_i}{\|p_j - p_i\|} \right|.$$

But this contradicts Lemma 5.7. The existence of a point $a$ that satisfies (4.17) is proved. The uniqueness follows from the concavity of the function $f$. □

If the numbers $\kappa_i(1)$ are sufficiently close to $\delta_i$, then Lemma 5.7 and hence Lemma 4.13 hold also with $\delta_i$ replaced by $\kappa_i(1)$. Due to Lemma 4.12 this implies that for any sequence $(t_n)$ such that $P(t_n)$ converges to a convex polytope $P$, the apex of $P(t_n)$ converges to the point $a \in P$ characterized by (4.2). Theorem 4.1 is proved.

## 5. Lemmas from spherical geometry

The Lemmas of this section deal with spherical polyhedral metrics on the disk or on the sphere. We consider only spherical polyhedral metrics with positive curvatures at singularities.

LEMMA 5.1. — *The angles $\alpha, \beta, \gamma$ of a spherical triangle as functions of side lengths $a, b, c$ have the following partial derivatives:*

$$\frac{\partial \alpha}{\partial a} = \frac{1}{\sin b \sin \gamma}, \tag{5.1}$$

$$\frac{\partial \alpha}{\partial b} = -\frac{\cot \gamma}{\sin b}. \tag{5.2}$$

*The angles $\alpha, \beta, \gamma$ of a Euclidean triangle as functions of side lengths $a, b, c$ have the following partial derivatives:*

$$\frac{\partial \alpha}{\partial a} = \frac{1}{b \sin \gamma},$$

$$\frac{\partial \alpha}{\partial b} = -\frac{\cot \gamma}{b}.$$

*Proof.* — Direct calculation using spherical and Euclidean cosine and sine theorems. □

As a technical tool we will use the so-called *merging of singularities*. Consider two cone points $A$ and $B$ with curvatures $\alpha$ and $\beta$ on a surface with a spherical polyhedral metric. Suppose that there is a geodesic arc joining $A$ and $B$. Such an arc always exists if all cone points have positive curvature and the boundary of the surface is convex. Cut the surface along this arc and paste in a singular digon that is formed by two copies of a triangle $ABC$ with angles $\frac{\alpha}{2}$ and $\frac{\beta}{2}$ at vertices $A$ and $B$, identified along the sides $AC$ and $BC$. The points $A$ and $B$ become regular, but there appears a new cone point $C$. Note that the curvature of $C$ is less than $\alpha + \beta$.

DEFINITION 5.2. — *A singular spherical polygon is a 2-disk equipped with a spherical polyhedral metric with piecewise geodesic boundary. It is called convex if the angles at the boundary vertices don't exceed $\pi$. A convex star polygon is a convex singular spherical polygon with a unique singularity.*

The area of a singular spherical polygon is

$$\text{area} = 2\pi - \sum_i \kappa_i - \sum_j (\pi - \theta_j), \tag{5.3}$$

where the $\kappa_i$ are the curvatures of singularities, and the $\theta_j$ are the angles of the polygon, $\pi - \theta_j$ thus being the exterior angles.

A convex star polygon can be cut into a collection of spherical triangles by a set of geodesic arcs joining the singularity to the boundary. Conversely, a convex star polygon may be defined as the result of gluing spherical triangles around a common vertex. In Fig. 5.1 we introduce notation for arc lengths and angles of a star polygon defined by a gluing procedure. We allow $\alpha_j + \beta_j = \pi$ for some $j$. We also allow a polygon to be glued from a single triangle by gluing two sides of the triangle together. Let

$$\kappa = 2\pi - \sum_j \omega_{j-1,j}$$

be the curvature of the singularity, and let

$$\delta = 2\pi - \sum_j \lambda_{j-1,j} = 2\pi - \mathrm{per},$$

where per is the perimeter.

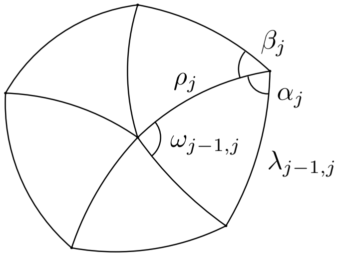


*Figure 5.1. Angles and lengths in a convex star polygon.*

Lemma 5.3. — *If in a convex star polygon $\rho_j < \frac{\pi}{2}$ holds for all $j$, then $\kappa < \delta$.*

*Proof.* — Consider the angles as functions of arc lengts. By equation (5.2),

$$\frac{\partial \kappa}{\partial \rho_j} = \frac{\cot \alpha_j + \cot \beta_j}{\sin \rho_j} \geqslant 0.$$

It is not hard to show that by increasing the $\rho_j$ and leaving boundary edges constant we can deform any polygon with $\rho_j < \frac{\pi}{2}$ into a polygon with $\rho_j = \frac{\pi}{2}$ for all $j$. If $\rho_j = \frac{\pi}{2}$ for all $j$, then $\kappa = \delta$. This implies the statement of the Lemma. □

Lemma 5.4. — *If $0 < \kappa < \delta$ for a convex star polygon $P$, then its boundary cannot contain a geodesic arc of length $\pi$.*

*Proof.* — Let us assume the opposite. Choose on the boundary of $P$ two points $A$ and $B$ that divide the boundary into two curves $L$ and $L'$ such that $L$ is geodesic and has length $|L| = \pi$. Cut $P$ into two spherical polygons $C$ and $C'$ by geodesic arcs $OA$ and $OB$, where $O$ is the singularity of $P$ and $C$ contains $L$ in its boundary. Clearly, the polygon $C$ is a spherical lune. Thus it has angle $\pi$ at the vertex $O$. Denote by $\gamma$ the angle of the polygon $C'$ at the vertex $O$. We have $\gamma = \pi - \kappa < \pi$. Thus $C'$ is convex and contains a geodesic arc $AB$. Consider the triangle $ABO$ contained in $C'$, and denote its side lengths by $a, b$, and $c$, see Figure 5.2.

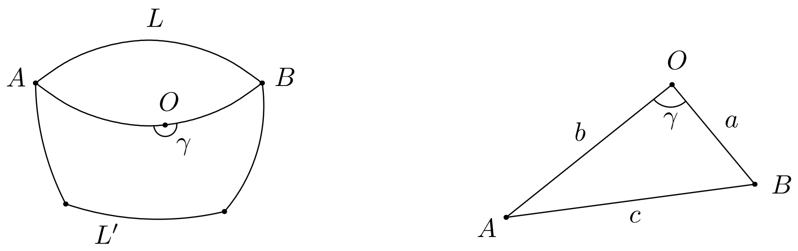


*Figure 5.2. Notation used in the proof of Lemma 5.4.*

We have

$$\kappa = \pi - \gamma,$$
$$\delta = \pi - |L'| \leqslant \pi - c.$$

Also $a + b = \pi$. By the spherical cosine theorem,

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$$
$$= -\cos^2 a + \sin^2 a \cos \gamma.$$

It follows that

$$1 + \cos c = \sin^2 a(1 + \cos \gamma) < 1 + \cos \gamma.$$

Thus $c > \gamma$ and $\kappa > \delta$ which is a contradiction. □

Lemma 5.5. — *For any $0 < c_1 < c_2 < \pi$ and $\gamma > 0$ there exists an $\varepsilon > 0$ such that the following holds. Any convex singular spherical polygon with positive singular curvatures, all side lengths between $c_1$ and $c_2$, and all exterior angles greater than $\gamma$ has perimeter less than $2\pi - \varepsilon$.*

In other words, a polygon with outer angles not too small and sides neither too large nor too small cannot have perimeter too close to $2\pi$. Note that the lemma is false if the polygon is allowed to have singularities of negative curvature.

*Proof.* — First consider the case when the polygon is non-singular. Let $a$ be the length of one of its sides. Then the polygon is contained in the spherical triangle with one side of length $a$ and both adjacent angles $\pi - \gamma$, see the left part of Figure 5.3; hence its perimeter is less than that of this triangle. For $a$ between $c_1$ and $c_2$ the perimeters of corresponding triangles are bounded from above by $2\pi - \varepsilon$ for some $\varepsilon > 0$. Thus in the non-singular case the assumptions of the Lemma can be weakened: the inequalities for one side and two adjacent angles already suffice.


*Figure 5.3. Notation used in the proof of Lemma 5.5.*

If the polygon is singular, merge all of its singularities consecutively. Let $\kappa$ be the curvature of the resulting star polygon $P$. The merging changes neither the sides nor the angles of the polygon, so it suffices to prove the Lemma for the polygon $P$. If $P$ has only one or two sides, then its perimeter is less than $2c_2$. Otherwise draw the shortest arc joining the singularity to the boundary of the polygon $P$; let $d$ be its length. We claim that $d < \frac{\pi}{2}$. Indeed, embed $P$ into the sphere with two singularities of curvature $\kappa$. This sphere is obtained from the ordinary sphere by removing a lune of angle $\kappa$ and identifying the boundary semicircles. If $d \geqslant \frac{\pi}{2}$, then the $\kappa$-sphere can be covered by two copies of $P$. Hence the area of $P$ has to be at least $2\pi - \kappa$. But it is smaller than this by formula (5.3). Thus $d < \frac{\pi}{2}$. Cut along the shortest arc and glue to the sides of the cut an isosceles triangle with the angle $\kappa$ at the vertex, see the right part of Figure 5.3. Due to $d < \frac{\pi}{2}$ the angles at its base are less than $\frac{\pi}{2}$. Thus the resulting non-singular polygon is convex. Furthermore, at least one of its sides together with the adjacent angles satisfy the assumptions of the lemma. By the first part of the proof, the perimeter of the resulting polygon is less than $2\pi - \varepsilon$. Then so is the perimeter of the initial singular polygon. □

Lemma 5.6. — *Let a spherical polyhedral metric on the sphere be given, with all of the singularities of positive curvature. Let one of the singularities be labeled by* 0*. Then*

$$\kappa_0 \leqslant \sum_{i \neq 0} \kappa_i.$$

*Suppose additionally that the singularity* 0 *is the center of a star polygon* $C$ *with vertices at some of the other singularities and all angles at least* $\pi$*. Let* $\mathrm{per}(C)$ *denote the perimeter of* $C$*. Then the following inequality holds:*

$$\kappa_0 \geqslant \left(1 - \frac{\mathrm{per}(C)}{2\pi}\right) \sum_{i \neq 0} \kappa_i. \tag{5.4}$$

*Proof.* — To prove the first inequality, merge consecutively all the singularities except 0. The result is the sphere with two singularities of curvature $\kappa_0$. Since merging decreases the total curvature, the result follows.

The proof of the second part of the Lemma is much more involved. Denote by $D$ the convex singular polygon that is the complement of the star polygon $C$. We make $D$ non-singular, increasing at the same time the right hand side of (5.4). This is done by a sequence of operations inverse to the merging of singularities. Choose a vertex of $D$, denote it by 1 and join it to a singularity $i$ in the interior of $D$ by the shortest arc $a_i$. Extend the arc $a_i$ beyond $i$ by a geodesic $b_i$ such that the two angles between $a_i$ and $b_i$ are equal. The arc $b_i$ ends either at some other singularity $j$ or on the boundary of $D$. If the latter is the case and if $D$ contains two spherical triangles with the sides $a_i$ and $b_i$, then we say that we have good luck. In this case we cut both triangles out and glue their remaining sides together, see Figure 5.4. This splitting of the singularity $i$ increases the right hand side of (5.4) because it does not change the perimeter and increases the sum $\sum_{i\neq 0} \kappa_i$. On the other hand it decreases the number of singularities inside $D$, so that we can proceed inductively. In the case of bad luck draw the arcs $a_j$ and $b_j$ for all singularities $j$ from the vertex 1. Then the arc $b_i$ crosses some $a_j$ or $b_j$. The former is not possible because then $a_j$ is not the shortest geodesic. If $b_i$ ends at the singularity $j$ without crossing any $b_k$ before this, then we cut out a pair of equal triangles with vertices $i, j$ and 1, thus decreasing the number of singularities. Finally, if $b_i$ crosses the interior $b_j$ at the point $B_{ij}$, we can assume without loss of generality that neither of the arcs $iB_{ij}$ and $jB_{ij}$ is crossed by some $b_k$. Then we can cut out two pairs of equal triangles with the common vertex $B_{ij}$ and again reduce the number of singularities.

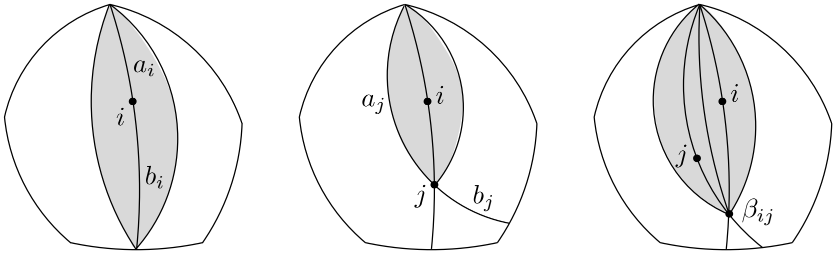


*Figure 5.4. Splitting the singularities inside the singular polygon $D$.*

Now the Lemma is reduced to the case of non-singular $D$. Embed the star polygon $C$ into the sphere with two singularities of curvature $\kappa_0$ and denote by $D'$ its complement. Denote by $\theta_i$ and $\theta'_i$ the corresponding angles of $D$ and $D'$. We have $\theta'_i - \theta_i = \kappa_i > 0$, whereas the corresponding sides of $D$ and $D'$ are equal, see Figure 5.5. Therefore the inequality (5.4) can be viewed as a comparison inequality between a usual spherical polygon $D$ and a star polygon $D'$ with both $D$ and $D'$ convex.

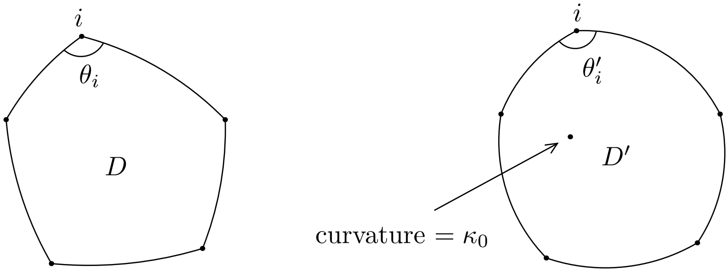


*Figure 5.5. A convex polygon $D$ and a convex singular polygon $D'$; $\theta'_i - \theta_i = \kappa_i$.*

Note that the star polygon $D'$ is completely determined by the polygon $D$ and angle differences $\kappa_i$. To see this, cut $D'$ along a geodesic joining the central singularity with any vertex and develop the result on the sphere. On the other hand, a procedure similar to the merging of singularities allows us to obtain $D'$ from $D$ as follows. Cut $D$ along a diagonal $ij$ and glue in a digon with the angles $\kappa_i$ and $\kappa_j$. The result is a star polygon with the angles $\theta'_i$ and $\theta'_j$ at the vertices $i$ and $j$, respectively. Then glue in a digon between a boundary vertex $k$ and the central singularity so that the singularity disappears and the angle at $k$ becomes $\theta'_k$. This yields a new convex star polygon, and so on.

If $\theta_i = \theta'_i$ for some $i$, then the above transformation of $D$ into $D'$ does not involve the vertex $i$ at all. If $D$ and $D'$ have more than two sides, this implies that the polygon $D'$ contains a non-singular triangle $\Delta'_i$ spanned by the sides incident to the vertex $i$. The triangle $\Delta'_i$ is congruent to the corresponding triangle $\Delta_i$ in $D$. If we cut off both triangles, then we get a new pair of polygons with the same $\kappa_0$ and $\sum_{i\neq 0}\kappa_i$ but with a smaller perimeter. Thus the inequality (5.4) for the new pair implies (5.4) for the old one.

The rest of the proof proceeds by induction on the number of sides: we carefully deform $D$ until $\theta_i$ becomes equal to $\theta'_i$ for some $i$, and cut off the triangles $\Delta$ and $\Delta'$ as in the previous paragraph. The induction base is two sides: $D$ is a doubly covered segment, $D'$ is a symmetric star digon. Let us deal with the induction step first. Suppose that $D$ has three or more sides, and $\theta'_i > \theta_i$ for all $i$. Choose a vertex $i$ and denote by $a$ and $b$ the lengths of the adjacent sides. Denote by $c$ the length of the diagonal in $D$ (or side, if $D$ is a triangle) that forms a triangle together with these two sides. Without loss of generality, $a \leqslant b$. Consider the following deformation of $D$. All of the vertices except of $i$ stay fixed, and $i$ moves so that the sum $a+b$ remains constant and $a$ decreases. Define the corresponding deformation of $D'$ as follows. Develop $D'$ onto the sphere by cutting it along a radius that ends at a vertex other than $i$. The deformation leaves all of the vertices of the development fixed, except for $i$ which moves so that $a+b$ remains constant and $a$ decreases, as in $D$. Denote by $c'$ the length of the diagonal that corresponds to the diagonal $c$ in $D$. See Figure 5.6.

Clearly, the deformation of $D'$ does not change $\kappa_0$. By construction, the perimeter of $D$ and $D'$ is also constant. As we will show, the deformation increases the sum $\sum \kappa_i$. This implies that during the deformation the left hand side of (5.4) remains constant, whereas the right hand side increases. The deformation stops in one of the following cases.

*We get $\theta_j = \theta'_j$ for some $j$.* Then we cut off the triangles $\Delta$ and $\Delta'$ at the vertex $j$ and use the induction assumption.

*We get $\theta'_j = \pi$ for some $j$.* In this case we start the deformation that affects the vertex $j$. It does not change the angles of $D'$, so it has to end for a different reason.

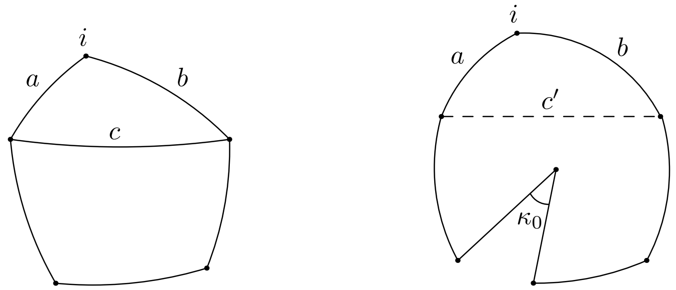


*Figure 5.6. Polygon $D$ and singular polygon $D'$ cut along a radius.*

*The polygon $D$ is a triangle and it degenerates so that $b = a + c$.* In this case $D'$ degenerates to a symmetric digon. This is the induction base.

*The side $b$ hits the image of the singularity in the development of $D'$.* This is impossible, because if the singularity is close to the boundary of $D'$ but far from its vertices, then $\kappa_0$ is small. This follows from the transformation of $D$ into $D'$ by gluing in digons as described before.

In order to show that $\sum \kappa_i$ increases during the deformation of $D$ and $D'$, consider the triangle with side lengths $a, b, c$ and the triangle with side lengths $a, b, c'$. It is not hard to see that

$$d \sum \kappa_i = d\text{area}' - d\text{area},$$

where area and area$'$ are the areas of the first and of the second triangle, respectively. If $\alpha, \beta, \gamma$ are the angles of the first triangle, then we have

$$d\text{area} = \frac{\partial \text{area}}{\partial b} - \frac{\partial \text{area}}{\partial a} = \left( \frac{\partial}{\partial b} - \frac{\partial}{\partial a} \right) (\alpha + \beta + \gamma).$$

Since $c' > c$, for $d\text{area}' > d\text{area}$ it suffices to show $\frac{\partial}{\partial c} d\text{area} > 0$. With the help of Lemma 5.1 one obtains

$$\frac{\partial^2 \text{area}}{\partial b \partial c} = \frac{\sin \alpha - \sin(\beta + \gamma)}{\sin^2 \alpha \sin \beta \sin \gamma \sin b \sin c}.$$

The denominator is invariant under permutations of $a, b$, and $c$. Thus we have

$$\begin{aligned} d \sum \kappa_i &= \frac{\sin \alpha - \sin \beta + \sin(\alpha + \gamma) - \sin(\beta + \gamma)}{\sin^2 \alpha \sin \beta \sin \gamma \sin b \sin c} \\ &= \frac{2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta + \gamma}{2} \cos \frac{\gamma}{2}}{\sin^2 \alpha \sin \beta \sin \gamma \sin b \sin c}. \end{aligned}$$

This is easily seen to be positive.

Now let us prove the induction base. The digon $D'$ is glued from two equal triangles. If we use the standard notation for the sides and angles of a triangle, the inequality (5.4) becomes

$$\pi - \gamma > \left(1 - \frac{c}{\pi}\right)(\alpha + \beta), \tag{5.5}$$

where $\alpha, \beta \leqslant \frac{\pi}{2}$ since $D'$ is convex. Let us show that for $c$ and $\alpha+\beta$ constant, the left hand side of (5.5) attains its minimum at $\alpha = \beta$. Assume $\alpha < \beta$. Then $a < b$. It follows that a simultaneous increase of $\alpha$ and decrease of $\beta$ by the same amount results in an increase of the area of the triangle. Hence the left hand side of (5.5) decreases. For $\alpha = \beta$, we have to show

$$\frac{\pi - \gamma}{\pi - c} > \frac{2\alpha}{\pi}, \tag{5.6}$$

for an isosceles triangle based on $c$. We have $\cos\frac{\gamma}{2} = \sin\alpha\cos\frac{c}{2}$. For $\alpha$ constant, the left hand side of (5.6) can be shown to be a monotonically increasing function of $c$. As $c \to 0$, it tends to $\frac{2\alpha}{\pi}$. This completes the proof of the Lemma. □

Lemma 5.7. — *Let $P \subset \mathbb{R}^3$ be a convex polytope with vertices $(p_i)$ and angle defects $(\delta_i)$. Then for any $i$*

$$\left| \sum_{j \neq i} \delta_j \frac{p_j - p_i}{\|p_j - p_i\|} \right| > \delta_i.$$

*Proof.* — Let $C \subset \mathbb{S}^2$ be the spherical section of the polytope $P$ at the vertex $i$. For any $j \neq i$ we have

$$v_j = \frac{p_j - p_i}{\|p_j - p_i\|} \in C.$$

The perimeter of $C$ equals $2\pi - \delta_i$. By Lemma 5.8, there exists a point $c \in C$ such that

$$\operatorname{dist}(v_j, c) \leqslant \frac{\pi}{2} - \frac{\delta_i}{4} \quad \text{for all } j.$$

Thus we have

$$\left\langle \sum_{j \neq i} \delta_j v_j, c \right\rangle = \sum_{j \neq i} \delta_j \langle v_j, c\rangle \geqslant \sum_{j \neq i} \delta_j \cdot \sin\frac{\delta_i}{4} = (4\pi - \delta_i)\sin\frac{\delta_i}{4} > \delta_i,$$

where the last inequality follows easily from $\sin x > \frac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$. □

Lemma 5.8. — *Let $C$ be a convex spherical polygon of perimeter* per. *Then $C$ is contained in a circle of radius $\frac{\mathrm{per}}{4}$. That is, there exists a point $c \in \mathbb{S}^2$ such that*

$$\operatorname{dist}(x, c) \leqslant \frac{\mathrm{per}}{4} \quad \textit{for all } x \in C$$

*in the intrinsic metric of the sphere.*

*Proof.* — Let $O$ be the circle of the smallest radius $r$ that contains $C$. Then there are three vertices $v_1, v_2$ and $v_3$ of $C$ such that $v_1, v_2, v_3$ lie on $O$, and the center of $O$ lies in the triangle $\Delta$ with the vertices $v_1, v_2, v_3$. Since $\mathrm{per}(\Delta) \leqslant \mathrm{per}$, it suffices to prove

$$r \leqslant \frac{\mathrm{per}(\Delta)}{4}. \tag{5.7}$$

The triangle $\Delta$ is defined by the radius $r$ and the angles $\alpha, \beta, \gamma \in (0, \pi)$ between the radii drawn from the center of $O$ to the vertices. By computing the derivatives $\frac{\partial \mathrm{per}(\Delta)}{\partial \alpha} - \frac{\partial \mathrm{per}(\Delta)}{\partial \beta}$ one sees that the minimum of $\mathrm{per}(\Delta)$ is achieved when $\Delta$ degenerates to an arc. In this case we have equality in (5.7). The lemma is proved. □

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