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A linear extension operator for Whitney fields on closed o-minimal sets

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A LINEAR EXTENSION OPERATOR FOR WHITNEY FIELDS ON CLOSED O-MINIMAL SETS

by Wiesław PAWŁUCKI (*)

1. Introduction

In 1997 K. Kurdyka and the author gave in [6] the following o-minimal version of the Whitney extension theorem:

**Theorem 1.1** ([6]). — **Given any o-minimal structure on the ordered field of real numbers \( \mathbb{R} \), a compact definable subset \( E \subset \mathbb{R}^n \), a definable \( C^p \)-Whitney field \( F \) on \( E \), where \( p \in \mathbb{N} \setminus \{0\} \), then for any integer \( q \geq p \), there exists a definable \( C^p \)-extension \( f : \mathbb{R}^n \to \mathbb{R} \) of \( F \) which is \( C^q \) on \( \mathbb{R}^n \setminus E \).**

However, the extension operator \( F \mapsto f \) from [6] is not linear and it was not clear how the construction from [6] based on o-minimal geometry could be adapted to get an extension operator for all Whitney fields on
any compact (or more generally closed) o-minimal subset \( E \) of \( \mathbb{R}^n \). The present paper is devoted to this question. The main goal here is to prove the following

**Theorem 1.2.** — Let \( E \) be a closed o-minimal subset of \( \mathbb{R}^n \) and \( p \in \mathbb{N} \). Let \( \mathcal{E}^p(E) \) denote the Fréchet algebra of all \( C^p \)-Whitney fields on \( E \).

Then there exists a continuous linear extension operator \( \mathcal{L} : \mathcal{E}^p(E) \rightarrow C^p(\mathbb{R}^n) \) which has the following properties

1. \( \mathcal{L} \) is a finite composition of operators each of which either preserves definability or (only if \( p > 0 \)) is an integration with respect to a parameter;
2. operators preserving definability in (1) are only of the following five types: substituting with a definable mapping; taking a linear combination with definable coefficients; differentiation; restriction to a definable subset and extending by zero;
3. there exists a constant \( M > 0 \) such that if \( \omega \) is a modulus of continuity of a field \( F \), then \( M\omega \) is a modulus of continuity of \( \mathcal{L} F \).

Since \( \mathcal{L} \) involves integration, it may not preserve definability in the initial o-minimal structure where \( E \) is definable. For example, if \( F \) is a (globally) subanalytic \( C^p \)-Whitney field, then \( \mathcal{L} F \) can \textit{a priori} involve the function \( t \mapsto \log t \), not subanalytic at 0. By a result of Lion and Rolin [7], we get in this case the following

**Corollary 1.3.** — Let \( \mathcal{A} \) denote the algebra of real functions generated by (globally) subanalytic functions and their logarithms; i.e. \( \mathcal{A} \) consists of all functions of the form \( P(h_1, \ldots, h_m, \log h_1, \ldots, \log h_m) \), where \( h_i : \mathbb{R}^n \rightarrow \mathbb{R} \) \( (i = 1, \ldots, m) \) are subanalytic, \( m \in \mathbb{N} \setminus \{0\} \), \( P \in \mathbb{R}[Y_1, \ldots, Y_{2m}] \), and where we adopt the convention: \( \log t = 0 \), for \( t \leq 0 \). Let \( E \) be a closed subanalytic subset of \( \mathbb{R}^n \) and \( p \in \mathbb{N} \).

Then there exists a continuous linear extension operator \( \mathcal{L} : \mathcal{E}^p(E) \rightarrow C^p(\mathbb{R}^n) \) which has the following properties:

1. if \( F \) is a \( C^p \)-Whitney field on \( E \) all derivatives of which \( F^\kappa \) are (restrictions to \( E \) of) functions in \( \mathcal{A} \), then \( \mathcal{L} F \in \mathcal{A} \);
2. there exists a constant \( M > 0 \) such that if \( \omega \) is a modulus of continuity of a field \( F \), then \( M\omega \) is a modulus of continuity of \( \mathcal{L} F \).

The case \( p = 0 \) in Theorem 1.2, when integration is not used seems worth being stated separately

**Corollary 1.4.** — Let \( E \) be a closed o-minimal subset of \( \mathbb{R}^n \) and let \( \mathcal{C}(E) \) denote the Fréchet space of all real continuous functions on \( E \).
Then there exists a continuous linear extension operator \( L : C(E) \rightarrow C(\mathbb{R}^n) \) preserving definability and such that there exists \( M > 0 \) such that, if \( \omega \) is a modulus of continuity for \( F \in C(E) \), then \( M \omega \) is a modulus of continuity for \( LF \).

By an o-minimal subset of an Euclidean space \( \mathbb{R}^n \) we mean a subset definable in any o-minimal structure on the ordered field of real numbers \( \mathbb{R} \) (see [2, 3] for the definition and fundamental properties).

We refer the reader to [13], [4], [8], [11] or/and [12] for basic facts on Whitney fields. It will be convenient for us to adopt the following definition of a Whitney field.

Let \( p \in \mathbb{N} \setminus \{0\} \) and let \( A \) be a locally closed subset of \( \mathbb{R}^n \); i.e. contained and closed in some open subset \( G \subset \mathbb{R}^n \). A \( C^p \)-Whitney field on \( A \) is a polynomial

\[
F(u, X) = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} F^\alpha(u) X^\alpha \in C(A)[X] = C(A)[X_1, \ldots, X_n],
\]

which fulfills the following condition

\((*)\) for each \( c \in A \) and each \( \alpha \in \mathbb{N}^n \) such that \( |\alpha| \leq p \)

\[
D_X^\alpha F(a, 0) - D_X^\alpha F(b, a-b) = o(|a-b|^{p-|\alpha|}), \quad \text{when } A \ni a \to c, A \ni b \to c,
\]

or equivalently (see [8], Chapter I, Theorem 2.2) - the condition

\((**)*\) for each \( c \in A \)

\[
F(a, x-a) - F(b, x-b) = o(|x-a|^p + |x-b|^p),
\]

uniformly with respect to \( x \in \mathbb{R}^n \), when \( A \ni a \to c, A \ni b \to c \).

We will denote by \( \mathcal{E}^p(A) \) the real algebra of all \( C^p \)-Whitney fields on \( A \). It is a Fréchet algebra with the topology defined by the following system of seminorms

\[
||F||_p^K = |F|_p^K + \sup_{a \neq b \atop |\alpha| \leq p} \frac{|D_X^\alpha F(a, 0) - D_X^\alpha F(b, a-b)|}{|a-b|^{p-|\alpha|}},
\]

where \( K \) is a compact subset of \( A \) and \( |.|_p^K \) is a seminorm defined by

\[
|F|_p^K = \sup_{a \in K} |F^\alpha(a)|.
\]

Let \( C^p(G) \) denote the usual Fréchet algebra of real functions of class \( C^p \) (\( C^p \)-functions) on \( G \). Then we have the following homomorphism of Fréchet
algebras

\[ T : \mathcal{C}^p(G) \longrightarrow \mathcal{E}^p(A), \quad T f(a, X) = T^p_a f(X) = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} D^\alpha f(a) X^\alpha, \]

and the Whitney extension theorem [13] says that there exists a linear continuous mapping

\[ W : \mathcal{E}^p(A) \longrightarrow \mathcal{C}^p(G) \]

such that \( T \circ W = \text{id}_{\mathcal{E}^p(A)} \), called an extension operator.

A subset \( E \) of \( \mathbb{R}^n \) is said to be 1-regular (with a constant \( C \geq 1 \)) if any two points \( a, b \) of \( E \) can be joined in \( E \) by a rectifiable arc \( \gamma : [0, 1] \longrightarrow E \) of length \( |\gamma| \leq C|a - b| \).

If \( F \in \mathcal{E}^p(A) \) and \( K \) is a compact 1-regular subset of \( A \) with a constant \( C \), then

\[ |F|^K_p \leq ||F||^K_p \leq 2n^2 C^p |F|^K_p \]

(See [12], p.76, (2.5.1)). Consequently, if every compact subset \( L \) of \( A \) is contained in a 1-regular compact subset \( K \) of \( A \), then the topology of \( \mathcal{E}^p(A) \) is defined by the system of seminorms \( |\ .\ |^K_p \).

As was shown by Glaeser [4] (see also [8], [12] or [11]) it is convenient to use a notion of a modulus of continuity in connection with Whitney fields. By a modulus of continuity we will understand any continuous, increasing and concave function \( \omega : [0, +\infty) \longrightarrow [0, +\infty), \) vanishing at 0. By a modulus of continuity of a \( \mathcal{C}^p \)-Whitney field

\[ F(u, X) = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} F^\alpha(u) X^\alpha \]

on a subset \( A \) of \( \mathbb{R}^n \) we will understand such a modulus of continuity \( \omega \) that

\[ |D^\alpha_X(a, 0) - D^\alpha_X(b, a - b)| \leq \omega(|a - b||a - b|^{p - |\alpha|}), \]

whenever \( |\alpha| \leq p \) and \( a, b \in A \). For a \( \mathcal{C}^p \)-function \( f \in \mathcal{C}^p(G) \) on an open subset \( G \), by its modulus of continuity we will understand a modulus of continuity of the \( \mathcal{C}^p \)-Whitney field \( Tf \) on \( G \).

Every \( \mathcal{C}^p \)-Whitney field on a compact subset of \( \mathbb{R}^n \) admits a modulus of continuity. If a \( \mathcal{C}^p \)-Whitney field \( F \) on a subset \( A \) has a modulus of continuity \( \omega \), then it is easily seen that \( F \) extends by uniform continuity to a \( \mathcal{C}^p \)-Whitney field on \( \overline{A} \) with the same modulus of continuity. Whitney’s extension operator [13] has the following property (see [4]):
There exists a constant $M$ depending only on $p$ and $n$ such that, for every $F \in \mathcal{E}^p(A)$ admitting a modulus of continuity $\omega$, $M\omega$ is a modulus of continuity for $WF$. (In fact a localization by a partition of unity is necessary.)

We have also the following

**Proposition 1.5.** — Let $F$ be a $C^p$-Whitney field on a (locally) closed 1-regular with constant $C$ subset $A$.

1. If $\omega$ is a modulus of continuity of $F$ on $A$, then $|F^\alpha(a) - F^\alpha(b)| \leq \omega(|a - b|)$, whenever $|\alpha| = p, a, b \in A$.
2. If $\omega$ is a modulus of continuity such that $|F^\alpha(a) - F^\alpha(b)| \leq \omega(|a - b|)$, whenever $|\alpha| = p, a, b \in A$, then $n^{\frac{p}{2}}C^p\omega$ is a modulus of continuity of $F$ on $A$.

**Proof.** — (1) being trivial, for (2) see again [12], (2.5.1), p.76. □

Shortly, our construction of the extension operator $\mathcal{L}$ is as follows. First we show how to extend $C^p$-Whitney fields from a linear subspace $R^k \times 0$ of $R^n$. Then we generalize the construction to the set of the form $\overline{\Omega} \times 0$, where $\Omega$ is open in $R^k$ for fields flat on $\partial \Omega \times 0$, simply by Hestenes Lemma. Using induction on dimension of $A$, this gives an extension operator for $A = \Gamma$, where $\Gamma = \Omega \times 0$ assuming we have it already built for the boundary $\partial \Gamma = \Gamma \setminus \Omega$ of $\Gamma$ which in this case is $\partial \Omega \times 0$. The next generalization is by taking $A = \Gamma$, where $\Gamma$ is a $\Lambda_p$-regular leaf of dimension $k$ in the sense of [6], and again assuming the fields are flat on $\partial \Gamma$. Additionally, the extension can be chosen vanishing outside a conical neighbourhood of $\Gamma$; i.e. the set $\{x \in \Omega \times R^{n-k} : d(x,\Gamma) < \varepsilon d(x,\partial \Gamma)\}$, where $\Omega$ is the orthogonal projection of $\Gamma$ to $R^k \times 0$ and $\varepsilon$ is a positive arbitrary constant. The next generalization is to the closure of a finite tower of $\Lambda_p$-regular leaves lying over a common open $\Lambda_p$-regular cell in $R^k$. To finish the construction we will prove that every closed definable $k$-dimensional subset $A$ admits a finite decomposition $A = M_0 \cup \cdots \cup M_s$ such that each $M_i$ is a finite tower of definable $\Lambda_p$-regular leaves in a suitable linear coordinate system and for any $i, j \in \{0, \ldots, s\}$, where $i \neq j$, $\overline{M_i}$ and $\overline{M_j}$ are simply separated relative to $\partial M_i$; i.e. $d(x, M_j) \geq Cd(x, \partial M_i)$, for each $x \in M_i$, with some positive constant $C$. (The proof of this $\Lambda_p$-regular Decomposition Theorem is based on [6] and [10].)
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2. Extension operator for a linear subspace

Observe that if $\Omega$ is an open subset of $\mathbb{R}^k$ and $A = \Omega \times 0 \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$, then the algebra $\mathcal{E}^p(A)$ can be identified with the algebra of polynomials

$$F(u,W) = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} F^\alpha(u) W^\alpha = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} F^\alpha(u) W_1^{\alpha_1} \ldots W_l^{\alpha_l},$$

where $l = n - k$ and $F^\alpha \in C^{p-|\alpha|}(\Omega)$, for each $\alpha \in \mathbb{N}^l$ such that $|\alpha| \leq p$ (cf. [4], Chap.III, (8.4)).

Let us now consider the case $k = n - 1$ and $A = \mathbb{R}^k \times 0$. Then the extension operator will be produced using regularization of functions $F^\alpha$ by convolution. Strictly, we have the following

**Proposition 2.1.** — Let $\sigma \in \{0, \ldots, p\}, g \in C^{p-\sigma}(\mathbb{R}^k), \varphi \in C^p(\mathbb{R}^k)$. Assume that $\text{supp} \varphi$ is compact and put

$$\varphi_w(v) = \frac{1}{w^k} \varphi\left(\frac{v}{w}\right)$$

and

$$G(u,w) = \frac{1}{\sigma!} (g \ast \varphi_w)(u) w^\sigma = \frac{1}{\sigma!} \int_{\mathbb{R}^k} g(u-v) \varphi_w(v) w^\sigma dv,$$

for $u \in \mathbb{R}^k$ and $w \in \mathbb{R}, w > 0$.

Then $G : \mathbb{R}^k \times (0, +\infty) \rightarrow \mathbb{R}$ is a $C^p$-function and for every $(\alpha, \beta) \in \mathbb{N}^k \times \mathbb{N}$ such that $|\alpha| + |\beta| \leq p$

$$\lim_{w \to 0} D^{(\alpha,\beta)} G(u,w) = \begin{cases} 0, & \text{if } \beta < \sigma \\ D^\alpha g(u) \int \varphi, & \text{if } \beta = \sigma \\ \sum_{|\gamma| = \beta - \sigma} \omega_{\gamma\sigma} D^{\alpha+\gamma} g(u), & \text{if } \beta > \sigma \end{cases}$$

uniformly on compact subsets with respect to $u$, where $\omega_{\gamma\sigma}$ are some constants depending only on $\gamma$, $\sigma$ and $\varphi$.

To prove Proposition 2.1 one needs the following
LEMMA 2.2. — For any $r \in \mathbb{R}$ and $\lambda \in \{0, \ldots, p\}$

$$\frac{\partial^\lambda}{\partial w^\lambda} \left[ w^r \varphi \left( \frac{v}{w} \right) \right] = w^{r-\lambda} \varphi_\lambda \left( \frac{v}{w} \right),$$

where $\varphi_\lambda$ is a $C^{p-\lambda}$-function on $\mathbb{R}^k$ with a compact support and $\int \varphi_\lambda = (k+r)(k+r-1) \cdots (k+r-\lambda+1) \int \varphi$.

Proof of Lemma 2.2. —

$$\frac{\partial}{\partial w} \left[ w^r \varphi \left( \frac{v}{w} \right) \right] = r w^{r-1} \varphi \left( \frac{v}{w} \right) + w^r \sum_{i=1}^k \frac{\partial \varphi}{\partial v_i} \left( \frac{v}{w} \right) \left( - \frac{v_i}{w^2} \right) = w^{r-1} \varphi_1 \left( \frac{v}{w} \right),$$

where $\varphi_1(v) = r \varphi(v) - \sum_{i=1}^k v_i \frac{\partial \varphi}{\partial v_i}(v)$. Moreover, integrating by parts,

$$\int \varphi_1 = r \int \varphi - \sum_{i=1}^k \int v_i \frac{\partial \varphi}{\partial v_i} = (r+k) \int \varphi$$

and Lemma 2.2 follows by induction. □

Proof of Proposition 2.1. — $G$ is of class $C^p$ on $\mathbb{R}^k \times (0, +\infty)$, because

$$G(u, w) = \int g(v) \frac{1}{w^k} \varphi \left( \frac{u-v}{w} \right) w^\sigma dv.$$  

(I) Assume first that $\beta \leq \sigma$ and $|\alpha| \leq p - \sigma$. Then

$$D^{(\alpha, \beta)} G(u, w) = \frac{1}{\sigma!} \int D^\alpha g(u-v) w^{\sigma-\beta-k} \varphi_\beta \left( \frac{v}{w} \right) dv =$$

$$\frac{1}{\sigma!} \int D^\alpha g(u-v) \frac{1}{w^k} \varphi_\beta \left( \frac{v}{w} \right) dv \rightarrow \frac{1}{\sigma!} \int 0^\sigma \beta D^\alpha g(u) \int \varphi_\beta,$$

when $w \to 0$, the convergence being uniform on compact subsets with respect to $u$. Consequently, the limit is 0, if $\beta < \sigma$ and $D^\alpha g \int \varphi$, if $\beta = \sigma$.

(II) Now assume that $\beta \leq \sigma$ and $|\alpha| > p - \sigma$. Then $\alpha = \gamma + \delta$, where $|\gamma| = p - \sigma$ and $\delta \neq 0$.

$$D^{(\gamma, \beta)} G(u, w) = \frac{1}{\sigma!} \int D^\gamma g(u-v) w^{\sigma-\beta-k} \varphi_\beta \left( \frac{v}{w} \right) dv =$$

$$\frac{1}{\sigma!} \int D^\gamma g(v) w^{\sigma-\beta-k} \varphi_\beta \left( \frac{u-v}{w} \right) dv.$$

$$D^{(\alpha, \beta)} G(u, w) = \frac{1}{\sigma!} \int D^\gamma g(v) w^{\sigma-\beta-k} w^{-|\delta|} D^\delta \varphi_\beta \left( \frac{u-v}{w} \right) dv =$$

$$\frac{1}{\sigma!} \int w^{\sigma-\beta-|\delta|} \int D^\gamma g(u-wv) D^\delta \varphi_\beta(v) dv.$$  

Notice that $\sigma - \beta - |\delta| = p - |\alpha| - \beta \geq 0$ and $\int D^\delta \varphi_\beta(v) dv = 0$, since $\varphi_\beta$ has a compact support. Consequently, $D^{(\alpha, \beta)} G(u, w) \to 0$, when $w \to 0$. 

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Finally, let $\beta > \sigma$. Then $|\alpha| \leq p - \beta < p - \sigma$ and $\beta = \sigma + \rho$, where $\rho > 0$. By the case (I),

$$D^{(\alpha, \sigma)} G(u, w) = \frac{1}{\sigma!} \int D^{\alpha} g(u - vw) \varphi_\sigma(v) dv.$$ 

$D^{\alpha} g$ being of class $p - \sigma - |\alpha| \geq \rho$, one obtains

$$D^{(\alpha, \beta)} G(u, w) = D^{(0, \rho)} (D^{(\alpha, \sigma)} G)(u, w) = \frac{1}{\sigma!} \sum_{|\mu| = \rho} \int D^{\alpha + \mu} g(u - vw)(-v)^\mu \varphi_\sigma(v) dv,$$

which tends to $\sum_{|\mu| = \rho} \omega_{\mu \sigma} D^{\alpha + \mu} g(u)$ uniformly on compact subsets with respect to $u$, when $w \to 0$, where $\omega_{\mu \sigma} = \frac{1}{\sigma!} \int (-v)^\mu \varphi_\sigma(v) dv$. \hfill $\square$

**Proposition 2.3.** — Let $\varphi \in C^p(\mathbb{R}^k)$ be with compact support and such that $\int \varphi = 1$. Then the formula

$$L(g W^\sigma)(u, w) = \left( \frac{w}{|w|} \right)^\sigma \left[ \frac{1}{\sigma!} (g \star \varphi_{|w|})(u)|w|^{\sigma} - \sum_{0 < |\gamma| \leq p - \sigma} \frac{1}{(\sigma + |\gamma|)!} \omega_{\gamma \sigma} L(D^\gamma g W^{\sigma + |\gamma|})(u, |w|) \right],$$

for $\sigma \in \{1, \ldots, p\}$, $g \in C^{p - \sigma}(\mathbb{R}^k)$, $u \in \mathbb{R}^k$ and $w \in \mathbb{R} \setminus \{0\}$, completed by putting

$$L(g W^\sigma)(u, 0) = 0, \quad \text{and} \quad L(g W^0) = L(g) = g, \quad \text{for } g \in C^p(\mathbb{R}^k),$$

defines (inductively) a continuous linear extension operator $L = L_p : \mathcal{E}^p(\mathbb{R}^k \times 0) \to C^p(\mathbb{R}^{k+1})$.

Moreover, there exists a constant $M > 0$ (depending only on $k, p$ and $\varphi$) such that if $\omega$ is a modulus of continuity of a field $F \in \mathcal{E}^p(\mathbb{R}^k \times 0)$, then $M \omega$ is a modulus of continuity of the $C^p$-function $LF$.

**Proof.** — This follows immediately from Proposition 2.1. \hfill $\square$

Now we generalize our extension operator to any linear subspace of $\mathbb{R}^n$.

**Proposition 2.4.** — Let $\mathbb{R}^k \times 0 \subset \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$, where $l > 1$. Then the formula

$$L_p(g W_1^{\alpha_1} \cdots W_l^{\alpha_l}) = L_p(L_{p-\alpha_l}(g W_1^{\alpha_1} \cdots W_{l-1}^{\alpha_{l-1}}) W_l^{\alpha_l}),$$

defines (inductively) a continuous linear extension operator $L = L_p : \mathcal{E}^p(\mathbb{R}^k \times 0) \to C^p(\mathbb{R}^{k+l})$. ANNALES DE L’INSTITUT FOURIER
where \( \alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l, |\alpha| \leq p \) and \( g \in C^p - |\alpha| (\mathbb{R}^k) \), defines by induction on \( l \) a linear continuous extension operator \( L = L_p : \mathcal{E}^p(\mathbb{R}^k \times 0) \to C^p(\mathbb{R}^n) \).

Moreover, there is a constant \( M > 0 \) such that if \( \omega \) is a modulus of continuity for \( F \in \mathcal{E}^p(\mathbb{R}^k \times 0) \), then \( M \omega \) is a modulus of continuity for \( LF \).

**Proof.** — This follows easily by induction from Proposition 2.3. \( \square \)

### 3. A generalization to the ideal of \( C^p \)-Whitney fields on \( \Omega \times 0 \) p-flat on \( \partial \Omega \times 0 \) (\( \Omega \) an open \( \Lambda_p \)-regular cell in \( \mathbb{R}^k = \mathbb{R}^k \times 0 \subset \mathbb{R}^k \times \mathbb{R}^l \))

If \( A \) is any locally closed subset of \( \mathbb{R}^n \) and \( B \) any closed subset of \( A \), \( \mathcal{E}^p(A, B) \) will denote the ideal of all \( C^p \)-Whitney fields \( F \) on \( A \) p-flat on \( B \); i.e. \( F^\alpha(u) = 0 \), when \( |\alpha| \leq p \) and \( u \in B \). It is closed in \( \mathcal{E}^p(A) \).

Let first \( \Omega \) be any open subset of \( \mathbb{R}^k \). By the Hestenes Lemma (see [12], Lemma 4.3, p.80)

\[
\mathcal{E}^p(\Omega \times 0, \partial \Omega \times 0) = \{ F = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} F^\alpha W^\alpha : F^\alpha \in C^p - |\alpha| (\Omega), \lim_{u \to a} D^\beta F^\alpha(u) = 0, \text{ if } a \in \partial \Omega, |\beta| \leq p - |\alpha| \},
\]

and putting

\[
\tilde{F}^\alpha(u) = \begin{cases} F^\alpha(u), & \text{if } u \in \Omega \\ 0, & \text{if } u \in \mathbb{R}^k \setminus \Omega \end{cases} \quad \text{and} \quad \tilde{F} = \sum_{\alpha} \frac{1}{\alpha!} \tilde{F}^\alpha W^\alpha,
\]

one obtains a linear continuous extension operator

\[
\mathcal{E}^p(\Omega \times 0, \partial \Omega \times 0) \ni F \to \tilde{F} \in \mathcal{E}^p(\mathbb{R}^k \times 0, (\mathbb{R}^k \setminus \Omega) \times 0)
\]

preserving modulus of continuity.

Now we will consider the case when \( \Omega \) is an open \( \Lambda_p \)-regular cell in \( \mathbb{R}^k \) (cf. [6]). We will first recall the notion of \( \Lambda_p \)-regular mapping. Let \( \psi : D \to \mathbb{R}^m \) be a mapping on an open subset \( D \subset \mathbb{R}^n \). We say that \( \psi \) is \( \Lambda_p \)-regular (on \( D \)) if it is of class \( C^p \) and there is a constant \( C \geq 0 \) such that

\[
|D^\kappa \psi(x)| \leq C/(d(x, \partial D)^{|\kappa| - 1}, \text{ whenever } 1 \leq |\kappa| \leq p \text{ and } x \in D.
\]

**Remark 3.1.** — Let \( \psi \) be \( \Lambda_p \)-regular on \( D \). Then

1. it is \( \Lambda_p \)-regular on every open \( D' \subset D \);
(2) if $A \subset \Omega$ is a 1-regular subset, then the restriction $\psi \vert A$ is Lipschitz and thus it has a continuous extension $\overline{\psi} \vert A$ to $\overline{A}$.

We shall say (after [6]) that $S$ is an open $\Lambda_p$-regular (definable in a given $o$-minimal structure) cell in $\mathbb{R}^n$ iff

1. $S$ is an open interval in $\mathbb{R}$, when $n = 1$;
2. $S = \{(x', x_n) : x' \in T, \quad \psi_1(x') < x_n < \psi_2(x')\}$, where $T$ is an open $\Lambda_p$-regular (definable) cell in $\mathbb{R}^{n-1}$ and each $\psi_i$ ($i = 1, 2$) is a function on $T$ being either real $\Lambda_p$-regular (definable) function on $T$, or identically equal to $-\infty$, or identically equal to $+\infty$, and $\psi_1(x') < \psi_2(x')$, for all $x' \in T$, when $n > 1$.

Remark 3.2. — Such a cell $S$ is 1-regular and if $\psi_i$ is finite it is Lipschitz on $T$, thus it admits a continuous extension $\overline{\psi}_i$ to $\overline{T}$.

For any open (definable) $\Lambda_p$-regular cell in $\mathbb{R}^n$, one defines, by induction on $n$, a sequence $\rho_j : \overline{S} \longrightarrow \mathbb{R} \cup \{+\infty\}$ ($j = 1, \ldots, 2n$) of the functions associated with the cell $S$:

1. When $n = 1$ and $S = (a_1, a_2)$, we put
   \[
   \rho_1(x) = \begin{cases} 
   x - a_1, & \text{if } a_1 \in \mathbb{R} \\
   +\infty, & \text{if } a_1 = -\infty
   \end{cases}
   \quad \text{and} \quad
   \rho_2(x) = a_2 - x, \quad \text{if } a_2 \in \mathbb{R}
   \]
   \[
   +\infty, \quad \text{if } a_2 = +\infty.
   \]
2. When $n > 1$ and $S = \{(x', x_n) : x' \in T, \quad \psi_1(x') < x_n < \psi_2(x')\}$, let $\sigma_j$ ($j = 1, \ldots, 2n - 2$) be the functions associated with $T$. We put, for any $x = (x', x_n) \in \overline{S}$, $\rho_j(x) = \sigma_j(x')$ for $j = 1, \ldots, 2n - 2$ and
   \[
   \rho_{2n-1}(x) = \begin{cases} 
   x_n - \overline{\psi}_1(x'), & \text{if } \psi_1 : T \to \mathbb{R} \\
   +\infty, & \text{if } \psi_1 \equiv -\infty
   \end{cases}
   \quad \text{and} \quad
   \rho_{2n}(x) = \begin{cases} 
   -\overline{\psi}_2(x') - x_n, & \text{if } \psi_2 : T \to \mathbb{R} \\
   +\infty, & \text{if } \psi_2 \equiv +\infty.
   \end{cases}
   \]

Remark 3.3 ([6], Lemma 3). — There exists a constant $\Theta > 0$ such that
   \[
   \Theta \min_j \rho_j(x) \leq d(x, \partial S) \leq \min_j \rho_j(x), \quad \text{for } x \in \overline{S}.
   \]
(We adopt the convention: $d(x, \emptyset) = +\infty$.)

Remark 3.4 ([6], Lemma 4). — The functions $\rho_j$ which are finite are $\Lambda_p$-regular on $S$, Lipschitz on $\overline{S}$ and definable, if $S$ is so.

Lemma 3.5 (cf. [6], Lemma 5). — Let $\varphi_{\nu} : \Omega \longrightarrow \mathbb{R}$ ($\nu = 1, \ldots, m$) be $\Lambda_p$-regular functions on an open subset $\Omega \subset \mathbb{R}^k$. Assume that $r(u) :=\]
Let $m \sum_{y=1}^m \varphi^2_y(u) \frac{1}{r} \neq 0$ for each $u \in \Omega$. Then there exists a constant $\tilde{C} > 0$ such that for each $u \in \Omega$

$$\left| D^\alpha \left( \frac{1}{r} \right)(u) \right| \leq \frac{\tilde{C}}{r(u) \min(r(u), d(u, \partial \Omega)) |\alpha|}, \text{ where } 0 \leq |\alpha| \leq p;$$

consequently $\left| D^\alpha \left( \frac{1}{r} \right)(u) \right| \leq \frac{\tilde{C}}{\min(r(u), d(u, \partial \Omega)) |\alpha| + 1}$.

**Proof.** — Induction on $|\alpha|$. \hfill \square

**PROPOSITION 3.6** (cf. [6], Lemmas 6-7). — Let $\Omega$ be an open subset of $\mathbb{R}^k$, let $f \in C^p(\Omega \times \mathbb{R}^l)$ and $r \in C^p(\Omega)$, and let $t : \Omega \rightarrow (0, +\infty)$ be any positive function such that $t(u) \leq d(u, \partial \Omega)$ for any $u \in \Omega$. Let $\varepsilon > 0$ and put

$$\Delta_\varepsilon := \{(u, w) \in \Omega \times \mathbb{R}^l : |w| < \varepsilon t(u)\}.$$

Assume that there exists a constant $\tilde{C} > 0$ such that $|D^\alpha \left( \frac{1}{r} \right)| \leq \frac{\tilde{C}}{t|\alpha| + 1}$, when $\alpha \in \mathbb{N}^k$, and for each $c \in \partial \Omega$, $D^\alpha f(u, w) = o(t(u)^{p - |\alpha|})$, when $\Delta_\varepsilon \ni (u, w) \rightarrow (c, 0)$ and $\alpha \in \mathbb{N}^k \times \mathbb{N}^l$, $|\alpha| \leq p$.

Let $\xi : \mathbb{R} \rightarrow \mathbb{R}$ be any $C^p$-function. Fix $i \in \{1, \ldots, l\}$ and put

$$g(u, w) := \xi \left( \frac{w_i}{r(u)} \right) f(u, w), \text{ for } (u, w) \in \Omega \times \mathbb{R}^l.$$

Then for each $c \in \partial \Omega$, $D^\alpha g(u, w) = o(t(u)^{p - |\alpha|})$, when $\Delta_\varepsilon \ni (u, w) \rightarrow (c, 0)$ and $\alpha \in \mathbb{N}^k \times \mathbb{N}^l$, $|\alpha| \leq p$.

**Proof.** — Put $h(u, w) = \xi \left( \frac{w_i}{r(u)} \right)$. By the Leibniz formula

$$D^\alpha g = \sum_{\lambda \leq \alpha} \binom{\alpha}{\lambda} D^\lambda h D^{\alpha - \lambda} f,$$

so it suffices to check that there exists a constant $C'_\varepsilon > 0$ such that $|D^\lambda h(u, w)| \leq C'_\varepsilon t(u)^{-|\lambda|}$, when $(u, w) \in \Delta_\varepsilon$ and $|\lambda| \leq p$. First, it is easy to see this for $h_0(u, w) := \frac{w_i}{r(u)}$ using Lemma 3.5.

Then for $h = \xi \circ h_0$ we have

$$\frac{\partial h}{\partial x_j} = (\xi' \circ h_0) \frac{\partial h_0}{\partial x_j}, \text{ where } (x_1, \ldots, x_n) = (u_1, \ldots, u_k, w_1, \ldots, w_l)$$

and $D^\lambda \left( \frac{\partial h}{\partial x_j} \right) = \sum_{\mu \leq \lambda} \binom{\lambda}{\mu} D^\mu (\xi' \circ h_0) D^{\lambda - \mu} \left( \frac{\partial h_0}{\partial x_j} \right)$, if $|\lambda| \leq p - 1$, so we conclude by induction. \hfill \square
Remark 3.7. — Suppose that \( f \) is a \( C^p \)-function on the whole space \( \mathbb{R}^k \times \mathbb{R}^l \) and such that for each \( c \in \partial \Omega \), \( D^\varepsilon f(u,0) = o(t(u)^{p-|\varepsilon|}) \), when \( \Omega \ni u \to c \) and \( \varepsilon \in \mathbb{N}^k \times \mathbb{N}^l, |\varepsilon| \leq p \).

Then for each \( c \in \partial \Omega \), \( D^\varepsilon f(u, w) = o(t(u)^{p-|\varepsilon|}) \), when \( \Delta \varepsilon \ni (u, w) \to (c, 0) \) and \( \varepsilon \in \mathbb{N}^k \times \mathbb{N}^l, |\varepsilon| \leq p \). This follows immediately from the Taylor formula

\[
D^\varepsilon f(u, w) = \sum_{|\lambda| \leq p-|\varepsilon|} \frac{1}{\lambda!} D^{\varepsilon+(0, \lambda)} f(u,0) w^\lambda + o(|w|^{p-|\varepsilon|}),
\]

when \( u \to c, w \to 0 \).

Let now \( \Omega \) be an open \( \Lambda_p \)-regular cell in \( \mathbb{R}^k \) and \( \rho_j \) \( (j = 1, \ldots, 2k) \) - the functions associated with \( \Omega \). We define an extension operator

\[
\mathcal{L} : \mathcal{E}^p(\overline{\Omega} \times 0, \partial \Omega \times 0) \longrightarrow C^p(\mathbb{R}^n), \text{ where } \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l,
\]

by the following formula

\[
\mathcal{L} F(u, w) = \begin{cases} 
\prod_{i=1}^l \prod_{j=1}^{2k} \xi \left( Q \frac{w_i}{\rho_j(u)} \right) (L \tilde{F})(u, w), & \text{if } u \in \Omega \\
0, & \text{if } u \in \mathbb{R}^k \setminus \Omega,
\end{cases}
\]

where \( Q \) is any real number \( > \sqrt{l} \Theta^{-1} \), \( \Theta \) is a constant from Remark 3.3 and \( \xi : \mathbb{R} \longrightarrow \mathbb{R} \) is a (definable, if we wish) \( C^p \)-function equal to 1 in a neighborhood of 0, and equal to 0 outside the open interval \((-1, 1)\).

To check that \( \mathcal{L} F \in C^p(\mathbb{R}^n) \) we use repeatedly Proposition 3.6 with \( r = \rho_j \neq +\infty \) and \( t(u) = d(u, \partial \Omega) \) (at the beginning we take \( f = L \tilde{F} \) as in Remark 3.7) and the Hestenes Lemma. The factors involving \( \rho_j \equiv +\infty \) being obviously 1 can be omitted in the above formula.

Observe that if \( \varepsilon \) is any constant from \((0, 1)\), we can choose \( Q \) in such a way that \( \mathcal{L} F \) is \( p \)-flat outside the set

\[
\Delta \varepsilon(\Omega \times 0) := \{ x \in \mathbb{R}^n : d(x, \overline{\Omega} \times 0) < \varepsilon d(x, \partial \Omega \times 0) \} \\
= \{ (u, w) \in \Omega \times \mathbb{R}^l : |w| < \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} d(u, \partial \Omega) \}.
\]

Remark 3.8. — If \( r \) and \( t \) are as in Proposition 3.6 and \( F \in \mathcal{E}^p(\overline{\Omega} \times 0, \partial \Omega \times 0) \) is such that, for each \( c \in \partial \Omega \), \( F^\varepsilon(u,0) = o(t(u)^{p-|\varepsilon|}) \), when \( \Omega \ni u \to c \) and \( |\varepsilon| \leq p \), the above formula for an extension of \( F \) can be modified by putting

\[
\mathcal{L}' F(u, w) = \begin{cases} 
\prod_{i=1}^l \xi \left( \sqrt{t} \frac{w_i}{r(u)} \right) \mathcal{L} F(u, w), & \text{if } u \in \Omega \\
0, & \text{if } u \in \mathbb{R}^k \setminus \Omega.
\end{cases}
\]
Then $\mathcal{L}' F$ is $p$-flat, outside the neighborhood \{(u, w) \in \Omega \times \mathbb{R}^l : |w| < r(u)\} of \Omega \times 0$ and outside $\Delta_\varepsilon (\Omega \times 0)$.

In order that $\mathcal{L} F$ (or $\mathcal{L}' F$) and $F$ have the same (up to a multiplicative constant) modulus of continuity we will prove the following

**Proposition 3.9. —** Under the assumptions of Proposition 3.6 assume additionally that $\Omega$ is 1-regular, $r \in C^{p+1}(\Omega)$ such that

$$|D^\alpha (\frac{1}{r})| \leq \frac{\widehat{c}}{|t|^{\alpha}+1}, \text{ when } \alpha \in \mathbb{N}^l, |\alpha| \leq p + 1$$

and $t$ is Lipschitz. Then there exists a constant $M > 0$ such that if $\omega$ is a modulus of continuity for $f$ on $\Delta_\varepsilon$ satisfying

$$|D^\kappa f(u, w)| \leq \omega(t(u)) t(u)^{p-|\kappa|},$$

when $(u, w) \in \Delta_\varepsilon$ and $|\kappa| \leq p$, then $M \omega$ is a modulus of continuity for $g$ on $\Delta_\varepsilon$ satisfying

$$|D^\kappa g(u, w)| \leq M \omega(t(u)) t(u)^{p-|\kappa|},$$

when $(u, w) \in \Delta_\varepsilon$ and $|\kappa| \leq p$.

**Proof. —** In view of the proof of Proposition 3.6, it suffices to check that, for a constant $M > 0$, $M \omega$ is a modulus of continuity for $g$ on $\Delta_\varepsilon$. First observe that $\Delta_\varepsilon$ is 1-regular, because $\Omega$ is so and the function $t$ is Lipschitz. There exists a constant $C \geq 1$ such that $|t(u_1) - t(u_2)| \leq C|u_1 - u_2|$, for any $u_1, u_2 \in \Omega$.

Fix any $\kappa \in \mathbb{N}^{k+l}$ such that $|\kappa| = p$, any $\lambda \leq \kappa$ and any two points $x_i = (u_i, w_i) \in \Delta_\varepsilon$ $(i = 1, 2)$. We have to estimate

$$|D^\lambda h(x_1) D^{\kappa-\lambda} f(x_1) - D^\lambda h(x_2) D^{\kappa-\lambda} f(x_2)|.$$

**Case I:** $t(u_i) \leq 2C|x_1 - x_2|$ $(i = 1, 2)$.

Then $|D^\lambda h(x_1) D^{\kappa-\lambda} f(x_1)| \leq C'_\varepsilon t(u_i)^{\kappa-\lambda} \omega(t(u_i)) t(u_i)^{p-|\kappa-\lambda|}$

$$\leq C'_\varepsilon \omega(2C|x_1 - x_2|) \leq 2CC'_\varepsilon \omega(|x_1 - x_2|).$$

**Case II:** $t(u_1) > 2C|x_1 - x_2|$. Then $|u_1 - u_2| \leq C|x_1 - x_2| < \frac{1}{2} t(u_1) \leq \frac{1}{2} d(u_1, \Omega)$; thus $[x_1, x_2] \subset \Omega \times \mathbb{R}^l$.

We have $|D^\lambda h(x_1) [D^{\kappa-\lambda} f(x_1) - D^{\kappa-\lambda} f(x_2)]| \leq |D^\lambda h(x_1)| \times$

$$\left[ \sum_{1 \leq |\mu| \leq p-|\kappa-\lambda|} \frac{1}{|\mu|!} |D^{\kappa-\lambda+\mu} f(x_1)||x_1 - x_2|^{|\mu|+\omega(|x_1 - x_2|)|x_1 - x_2|^{p-|\kappa-\lambda|}} \right] \leq M_1 \omega(t(u_1)) t(u_1)^{-1} |x_1 - x_2| + M_2 \omega(|x_1 - x_2|) \leq M' \omega(|x_1 - x_2|),$$

where $M_1, M_2$ and $M'$ are some constants depending on $\delta$. 

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where $M_1, M_2$ and $M'$ are positive constants and we use: $\omega(s)t \leq \omega(t)s$ if $t \leq s$.

On the other hand $|\langle D^\lambda h(x_1) - D^\lambda h(x_2) \rangle D^{\lambda - \lambda} f(x_2)| \leq \sup_{x \in [x_1, x_2]} \sum_{j=1}^{k+l} |D^\lambda(j) h(x)||x_1 - x_2||D^{\lambda - \lambda} f(x_2)|$.

For any $x = (u, w) \in [x_1, x_2], \quad 2|t(u_1) - t(u)| \leq 2C|u_1 - u| \leq 2C|x_1 - x_2| < t(u_1)$ and $2|w_1 - w| \leq 2C|x_1 - x_2| < t(u_1)$; thus $\frac{1}{2}t(u_1) < t(u) < \frac{3}{2}t(u_1)$ and $|w| \leq |w_1| + |w_1 - w| < \varepsilon t(u_1) + t(u) \leq (2\varepsilon + 1)t(u)$.

Consequently $x \in \Delta_{2\varepsilon + 1}$ and

$$|D^\lambda(j) h(x)| \leq C'_{2\varepsilon + 1} t(u)^{-|\lambda|-1} \leq 2^{|\lambda|+1} C'_{2\varepsilon + 1} t(u_1)^{-|\lambda|-1}$$

and

$$|D^{\lambda - \lambda} f(x_2)| \leq \omega(t(u_2))t(u_2)^{|\lambda|} \leq \left(\frac{3}{2}\right)^{|\lambda|+1} \omega(t(u_1))t(u_1)^{|\lambda|}.$$  

The needed inequality follows. $\square$

**Remark 3.10.** — Suppose that $f$ is a $C^p$-function on the whole space $\mathbb{R}^k \times \mathbb{R}^l$ and $\omega$ is its modulus of continuity such that

$$|D^\kappa f(u, 0)| \leq \omega(t(u))t(u)^{p-|\kappa|},$$

when $u \in \Omega$ and $\kappa \in \mathbb{N}^{k+l}, |\kappa| \leq p$.

Then there exists a constant $M'' > 0$ such that

$$|D^\kappa f(u, w)| \leq M'' \omega(t(u))t(u)^{p-|\kappa|},$$

when $(u, w) \in \Delta_\varepsilon$, and $\kappa \in \mathbb{N}^{k+l}, |\kappa| \leq p$.

Indeed, this follows immediately from

$$|D^\kappa f(u, w) - \sum_{|\lambda| \leq p-|\kappa|} \frac{1}{\lambda!} D^{\kappa(0, \lambda)} f(u, 0) w^\lambda| \leq \omega(|w|)|w|^{p-|\kappa|}.$$  

**Remark 3.11.** — If $\Omega$ is an open $\Lambda_{p+1}$-regular cell in $\mathbb{R}^k$ and $\xi$ is a $C^{p+1}$-function, then there exists a positive constant $M$, such that, for any $F \in \mathcal{E}^p(\Omega \times 0, \partial \Omega \times 0)$ (respectively, fulfilling additional conditions: $|F^\kappa(u, 0)| \leq \omega(r(u))r(u)^{p-|\kappa|}$, when $u \in \Omega, \quad \kappa \in \mathbb{N}^{k+l}, |\kappa| \leq p$) if $\omega$ is a modulus of continuity for $F$, then $M\omega$ is a modulus of continuity for $\mathcal{L}F$ (respectively, for $\mathcal{L'}F$).
4. A generalization to the ideal of $C^p$-Whitney fields on the closure of a $\Lambda_p$-regular leaf $p$-flat on its boundary

Now we will transpose the extension operator $L$ to the closure of any $\Lambda_p$-regular leaf. A subset $E \subset \mathbb{R}^n$ is called a (definable) $\Lambda_p$-regular leaf of dimension $k$ in $\mathbb{R}^n$ if it is the graph $E = \{(u, \varphi(u)) : u \in \Omega\}$ of a (definable) $\Lambda_p$-regular mapping $\varphi : \Omega \rightarrow \mathbb{R}^l$ defined on an open (definable) $\Lambda_p$-regular cell $\Omega$ in $\mathbb{R}^k$. A reduction of this case to the previous one will be by the following Lipschitz automorphism

$$\overline{\Omega} \times \mathbb{R}^l \ni (u, w) \mapsto (u, w + \varphi(u)) \in \overline{\Omega} \times \mathbb{R}^l$$

and the following

**Proposition 4.1** (cf. [6], Proposition 3). — *Let $\varphi : \Omega \rightarrow \mathbb{R}^l$ be a $\Lambda_p$-regular mapping defined on an open subset $\Omega \subset \mathbb{R}^k$. Let $t : \Omega \rightarrow (0, +\infty)$ be any function such that $t(u) \leq d(u, \partial \Omega)$, for each $u \in \Omega$. Let $E$ be any closed subset of $\Omega \times \mathbb{R}^l$ and

$$F(u, w; U, W) = \sum_{|\alpha| + |\beta| \leq p} \frac{1}{\alpha! \beta!} F^{(\alpha, \beta)}(u, w) U^\alpha W^\beta \begin{cases} U = (U_1, \ldots, U_k), \\ W = (W_1, \ldots, W_l) \end{cases}$$

a $C^p$-Whitney field on $E$ such that, for any $c \in \partial \Omega$

$$F^{(\alpha, \beta)}(u, w) = o(t(u)^{p-|\alpha|-|\beta|}),$$

when $u \rightarrow c$ and $|\alpha| + |\beta| \leq p$.

Let $F_\varphi(u, v; U, V)$ be a polynomial in $(U, V)$ of degree $\leq p$ such that

$$F_\varphi(u, v; U, V) = \sum_{|\alpha| + |\beta| \leq p} \frac{1}{\alpha! \beta!} F^{(\alpha, \beta)}(u, v + \varphi(u)) U^\alpha$$

$$\left(V + \sum_{1 \leq |\kappa| \leq p} \frac{1}{\kappa!} D^\kappa \varphi(u) U^{\kappa}\right)^\beta \mod(U, V)^{p+1}$$

defined for $(u, v) \in E_\varphi$, where $E_\varphi = \{(u, v) \in \Omega \times \mathbb{R}^l : (u, v + \varphi(u)) \in E\}$.

Then $F_\varphi$ is a $C^p$-Whitney field on $E_\varphi$ such that, for any $c \in \partial \Omega$

$$F^{(\alpha, \beta)}_\varphi(u, v) = o(t(u)^{p-|\alpha|-|\beta|}),$$

when $u \rightarrow c$ and $|\alpha| + |\beta| \leq p$.

**Proof.** — It is easy to check that $F_\varphi$ fulfills the condition (***) from Introduction, thus it is a $C^p$-Whitney field on $E_\varphi$. Besides

$$F_\varphi(u, v; U, V) = \sum_{|\alpha| + |\beta| \leq p} \frac{1}{\alpha! \beta!} F^{(\alpha, \beta)}(u, v + \varphi(u)) U^\alpha \times$$

$$\sum_{\gamma + \sum_\delta w = \beta} \frac{\beta!}{\gamma! \prod_\delta \delta_w!} V^\gamma \prod_\kappa \left[\frac{1}{\kappa! \delta_w!} U^{\delta_w} (D^\kappa \varphi(u))^{\delta_w}\right] \mod(U, V)^{p+1},$$

where $w = (\delta_w : \delta_w \in \mathbb{N}^n)$.
thus

\[ F_{\varphi}^{(\sigma,\gamma)}(u, v) = \sum_{\alpha+\sum_{\kappa} \delta_\kappa = \sigma} \left[ \right] F^{(\alpha, \gamma + \sum_\kappa \delta_\kappa)}(u, v + \varphi(u)) \prod_\kappa (D^\kappa \varphi(u))^{\delta_\kappa}, \]

where \([.]\) denotes constants. To conclude notice that

\[ F^{(\alpha, \gamma + \sum_\kappa \delta_\kappa)}(u, v + \varphi(u)) \prod_\kappa (D^\kappa \varphi(u))^{\delta_\kappa} = \]

\[ o(1) t(u)^{p - |\alpha| - |\gamma| - \sum_\kappa |\delta_\kappa|} C \prod_\kappa d(u, \partial \Omega)^{-|\delta_\kappa|} |\alpha| + |\delta_\kappa| = \]

\[ o(t(u)^{p - |\sigma| - |\gamma|}). \]

\[ \square \]

**Remark 4.2.** — If \( E = \{(u, \varphi(u)) : u \in \Omega\} \) (resp. \( E = \Omega \times \mathbb{R}^l \)), then \( F_\varphi \) extends to a \( C^p \)-Whitney field on \( \overline{E_\varphi} = \overline{\Omega} \times 0 \) (resp. \( \overline{E_\varphi} = \overline{\Omega} \times \mathbb{R}^l \)) \( p \)-flat on \( \partial E_\varphi = \partial \Omega \times 0 \) (resp. \( \partial E_\varphi = \partial \Omega \times \mathbb{R}^l \)).

**Proof.** — The both cases follow from the Hestenes Lemma. \( \square \)

**Proposition 4.3.** — Under the assumptions of Proposition 4.1, assume additionally that the mapping \( \varphi \) is \( \Lambda_{p+1} \)-regular, \( E \) and \( \Omega \) are both 1-regular and \( \overline{E} \) and \( \partial \Omega \times \mathbb{R}^l \) are simply separated \((*)\). Then there exists a constant \( M > 0 \) such that, for each \( F \in \mathcal{C}^p(\overline{E}, \partial E) \), if \( \omega \) is a modulus of continuity of \( F \), then \( M \omega \) is a modulus of continuity of \( F_\varphi \).

Moreover, if \( |F^\kappa(u, w)| \leq \omega(t(u)) t(u)^{p - |\kappa|} \), when \( (u, w) \in E \) and \( |\kappa| \leq p \), then \( |F^\kappa_\varphi(u, v)| \leq M \omega(t(u)) t(u)^{p - |\kappa|} \), when \( (u, v) \in E_\varphi \) and \( |\kappa| \leq p \).

**Proof.** — Observe that \( E_\varphi \) is 1-regular. Let \( \sigma \in \mathbb{N}^k \), \( \gamma \in \mathbb{N}^l \) be such that \( |\sigma| + |\gamma| = p \) and let \( (u_i, v_i) \in E_\varphi \), \((i = 1, 2)\). We have to estimate

\[ |F^{(\sigma, \gamma)}_\varphi(u_1, v_1) - F^{(\sigma, \gamma)}_\varphi(u_2, v_2)| \leq \]

\[ \sum_{\alpha + \sum_{\kappa} \delta_\kappa = \sigma} \left[ . \right] F^{(\alpha, \gamma + \sum_\kappa \delta_\kappa)}(u_1, v_1 + \varphi(u_1)) \prod_\kappa (D^\kappa \varphi(u_1))^{\delta_\kappa} - \]

\[ F^{(\alpha, \gamma + \sum_\kappa \delta_\kappa)}(u_2, v_2 + \varphi(u_2)) \prod_\kappa (D^\kappa \varphi(u_2))^{\delta_\kappa}. \]

Fix \( \lambda = (\alpha, \gamma + \sum_\kappa \delta_\kappa) \) and put \( x_i = (u_i, v_i + \varphi(u_i)) \) and

\[ \theta(u) = \prod_\kappa (D^\kappa \varphi(u))^{\delta_\kappa}. \]

\((*)\) See the beginning of Section 5 for the definition of simple separation.
Case I: $|x_1 - x_2| \geq \frac{1}{2} d(u_i, \partial \Omega)$ for $i = 1, 2$.

$$|F^\lambda(x_i)\theta(u_i)| \leq \omega(d(x_i, \partial E))d(x_i, \partial E)^{p-|\lambda|}|\theta(u_i)| \leq$$

$$\omega(Cd(u_i, \partial \Omega))[Cd(u_i, \partial \Omega)]^{p-|\lambda|}|\theta(u_i)| \leq$$

$$\omega(2C|x_1 - x_2|)[Cd(u_i, \partial \Omega)]^{p-|\lambda|} \prod_{\kappa} d(u_i, \partial \Omega)^{-|\delta_{\kappa}||x_i|+|\delta_{\kappa}|} \leq M\omega(|x_1 - x_2|).$$

Case II: $|x_1 - x_2| \leq \frac{1}{2} d(u_i, \partial \Omega)$.

$$|F^\lambda(x_1)\theta(u_1) - F^\lambda(x_2)\theta(u_2)| \leq$$

$$\left[ \sum_{1 \leq |\mu| \leq p-|\lambda|} \frac{1}{\mu!} |F^{\lambda+\mu}(x_1)||x_2 - x_1|^{|\mu|} + \omega(|x_1 - x_2|)|x_1 - x_2|^{p-|\lambda|} \right] \theta(u_2) +$$

$$|F^\lambda(x_1)| \sup_{z \in [u_1, u_2]} \sum_{j=1}^k |D^{(j)}\theta(z)||u_1 - u_2| \leq$$

$$\left[ \sum_{1 \leq |\mu| \leq p-|\lambda|} \frac{1}{\mu!} \omega(d(x_1, \partial E))d(x_1, \partial E)^{p-|\lambda|-|\mu|}|x_1 - x_2|d(u_1, \partial \Omega)^{|\mu|-1} +$$

$$\omega(|x_1 - x_2|)d(u_1, \partial \Omega)^{p-|\lambda|} \right] \theta(u_2) +$$

$$\omega(d(x_1, \partial \Omega))|x_1 - x_2| \sup_{z \in [u_1, u_2]} \sum_{j=1}^k |D^{(j)}\theta(z)|$$

$$\left[ C_1 \omega(d(u_1, \partial \Omega))|x_1 - x_2|d(u_1, \partial \Omega)^{p-|\lambda|-1} +$$

$$\omega(|x_1 - x_2|)d(u_1, \partial \Omega)^{p-|\lambda|} \right] \theta(u_2) +$$

$$C_2\omega(d(u_1, \partial \Omega))|x_1 - x_2| \sup_{z \in [u_1, u_2]} \prod_{\kappa} d(z, \partial \Omega)^{-|\delta_{\kappa}||x_i|+|\delta_{\kappa}|-1}.$$
For any constant $\varepsilon > 0$, we can specify this operator in such a way that for each $F \in \mathcal{E}^p(\overline{E}, \partial E)$, $LF$ is flat outside the neighborhood $\Delta_{\varepsilon}(E) := \{ x \in \mathbb{R}^n : d(x, E) < \varepsilon d(x, \partial E) \}$.

5. A generalization to a finite tower of $\Lambda_p$-regular leaves

Here we will generalize the extension operator $L$ to the ideal $\mathcal{E}^p(\overline{E}, \partial E)$, where $E$ is a finite disjoint union $E = E_1 \cup \cdots \cup E_s$ of graphs of $\Lambda_p$-regular mappings $\varphi_\sigma : \Omega \to \mathbb{R}^l$ ($\sigma = 1, \ldots, s$) defined on a common open $\Lambda_p$-regular cell $\Omega \subset \mathbb{R}^k$. Put $r_\sigma(u) := |\varphi_\sigma(u) - \varphi_s(u)|$ for $\sigma = 1, \ldots, s - 1$ and $u \in \Omega$.

We first define $LF$ for any $F \in \mathcal{E}^p(\overline{E}, \overline{E_1} \cup \cdots \cup \overline{E_{s-1}} \cup \partial E_s)$.

Then we put

$$LF = \begin{cases} \prod_{i=1}^{s-1} \prod_{\sigma=1}^l \xi \left( \sqrt{1 - \frac{w_i}{r_\sigma(u)}} \right) \mathcal{L}( (F|\overline{E_s})_{\varphi_s} ) - \varphi_s, & \text{on } \Omega \times \mathbb{R}^l \\ 0, & \text{on } (\mathbb{R}^k \setminus \Omega) \times \mathbb{R}^l, \end{cases}$$

which gives an extension operator according to Proposition 3.6 (used repeatedly with $t(u) := \min\{ \{r_\sigma(u)\}, d(u, \partial \Omega) \}$), Remark 3.8 and Proposition 4.1.

Let now consider a general case where $F$ is any element of $\mathcal{E}^p(\overline{E}, \partial E)$. Proceeding by induction, assume that $\mathcal{L}(F|\overline{E_1} \cup \cdots \cup \overline{E_{s-1}})$ has already been defined. Then $H := F - TL(F|\overline{E_1} \cup \cdots \cup \overline{E_{s-1}})|\overline{E} \in \mathcal{E}^p(\overline{E}, \overline{E_1} \cup \cdots \cup \overline{E_{s-1}} \cup \partial E_s)$ and we put

$$LF = LH + \mathcal{L}(F|\overline{E_1} \cup \cdots \cup \overline{E_{s-1}}).$$

For any $\varepsilon > 0$, we can specify this operator in such a way that $LF$ is $p$-flat outside the set $\Delta_{\varepsilon}(E) := \{ x \in \mathbb{R}^n : d(x, E) < \varepsilon d(x, \partial E) \}$.

6. Extension operator for a closed definable subset of $\mathbb{R}^n$

DEFINITION 6.1 (cf. [10]). — Let $A, B, Z \subset \mathbb{R}^n$. We say that $A$ and $B$ are simply $Z$-separated if one of the following equivalent conditions holds

1. $\exists M > 0 \forall x \in A, \; d(x, B) \geq Md(x, Z)$;
2. $\exists C > 0 \forall x \in \mathbb{R}^n, \; d(x, A) + d(x, B) \geq Cd(x, Z)$. (If (1) holds, one can take $C = M/(M + 1)$.)

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We say that $A$ and $B$ are simply separated if they are simply $A \cap B$-separated.

**Proposition 6.2.** — Let $E_i \supset E'_i$ ($i = 1, \ldots, s$) be closed subsets of $\mathbb{R}^n$ and let $C > 0$ be a constant such that, for any $i, j \in \{1, \ldots, s\}, i \neq j$ and any $x \in \mathbb{R}^n$

$$d(x, E_i) + d(x, E_j) \geq Cd(x, E'_i).$$

Let $\varepsilon \in (0, C/2]$. Put $\Gamma_\varepsilon(E_i, E'_i) := \{x \in \mathbb{R}^n : d(x, E_i) < \varepsilon d(x, E'_i)\}$. Suppose that, for each $i = 1, \ldots, s$

$$L_i : \mathcal{E}^p(E_i, E'_i) \longrightarrow \mathcal{C}^p(\mathbb{R}^n)$$

is an extension operator such that $L_i F$ is $p$-flat outside $\Gamma_\varepsilon(E_i, E'_i)$, for any $F \in \mathcal{E}^p(E_i, E'_i)$.

Then the formula

$$LF = \sum_{i=1}^s L_i(F|E_i)$$

defines an extension operator $L : \mathcal{E}^p(\bigcup_i E_i, \bigcup_i E'_i) \longrightarrow \mathcal{C}^p(\mathbb{R}^n)$. Moreover, if each $L_i$ preserves (up to a multiplicative constant) a modulus of continuity, then $L$ has the same property.

**Proof.** — It suffices to check that $\Gamma_\varepsilon(E_i, E'_i) \cap \Gamma_\varepsilon(E_j, E'_j) = \emptyset$, if $i \neq j$. If there were $x \in \Gamma_\varepsilon(E_i, E'_i) \cap \Gamma_\varepsilon(E_j, E'_j)$, then

$$2\varepsilon[d(x, E'_i) + d(x, E'_j)] > 2[d(x, E_i) + d(x, E_j)] \geq C[d(x, E'_i) + d(x, E'_j)],$$

a contradiction. $\square$

A proof of the following theorem will be given in the next section.

**$\Lambda_p$-regular Decomposition Theorem 6.3.** — Let $E$ be a closed subset of $\mathbb{R}^n$ definable in some fixed o-minimal structure on the ordered field of the real numbers $\mathbb{R}$. Let $k = \dim E$. Let $Z$ be any definable subset of $E$ of dimension $< k$.

Then there exists a finite decomposition

$$E = M_1 \cup \cdots \cup M_s \cup A$$

such that each $M_i$ is a finite tower of $\Lambda_p$-regular $k$-dimensional definable leaves in an appropriate linear coordinate system, $A$ is a closed definable subset of $\dim < k$ containing $Z$ and, for any $i, j \in \{1, \ldots, s\}$ ($i \neq j$), $M_i$ and $M_j$ are simply $\partial M_i$-separated and, for any $i$, $M_i$ and $A$ are simply $\partial M_i$-separated.
In order to define an extension operator for any closed definable subset $E \subset \mathbb{R}^n$ we will use induction on $\dim E$. By the induction hypothesis we have an extension operator

$$L_0 : \mathcal{E}^p(\bigcup_{i=1}^q \partial M_i \cup A) \rightarrow \mathcal{C}^p(\mathbb{R}^n),$$

and by Section 5 combined with Proposition 6.2 we have an extension operator

$$L_1 : \mathcal{E}^p(E, \bigcup_{i=1}^q \partial M_i \cup A) \rightarrow \mathcal{C}^p(\mathbb{R}^n).$$

Now an extension operator for $E$ is defined by the formula

$$L_F = L_1[F - TL_0(\bigcup \partial M_i \cup A)|E] + L_0(\bigcup \partial M_i \cup A).$$

7. Proof of $\Lambda_p$-regular Decomposition Theorem

Let $P \subset \mathbb{R}^n$ be any definable subset and $V$ - a linear subspace of $\mathbb{R}^n$ of dimension $n - k$. Following [10], we will say that $P$ is perfectly situated relative to $V$ if, for a/any linear complement $W$ of $V$ in $\mathbb{R}^n$, $P$ can be represented as a disjoint union

$$P = \bigcup \{\hat{\varphi} : \varphi \in \mathcal{F}\}$$

of a finite family $\mathcal{F}$ of definable $C^1$-mappings $\varphi : \Delta\varphi \rightarrow V$ defined on connected $C^1$-submanifolds $\Delta\varphi \subset W$ and with bounded derivatives ($\hat{\varphi}$ stands here for the graph $\{u + \varphi(u) : u \in \Delta\varphi\}$ of $\varphi$).

We will use the following

**Theorem 7.1** (cf. [10], Theorem 0). — Let $\Sigma = \{\sigma \subset \{1, \ldots, n\} : \text{card } \sigma = n - k\} = \{\sigma_1, \ldots, \sigma_q\}$, where $q = \binom{n}{k}$.

Let $V_i = \bigoplus_{\nu \in \sigma_i} \mathbb{R}e_{\nu}$ $(i = 1, \ldots, q)$, where $e_1, \ldots, e_n$ is the canonical basis in $\mathbb{R}^n$.

Any definable closed subset $E \subset \mathbb{R}^n$ of dimension $k$ is a union $E = \bigcup_{i=1}^q E_i$ of definable closed subsets $E_i$ such that, for each $i$, $E_i$ is perfectly situated relative to $V_i$ and, for each $j \neq i$, $E_i$ and $E_j$ are simply separated and $\dim(E_i \cap E_j) < k$.

From the last theorem and easy properties of simply separated sets (see [10], Proposition 2; (1) and (3)), it follows that it suffices to prove $\Lambda_p$-regular Decomposition Theorem for each $E_i$ and $Z_i = (Z \setminus E_i) \cup (\bigcup_{j \neq i} E_i \cap E_j)$ separately, therefore - up to a permutation of variables - it suffices to prove it assuming that $E$ is perfectly situated relative to $0 \times \mathbb{R}^l$, where $l = n - k$.

The proof in this case is based on the following two propositions.
Proposition 7.2 ([6], Proposition 2). — If \( \varphi : \Omega \to \mathbb{R}^l \) is a definable \( \Lambda_1 \)-regular mapping defined on an open \( \Omega \subset \mathbb{R}^k \), then there exists a closed definable subset \( Z \) of \( \Omega \) such that \( \dim Z < k \) and \( \varphi|\Omega \setminus Z \) is \( \Lambda_p \)-regular mapping on \( \Omega \setminus Z \).

Proposition 7.3 ([6], Proposition 4). — For any definable open subset \( \Omega \subset \mathbb{R}^k \), there exists a finite family \( S \) of disjoint subsets of \( \Omega \) such that \( \dim(\Omega \setminus \bigcup S) < k \) and each \( S \in S \) is an open definable \( \Lambda_p \)-regular cell in an appropriate linear system of coordinates in \( \mathbb{R}^k \).

Proof of Proposition 7.3. — See [6], Proposition 4, where the set is assumed bounded, but this assumption is not essential. Alternatively, first one can apply [10]; Theorem 1, \((B_k)\) to get the case \( p = 1 \) of Proposition 7.3, which is the theorem of Kurdyka [5] and Parusiński [9], and then by induction on \( k \) one gets the case of any \( p \geq 1 \), applying Proposition 7.2. \( \square \)

To finish the proof of the theorem, first represent \( E \) as union of graphs with bounded derivatives:

\[ E = \bigcup \{ \hat{\varphi} : \varphi \in \mathcal{F} \}, \]

as in the beginning of the section. Adding to \( Z \) all the graphs with \( \dim \Delta \varphi < k \), one can assume that

\[ E = Z \cup \bigcup \{ \hat{\varphi} : \varphi \in \mathcal{F}_* \}, \]

where \( \mathcal{F}_* = \{ \varphi \in \mathcal{F} : \Delta \varphi \text{ non-empty open in } \mathbb{R}^k \} \). By Proposition 7.2, for each \( \varphi \in \mathcal{F}_* \) there exists a closed definable subset \( K_\varphi \) of \( \Delta \varphi \) of \( \dim < k \) such that \( \varphi|\Delta \varphi \setminus K_\varphi \) is \( \Lambda_p \)-regular. Let

\[ \Theta := \pi(Z) \cup \bigcup \{ \partial \Delta \varphi \cup K_\varphi : \varphi \in \mathcal{F}_* \}, \]

where \( \pi : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k \) is the canonical projection. Take a family \( S \) as in Proposition 7.3 for the open subset

\[ \Omega := \bigcup \{ \Delta \varphi : \varphi \in \mathcal{F}_* \} \setminus \Theta. \]

Now it suffices to define, for each \( S \in S \)

\[ M_S := E \cup \pi^{-1}(S) \quad \text{and} \quad A := E \setminus \bigcup \{ M_S : S \in S \}. \]
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