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Tome 58, nº 2 (2008), p. 383-404.

<http://aif.cedram.org/item?id=AIF_2008__58_2_383_0>

Ann. Inst. Fourier, Grenoble
58, 2 (2008) 383-404

A LINEAR EXTENSION OPERATOR FOR WHITNEY FIELDS ON CLOSED O-MINIMAL SETS

by Wiesław PAWŁUCKI (*)

Dedicated to my wife Jolanta

Abstract. — A continuous linear extension operator, different from Whitney's, for $\mathcal{C}^p$-Whitney fields (p finite) on a closed o-minimal subset of $\mathbb{R}^n$ is constructed. The construction is based on special geometrical properties of o-minimal sets earlier studied by K. Kurdyka with the author.

Résumé. — On construit un opérateur d'extension linéaire et continu pour les champs de Whitney de classe $\mathcal{C}^p$ (p fini) sur un sous-ensemble fermé o-minimal de $\mathbb{R}^n$. La construction, différente de celle de Whitney, est basée sur des propriétés géométriques spéciales des ensembles o-minimaux, étudiées avant par K. Kurdyka et l'auteur.

1. Introduction

In 1997 K. Kurdyka and the author gave in [6] the following o-minimal version of the Whitney extension theorem:

Theorem 1.1 ([6]). — *Given any o-minimal structure on the ordered field of real numbers $\mathbb{R}$, a compact definable subset $E \subset \mathbb{R}^n$, a definable $\mathcal{C}^p$-Whitney field F on E, where $p \in \mathbb{N} \setminus \{0\}$, then for any integer $q \geqslant p$, there exists a definable $\mathcal{C}^p$-extension $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of F which is $\mathcal{C}^q$ on $\mathbb{R}^n \setminus E$.*

However, the extension operator $F \mapsto f$ from [6] is not linear and it was not clear how the construction from [6] based on o-minimal geometry could be adapted to get an extension operator for *all* Whitney fields on

Keywords: Whitney field, extension operator, o-minimal structure, subanalytic set.
Math. classification: Primary 26B05, 14P10. Secondary 32B20, 03C64.
(*) Research partially supported by the KBN grant 5 PO3A 005 21 and the European Community IHP-Network RAAG (HPRN-CT-2001-00271).

any compact (or more generally closed) o-minimal subset E of $\mathbb{R}^n$. The present paper is devoted to this question. The main goal here is to prove the following

Theorem 1.2. — *Let E be a closed o-minimal subset of $\mathbb{R}^n$ and $p \in \mathbb{N}$. Let $\mathcal{E}^p(E)$ denote the Fréchet algebra of all $\mathcal{C}^p$-Whitney fields on E.*

Then there exists a continuous linear extension operator $\mathcal{L} : \mathcal{E}^p(E) \longrightarrow \mathcal{C}^p(\mathbb{R}^n)$ which has the following properties

1. *$\mathcal{L}$ is a finite composition of operators each of which either preserves definability or (only if $p > 0$) is an integration with respect to a parameter;*
2. *operators preserving definability in (1) are only of the following five types: substituting with a definable mapping; taking a linear combination with definable coefficients; differentiation; restriction to a definable subset and extending by zero;*
3. *there exists a constant $M > 0$ such that if ω is a modulus of continuity of a field F, then $M\omega$ is a modulus of continuity of $\mathcal{L}F$.*

Since $\mathcal{L}$ involves integration, it may not preserve definability in the initial o-minimal structure where E is definable. For example, if F is a (globally) subanalytic $\mathcal{C}^p$-Whitney field, then $\mathcal{L}F$ can *a priori* involve the function $t \mapsto t \log t$, not subanalytic at 0. By a result of Lion and Rolin [7], we get in this case the following

Corollary 1.3. — *Let $\mathcal{A}$ denote the algebra of real functions generated by (globally) subanalytic functions and their logarithms; i.e. $\mathcal{A}$ consists of all functions of the form $P(h_1, \dots, h_m, \log h_1, \dots, \log h_m)$, where $h_i : \mathbb{R}^n \longrightarrow \mathbb{R} \quad (i = 1, \dots, m)$ are subanalytic, $m \in \mathbb{N} \setminus \{0\}$, $P \in \mathbb{R}[Y_1, \dots, Y_{2m}]$, and where we adopt the convention: $\log t = 0$, for $t \leqslant 0$. Let E be a closed subanalytic subset of $\mathbb{R}^n$ and $p \in \mathbb{N}$.*

Then there exists a continuous linear extension operator $\mathcal{L} : \mathcal{E}^p(E) \longrightarrow \mathcal{C}^p(\mathbb{R}^n)$ which has the following properties:

1. *if F is a $\mathcal{C}^p$-Whitney field on E all derivatives of which $F^\varkappa$ are (restrictions to E of) functions in $\mathcal{A}$, then $\mathcal{L}F \in \mathcal{A}$;*
2. *there exists a constant $M > 0$ such that if ω is a modulus of continuity of a field F, then $M\omega$ is a modulus of continuity of $\mathcal{L}F$.*

The case $p = 0$ in Theorem 1.2, when integration is not used seems worth being stated separately

Corollary 1.4. — *Let E be a closed o-minimal subset of $\mathbb{R}^n$ and let $\mathcal{C}(E)$ denote the Fréchet space of all real continuous functions on E*

Then there exists a continuous linear extension operator $\mathcal{L} : \mathcal{C}(E) \longrightarrow \mathcal{C}(\mathbb{R}^n)$ preserving definability and such that there exists $M > 0$ such that, if ω is a modulus of continuity for $F \in \mathcal{C}(E)$, then $M\omega$ is a modulus of continuity for $\mathcal{L}F$.

By an o-minimal subset of an Euclidean space $\mathbb{R}^n$ we mean a subset definable in any o-minimal structure on the ordered field of real numbers $\mathbb{R}$ (see [2, 3] for the definition and fundamental properties).

We refer the reader to [13], [4], [8], [11] or/and [12] for basic facts on Whitney fields. It will be convenient for us to adopt the following definition of a Whitney field.

Let $p \in \mathbb{N} \setminus \{0\}$ and let A be a locally closed subset of $\mathbb{R}^n$; i.e. contained and closed in some open subset $G \subset \mathbb{R}^n$. A $\mathcal{C}^p$-*Whitney field* on A is a polynomial

$$F(u, X) = \sum_{|\varkappa| \leqslant p} \frac{1}{\varkappa!} F^{\varkappa}(u) X^{\varkappa} \in \mathcal{C}(A)[X] = \mathcal{C}(A)[X_1, \ldots, X_n],$$

which fulfills the following condition

$(*)$ for each $c \in A$ and each $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leqslant p$

$$D_X^{\alpha} F(a, 0) - D_X^{\alpha} F(b, a-b) = o(|a-b|^{p-|\alpha|}), \quad \text{when } A \ni a \to c,\ A \ni b \to c,$$

or equivalently (see [8], Chapter I, Theorem 2.2) - the condition

$(**)$ for each $c \in A$

$$F(a, x-a) - F(b, x-b) = o(|x-a|^p + |x-b|^p),$$

uniformly with respect to $x \in \mathbb{R}^n$, when $A \ni a \to c$, $A \ni b \to c$.

We will denote by $\mathcal{E}^p(A)$ the real algebra of all $\mathcal{C}^p$-Whitney fields on A. It is a Fréchet algebra with the topology defined by the following system of seminorms

$$||F||_p^K = |F|_p^K + \sup_{\substack{a,b \in K \\ a \neq b \\ |\alpha| \leqslant p}} \frac{|D_X^{\alpha} F(a, 0) - D_X^{\alpha} F(b, a-b)|}{|a-b|^{p-|\alpha|}},$$

where K is a compact subset of A and $|\ .\ |_p^K$ is a seminorm defined by

$$|F|_p^K = \sup_{\substack{a \in K \\ |\alpha| \leqslant p}} |F^{\alpha}(a)|.$$

Let $\mathcal{C}^p(G)$ denote the usual Fréchet algebra of real functions of class $\mathcal{C}^p$ ($\mathcal{C}^p$-functions) on G. Then we have the following homomorphism of Fréchet algebras

$$T : \mathcal{C}^p(G) \longrightarrow \mathcal{E}^p(A), \quad Tf(a,X) = T_a^p f(X) = \sum_{|\varkappa| \leqslant p} \frac{1}{\varkappa!} D^{\varkappa} f(a) X^{\varkappa},$$

and the Whitney extension theorem [13] says that there exists a linear continuous mapping

$$W : \mathcal{E}^p(A) \longrightarrow \mathcal{C}^p(G) \qquad \text{such that} \qquad T \circ W = id_{\mathcal{E}^p(A)},$$

called an *extension operator*.

A subset E of $\mathbb{R}^n$ is said to be *1-regular* (*with a constant* $C \geqslant 1$) if any two points a, b of E can be joined in E by a rectifiable arc $\gamma : [0,1] \longrightarrow E$ of length $|\gamma| \leqslant C|a-b|$.

If $F \in \mathcal{E}^p(A)$ and K is a compact 1-regular subset of A with a constant C, then

$$|F|_p^K \leqslant ||F||_p^K \leqslant 2n^{\frac{p}{2}} C^p |F|_p^K \qquad \text{(See [12], p.76, (2.5.1)).}$$

Consequently, if every compact subset L of A is contained in a 1-regular compact subset K of A, then the topology of $\mathcal{E}^p(A)$ is defined by the system of seminorms $| \,.\, |_p^K$.

As was shown by Glaeser [4] (see also [8], [12] or [11]) it is convenient to use a notion of a modulus of continuity in connection with Whitney fields. By a *modulus of continuity* we will understand any continuous, increasing and concave function $\omega : [0,+\infty) \longrightarrow [0,+\infty)$, vanishing at 0. By a modulus of continuity of a $\mathcal{C}^p$-Whitney field

$$F(u,X) = \sum_{|\varkappa| \leqslant p} \frac{1}{\varkappa!} F^{\varkappa}(u) X^{\varkappa}$$

on a subset A of $\mathbb{R}^n$ we will understand such a modulus of continuity ω that

$$|D_X^{\alpha}(a,0) - D_X^{\alpha}(b,a-b)| \leqslant \omega(|a-b|)|a-b|^{p-|\alpha|},$$

whenever $|\alpha| \leqslant p$ and $a,b \in A$. For a $\mathcal{C}^p$-function $f \in \mathcal{C}^p(G)$ on an open subset G, by its modulus of continuity we will understand a modulus of continuity of the $\mathcal{C}^p$-Whitney field Tf on G.

Every $\mathcal{C}^p$-Whitney field on a compact subset of $\mathbb{R}^n$ admits a modulus of continuity. If a $\mathcal{C}^p$-Whitney field F on a subset A has a modulus of continuity ω, then it is easily seen that F extends by uniform continuity to a $\mathcal{C}^p$-Whitney field on $\overline{A}$ with the same modulus of continuity. Whitney's extension operator [13] has the following property (see [4]):

There exists a constant M depending only on p and n such that, for every $F \in \mathcal{E}^p(A)$ admitting a modulus of continuity ω, $M\omega$ is a modulus of continuity for WF. (In fact a localization by a partition of unity is necessary.)

We have also the following

Proposition 1.5. — *Let F be a $\mathcal{C}^p$-Whitney field on a (locally) closed 1-regular with constant C subset A.*

(1) *If ω is a modulus of continuity of F on A, then $|F^\alpha(a) - F^\alpha(b)| \leqslant \omega(|a-b|)$, whenever $|\alpha| = p$, $a, b \in A$.*
(2) *If ω is a modulus of continuity such that $|F^\alpha(a) - F^\alpha(b)| \leqslant \omega(|a-b|)$, whenever $|\alpha| = p$, $a, b \in A$, then $n^{\frac{p}{2}} C^p \omega$ is a modulus of continuity of F on A.*

Proof. — (1) being trivial, for (2) see again [12], (2.5.1), p.76. □

Shortly, our construction of the extension operator $\mathcal{L}$ is as follows. First we show how to extend $\mathcal{C}^p$-Whitney fields from a linear subspace $\mathbb{R}^k \times 0$ of $\mathbb{R}^n$. Then we generalize the construction to the set of the form $\overline{\Omega} \times 0$, where Ω is open in $\mathbb{R}^k$ for fields flat on $\partial\Omega \times 0$, simply by Hestenes Lemma. Using induction on dimension of A, this gives an extension operator for $A = \overline{\Gamma}$, where $\Gamma = \Omega \times 0$ assuming we have it already built for the *boundary* $\partial\Gamma = \overline{\Gamma} \setminus \Gamma$ of Γ which in this case is $\partial\Omega \times 0$. The next generalization is by taking $A = \overline{\Gamma}$, where Γ is a Λ_p-*regular leaf* of dimension k in the sense of [6], and again assuming the fields are flat on $\partial\Gamma$. Additionally, the extension can be chosen vanishing outside a *conical neighbourhood* of Γ; i.e. the set $\{x \in \Omega \times \mathbb{R}^{n-k} : d(x, \Gamma) < \varepsilon d(x, \partial\Gamma)\}$, where Ω is the orthogonal projection of Γ to $\mathbb{R}^k \times 0$ and ε is a positive arbitrary constant. The next generalization is to the closure of a *finite tower* of Λ_p-regular leaves lying over a common *open* Λ_p-*regular cell* in $\mathbb{R}^k$. To finish the construction we will prove that every closed definable k-dimensional subset A admits a finite decomposition $A = M_0 \cup \cdots \cup M_s$ such that each M_i is a finite tower of definable Λ_p-regular leaves in a suitable linear coordinate system and for any $i, j \in \{0, \ldots, s\}$, where $i \neq j$, $\overline{M}_i$ and $\overline{M}_j$ are *simply separated relative to* ∂M_i; i.e. $d(x, M_j) \geqslant C d(x, \partial M_i)$, for each $x \in M_i$, with some positive constant C. (The proof of this Λ_p-regular Decomposition Theorem is based on [6] and [10].)

Acknowledgements. The present paper in its initial form was turned out in August 2003 when the author participated in Short Programme on Analysis and Resolution of Singularities at the Centre de Recherches Mathématiques of the Université de Montréal. He wishes to thank the CRM, the organizers and especially Edward Bierstone and Pierre D. Milman for stimulating discussions. He also thanks the anonymous referee for valuable remarks.

2. Extension operator for a linear subspace

Observe that if Ω is an open subset of $\mathbb{R}^k$ and $A = \Omega \times 0 \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$, then the algebra $\mathcal{E}^p(A)$ can be identified with the algebra of polynomials

$$F(u,W) = \sum_{|\alpha| \leqslant p} \frac{1}{\alpha!} F^\alpha(u) W^\alpha = \sum_{|\alpha| \leqslant p} \frac{1}{\alpha!} F^\alpha(u) W_1^{\alpha_1} \dots W_l^{\alpha_l},$$

where $l = n-k$ and $F^\alpha \in \mathcal{C}^{p-|\alpha|}(\Omega)$, for each $\alpha \in \mathbb{N}^l$ such that $|\alpha| \leqslant p$ (cf. [4], Chap.III, (8.4)).

Let us now consider the case $k = n-1$ and $A = \mathbb{R}^k \times 0$. Then the extension operator will be produced using regularization of functions F^α by convolution. Strictly, we have the following

Proposition 2.1. — *Let $\sigma \in \{0, \dots, p\}, g \in \mathcal{C}^{p-\sigma}(\mathbb{R}^k), \varphi \in \mathcal{C}^p(\mathbb{R}^k)$. Assume that supp$\varphi$ is compact and put*

$$\varphi_w(v) = \frac{1}{w^k} \varphi(\frac{v}{w})$$

$$\text{and} \quad G(u,w) = \frac{1}{\sigma!} (g \star \varphi_w)(u) w^\sigma = \frac{1}{\sigma!} \int_{\mathbb{R}^k} g(u-v) \varphi_w(v) w^\sigma dv,$$

for $u \in \mathbb{R}^k$ and $w \in \mathbb{R}, w > 0$.

Then $G : \mathbb{R}^k \times (0, +\infty) \longrightarrow \mathbb{R}$ is a $\mathcal{C}^p$-function and for every $(\alpha, \beta) \in \mathbb{N}^k \times \mathbb{N}$ such that $|\alpha| + \beta \leqslant p$

$$\lim_{w \to 0} D^{(\alpha,\beta)} G(u,w) = \begin{cases} 0\,, & \text{if } \beta < \sigma \\ D^\alpha g(u) \int \varphi\,, & \text{if } \beta = \sigma \\ \displaystyle\sum_{|\gamma| = \beta - \sigma} \omega_{\gamma\sigma} D^{\alpha+\gamma} g(u)\,, & \text{if } \beta > \sigma \end{cases}$$

uniformly on compact subsets with respect to u, where $\omega_{\gamma\sigma}$ are some constants depending only on γ, σ and φ.

To prove Proposition 2.1 one needs the following

Lemma 2.2. — *For any $r \in \mathbb{R}$ and $\lambda \in \{0, \dots, p\}$*

$$\frac{\partial^\lambda}{\partial w^\lambda}\left[w^r \varphi\left(\frac{v}{w}\right)\right] = w^{r-\lambda}\varphi_\lambda(\frac{v}{w}),$$

where φ_λ is a $\mathcal{C}^{p-\lambda}$-function on $\mathbb{R}^k$ with a compact support and $\int \varphi_\lambda = (k+r)(k+r-1)\cdots(k+r-\lambda+1)\int \varphi$.

Proof of Lemma 2.2. —

$$\frac{\partial}{\partial w}\left[w^r \varphi\left(\frac{v}{w}\right)\right] = rw^{r-1}\varphi\left(\frac{v}{w}\right)+w^r\sum_{i=1}^k \frac{\partial \varphi}{\partial v_i}\left(\frac{v}{w}\right)\left(-\frac{v_i}{w^2}\right) = w^{r-1}\varphi_1\left(\frac{v}{w}\right),$$

where $\varphi_1(v) = r\varphi(v) - \sum_{i=1}^k v_i \frac{\partial \varphi}{\partial v_i}(v)$. Moreover, integrating by parts,

$$\int \varphi_1 = r\int \varphi - \sum_{i=1}^k \int v_i \frac{\partial \varphi}{\partial v_i} = (r+k)\int \varphi$$

and Lemma 2.2 follows by induction. □

Proof of Proposition 2.1. — G is of class $\mathcal{C}^p$ on $\mathbb{R}^k \times (0, +\infty)$, because

$$G(u,w) = \int g(v)\frac{1}{w^k}\varphi\left(\frac{u-v}{w}\right)w^\sigma dv.$$

(I) Assume first that $\beta \leqslant \sigma$ and $|\alpha| \leqslant p - \sigma$. Then

$$D^{(\alpha,\beta)}G(u,w) = \frac{1}{\sigma!}\int D^\alpha g(u-v)w^{\sigma-\beta-k}\varphi_\beta\left(\frac{v}{w}\right)dv =$$

$$\frac{1}{\sigma!}w^{\sigma-\beta}\int D^\alpha g(u-v)\frac{1}{w^k}\varphi_\beta\left(\frac{v}{w}\right)dv \longrightarrow \frac{1}{\sigma!}0^{\sigma-\beta}D^\alpha g(u)\int \varphi_\beta,$$

when $w \to 0$, the convergence being uniform on compact subsets with respect to u. Consequently, the limit is 0, if $\beta < \sigma$ and $D^\alpha g \int \varphi$, if $\beta = \sigma$.

(II) Now assume that $\beta \leqslant \sigma$ and $|\alpha| > p - \sigma$. Then $\alpha = \gamma + \delta$, where $|\gamma| = p - \sigma$ and $\delta \neq 0$.

$$D^{(\gamma,\beta)}G(u,w) = \frac{1}{\sigma!}\int D^\gamma g(u-v)w^{\sigma-\beta-k}\varphi_\beta\left(\frac{v}{w}\right)dv =$$

$$\frac{1}{\sigma!}\int D^\gamma g(v)w^{\sigma-\beta-k}\varphi_\beta\left(\frac{u-v}{w}\right)dv.$$

$$D^{(\alpha,\beta)}G(u,w) = \frac{1}{\sigma!}\int D^\gamma g(v)w^{\sigma-\beta-k}w^{-|\delta|}D^\delta\varphi_\beta\left(\frac{u-v}{w}\right)dv =$$

$$\frac{1}{\sigma!}w^{\sigma-\beta-|\delta|}\int D^\gamma g(u-wv)D^\delta\varphi_\beta(v)dv.$$

Notice that $\sigma - \beta - |\delta| = p - |\alpha| - \beta \geqslant 0$ and $\int D^\delta \varphi_\beta(v)dv = 0$, since φ_β has a compact support. Consequently, $D^{(\alpha,\beta)}G(u,w) \longrightarrow 0$, when $w \to 0$.

(III) Finally, let $\beta > \sigma$. Then $|\alpha| \leqslant p - \beta < p - \sigma$ and $\beta = \sigma + \rho$, where $\rho > 0$. By the case (I),

$$D^{(\alpha,\sigma)}G(u,w) = \frac{1}{\sigma!}\int D^{\alpha}g(u - vw)\varphi_{\sigma}(v)dv.$$

$D^{\alpha}g$ being of class $p - \sigma - |\alpha| \geqslant \rho$, one obtains

$$\begin{aligned} D^{(\alpha,\beta)}G(u,w) =& D^{(0,\rho)}(D^{(\alpha,\sigma)}G)(u,w) \\ =& \frac{1}{\sigma!}\sum_{|\mu|=\rho}\int D^{\alpha+\mu}g(u - vw)(-v)^{\mu}\varphi_{\sigma}(v)dv, \end{aligned}$$

which tends to $\sum_{|\mu|=\rho}\omega_{\mu\sigma}D^{\alpha+\mu}g(u)$ uniformly on compact subsets with respect to u, when $w \to 0$, where $\omega_{\mu\sigma} = \dfrac{1}{\sigma!}\displaystyle\int (-v)^{\mu}\varphi_{\sigma}(v)dv$. □

Proposition 2.3. — *Let $\varphi \in \mathcal{C}^p(\mathbb{R}^k)$ be with compact support and such that $\int \varphi = 1$. Then the formula*

$$\begin{aligned} L(gW^{\sigma})(u,w) = \left(\frac{w}{|w|}\right)^{\sigma} \Bigg[& \frac{1}{\sigma!}(g \star \varphi_{|w|})(u)|w|^{\sigma} \\ & - \sum_{0<|\gamma|\leqslant p-\sigma} \frac{1}{(\sigma + |\gamma|)!}\omega_{\gamma\sigma}L(D^{\gamma}gW^{\sigma+|\gamma|})(u,|w|)\Bigg], \end{aligned}$$

for $\sigma \in \{1,\ldots,p\}$, $g \in \mathcal{C}^{p-\sigma}(\mathbb{R}^k)$, $u \in \mathbb{R}^k$ and $w \in \mathbb{R}\setminus\{0\}$, completed by putting

$$L(gW^{\sigma})(u,0) = 0, \quad \text{and} \quad L(gW^0) = L(g) = g, \quad \text{for } g \in \mathcal{C}^p(\mathbb{R}^k),$$

defines (inductively) a continuous linear extension operator $L = L_p : \mathcal{E}^p(\mathbb{R}^k \times 0) \longrightarrow \mathcal{C}^p(\mathbb{R}^{k+1})$.

Moreover, there exists a constant $M > 0$ (depending only on k, p and φ) such that if ω is a modulus of continuity of a field $F \in \mathcal{E}^p(\mathbb{R}^k \times 0)$, then $M\omega$ is a modulus of continuity of the $\mathcal{C}^p$-function LF.

Proof. — This follows immediately from Proposition 2.1. □

Now we generalize our extension operator to any linear subspace of $\mathbb{R}^n$.

Proposition 2.4. — *Let $\mathbb{R}^k \times 0 \subset \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$, where $l > 1$. Then the formula*

$$L_p(gW_1^{\alpha_1}\cdots W_l^{\alpha_l}) = L_p(L_{p-\alpha_l}(gW_1^{\alpha_1}\cdots W_{l-1}^{\alpha_{l-1}})W_l^{\alpha_l}),$$

where $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l, |\alpha| \leqslant p$ *and* $g \in \mathcal{C}^{p-|\alpha|}(\mathbb{R}^k)$*, defines by induction on* l *a linear continuous extension operator* $L = L_p : \mathcal{E}^p(\mathbb{R}^k \times 0) \longrightarrow \mathcal{C}^p(\mathbb{R}^n)$.

Moreover, there is a constant $M > 0$ *such that if* ω *is a modulus of continuity for* $F \in \mathcal{E}^p(\mathbb{R}^k \times 0)$*, then* $M\omega$ *is a modulus of continuity for* LF.

Proof. — This follows easily by induction from Proposition 2.3. □

3. A generalization to the ideal of $\mathcal{C}^p$-Whitney fields on $\overline{\Omega} \times 0$ p-flat on $\partial\Omega \times 0$ (Ω- an open Λ_p-regular cell in $\mathbb{R}^k = \mathbb{R}^k \times 0 \subset \mathbb{R}^k \times \mathbb{R}^l$)

If A is any locally closed subset of $\mathbb{R}^n$ and B any closed subset of A, $\mathcal{E}^p(A, B)$ will denote the ideal of all $\mathcal{C}^p$-Whitney fields F on A *p-flat* on B; i.e. $F^\alpha(u) = 0$, when $|\alpha| \leqslant p$ and $u \in B$. It is closed in $\mathcal{E}^p(A)$.

Let first Ω be any open subset of $\mathbb{R}^k$. By the Hestenes Lemma (see [12], Lemma 4.3, p.80)

$$\mathcal{E}^p(\overline{\Omega} \times 0, \partial\Omega \times 0) = \{F = \sum_{|\alpha| \leqslant p} \frac{1}{\alpha!} F^\alpha W^\alpha : F^\alpha \in \mathcal{C}^{p-|\alpha|}(\Omega),$$
$$\lim_{u \to a} D^\beta F^\alpha(u) = 0, \text{ if } a \in \partial\Omega, |\beta| \leqslant p - |\alpha|\},$$

and putting

$$\widetilde{F}^\alpha(u) = \begin{cases} F^\alpha(u)\,, & \text{if } u \in \Omega \\ 0\,, & \text{if } u \in \mathbb{R}^k \setminus \Omega \end{cases} \qquad \text{and} \quad \widetilde{F} = \sum_\alpha \frac{1}{\alpha!} \widetilde{F}^\alpha W^\alpha,$$

one obtains a linear continuous extension operator

$$\mathcal{E}^p(\overline{\Omega} \times 0, \partial\Omega \times 0) \ni F \longrightarrow \widetilde{F} \in \mathcal{E}^p(\mathbb{R}^k \times 0, (\mathbb{R}^k \setminus \Omega) \times 0)$$

preserving modulus of continuity.

Now we will consider the case when Ω is an open Λ_p-regular cell in $\mathbb{R}^k$ (cf. [6]). We will first recall the notion of Λ_p-regular mapping. Let $\psi : D \longrightarrow \mathbb{R}^m$ be a mapping on an open subset $D \subset \mathbb{R}^n$. We say that ψ is Λ_p-*regular* (on D) if it is of class $\mathcal{C}^p$ and there is a constant $C \geqslant 0$ such that

$$|D^\varkappa \psi(x)| \leqslant C/d(x, \partial D)^{|\varkappa|-1}, \quad \text{whenever} \quad 1 \leqslant |\varkappa| \leqslant p \quad \text{and} \quad x \in D.$$

Remark 3.1. — Let ψ be Λ_p-regular on D. Then

(1) it is Λ_p-regular on every open $D' \subset D$;

(2) if $A \subset \Omega$ is a 1-regular subset, then the restriction $\psi|A$ is Lipschitz and thus it has a continuous extension $\overline{\psi|A}$ to $\overline{A}$.

We shall say (after [6]) that S is an *open* Λ_p*-regular* (*definable in a given o-minimal structure*) *cell* in $\mathbb{R}^n$ iff

(1) S is an open interval in $\mathbb{R}$, when $n = 1$;

(2) $S = \{(x', x_n) : x' \in T, \quad \psi_1(x') < x_n < \psi_2(x')\}$, where T is an open Λ_p-regular (definable) cell in $\mathbb{R}^{n-1}$ and each ψ_i $(i = 1, 2)$ is a function on T being either real Λ_p-regular (definable) function on T, or identically equal to $-\infty$, or identically equal to $+\infty$, and $\psi_1(x') < \psi_2(x')$, for all $x' \in T$, when $n > 1$.

Remark 3.2. — Such a cell S is 1-regular and if ψ_i is finite it is Lipschitz on T, thus it admits a continuous extension $\overline{\psi}_i$ to $\overline{T}$.

For any open (definable) Λ_p-regular cell in $\mathbb{R}^n$, one defines, by induction on n, a sequence $\rho_j : \overline{S} \longrightarrow \mathbb{R} \cup \{+\infty\}$ $(j = 1, \dots, 2n)$ of the *functions associated with the cell* S:

(1) When $n = 1$ and $S = (a_1, a_2)$, we put

$$\rho_1(x) = \begin{cases} x - a_1\,, & \text{if } a_1 \in \mathbb{R} \\ +\infty\,, & \text{if } a_1 = -\infty \end{cases} \quad \text{and} \quad \begin{cases} \rho_2(x) = a_2 - x\,, & \text{if } a_2 \in \mathbb{R} \\ +\infty\,, & \text{if } a_2 = +\infty. \end{cases}$$

(2) When $n > 1$ and $S = \{(x', x_n) : x' \in T \quad \psi_1(x') < x_n < \psi_2(x')\}$, let σ_j $(j = 1, \dots, 2n-2)$ be the functions associated with T. We put, for any $x = (x', x_n) \in \overline{S}$, $\rho_j(x) = \sigma_j(x')$ for $j = 1, \dots, 2n-2$ and

$$\rho_{2n-1}(x) = \begin{cases} x_n - \overline{\psi}_1(x')\,, & \text{if } \psi_1 : T \to \mathbb{R} \\ +\infty\,, & \text{if } \psi_1 \equiv -\infty \end{cases} \qquad \text{and}$$

$$\rho_{2n}(x) = \begin{cases} \overline{\psi}_2(x') - x_n\,, & \text{if } \psi_2 : T \to \mathbb{R} \\ +\infty\,, & \text{if } \psi_2 \equiv +\infty. \end{cases}$$

Remark 3.3 ([6]*, Lemma 3).* — There exists a constant $\Theta > 0$ such that

$$\Theta \min_j \rho_j(x) \leqslant d(x, \partial S) \leqslant \min_j \rho_j(x), \qquad \text{for} \quad x \in \overline{S}.$$

(We adopt the convention: $d(x, \emptyset) = +\infty$.)

Remark 3.4 ([6]*, Lemma 4).* — The functions ρ_j which are finite are Λ_p-regular on S, Lipschitz on $\overline{S}$ and definable, if S is so.

Lemma 3.5 (cf. [6], Lemma 5). — *Let* $\varphi_\nu : \Omega \longrightarrow \mathbb{R}$ $(\nu = 1, \dots, m)$ *be* Λ_p*-regular functions on an open subset* $\Omega \subset \mathbb{R}^k$*. Assume that* $r(u) :=$

$(\sum_{\nu=1}^{m}\varphi_\nu^2(u))^{\frac{1}{2}} \neq 0$ *for each* $u \in \Omega$. *Then there exists a constant* $\widetilde{C} > 0$ *such that for each* $u \in \Omega$

$$\left|D^\alpha\left(\frac{1}{r}\right)(u)\right| \leqslant \frac{\widetilde{C}}{r(u)\min(r(u), d(u,\partial\Omega))^{|\alpha|}}, \quad \text{where } 0 \leqslant |\alpha| \leqslant p\,;$$

$$\text{consequently} \quad \left|D^\alpha\left(\frac{1}{r}\right)(u)\right| \leqslant \frac{\widetilde{C}}{\min(r(u), d(u,\partial\Omega))^{|\alpha|+1}}.$$

Proof. — Induction on $|\alpha|$. □

Proposition 3.6 (cf. [6], Lemmas 6-7). — *Let* Ω *be an open subset of* $\mathbb{R}^k$, *let* $f \in \mathcal{C}^p(\Omega\times\mathbb{R}^l)$ *and* $r \in \mathcal{C}^p(\Omega)$, *and let* $t : \Omega \longrightarrow (0,+\infty)$ *be any positive function such that* $t(u) \leqslant d(u,\partial\Omega)$ *for any* $u \in \Omega$. *Let* $\varepsilon > 0$ *and put*

$$\Delta_\varepsilon := \{(u,w) \in \Omega\times\mathbb{R}^l : |w| < \varepsilon t(u)\}.$$

Assume that there exists a constant $\widetilde{C} > 0$ *such that* $|D^\alpha(\frac{1}{r})| \leqslant \frac{\widetilde{C}}{t^{|\alpha|+1}}$, *when* $\alpha \in \mathbb{N}^k$, *and for each* $c \in \partial\Omega$, $D^\varkappa f(u,w) = o(t(u)^{p-|\varkappa|})$, *when* $\Delta_\varepsilon \ni (u,w) \to (c,0)$ *and* $\varkappa \in \mathbb{N}^k\times\mathbb{N}^l$, $|\varkappa| \leqslant p$.

Let $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ *be any* $\mathcal{C}^p$*-function. Fix* $i \in \{1,\dots,l\}$ *and put*

$$g(u,w) := \xi\Big(\frac{w_i}{r(u)}\Big)f(u,w), \quad \text{for } (u,w) \in \Omega\times\mathbb{R}^l.$$

Then for each $c \in \partial\Omega$, $D^\varkappa g(u,w) = o(t(u)^{p-|\varkappa|})$, *when* $\Delta_\varepsilon \ni (u,w) \to (c,0)$ *and* $\varkappa \in \mathbb{N}^k\times\mathbb{N}^l, |\varkappa| \leqslant p$.

Proof. — Put $h(u,w) = \xi\big(\frac{w_i}{r(u)}\big)$. By the Leibniz formula $D^\varkappa g = \sum_{\lambda\leqslant\varkappa}\binom{\varkappa}{\lambda}D^\lambda h D^{\varkappa-\lambda}f$, so it suffices to check that there exists a constant $C'_\varepsilon > 0$ such that $|D^\lambda h(u,w)| \leqslant C'_\varepsilon t(u)^{-|\lambda|}$, when $(u,w) \in \Delta_\varepsilon$ and $|\lambda| \leqslant p$. First, it is easy to see this for $h_0(u,w) := \dfrac{w_i}{r(u)}$ using Lemma 3.5. Then for $h = \xi\circ h_0$ we have

$$\frac{\partial h}{\partial x_j} = (\xi'\circ h_0)\frac{\partial h_0}{\partial x_j}, \quad \text{where } (x_1,\dots,x_n) = (u_1,\dots,u_k,w_1,\dots,w_l)$$

and $D^\lambda\left(\dfrac{\partial h}{\partial x_j}\right) = \sum_{\mu\leqslant\lambda}\binom{\lambda}{\mu}D^\mu(\xi'\circ h_0)D^{\lambda-\mu}\left(\dfrac{\partial h_0}{\partial x_j}\right)$, if $|\lambda| \leqslant p-1$, so we conclude by induction. □

Remark 3.7. — Suppose that f is a $\mathcal{C}^p$-function on the whole space $\mathbb{R}^k \times \mathbb{R}^l$ and such that for each $c \in \partial\Omega$, $D^\varkappa f(u,0) = o(t(u)^{p-|\varkappa|})$, when $\Omega \ni u \to c$ and $\varkappa \in \mathbb{N}^k \times \mathbb{N}^l, |\varkappa| \leqslant p$.

Then for each $c \in \partial\Omega$, $D^\varkappa f(u,w) = o(t(u)^{p-|\varkappa|})$, when $\Delta_\varepsilon \ni (u,w) \to (c,0)$ and $\varkappa \in \mathbb{N}^k \times \mathbb{N}^l, |\varkappa| \leqslant p$. This follows immediately from the Taylor formula

$$D^\varkappa f(u,w) = \sum_{|\lambda| \leqslant p-|\varkappa|} \frac{1}{\lambda!} D^{\varkappa+(0,\lambda)} f(u,0) w^\lambda + o(|w|^{p-|\varkappa|}),$$

when $u \to c, w \to 0$.

Let now Ω be an open Λ_p-regular cell in $\mathbb{R}^k$ and ρ_j $(j = 1, \ldots, 2k)$ - the functions associated with Ω. We define an extension operator

$$\mathcal{L} : \mathcal{E}^p(\overline{\Omega} \times 0, \partial\Omega \times 0) \longrightarrow \mathcal{C}^p(\mathbb{R}^n)\,, \quad \text{where} \quad \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l,$$

by the following formula

$$\mathcal{L}F(u,w) = \begin{cases} \prod_{i=1}^{l} \prod_{j=1}^{2k} \xi\Big(Q \dfrac{w_i}{\rho_j(u)}\Big)(L\widetilde{F})(u,w), & \text{if } u \in \Omega \\ 0, & \text{if } u \in \mathbb{R}^k \setminus \Omega, \end{cases}$$

where Q is any real number $> \sqrt{l}\Theta^{-1}$, Θ is a constant from Remark 3.3 and $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ is a (definable, if we wish) $\mathcal{C}^p$-function equal to 1 in a neighborhood of 0, and equal to 0 outside the open interval $(-1,1)$.

To check that $\mathcal{L}F \in \mathcal{C}^p(\mathbb{R}^n)$ we use repeatedly Proposition 3.6 with $r = \rho_j \not\equiv +\infty$ and $t(u) = d(u, \partial\Omega)$(at the beginning we take $f = L\widetilde{F}$ as in Remark 3.7) and the Hestenes Lemma. The factors involving $\rho_j \equiv +\infty$ being obviously 1 can be omitted in the above formula.

Observe that if ε is any constant from $(0,1)$, we can choose Q in such a way that $\mathcal{L}F$ is p-flat outside the set

$$\begin{aligned}\Delta_\varepsilon(\Omega \times 0) :=& \{x \in \mathbb{R}^n : d(x, \overline{\Omega} \times 0) < \varepsilon d(x, \partial\Omega \times 0)\} \\ =& \{(u,w) \in \Omega \times \mathbb{R}^l : |w| < \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} d(u, \partial\Omega)\}.\end{aligned}$$

Remark 3.8. — If r and t are as in Proposition 3.6 and $F \in \mathcal{E}^p(\overline{\Omega} \times 0, \partial\Omega \times 0)$ is such that, for each $c \in \partial\Omega$, $F^\varkappa(u,0) = o(t(u)^{p-|\varkappa|})$, when $\Omega \ni u \to c$ and $|\varkappa| \leqslant p$, the above formula for an extension of F can be modified by putting

$$\mathcal{L}'F(u,w) = \begin{cases} \prod_{i=1}^{l} \xi\Big(\sqrt{l} \dfrac{w_i}{r(u)}\Big) \mathcal{L}F(u,w), & \text{if } u \in \Omega \\ 0, & \text{if } u \in \mathbb{R}^k \setminus \Omega. \end{cases}$$

Then $\mathcal{L}'F$ is p-flat, outside the neighborhood $\{(u,w)\in\Omega\times\mathbb{R}^l : |w|<r(u)\}$ of $\Omega\times 0$ and outside $\Delta_\varepsilon(\Omega\times 0)$.

In order that $\mathcal{L}F$ (or $\mathcal{L}'F$) and F have the same (up to a multiplicative constant) modulus of continuity we will prove the following

PROPOSITION 3.9. — *Under the assumptions of Proposition 3.6 assume additionally that Ω is 1-regular, $r\in\mathcal{C}^{p+1}(\Omega)$ such that*

$$|D^\alpha(\frac{1}{r})|\leqslant\frac{\widetilde{c}}{t^{|\alpha|+1}},\quad \textit{when}\quad \alpha\in\mathbb{N}^k, |\alpha|\leqslant p+1$$

and t is Lipschitz. Then there exists a constant $M>0$ such that if ω is a modulus of continuity for f on Δ_ε satisfying

$$|D^\varkappa f(u,w)|\leqslant\omega(t(u))t(u)^{p-|\varkappa|},$$

when $(u,w)\in\Delta_\varepsilon$ and $|\varkappa|\leqslant p$, then $M\omega$ is a modulus of continuity for g on Δ_ε satisfying

$$|D^\varkappa g(u,w)|\leqslant M\omega(t(u))t(u)^{p-|\varkappa|},$$

when $(u,w)\in\Delta_\varepsilon$ and $|\varkappa|\leqslant p$.

Proof. — In view of the proof of Proposition 3.6, it suffices to check that, for a constant $M>0$, $M\omega$ is a modulus of continuity for g on Δ_ε. First observe that Δ_ε is 1-regular, because Ω is so and the function t is Lipschitz. There exists a constant $C\geqslant 1$ such that $|t(u_1)-t(u_2)|\leqslant C|u_1-u_2|$, for any $u_1,u_2\in\Omega$.

Fix any $\varkappa\in\mathbb{N}^{k+l}$ such that $|\varkappa|=p$, any $\lambda\leqslant\varkappa$ and any two points $x_i=(u_i,w_i)\in\Delta_\varepsilon$ $(i=1,2)$. We have to estimate

$$|D^\lambda h(x_1)D^{\varkappa-\lambda}f(x_1)-D^\lambda h(x_2)D^{\varkappa-\lambda}f(x_2)|.$$

Case I: $\quad t(u_i)\leqslant 2C|x_1-x_2|$ $(i=1,2)$.

Then $|D^\lambda h(x_i)D^{\varkappa-\lambda}f(x_i)|\leqslant C'_\varepsilon t(u_i)^{-|\lambda|}\omega(t(u_i))t(u_i)^{p-|\varkappa-\lambda|}$

$$\leqslant C'_\varepsilon\omega(2C|x_1-x_2|)\leqslant 2CC'_\varepsilon\omega(|x_1-x_2|).$$

Case II: $\quad t(u_1)>2C|x_1-x_2|$.

Then $|u_1-u_2|\leqslant C|x_1-x_2|<\frac{1}{2}t(u_1)\leqslant\frac{1}{2}d(u_1,\Omega)$; thus $[x_1,x_2]\subset\Omega\times\mathbb{R}^l$.

We have $|D^\lambda h(x_1)[D^{\varkappa-\lambda}f(x_1)-D^{\varkappa-\lambda}f(x_2)]|\leqslant|D^\lambda h(x_1)|\times$

$$\Big[\sum_{1\leqslant|\mu|\leqslant p-|\varkappa-\lambda|}\frac{1}{\mu!}|D^{\varkappa-\lambda+\mu}f(x_1)||x_1-x_2|^{|\mu|}+\omega(|x_1-x_2|)|x_1-x_2|^{p-|\varkappa-\lambda|}\Big]\leqslant$$

$$M_1\omega(t(u_1))t(u_1)^{-1}|x_1-x_2|+M_2\omega(|x_1-x_2|)\leqslant M'\omega(|x_1-x_2|),$$

where M_1, M_2 and M' are positive constants and we use: $\omega(s)t \leqslant \omega(t)s$ if $t \leqslant s$.

On the other hand $|[D^\lambda h(x_1) - D^\lambda h(x_2)]D^{\varkappa-\lambda} f(x_2)| \leqslant$

$$\sup_{x\in[x_1,x_2]} \sum_{j=1}^{k+l} |D^{\lambda+(j)} h(x)||x_1 - x_2||D^{\varkappa-\lambda} f(x_2)|.$$

For any $x = (u, w) \in [x_1, x_2]$, $\quad 2|t(u_1) - t(u)| \leqslant 2C|u_1 - u| \leqslant 2C|x_1 - x_2| < t(u_1)$ and $2|w_1 - w| \leqslant 2C|x_1 - x_2| < t(u_1)$; thus $\frac{1}{2}t(u_1) < t(u) < \frac{3}{2}t(u_1)$ and $|w| \leqslant |w_1| + |w_1 - w| < \varepsilon t(u_1) + t(u) \leqslant (2\varepsilon + 1)t(u)$.

Consequently $x \in \Delta_{2\varepsilon+1}$ and

$$|D^{\lambda+(j)} h(x)| \leqslant C'_{2\varepsilon+1} t(u)^{-|\lambda|-1} \leqslant 2^{|\lambda|+1} C'_{2\varepsilon+1} t(u_1)^{-|\lambda|-1}$$

and

$$|D^{\varkappa-\lambda} f(x_2)| \leqslant \omega(t(u_2))t(u_2)^{|\lambda|} \leqslant \left(\frac{3}{2}\right)^{|\lambda|+1} \omega(t(u_1)))t(u_1)^{|\lambda|}.$$

The needed inequality follows. □

Remark 3.10. — Suppose that f is a $\mathcal{C}^p$-function on the whole space $\mathbb{R}^k \times \mathbb{R}^l$ and ω is its modulus of continuity such that

$$|D^\varkappa f(u, 0)| \leqslant \omega(t(u))t(u)^{p-|\varkappa|},$$

when $u \in \Omega$ and $\varkappa \in \mathbb{N}^{k+l}, |\varkappa| \leqslant p$.

Then there exists a constant $M'' > 0$ such that

$$|D^\varkappa f(u, w)| \leqslant M''\omega(t(u))t(u)^{p-|\varkappa|},$$

when $(u, w) \in \Delta_\varepsilon$, and $\varkappa \in \mathbb{N}^{k+l}$, $|\varkappa| \leqslant p$.

Indeed, this follows immediately from

$$|D^\varkappa f(u, w) - \sum_{|\lambda|\leqslant p-|\varkappa|} \frac{1}{\lambda!} D^{\varkappa+(0,\lambda)} f(u, 0)w^\lambda| \leqslant \omega(|w|)|w|^{p-|\varkappa|}.$$

Remark 3.11. — If Ω is an open Λ_{p+1}-regular cell in $\mathbb{R}^k$ and ξ is a $\mathcal{C}^{p+1}$-function, then there exists a positive constant M, such that, for any $F \in \mathcal{E}^p(\overline{\Omega} \times 0, \partial\Omega \times 0)$ (respectively, fulfilling additional conditions: $|F^\varkappa(u, 0)| \leqslant \omega(r(u))r(u)^{p-|\varkappa|}$, when $u \in \Omega$, $\varkappa \in \mathbb{N}^{k+l}$, $|\varkappa| \leqslant p$) if ω is a modulus of continuity for F, then $M\omega$ is a modulus of continuity for $\mathcal{L}F$ (respectively, for $\mathcal{L}'F$).

4. A generalization to the ideal of $\mathcal{C}^p$-Whitney fields on the closure of a Λ_p-regular leaf p-flat on its boundary

Now we will transpose the extension operator $\mathcal{L}$ to the closure of any Λ_p-regular leaf. A subset $E \subset R^n$ is called a (definable) Λ_p-*regular leaf* of dimension k in $\mathbb{R}^n$ if it is the graph $E = \{(u, \varphi(u)) : u \in \Omega\}$ of a (definable) Λ_p-regular mapping $\varphi : \Omega \longrightarrow \mathbb{R}^l$ defined on an open (definable) Λ_p-regular cell Ω in $\mathbb{R}^k$. A reduction of this case to the previous one will be by the following Lipschitz automorphism

$$\overline{\Omega} \times \mathbb{R}^l \ni (u, w) \longmapsto (u, w + \overline{\varphi}(u)) \in \overline{\Omega} \times \mathbb{R}^l$$

and the following

PROPOSITION 4.1 (cf. [6], Proposition 3). — *Let $\varphi : \Omega \longrightarrow \mathbb{R}^l$ be a Λ_p-regular mapping defined on an open subset $\Omega \subset \mathbb{R}^k$. Let $t : \Omega \longrightarrow (0, +\infty)$ be any function such that $t(u) \leqslant d(u, \partial\Omega)$, for each $u \in \Omega$. Let E be any closed subset of $\Omega \times \mathbb{R}^l$ and*

$$F(u, w; U, W) = \sum_{|\alpha|+|\beta|\leqslant p} \frac{1}{\alpha!\beta!} F^{(\alpha,\beta)}(u, w) U^\alpha W^\beta \quad \begin{cases} U = (U_1, \dots, U_k), \\ W = (W_1, \dots, W_l) \end{cases}$$

a $\mathcal{C}^p$-Whitney field on E such that, for any $c \in \partial\Omega$
$F^{(\alpha,\beta)}(u, w) = o(t(u)^{p-|\alpha|-|\beta|})$, when $u \to c$ and $|\alpha| + |\beta| \leqslant p$.

Let $F_\varphi(u, v; U, V)$ be a polynomial in (U, V) of degree $\leqslant p$ such that

$$F_\varphi(u, v; U, V) = \sum_{|\alpha|+|\beta|\leqslant p} \frac{1}{\alpha!\beta!} F^{(\alpha,\beta)}(u, v + \varphi(u)) U^\alpha$$
$$\Big(V + \sum_{1\leqslant|\varkappa|\leqslant p} \frac{1}{\varkappa!} D^\varkappa\varphi(u) U^\varkappa\Big)^\beta \, mod(U, V)^{p+1}$$

defined for $(u, v) \in E_\varphi$, where $E_\varphi = \{(u, v) \in \Omega \times \mathbb{R}^l : (u, v + \varphi(u)) \in E\}$.

Then F_φ is a $\mathcal{C}^p$-Whitney field on E_φ such that, for any $c \in \partial\Omega$
$F_\varphi^{(\alpha,\beta)}(u, v) = o(t(u)^{p-|\alpha|-|\beta|})$, when $u \to c$ and $|\alpha| + |\beta| \leqslant p$.

Proof. — It is easy to check that F_φ fulfills the condition $(**)$ from Introduction, thus it is a $\mathcal{C}^p$-Whitney field on E_φ. Besides

$$F_\varphi(u, v; U, V) = \sum_{|\alpha|+|\beta|\leqslant p} \frac{1}{\alpha!\beta!} F^{(\alpha,\beta)}(u, v + \varphi(u)) U^\alpha \times$$
$$\sum_{\gamma+\sum_\varkappa \delta_\varkappa=\beta} \frac{\beta!}{\gamma! \prod \delta_\varkappa!} V^\gamma \prod_\varkappa \left[\frac{1}{\varkappa!^{|\delta_\varkappa|}} U^{|\delta_\varkappa|\varkappa} (D^\varkappa\varphi(u))^{\delta_\varkappa}\right] \quad mod(U, V)^{p+1},$$

thus

$$F_\varphi^{(\sigma,\gamma)}(u,v) = \sum_{\alpha+\sum_\varkappa|\delta_\varkappa|\varkappa=\sigma} [\,.\,]F^{(\alpha,\gamma+\sum_\varkappa \delta_\varkappa)}(u, v+\varphi(u)) \prod_\varkappa (D^\varkappa\varphi(u))^{\delta_\varkappa},$$

where $[\,.\,]$ denotes constants. To conclude notice that

$$F^{(\alpha,\gamma+\sum_\varkappa \delta_\varkappa)}(u, v+\varphi(u)) \prod_\varkappa (D^\varkappa\varphi(u))^{\delta_\varkappa} =$$

$$o(1)t(u)^{p-|\alpha|-|\gamma|-\sum_\varkappa|\delta_\varkappa|}C\prod_\varkappa d(u,\partial\Omega)^{-|\delta_\varkappa||\varkappa|+|\delta_\varkappa|} =$$

$$o\big(t(u)^{p-|\sigma|-|\gamma|}\big).$$

□

Remark 4.2. — If $E = \{(u,\varphi(u)) : u \in \Omega\}$ (resp. $E = \Omega\times\mathbb{R}^l$), then F_φ extends to a $\mathcal{C}^p$-Whitney field on $\overline{E_\varphi} = \overline{\Omega}\times 0$ (resp. $\overline{E_\varphi} = \overline{\Omega}\times\mathbb{R}^l$) p-flat on $\partial E_\varphi = \partial\Omega\times 0$ (resp. $\partial E_\varphi = \partial\Omega\times\mathbb{R}^l$).

Proof. — The both cases follow from the Hestenes Lemma. □

Proposition 4.3. — *Under the assumptions of Proposition 4.1, assume additionally that the mapping φ is Λ_{p+1}-regular, E and Ω are both 1-regular and $\overline{E}$ and $\partial\Omega\times\mathbb{R}^l$ are simply separated* (*). *Then there exists a constant $M > 0$ such that, for each $F \in \mathcal{E}^p(\overline{E},\partial E)$, if ω is a modulus of continuity of F, then $M\omega$ is a modulus of continuity of F_φ.*

Moreover, if $|F^\varkappa(u,w)| \leqslant \omega(t(u))t(u)^{p-|\varkappa|}$, when $(u,w)\in E$ and $|\varkappa|\leqslant p$, then $|F_\varphi^\varkappa(u,v)| \leqslant M\omega(t(u))t(u)^{p-|\varkappa|}$, when $(u,v)\in E_\varphi$ and $|\varkappa|\leqslant p$.

Proof. — Observe that E_φ is 1-regular. Let $\sigma\in\mathbb{N}^k$, $\gamma\in\mathbb{N}^l$ be such that $|\sigma|+|\gamma| = p$ and let $(u_i,v_i)\in E_\varphi, (i=1,2)$. We have to estimate

$$|F_\varphi^{(\sigma,\gamma)}(u_1,v_1) - F_\varphi^{(\sigma,\gamma)}(u_2,v_2)| \leqslant$$

$$\sum_{\alpha+\sum_\varkappa|\delta_\varkappa|\varkappa=\sigma} [\,.\,]|F^{(\alpha,\gamma+\sum_\varkappa \delta_\varkappa)}(u_1, v_1+\varphi(u_1)) \prod_\varkappa (D^\varkappa\varphi(u_1))^{\delta_\varkappa} -$$

$$F^{(\alpha,\gamma+\sum_\varkappa \delta_\varkappa)}(u_2, v_2+\varphi(u_2)) \prod_\varkappa (D^\varkappa\varphi(u_2))^{\delta_\varkappa}|.$$

Fix $\lambda = (\alpha, \gamma+\sum_\varkappa\delta_\varkappa)$ and put $x_i = (u_i, v_i+\varphi(u_i))$ and

$$\theta(u) = \prod_\varkappa (D^\varkappa\varphi(u))^{\delta_\varkappa}.$$

(*) See the beginning of Section 5 for the definition of simple separation.

Case I: $|x_1 - x_2| \geqslant \frac{1}{2} d(u_i, \partial\Omega)$ for $i = 1, 2$.

$$|F^\lambda(x_i)\theta(u_i)| \leqslant \omega(d(x_i, \partial E))d(x_i, \partial E)^{p-|\lambda|}|\theta(u_i)| \leqslant$$
$$\omega(Cd(u_i, \partial\Omega))[Cd(u_i, \partial\Omega)]^{p-|\lambda|}|\theta(u_i)| \leqslant$$
$$\omega(2C|x_1 - x_2|)[Cd(u_i, \partial\Omega)]^{p-|\lambda|} \prod_{\varkappa} d(u_i, \partial\Omega)^{-|\delta_\varkappa||\varkappa|+|\delta_\varkappa|} \leqslant M\omega(|x_1 - x_2|).$$

Case II: $|x_1 - x_2| \leqslant \frac{1}{2} d(u_1, \partial\Omega)$.

$$|F^\lambda(x_1)\theta(u_1) - F^\lambda(x_2)\theta(u_2)| \leqslant$$
$$|F^\lambda(x_1) - F^\lambda(x_2)||\theta(u_2)| + |F^\lambda(x_1)||\theta(u_1) - \theta(u_2)| \leqslant$$
$$\Bigg[\sum_{1 \leqslant |\mu| \leqslant p-|\lambda|} \frac{1}{\mu!}|F^{\lambda+\mu}(x_1)||x_2 - x_1|^{|\mu|} + \omega(|x_1 - x_2|)|x_1 - x_2|^{p-|\lambda|}\Bigg]|\theta(u_2)| +$$
$$|F^\lambda(x_1)| \sup_{z \in [u_1, u_2]} \sum_{j=1}^{k} |D^{(j)}\theta(z)||u_1 - u_2| \leqslant$$
$$\Bigg[\sum_{1 \leqslant |\mu| \leqslant p-|\lambda|} \frac{1}{\mu!}\omega(d(x_1, \partial E))d(x_1, \partial E)^{p-|\lambda|-|\mu|}|x_1 - x_2|d(u_1, \partial\Omega)^{|\mu|-1} +$$
$$\omega(|x_1 - x_2|)d(u_1, \partial\Omega)^{p-|\lambda|}\Bigg]|\theta(u_2)| +$$
$$\omega(d(x_1, \partial\Omega))|x_1 - x_2| \sup_{z \in [u_1, u_2]} \sum_{j=1}^{k} |D^{(j)}\theta(z)|$$
$$\Bigg[C_1\omega(d(u_1, \partial\Omega))|x_1 - x_2|d(u_1, \partial\Omega)|^{p-|\lambda|-1} +$$
$$\omega(|x_1 - x_2|)d(u_1, \partial\Omega)^{p-|\lambda|}\Bigg]|\theta(u_2)| +$$
$$C_2\omega(d(u_1, \partial\Omega))|x_1 - x_2| \sup_{z \in [u_1, u_2]} \prod_{\varkappa} d(z, \partial\Omega)^{-|\delta_\varkappa||\varkappa|+|\delta_\varkappa|-1}.$$

Now it suffices to observe that $\omega(d(u_1, \partial\Omega))|x_1 - x_2| \leqslant \omega(|x_1 - x_2|)d(u_1, \partial\Omega)$ and $d(z, \partial\Omega) \geqslant d(u_1, \partial\Omega) - |z - u_1| \geqslant d(u_1, \partial\Omega) - |x_1 - x_2| \geqslant \frac{1}{2} d(u_1, \partial\Omega)$, if $z \in [u_1, u_2]$. $\square$

Assume now that $E = \{(u, \varphi(u)) : u \in \Omega\}$ is a Λ_p-regular leaf of dimension k in $\mathbb{R}^n$. We define an extension operator $\mathcal{L} : \mathcal{E}^p(\overline{E}, \partial E) \longrightarrow \mathcal{C}^p(\mathbb{R}^n)$ by the formula

$$\mathcal{L}F = \begin{cases} (\mathcal{L}F_\varphi)_{-\varphi}, & \text{on } \Omega \times \mathbb{R}^l \\ 0, & \text{on } (\mathbb{R}^k \setminus \Omega) \times \mathbb{R}^l. \end{cases}$$

For any constant $\varepsilon > 0$, we can specify this operator in such a way that for each $F \in \mathcal{E}^p(\overline{E}, \partial E)$, $\mathcal{L}F$ is flat outside the neighborhood $\Delta_\varepsilon(E) := \{x \in \mathbb{R}^n : d(x,E) < \varepsilon d(x, \partial E)\}$.

5. A generalization to a finite tower of Λ_p-regular leaves

Here we will generalize the extension operator $\mathcal{L}$ to the ideal $\mathcal{E}^p(\overline{E}, \partial E)$, where E is a finite disjoint union $E = E_1 \cup \cdots \cup E_s$ of graphs of Λ_p-regular mappings $\varphi_\sigma : \Omega \longrightarrow \mathbb{R}^l$ $(\sigma = 1, \ldots, s)$ defined on a common open Λ_p-regular cell $\Omega \subset \mathbb{R}^k$. Put $r_\sigma(u) := |\varphi_\sigma(u) - \varphi_s(u)|$ for $\sigma = 1, \ldots, s-1$ and $u \in \Omega$.

We first define $\mathcal{L}F$ for any $F \in \mathcal{E}^p(\overline{E}, \overline{E}_1 \cup \cdots \cup \overline{E}_{s-1} \cup \partial E_s)$.

Then we put

$$\mathcal{L}F = \begin{cases} \left[\displaystyle\prod_{\sigma=1}^{s-1} \prod_{i=1}^{l} \xi\left(\sqrt{l}\, \frac{w_i}{r_\sigma(u)}\right) \mathcal{L}\big((F|\overline{E}_s)_{\varphi_s}\big) \right]_{-\varphi_s}, & \text{on } \Omega \times \mathbb{R}^l \\ 0, & \text{on } (\mathbb{R}^k \setminus \Omega) \times \mathbb{R}^l, \end{cases}$$

which gives an extension operator according to Proposition 3.6 (used repeatedly with $t(u) := \min\big(\{r_\sigma(u)\}, d(u, \partial\Omega)\big)$), Remark 3.8 and Proposition 4.1.

Let now consider a general case where F is any element of $\mathcal{E}^p(\overline{E}, \partial E)$. Proceeding by induction, assume that $\mathcal{L}(F|\overline{E}_1 \cup \cdots \cup \overline{E}_{s-1})$ has already been defined. Then $H := F - T\mathcal{L}(F|\overline{E}_1 \cup \cdots \cup \overline{E}_{s-1})|\overline{E} \in \mathcal{E}^p(\overline{E}, \overline{E}_1 \cup \cdots \cup \overline{E}_{s-1} \cup \partial E_s)$ and we put

$$\mathcal{L}F = \mathcal{L}H + \mathcal{L}(F|\overline{E}_1 \cup \cdots \cup \overline{E}_{s-1}).$$

For any $\varepsilon > 0$, we can specify this operator in such a way that $\mathcal{L}F$ is p-flat outside the set $\Delta_\varepsilon(E) := \{x \in \mathbb{R}^n : d(x,E) < \varepsilon d(x, \partial E)\}$.

6. Extension operator for a closed definable subset of $\mathbb{R}^n$

Definition 6.1 (cf. [10]). — *Let $A, B, Z \subset \mathbb{R}^n$. We say that A and B are simply Z-separated if one of the following equivalent conditions holds*

(1) *$\exists M > 0 \forall x \in A$, $\quad d(x,B) \geqslant M d(x,Z)$;*
(2) *$\exists C > 0 \forall x \in \mathbb{R}^n$, $\quad d(x,A) + d(x,B) \geqslant C d(x,Z)$. (If (1) holds, one can take $C = M/(M+1)$.)*

We say that A and B are simply separated if they are simply $A \cap B$-separated.

Proposition 6.2. — *Let $E_i \supset E_i'$ $(i = 1, \ldots, s)$ be closed subsets of $\mathbb{R}^n$ and let $C > 0$ be a constant such that, for any $i, j \in \{1, \ldots, s\}, i \neq j$ and any $x \in \mathbb{R}^n$*

$$d(x, E_i) + d(x, E_j) \geqslant C d(x, E_i').$$

Let $\varepsilon \in (0, C/2]$. Put $\Gamma_\varepsilon(E_i, E_i') := \{x \in \mathbb{R}^n : d(x, E_i) < \varepsilon d(x, E_i')\}$. Suppose that, for each $i = 1, \ldots, s$

$$\mathcal{L}_i : \mathcal{E}^p(E_i, E_i') \longrightarrow \mathcal{C}^p(\mathbb{R}^n)$$

is an extension operator such that $\mathcal{L}_i F$ is p-flat outside $\Gamma_\varepsilon(E_i, E_i')$, for any $F \in \mathcal{E}^p(E_i, E_i')$.

Then the formula

$$\mathcal{L}F = \sum_{i=1}^{s} \mathcal{L}_i(F|E_i)$$

defines an extension operator $\mathcal{L} : \mathcal{E}^p(\bigcup_i E_i, \bigcup_i E_i') \longrightarrow \mathcal{C}^p(\mathbb{R}^n)$. Moreover, if each $\mathcal{L}_i$ preserves (up to a multiplicative constant) a modulus of continuity, then $\mathcal{L}$ has the same property.

Proof. — It suffices to check that $\Gamma_\varepsilon(E_i, E_i') \cap \Gamma_\varepsilon(E_j, E_j') = \emptyset$, if $i \neq j$. If there were $x \in \Gamma_\varepsilon(E_i, E_i') \cap \Gamma_\varepsilon(E_j, E_j')$, then

$$2\varepsilon[d(x, E_i') + d(x, E_j')] > 2[d(x, E_i) + d(x, E_j)] \geqslant C[d(x, E_i') + d(x, E_j')],$$

a contradiction. $\square$

A proof of the following theorem will be given in the next section.

Λ_p-regular Decomposition Theorem 6.3. — *Let E be a closed subset of $\mathbb{R}^n$ definable in some fixed o-minimal structure on the ordered field of the real numbers $\mathbb{R}$. Let $k = \dim E$. Let Z be any definable subset of E of dimension $< k$.*

Then there exists a finite decomposition

$$E = M_1 \cup \cdots \cup M_s \cup A$$

such that each M_i is a finite tower of Λ_p-regular k-dimensional definable leaves in an appropriate linear coordinate system, A is a closed definable subset of $\dim < k$ containing Z and, for any $i, j \in \{1, \ldots, s\}$ $(i \neq j)$, $\overline{M}_i$ and $\overline{M}_j$ are simply ∂M_i-separated and, for any i, $\overline{M}_i$ and A are simply ∂M_i-separated.

In order to define an extension operator for any closed definable subset $E \subset \mathbb{R}^n$ we will use induction on $\dim E$. By the induction hypothesis we have an extension operator

$$\mathcal{L}_0 : \mathcal{E}^p(\cup_{i=1}^s \partial M_i \cup A) \longrightarrow \mathcal{C}^p(\mathbb{R}^n),$$

and by Section 5 combined with Proposition 6.2 we have an extension operator

$$\mathcal{L}_1 : \mathcal{E}^p(E, \cup_{i=1}^s \partial M_i \cup A) \longrightarrow \mathcal{C}^p(\mathbb{R}^n).$$

Now an extension operator for E is defined by the formula

$$\mathcal{L}F = \mathcal{L}_1[F - T\mathcal{L}_0(F| \cup_i \partial M_i \cup A)|E] + \mathcal{L}_0(F| \cup_i \partial M_i \cup A).$$

7. Proof of Λ_p-regular Decomposition Theorem

Let $P \subset \mathbb{R}^n$ be any definable subset and V - a linear subspace of $\mathbb{R}^n$ of dimension $n-k$. Following [10], we will say that P is *perfectly situated relative to* V if, for a/any linear complement W of V in $\mathbb{R}^n$, P can be represented as a disjoint union

$$P = \bigcup \{\hat{\varphi} : \varphi \in \mathcal{F}\}$$

of a finite family $\mathcal{F}$ of definable $\mathcal{C}^1$-mappings $\varphi : \Delta_\varphi \longrightarrow V$ defined on connected $\mathcal{C}^1$-submanifolds $\Delta_\varphi \subset W$ and with bounded derivatives ($\hat{\varphi}$ stands here for the graph $\{u + \varphi(u) : u \in \Delta_\varphi\}$ of φ).

We will use the following

THEOREM 7.1 (cf. [10], Theorem 0). — *Let* $\Sigma = \{\sigma \subset \{1, \ldots, n\} : \operatorname{card} \sigma = n-k\} = \{\sigma_1, \ldots, \sigma_q\}$, *where* $q = \binom{n}{k}$.

Let $V_i = \bigoplus_{\nu \in \sigma_i} \mathbb{R}e_\nu$ $(i = 1, \ldots, q)$, *where* $e_1, \ldots, e_n$ *is the canonical basis in* $\mathbb{R}^n$.

Any definable closed subset $E \subset \mathbb{R}^n$ *of dimension* k *is a union* $E = \bigcup_{i=1}^q E_i$ *of definable closed subsets* E_i *such that, for each* i, E_i *is perfectly situated relative to* V_i *and, for each* $j \neq i$, E_i *and* E_j *are simply separated and* $\dim(E_i \cap E_j) < k$.

From the last theorem and easy properties of simply separated sets (see [10], Proposition 2; (1) and (3)), it follows that it suffices to prove Λ_p-regular Decomposition Theorem for each E_i and $Z_i = (Z \cap E_i) \cup (\bigcup_{j \neq i} E_i \cap E_j)$ separately, therefore - up to a permutation of variables - it suffices to prove it assuming that E is perfectly situated relative to $0 \times \mathbb{R}^l$, where $l = n-k$. The proof in this case is based on the following two propositions.

Proposition 7.2 ([6], Proposition 2). — *If $\varphi : \Omega \longrightarrow \mathbb{R}^l$ is a definable Λ_1-regular mapping defined on an open $\Omega \subset \mathbb{R}^k$, then there exists a closed definable subset Z of Ω such that $\dim Z < k$ and $\varphi | \Omega \setminus Z$ is Λ_p-regular mapping on $\Omega \setminus Z$.*

Proposition 7.3 ([6], Proposition 4). — *For any definable open subset $\Omega \subset \mathbb{R}^k$, there exists a finite family $\mathcal{S}$ of disjoint subsets of Ω such that $\dim(\Omega \setminus \bigcup \mathcal{S}) < k$ and each $S \in \mathcal{S}$ is an open definable Λ_p-regular cell in an appropriate linear system of coordinates in $\mathbb{R}^k$.*

Proof of Proposition 7.3. — See [6], Proposition 4, where the set is assumed bounded, but this assumption is not essential. Alternatively, first one can apply [10]; Theorem 1, (B_k) to get the case $p = 1$ of Proposition 7.3, which is the theorem of Kurdyka [5] and Parusiński [9], and then by induction on k one gets the case of any $p \geqslant 1$, applying Proposition 7.2. □

To finish the proof of the theorem, first represent E as union of graphs with bounded derivatives:

$$E = \bigcup \{\hat{\varphi} : \varphi \in \mathcal{F}\},$$

as in the beginning of the section. Adding to Z all the graphs with $\dim \Delta_\varphi < k$, one can assume that

$$E = Z \cup \bigcup \{\hat{\varphi} : \varphi \in \mathcal{F}_*\},$$

where $\mathcal{F}_* = \{\varphi \in \mathcal{F} : \Delta_\varphi \quad \text{non-empty open in} \quad \mathbb{R}^k\}$. By Proposition 7.2, for each $\varphi \in \mathcal{F}_*$ there exists a closed definable subset K_φ of Δ_φ of $\dim < k$ such that $\varphi | \Delta_\varphi \setminus K_\varphi$ is Λ_p-regular. Let

$$\boldsymbol{\Theta} := \overline{\pi(Z)} \cup \bigcup \{\partial \Delta_\varphi \cup K_\varphi : \varphi \in \mathcal{F}_*\},$$

where $\pi : \mathbb{R}^k \times \mathbb{R}^l \longrightarrow \mathbb{R}^k$ is the canonical projection. Take a family $\mathcal{S}$ as in Proposition 7.3 for the open subset

$$\Omega := \bigcup \{\Delta_\varphi : \varphi \in \mathcal{F}_*\} \setminus \boldsymbol{\Theta}.$$

Now it suffices to define, for each $S \in \mathcal{S}$

$$M_S := E \cup \pi^{-1}(S) \qquad \text{and} \qquad A := E \setminus \bigcup \{M_S : S \in \mathcal{S}\}.$$

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Manuscrit reçu le 26 août 2004,
révisé le 14 février 2007,
accepté le 5 avril 2007.

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