ANNALES

DE

L'INSTITUT FOURIER

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Tome 58, nº 1 (2008), p. 377-381.

<http://aif.cedram.org/item?id=AIF_2008__58_1_377_0>

Ann. Inst. Fourier, Grenoble
58, 1 (2008) 377-381

CORRIGENDUM TO: HOLOMORPHIC MORSE INEQUALITIES ON MANIFOLDS WITH BOUNDARY

Annales de l'institut Fourier, vol. 55 (2005), n° 4, p. 1055–1103

by Robert BERMAN

Abstract. — A statement in the paper "Holomorphic Morse inequalities on manifolds with boundary" saying that the holomorphic Morse inequalities for an hermitian line bundle L over X are sharp as long as L extends as semi-positive bundle over a Stein-filling is corrected, by adding certain assumptions. A more general situation is also treated.

Résumé. — Nous corrigeons l'énoncé qui affirme que les inégalités de Morse holomorphes pour un fibré hermitien en droites L sur X sont optimales tant que L s'étend comme un fibré semi-positif sur un remplissage Stein, paru dans l'article "Inégalités de Morse holomorphes sur des variétés à bord". Nous ajoutons certaines conditions et considérons une situation plus générale.

Let (L, ϕ) be a hermitian holomorphic line bundle over a closed (*i.e.* compact without boundary) complex hermitian manifold (Y, ω) of dimension n and let $X = \{\rho \leqslant 0\}$ be a strongly pseudoconcave domain in Y with smooth boundary. Let $d^c := i(-\partial + \overline{\partial})/4\pi$ and denote by $dd^c\phi (= i\partial\overline{\partial}\phi/2\pi)$ the normalized curvature form of ϕ. If η is a $(1,1)$-form we will write $\eta_p := \eta^p/p!$ and we will denote by $\{\eta\}_y$ the hermitian linear operator on $T^{1,0}(Y)_y$ (or on some specified subbundle) corresponding to η (using a fixed metric form ω).

The purpose here is to correct a statement in [1] saying that the holomorphic Morse inequalities on X obtained in [1] are always sharp when the curvature form $dd^c\phi$ is semi-positive on all of Y and $Y - X$ is a Stein manifold. The correct statement, as shown below, is obtained by adding

Keywords: Line bundles, cohomology, harmonic forms, holomorphic sections, Bergman kernel.
Math. classification: 32L10, 32A25, 32L20.

the assumption (3) about the compatibility between the curvature of the line bundle L and the curvature of the boundary ∂X. The (non-Stein) case when $Y - X$ contains an irreducible divisor will also be treated here. In fact, in a large class of examples (where Y is a ruled manifold and X is a disc bundle) the assumptions below are essentially necessary as will be further investigated in a forthcoming paper.

Recall that the boundary contribution to the holomorphic Morse inequalities in [1] (*i.e.* inequalities for $\limsup_k k^{-n} \dim H^0(X, L^k)$) is obtained by integrating the following form over ∂X:

$$\mu := \int_0^T (dd^c\phi + tdd^c\rho)^{n-1} \wedge d^c\rho)dt/(n-1)!, \tag{1}$$

where T is the following function on ∂X, that will be referred to as the *slope function*:

$$T = \sup \left\{ t \geqslant 0 \colon (dd^c\phi + tdd^c\rho)_x \geqslant 0 \ \text{ along } T^{1,0}(\partial X)_x \right\}. \tag{2}$$

The assumptions

First assume that the defining function $-\rho$ of the pseudoconvex manifold $Y - X$ may be chosen to be smooth with $dd^c(-\rho) > 0$ in $(Y - X) - Z$, where Z is either a point or an irreducible divisor in $Y - X$ [(1)]. Moreover, assume that there is a diffeomorphism between $(Y - X) - Z$ and $\partial(Y - X) \times]-1, 0]$ such that $-\rho$ corresponds to projection on the second factor. Then the slope function T extends to a function on $(Y - X) - Z$, by replacing ∂X in (2) with $\rho^{-1}(c)$ for any c in $[0, 1[$. Now assume the following

$$\text{compatibility:} \quad T \text{ is a } \mathcal{C}^1\text{-function of } \rho, \tag{3}$$

(in particular, T is constant on ∂X). Next, *if Z is a point* it is assumed that $-\rho$ extends to a smooth function on $X - Y$ such that $dd^c(-\rho) > 0$ on all of $Y - X$. *If Z is an irreducible divisor* it is assumed that T is bounded from above on $Y - X$, that

$$\int_Z (dd^c\phi)^{n-1} = 0, \tag{4}$$

and that

$$dd^c(-\rho) = [Z] + \beta, \tag{5}$$

in the sense of currents on $Y - X$, where β is a semi-positive smooth form.

(1) *i.e.* a (possibly singular) connected compact closed complex submanifold of codimension one in $Y - X$. Then the integration current $[Z]$ is well-defined.

Sharp Morse inequalities for fillings

Proposition 1. — *Assume that the (normalized) curvature $dd^c\phi$ of L is a smooth semi-positive form and that the defining function $-\rho$ of $Y - X$ satisfies the assumptions in the previous section. Then*

$$\int_{\partial X} \mu = \int_{Y-X} (dd^c\phi)_n.$$

Given the equality in the proposition above the holomorphic Morse inequalities on X are sharp as shown in [1]. To prove the proposition first note that the form μ (formula (1)) is well-defined on any hypersurface $\rho^{-1}(c)$ with c in $[0, 1[$ by using the extended function slope function T. The following lemma corrects a statement in section 7.1 (remark 7.4 and above) in [1], where the integral I_ϵ below was incorrectly assumed to vanish in the general case. In its statement and proof we will use the convention that the orientation on the level surfaces $\{\rho = c\}$ is such that the restriction of $d^c(-\rho)(dd^c(-\rho))_{n-1}$ is a volume form, *i.e.* the orientation obtained by realizing $\{\rho = c\}$ as the boundary of the domain $\{(-\rho) \leqslant -c\}$ in Y.

Lemma 2. — *For each fixed positive number ϵ the following identity holds:*

$$0 = \int_{\rho^{-1}(0)} \mu - \int_{\rho^{-1}(1-\epsilon)} \mu - \int_{\rho^{-1}([0,1-\epsilon]} (dd^c\phi)_n - I_\epsilon, \tag{6}$$

where

$$I_\epsilon := \int_{\rho^{-1}([0,1-\epsilon]} (dd^c\phi + T dd^c\rho)_{n-1} \wedge d^c(-\rho) \wedge dT.$$

Proof. — Denote by $\mathcal{X}_\varepsilon(0)$ the following manifold with boundary (and corners), realized as a subset of the real cotangent bundle of Y:

$$\mathcal{X}_\varepsilon(0) := \left\{ t d^c(-\rho)_y \colon y \in Y - X,\ 0 \leqslant t \leqslant T,\ 0 \leqslant \rho(y) \leqslant 1 - \epsilon \right\}.$$

Hence, $\mathcal{X}_\varepsilon(0)$ is topologically a fiber bundle of closed intervals over a subset of $X - Y$ and when ϵ tends to zero, the base of $\mathcal{X}_\varepsilon(0)$ tends to $Y - X$. Next, observe that the following form on $\mathcal{X}_\varepsilon(0)$ is closed:

$$\eta := (dd^c\phi + d(td^c\rho))_n.$$

Hence, Stokes theorem gives after expansion of η

$$\int_{\partial(\mathcal{X}_\varepsilon(0))} (dd^c\phi + tdd^c\rho)_n + (dd^c\phi + tdd^c\rho)_{n-1} \wedge d^c\rho \wedge dt = 0. \tag{7}$$

Decomposing the boundary of $\mathcal{X}_\varepsilon(0)$ into the following components

$$\partial(\mathcal{X}_\varepsilon(0)) = \{\rho = 0\} \cup \{\rho = 1 - \epsilon\} \cup \{t = 0\} \cup \{t = T\}$$

then proves the identity (6), by identifying the terms in (6) with the integrals over the components above (in the corresponding order). To see this, note that the first term in (7) gives no contribution on the first two components (for degree reasons). Moreover, the first term in (7) vanishes on the fourth component $\{t = T\}$. Indeed, at the point y the form $(dd^c\phi+Tdd^c\rho)_n$ is proportional to the determinant of the operator $\{dd^c\phi + Tdd^c\rho\}$ on $T^{1,0}(Y)_y$. The determinant vanishes since, by the definition of T, the operator has a zero-eigenvalue along the subspace $T^{1,0}(\rho^{-1}(\rho(y))$ of $T^{1,0}(Y)$. □

Now if T is assumed to be a function of ρ the integrand in I_ϵ, defined above, may be written as a function times the form $(dd^c\phi+Tdd^c\rho)_{n-1}\wedge\partial\rho\wedge\overline{\partial}\rho$. Since the latter form at the point y is proportional to the determinant of $\{dd^c\phi+Tdd^c\rho\}$ restricted to $T^{1,0}(\rho^{-1}(\rho(y))_y$, it follows from the definition of T that

$$I_\epsilon = 0 \tag{8}$$

vanishes for each ϵ. Next, we will show that the following "residue" vanishes:

$$\lim_{\epsilon\to 0}\int_{\rho^{-1}(1-\epsilon)}\mu = 0. \tag{9}$$

The case when Z is a point (denoted by y_0) : First observe that

$$\int_{\rho^{-1}(1-\epsilon)}\mu \leqslant T(1-\epsilon)\int_{\rho^{-1}(1-\epsilon)}(dd^c\phi)_{n-1}\wedge d^c(-\rho), \tag{10}$$

only using that $dd^c\rho$ is negative along $T^{1,0}(\rho^{-1}(c))$. Moreover, since the form $dd^c\rho$ is assumed to be strictly negative on all of $Y-X$, its eigenvalues are bounded from above by a negative constant, which shows that T is uniformly bounded from above by a positive constant C, when ϵ tends to zero. A direct application of Stokes theorem now gives

$$I_{1-\epsilon}\leqslant C\int_{\rho^{-1}[1-\epsilon,1]}(dd^c\phi)_{n-1}\wedge dd^c(-\rho).$$

Since, by assumption, $dd^c\rho$ extends smoothly over the neighbourhood $\rho^{-1}[1-\epsilon,1]$ of the point x_0 it follows that the right hand side tends to zero when ϵ tends to zero (since $\rho^{-1}[1-\epsilon,1]$ shrinks to the point y_0), proving (9).

The case when Z is a divisor: First observe that (10) still holds, by a similar argument, using the assumption that T is bounded from above on $Y-X$. Applying Stokes theorem again now gives, using the assumption (5) on $dd^c\rho$, that

$$\int_{\rho^{-1}(1-\epsilon)}\mu\leqslant C\int_Z(dd^c\phi)_{n-1}+\int_{\rho^{-1}[1-\epsilon,1]}(dd^c\phi)_{n-1}\wedge\beta.$$

Since β is assumed to be smooth on all of $Y - X$, it follows that

$$\lim_{\epsilon \to 0} \int_{\rho^{-1}(1-\epsilon)} \mu \leqslant C \int_Z (dd^c \phi)_{n-1}$$

and the right hand side vanishes by the assumption (4), proving (9).

Finally, combining the identity (6) in Lemma 2 with the vanishing in (8) and (9) finishes the proof of Proposition 1.

Remark. — A further correction to [1]: In formula (7.1) B_X^q should be replaced with $\lim_{k\to\infty} B_X^{q,k}/k^n$. Accordingly, in formula (7.2) B_X^q should be replaced with $\lim_{k\to\infty} B_X^{q,k}/k^n$ and the sum with an integral over t.

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