Masafumi YOSHINO & Todor GRAMCHEV

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SIMULTANEOUS REDUCTION TO NORMAL FORMS OF COMMUTING SINGULAR VECTOR FIELDS WITH LINEAR PARTS HAVING JORDAN BLOCKS

by Masafumi YOSHINO & Todor GRAMCHEV (*)

Abstract. — We study the simultaneous linearizability of $d$–actions (and the corresponding $d$-dimensional Lie algebras) defined by commuting singular vector fields in $\mathbb{C}^n$ fixing the origin with nontrivial Jordan blocks in the linear parts. We prove the analytic convergence of the formal linearizing transformations under a certain invariant geometric condition for the spectrum of $d$ vector fields generating a Lie algebra. If the condition fails and if we consider the situation where small denominators occur, then we show the existence of divergent solutions of an overdetermined system of linearized homological equations. In the $C^\infty$ category, the situation is completely different. We show Sternberg’s theorem for a commuting system of $C^\infty$ vector fields with a Jordan block although they do not satisfy the condition.

Résumé. — Nous étudions la linéarisation simultanée de $d$–actions (et les algèbres correspondants de Lie $d$–dimensionnelles) définie par des champs de vecteurs singuliers dans $\mathbb{C}^n$ fixant l’origine avec des parties linéaires ayant des blocs de Jordan. Nous montrons la convergence analytique des transformations linéarisantes formelles sous une condition d’invariance géométrique pour le spectre de $d$-champs de vecteurs qui engendrent une algèbre de Lie. Si la condition n’est pas satisfaite et si il y a des petits diviseurs, nous montrons l’existence de solutions divergentes pour un système sous déterminé d’équations linéarisées homologiques. Dans le cadre de fonctions $C^\infty$ la situation est complètement différente. Nous montrons le théorème de Sternberg pour une famille commutative de champs de vecteurs qui ne satisfait pas la condition.

Keywords: singular vector field, linearization, Jordan block, homological equation, Diophantine conditions, Gevrey spaces, decomposition.

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1. Simultaneous normalization

Let $\mathbb{K}$ be $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, and $B = \infty$, $B = \omega$ or $B = k$ for some $k > 0$. Let $G^n_B$ denote a $d$-dimensional Lie algebra of germs at $0 \in \mathbb{K}^n$ of $C^B$ vector fields vanishing at $0$. Let $\rho$ be a germ of singular infinitesimal $\mathbb{K}^d$–actions of class $C^B$ ($d \geq 2$)

$$\rho : \mathbb{K}^d \longrightarrow G^n_B.$$  

(1.1)

We denote by $\text{Act}^B(\mathbb{K}^d : \mathbb{K}^n)$ the set of germs of singular infinitesimal $\mathbb{K}^d$–actions of class $C^B$ at $0 \in \mathbb{K}^n$. By choosing a basis $e_1, \ldots, e_d \in \mathbb{K}^n$, the infinitesimal action can be identified with a $d$–tuple of germs at $0$ of commuting vector fields $X^j = \rho(e_j)$, $j = 1, \ldots, d$ (cf. [13], [17]). We can define, in view of the commutativity relation, the action

$$\tilde{\rho} : \mathbb{K}^d \times \mathbb{K}^n \longrightarrow \mathbb{K}^n,$$

(1.2)

$$\tilde{\rho}(s; z) = X_{s_1}^1 \circ \cdots \circ X_{s_d}^d(z) = X_{s_{s_1}}^{s_1} \circ \cdots \circ X_{s_{s_d}}^{s_d}(z), \quad s = (s_1, \ldots, s_d),$$

for all permutations $\sigma = (\sigma_1, \ldots, \sigma_d)$ of $\{1, \ldots, d\}$, where $X^j_t$ denotes the flow of $X^j$. We denote by $\rho_{\text{lin}}$ the linear action formed by the linear parts of the vector fields defining $\rho$.

We shall investigate necessary and sufficient conditions for the linearization of $\rho$, namely, whether there exists a $C^B$ diffeomorphism $g$ preserving $0$ such that $g$ conjugates $\tilde{\rho}$ and $\rho_{\text{lin}}$

$$\tilde{\rho}(s; g(z)) = g(\rho_{\text{lin}}(s, z)), \quad (s, z) \in \mathbb{K}^d \times \mathbb{K}^n.$$

(1.3)

We recall that in [13] and [24], the linear parts were supposed to be diagonalizable, while in [29] the existence of $n - d$ analytic first integrals was required. (See also [1], [15]). Following Katok's argument in [17], we take a positive integer $m \leq n$ such that $\mathbb{K}^n$ is decomposed into a direct sum of $m$ linear subspaces invariant under all $A^\ell = \nabla X_\ell(0)$ ($\ell = 1, \ldots, d$):

$$\mathbb{K}^n = \mathbb{I}^{s_1} + \cdots + \mathbb{I}^{s_m}, \quad \dim \mathbb{I}^{s_j} = s_j, \quad j = 1, \ldots, m,$$

$$s_1 + \cdots + s_m = n.$$  

(1.4)

The matrices $A^1, \ldots, A^d$ can be simultaneously brought in an upper triangular form, and we write again $A^\ell$ for the matrices,

$$A^\ell = \begin{pmatrix} A^\ell_1 & 0_{s_1 \times s_2} & \cdots & 0_{s_1 \times s_m} \\ 0_{s_2 \times s_1} & A^\ell_2 & \cdots & 0_{s_2 \times s_m} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{s_m \times s_1} & 0_{s_m \times s_2} & \cdots & A^\ell_m \end{pmatrix}, \quad \ell = 1, \ldots, d.$$  

(1.5)
If \( \mathbb{K} = \mathbb{C} \), the matrix \( A_j^\ell \) is given by

\[
A_j^\ell = \begin{pmatrix}
\lambda_j^\ell & A_{j,12}^\ell & \cdots & A_{j,1s_j}^\ell \\
0 & \lambda_j^\ell & \cdots & A_{j,2s_j}^\ell \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_j^\ell \\
\end{pmatrix}, \quad \ell = 1, \ldots, d, \quad j = 1, \ldots, m, \tag{1.6}
\]

with \( \lambda_j^\ell, A_{j, \nu \mu}^\ell \in \mathbb{C} \). On the other hand, if \( \mathbb{K} = \mathbb{R} \), then we have, for every \( 1 \leq j \leq m \) two possibilities: firstly, all \( A_j^\ell \) (\( \ell = 1, \ldots, d \)) are given by (1.6) with \( \lambda_j^\ell \in \mathbb{R} \). Secondly, \( s_j = 2 \tilde{s}_j \) is even and \( A_j^\ell \) is a \( \tilde{s}_j \times \tilde{s}_j \) square block matrix given by

\[
A_j^\ell = \begin{pmatrix}
R_2(\lambda_j^\ell, \mu_j^\ell) & A_{j,12}^{\ell \tilde{s}_j} & \cdots & A_{j,1s_j}^{\ell \tilde{s}_j} \\
0 & R_2(\lambda_j^\ell, \mu_j^\ell) & \cdots & A_{j,2s_j}^{\ell \tilde{s}_j} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & R_2(\lambda_j^\ell, \mu_j^\ell) \\
\end{pmatrix}, \quad \ell = 1, \ldots, d, \tag{1.7}
\]

where

\[
R_2(\lambda, \mu) := \begin{pmatrix}
\lambda & \mu \\
-\mu & \lambda \\
\end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}, \tag{1.8}
\]

and \( A_{j, \nu \mu}^{\ell \tilde{s}_j} \) are appropriate real matrices.

Following the decomposition (1.6) (respectively, (1.7)) we define \( \tilde{\lambda}^j \) by

\[
\tilde{\lambda}^j = t(\lambda_1^j, \ldots, \lambda_m^j) \in \mathbb{K}^m, \quad k = 1, \ldots, d. \tag{1.9}
\]

Then we assume

\[
\tilde{\lambda}^1, \ldots, \tilde{\lambda}^d \text{ are linearly independent in } \mathbb{K}^m. \tag{1.10}
\]

One can easily see that (1.10) is invariantly defined.

By (1.5) we define

\[
\tilde{\lambda}_j^k = t(\lambda_1^j, \ldots, \lambda_m^j) \in \mathbb{K}^d, \quad j = 1, \ldots, m, \tag{1.11}
\]

and

\[
\Lambda_m := \{ \tilde{\lambda}_1, \ldots, \tilde{\lambda}_m \}. \tag{1.12}
\]

We define the cone \( \Gamma[\Lambda_m] \) by

\[
\Gamma[\Lambda_m] = \left\{ \sum_{j=1}^m t_j \tilde{\lambda}_j^k \in \mathbb{K}^d; \ t_j \geq 0, \ j = 1, \ldots, m, \sum_{j=1}^m t_j \neq 0 \right\}. \tag{1.13}
\]
Definition 1.1. — We say that the $K^d$–action $\rho$ is a Poincaré morphism if there exists a basis $\Lambda_m \subset K^m$ such that $\Gamma[\Lambda_m]$ is a proper cone in $K^m$, namely it does not contain a straight real line. If the condition is not satisfied, then, we say that the $K^d$–action is in a Siegel domain.

Note that the definition is invariant under the choice of the basis $\Lambda_m$.

Remark 1.2. — As to the alternative definition of a Poincaré morphism we refer to [Definition 6.2.1, [24]].

Next, we introduce the notion of simultaneous resonance. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in K^m$, $\beta = (\beta_1, \ldots, \beta_m) \in K^m$, we set $\langle \alpha, \beta \rangle = \sum_{\nu=1}^{m} \alpha_\nu \beta_\nu$.

For a positive integer $k$ we define $Z_m^+(k) = \{ \alpha \in Z_m^+ ; |\alpha| \geqslant k \}$. Put

\begin{align}
\omega_j(\alpha) &= \sum_{\nu=1}^{d} |\langle \tilde{\lambda}_\nu, \alpha \rangle - \lambda_j^\nu|, \quad j = 1, \ldots, m, \\
\omega(\alpha) &= \min\{\omega_1(\alpha), \ldots, \omega_m(\alpha)\}.
\end{align}

Definition 1.3. — We say that $\Lambda_m$ is simultaneously nonresonant (or, in short $\rho$ is simultaneously nonresonant), if

\begin{equation}
\omega(\alpha) \neq 0, \quad \forall \alpha \in Z_m^+(2).
\end{equation}

If (1.16) does not hold, then we say that $\Lambda_m$ is simultaneously resonant.

Clearly, the simultaneously nonresonant condition (1.16) is invariant under the change of the basis $\Lambda_m$. We state the first main result of our paper

Theorem 1.4. — Let $\rho$ be a Poincaré morphism. Then $\rho$ is conjugated to a polynomial action by an holomorphic change of variables.

Remark 1.5. — In case $\rho$ has a semi simple linear part, then Theorem 1.4 is already known. (cf. [Theorem 2.1.4, [24]])

Example 1.6. — We compare our theorem with the results of Stolovitch [24] and Zung [29]. Let $\rho$ be a $R^2$–action in $R^n$, $n \geqslant 4$ with $m = 3$. We choose a basis $\Lambda_2$ of $R^3$ such that

\begin{equation}
\Lambda_2 = \{(1,1,\nu), (0,1,\mu)\}, \quad \nu, \mu \in R.
\end{equation}

(cf. [12] for similar and more general reductions of commuting vector fields on the torus).

We will characterize the set of $(\nu, \mu) \in R^2$ so that the action is a Poincaré morphism, and determine the simultaneous resonances. By (1.13), $\Gamma[\Lambda_2]$ is generated by the set of vectors $\{(1,0), (1,1), (\nu, \mu)\}$. Hence the action is a Poincaré morphism if and only if these vectors generate a proper cone,
namely $(\nu, \mu)$ is not in the set $\{(\nu, \mu) \in \mathbb{R}^2; \nu \leq \mu \leq 0\}$. We note that the interesting case is $\mu < \nu \leq 0$, where every generator in (1.17) is in a Siegel domain. Theorem 1.4 can be applied to such a case. In §3 we will show that if the action is not a Poincaré morphism, i.e., $\nu < \mu < 0$, then there exist $(\nu, \mu)$ with the density of continuum such that the linearized overdetermined system of two homological equations has a divergent solution.

Next we will determine $(\nu, \mu)$ so that a simultaneous resonance exists. If $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{Z}_+^3(2)$ is a simultaneous resonance, we have the following set of equations:

\begin{align*}
(1) & \quad \eta_1 + \eta_2 + \nu \eta_3 = 1, \quad \eta_2 + \mu \eta_3 = 0, \\
(2) & \quad \eta_1 + \eta_2 + \nu \eta_3 = 1, \quad \eta_2 + \mu \eta_3 = 1, \\
(3) & \quad \eta_1 + \eta_2 + \nu \eta_3 = \nu, \quad \eta_2 + \mu \eta_3 = \mu.
\end{align*}

Elementary computations imply that, in order that one of these equations has a solution $\eta$ the $(\nu, \mu)$ satisfies the following:

a) Case $\nu \leq \mu \leq 0$. The resonance exists iff $(\nu, \mu) \in \mathbb{Q}_- \times \mathbb{Q}_-$, where $\mathbb{Q}_-$ is the set of nonpositive rational numbers. The resonance is given by $(1+(\mu-\nu)k, -\mu k, k)$ and $((\mu-\nu)k, 1-k\mu, k)$ where $k \geq 1/(1-\nu), k \in \mathbb{Z}_+$, and $((\nu-\mu)(1-k), \mu(1-k), k)$, where $k \geq (2-\nu)(1-\nu), k \in \mathbb{Z}_+$.

b) Case $\nu > \mu$ and $\mu \leq 0$. The resonance is given by $(0, -\mu/(\nu-\mu), 1/(\nu-\mu))$, where $-\mu/(\nu-\mu) \in \mathbb{Z}_+, 1/(\nu-\mu) \in \mathbb{Z}_+$ and $2\nu-\mu \leq 1$.

c) Case $\mu > 0$, $\nu \leq \mu$. The resonance is given by $(0, 0, 1/\nu)$, when $\nu = \mu$, $\nu \leq 1/2, \nu^{-1} \in \mathbb{Z}_+$; $(0, \nu, 0)$, when $\nu = \mu > 2$, $\nu \in \mathbb{Z}_+$; $((\mu-\nu)/\mu, 0, 1/\mu)$, if otherwise, where $(\mu-\nu)/\mu \in \mathbb{Z}_+, 1/\mu \in \mathbb{Z}_+$ and $\nu + \mu \leq 1$.

d) Case $\nu > \mu$, $\mu \geq 0$. The resonance is given by $(\nu-\mu, \mu, 0)$, where $\nu-\mu \in \mathbb{Z}_+, \mu \in \mathbb{Z}_+$ and $\nu \geq 2$.

Let $\nu$ be a negative rational number, $\nu = -k_1/k_2, k_1, k_2 \in \mathbb{Z}_+, k_2 \neq 0$. Let $\mu$ be a rational number and satisfy $\mu < \nu$. Assume that the nonlinear part of $X^2$ is zero. If the nonlinear part of $X^1$ consists of the resonant terms of $X^2$, then we have $[X^1, X^2] = 0$. We can easily see that the linearizability of $X^1$ holds provided $\mu \neq \nu - 1/k_2 = -(k_1 + 1)/k_2$.

## 2. A Poincaré morphism

We start by showing equivalent forms of a Poincaré morphism.

**Proposition 2.1. —** The action is a Poincaré morphism if and only if each of the following conditions holds
i) there exist a positive constant $C$ and an integer $k_0$ such that

\[(2.1) \sum_{k=1}^{d} \left| \sum_{j=1}^{m} \lambda_j^k \alpha_j \right| \geq C |\alpha|, \quad \forall \alpha \in \mathbb{Z}_+^m(k_0). \]

ii) there exists a nonzero vector $c = (c_1, \ldots, c_d) \in \mathbb{C}^d$ if $\mathbb{K} = \mathbb{C}$ (respectively, $c = (c_1, \ldots, c_d) \in \mathbb{R}^d$ if $\mathbb{K} = \mathbb{R}$) such that

\[(2.2) c_1 \lambda_1^j + \cdots + c_d \lambda_d^j \text{ is in a Poincaré domain,} \]

namely, the convex hull of the set \{\sum_{j=1}^{d} c_j \lambda_k^j; k = 1, \ldots, m\} in $\mathbb{C}$ does not contain $0 \in \mathbb{C}$ (respectively,

\[(2.3) \text{ the real parts of } c_1 \lambda_j^1 + \cdots + c_d \lambda_j^d, j = 1, \ldots, m, \text{ are positive.} \]

**Proof.** — First we show (2.1). Suppose that (2.1) does not hold. Then there exists a sequence $\alpha^\ell \in \mathbb{Z}_+^m$, $\ell \in \mathbb{N}$ such that

\[(2.4) \sum_{k=1}^{d} \left| \sum_{j=1}^{m} \lambda_j^k \alpha_j^\ell \right| \leq \frac{|\alpha^\ell|}{\ell}, \quad \ell \in \mathbb{N}. \]

By taking a subsequence, if necessary, we may assume that $\alpha^\ell / |\alpha^\ell| \to t^0 = (t_1^0, \ldots, t_m^0) \in S^{1}_{\ell} \cap \mathbb{R}_m^+$ when $\ell \to \infty$, where $S^{1}_{\ell} := \{x \in \mathbb{K}^m; \|x\|_{\ell^1} = \sum_{j=1}^{m} |x_j| = 1\}$ stands for the $\ell^1$ unit sphere. By letting $\ell \to \infty$ in (2.4) we get

\[\sum_{k=1}^{d} \left| \sum_{j=1}^{m} \lambda_j^k t_j^0 \right| = 0.\]

It follows that $\sum_{j=1}^{m} t_j^0 \tilde{\lambda}_j = 0$. Let $J \subset \{1, \ldots, m\}$ be such that $\sum_{j \in J} t_j^0 \tilde{\lambda}_j \neq 0$. Such a set $J$ exists by (1.10). It follows that

\[0 \neq \sum_{j \in J} t_j^0 \tilde{\lambda}_j = - \sum_{j \in \{1, \ldots, m\} \setminus J} t_j^0 \tilde{\lambda}_j. \]

Hence $\Gamma[\Lambda_m]$ contains a straight line generated by $\sum_{j \in J} t_j^0 \tilde{\lambda}_j \neq 0$. This contradicts the assumption that $\Gamma[\Lambda_m]$ is a proper cone.

Conversely, suppose that (2.1) is satisfied. We shall show that $\Gamma[\Lambda_m]$ is proper. Indeed, if otherwise, we can find $t^0 = (t_1^0, \ldots, t_m^0) \in S^{1}_{\ell} \cap \mathbb{R}_m^+ \setminus 0$ such that

\[(2.5) \sum_{j=1}^{m} t_j^0 \lambda_j^k = 0, \quad k = 1, \ldots, d.\]
Because the set \( \{ \alpha/|\alpha|; \alpha \in \mathbb{Z}_+^m(2) \} \) is dense in \( S^1 \cap \mathbb{R}_+^m \), there exists a sequence \( \alpha^\ell \in \mathbb{Z}_+^m \), \( \ell \in \mathbb{N} \) such that \( |\alpha^\ell| \to \infty (\ell \to \infty) \) and \( \lim_{\ell \to \infty} \alpha^\ell/|\alpha^\ell| = t^0 \). Therefore, in view of (2.5), we get

\[
\lim_{\ell \to \infty} \left( \frac{1}{|\alpha^\ell|} \sum_{k=1}^{d} \left| \sum_{j=1}^{m} \lambda_j^k \alpha_j^\ell \right| \right) = 0,
\]

which contradicts (2.1).

Next, we shall show ii). Suppose that \( \Gamma[\Lambda_m] \) be a proper cone in \( \mathbb{K}^d \).

Then we can find \( c = (c_1, \ldots, c_d) \in \mathbb{C}^d \) such that \( \Gamma[\Lambda_m] \) is contained in the real half–space \( P_c := \{ z \in \mathbb{K}^d , \text{Re}(\sum_{k=1}^{d} c_k z_k) > 0 \} \). Therefore (2.6)

\[
0 < \text{Re}(\sum_{k=1}^{d} c_k \sum_{j=1}^{m} t_j \lambda_j^k) = \sum_{j=1}^{m} t_j \text{Re}(\sum_{k=1}^{d} c_k \lambda_j^k)
\]

for all \( t \in \mathbb{R}_+^m \setminus 0 \), which yields \( \text{Re}(\sum_{k=1}^{d} c_k \lambda_j^k) > 0 \) for \( j = 1, \ldots, m \). We note that, if \( \mathbb{K} = \mathbb{R} \), then the use of the real part in the definition of the half–space is superfluous. Finally, we readily see, from (2.2) that, if \( \mathbb{K} = \mathbb{C} \) (respectively, (2.3) if \( \mathbb{K} = \mathbb{R} \)), then the cone \( \Gamma[\Lambda_m] \) is contained in \( P_c \). Hence \( \Gamma[\Lambda_m] \) is proper. \( \square \)

Although the following proposition is known, we give an alternative proof for the sake of completeness. (cf. [Lemma 3.1, [25]].)

**Proposition 2.2.** — Let the action \( \rho \) be a Poincaré morphism. Then we can find a vector field in the corresponding Lie algebra which has the same resonance as the simultaneous resonance of \( \rho \) and is in the Poincaré domain.

**Proof.** — By ii) of Proposition 2.1 we can find a Poincaré vector field in the Lie algebra as a linear combination of a base corresponding to (2.2). Let \( c_\nu \) be the numbers in (2.2), and define \( \tilde{\lambda}^0 := (\lambda^0_1, \ldots, \lambda^0_m) = \sum_{\nu=1}^{d} c_\nu \tilde{\lambda}^\nu \).

Let \( S \) be a simultaneous resonance of \( \rho \). Consider

\[
\langle \tilde{\lambda}^0, \alpha \rangle - \lambda^0_j = \sum_{\nu=1}^{d} c_\nu \left( \langle \tilde{\lambda}^\nu, \alpha \rangle - \lambda^\nu_j \right).
\]

Because \( \sum_{\nu=1}^{d} |\langle \tilde{\lambda}^\nu, \alpha \rangle - \lambda^\nu_j| \neq 0 \) for every \( \alpha \in \mathbb{Z}_+^m(2) \setminus S \), it follows that the set \( \langle \tilde{\lambda}^0, \alpha \rangle - \lambda^0_j = 0 \) in \( c = (c_1, \ldots, c_d) \in \mathbb{C}^d \) is a hyperplane if \( \alpha \not\in S \).

It follows that the set

\[
\{ c = (c_1, \ldots, c_d) \in \mathbb{C}^d; \langle \tilde{\lambda}^0, \alpha \rangle - \lambda^0_j = 0, \exists j, 1 \leq j \leq m, \exists \alpha \in \mathbb{Z}_+^m(2) \setminus S \}
\]
is a countable union of nowhere dense closed set. Therefore we can find $c = (c_1, \ldots, c_d)$ for which $\sum_{\nu=1}^{d} c_{\nu} \lambda^\nu$ satisfies the Poincaré condition and has the resonance $S$. \qed

We propose a geometric expression of a Poincaré morphism.

**Definition 2.3.** — Let $r > 0$ and $g$ be a Riemannian metric on $\mathbb{R}^n$. We denote by $\langle \cdot, \cdot \rangle_g$ and $\| \cdot \|_g$ the inner product and the norm with respect to $g$, respectively. We say that $X^\nu := \sum_{j=1}^{n} X_j^\nu (x) \partial_{x_j}$, $(\nu = 1, \ldots, d)$ are simultaneously transversal to the sphere $\|x\|_g = r$ if, the vectors $X^\nu := (X_1^\nu, \ldots, X_d^\nu)$ $(\nu = 1, \ldots, d)$ satisfy

\begin{equation}
\sum_{\nu=1}^{d} |\langle X^\nu, x \rangle_g| \neq 0, \quad \forall x, \quad \|x\|_g = r.
\end{equation}

**Theorem 2.4.** — Let $r > 0$. Suppose that $B^\nu := \sum_{j=1}^{n} (A^\nu x)_j \partial_{x_j}$ $(\nu = 1, \ldots, d)$ be a commuting system of semi simple linear real vector fields in $\mathbb{R}^n$. Let $\rho$ be the action generated by $\{B^\nu\}$. We choose a real non-singular matrix $P$ such that $A^\nu = PA^\nu P^{-1}$ is a block diagonal matrix given by $\Lambda^\nu = \text{diag} \{R_2(\xi^\nu_1, \eta^\nu_1), \ldots, R_2(\xi^\nu_{n_1}, \eta^\nu_{n_1}, \lambda^\nu_{n_1+1}, \ldots, \lambda^\nu_{n})\}$ for some integer $n_1 \leq n$. Let $g$ be a Riemannian metric defined by $\rho PP$. Then the following conditions are equivalent.

(a) $B^\nu$ $(\nu = 1, \ldots, d)$ are simultaneously transversal to the sphere $\|x\|_g = r$.

(b) $\rho$ is a Poincaré morphism.

(c) There exist real numbers $c^\nu$ $(\nu = 1, \ldots, d)$ such that $\sum_{\nu=1}^{d} c^\nu B^\nu$ is transversal to the sphere $\|x\|_g = r$.

**Proof.** — We note that $\langle x, y \rangle_g = \langle Px, Py \rangle$ and $\|x\|_g = \|Px\|$. By inserting the relation $X^\nu = A^\nu x = P^{-1} \Lambda^\nu P x$ into (2.7) we can easily see that the simultaneous transversality condition is equivalent to

\begin{equation}
\sum_{\nu=1}^{d} |\langle \Lambda^\nu y, y \rangle| \neq 0, \quad \forall y = (y_1, \ldots, y_n), \quad \|y\| = r.
\end{equation}

By definition, (2.8) can be written in

\begin{equation}
\sum_{\nu=1}^{d} \left| \sum_{j=1}^{n_1} \xi^\nu_j (y_{2j-1}^2 + y_{2j}^2) + \sum_{j=n_1+1}^{n} y_j^2 \lambda_j^\nu \right| \neq 0, \quad \forall y, \quad \|y\| = r.
\end{equation}

We define $t = (t_1, \ldots, t_n)$, $t \in \mathbb{R}^n_+$, $\sum t_j = 1$ by $t_j = (y_{2j-1}^2 + y_{2j}^2)/2$ if $j \leq n_1$ and $t_j = y_j^2$ if $j > 2n$. Noting that $\xi^\nu_j (y_{2j-1}^2 + y_{2j}^2) = 2t_j \xi^\nu_j = t_j (\xi^\nu_j + i\eta^\nu_j + \xi^\nu_j - i\eta^\nu_j)$ we see that (2.9) is written in $\sum_{\nu=1}^{d} |\sum_{j=1}^{n} t_j \lambda_j^\nu| \neq 0$ for every $t \in \mathbb{R}^n_+$ and $\sum t_j = 1$. This is equivalent to (b) by definition. Hence we have proved the equivalence of (a) and (b).
By Proposition 2.1 the condition (b) is equivalent to the existence of real numbers \(c_{\nu} (\nu = 1, \ldots, d)\) such that \(\sum_{\nu=1}^{d} c_{\nu} B_{\nu}\) is a Poincaré vector field. By what we have proved in the above \((d = 1)\) this is equivalent to say that \(\sum_{\nu=1}^{d} c_{\nu} B_{\nu}\) is transversal to the sphere \(\|x\|_g = r\).

\[\Box\]

**Proof of Theorem 1.4.** By Proposition 2.2 there exists a Poincaré vector field \(\chi_0\) in \(\rho\) which is in a Poincaré domain and has the same resonance as \(\rho\). If \(\rho\) is not resonant, then we have Theorem 1.4. In case there is a resonance of \(\rho\), then it follows from Lemma 3.2 of [25] that, if \(\chi_0\) is normalized, then so is \(\rho\).

\[\Box\]

3. Divergent solutions of overdetermined systems of linearized homological equations

We now study the action \(\rho_{lin}\) which admits a Jordan block. We assume that the action is formally (simultaneously) linearizable and is not a Poincaré morphism and that the family of linear parts is Diophantine.

Let \(\mathbb{C}_2^n \{x\}\) be the set of \(n\) vector functions of convergent power series of \(x\) without constant and linear terms. We consider

\[(3.1) \quad L_A v = t(L_1 v, \ldots, L_d v) = f, \quad f := t(f^1, \ldots, f^d) \in (\mathbb{C}^n_2 \{x\})^d,\]

under the compatibility conditions

\[(3.2) \quad L_j f^k = L_k f^j, \quad j, k = 1, \ldots, d,\]

where \(L_j\) is the Lie derivative of the linear vector field \(A_j x \partial_x\)

\[L_j v = [A_j x, v] = (A_j x, \partial_x) v - A_j v, \quad j = 1, \ldots, d.\]

First we consider a 2–action studied in Example 1.6. We assume that there exists a vector field in the two-dimensional Lie algebra which is not semi simple. In view of Example 1.6 we can choose a base \(X_1, X_2\) with linear parts \(A_j \in GL(4; \mathbb{C})\) satisfying \(\text{Spec} (A_1) = \{1, 1, \nu, \nu\}\) and \(\text{Spec} (A_2) = \{0, 1, \mu, \mu\}\), respectively, where \(\nu \leq \mu \leq 0\), \((\nu, \mu) \not\in \mathbb{Q}^2\), and

\[(3.3) \quad A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \nu & \varepsilon \\ 0 & 0 & 0 & \nu \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu & \varepsilon_0 \varepsilon \\ 0 & 0 & 0 & \mu \end{pmatrix},\]

where \(\varepsilon \neq 0\) and \(\varepsilon_0 \in \mathbb{C}\). We can make \(|\varepsilon| > 0\) arbitrarily small by an appropriate linear change of variables.
Let \( \omega(\alpha) \) be defined by (1.15). We say that the simultaneous Diophantine order of \( \{ \text{Spec}(A_1), \text{Spec}(A_2) \} \) is \( \tau_0 \), if, for every \( \tau > \tau_0 \) there exists \( C = C_\tau > 0 \) such that

\[
(3.4) \quad \omega(\alpha) \geq C|\alpha|^{-\tau}, \quad \forall \alpha \in \mathbb{Z}_+^4(2),
\]

while, for every \( \tau < \tau_0 \) there exist \( C' > 0 \) and a subsequence \( \alpha^\ell \in \mathbb{Z}_+^4(2) \) \((\ell = 1, 2, \ldots)\) such that

\[
(3.5) \quad \omega(\alpha^\ell) \leq C'|\alpha^\ell|^{-\tau}, \quad \ell \in \mathbb{N}.
\]

First we note that the conditions (3.4) and (3.5) for \( \omega(\alpha) \) are equivalent to the corresponding ones for \( \|q\nu\| + \|q\mu\| \) when \( q \in \mathbb{N}, q \to \infty \), where \( \|t\| = \min_{p \in \mathbb{Z}} |p - t| \). Hence the number \( \tau_0 \) in (3.4) and (3.5) is equal to the speed of the simultaneous approximation of \( \nu \) and \( \mu \), namely \( \|q\nu\| + \|q\mu\| \sim Cq^{-\tau_0} \) for some constant \( C > 0 \) independent of \( q \). Clearly, if (3.4) holds, then we have an upper bound of \( \tau_0 \). By the result of M. Herman, [16], we have an upper bound \( 2 + \varepsilon \) for every \( \varepsilon > 0 \) for almost all \( \nu \) and \( \mu \). On the other hand, Moser showed that there exist Liouville numbers \( \nu \) and \( \mu \) such that \( \|q\nu\| + \|q\mu\| \geq cq^{-\tau} \) for any given \( \tau > 2 \). (See [Theorem 2, [20]]). This implies that for every \( \tau > 2 \), there exist Liouville numbers \( \nu \) and \( \mu \) such that \( \tau_0 \leq \tau \). We have another upper bound of \( \tau_0 \) if either \( \nu \) or \( \mu \) is an algebraic number. Indeed, by Roth’s theorem, for any given \( \tau > 1 \) there exists \( c > 0 \) such that \( \|q\nu\| + \|q\mu\| \geq cq^{-\tau} \). Hence we have \( \tau_0 \leq 1 \). Finally, by [Corollary 1B, p.27, [22]], if either \( \nu \) or \( \mu \) is an irrational number, then we have a lower bound \( \tau_0 \geq 1/2 \).

We say that \( \nu \) and \( \mu \) are simultaneously Liouville, if (3.5) holds for every \( \tau > 0 \).

Let \( \sigma \geq 1 \). We say that a formal power series \( f(x) = \sum \alpha f_\alpha x^\alpha \) is in a Gevrey space \( G_2^\sigma(\mathbb{C}^4) \) if \( f_\alpha = 0 \) for \( |\alpha| \leq 1 \) and, there exist \( C > 0 \) and \( R > 0 \) such that \( |f_\alpha| \leq CR^{||\alpha||}\alpha!^\sigma - 1, (\forall \alpha \in \mathbb{Z}_+^4) \).

We consider the following equation

\[
(3.6) \quad L_Av := t(L_1v, L_2v) = f, \quad f = t(f^1, f^2) \in (\mathbb{C}_+^4(x))^2, \quad x \in \mathbb{C}^4,
\]

where \( t(f^1, f^2) \) satisfies the compatibility condition \( L_1 f^2 = L_2 f^1 \). Then we have:

**Theorem 3.1.** — Assume that \( \varepsilon_0 \in \mathbb{R} \setminus \{0\} \). Let \( 1 < \tau_0 < \infty \) be given. Then there exists \( E_0 \subset \{(\nu, \mu) \in \mathbb{R}^2; \nu < \mu \leq 0\} \) with the density of continuum satisfying \( \{(\nu, \mu) \in \mathbb{Q}^2; \nu < \mu \leq 0\} \subset E_0 \) such that for every \( (\nu, \mu) \in E_0 \), there exists an \( f = t(f^1, f^2) \in (\mathbb{C}_+^4(x))^2 \) such that \( L_1 f^2 = L_2 f^1 \) and Eq. (3.6) has a formal power series solution \( v \notin \bigcup_{1 \leq \sigma < 2 + \tau_0} G_2^\sigma(\mathbb{C}^4) \).
Furthermore, if the conditions \((\nu, \mu) \not\in \mathbb{Q}^2\), (3.4) and \(\tau_0 < +\infty\) hold, then (3.6) has a unique solution \(v \in \bigcap_{\sigma > 2 + \tau_0} G_2^2(C^4)\) for every \(t(f^1, f^2) \in (C^4_\alpha \{x\})^2\) satisfying \(L_1 f^2 = L_2 f^1\).

In order to prove Theorem 3.1, we need a function space \(G\) which is a subspace of a set of holomorphic functions in a neighborhood of the origin. First we give the definition in the case \(\varepsilon_0 = 1\), i.e., the nilpotent parts of \(A_1\) and \(A_2\) coincide. We define \(G\) by

\[
G := \left\{ f = t(f_1, f_2, f_3, f_4); \ f_j = f_j(x) = \sum_{\alpha \in C_j} f_{\alpha,j} x^{\alpha}, \ j = 1, 2, 3, 4 \right\},
\]

where \(C_j \subset \mathbb{Z}^4_+(2)\) satisfies the following two conditions.  

1. There exist \(c_0 > 0\) and \(\tau > \tau_0\) such that for every \(\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in C_j\), we have \(N = \alpha_3 + \alpha_4 \neq 0\), \(|\alpha| \geq 2\), and

\[
|\alpha_1 - 1 + (\nu - \mu) N| < N^{-N(\tau + 1)} c_0^N, \quad \forall \alpha \in C_j, (j = 1, 2),
\]

\[
|\alpha_1 + (\nu - \mu)(N - 1)| < N^{-N(\tau + 1)} c_0^N, \quad \forall \alpha \in C_j, (j = 3, 4).
\]

2. The Diophantine condition for \(Spec(A_1)\) holds: namely for every \(\tau' < \tau_0 < \tau''\), there exist \(c_1 > 0\) and \(c_2 > 0\) such that

\[
c_1 N^{-\tau''} < |\alpha_1 + \alpha_2 - 1 + \nu N| < c_2 N^{-\tau'} \quad \text{if} \quad \alpha \in C_j, j = 1, 2,
\]

\[
c_1 N^{-\tau''} < |\alpha_1 + \alpha_2 + \nu N - \nu| < c_2 N^{-\tau'} \quad \text{if} \quad \alpha \in C_j, j = 3, 4,
\]

where \(N = \alpha_3 + \alpha_4 \neq 0\).

Remark 3.2. — If \(\varepsilon_0 \neq 1\), we replace (1) with the following (1)'.

1'. There exist \(c_0 > 0\) and \(\tau > \tau_0\) such that for every \(\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in C_j\), we have \(N = \alpha_3 + \alpha_4 \neq 0\), and

\[
|\varepsilon_0 (\alpha_1 - 1 + \alpha_2 + \nu N) - (\alpha_2 + \mu N)| < N^{-N(\tau + 1)} c_0^N, \quad \text{if} \quad \alpha \in C_1.
\]

In the case \(\alpha \in C_2\), we replace \(\alpha_1\) and \(\alpha_2\) in the left-hand side of the above inequality with \(\alpha_1 + 1\) and \(\alpha_2 - 1\), respectively. Similarly, in the case \(\alpha \in C_3\) or \(\alpha \in C_4\), we replace \(\alpha_1\) and \(N\) in the left-hand side of the above inequality with \(\alpha_1 + 1\) and \(N - 1\), respectively.

Remark 3.3. — The space \(G\) is a normed space with the norm \(\|f\| := \sum_\alpha |f_\alpha|\), where \(|f_\alpha| = \sum_j |f_{\alpha,j}|\) \((f \in G)\). If the conditions (1) and (2) in the definition of \(G\) hold, then the Diophantine condition for \(Spec(A_2)\) holds. Hence we have a simultaneous Diophantine condition for \(Spec(A_1)\) and \(Spec(A_2)\). In the following, we will show that on the support \(C_j\) of \(G\), the divergence of the solutions of \(L_A\) occurs, with a sharp Gevrey loss equal to \(1 + \tau\) \((G^1 \rightarrow G^{2+\tau})\).
Remark 3.4. — The space \( \mathcal{G} \) is not empty for an appropriate choice of \( \nu \) and \( \mu \) such that \( \nu < \mu < 0 \), i.e., the action is not a Poincaré morphism. We first consider the case \( \varepsilon_0 = 1 \) for the sake of simplicity. If we construct \( \nu \) and \( \mu \) so as to satisfy the conditions (1) and (2) for \( C_1 = C_2 \), then (1) and (2) for \( C_3 (= C_4) \) hold if we replace \( \alpha_1 \) and \( N \) in \( C_1 \) with \( \alpha_1 + 1 \) and \( N - 1 \), respectively. Hence we will consider \( C_1 \).

We can easily construct an irrational number \( \nu < 0 \) which satisfies (2). In fact, \( \alpha_1 + \alpha_2 \) and \( N \) are given by a continued fraction expansion of \( \nu \). Note that \( \alpha_1 \) can be taken arbitrarily. Next, by the standard measure theoretic argument, we can show that there exist an irrational number \( \mu \) with \( \nu - \mu < 0 \) and the sequence \( \{ \alpha_1 \} \) such that (1) holds. By the construction, we can also choose \( \mu < 0 \) such that \( \nu < \mu < 0 \). It follows that the action is not a Poincaré morphism. Moreover, we can easily see that the set of \( \nu \) and \( \mu \) satisfying (1) and (2) has the density of continuum.

Next we consider the case \( \varepsilon_0 \neq 1 \). For the sake of simplicity, we give the sketch of the proof for \( C_1 \) in the case \( 0 < \varepsilon_0 < 1 \). The other cases can be treated similarly. First we construct \( \nu \) so as to satisfy (2). Then the sequence of the integers \( k \equiv \alpha_1 + \alpha_2 - 1 \) and \( N \) are also given. In order to show that there exists \( \mu \) satisfying (1)', we consider the inequality

\[
\left| \frac{\alpha_1 - 1 + (1 - \varepsilon_0^{-1})\alpha_2}{N} - (\varepsilon_0^{-1}\mu - \nu) \right| < N^{-N(\tau + 1) - 1} c_0^{N} \varepsilon_0^{-1}.
\]

We consider closed intervals of length \( 2N^{-N(\tau + 1) - 1} c_0^{N} \varepsilon_0^{-1} \) with the centers at \( \frac{\alpha_1 - 1 + (1 - \varepsilon_0^{-1})\alpha_2}{N} \), \( (\alpha_1 + \alpha_2 = k + 1) \). Let \( N \) and one of these intervals \( I_N \) are given. Then we can choose \( N' > N \) and \( I_{N'} \) such that \( I_N \) contains \( I_{N'} \).

Hence we can construct a sequence of monotone decreasing intervals. By taking a subsequence, if necessary, we see that there exists \( \mu \) which satisfies (1)'. By construction the set of \( \mu \) has the density of continuum. We remark that we can take \( \tilde{v} := \varepsilon_0^{-1}\mu - \nu > 0 \) if \( 0 < \varepsilon_0 < 1 \). Indeed, since \( 1 - \varepsilon_0^{-1} < 0 \) and \( k/N \to -\nu > 0 \) as \( k, N \to \infty \), it follows that one can take the interval \( I_N \) so that \( I_N \) is contained in the positive real axis and it is arbitrarily close to the origin. Hence we have \( \mu = \varepsilon_0(\tilde{v} + \nu) > \varepsilon_0\nu > \nu \), which implies that the action is not a Poincaré morphism. Similarly, we can show that there exists \( \mu \) such that the condition does not hold in other cases.

The proof of Theorem 3.1 follows from the following propositions.

**Proposition 3.5.** — Assume that \( \varepsilon_0 \in \mathbb{R} \setminus \{0\} \). Let \( 1 < \tau_0 < \infty \). Then there exists \( E'_0 \subset \{ (\nu, \mu) \in \mathbb{R}^2 \setminus \mathbb{Q}^2; \nu < \mu < 0 \} \) with the density of continuum such that the following property holds. For every \( (\nu, \mu) \in E'_0 \) there exist real numbers \( c_1, c_2 \) and \( k_0 > 0 \) such that for every \( g \in \mathcal{G} \),
Assume that our divergence results imply in the case of a single vector field, that generically vector fields obtained by nonlinear holomorphic perturbations are nonlinearizable (see R. Pérez Marco [19] for more details). We point out that our results generalize those for single vector fields in the presence of nontrivial Jordan blocks (see [15]). As to the case of smooth $C^\infty$ hyperbolic $\mathbb{R}^2$–actions we refer [13].

First we will prove Theorem 3.1, assuming Propositions 3.5 and 3.6.

**Proof of Theorem 3.1.** — We will prove the former half. Let $E'_0$ be the set given by Proposition 3.5. We define $E_0 := E'_0 \cup \{(\nu, \mu) \in \mathbb{Q}^2; \nu < \mu \leq 0\}$. By the result of Example 1.6 (a), we know that if $(\nu, \mu) \in \mathbb{Q}^2$, $\nu < \mu \leq 0$, then (3.6) has an infinite resonance, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\alpha_1 = 1 + (\mu - \nu)k$, $\alpha_2 = -\mu k$, $\alpha_3 + \alpha_4 = k$, where $k \geq (1 - \nu)^{-1}$, $k \in \mathbb{Z}_+$. Hence $v = \sum_\alpha v_\alpha x^\alpha$ is a formal solution of (3.6) with $f = 0$ for $v_\alpha \in \mathbb{C}^4$, where the summation with respect to $\alpha$ is taken over the resonances $\alpha$ in the above. Because $|v_\alpha|$ is arbitrary, we take $v_\alpha$ such that $|v_\alpha| = |\alpha|^2 + \tau_0$ ($|\alpha| \to \infty$), which implies $v \not\in \bigcup_{1 \leq \sigma < 2 + \tau_0} G^\sigma_2(\mathbb{C}^4)$.

Next we study the case $(\nu, \mu) \in E'_0$. By Proposition 3.6 there exists $g \in \mathcal{G}$ such that the unique solution $v$ of $L_B v = g$ with $B = c_1 A_1 + c_2 A_2$ is not contained in $\bigcup_{1 \leq \sigma < 2 + \tau_0} G^\sigma_2(\mathbb{C}^4)$. By Proposition 3.5 we can choose $f^j \in \mathcal{G}$ ($j = 1, 2$) such that $L_1 f^2 = L_2 f^1$ and $g = c_1 f^1 + c_2 f^2$. Because the solution $v$ of $L_A v = f$ is a unique solution of $L_B v = g$, we see that $v \not\in \bigcup_{1 \leq \sigma < 2 + \tau_0} G^\sigma_2(\mathbb{C}^4)$.

We will prove the latter half. We consider the system of equations $L_j v = f^j (j = 1, 2)$, where $L_1 f^2 = L_2 f^1$. Let $B$ denote either $A_1$ or $A_2$. For the sake of simplicity, we assume that $B$ is put in a Jordan normal form with the diagonal part $B^0 := \text{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_3\}$ and the off-diagonal element $\varepsilon_1$. We note that, for the equation $L_1 v = f^1$ we have $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \nu$, $\varepsilon_1 = \varepsilon$, while for $L_2 v = f^2$ we have $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = \mu$, $\varepsilon_1 = \varepsilon_0 \varepsilon$. The
homology operator $L_B$ corresponding to $B$ is given by

\begin{align}
(3.7) \quad L_B v &= \langle B^0 x, \partial_x \rangle v + \varepsilon_1 R[v] - B v, \quad v \in \mathbb{C}_2^4 \{x\}, \\
(3.8) \quad \langle B^0 x, \partial_x \rangle v &= \sum_{|\alpha| \geq 2} (\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 (\alpha_3 + \alpha_4)) v_\alpha x^\alpha,
\end{align}

where $v(x) = \sum_{|\alpha| \geq 2} v_\alpha x^\alpha$ and

\begin{equation}
(3.9) \quad R[v] = \sum_{|\alpha| \geq 2} (\alpha_3 + 1) v_{(\alpha_1, \alpha_2, \alpha_3+1, \alpha_4-1)} x^\alpha.
\end{equation}

For $g(x) = t(g_1, g_2, g_3, g_4) \in \mathbb{C}_2^4 \{x\}$ we expand $g_k(x)$ in the Taylor series $g_k(x) = \sum_\alpha g_\alpha k x^\alpha$. For nonnegative integers $N$, $\alpha_1$ and $\alpha_2$ we define $V_k$ and $G_k$ by

\begin{equation}
V_k := t\{v_{(\alpha_1, \alpha_2, N-\ell, \ell); k}\}_{\ell=0}^N, \quad G_k := t\{g_{(\alpha_1, \alpha_2, N-\ell, \ell); k}\}_{\ell=0}^N,
\end{equation}

where $k = 1, 2, 3, 4$. In view of (3.9), the equation $L_B v = g$ is equivalent to

\begin{align}
(3.10) \quad (\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 N - \lambda_1) V_1 + \varepsilon_1 M_N V_1 &= G_1, \\
(3.11) \quad (\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 N - \lambda_2) V_2 + \varepsilon_1 M_N V_2 &= G_2, \\
(3.12) \quad (\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 (N-1)) V_3 + \varepsilon_1 M_N V_3 &= G_3 + \varepsilon_1 V_4, \\
(3.13) \quad (\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 (N-1)) V_4 + \varepsilon_1 M_N V_4 &= G_4,
\end{align}

where $M_N$ is given by

\begin{equation}
(3.14) \quad M_N = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
N & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & N - 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & N - 2 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 2 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 & 0
\end{pmatrix}, \quad N \geq 1,
\end{equation}

and $M_0 = 0$.

Let $f^j(x) = t(f^j_1(x), \ldots, f^j_4(x))$ and let $f^j_k(x) = \sum_\alpha f^j_{\alpha,k} x^\alpha$ ($j = 1, 2; k = 1, \ldots, 4$) be the Taylor expansion of $f^j_k(x)$. We substitute the Taylor expansions of $v$ and $f^j$ into the equations $L_j v = f^j$. For every $(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ and $N \in \mathbb{Z}_+$ such that $\alpha_1 + \alpha_2 + N \geq 2$ we compare the coefficients of $x^\alpha$ ($\alpha_3 + \alpha_4 = N$) with homogeneous degree $\alpha_1 + \alpha_2 + N$. If we set

\begin{equation}
(3.15) \quad F^j = t(F^j_1, F^j_2, \ldots, F^j_4), \quad F^j_k = t\{f^j_{(\alpha_1, \alpha_2, N-r, r); k}\}_{r=0}^N,
\end{equation}

and $V = t(V_1, V_2, \ldots, V_4)$, $V_k := t\{v_{(\alpha_1, \alpha_2, N-r, r); k}\}_{r=0}^N$, $H = t(0, 0, V_4, 0)$, then we can write the system of equations $L_j v = f^j$ ($j = 1, 2$) in the
following form

\[ \mathcal{A}V = F^1 + \varepsilon H, \quad \mathcal{B}V = F^2 + \varepsilon \varepsilon_0 H, \]

where \( \mathcal{A} \) and \( \mathcal{B} \) are the block diagonal matrices given by

\[
\mathcal{A} := \text{diag}\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\} = \text{diag}
\begin{pmatrix}
(\alpha_1 + \alpha_2 + \nu N - 1)Id + \varepsilon M_N \\
(\alpha_1 + \alpha_2 + \nu N - 1)Id + \varepsilon M_N \\
(\alpha_1 + \alpha_2 + \nu N - \nu)Id + \varepsilon M_N \\
(\alpha_1 + \alpha_2 + \nu N - \nu)Id + \varepsilon M_N
\end{pmatrix},
\]

\[
\mathcal{B} := \text{diag}\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\} = \text{diag}
\begin{pmatrix}
(\alpha_2 + \mu N)Id + \varepsilon_0 \varepsilon M_N \\
(\alpha_2 + \mu N - 1)Id + \varepsilon_0 \varepsilon M_N \\
(\alpha_2 + \mu N - \mu)Id + \varepsilon_0 \varepsilon M_N \\
(\alpha_2 + \mu N - \mu)Id + \varepsilon_0 \varepsilon M_N
\end{pmatrix}.
\]

We will solve (3.16). Because either \( \nu \) or \( \mu \) is an irrational number, we suppose that \( \nu \notin \mathbb{Q} \). We will show that for each \( k = 1, \ldots, 4 \) either \( \mathcal{A}_k \) or \( \mathcal{B}_k \) is nonsingular. Indeed, suppose that \( |\alpha| = \alpha_1 + \alpha_2 + N \geq 2 \). If \( N \neq 0, 1 \), then by the irrationality of \( \nu \), the matrices \( \mathcal{A}_k \) \( k = 1, \ldots, 4 \) are nonsingular. If \( N = 0 \) or \( N = 1 \), then by the condition \( \alpha_1 + \alpha_2 + N \geq 2 \), \( \mathcal{A}_k \) \( k = 1, \ldots, 4 \) are nonsingular. We can similarly argue if \( \mu \notin \mathbb{Q} \).

We will determine \( V_4 \). By inductive arguments and \( L_1 f^2 = L_2 f^1 \) we get

\[
v_{(\alpha_1, \alpha_2, N-\ell, \ell); 4} = \sum_{r=0}^{\ell} (\varepsilon_1)^r (\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 (N-1))^{r+1} (N-\ell+r)! (N-\ell)! \cdot g_{(\alpha_1, \alpha_2, N-\ell+r, \ell-r); 4}
\]

for \( \ell = 0, 1, \ldots, N \), provided \( \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 (N-1) \neq 0 \). Note that, if \( \mathcal{A}_4 \) is nonsingular, then (3.19) is valid for \( \lambda_1 = 0, \lambda_2 = \lambda_3 = \nu, \varepsilon = \varepsilon_1 = \varepsilon, g_{(\alpha_1, \alpha_2, N-\ell+r, \ell-r); 4} = f_{(\alpha_1, \alpha_2, N-\ell+r, \ell-r); 4}, \) while if \( \mathcal{B}_4 \) is nonsingular, then (3.19) is valid for \( \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = \mu, \varepsilon = \varepsilon_0 \varepsilon, g_{(\alpha_1, \alpha_2, N-\ell+r, \ell-r); 4} = f_{(\alpha_1, \alpha_2, N-\ell+r, \ell-r); 4}. \) Similar explicit formulas are derived for \( v_{(\alpha_1, \alpha_2, N-\ell, \ell, k); 3} \), there appears the term \( \varepsilon_1 V_4^N \) in the right-hand side of (3.12).

By (3.4) we have

\[
|\alpha_1 + \alpha_2 + \nu N - \nu| + |\alpha_2 + \mu N - \mu| \geq C|\alpha_1 + \alpha_2 + N|^{-\tau}
\]

for some \( C > 0 \). It follows that either \( |\alpha_1 + \alpha_2 + \nu N - \nu| \geq C|\alpha_1 + \alpha_2 + N|^{-\tau}/2 \) or \( |\alpha_2 + \mu N - \mu| \geq C|\alpha_1 + \alpha_2 + N|^{-\tau}/2 \) holds. Suppose that the former
estimate holds. We have the same estimate in case the latter inequality holds. Without loss of generality we may assume that $C < 2$. Let $\tau$ be such that $\tau > \tau_0$. Then we have

$$
(3.21) \quad |\alpha_1 + \alpha_2 + \nu(N-1)|^{r+1} \geq (C/2)^{r+1}|\alpha_1 + \alpha_2 + N|^{-\tau(r+1)} \geq (C/2)^{N+1}|\alpha_1 + \alpha_2 + N|^{-\tau(N+1)}.
$$

Noting that $(N - \ell + r)!/(N - \ell)! \leq N!$, we see from (3.19) that if $g_{(\alpha_1, \alpha_2, N-\ell+r, \ell-r);4}$ has a $G^s$ estimate, namely, $g_{(\alpha_1, \alpha_2, N-\ell+r, \ell-r);4} = O((\alpha_1 + \alpha_2 + N)!^{s-1})$ modulo exponential factors, then $v_{(\alpha_1, \alpha_2, N-\ell, \ell);4} = O((\alpha_1 + \alpha_2 + N)!^{s+\tau})$. Especially, if $s = 1$, then we have $v_{(\alpha_1, \alpha_2, N-\ell, \ell);4} = O((\alpha_1 + \alpha_2 + N)!^{\tau+1})$. Similarly, we can easily see that $v_{(\alpha_1, \alpha_2, N-\ell, \ell);j}$ ($j = 1, 2, 4$) have the estimate $v_{(\alpha_1, \alpha_2, N-\ell, \ell);j} = O((\alpha_1 + \alpha_2 + N)!^{\tau+1})$.

Next we determine $v_{(\alpha_1, \alpha_2, N-\ell, \ell);3}$ by a similar relation like (3.19). We note that there appears $v_{(\alpha_1, \alpha_2, N-\ell+r, \ell-r);4}$ in the term $g_{(\alpha_1, \alpha_2, N-\ell+r, \ell-r);3}$ of (3.19). By (3.19) we can easily see that $v_{(\alpha_1, \alpha_2, N-\ell+r, \ell-r);4} = O(N^{(\ell-r+1)}|\ell - r|!)$ modulo terms of exponential growth $C^N$ for some $C > 0$. Substituting the estimate into (3.19), we see that the right-hand side of (3.19) is estimated by $N^{(\ell-r+1)}|\ell - r|!(N-\ell)!/(N-\ell)! = N^{(\ell-r+1)}|\ell - r|!/(N-\ell)!$ modulo terms of exponential growth. Because $\ell \leq N$ and $\tau \geq 1$, we see that $v_{(\alpha_1, \alpha_2, N-\ell, \ell);3} = O(N^{N\tau}N!)$ modulo terms of exponential growth. Since $\tau > \tau_0$ is arbitrary, it follows that $v_{(\alpha_1, \alpha_2, N-\ell, \ell);3} = O((\alpha_1 + \alpha_2 + N)!^\sigma)$ for every $\sigma > 1 + \tau_0$.

**Proof of Proposition 3.5.** — Let $E'_0$ be the set of $(\nu, \mu) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$, $\nu < \mu \leq 0$ such that $G \neq \emptyset$. The set $E'_0$ has the density of continuum. (cf. Remark 3.4.) We shall show that if $(c_1, c_2) \notin \mathbb{R}^2$ is not contained in the some set $E$ with Lebesgue measure zero, then $B := c_1A_1 + c_2A_2$ is nonresonant. Indeed, the eigenvalues of $B$ are given by $c_1 + c_2, c_1\nu + c_2\mu$ with multiplicity. For every $\alpha = (\alpha_1, \ldots, \alpha_4) \in \mathbb{Z}^4_+(2)$, the resonance relations are given by

$$
(3.22) \quad c_1\alpha_1 + (c_1 + c_2)\alpha_2 + (c_1\nu + c_2\mu)(\alpha_3 + \alpha_4) = c_1,
$$

and the ones with $c_1$ in the right-hand side replaced by $c_1 + c_2$ and $c_1\nu + c_2\mu$, respectively. Because the argument is similar, we consider the first relation. It follows from (3.22) that

$$
(c_1(\alpha_1 + \alpha_2 + \nu(\alpha_3 + \alpha_4) - 1) + c_2(\alpha_2 + \mu(\alpha_3 + \alpha_4)) = 0.
$$

Because $(\nu, \mu) \notin \mathbb{Q}^2$ and $|\alpha_1| \geq 2$, we can easily see that either $\alpha_1 + \alpha_2 + \nu(\alpha_3 + \alpha_4) - 1 \neq 0$ or $\alpha_2 + \mu(\alpha_3 + \alpha_4) \neq 0$ holds. Hence the set of $(c_1, c_2) \in \mathbb{R}^2$ satisfying (3.22) is a straight line. Therefore the set $E$ of all $(c_1, c_2)$ satisfying a resonance relation for some $\alpha$ has Lebesgue measure zero.
In order to show that $Spec(B)$ satisfies (3.5), let $\tilde{\omega}_j(\alpha)$ and $\tilde{\omega}(\alpha)$ ($\alpha \in \mathbb{Z}_+^4$) be defined by (1.14) and (1.15) for $B$, respectively. Then we have $\tilde{\omega}(\alpha) \leq \tilde{\omega}_j(\alpha) \leq \max\{c_1, c_2\} \omega_j(\alpha)$ for $j = 1, \ldots, 4$ and all $\alpha \in \mathbb{Z}_+^4$. Next, we will estimate $\omega_4(\alpha)$ for $\alpha \in C_4$, where $C_4$ is given in the definition of $G$. We note $\omega_4(\alpha) = |\alpha_1 + \alpha_2 + \nu(N - 1)| + |\alpha_2 + \mu(N - 1)|$ and $|\alpha|/N \to \nu (N, |\alpha| \to \infty)$. By the conditions (1) and (2) in the definition of $G$ we have that for every $\tau' < \tau_0$ there exists $c_0 > 0$ such that $\omega_4(\alpha) \leq c_0|\alpha|^{-\tau'}$, when $|\alpha| \to \infty$, $\alpha \in C_4$. This proves (3.5).

Let $(c_1, c_2) \notin E$ and $g \in G$ be given. We consider

$$L_1f^2 = L_2f^1, \quad c_1f^1 + c_2f^2 = g. \quad (3.23)$$

By expanding $f^j(x) = t(f^j_1, f^j_2, f^j_3, f^j_4)$ into the Taylor series we define $F^j$ by (3.15). We similarly define

$$G = t(G_1, G_2, G_3, G_4), \quad G_k = t\{g(\alpha_1, \alpha_2, -r, -r): k\}_{r=0}^N,$$

where $g(x) = t(g_1(x), g_2(x), g_3(x), g_4(x))$, $g_k(x) = \sum g_{\alpha}x^\alpha$. We set $H^1 := t(0, 0, F^1_1, 0)$ and $H^2 := t(0, 0, F^2_1, 0)$. We substitute the expansions of $f^j$ and $g$ into (3.23). For every $(\alpha_1, \alpha_2) \in \mathbb{Z}_+^4$ and $N \in \mathbb{Z}_+$ such that $\alpha_1 + \alpha_2 + N \geq 2$ we compare the coefficients of $x^\alpha$ of homogeneous degree $\alpha_1 + \alpha_2 + N$. Then (3.23) is equivalent to

$$AF^2 - BF^1 - \varepsilon H^2 + \varepsilon \varepsilon_0 H^1 = 0, \quad c_1F^1 + c_2F^2 = G, \quad (3.24)$$

where $A$ and $B$ are given by (3.17) and (3.18).

First we will construct a formal power series solution $F^j$ ($j = 1, 2$) of (3.24). It follows from (3.24) and the definition of $H^j$ that

$$A_kF^1_k - B_kF^2_k = 0, \quad c_1F^1_k + c_2F^2_k = G_k, \quad k = 1, 2, 4. \quad (3.25)$$

We recall that (cf. the proof of Theorem 3.1) either $A_k$ or $B_k$ is nonsingular for each $k = 1, 2, \ldots, 4$. Assuming that $A_k$ is nonsingular we obtain $F^2_k = A_k^{-1}B_kF^1_k$, and hence $c_1F^1_k + c_2A_k^{-1}B_kF^1_k = G_k$. It follows that

$$F^1_k = (c_1Id + c_2A_k^{-1}B_k)^{-1}G_k = (c_1A_k + c_2B_k)^{-1}A_kG_k,$$

if $c_1A_k + c_2B_k$ is nonsingular. The last condition holds if $(c_1, c_2)$ is not contained in a set of Lebesgue measure zero in $\mathbb{R}^2$, which may depend on $\alpha_1, \alpha_2, N$. We have similar relations if $B_k$ is nonsingular.

In case $k = 3$, we obtain $A_3F^2_3 - B_3F^1_3 = \varepsilon (F^2_4 - \varepsilon_0 F^1_4)$ instead of $A_kF^2_k - B_kF^1_k = 0$. A simple computation yields that

$$F^3_1 = (c_1A_3 + c_2B_3)^{-1}A_3G_3 - \varepsilon c_2(c_1A_3 + c_2B_3)^{-1}(F^2_4 - \varepsilon_0 F^1_4).$$

By taking the union of all exceptional sets of $(c_1, c_2)$ with $\alpha_1, \alpha_2$ and $N$ in the set of nonnegative integers such that $\alpha_1 + \alpha_2 + N \geq 2$, we see that
there exists a unique formal power series solution \( f^j(x) \) \((j = 1, 2)\) of (3.23), provided \((c_1, c_2)\) is not in an exceptional set of Lebesgue measure zero.

We will show the convergence of \( f^j(x) \) \((j = 1, 2)\). It is sufficient to prove the convergence of \( f^1(x) \) since we may take \( c_2 \neq 0 \) in view of the choice of \( c_1 \) in the above argument. By the condition (2) in the definition of \( \mathcal{G} \), we can easily see that \( L_1^{-1} \) exists on \( \mathcal{G} \), namely \( L_1^{-1}L_1 = L_1L_1^{-1} = Id. \) Let \( g \in \mathcal{G} \). Then it follows from (3.23) and the relation \( L_1f^2 = L_2f^1 \) that \( L_1g = c_1L_1f^1 + c_2L_2f^1. \) Hence we have

\[
g = c_1f^1 + c_2L_1^{-1}L_2f^1.\]

Now we have

\[
L_1^{-1}L_2 = L_2L_1^{-1} = (L_2 - \varepsilon_0 L_1)L_1^{-1} + \varepsilon_0 Id, \quad \text{on } \mathcal{G}.
\]

By definition, we have

\[
L_2 - \varepsilon_0 L_1 = (A_2x, \partial_x) - \varepsilon_0(A_1x, \partial_x) + \varepsilon_0 A_1 - A_2.
\]

Hence, \( L_2 - \varepsilon_0 L_1 \) is semi-simple. By the condition (2) and the proof of the latter half of Theorem 3.1, it follows that the absolute value of the coefficient of \( x^\alpha \) of \( L_1^{-1}g \) \((g = \sum_\alpha g_\alpha x^\alpha)\) is bounded by \( N^{(\tau'' + 1)N}C^N|g_\alpha| \) for some \( C > 0 \), where \( \tau'' > \tau_0 \) can be taken arbitrarily close to \( \tau_0 \). On the other hand, the operator \((L_2 - \varepsilon_0 L_1)\) is the one which multiplies the coefficients of \( x^\alpha \) with \((\alpha_2 + \mu N - \varepsilon_0(\alpha_1 + \alpha_2 - 1 + \nu N))\) for the first component. We have similar expressions for other components. By the condition (1)', the absolute value of the term is bounded by \( N^{-N(\tau + 1)C_0^N} \) for some \( \tau > \tau_0 \). Because \( \tau'' > \tau_0 \) can be taken arbitrarily close to \( \tau_0 \), the growth \( N^N(\tau'' + 1)C^N \) which comes from \( L_1^{-1} \) is absorbed by the term \( N^{-N(\tau + 1)C_0^N} \). Therefore, the operator \((L_2 - \varepsilon_0 L_1)\) maps \( \mathcal{G} \) to \( \mathcal{G} \). By taking \( k_0 \) sufficiently large, the norm of \((L_2 - \varepsilon_0 L_1)L_1^{-1}\) on the space \( \mathcal{G} \cap \{g = \sum_\alpha g_\alpha x^\alpha; |\alpha| > k_0\} \) can be made arbitrarily small.

In view of the construction of \( c_1 \) and \( c_2 \) we may assume that \( c_1 + c_2\varepsilon_0 \neq 0 \). Writing

\[
g = c_1f^1 + c_2L_1^{-1}L_2f^1 = (c_1 Id + c_2\varepsilon_0 Id + R)f^1,
\]

where \( R = (\varepsilon_0 L_1 - L_2)L_1^{-1}, \) and by noting that \( R \) preserves homogeneous polynomials, we see that \((c_1 Id + c_2\varepsilon_0 Id + R)^{-1}\) exists as a map from \( \mathcal{G} \) to \( \mathcal{G} \). Therefore we have \( f^1 \in \mathcal{G} \).

**Proof of Proposition 3.6.** — Because \( \text{Spec} (B) \) satisfies (3.5) by Proposition 3.5 we may assume, by taking a subsequence if necessary, that \( \bar{\omega}_j(\alpha) \) satisfies (3.5) for \( \alpha = \alpha_\ell \). Without loss of generality we may assume that \( j = 4 \). We consider \( \bar{\omega}_4(\alpha) \). Let \( g = \ell(g_1, \ldots, g_4); g_k = \sum_\beta g_{\beta,k}x^\beta \) be the convergent power series defined by \( g_{\beta,k} = 0 \) for \( k = 1, 2, 3 \) and all \( \beta \in \mathbb{Z}_+^4(2); \)
We have the formula \( g(\beta_1, \beta_2, \beta_3, \beta_4) = 0 \) if \( \beta_4 \geq 1 \); \( g(\alpha_1, \alpha_2, 0, 0) = 1 \) for \( \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_\ell, \quad (\ell = 1, 2, \ldots) \), \( N = \alpha_3 + \alpha_4 \); \( g(\beta_1, \beta_2, \beta_3, 0) = 0 \), if otherwise. We want to solve \( L_B v = g \). Let \( \lambda_j \) be the eigenvalues of \( B \). By the same argument as in the proof of Theorem 3.1 we have the formula (3.19). Then we have

\[
(3.26) \quad v(\alpha_1, \alpha_2, 0, N); 4 = (-\varepsilon_1)^N (\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \lambda_3 (N - 1))^{-N-1} N!,
\]

for all \( \alpha = \alpha_\ell, \ell = 1, 2, \ldots \) By (3.5) we have: for every \( \tau' < \tau_0 \) we can find a constant \( C > 0 \) such that

\[
|((\lambda_1 \alpha_1^\ell + \lambda_2 \alpha_2^\ell + \lambda_3 (N_\ell - 1))^{-1}| \geq C N_\ell^{\tau'}, \quad \forall \ell \in \mathbb{N},
\]

where \( \alpha^\ell = (\alpha_1^\ell, \alpha_2^\ell, \alpha_3^\ell, \alpha_4^\ell) \), \( N_\ell = \alpha_3^\ell + \alpha_4^\ell \). Therefore, by (3.26)

\[
(3.27) \quad |v(\alpha_1^\ell, \alpha_2^\ell, 0, N_\ell); 4| \geq (C|\varepsilon_1|)^N N_\ell^{(N_\ell+1)} N!^\tau', \quad \ell \in \mathbb{N}, \alpha_1 \in \mathbb{Z}_+(2).
\]

Because \( \varepsilon_1 \neq 0 \) and \( \tau' < \tau_0 \) can be taken arbitrarily close to \( \tau_0 \), (3.27), Stirling’s formula and the inequality \( N! \geq C^N N^N, \forall N \in \mathbb{Z}_+ \) yield the assertion.

\[ \square \]

**Example 3.8.** We give an example of a formal Gevrey linearization. (cf. Theorem 3.1.) We consider

\[
(3.28) \quad L_\Lambda u = R(x + u), \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\tau & -1 \\ 0 & 0 & -\tau \end{pmatrix},
\]

where \( \tau > 0 \) is an irrational number. For \( C \gg 1 \), let \( f \) be an analytic function \( f(x_1, x_2) = \sum \alpha f_\alpha x_1^\alpha x_2^\alpha \), where the summation with respect to \( \alpha \) is taken for \( \alpha \in \mathbb{Z}_+^2 \) such that \( 1 < \alpha_1 - \tau \alpha_2 < C \). We define \( R(x) = t(0, x_3 f(x_1, x_2), 0) \). We shall show that the unique solution of (3.28) is in \( G^2 \). Indeed, we may look for the solution \( u \) of (3.28) in the form \( u = t(0, x_3 w(x), 0) \). We can easily see that \( w \) satisfies

\[
(3.29) \quad \mathcal{L} w = (x_1 \partial x_1 - \tau x_2 \partial x_2 - \tau x_3 \partial x_3 + x_3 \partial x_2) w = f(x_1, x_2 + x_3 w) \equiv g(x).
\]

We substitute the expansion \( w(x) = \sum \alpha w_\alpha x^\alpha \) into (3.29). We can easily see that the summation in the expansion of \( w(x) \) can be taken for \( \alpha \) such that \( \alpha_1 - \tau (\alpha_2 + \alpha_3) > 1 \). Indeed, by the definition the support of \( g(x) \) satisfies \( \alpha_1 - \tau (\alpha_2 + \alpha_3) > 1 \) if the support of \( w(x) \) satisfies the same condition. On the other hand, by simple computations the support of \( \mathcal{L}^{-1} w \) satisfies \( \alpha_1 - \tau (\alpha_2 + \alpha_3) > 1 \) if the support of \( w \) satisfies the condition. From these properties we can show the assertion for the homogeneous part 2 of \( w \) because there appears no term form \( w \) in \( g \). Inductively, we can prove the
assertion. If we expand $g(x) = \sum_\alpha g_\alpha x^\alpha$, then by the same calculations as in (3.19) we obtain

$$w_{(\alpha_1, N-\ell, \ell)} = \sum_{\ell=0}^{\ell} \frac{g_{(\alpha_1, N-\ell+r, \ell-r)}}{(\alpha_1 - \tau N)^{r+1}} \frac{(N-\ell+r)!}{(N-\ell)!}, \quad \ell = 0, 1, \ldots, N. \tag{3.30}$$

If we can show that $g_{(\alpha_1, N-\ell+r, \ell-r)} = O((\ell - r)!) \bmod \text{terms of order } K^{\alpha_1+N}$ ($K > 0$), then we can easily see that $w_{(\alpha_1, N-\ell, \ell)} = O(\ell!)$. This proves that the solution $u$ of (3.29) is in $G^2$.

If $\alpha_1 + N = 2$, then no term from the expansion of $w$ appears in $g_{(\alpha_1, N-\ell+r, \ell-r)}$ in (3.29). Hence, by the analyticity assumption of $f$, we obtain the desired estimate for $w_\alpha$ with $\alpha_1 + N = 2$, $\alpha_2 + \alpha_3 = N$.

We will briefly mention the general case of $d$–actions. We suppose that there exist $j, 1 \leq j \leq m$ and $\ell_0, 1 \leq \ell_0 \leq d$ such that $A_{\ell_0}^{\ell_0}$ in (1.6) admits only one dimensional eigenspace, i.e., the geometric multiplicity of $\lambda_j^\ell$ is one. For a positive integer $r$ we define the $r$– square nilpotent matrix $N_r$ by

$$N_r = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{3.31}$$

By assumption we have

$$A_{\ell_0}^{\ell_0} = \lambda_{\ell_0}^{\ell_0}Id + \varepsilon N_s, \quad \varepsilon \neq 0. \tag{3.32}$$

By the explicit description of the centralizers of matrices (cf. [14]) all other matrices have the following form

$$A_{\ell_0}^{e_0} = \lambda_{\ell_0}^{e_0}Id + \sum_{k=1}^{s_j-1} \varepsilon_k^{e_0}(N_s)^k, \quad \varepsilon_k^{e_0} \in \mathbb{C}, k = 1, \ldots, s_j - 1. \tag{3.33}$$

We have

**Theorem 3.9.** — Assume (3.32). Then there exist $\varepsilon_k^{e_0}$ in (3.33), $\lambda_{\ell_0}^{e_0}$, ($\ell = 1, 2, \ldots, d; j = 1, 2, \ldots, n$) with the density of continuum such that the followings hold;
(i) The simultaneously nonresonant condition \((\ref{1.16})\) holds, and there exists a sequence \(\alpha^\ell \in \mathbb{Z}^n_+ (2), \, \ell \in \mathbb{N}\) and a positive number \(c_0 > 0\) such that \(|\alpha^\ell| \to \infty (\ell \to \infty)\) and \(0 < \omega(\alpha^\ell) \leq c_0, \, \ell \in \mathbb{N}\).

(ii) There exists an \(f := t(f_1, f_2, \ldots, f^d) \in (\mathbb{C}_2^\mathbb{Z}\{x\})^d\) satisfying \((\ref{3.2})\) such that \(v := L_A^{-1} f\) satisfies \(v \notin \bigcup_{1 \leq \sigma < 2} G_\sigma^2(\mathbb{C}^n)\).

4. Sternberg’s theorem for commuting vector fields

The results in section 2 imply that the simultaneous linearization of a Poincaré morphism with a Jordan block is reduced essentially to the Poincaré–Dulac theorem for a single vector field in an analytic category. On the other hand, in view of the results in section 3, the reduction seems impossible if the action is not a Poincaré morphism.

In this section we shall illustrate that the situation is completely different in a smooth category. We consider two commuting vector field in \(\mathbb{R}^4\) which are in a Siegel domain and only one of the two has a linear part with nontrivial Jordan block. Obviously, the action is not a Poincaré morphism. We will show that they are simultaneously linearizable in \(C^k\) for every \(k \geq 1\).

Let \(X(y)\) and \(Y(y)\) be commuting \(C^\infty\) vector fields with the common singular point at the origin \(0 \in \mathbb{R}^4\). Suppose that \(\nabla X(0) = A, \nabla Y(0) = B\), where

\[
(4.1) \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\nu & 0 \\ 0 & 0 & 0 & -\nu \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\mu & \varepsilon \\ 0 & 0 & 0 & -\mu \end{pmatrix},
\]

where \(\varepsilon \neq 0\). We assume that the action is not a Poincaré morphism, namely, (cf. Example 1.6)

\[
(4.2) \quad \nu > \mu > 0, \nu \in \mathbb{R} \setminus \mathbb{Q}.
\]

We also note that the irrationality of \(\nu\) implies that the pair \((X, Y)\) is nonresonant. Then we have

**Theorem 4.1. —** Suppose that \((4.1)\) and \((4.2)\) are verified. Let \(m \geq 1\) be an integer. Then there exists a \(C^m\)– change of the variables \(y = u(x) = x + v(x), \, v(0) = 0, \nabla v(0) = 0\) in some neighborhood of the origin which transforms both \(X\) and \(Y\) to their linear parts.
We need to prepare lemmas in order to prove our theorem. By Sternberg’s theorem we may assume that $X$ is linear, i.e. $Xv(y) = \nabla v(y) Ay$. Let $R(y) = t(R_1(y), R_2(y), R_3(y), R_4(y))$ be the nonlinear part of $Y$

(4.3) $Yf(y) = \nabla f(y)(By + R(y))$.

Suppose that the change of variables $y = u(x) = x + v(x), v(0) = 0, \nabla v(0) = 0$ linearizes both $X$ and $Y$. Then $v(x)$ satisfies the system of homology equations

(4.4) $\nabla v(x), Bx - Bv = R(x + v(x))$,

and

(4.5) $\nabla v(x)Ax - Av = 0$.

We write $x = (x_1, x_2, x''')$ and $z = (z_1, z')$. Let $c_1 > 0$ and $0 < c_2 \leq 1$ be constants. Then we define

(4.6) $\Omega = \{x' = (x_2, x_3, x_4) = (x_2, x''') \in \mathbb{R}^3; |x_2| < c_1, |x'''| < c_2\}$,

(4.7) $\Omega_1 = \{x_1 \in \mathbb{R}; |x_1| < 1\} \times \Omega$.

Then we have

**Lemma 4.2.** — Let $k = \infty$ or $k \geq 1$ be an integer. Let $L$ be given by

$$L = \sum_{j=1}^{2} x_j \partial_{x_j} - \nu \sum_{k=3}^{4} x_k \partial_{x_k}.$$  

Then the $C^k$ solution of

(4.8) $Lf(x) - f(x) = 0, \ x = (x_1, x_2, x_3, x_4) \in \Omega_1$,  

(respectively,

(4.9) $Lw(x) + \nu w(x) = 0 \ x = (x_1, x_2, x_3, x_4) \in \Omega_1$)

is given by

(4.10) $f(x) = x_1 \varphi_{\pm}^{\nu}(\frac{x_2}{x_1}, x_3|x_1|^{\nu}, x_4|x_1|^{\nu}), \ \text{for} \ \pm x_1 > 0$,  

or

(4.11) $f(x) = x_2 \varphi_{\pm}^{\nu}(\frac{x_2}{x_1}, x_3|x_1|^{\nu}, x_4|x_1|^{\nu}), \ \text{for} \ \pm x_1 > 0$,  

(respectively, by

(4.12) $w(x) = |x_1|^{-\nu} \psi_{\pm}^{\nu}(\frac{x_2}{x_1}, x_3|x_1|^{\nu}, x_4|x_1|^{\nu}), \ \text{for} \ \pm x_1 > 0$),

where $\varphi_{\pm}(z) \in C^k(\Omega)$ (respectively $\psi_{\pm}(z) \in C^k(\Omega)$).
Proof. — Let $L$ be the operator given in the lemma. We want to solve (4.8) and (4.9). First we solve (4.8) in the region $x_1 > 0$. If we set $f(x) = x_1 \varphi(x)$ (resp. $f(x) = x_2 \psi(x)$), then we have that

$$L \varphi(x) = 0, \quad (\text{resp. } L \psi(x) = 0).$$

By the theorem in page 61 of [2], the solutions of (4.13) are given by the first integral of the corresponding characteristic equation. For the sake of simplicity, we consider the equation $L \varphi(x) = 0$. The characteristic equation is given by

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2} = - \frac{dx_3}{\nu x_3} = - \frac{dx_4}{\nu x_4}.$$  

If we integrate (4.14) by taking $x_1$ as an independent variable, then we obtain

$$x_2 = x_1 x_2^0, \quad x_3 = x_1^{-\nu} x_3^0, \quad x_4 = x_1^{-\nu} x_4^0,$$

where $x_2^0, x_3^0, x_4^0$ are certain constants. It follows that the first integral $\varphi_+(x)$ is given by

$$\varphi_+(x) = \tilde{\varphi}_+ \left( \frac{x_2}{x_1}, x_3 x_1^\nu, x_4 x_1^\nu \right) = \tilde{\varphi}_+(x_2^0, x_3^0, x_4^0),$$

for some differentiable function $\tilde{\varphi}_+$. Hence, the general solution of (4.8) in $x_1 > 0$ is given by $f(x) = x_1 \varphi_+(x)$ (resp. $f(x) = x_2 \varphi_+(x)$ for possibly different $\varphi_+$).

In case $x_1 < 0$ we make the same argument by replacing $x_1$ with $-x_1$. We see that there exists $\varphi_-(x)$ such that $f(x) = x_1 \varphi_-(x)$ (resp. $f(x) = x_2 \varphi_-(x)$ for possibly different $\varphi_-$).

Next we consider the equation (4.9). We set $w(x) = |x_1|^{-\nu} \psi(x)$. For the sake of simplicity we consider the case $x_1 > 0$. The case $x_1 < 0$ can be treated similarly if we replace $x_1$ with $-x_1$. We can easily see that $\psi$ satisfies $L \psi = 0$. Hence it follows from the above argument that

$$w(x) = x_1^{-\nu} \psi_+(x) = x_1^{-\nu} \tilde{\psi}_+ \left( \frac{x_2}{x_1}, x_3 x_1^\nu, x_4 x_1^\nu \right).$$

By the commutativity we see that every component of $v = R(x) = (R_1, \ldots, R_4)$ satisfies either (4.8) or (4.9). Hence, by Lemma 4.1 we have, for $\pm x_1 > 0$,

$$R_j(x) = x_j \Psi_j^\pm \left( \frac{x_2}{x_1}, x_3 |x_1|^{\nu}, x_4 |x_1|^{\nu} \right), \quad j = 1, 2,$$

$$R_j(x) = |x_1|^{-\nu} \Psi_j^\pm \left( \frac{x_2}{x_1}, x_3 |x_1|^{\nu}, x_4 |x_1|^{\nu} \right), \quad j = 3, 4.$$
for some functions $\Psi^j_{\pm}$. In the following we will cut off $R_j(x)$ with a smooth function being identically equal to 1 in some neighborhood of the origin and with support contained in a small neighborhood of the origin, which we give in the proof of Theorem 4.1. For the sake of simplicity, we denote the modified $R_j(x)$ with the same letter. We set

$$(4.20) \quad z_1 = x_2/x_1, \; z_2 = x_3|x_1|^{\nu}, \; z_3 = x_4|x_1|^{\nu}.$$  

For every $x_1 \neq 0$, we define $\Psi^j_{\pm}(z)$ by (4.18) and (4.19), namely, for $\pm x_1 > 0$,

$$(4.21) \quad \Psi^j_{\pm}(z) = x_j^{-1}R_j(x_1, x_1z_1, |x_1|^{-\nu}z_2, |x_1|^{-\nu}z_3), \quad j = 1, 2,$$

$$(4.22) \quad \Psi^j_{\pm}(z) = |x_1|^{\nu}R_j(x_1, x_1z_1, |x_1|^{-\nu}z_2, |x_1|^{-\nu}z_3), \quad j = 3, 4.$$  

We can easily see that $\Psi^j_{\pm} \in C^\infty(\mathbb{R}^3)$ ($j = 1, 2, 3, 4$).

By (4.5) and simple computations we see that every component of $v(x) = (v_1(x), \ldots, v_4(x))$ satisfies either (4.8) or (4.9). It follows from Lemma 4.1 that every component of $v$ has an expression

$$(4.23) \quad v_j(x) = x_j\varphi^j_{\pm}(x_1/x_2, x_3|x_1|^{\nu}, x_4|x_1|^{\nu}), \quad j = 1, 2,$$

and

$$(4.24) \quad v_j(x) = |x_1|^{-\nu}\varphi^j_{\pm}(x_1/x_2, x_3|x_1|^{\nu}, x_4|x_1|^{\nu}), \quad j = 3, 4,$$

for some $\varphi^j_{\pm}$ with $\pm x_1 > 0$.

We substitute the transformation (4.20) and (4.23), (4.24) into (4.4), and we rewrite (4.4) as an equation of $z$ for the unknown functions $\varphi^j_{\pm}(z)$ with a parameter $x_1$. Recalling that $v_j = x_j\varphi^j_{\pm}$ and $v_j = |x_1|^{-\nu}\varphi^j_{\pm}$ we obtain

$$(4.25) \quad x_2\partial_{x_2}v_1 = x_1z_1\partial_{z_1}\varphi^1_{\pm}(z), \quad x_3\partial_{x_3}v_1 = x_1z_2\partial_{z_2}\varphi^1_{\pm}(z),$$

$$(4.26) \quad x_4\partial_{x_4}v_1 = x_1z_3\partial_{z_3}\varphi^1_{\pm}(z), \quad x_4\partial_{x_4}v_1 = x_1z_3\partial_{z_3}\varphi^1_{\pm}(z),$$

and we have similar relations for $v_2 = x_2\varphi^2_{\pm}(x)$ and $v_j = |x_1|^{-\nu}\varphi^j_{\pm}(x)$. In fact we have

$$(4.27) \quad \langle \nabla v_1(x), Bx \rangle = x_1\mathcal{L}\varphi^1_{\pm}(z), \quad \text{for} \; \pm x_1 > 0,$$

$$(4.28) \quad \langle \nabla v_2(x), Bx \rangle - v_2(x) = x_2\mathcal{L}\varphi^2_{\pm}(z), \quad \text{for} \; \pm x_1 > 0,$$

$$(4.29) \quad \langle \nabla v_j(x), Bx \rangle = |x_1|^{-\nu}\mathcal{L}\varphi^j_{\pm}(z), \quad \text{for} \; \pm x_1 > 0, \; j = 3, 4,$$

where

$$(4.30) \quad \mathcal{L}f(z) = z_1\partial_{z_1}f(z) - (\mu z_2 - \varepsilon z_3)\partial_{z_2}f(z) - \mu z_3\partial_{z_3}f(z).$$

We define $\varphi_{\pm}(z) = \varphi^1_{\pm}(z), \varphi^2_{\pm}(z), \varphi^3_{\pm}(z), \varphi^4_{\pm}(z))$. 

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**Lemma 4.3.** — We have the expression

\[(4.31) \quad R_j(x + v(x)) = x_j E^j_\pm (z, \phi_\pm(z)), \quad \text{for } \pm x_1 > 0, \ j = 1, 2,\]

where \(E^j_\pm(z,w)\) is given by

\[(4.32) \quad E^j_\pm(z,w) = (1 + w^j) \Psi^j_\pm \left( z \frac{1 + w_2}{1 + w_1}, (z_2 + w_3)|1 + w_1|^{\nu}, (z_3 + w_4)|1 + w_1|^{\nu} \right) \]

and

\[(4.33) \quad R_j(x + v(x)) = |x_1|^{-\nu} E^j_\pm (z, \phi_\pm(z)) \quad \text{for } \pm x_1 > 0, \ j = 3, 4, \]

with

\[(4.34) \quad E^j_\pm(z,w) = |1 + w_1|^{-\nu} \times \Psi^j_\pm \left( z \frac{1 + w_2}{1 + w_1}, (z_2 + w_3)|1 + w_1|^{\nu}, (z_3 + w_4)|1 + w_1|^{\nu} \right). \]

**Proof.** — We have

\[(4.35) \quad \frac{x_2 + v_2(x)}{x_1 + v_1(x)} = \frac{x_2(1 + \phi^2_\pm(z))}{x_1(1 + \phi^1_\pm(z))} \]

\[= \frac{x_2 1 + \phi^2_\pm(z)}{x_1 1 + \phi^1_\pm(z)} = z_1 \frac{1 + \phi^2_\pm(z)}{1 + \phi^1_\pm(z)}. \]

\[(4.36) \quad (x_3 + v_3(x))|x_1 + v_1|^{\nu} \]

\[= (x_3 + |x_1|^{-\nu} \phi^3_\pm(z)|x_1|^{\nu}|1 + \phi^1_\pm(z)|^{\nu} \]

\[= (x_3|x_1|^{\nu} + \phi^3_\pm(z)|1 + \phi^1_\pm(z)|^{\nu} \]

\[= (z_2 + \phi^3_\pm(z)|1 + \phi^1_\pm(z)|^{\nu}. \]

\[(4.37) \quad (x_4 + v_4(x))|x_1 + v_1|^{\nu} \]

\[= (x_4 + |x_1|^{-\nu} \phi^4_\pm(z)|x_1|^{\nu}|1 + \phi^1_\pm(z)|^{\nu} \]

\[= (x_4|x_1|^{\nu} + \phi^4_\pm(z)|1 + \phi^1_\pm(z)|^{\nu} \]

\[= (z_3 + \phi^4_\pm(z)|1 + \phi^1_\pm(z)|^{\nu}. \]

Hence, if \(j = 1, 2\), we get

\[(4.38) \quad R_j(x + v(x)) = (x_j + v_j(x)) \]

\[\times \Psi^j_\pm \left( \frac{x_2 + v_2(x)}{x_1 + v_1(x)}, (x_3 + v_3(x))|x_1 + v_1|^{\nu}, (x_4 + v_4(x))|x_1 + v_1(x)|^{\nu} \right) \]

\[= x_j(1 + \phi^j_\pm) \]

\[\times \Psi^j_\pm \left( z_1 \frac{1 + \phi^2_\pm}{1 + \phi^1_\pm}, (z_2 + \phi^3_\pm)|1 + \phi^1_\pm|^{\nu}, (z_3 + \phi^4_\pm)|1 + \phi^1_\pm|^{\nu} \right), \]
which yields (4.31). Similarly, we can readily prove (4.33). □

Now we are ready to write explicitly the reduction of the overdetermined system for $v$: $(X_A - A)v = 0$, $(X_B - B)v = R(x + v(x))$ into a $4 \times 4$ system of equations for $\varphi_{\pm}(z)$ in $z \in \Omega$ with a parameter $x_1$. Then the new system of semilinear homological equations for $\varphi_{\pm}$ is written as follows

\[(4.39) \quad (\mathcal{L} - \tilde{B})(\varphi_{\pm}) = E_{\pm}(z, \varphi_{\pm}(z)),\]

\[E_{\pm}(z, w) = (E_{1\pm}(z, w), \ldots, E_{4\pm}(z, w)),\]

where $E_{j\pm}(z, w)$ are given by (4.32) and (4.34) and

\[(4.40) \quad \tilde{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & \varepsilon \\ 0 & 0 & 0 & -\mu \end{pmatrix}.

We prepare a lemma.

**Lemma 4.4.** — Let $\nu > 0$ be an irrational number. Let $f(x)$ and $w(x)$ be smooth solutions of (4.8) and (4.9) in $\Omega_1$, respectively satisfying that

\[(4.41) \quad f(0) = w(0) = 0,\]

\[(4.42) \quad \nabla f(0) = \nabla w(0) = 0.\]

We cut off $f(x)$ and $w(x)$ with a smooth function being identically equal to 1 in some neighborhood of the origin and with support contained in a small neighborhood of the origin. For the sake of simplicity we denote the modified functions with the same letter. Let $\varphi_{\pm}(z)$ and $\psi_{\pm}(z)$ be defined by (4.10), (4.11) and (4.12), respectively by the same way as (4.21) and (4.22). Then, for every $\alpha \in \mathbb{Z}_+^3$, we have

\[(4.43) \quad \partial_2^\alpha \Theta(z_1, 0) = 0, \quad \forall z = (z_1, 0) \in \Omega,

with $\Theta = \varphi_{\pm}$ and $\Theta = \psi_{\pm}$.

**Proof.** — Because $\nu$ is an irrational number we can easily see, from (4.8) and (4.9) that every $f(x)$ and $w(x)$ satisfying (4.41) and (4.42) are flat at the origin, namely all derivatives $\partial_x^\alpha f(x)$, $\partial_x^\alpha w(x)$ ($\alpha \in \mathbb{N}^4$) vanish at the origin $x = 0$. Let $\Theta(z) = \varphi_{\pm}(z)$, and set $f(x) = x_1 \varphi_{\pm}(x_2/x_1, x_3|x_1|^\nu, x_4|x_1|^\nu)$, $x_1 \neq 0$. Then we have

\[(4.44) \quad \partial_x^\alpha f(x) = x_1^{-\alpha_2} |x_1|^\nu(\alpha_3 + \alpha_4) \partial_{z_1}^{\alpha_2} \partial_{z_2}^{\alpha_3} \partial_{z_3}^{\alpha_4} \varphi_{\pm}(z) \bigg|_{z_1 = x_2/x_1, z_2 = x_3|x_1|^\nu, z_3 = x_4|x_1|^\nu}.

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We let \( x \) tend to zero so as to satisfy \( x_2/x_1 = z_1, z_2 = x_3|x_1|^{\nu} = 0 \) and \( z_3 = x_4|x_1|^{\nu} = 0 \). Then we have

\[
\begin{align*}
\frac{\partial^{\alpha_2} \partial^{\alpha_3} \partial^{\alpha_4} \varphi_\pm (z_1, 0, 0)}{\partial \varphi (x_1, 2, 0, 0)} & = \lim_{x \to 0} x^\alpha |x|^{-\nu(\alpha_3 + \alpha_4)} \partial_x^\alpha \left( x_1^{-1} f(x_1, x_2, 0, 0) \right) = 0,
\end{align*}
\]

because \( f(x) \) is flat at the origin. The other cases will be proved similarly.

\[\square\]

**Remark.** — Let \( \varphi_\pm (z) \in C^k(\Omega) \) be given. Assume that (4.43) is satisfied for \( \Theta = \varphi_\pm \) up to some finite \(|\alpha|\). Then the function \( f(x) \) defined by (4.10) gives a finitely smooth solution of (4.8) if \( \nu \) is an irrational number. Indeed, the finite smoothness at \( x_1 = 0 \) follows from the argument of Lemma 4.4.

In order to solve (4.39) we introduce a function space. Let \( N \geq 1 \) and \( k \leq N \) be integers. Let \( 0 < c'_2 < c_2 \leq 1 \) be a constant. Then we define

\[
\|V\|_{k;N} = \sup_{z \in \mathbb{R}^3, 0 < |z'| \leq c'_2} \sum_{|\alpha| \leq k} |z'|^{|\alpha|} |\partial_z^\alpha (|z'|^{-N}V(z))|,
\]

\[|V(z)| = \left( \sum_{j=1}^{4} |V_j(z)|^2 \right)^{1/2}, \quad V(z) = (V_1(z), V_2(z), V_3(z), V_4(z)).\]

The set of all \( C^k \) functions \( V(z) \) such that \( \|V\|_{k;N} < \infty \) is a Banach space \( B_{k;N} \) with the norm \( \|\cdot\|_{k;N} \). Then we have

**Lemma 4.5.** —

i) For any integers \( k \geq 0 \) and \( 0 \leq \ell \leq N \), there exists a constant \( C_{k,N} > 0 \) such that

\[
\|u\|_{k;\ell} \leq C_{k,N} \|u\|_{k;N}, \quad \forall u \in B_{k;N}.
\]

ii) For every \( f, g \in B_{k;N} \) we have \( fg \in B_{k;N} \) and there exists a constant \( C_{k,N} > 0 \) such that

\[
\|fg\|_{k;N} \leq C_{k,N} \|f\|_{k;N} \|g\|_{k;N}, \quad \forall f, g \in B_{k;N}.
\]

**Proof.** — Because \( |z'| \leq 1 \), we have, for \( |\alpha| \leq k \)

\[
|z'|^{|\alpha|} |\partial^\alpha (|z'|^{-\ell} u(z))| = |z'|^{|\alpha|} |\partial^\alpha (|z'|^{N-\ell} |z'|^{-N} u(z))|
\]

\[
= \sum_{\beta + \gamma = \alpha} |\alpha| \partial^\beta |z'|^{N-\ell} \partial^\gamma (|z'|^{-N} u(z))
\]

\[
\leq C_1 \sup |z'|^{|\gamma|} |\partial^\gamma (|z'|^{-N} u(z))|
\]

for some \( C_1 > 0 \). This proves i).
In order to prove ii) we have, for $|\alpha| \leq k$

\begin{equation}
|z'|^{\alpha}|\partial^\alpha(|z'|^{-N}fg)|
\leq \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} |z'|^{\beta}||\partial^\beta(|z'|^{-N}f)||z'|^{\gamma}||\partial^\gamma g|
\leq C_2\|f\|_{k:N}\|g\|_{k,0} \leq C_3\|f\|_{k:N}\|g\|_{k,N}.
\end{equation}

Here $C_2 > 0$ and $C_3 > 0$ are constants. This proves ii) .

We define the operator $Q$ by

\begin{equation}
QV = -\int_0^\infty e^{-t\tilde{B}}V(e^{tC}z)dt,
\end{equation}

$V = (V_1, \ldots, V_4) = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$,

where

\begin{equation}
C = \begin{pmatrix}
1 & 0 & 0 \\
0 & -\mu & \varepsilon \\
0 & 0 & -\mu
\end{pmatrix}.
\end{equation}

We can easily see that $U = QV$ gives the solution of $(L - \tilde{B})U = V$. Then we have

**Lemma 4.6.** — Let the integers $k$ and $N$ satisfy that $0 \leq 2k < N - \mu$ and $\mu(k + 1 - N) + k < 0$. Then there exists $C_{k,N}(\Omega) > 0$ such that

\begin{equation}
\|QV\|_{k:N} \leq C_{k,N}(\Omega) \|V\|_{k:N}, \quad \forall V \in B_{k:N}.
\end{equation}

**Proof.** — First we note that

\begin{align*}
e^{tC}z &= (e^t z_1, e^{-\mu t} z_2 + e^{-\mu t} \varepsilon tz_3, e^{-\mu t} z_3), \\
e^{-t\tilde{B}}V &= (V_1, V_2, e^{\mu t}(V_3 - \varepsilon tV_4), e^{\mu t}V_4).
\end{align*}

Hence we have

\begin{equation}
V(e^{tC}z) = V(e^t z_1, e^{-\mu t}(z_2 + \varepsilon tz_3), e^{-\mu t} z_3) = e^{-\mu N t}(z_2 + \varepsilon tz_3)^2 + z_3^2)^{N/2}\tilde{V}(e^{tC}z),
\end{equation}

where $\tilde{V}(\zeta) = V(\zeta)/|\zeta'|^N$. It follows that the right-hand side integral of (4.50) converges, because the growing term $e^{\mu t}$ in $e^{-tB}$ can be absorbed by
\( e^{-\mu N t} \), \((\mu > 0)\). First we estimate \( \|QV\|_{0,N} \). By (4.54) and (4.55) we have

\[
(4.56) \quad \|QV\|_{0,N} = \sup_{z \in \mathbb{R}^3, 0 < |z'| \leq c'_2} \left( \frac{1}{|z'|^N} \int_0^\infty \left| e^{-tB} V(e^{tC} z) \right| dt \right) \leq \sup_{z \in \mathbb{R}^3, 0 < |z'| \leq c'_2} \left( \frac{1}{|z'|^N} \int_0^\infty (1 + |\varepsilon|t)e^{\mu t} \left| V(e^{tC} z) \right| dt \right) \leq \sup \left( \frac{1}{|z'|^N} \int_0^\infty (1 + |\varepsilon|t)e^{\mu(1-N)t} \times ((z_2 + |\varepsilon|t z_3)^2 + z_3^2)^{N/2} \left| \tilde{V}(e^{tC} z) \right| dt \right).
\]

On the other hand we note that

\[
(4.57) \quad |z'|^{-N}((z_2 + |\varepsilon|t z_3)^2 + z_3^2)^{N/2} \leq |z'|^{-N}(|z'| + |\varepsilon|t z_3))^N \leq (1 + |\varepsilon|t)^N.
\]

In order to estimate \( \tilde{V}(e^{tC} z) \) we note the following inequality

\[
(4.58) \quad e^{-\mu t}(|z_2 + \varepsilon t z_3|^2 + z_3^2)^{1/2} \leq |z'|(|1 + |\varepsilon|t)e^{-\mu t} \leq |z'| \leq c'_2,
\]

because we have \(|\varepsilon| < \mu\). It follows that

\[
(4.59) \quad |\tilde{V}(e^{tC} z)| \leq \sup_{z \in \mathbb{R}^3, 0 < |z'| \leq c'_2} |\tilde{V}(z)|.
\]

Hence the right-hand side of (4.56) is estimated in the following way

\[
(4.60) \quad \leq \sup_{z \in \mathbb{R}^3, 0 < |z'| \leq c'_2} |\tilde{V}(z)| \int_0^\infty (1 + |\varepsilon|t)^{N+1}e^{\mu(1-N)t} dt \leq C \|V\|_{0,N}
\]

for some \( C > 0 \) independent of \( V \). It follows that \( \|QV\|_{0,N} \leq C \|V\|_{0,N} \) for some \( C > 0 \).

Next we will estimate the derivative \(|z'|^|\alpha|\partial^\alpha (|z'|^{-N}QV)\). By Leibnitz rule it is sufficient to estimate the term \(|z'|^|\alpha|\partial^\alpha |z'|^{-N} \partial^{\alpha-\gamma}(QV)\), where \( \alpha \geq \gamma \). By simple computations, we have \(|z'|^|\alpha|\partial^\alpha |z'|^{-N} \leq C_1 |z'|^{-N+|\alpha|-|\gamma|}\) for some \( C_1 > 0 \) independent of \( z' \). On the other hand, we have

\[
\partial^{\alpha-\gamma}(QV) = -\partial^{\alpha-\gamma} \int_0^\infty e^{-tB}((z_2 + \varepsilon t z_3)^2 + z_3^2)^{N/2} e^{-\mu N t} \tilde{V}(e^{tC} z) dt
\]

\[
= - \sum_{\beta \leq \alpha-\gamma} \binom{\alpha-\gamma}{\beta} \int e^{-tB-\mu N t} \partial^\beta((z_2 + \varepsilon t z_3)^2 + z_3^2)^{N/2} \partial^{\alpha-\gamma-\beta} \tilde{V}(e^{tC} z) dt.
\]
We can easily see
\begin{equation}
(4.61) \quad \left| \partial_z^\beta ((z_2 + \epsilon tz_3)^2 + z_3^3)^{N/2} \right| \leq C_2 (1 + |\epsilon| t)^N |z'|^{N-|\beta|}
\end{equation}
for some $C_2 > 0$. If we set $\alpha - \beta - \gamma = \delta$, $\delta = (\delta_1, \delta_2, \delta_3)$, then we have
\begin{equation}
(4.62) \quad \partial^\alpha - \gamma \widetilde{\psi}(e^{tC} z) = e^{t\delta_1 - \mu(\delta_2 + \delta_3) t} (\partial_1 \delta_1 \partial_2 \delta_2 (\epsilon t \partial_2 + \partial_3 \delta_3) \widetilde{\psi})(e^{tC} z).
\end{equation}
It follows that
\begin{equation}
(4.63) \quad |z'|^{\alpha_1} |\partial_1^\gamma |z'|^{-N} |\partial^\alpha - \gamma (QV)|
\end{equation}
\begin{equation}
\leq C_3 |z'|^{-N+|\alpha|-|\gamma|} \int_0^\infty e^{\mu t - \mu N t} \times |\partial_1^\beta ((z_2 + \epsilon tz_3)^2 + z_3^3)^{N/2}||\partial^\alpha - \gamma \widetilde{\psi}(e^{tC} z)| dt
\end{equation}
\begin{equation}
\leq C_3 |z'|^{-N+|\alpha|-|\gamma|} \int_0^\infty e^{\mu t - \mu N t} (1 + |\epsilon| t)^N+1 \times \sum_\beta |z'|^{-|\beta|} |\partial^\alpha - \gamma \widetilde{\psi}(e^{tC} z)| dt
\end{equation}
\begin{equation}
\leq C_4 \int_0^\infty \sum_{|\xi|=|\alpha-\gamma|} |z'|^{\xi} \left| (\partial_2^\xi \widetilde{\psi})(e^{tC} z) \right|
\times (1 + |\epsilon| t)^N+1 + |\xi| e^{\mu t - \mu N t + |\alpha| t} dt.
\end{equation}
In order to estimate $|z'|^{\xi} \left| (\partial_2^\xi \widetilde{\psi})(e^{tC} z) \right|$, we set $\zeta = e^{tC} z$. Then we have
\begin{equation}
(4.64) \quad |z'|^{\xi} \left| (\partial_2^\xi \widetilde{\psi})(e^{tC} z) \right| = |(e^{-tC} \zeta')^{\xi} \left| \left| (\partial_2^\xi \widetilde{\psi})(\zeta) \right| \right|
\leq e^{\mu |\xi| t} \left| \left| (\partial_2^\xi \widetilde{\psi})(\zeta) \right| \right| \left| ((\zeta_2 + \epsilon t \zeta_3)^2 + \zeta_3^3)^{\xi/2} \right|
\leq e^{\mu k t} (1 + |\epsilon| t)^k |\zeta'|^{\xi} \left| \left| (\partial_2^\xi \widetilde{\psi})(\zeta) \right| \right|
\leq \|V\|_{k;N} e^{\mu k t} (1 + |\epsilon| t)^k.
\end{equation}
By assumption we have $(1 + k - N)\mu + |\alpha| \leq (1 + k - N)\mu + k < 0$. Hence the right-hand side integral in (4.63) converges. Therefore we see that the right-hand side of (4.63) can be estimated by $C_5 \|V\|_{k;N}$. \hfill \Box

**Proof of Theorem 4.1.** — Let the integers $k$ and $N$ satisfy that $0 \leq 2k < N - \mu$ and $n \geq 2$, $\mu(k + 1 - N) + k < 0$. By setting $\varphi_\pm = QV$, (4.39) is equivalent to
\begin{equation}
(4.65) \quad V = E_\pm (z, QV).
\end{equation}
We define the sequence $V_{\pm}^j$ ($j = 0, 1, 2, \ldots$) by
\begin{equation}
(4.66) \quad V_{\pm}^0 = E_\pm (z, 0), \quad V_{\pm}^1 = E_\pm (z, QV_{\pm}^0) - E_\pm (z, 0),
\end{equation}
and
\begin{equation}
V_{\pm}^{j+1} = E_{\pm}(z, Q(V_{\pm}^0 + \cdots + V_{\pm}^j)) - E_{\pm}(z, Q(V_{\pm}^0 + \cdots + V_{\pm}^{j-1})), \quad j = 1, 2, \ldots
\end{equation}

We will show the convergence of \(\sum_{j=0}^{\infty} V_{\pm}^j =: V_{\pm}\). By the definition and Lemma 4.3 we have \(V_{\pm}^0 = E_{\pm}(z, 0) = \Psi_{\pm}(z)\). Next we have
\begin{equation}
V_{\pm}^1 = E_{\pm}(z, QV_{\pm}^0) - E_{\pm}(z, 0) = QV_{\pm}^0 \int_0^1 \nabla_w E_{\pm}(z, \tau QV_{\pm}^0) d\tau.
\end{equation}

Let \(\varepsilon' > 0\) be a small constant chosen later, and suppose that
\begin{equation}
\|\Psi_{\pm}\|_{k; N} < \varepsilon', \quad \|\nabla \Psi_{\pm}\|_{k; N} < \varepsilon'.
\end{equation}

Then, by Lemma 4.6 and the definition of \(V_{\pm}^0\) we have
\begin{equation}
\|\tau QV_{\pm}^0\|_{k; N} \leq c_1 \|V_{\pm}^0\|_{k; N} = c_1 \|\Psi_{\pm}\|_{k; N} < c_1 \varepsilon'
\end{equation}

for some \(c_1 > 0\) independent of \(\Psi_{\pm}\). Here we recall from (4.66) that \(V_{\pm}^0 = E_{\pm}(z, 0)\) and \(E_{\pm}(z, 0) = \Psi_{\pm}(z)\) by (4.32) and (4.34).

In order to estimate \(\|\nabla_w E_{\pm}(\cdot, \tau QV_{\pm}^0)\|_{k; N}\), we set \(w = (w_1, \ldots, w_4) = \tau QV_{\pm}^0\) and
\[\zeta = (\zeta_1, \zeta') = \left( z_1 \frac{1+w_2}{1+w_1}, (z_2+w_3)|1+w_1|^\nu, (z_3+w_4)|1+w_1|^\nu \right).\]

The differentiation \(\partial_z^\alpha (\nabla_w E_{\pm}(z, \tau QV_{\pm}^0))\) consists of terms which are product of \(\partial^\beta \nabla \Psi_{\pm}(\zeta)\) (\(\alpha \geq \beta\)) and the differentiations of \(w\). First, the product of differentiations of \(w\) is bounded by a constant in view of (4.70). On the other hand, in order to estimate \(\|\partial_\zeta^\beta \nabla \Psi_{\pm}(\zeta)\|\), we note
\[|\partial_\zeta^\beta \nabla \Psi_{\pm}(\zeta)| = |\partial_\zeta^\beta (|\zeta'|^N |\zeta'|^{-N} \nabla \Psi_{\pm}(\zeta))| \leq C_0 \sum_{\gamma \leq \beta} |\partial_\zeta^\beta |\zeta'|^N|\partial_\zeta^{\beta-\gamma} (|\zeta'|^{-N} \nabla \Psi_{\pm}(\zeta))|\]

for some constant \(C_0 > 0\). Because \(N \geq 2k \geq 2|\beta| \geq 2|\gamma|\) and \(|\zeta'| \leq 1\), we have \(\|\partial_\zeta^\beta |\zeta'|^N\| \leq C_1 |\zeta'|^{N-|\gamma|} \leq C_1 |\zeta'|^{|\beta|-|\gamma|}\) for some \(C_1 > 0\). It follows from the definition of the norm that \(\|\partial_\zeta^\beta \nabla \Psi_{\pm}(\zeta)\| \leq C_2 \|\nabla \Psi_{\pm}\|_{k; N}\) for some \(C_2 > 0\). Hence, if \(\varepsilon' > 0\) is sufficiently small, then we obtain, by the definition of \(E_{\pm}(z, w)\) in (4.39), (4.32) and (4.34),
\begin{equation}
\|\nabla_w E_{\pm}(\cdot, \tau QV_{\pm}^0)\|_{k; N} \leq c_2 \|\nabla \Psi_{\pm}\|_{k; N} < c_2 \varepsilon',
\end{equation}

for some \(c_2 > 0\) independent of \(\varepsilon'\) and \(\Psi_{\pm}\).
It follows from (4.68) that
\[ \|V_{\pm}^1\|_{k;N} \leq \|QV_{\pm}^0\|_{k;N} \int_0^1 \|\nabla_w E_{\pm}(z, \tau QV_{\pm}^0)\|_{k;N} \, d\tau \leq c_1 c_2 \varepsilon'^2. \]

In order to show the general case, we assume that \( \|V_{\pm}^j\|_{k;N} \leq c_1^j c_2^j \varepsilon'^j+1 \) for \( j = 0, 1, 2, \ldots, k \). Then we have
\[
(4.72) \quad \left\| \sum_{j=0}^k V_{\pm}^j \right\|_{k;N} \leq \sum_{j=0}^k c_1^j c_2^j \varepsilon'^j+1 \leq \frac{\varepsilon'}{1 - c_1 c_2 \varepsilon'}. \]

By the definition we have
\[
(4.73) \quad V_{\pm}^{k+1} = E_{\pm}(z, Q(V_{\pm}^0 + \cdots + V_{\pm}^k)) - E_{\pm}(z, Q(V_{\pm}^0 + \cdots + V_{\pm}^{k-1})) = QV_{\pm}^k \int_0^1 \nabla_w E_{\pm}(z, Q(V_{\pm}^0 + \cdots + V_{\pm}^{k-1}) + \tau QV_{\pm}^k) \, d\tau. \]

By the apriori estimate (4.72) and the boundedness of \( Q \), the substitution in the right-hand side of (4.73) is well–defined. Moreover, by the same argument as in the proof of (4.71) we see that
\[ \|\nabla_w E_{\pm}(z, Q(V_{\pm}^0 + \cdots + V_{\pm}^{k-1}) + \tau QV_{\pm}^k)\|_{k;N} \leq c_2 \varepsilon'. \]

It follows from (4.73) that
\[ \|V_{\pm}^{k+1}\|_{k;N} \leq \|QV_{\pm}^k\|_{k;N} c_2 \varepsilon' \int_0^1 \, d\tau \leq c_1^{k+1} c_2^{k+1} \varepsilon'^{k+2}. \]

Hence we have the estimate of \( V_{\pm}^j \) for \( j = k+1 \). It follows that the series \( V_{\pm} := \sum_{j=0}^{\infty} V_{\pm}^j \) converges in \( B_{k;N} \) and \( V_{\pm} \) is a solution of (4.65). We note that, by (4.72) \( V_{\pm} \) satisfies the estimate \( \|V_{\pm}\|_{k;N} \leq \varepsilon'(1 - c_1 c_2 \varepsilon')^{-1} \), and \( V_{\pm} \) is divisible by \( |z'|^2 \).

Next we verify the smallness assumption (4.69) uniformly with respect to \( x_1 \neq 0 \) in some neighborhood of \( x_1 = 0 \). Because the argument is similar we consider the condition \( \|\Psi_{\pm}\|_{k;N} < \varepsilon' \). In view of the definition of \( \Psi_{\pm} \) in (4.21) and (4.22), we shall estimate
\[ x_j^{-1} R_j(x_1, x_1 z_1, |x_1|^{-\nu} z_2, |x_1|^{-\nu} z_3), \quad j = 1, 2 \]
and
\[ |x_1|^\nu R_j(x_1, x_1 z_1, |x_1|^{-\nu} z_2, |x_1|^{-\nu} z_3), \quad j = 3, 4 \]
with \( x_1 \neq 0 \) close to 0. Because the argument is similar, we consider the case \( j = 1 \). We have, for \( |\alpha| \leq k \)
\[
(4.74) \quad |z'|^{|\alpha|} |\partial_{z'}^{(|\alpha|)(|z'|^{-N} \Psi_{\pm}^1(z))}| = x_1^{-1} |z'|^{|\alpha|} |\partial_{z'}^{(|\alpha|)(|z'|^{-N} R_1(x_1, x_1 z_1, |x_1|^{-\nu} z_2, |x_1|^{-\nu} z_3))}|. \]
By (4.43) we have that, for every positive integer $p$, the term
\[ R_1(x_1, x_1 z_1, |x_1|^{-\nu} z_2, |x_1|^{-\nu} z_3)|z'|^{-p} \]
is smooth at $z = 0$. Because
\[ |z'|^p = (|x_1|^p |x_1|^{-\nu} |z'|)^p = (|x_1|^p |x''|^p), \quad x'' = (x_3, x_4), \]
and $|x''|$ is bounded by the support condition of $R_j$, the negative power $|z'|^{-N}$ in the right-hand side of (4.74) is absorbed by $|z'|^p$ if $p$ is sufficiently large. On the other hand, if the differentiation $\partial^\nu_z$ is applied to $R_1(x_1, x_1 z_1, |x_1|^{-\nu} z_2, |x_1|^{-\nu} z_3)$, then the negative power of $|x_1|$ appears. These terms are also uniformly bounded when $x_1 \to 0$, because there appears positive power of $|x_1|$ from $|z'|^p$. Because all derivatives of $R(x)$ at the origin vanish, we see that the right-hand side of (4.74) can be made arbitrarily small if we cut off $R(x)$ in a sufficiently small neighborhood of the origin. This proves that we have (4.69).

We set $\varphi_{\pm} = QV_{\pm} \in B_{k,N}$, and $\varphi_{\pm}(z) = (\varphi^1_{\pm}(z), \varphi^2_{\pm}(z), \varphi^3_{\pm}(z), \varphi^4_{\pm}(z))$. The function $\varphi_{\pm}$ is a solution of (4.39). Then we define $v^j(x)$ ($j = 1, 2, 3, 4$) by (4.23) and (4.24). For a given integer $m$, we can easily see that $v^j(x)$ is a $C^m$ function if we take $k$ and $N$ in $B_{k,N}$ sufficiently large. If we rewrite (4.39) with the variable $x$, then we see that $v$ is a solution of (4.4), where the nonlinear part $R$ is modified by a cutoff function. In order to show that $v$ is a solution of the original (4.4) we will show the apriori estimate of $v$. Indeed, if $|x + v| < \varepsilon''$ for sufficiently small $\varepsilon''$, then $v$ is a solution of (4.4). By Lemma 4.6 and the uniform estimate of $V_{\pm}$ in $x_1$ we know that $\varphi^1_{\pm}(z)$ is uniformly bounded in $z$ and $x_1$. It follows that $v_1(x) = x_1 \varphi^1_{\pm}$ is arbitrarily small if $x_1$ is sufficiently small. Similarly we can show that $v_2(x) = x_2 \varphi^2$ is small by the estimate of $V_{\pm}$. On the other hand, we have $x_3 + v_3(x) = x_3 + |x_1|^{-\nu} \varphi^3_{\pm}(z)$. Because $\varphi^3_{\pm}$ is divisible by $|z'|^2$ and $|z'| = |x_1|^\nu |x''|$, by Lemma 4.4 we see that $|x_3 + v_3(x)| < \varepsilon''$ uniformly in $x_1$. Similarly we can show the same estimate for $x_4 + v_4$. Therefore we see that $v$ is a solution of (4.4).

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Masafumi YOSHINO
Graduate School of Science
Hiroshima University
Higashi-Hiroshima, 739-8526 (Japan)
yoshino@math.sci.hiroshima-u.ac.jp

Todor GRAMCHEV
Università di Cagliari
Dipartimento di Matematica
via Ospedale 72
09124 Cagliari (Italy)
todor@unica.it