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RIESZ TRANSFORMS ON CONNECTED SUMS

by Gilles CARRON

Abstract. — Assume that $M_0$ is a complete Riemannian manifold with Ricci curvature bounded from below and that $M_0$ satisfies a Sobolev inequality of dimension $\nu > 3$. Let $M$ be a complete Riemannian manifold isometric at infinity to $M_0$ and let $p \in (\nu/(\nu - 1), \nu)$. The boundedness of the Riesz transform of $L^p(M_0)$ then implies the boundedness of the Riesz transform of $L^p(M)$.

Résumé. — Soit $M_0$ une variété riemannienne complète à courbure de Ricci bornée inférieurement et qui vérifie l’inégalité Sobolev de dimension $\nu > 3$. Si $M$ est une variété riemannienne complète isométrique à $M_0$ en dehors d’un compact et si $p \in (\nu/(\nu - 1), \nu)$ alors lorsque la transformée de Riesz est bornée sur $L^p(M_0)$ elle est également bornée sur $L^p(M)$.

1. Introduction

Let $(M, g)$ be a complete Riemannian manifold with infinite volume, we denote by $\Delta = \Delta^g$ its Laplace operator, it has an unique self-adjoint extension on $L^2(M, d\text{vol}_g)$ which is also denoted by $\Delta$. The Green formula and the spectral theorem show that for any $\varphi \in C^\infty_0(M)$:

$$\|d\varphi\|^2_{L^2} = \langle \Delta \varphi, \varphi \rangle = \|\Delta^{1/2} \varphi\|^2_{L^2};$$

hence the Riesz transform $T := d\Delta^{-1/2}$ extends to a bounded operator

$$T : L^2(M) \to L^2(M; T^*M).$$

On the Euclidean space, it is well known that the Riesz transform has also a bounded extension $L^p(M) \to L^p(M; TM)$ for any $p \in ]1, \infty[$. However, this is not a general feature of the Riesz transform on complete Riemannian manifolds, as a matter of fact, on the connected sum of two copies of the Euclidean space $\mathbb{R}^n$, the Riesz transform is not bounded on $L^p$ for any

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\( p \in [n, \infty]\cap [2, \infty] \) (\([7, 5]\)). It is of interest to figure out the range of \( p \) for which \( T \) extends to a bounded map \( L^p(M) \to L^p(M; T^*M) \). The main result of [5] answered to this question for manifolds with Euclidean ends:

**Theorem 1.1.** — Let \( M \) be a complete Riemannian manifold of dimension \( n \geq 3 \) which is the union of a compact part and a finite number of Euclidean ends. Then the Riesz transform is bounded from \( L^p(M) \) to \( L^p(M; T^*M) \) for \( 1 < p < n \), and is unbounded on \( L^p \) for all other values of \( p \) if the number of ends is at least two.

The proof of this result used an asymptotic expansion of the Schwarz kernel of the resolvent \( (\Delta + k^2)^{-1} \) near \( k \to 0 \). In [5] using \( L^p \) cohomology, we also find a criterion which insures that the Riesz transform is unbounded on \( L^p \):

**Theorem 1.2.** — Assume that \((M, g)\) is a complete Riemannian manifold with Ricci curvature bounded from below such that for some \( \nu > 2 \) and \( C > 0 \), \((M, g)\) satisfies the Sobolev inequality

\[
\forall \varphi \in C^\infty_0(M), \|\varphi\|_{L^{2\nu}} \leq C\|d\varphi\|_{L^2}
\]

and

\[
\forall x \in M, \forall r > 1, \ \text{vol} \ B(x, r) \leq Cr^\nu.
\]

If \( M \) has at least two ends, then the Riesz transform is not bounded on \( L^p \) for any \( p \geq \nu \).

Let \((N, g_0)\) be a simply connected nilpotent Lie group of dimension \( n > 2 \) (endowed with a left invariant metric). According to [1] we know that the Riesz transform on \((N, g_0)\) is bounded on \( L^p \) for every \( p \in ]1, \infty[ \). Let \( \nu \) be the homogeneous dimension of \( N \); for instance we can set

\[
\nu = \lim_{R \to \infty} \frac{\log \text{vol} \ B(o, R)}{\log R},
\]

\( o \in N \) being a fixed point. Let \((M, g)\) be a manifold isometric at infinity to \( k \geq 1 \) copies of \((N, g_0)\). That is to say there are compact sets \( K \subset M \) and \( K_0 \subset N \) such that \((M \setminus K, g)\) is isometric to \( k \) copies of \((N \setminus K_0, g_0)\). According to [7] we know that on \((M, g)\) the Riesz transform is bounded on \( L^p \) for \( p \in ]1, 2[ \). And the theorem 1.2 says that the Riesz transform is not bounded on \( L^p \) when \( p \geq \nu \). In [5], we make the following conjecture:

*show that the Riesz transform on \((M, g)\) is bounded on \( L^p \) for \( p \in ]1, \nu[ \).*

The main result of this paper gives a positive answer to this conjecture; in fact we obtain a more general result concerning the boundedness of Riesz transform for connected sums, under some mild geometrical conditions:
Theorem 1.3. — Let $(M_0, g_0)$ be a complete Riemannian manifold, we assume that the Ricci curvature of $(M_0, g_0)$ is bounded from below and that for some $\nu > 3$ and $C > 0$, $(M_0, g_0)$ satisfies the Sobolev inequality
\[
\forall \varphi \in C_0^\infty (M_0), \quad \|\varphi\|_{L^{\frac{2\nu}{\nu - 2}}} \leq C\|\varphi\|_{L^2}.
\]
Let $p \in ]\nu/\(\nu - 1\), \nu[$, if on $(M_0, g_0)$ the Riesz transform is bounded on $L^p$ then the Riesz transform is also bounded on $L^p$ for any manifold $M$ isometric at infinity to several copies of $(M_0, g_0)$.

Moreover under a uniform upper growth control of the volume of geodesic balls (such as (1.1)), the result of [7] implies that under the assumption of the theorem 1.3, the Riesz transform is bounded on $M$ for any $p \in ]1, 2]$ ; hence the restriction of $p > \nu/\(\nu - 1\)$ is not really a serious one. Our method is here less elaborate than the one of [5], its gives a more general result but it is less sharp ; there are two restrictions : the first one is the dimension restriction $\nu > 3$ which is unsatisfactory, and the second concerns the limitation $p < \nu$ which is perhaps also unsatisfactory when $M$ has only one end. However there are recent results of T. Coulhon and N. Dungey in this direction [6].

There is now a long list of complete Riemannian manifolds $(M_0, g_0)$ satisfying our hypothesis and on which the Riesz transform is known to be bounded on $L^p$ for every $p \in ]1, \infty[$. For instance Cartan-Hadamard manifolds with a spectral gap [17], non-compact symmetric spaces [2] and Lie groups of polynomial growth [1], manifolds with nonnegative Ricci curvature and maximal volume growth [3] (see the discussion at the end of the proof of theorem 1.3 about the case of manifolds with nonnegative Ricci curvature and non maximal volume growth). Also H.-Q. Li [16] proved that the Riesz transform on $n$-dimensional cones with compact basis is bounded on $L^p$ for $p < p_0$, where
\[
p_0 = \begin{cases} 
    n\left(\frac{n}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1}\right)^{-1}, & \lambda_1 < n - 1 \\
    +\infty, & \lambda_1 \geq n - 1,
\end{cases}
\]
where $\lambda_1$ is the smallest nonzero eigenvalue of the Laplacian on the basis. Note that $p_0 > n$. Our proof also applies to a manifold isometric at infinity to several copies of cones, hence our theorem 1.3 also gives a partial answer to the open problem 8.1 of [5] :

Corollary 1.4. — If $(M, g)$ is a smooth Riemannian $n-$manifold of dimension $n \geq 4$ with conic ends, then the Riesz transform is bounded on $L^p$ for any $p \in ]1, n[$.
Our manifold \((M_0, g_0)\) is not assumed to be connected, for instance the theorem 1.3 implies that on the connected sum of a hyperbolic space and a euclidean space of dimension \(n > 3\), the Riesz transform is bounded on \(L^p\), for \(p \in \left]\frac{n}{n-1}, n\right]\.

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2. Analytic preliminaries

2.1. A Sobolev inequality

**Proposition 2.1.** — Let \((M, g)\) be a complete Riemannian manifold with Ricci curvature bounded from below then for any \(p \in [1, \infty[\), there is a constant \(C\) such that for all \(\varphi \in C_0^\infty(M)\)

\[
\| df \|_{L^p} \leq C \left[ \| \Delta f \|_{L^p} + \| f \|_{L^p} \right].
\]

**Remark 2.2.**

i) In \([8]\), T. Coulhon and X. Duong have shown that for every complete Riemannian manifolds and any \(p \in [1, 2]\), there is a constant \(C\) such that

\[
\forall f \in C_0^\infty(M), \| df \|_{L^p}^2 \leq C \| \Delta f \|_{L^p} \| f \|_{L^p}.
\]

When \(p \in [1, 2]\), this is clearly a stronger result.

ii) When the injectivity radius is assumed moreover to be positive, this result is due to B. Davies (see corollary 10 in \([10]\)) ; in this setting, another proof along the idea of \([14]\) can be given.

**Proof.** — According to (theorem 4.1 in \([3]\)) we know that if \((M, g)\) is a complete manifold with Ricci curvature bounded from below then for any \(p \in [1, \infty[\) there is a constant \(C\) such that

\[
\forall f \in C_0^\infty(M), \| df \|_{L^p} \leq C \left[ \| \Delta^{1/2} f \|_{L^p} + \| f \|_{L^p} \right].
\]

Then an interpolation argument (see for instance proposition 5.5 in \([15]\)) implies that

\[
\| \Delta^{1/2} f \|_{L^p}^2 \leq \| \Delta f \|_{L^p} \| f \|_{L^p},
\]

the proposition is now straightforward. \(\square\)
2.2. Some estimates on the Poisson operator

Lemma 2.3. — Let $(M, g)$ be a complete Riemannian manifold which for some $\nu > 2$ and $C > 0$ satisfies the Sobolev inequality:
\[
\forall \varphi \in C_0^{\infty}(M), \|\varphi\|_{L^{2\nu/\nu - 2}} \leq C\|d\varphi\|_{L^2},
\]
then the Schwarz kernel $P_\sigma(x, y)$ of the Poisson operator $e^{-\sigma\sqrt{\Delta}}$ satisfies
\[
P_\sigma(x, y) \leq \frac{C\sigma}{(\sigma^2 + d(x, y)^2)^{\frac{\nu}{2}}}.
\]
Moreover if $1 \leq r \leq p \leq +\infty$ then
\[
\left\|e^{-\sigma\sqrt{\Delta}}\right\|_{L^r \to L^p} \leq \frac{C}{\sigma^{\nu(\frac{1}{r} - \frac{1}{p})}}.
\]

We know that the heat operator $e^{-t\Delta}$ and the Poisson operator are related through the subordination identity:
\[
e^{-\sigma\sqrt{\Delta}} = \frac{\sigma}{2\sqrt{\pi}} \int_0^\infty e^{\frac{\sigma^2}{4t}} e^{-t\Delta} \frac{dt}{t^{3/2}}.
\]
Hence these properties follow directly from the corresponding ones for the heat operator $e^{-t\Delta}$ and its Schwarz kernel $H_t(x, y)$:
\[
H_t(x, y) \leq \frac{c}{t^{\frac{1}{2}}} e^{-\frac{d(x, y)^2}{4t}}.
\]
and if $1 \leq r \leq p \leq +\infty$ then
\[
\left\|e^{-t\Delta}\right\|_{L^r \to L^p} \leq \frac{C}{t^{\frac{1}{2}(\frac{1}{r} - \frac{1}{p})}},
\]
which are consequences of the Sobolev inequality [18, 9].

We will also need an estimate for the derivative of the Poisson kernel:

Lemma 2.4. — Under the assumptions of lemma (2.3), let $\Omega \subset M$ be an open subset and $K$ be a compact set in the interior of $M \setminus \Omega$ then
\[
\left\|e^{-\sigma\sqrt{\Delta}}\right\|_{L^p(\Omega) \to L^\infty(K)} \leq \frac{C}{(1 + \sigma)^{\nu/p}},
\]
\[
\left\|\nabla e^{-\sigma\sqrt{\Delta}}\right\|_{L^p(\Omega) \to L^\infty(K)} \leq \frac{C}{(1 + \sigma)^{\nu/p}}.
\]

Proof. — The first estimate is only a consequence of the lemma 2.3 because by assumption there is a constant $\varepsilon > 0$ such that
\[
(x, y) \in K \times \Omega \Rightarrow d(x, y) \geq \varepsilon.
\]
To prove the second inequality, we will again only show the corresponding estimate for the heat operator. First, according to the local Harnack
inequality (see V.4.2 in [9]), there is a constant $C$ such that for any $x \in K$, $t \in [0, 1]$ and $y \in M$:

$$\tag{2.3} |\nabla_x H_t(x, y)| \leq C \sqrt{t} H_{2t}(x, y).$$

But hence by (2.2) and (2.1), we get: for all $(x, y) \in K \times \Omega$ then

$$H_{2t}(x, y) \leq \frac{c}{t^{\nu/2}} e^{-\frac{t^2}{10t}}.$$

It follows easily that there is a certain constant $C$ such that

$$\forall t \in [0, 1] : \| \nabla e^{-t\Delta} \|_{L^p(\Omega) \to L^\infty(K)} \leq C.$$

Now assume that $t > 1$:

$$\| \nabla e^{-t\Delta} \|_{L^p(\Omega) \to L^\infty(K)} \leq \sup_x \int_M |\nabla_x H_{1/2}(x, y)| dy \leq C \sup_x \int_M H_1(x, y) dy \leq C.$$

Hence for all $t > 0$, we obtain

$$\| \nabla e^{-t\Delta} \|_{L^p(\Omega) \to L^\infty(K)} \leq \frac{C}{(1 + t)^{\frac{\nu}{2}}}$$

and the second estimate follows from the subordination identity. \(\square\)

### 3. Proof of the main theorem

Let $(M_0, g_0)$ be a complete Riemannian manifold, we assume that the Ricci curvature of $(M_0, g_0)$ is bounded from below and that for some $\nu > 3$ and $C > 0$, that $(M, g)$ satisfies the Sobolev inequality

$$\forall \phi \in C^\infty_0(M_0), \| \phi \|_{L^{2p}} \leq C \| d\phi \|_{L^2}.$$

We assume that on $(M_0, g_0)$ the Riesz transform is bounded on $L^p$ for some $p \in [\nu/(\nu - 1), \nu[$. And we consider $M$ a complete Riemannian manifold such that outside compact sets $K \subset M$ and $K_0 \subset M_0$, $M \setminus K$ is isometric to $M_0 \setminus K_0$, the case where $M \setminus K$ is isometric to several copies of $M_0 \setminus K_0$ can

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be done similarly by considering the disjoint union of \( k \) copies of \( M_0 \). We are going to prove that on \( M \) the Riesz transform is also bounded on \( L^p \).

The first step is to build a good parametrix for the Poisson operator on \( M \). The first problem is that the operator \( \sqrt{\Delta} \) is not a differential operator, we circumvent these difficulties by working on \( \mathbb{R}_+ \times M \). As a matter of fact, the Poisson operator solves the Dirichlet problem:

\[
\begin{align*}
\left( -\frac{\partial^2}{\partial \sigma^2} + \Delta \right) u(\sigma, x) &= 0 \quad \text{on } ]0, \infty[ \times M \\
u(0, x) &= u(x) \\
\lim_{\sigma \to \infty} u(\sigma, .) &= 0.
\end{align*}
\]

The construction of the parametrix will be standard, the non locality nature of the operator \( \sqrt{\Delta} \) implies that we cannot use the Duhamel formula, instead we used the Green operator. The idea is to find \( E_\sigma(u) \) an approximate solution for (3.1) and then to use the fact that, if \( G \) is the Green operator of the operator \( -\frac{\partial^2}{\partial \sigma^2} + \Delta \) for the Dirichlet boundary condition, then

\[
e^{-\sigma \sqrt{\Delta}} u = E_\sigma(u) - G \left( -\frac{\partial^2}{\partial \sigma^2} + \Delta \right) E_\sigma(u).
\]

### 3.1. The parametrix construction

Let \( \tilde{K} \) be another compact set in \( M \) containing \( K \) in its interior. We identify

\[\Omega = M \setminus K = M_0 \setminus K_0.\]

Let \( \rho_0, \rho_1 \) a smooth partition of unity such that

\[\text{supp } \rho_0 \subset \Omega \quad \text{and} \quad \text{supp } \rho_1 \subset \tilde{K},\]

let also \( \varphi_0, \varphi_1 \) be smooth functions, such that

\[\text{supp } \varphi_0 \subset \Omega \quad \text{and} \quad \text{supp } \varphi_1 \subset \tilde{K}\]

Moreover we require that \( \varphi_i = 1 \) on a neighborhood of the support of \( \rho_i \) so that we have:

\[\varphi_i \rho_i = \rho_i.\]

Let \( \Delta_1 \) be the realization of the Laplace operator on \( \tilde{K} \) for the Dirichlet boundary condition and let \( \Delta_0 \) be the Laplace operator on \( M_0 \). Let \( e^{-\sigma \sqrt{\Delta_i}} \) their associated Poisson operator then we define for \( u \in L^p(M) \):

\[
E_\sigma(u) = \sum_{i=0}^1 \varphi_i(e^{-\sigma \sqrt{\Delta_i}} \rho_i u),
\]
where we think of $\rho_0 u$ as a function on $\Omega \subset M_0$ and of $\varphi_0(e^{-\sigma\sqrt{\Delta}}\rho_0 u)$ as a function on $\Omega \subset M$.

We can easily compute:

$$
\left(-\frac{\partial^2}{\partial \sigma^2} + \Delta\right) E_\sigma(u) = \sum_{i=0}^{\infty} [\Delta, \varphi_i](e^{-\sigma\sqrt{\Delta}}\rho_i u) = f(\sigma, x) = \sum_{i=0}^{\infty} f_i(\sigma, x),
$$

where

$$
(3.2) \quad f_i(\sigma, x) = [\Delta, \varphi_i](e^{-\sigma\sqrt{\Delta}}\rho_i u)(x)
= \Delta \varphi_i(x)(e^{-\sigma\sqrt{\Delta}}\rho_i u)(x) - 2 \left\langle d\varphi_i(x), d(e^{-\sigma\sqrt{\Delta}}\rho_i u)(x) \right\rangle.
$$

From lemma 2.4 and the fact that the support of $d\varphi_0$ and $\rho_0$ are disjoint, we easily get that for all $\sigma \geq 0$:

$$
(3.3) \quad \|f_0(\sigma)\|_{L^1} + \|f_0(\sigma)\|_{L^p} \leq \frac{C}{(1 + \sigma)^\nu/p} \|\rho_0 u\|_{L^p}.
$$

Let us explain why this estimate also holds for $f_i$. Note that the operator

$$
S(\sigma) = [\Delta, \varphi_1]e^{-\sigma\sqrt{\Delta}}\rho_1
$$

is an operator with smooth Schwarz kernel and compact support, moreover because the corresponding estimate of the lemma (2.4) also holds for $\sigma \in [0, 1]$ on a compact manifold, the Schwarz kernel of $S(\sigma)$ is uniformly bounded when $\sigma \to 0$. Hence there is a constant $C$ such that

$$
\forall \sigma \in [0, 1], \|S(\sigma)u\|_{L^\infty} \leq C\|\rho_1 u\|_{L^p}.
$$

Now the operator $\Delta_1$ has a spectral gap on $L^p$ (its $L^p$ spectrum is also its $L^2$ spectrum), hence there is a constant $C$ such that for all $\sigma \geq 0$ then

$$
\|e^{-\sigma\sqrt{\Delta}}\|_{L^p \to L^p} \leq Ce^{-\sigma/C}.
$$

Hence for $\sigma \geq 1$:

$$
\|S(\sigma)u\|_{L^\infty} \leq \|[\Delta, \varphi_1]e^{\frac{1}{2}\sqrt{\Delta}}\|_{L^p \to L^\infty} \|e^{-(\sigma-1/2)\sqrt{\Delta}}\rho_1 u\|_{L^p} \leq Ce^{-\sigma/C}\|\rho_1 u\|_{L^p}.
$$

The result follows by noticing that the $f_i$'s have compact support in $\tilde{K} \setminus K$.

Eventually we obtain the estimate:

**Lemma 3.1.** — When $u \in L^p(M)$ and we define an operator $S_\sigma$ by $S_\sigma u = f = f_0 + f_1$ where $f_0, f_1$ are defined by (3.2) then

$$
\forall \sigma \geq 0, \|S_\sigma(u)\|_{L^1} + \|S_\sigma(u)\|_{L^p} \leq \frac{C}{(1 + \sigma)^\nu/p} \|u\|_{L^p}.
$$
3.2. The Riesz transform on $M$

We introduce now $G$, the Green operator of the operator $\left( -\frac{\partial^2}{\partial \sigma^2} + \Delta \right)$ on $\mathbb{R}_+ \times M$ for the Dirichlet boundary condition. Its Schwarz kernel is given by

$$G(\sigma, s, x, y) = \int_0^\infty \left[ e^{-\frac{(\sigma-s)^2}{4t}} - e^{-\frac{(\sigma+s)^2}{4t}} \right] \frac{1}{\sqrt{4\pi t}} H_t(x, y) dt$$

where $H_t$ is the heat kernel on $M$ and

$$e^{-\frac{(\sigma-s)^2}{4t}} - e^{-\frac{(\sigma+s)^2}{4t}}$$

the heat kernel on the half-line $\mathbb{R}_+$ for the Dirichlet boundary condition.

We have

$$e^{-\sigma \sqrt{\Delta}} u = E_\sigma(u) - G(S_\sigma(u)).$$

Hence

$$\Delta^{-1/2} u = \int_0^\infty e^{-\sigma \sqrt{\Delta}} u d\sigma = \sum_{i=0}^{1} \varphi_i \Delta_i^{-1/2} \rho_i u$$

\[ - \int_{\mathbb{R}_+^2 \times M} G(\sigma, s, x, y) f(s, y) d\sigma ds dy. \]

Let

$$g(x) = \int_{\mathbb{R}_+^2 \times M} G(\sigma, s, x, y) f(s, y) d\sigma ds dy$$

then we have

$$\Delta^{-1/2} u = \sum_{i=0}^{1} \varphi_i \Delta_i^{-1/2} \rho_i u - g.$$  

But

$$\int_0^\infty G(\sigma, s, x, y) d\sigma = \frac{1}{\sqrt{4\pi}} \int_0^\infty \left[ \int_0^s e^{-\frac{v^2}{4t}} dv \right] H_t(x, y) \frac{dt}{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-r^2} \left[ \int_0^{4t^2} H_t(x, y) dt \right] dr.$$  

It follows from the above computation that

$$g(x) = \frac{2}{\sqrt{\pi}} \int_{\mathbb{R}_+^2} e^{-r^2} \left[ \int_0^{4t^2} (e^{-t\Delta} f(s))(x) dt \right] dr ds.$$

The following lemma is now the last crucial estimate: 

\[ \text{TOME 57 (2007), FASCICULE 7} \]
Lemma 3.2. — There is a constant $C$ such that
\[ \| \Delta g \|_{L^p} + \| g \|_{L^p} \leq C \| u \|_{L^p}. \]

Proof. — Recall that according to [4], $(M, g)$ itself satisfies the same Sobolev inequality:
\[ \forall \varphi \in C_0^\infty(M), \| \varphi \|_{L^{2\nu}} \leq C \| d\varphi \|_{L^2}. \]

Hence the heat operator satisfies the following mapping properties: for $1 \leq q \leq p \leq +\infty$ we have
\[ \| e^{-t\Delta} \|_{L^q \to L^p} \leq \frac{C}{t^{\frac{\nu}{2}(\frac{1}{q} - \frac{1}{p})}}. \]

As a consequence, for all $t \in [0, 1]$, then
\[ \| (e^{-t\Delta} f(s)) \|_{L^p} \leq \| f(s) \|_{L^p} \leq \frac{C}{(1 + s)^{\nu/p}} \| u \|_{L^p} \]
and if $t > 1$, then
\[ \| (e^{-t\Delta} f(s)) \|_{L^p} \leq \| e^{-t\Delta} \|_{L^1 \to L^p} \| f(s) \|_{L^1} \leq \frac{1}{t^{\frac{\nu}{2}(1 - \frac{1}{p})}} \frac{C}{(1 + s)^{\nu/p}} \| u \|_{L^p}. \]

Hence
\[ g \|_{L^p} \leq \frac{2}{\sqrt{\pi}} \int_{\mathbb{R}_+^2} e^{-r^2} \left[ \int_0^{2\pi} \| (e^{-t\Delta} f(s)) \|_{L^p} dt \right] dsdr \]
\[ \leq \frac{2}{\sqrt{\pi}} \left( \int_{\mathbb{R}_+^2} e^{-r^2} \left[ \int_0^{2\pi} \frac{C}{\max \left( 1, t^{\frac{\nu}{2}(1 - \frac{1}{p})} \right)} (1 + 2r^2)^{\nu/p - 1} dt \right] dsdr \right) \| u \|_{L^p}. \]

But because $p < \nu$, we have
\[ \int_{\{2r\sqrt{t} \leq s\}} e^{-r^2} \frac{1}{\max \left( 1, t^{\frac{\nu}{2}(1 - \frac{1}{p})} \right)} (1 + s)^{\nu/p} dsdt \]
\[ = \frac{\nu}{\nu - p} \int_{\mathbb{R}_+^2} e^{-r^2} \frac{1}{\max \left( 1, t^{\frac{\nu}{2}(1 - \frac{1}{p})} \right)} (1 + 2r^2)^{\nu/p - 1} dt \]
and this integral is finite exactly when $p > \nu/(\nu - 1)$ and $\nu > 3$. 

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It remains to estimate $\|\Delta g\|_{L^p}$, which is easier because
\[
\Delta g = \frac{2}{\sqrt{\pi}} \int_{\mathbb{R}^2_+} e^{-r^2} \left[ \int_0^{\frac{4\pi^2}{s^2}} \Delta (e^{-t\Delta} f(s)) \, dt \right] \, dr \, ds
\]
\[
= -\frac{2}{\sqrt{\pi}} \int_{\mathbb{R}^2_+} e^{-r^2} \left[ \int_0^{\frac{4\pi^2}{s^2}} \frac{d}{dt} (e^{-t\Delta} f(s)) \, dt \right] \, dr \, ds
\]
\[
= \frac{2}{\sqrt{\pi}} \int_{\mathbb{R}^2_+} e^{-r^2} \left[ f(s) - (e^{-\frac{s^2}{4\pi^2}} \Delta f(s)) \right] \, dr \, ds.
\]
Hence
\[
\|\Delta g\|_{L^p} \leq \frac{4}{\sqrt{\pi}} \int_{\mathbb{R}^2_+} e^{-r^2} \|f(s)\|_{L^p} \, dr \, ds
\]
\[
\leq \frac{4}{\sqrt{\pi}} \left( \int_{\mathbb{R}^2_+} e^{-r^2} \frac{C}{(1 + s)^{\nu/p}} \, dr \, ds \right) \|u\|_{L^p}.
\]

Now we can finish the proof of the main theorem: let $T_i$ be the Riesz transform associated with the operator $\Delta_i$. With the formula (3.4), we obtain
\[
d\Delta^{-1/2} u = \sum_{i=0}^1 \varphi_i T_i \rho_i u + \sum_{i=0}^1 d\varphi_i (\Delta_i^{-1/2} \rho_i u) - dg.
\]
By hypothesis, $T_0$ is bounded on $L^p$. Moreover since $\varphi_1 T_1 \rho_1$ is a pseudo differential operator of order 0 with compact support it is also bounded on $L^p$. The operator $d\varphi_1 (\Delta_i^{-1/2} \rho_i u)$ has a smooth kernel with compact support, hence it is bounded on $L^p$. Moreover, the Sobolev inequality
\[
\forall \varphi \in C_0^\infty (M_0), \quad \|\varphi\|_{L^{\frac{2p}{s+2}}} \leq C \|d\varphi\|_{L^2}.
\]
also implies the following mapping properties of the $\Delta_0^{-1/2}$ ([18]):
\[
\left\| \Delta_0^{-1/2} \right\|_{L^p \to L^{\frac{p\nu}{p-\nu}}} \leq C.
\]
Hence
\[
\left\| d\varphi_0 (\Delta_0^{-1/2} \rho_0 u) \right\|_{L^p} \leq C \|\Delta_0^{-1/2} \rho_0 u\|_{L^p(K \setminus K)} \leq C' \|\Delta_0^{-1/2} \rho_0 u\|_{L^{\frac{p\nu}{p-\nu}}(\tilde{K})}
\]
\[
\leq C' \|\rho_0 u\|_{L^p}.
\]
Moreover the lemmas (3.2) and (2.1) imply that
\[
\|dg\|_{L^p} \leq C \|u\|_{L^p}.
\]
All these estimates yield the fact that the Riesz transform is bounded on $L^p$. 

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3.3. A comment on manifolds with non negative Ricci curvature

The proof of theorem 1.3 is fairly general, we can easily make a list of the properties which makes it runs ; let \((M_i, g_i) \ i = 1, \ldots, b\) be complete Riemannian manifolds and let \((M, g)\) be isometric at infinity to the disjoint union \(M_1 \cup \ldots \cup M_b\). That is to say there are compact sets \(K \subset M, K_i \subset M_i\) such that \(M \setminus K\) is isometric to \((M_1 \setminus K_1) \cup \ldots \cup (M_b \setminus K_b)\). Let \(\hat{K} \subset K\) such that \(\hat{K}\) (resp. \(\hat{K}\)) contains \(K\) in its interior (resp. \(\hat{K}\)). And let \(\hat{K}, \hat{K}_i \subset M_i\) such that :

\[
\forall i, M \setminus \hat{K}_i \simeq (M_1 \setminus \hat{K}_1) \cup \ldots \cup (M_b \setminus \hat{K}_b),
\]

let \(\Delta_i\) be the Laplace operator on \(M_i\). We assume that on each \(M_i\), the Ricci curvature is bounded from below such that on each \(M_i\) and \(M\), we get the estimate induced by the Sobolev inequality \((2.1)\). Assume that for some functions \(f, g : \mathbb{R}_+ \to \mathbb{R}_+^*\) we have the estimate :

\[
\|e^{-\sigma \sqrt{\Delta_i}}\|_{L^p(M_i \setminus \hat{K}_i) \to L^\infty(\hat{K}_i)} + \|\nabla e^{-\sigma \sqrt{\Delta_i}}\|_{L^p(M_i \setminus \hat{K}_i) \to L^\infty(\hat{K}_i)} \leq \frac{1}{f(\sigma)},
\]

and that on the manifold \(M\) :

\[
\|e^{-t \Delta}\|_{L^1(\hat{K}) \to L^p(M)} \leq \frac{1}{g(t)}.
\]

with

\[
\int_0^\infty \frac{ds}{f(s)} < \infty
\]

\[
\int_{\mathbb{R}^2_+} e^{-u^2} \min \left(1, \frac{1}{g(t)}\right) \left[\int_0^\infty \frac{ds}{f(s)}\right] \, du \, dt < \infty.
\]

Then if for all \(i\), the Riesz transform \(T_i := d \Delta_i^{-1/2}\) is bounded on \(L^p\), then on \(M\), the Riesz transform is also bounded on \(L^p\).

A natural and well study class of manifolds satisfying such estimates are manifolds satisfying the so called relative Faber-Krahn inequality : for some \(\alpha > 0\) and \(c > 0\), we have :

\[
\forall B(x, R), \forall \Omega \subset B(x, R), \lambda_1(\Omega) \geq \frac{c}{R^2} \left(\frac{\text{vol } \Omega}{\text{vol } B(x, R)}\right)^{-\alpha}
\]

where

\[
\lambda_1(\Omega) = \inf_{f \in C_c^\infty(\Omega)} \frac{\int_\Omega |df|^2}{\int_\Omega f^2}
\]

is the first eigenvalue of the Laplace operator on \(\Omega\) for the Dirichlet boundary condition. According to A. Grigor’yan [11] this inequality is equivalent
to the conjunction of the doubling property: uniformly in $x$ and $R > 0$ we have

$$\frac{\text{vol } B(x, 2R)}{\text{vol } B(x, R)} \leq C$$

and of the upper bound on the heat operator

$$H_t(x, y) \leq \frac{C}{\text{vol } B(x, \sqrt{t})} e^{-\frac{d(x, y)^2}{4t}}.$$

Manifolds with non-negative Ricci curvature are examples of manifolds satisfying this relative Faber-Krahn inequalities.

Assume that each $M_i$ satisfies this relative Faber-Krahn inequality and if we assume that for $i = 1, \ldots, b$, there is a point $o_i \in K_i$ and all $R \geq 1$

$$\text{vol } B(o_i, R) := V_i(R) \geq CR^\nu$$

then we get easily from the subordination identity:

$$\left\| e^{-\sigma \sqrt{\Delta}} \right\|_{L^p(M_i \setminus \hat{K}_i) \rightarrow L^\infty(\hat{K}_i)} + \left\| \nabla e^{-\sigma \sqrt{\Delta}} \right\|_{L^p(M_i \setminus \hat{K}_i) \rightarrow L^\infty(\hat{K}_i)} \leq \frac{1}{(1 + \sigma)^{\nu/p}}.$$

Now the problem comes from the fact that we don’t know how to obtain a relative Faber-Krahn inequality on $M$ from the one we assume on the $M_i$’s. However, recently in (page 877 of [13]), A. Grigor’yan and L. Saloff-Coste have announced the following very useful result (see also [12]): when the $M_i$’s satisfy the relative Faber-Krahn inequality then

$$\forall B(x, R) \subset M, \forall \Omega \subset B(x, R), \quad \lambda_1(\Omega) \geq \frac{c}{R^2} \left( \frac{\text{vol } \Omega}{\mu(x, R)} \right)^{\alpha},$$

where

$$\mu(x, R) = \begin{cases} 
\text{vol } B(x, R) & \text{if } B(x, R) \subset M \setminus K \\
\inf_i V_i(R) & \text{else.}
\end{cases}$$

Hence from our volume growth estimate, we will obtain (see [11]) when $t \geq 1$:

$$\left\| e^{-t \Delta} \right\|_{L^1(\hat{K}) \rightarrow L^p(M)} \leq \frac{C}{t^{\frac{p}{2} - \frac{1}{p}}}.$$

With this result of A. Grigor’yan and L. Saloff-Coste and with the result of D. Bakry [3], we will obtain:

**Proposition 3.3.** — Let $(M_1, g_1), \ldots, (M, g_b)$ be complete Riemannian manifolds with non negative Ricci curvature. Assume that on all $M_i$’s we have the volume growth lower bound:

$$\text{vol } B(o_i, R) \geq CR^\nu.$$
Then assume that $\nu > 3$ and $p \in ]\nu/(\nu - 1), \nu[$ then on any manifold isometric at infinity to the disjoint union of the $M'_i$'s, the Riesz transform is bounded on $L^p$.

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